AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **40** (2008), Pages 115–136

Parity and tiling by trominoes

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Abstract

The problem of counting tilings by dominoes and other dimers and finding arithmetic significance in these numbers has received considerable attention. In contrast, little attention has been paid to the number of tilings by more complex shapes. In this paper, we consider tilings by trominoes and the parity of the number of tilings. We mostly consider reptilings and tilings of related shapes by the L tromino. We were led to this by revisiting a theorem of Hochberg and Reid (*Discrete Math.* 214 (2000), 255–261) about tiling with d-dimensional notched cubes, for $d \geq 3$; the L tromino is the 2-dimensional notched cube. We conjecture that the number of tilings of a region shaped like an L tromino, but scaled by a factor of n, is odd if and only if n is a power of 2. More generally, we conjecture the the number of tilings of a region obtained by scaling an L tromino by a factor of m in the x direction and a factor of n in the y direction, is odd if and only if $m = n$ and the common value is a power of 2. The conjecture is proved for odd values of m and n, and also for several small even values. In the final section, we briefly consider tilings by other shapes.

1 Introduction

In this paper, we consider the number of tilings of certain regions by L trominoes, and try to understand the parity of this number. The regions we consider are geometrically similar to an L tromino, but enlarged. More generally, we consider regions formed by scaling an L tromino by one factor along the x-axis and by another factor along the y-axis. We are led to consider this by examining an earlier result of Hochberg and Reid ([2], Theorem 2). It is conceivable that there are other regions for which the number of tilings has interesting arithmetic significance; however, we do not consider this here. The corresponding scenario for domino and other dimer tilings has received considerable attention, as will be discussed below.

We now introduce some standard terminology that we will use throughout. A *self-replicating tile* (or *reptile*, for short) is a figure that can tile a larger shape similar to itself. Such a tiling is called a *reptiling*, or an N*-reptiling* if it uses N tiles. In such a case, we say that the tile is *rep-*N. A well-known example is shown, which illustrates the terminology.

Example 1.1. The L tromino.

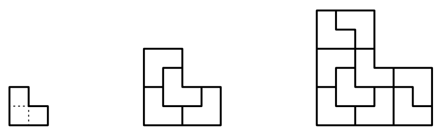


Figure 1.2. The L tromino, a 4-reptiling and a 9-reptiling.

Suppose we have a reptile, and a fixed reptiling by that shape. Given any tiling by the shape, we may "inflate" the tiling by the reptiling, as follows. First, scale the tiling by the ratio of similitude of the reptiling. Then replace each enlarged tile by the given reptiling. The result is a tiling of a figure that is similar to the figure of the original tiling. An important special case of this is when the tiling is also a reptiling; in that case, the inflated tiling is again a reptiling.

Example 1.3.

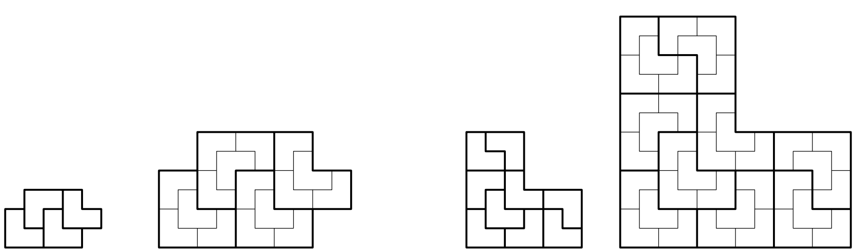


Figure 1.4. Inflating tilings by the 4-reptiling of Figure 1.2.

Thus the notion of inflation of one reptiling by another defines a binary operation, called *composition*, on the set of all reptilings by a fixed tile. It is easy to show that composition of reptilings is associative, but in general it is not commutative. The volume of a d-dimensional reptiling, with ratio of similitude r, is r^d times the volume of the tile. In particular, r^d, which is the number of tiles used, must be an integer. In this paper, we will mainly consider polyominoes (in 2 and higher dimensions), and for such tiles, the ratio of similitude is the quotient of two edge lengths, so is rational. Since its d-th power is an integer, the ratio of similitude must be an integer. Thus we need only consider n^d-reptilings, where d is the dimension of the tile.

The L tromino of Figure 1.2 above has a natural generalization to a d-dimensional tile, which we call a “notched cube”. Examples are shown for 2 and 3 dimensions.

$d = 2$ $\qquad$ $d = 3$

Figure 1.5. Notched cubes.

Notched cubes are considered in some detail in [2]. For all d, the d-dimensional notched cube has a 2^d-reptiling, which is more or less an obvious generalization of the reptiling in Figure 1.2. It then follows, by inflation, that it is rep-2^{kd} for all positive k. Far less obvious is the fact that (for $d \geq 3$) it is not rep-N for any other value of N. Moreover, Hochberg and Reid prove the following.

Theorem 1.6. (Hochberg and Reid [2], Theorem 2) *Suppose $d \geq 3$. If the d-dimensional notched cube has an m^d-reptiling, then m is a power of 2, and the reptiling is unique; it is the repeated composition of the basic 2^d-reptiling with itself.*

This result shows that, for $d \geq 3$, although the notched cube is a reptile, it is barely so, in that it possesses as few reptilings as possible: one minimal reptiling, and all compositions of that reptiling with itself. Because these are the only reptilings, this is an example of a shape for which the operation of composition of reptilings is commutative.

2 Conjectures

Let $d \geq 2$, and consider reptilings of the d-dimensional notched cube. As noted above, the ratio of similitude for a reptiling must be an integer, n, so the reptiling is a n^d-reptiling. For $n \geq 1$, let $R_d(n)$ denote the number of n^d-reptilings of the d-dimensional notched cube. With this notation, Theorem 1.6 above has a simple reformulation.

Theorem 2.1. *If $d \geq 3$, then* $R_d(n) = \begin{cases} 1 & \text{if } n \text{ is a power of } 2, \\ 0 & \text{otherwise.} \end{cases}$

As noted in [2], this result does not hold for $d = 2$, but it suggests consideration of the function $R(n) = R_2(n)$. Some values of $R(n)$ are shown in Table 2.2.

n	$R(n)$
1	1
2	1
3	4
4	409
5	108388
6	104574902
7	608850350072
8	19464993703121249
9	3058588688924405306744
10	2667688636188332437795588320
11	11779489664021227770290904703308312
12	279652174829276379737422154227421710684110
13	34733463898947150523805900066147780144341745331036
14	22720195678464510908686881688825214686704062736465051670450
15	7844489704990260645554878546109787 7046139992379383750651894344403136
16	1424978701125400427681685418503575427223 371410038937838057196135692816195590721
17	136393199026254212596712869169517361069408596 073683723879873796407144984113947789743716028
18	687837885676813402518932012838940274043564044318508 53693591538655067859513438824683463023133608135932
19	182680551514571989252472166264511772373870530542851697215 904119389883749080595647067236655224960662434046555666212

Table 2.2. Values of $R(n)$.

We remark that, at the time of writing, the sequence $1, 1, 4, 409, 108388, \ldots$ is not in the Encyclopedia of Integer Sequences [12], although we expect that status to change. Without further ado, we make the following conjecture.

Conjecture A. *$R_d(n)$ is odd if and only if n is a power of 2.*

The conjecture is true in $d \geq 3$ dimensions, so the remaining question is what happens in 2 dimensions.

We can utilize symmetry of the L tromino to pair up most of its reptilings. Given an n^2-reptiling, reflect it over the axis of symmetry of the region to get another n^2-reptiling. Only those reptilings that are symmetric do not get paired up. Accordingly, let $S(n)$ denote the number of symmetric n^2-reptilings of the L tromino.

Conjecture A'. *$S(n)$ is odd if and only if n is a power of 2.*

The preceding discussion shows that $R(n)$ and $S(n)$ have the same parity, so that Conjecture A' is equivalent to Conjecture A. Some values of $S(n)$ are shown.

n	$S(n)$
1	1
2	1
3	2
4	1
5	38
6	240
7	11536
8	1003499
9	186338372
10	80417382822
11	77271184273892
12	171787394401053106
13	874293316752182144666
14	10213210340141048167584498
15	273982951274411338241538348532
16	16862661807587072571123221812023233
17	2382176902989403164248265067315724864806
18	772445362142597099327396850933337596231352746
19	574816740286855552790285853921172382090309084261590
20	981804458521661511443845747259580873344975040903910566832
21	3848771278006007406986505278503065923641515299969767023671119088

Table 2.3. Values of $S(n)$.

The sequence $1, 1, 2, 1, 38, 240, \ldots$ is also not in the Encyclopedia of Integer Sequences [12] at the present time.

We now consider tilings of "stretched" notched cubes by notched cubes. For positive integers $a_1, a_2, \ldots, a_d$, let $L_{a_1,\ldots,a_d}$ denote the region obtained by starting with a d-dimensional notched cube, and for each i, stretching it by a factor of a_i along the i-th coordinate axis. Let $T_d(a_1, a_2, \ldots, a_d)$ denote the number of tilings of $L_{a_1,\ldots,a_d}$ by notched cubes. In $d \geq 3$ dimensions, we can count the number of such tilings.

Theorem 2.4. *Suppose $d \geq 3$. Then $T_d(a_1, a_2, \ldots, a_d) = 0$ unless $a_1 = a_2 = \cdots = a_d$ and the common value is a power of 2, in which case $T_d(a_1, a_2, \ldots, a_d) = 1$.*

Proof. Suppose that $T_d(a_1, a_2, \ldots, a_d) > 0$. Since the notched cube tiles an orthant, this tiling can be stretched to give a tiling of the orthant by $L_{a_1,\ldots,a_d}$. By replacing each $L_{a_1,\ldots,a_d}$ with its tiling by notched cubes, we get a tiling of the orthant by notched cubes. This shows that the tiling of $L_{a_1,\ldots,a_d}$ can be placed in the corner of the orthant, and then extended to a tiling of the entire orthant by notched cubes. However, Hochberg and Reid show that, for $d \geq 3$, there is a unique tiling of the orthant by notched cubes ([2], Theorem 1). This tiling is formed by placing a single tile in the corner of the orthant, oriented so that its notch is opposite the corner, and then repeatedly inflating by the basic 2^d-reptiling. For the positive orthant, this tiling has the property that, for each n, there is a notched cube with vertex at $(n, n, \ldots, n)$, oriented with its notch diametrically opposite the corner of the orthant. (Each such tile sits in the notch of the previous one.) Consider the first of these that is not contained in the tiling of $L_{a_1,\ldots,a_d}$; suppose its vertex is at $(n, n, \ldots, n)$. Since $L_{a_1,\ldots,a_d}$ contains the previous tile, it contains all but 1 of the 2^d cells with a vertex at $(n, n, \ldots, n)$. Therefore this point must be the corner of the notch of $L_{a_1,\ldots,a_d}$, so that each $a_i = n$. We now have $T_d(a_1, a_2, \ldots, a_d) = R_d(n)$, which is either 1 or 0, depending on whether n is a power of 2 or not. □

In 2 dimensions, the number of such tilings is more complicated. As with Conjecture A, we expect that the parity of the number of tilings in 2 dimensions is consistent with the behavior in higher dimensions. Let $T(m, n)$ denote $T_2(m, n)$; we then make the following conjecture.

Conjecture B. *$T(m, n)$ is odd if and only if $m = n$ and the common value is a power of 2.*

Note that Conjecture B implies Conjecture A, and therefore also Conjecture A', because $R(n) = T(n, n)$. Some values of $T(m, n)$ have been computed and appear in the appendix.

Some comments are appropriate here. Kasteleyn [4], Temperley and Fisher [15], and Percus [9], gave methods for counting domino tilings, using Pfaffians and determinants. These results have spawned considerable interest in finding arithmetic information in the number of dimer tilings of regions; for example, see [1, 3, 6, 8, 16]. See also Propp [10] for a broad overview. It is generally recognized that the techniques of Kasteleyn and others are particular to dimer tilings and do not generalize to tilings by other shapes. This perhaps accounts for the comparative lack of attention given to counts of tilings by other shapes.

We find it surprising to observe some arithmetic information in counts of tilings by trominoes, even if only conjecturally. Moreover, we will see below that our conjectures have a direct implication for counting tilings by another shape.

There is at least one previously known situation in which arithmetic information occurs in the counts of tilings by other shapes. This comes from the "transfer matrix" method of counting the number of tilings of a rectangle of fixed width and varying length, possibly with appendages on either end. For example, see [5], [7], [11] (Prop. 2.1), [13], [14] (Section 4.7), as well as Proposition 3.10 below.

3 Results

In this section, we present partial results in support of the conjectures of the previous section. We continue to use the notation $L_{m,n}$ to denote the "stretched" L tromino.

Lemma 3.1. *If R is a polyomino region, then the number of tilings of R by L trominoes that contain a subrectangle is even.*

Proof. We will exhibit a pairing on the set of tilings that contain a subrectangle. Consider the smallest (in area) subrectangle. Of these, consider the one with highest top edge, and of those, the one with leftmost left edge. There is exactly one such of these; for if there were two, their intersection would be a smaller subrectangle.

Reflect the tiling of this subrectangle vertically. Note that its tiling cannot be invariant under this reflection, because if so, it would necessarily be composed of two symmetric halves which would be smaller subrectangles.

We claim that the new tiling has exactly the same set of subrectangles as the original tiling. For if there was a new one, it would necessarily intersect the rectangle we just flipped, which would give a smaller subrectangle that would also be in the original tiling, which is a contradiction. Therefore, applying the same procedure to the new tiling flips the same subrectangle, which returns us to the original tiling. Thus we have a true pairing on the set of tilings that contain a subrectangle. □

Proposition 3.2. (a) *Conjecture B holds if m or n is odd.*
(b) *Conjectures A and A' hold if n is odd.*

Proof. (a) By symmetry, we may assume m is odd. For $m = 1$, we have $T(1,1) = 1$ and $T(1,n) = 0$ for $n > 1$, so Conjecture B holds, and similarly, it holds if $n = 1$. Now we may assume that $m, n > 1$ and m is odd. Consider the m squares in the top row of $L_{m,n}$. Some L tromino must cover an odd number of these, so it covers exactly 1 of these squares. This forces another L tromino to pair with it to form a 2×3 subrectangle.

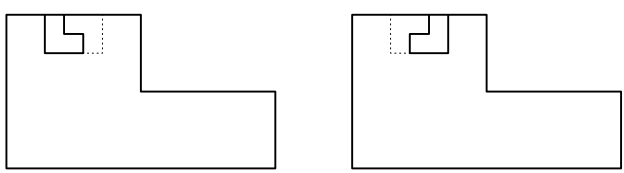


Figure 3.3. A 2×3 rectangle must occur along the top edge.

Thus every tiling has a subrectangle, whence the number of tilings is even.

(b) This follows from (a) since $R(n) = T(n,n)$, and $S(n)$ has the same parity. □

Lemma 3.1 suggests considering only tilings of $L_{m,n}$ that do not contain subrectangles. Unfortunately, we do not have a good way to count these. Figure 3.5 below illustrates several of these, which shows that there are such rectangle free tilings besides the basic 2^2-reptiling and compositions of it with itself.

Proposition 3.4. *Conjecture B holds if m or n is either 2 or 4.*

Proof. Consider first the case $m = 2$. We easily calculate $T(2,1) = 0$ and $T(2,2) = 1$, so we need only consider $n \geq 3$. In this case, the top three rows of a tiling of $L_{2,n}$ must be filled with two L trominoes forming a 3×2 rectangle.

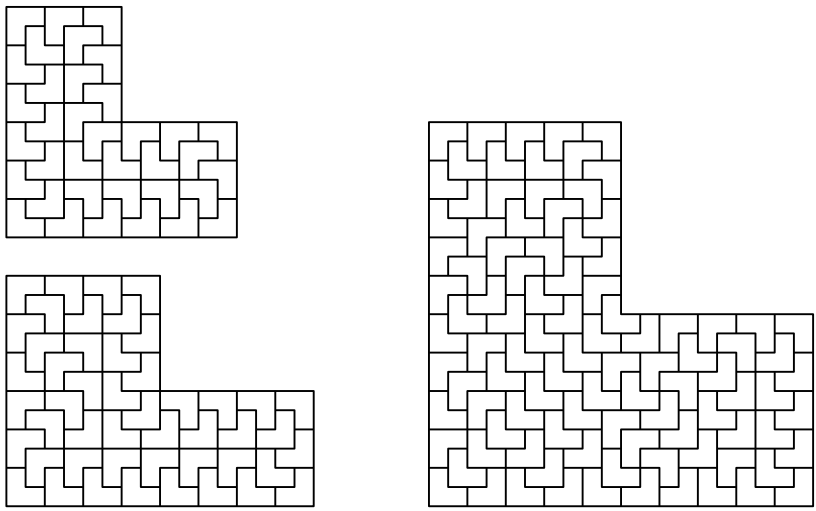


Figure 3.5. Rectangle free tilings. The tiling of $L_{10,10}$ is symmetric.

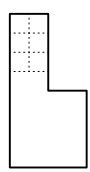


Figure 3.6. The top 3 rows contain a 3×2 subrectangle.

Now Lemma 3.1 implies that the number of tilings is even. This proves the case $m = 2$, and $n = 2$ holds by symmetry.

Now consider the case $m = 4$. We calculate $T(4, 1) = 0$, $T(4, 2) = 4$, $T(4, 3) = 72$ and $T(4, 4) = 409$, so it remains to consider $n \geq 5$. Consider all trominoes that cover some cell in the top four rows. (See Figure 3.8 below.) No such tromino can extend past the fifth row, so these trominoes cover all of the first four rows, and some of the fifth row.

There are six possible shapes for the region they cover, and in each case, the number of ways to tile this region is even. Therefore the total number of tilings, $T(4, n)$, is even for $n \geq 5$. □

Theorem 3.7. *For fixed m, $T(m, n)$ satisfies a homogeneous linear recurrence in n,* *with integer coefficients. The degree of this recurrence is at most*

$$\begin{array}{ll} \frac{1}{9}(2^{2m-1}+2^{m-1}+2)(2^{m-1}+2^{(m-2)/2}+(-1)^{m/2}+1) & \text{if } m \equiv 0 \bmod 6, \\ \frac{1}{9}(2^{2m-1}+2^{m-1})(2^{m-1}+2^{(m-1)/2}+(-1)^{(m+1)/2}-1) & \text{if } m \equiv 3 \bmod 6, \\ (2^{2m-1}+2^{m-1})(2^{m-1}+2^{(m-2)/2}) & \text{if } m \equiv \pm 2 \bmod 6, \\ (2^{2m-1}+2^{m-1})(2^{m-1}+2^{(m-1)/2}) & \text{if } m \equiv \pm 1 \bmod 6. \end{array}$$

In particular, for fixed m, the parity of $T(m,n)$ is eventually periodic in n.

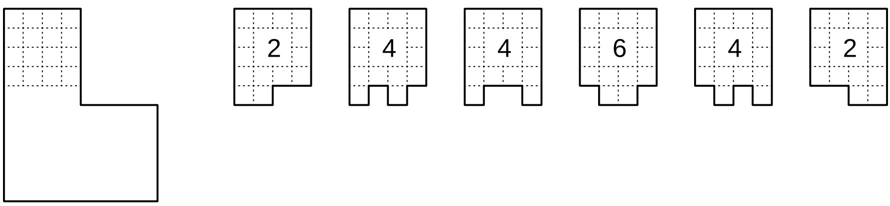


Figure 3.8. Different ways to cover the top 4 rows of $L_{4,n}$.

In order to prove Theorem 3.7, we first need two preliminary results.

Proposition 3.9. *Suppose that the sequences $\{a_n\}$ and $\{b_n\}$ satisfy homogeneous linear recurrences with constant coefficients, of degrees r and s, respectively. Then the sequence $\{a_nb_n\}$ satisfies a homogeneous, degree rs linear recurrence with constant coefficients. Moreover, this recurrence only depends upon the recurrences for $\{a_n\}$ and $\{b_n\}$. If the recurrences for $\{a_n\}$ and $\{b_n\}$ have integer coefficients, then so does the recurrence for $\{a_nb_n\}$.*

Proof. We have $a_n = \sum_{i=1}^{r} c_i a_{n-i}$ for some coefficients c_i, and $b_n = \sum_{j=1}^{s} d_j b_{n-j}$ for some coefficients d_j. The recurrence for the a_n's can be written in matrix form as

$$\begin{pmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_{n-r+1} \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_{r-1} & c_r \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n-1} \\ a_{n-2} \\ \vdots \\ a_{n-r} \end{pmatrix},$$

and there is a similar expression for the b_n's. Let A and B denote the corresponding coefficient matrices. We have

$$a_nb_n = \sum_{i=1}^{r}\sum_{j=1}^{s} c_i d_j a_{n-i} b_{n-j},$$

$$a_nb_{n-j} = \sum_{i=1}^{r} c_i a_{n-i} b_{n-j}, \quad \text{and} \quad a_{n-i}b_n = \sum_{j=1}^{s} d_j a_{n-i} b_{n-j},$$

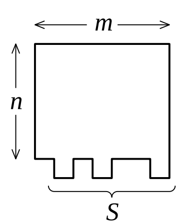
which can be expressed in matrix form as

$$\begin{pmatrix} a_n b_n \\ a_n b_{n-1} \\ a_n b_{n-2} \\ \vdots \\ a_{n-r+1} b_{n-s+1} \end{pmatrix} = (A * B) \begin{pmatrix} a_{n-1} b_{n-1} \\ a_{n-1} b_{n-2} \\ a_{n-1} b_{n-3} \\ \vdots \\ a_{n-r} b_{n-s} \end{pmatrix},$$

where $(A*B)$ is the Kronecker product of the coefficient matrices A and B. It follows that $\{a_n b_n\}$ satisfies a degree rs linear recurrence with constant coefficients, whose characteristic polynomial is the characteristic polynomial of the matrix $(A*B)$. The matrix $(A*B)$ and its characteristic polynomial only depend on the coefficients c_i and d_j. This proves the "moreover" statement. Finally, if all the c_i's and d_j's are integers, then so are all entries of $(A*B)$, as well as all coefficients of its characteristic polynomial, which means that the recurrence for $\{a_n b_n\}$ has integer coefficients. □

Remark. A weaker form of this result can be found in [14], Proposition 4.2.5, but we require a stronger statement here. We can be more specific about the recurrence for $\{a_n b_n\}$; if the characteristic polynomials for the recurrences of $\{a_n\}$ and $\{b_n\}$ are $\prod_i (T - \alpha_i)$ and $\prod_j (T - \beta_j)$, then the characteristic polynomial for the $\{a_n b_n\}$ recurrence is $\prod_{i,j} (T - \alpha_i \beta_j)$. This is immediate from the corresponding relation between the characteristic polynomial of $(A*B)$ and those of A and B.

Proposition 3.10. *Fix $m \geq 1$, and let a_n denote the number of tilings by L trominoes of a region which is an $n \times m$ rectangle with a fixed subset of squares from the $(n+1)$-st row as shown.*



Then $\{a_n\}$ satisfies a homogeneous linear recurrence with integer coefficients. The recurrence depends upon m, but it is independent of the fixed subset of extra squares. Moreover, the degree of the recurrence is at most

$$\begin{array}{ll} \frac{1}{3}(2^{m-1} + 2^{(m-2)/2} + (-1)^{m/2} + 1) & \text{if } m \equiv 0 \bmod 6, \\ \frac{1}{3}(2^{m-1} + 2^{(m-1)/2} + (-1)^{(m+1)/2} - 1) & \text{if } m \equiv 3 \bmod 6, \\ 2^{m-1} + 2^{(m-2)/2} & \text{if } m \equiv \pm 2 \bmod 6, \\ 2^{m-1} + 2^{(m-1)/2} & \text{if } m \equiv \pm 1 \bmod 6. \end{array}$$

Proof. For a subset S of extra squares, let R_n^S denote the region which is an $n \times m$ rectangle with these extra squares on the $(n+1)$-st row, and let a_n^S denote the number

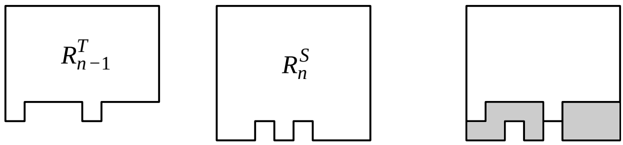


Figure 3.11. Region that extends R^T_{n-1} to R^S_n.

of tilings of this region by L trominoes. By considering how the first $n-1$ rows of R^S_n can be tiled, we have

$$a^S_n = \sum_T c(S,T)a^T_{n-1}, \qquad (*)$$

where the sum is over all sets T of extra squares, and the coefficient $c(S,T)$ is the number of ways to extend a tiling of R^T_{n-1} to a tiling of R^S_n.
The region that extends R^T_{n-1} to R^S_n is independent of n, and therefore so is $c(S,T)$. It follows from equation $(*)$ that $\left\{a^S_n\right\}$ satisfies the homogeneous linear recurrence whose characteristic polynomial is the same as the characteristic polynomial of the transfer matrix $(c(S,T))$. The coefficients of the characteristic polynomial are integers, and its degree is the number of subsets S, i.e. 2^m. We can reduce this degree by utilizing the following observations. Firstly, the number of tilings of a region is the same if the region is reflected vertically. This means that if S' is the reflection of S, then $a^S_n = a^{S'}_n$, so we do not need to consider all subsets of squares on the $(n+1)$-st row. Secondly, if m is a multiple of 3, then $a^S_n = 0$ for all S whose cardinality is not a multiple of 3, so automatically satisfies any homogeneous linear recurrence.

If $m \equiv 0 \bmod 6$, then there are $\frac{1}{3}(2^m+2)$ subsets S whose cardinality is a multiple of 3. Up to left-right symmetry, there are $\frac{1}{3}(2^{m-1} + 2^{(m-2)/2} + (-1)^{(m/2)} + 1)$ such subsets. If $m \equiv 3 \bmod 6$, then there are $\frac{1}{3}(2^m - 2)$ subsets S whose cardinality is a multiple of 3, and up to symmetry, there are $\frac{1}{3}(2^{m-1} + 2^{(m-1)/2} + (-1)^{(m+1)/2} - 1)$ such subsets. If $m \equiv \pm 2 \bmod 6$, then there are 2^m subsets S, and $2^{m-1} + 2^{(m-2)/2}$ up to symmetry. If $m \equiv \pm 1 \bmod 6$, then there are 2^m subsets, and $2^{m-1} + 2^{(m-1)/2}$ up to symmetry. This gives the smaller degrees in the statement of the proposition. □

Proof of Theorem 3.7. Every tiling of $L_{m,n}$ can be split into two components; those tiles that cover some squares in the top $n-1$ rows, and the remaining tiles.

Note that the remaining tiles are exactly those that cover some square from the bottom n rows. Moreover, this decomposition is uniquely determined by the tiling. Therefore we have $T(m,n) = \sum_T a^T_{n-1} b^{T'}_n$, where T ranges over all subsets of squares of the n-th row, T' is the complement of T, a^T_n is as in Proposition 3.10, and b^T_n is the same, except for width $2m$. Proposition 3.10 shows that each a^T_{n-1} and $b^{T'}_n$ satisfies a linear recurrence with constant coefficients, and they are independent of T and T' respectively. Proposition 3.9 then shows that each $a^T_{n-1} b^{T'}_n$ satisfies a linear recurrence with integer coefficients, which are independent of T. Therefore their sum satisfies the same recurrence. Finally, the degree of this recurrence is at most

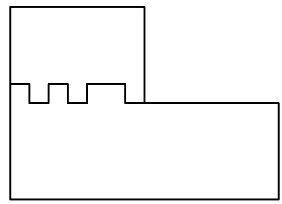


Figure 3.12. Splitting $L_{m,n}$ into two components.

the product of the degrees of the recurrences for a_{n-1}^T and $b_n^{T'}$, which is provided by Proposition 3.10. □

Based upon Theorem 3.7, we are able to verify Conjecture B in more cases.

Proposition 3.13. *Conjecture B holds if m or n is either 6 or 8.*

Proof. We verified by computer that $T(6,n)$ is even for $1 \leq n \leq 8324$. Since $T(6,n)$ satisfies a degree 8324 linear recurrence with integer coefficients, it follows by induction that $T(6,n)$ is even for all larger values of n. For $m = 8$, we verified by computer that $T(8,n)$ is even for $1 \leq n \leq 7$, is odd for $n = 8$, and is even for $9 \leq n \leq 4473864$. In this case, the degree of the recurrence is 4473856, so again induction shows that $T(8,n)$ is even for all larger values of n. □

For $m = 8$, an interesting phenomenon occurs. It turns out that for each possible way to tile the first 8 rows and some extra squares in the ninth row, the number of ways to tile the region is even. This is similar to what happens for $m = 4$, as in the proof of Proposition 3.4, and also what happens for $m = 1$ and 2. Curiously, it appears that this pattern does not continue for $m = 16$; the region in Figure 3.16 below, consisting of the first 16 rows and two squares of the seventeenth row has an odd number of tilings by the L tromino.

We now give some bounds on the growth of the functions $R(n)$, $S(n)$ and $T(m,n)$.

Lemma 3.14. *A polyomino region of area $3k$ has at most 4^k tilings by L trominoes.*

Proof. We induct on k. For $k = 0$, the result is trivial.

Now suppose the result holds for $k-1$. There are at most 4 ways to fill the leftmost square in the top row (corresponding to the four orientations of the L tromino). For each, there are at most 4^{k-1} ways to tile the rest of the region, by the induction hypothesis. Therefore, the region has at most 4^k tilings. This completes the induction. □

Proposition 3.15. *There are positive constants c and C such that $e^{cn^2} \leq R(n) \leq e^{Cn^2}$ for all sufficiently large n. Specifically, we have $R(n) \leq 4^{n^2}$ for all n, and $R(n) \geq 2^{n^2/2}$ for $n \geq 4$.*

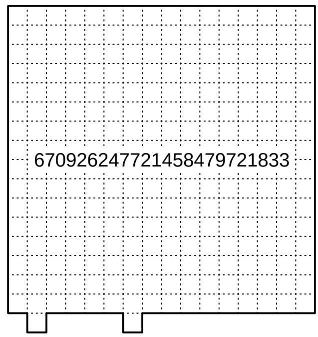


Figure 3.16. Covering the first 16 rows of $L_{16,n}$.
This shape has an odd number of tilings.

Proof. The upper bound is immediate from Lemma 3.14. The lower bound is a special case of Proposition 3.20 below. □

Proposition 3.17. *There are positive constants, c and C such that $e^{cn^2} \le S(n) \le e^{Cn^2}$ for sufficiently large n. More precisely, we have $S(n) < 2^{n^2}$ for all n, and $S(n) \ge 38^{n^2/25}$ for $n \ge 5$.*

Proof. In a symmetric n^2-reptiling, the tiles along the axis of symmetry must be placed as shown in Figure 3.18, which splits the remainder of the region into two components.

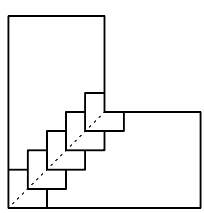


Figure 3.18. Tiles placed along axis of symmetry.

The tiling is then determined by the tiling of either component, so we have $S(n) \le 4^{(n^2-n)/2} < 2^{n^2}$ from Lemma 3.14.

For the lower bound, we first observe that $S(n) \ge 38^{n^2/25}$ for $5 \le n \le 10$. A symmetric n^2-reptiling extends to a symmetric $(n+6)^2$-reptiling, as shown in Figure 3.19.

In the diagram, there are $S(6)$ ways to tile the rep-6^2 region symmetrically. For $n > 1$, a $6 \times n$ rectangle can be partitioned into 2×3 rectangles, each of which can be tiled in 2 ways. Thus one of the $6 \times n$ rectangles can be tiled in 2^n ways, and

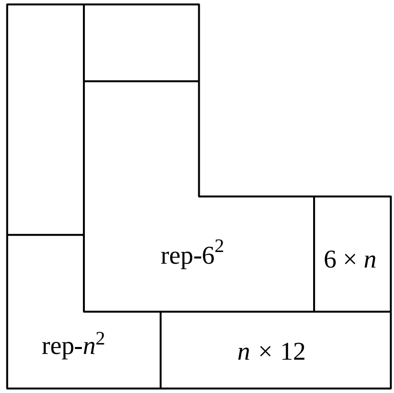


Figure 3.19. Extending a symmetric reptiling.

the tiling of the other is determined by symmetry. Similarly, there are at least 2^{2n} ways to tile one of the $n \times 12$ rectangles. This shows that $S(n+6) \geq 2^{3n} S(6) S(n)$, for $n \geq 1$, and the lower bound $S(n) \geq 38^{n^2/25}$ for all $n \geq 5$ now follows easily by induction. □

Proposition 3.20. *There are positive constants, c and C such that $e^{cmn} \leq T(m,n) \leq e^{Cmn}$ for all sufficiently large m, n. More precisely, we have $T(m,n) \leq 4^{mn}$ for all m, n, and $T(m,n) \geq 2^{mn/2}$ for $m, n \geq 4$.*

Proof. The upper bound is immediate from Lemma 3.14. For the lower bound, we first show by computation that $T(m,n) \geq 2^{mn/2}$ for $4 \leq m, n \leq 9$. A tiling of $L_{m,n}$ can be extended to a tiling of $L_{m+6,n}$ as shown in Figure 3.21.

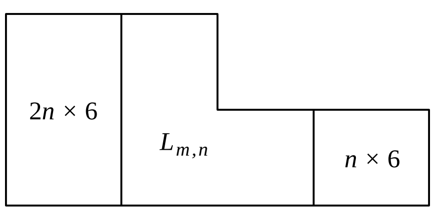


Figure 3.21. Extending a tiling of $L_{m,n}$ to a tiling of $L_{m+6,n}$.

Since there are (at least) 2^n ways to tile the $n \times 6$ rectangle, and 2^{2n} ways to tile the $2n \times 6$ rectangle, we have $T(m+6,n) \geq 2^{3n} T(m,n)$. Similarly, $T(m,n+6) \geq 2^{3m} T(m,n)$. Now the lower bound $T(m,n) \geq 2^{mn/2}$ for all $m, n \geq 4$ follows by induction. □

These bounds can certainly be improved by a more delicate analysis. It would be of interest to prove that the limits $\lim_{n\to\infty} \log(R(n))/n^2$, $\lim_{n\to\infty} \log(S(n))/n^2$, and

$\lim_{m,n\to\infty} \log(T(m,n))/mn$ exist, and to determine their exact values. See [7] for the analogous question for tilings of rectangles.

4 Method of Counting

In this section, we briefly describe our technique for enumerating $R(n)$, $S(n)$ and $T(m,n)$.

A tiling of $L_{m,n}$ by L trominoes splits into three pieces: the tiles covering the top $n-1$ rows, the tiles covering the rightmost $m-1$ columns, and the remaining tiles which cover the "elbow". For each possible partition, we count the number of tilings of each part and multiply them. Then we sum over all partitions to get the total number of tilings of $L_{m,n}$.

To count the number of tilings of the top $n-1$ rows, we use the "transfer matrix" method. For a subset S of squares on a row of width m, let a_k^S denote the number of tilings of k rows of width m, with the extra squares in S adjoined along the $(k+1)$-st row. For $k=0$, we have $a_0^{\varnothing}=1$, and $a_0^S=0$ if S is non-empty. As in the proof of Proposition 3.10, we have $a_k^S=\sum_T c(S,T)a_{k-1}^T$. Using this relation, we iteratively calculate a_k^S for all subsets S simultaneously from the values a_{k-1}^S.

Most of the coefficients $c(S,T)$ are 0. The non-zero values can be determined as follows. Recall that $c(S,T)$ counts the number of ways to tile an extension from R_0^T to R_1^S (using the notation of 3.10). Such a tiling decomposes horizontally into several types of primitive pieces: a 2×3 rectangle, a single L tromino shape (occurring in any of four orientations), and an empty region of width 1.

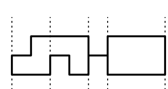


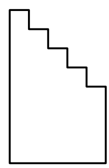
Figure 4.1. Decomposition of extension shape into primitive pieces.

We encode the subset S as a string of 0's and 1's in the natural way; the digit in the i-th position is 1 if and only if the square in the i-th position is in the set S. The non-zero coefficients $c(S,T)$ can be generated recursively from $c(\varnothing,\varnothing)=1$, $c(0x,1y)=c(x,y)$, $c(01x,00y)=c(x,y)$, $c(10x,00y)=c(x,y)$, $c(11x,01y)=c(x,y)$, $c(11x,10y)=c(x,y)$, and $c(111x,000y)=2c(x,y)$. These correspond to the facts that a 2×3 rectangle can be tiled in 2 ways, and the other primitive pieces can be tiled in exactly 1 way. In the example of Figure 4.1 above, we have $c(11010111,10001000)=c(010111,001000)=c(0111,1000)=c(111,000)=2c(\varnothing,\varnothing)=2$.

Counting the number of tilings of the rightmost $m-1$ columns proceeds in the same way. To count the number of tilings of the elbow, we use a simple modification of this method. For convenience, reflect the elbow horizontally. Let S be a subset of extra squares of a width $m+1$ row, and let b_k^S denote the number of tilings of the top k rows along with the extra squares in S. This is similar to counting partial tilings of the top $n-1$ rows, except the width here is $m+1$, and more importantly, some squares have been deleted from the rightmost edge. As before, we have $b_k^S = \sum_T c(S,T) b_{k-1}^T$ if the k-th row contains the rightmost square. If the rightmost square has been deleted from the k-th row, this needs to be modified to $b_k^S = \sum_T c(S,T') b_{k-1}^T$, where T' denotes T with the rightmost square included. (Here the coefficients $c(S,T)$ are from the transfer matrix for width $m+1$.)

Note that we do not need to calculate these numbers from scratch for each possible set of squares deleted from the rightmost edge; we can reuse the partial computations for shapes that agree along the top several rows.

To calculate $S(n)$, recall that this is the number of tilings of the region

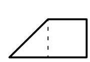


of width n. We compute these numbers using a similar modification of the transfer matrix method that accounts for the missing squares in the upper right corner.

The parity of $T(m,n)$ and $S(n)$ can be computed in the same way, with little modification. For example, in Proposition 3.13, we only computed the parity of $T(6,n)$. We also computed the parity of $T(10,n)$ for $n \leq 2000000$; all were found to be even. This computation took 189 hours of CPU time. To verify Conjecture B for $m = 10$ would require computing the parity of $T(10,n)$ for $n \leq 277094400$.

5 Other shapes

A *polyabolo* is a shape made by joining congruent right isosceles triangles so that they are "aligned" in a natural way. (Specifically, if the legs of the triangles have length 1, then the shape can be positioned in the plane so that the coordinates of all vertices are integers.) For example, consider the *triabolo*



Let $U(n)$ denote the number of n^2-reptilings by this shape. For $n = 1, 2, \ldots$, we have $U(n) = 1, 1, 10, 721, 96158, 94484630, 488195932976, \ldots$.

Conjecture C. *$U(n)$ is odd if and only if n is a power of 2.*

Theorem 5.1. *Conjecture C is equivalent to Conjectures A and A'.*

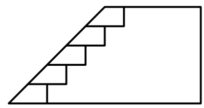


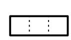
Figure 5.2. Placement of tiles along diagonal edge.

Proof. In an n^2-reptiling, tiles along the diagonal edge must be placed as shown in Figure 5.2.

Thus $U(n)$ equals the number of tilings of the remaining polyomino region. Tiles in this region pair up along their diagonal edges into either L trominoes or straight trominoes (1×3 rectangles). Conversely, a tiling a the polyomino region by L trominoes and straight trominoes can be decomposed into a tiling by the triabolo. Moreover, a straight tromino can be tiled by the triabolo in 2 different ways. This means that each tiling of the polyomino region by L trominoes and straight trominoes corresponds to 2^k tilings by the triabolo, where k is the number of straight trominoes. Thus the parity of $U(n)$ is the same as the number of tilings of the polyomino region by L trominoes, which we have seen is $S(n)$. □

For a polyabolo, there is a possibility that it is rep-$2n^2$ for some n, in other words, there might be a reptiling with ratio of similitude $n\sqrt{2}$. However, for this triabolo, it is easy to show that it is not rep-$2n^2$ for any n.

This example suggests counting reptilings by other shapes. Although he did not phrase it in terms of reptilings, Propp ([10], Problem 22) notes that Kasteleyn's formula implies that the number of n^2-reptilings by the domino is $\equiv 1 \bmod 4$, and he asks for a combinatorial proof of this.

The *straight tromino* is a 1×3 rectangle:  . The number of n^2-reptilings by the straight tromino, for $n = 1, 2, \ldots$, is $1, 1, 19, 249,\ 3643,\ 1600185, 329097125, \ldots$. We can prove that the number of n^2-reptilings has the same parity as the number of $(2n)^2$-reptilings, so the next term in the sequence is also odd. This is worthy of further attention.

Acknowledgment

This work was partially supported by UCF In-House Research Grant #11-60-9006.

Appendix

We give here some computed values of $T(m,n)$. Although $T(m,n)$ is symmetric in m and n, the method of computing these values is not symmetric in the counting tilings of the "elbow", For all $m, n \leq 18$, we computed both $T(m,n)$ and $T(n,m)$, and in all cases, the computed values agreed. This gives us some confidence in the correctness of the results. Because of this symmetry, we need only give values for $m < n$; the values for $m = n$ are given above in Table 2.2. Moreover, $T(1, n) = 0$ for $n > 1$, so we only consider $m \geq 2$ here. We have also calculated the parity of $T(m, 20)$ and $T(m, 22)$ for all $m \leq 22$; all were found to be even. (In the tables, the columns are indexed by m and the rows by n.)

Values of $T(m, n)$

	2	3	4	5
3	8	4	72	120
4	4	72	409	4168
5	16	120	4168	108388
6	72	1296	44046	2215560
7	80	3072	421716	47842016
8	232	24224	3826444	1023037984
9	704	72864	41106116	22171827216
10	1248	423744	405615924	483938669160
11	3200	1606144	3799947504	10447063869824
12	7488	7589504	40653384976	225771799645872
13	17792	33947776	400773856464	4880206632105920
14	43072	141305088	3837259802704	106071967859251008
15	85504	702597120	40905444293232	2310577253142103072
16	243456	2691361280	403177422839720	49627917494449314704
17	572416	14361788928	3894225930818624	1077324401627268655776
18	1028608	51983447040	41504751359473584	23319184057927134959248
19	3263488	291186573312	408978735029551792	502705706321944502435936
20	7555584	1014550358016	3965868711622647456	10927535898450558839362176
21	12808192	5872677455872	42283060666385238160	237356840611127805481845984

	6	7
7	5328885922	608850350072
8	276408992770	71324156785552
9	13933343444778	8141004894379048
10	722908373529706	951328813777052244
11	36868626800299334	108462137456648779432
12	1894921144730134674	12577357132484337185736
13	97356328787643248644	1449730609072010690217528
14	4997314715104212563040	166263613339328200790749352
15	256931348295412047167732	19252498319294212296641292824
16	13205049021156776464061514	2218544464014728584248776736856
17	678939634575534704742741310	254569908899407051712982885745696
18	34916472343609869412634711494	29464746047086786588382566505666120
19	1795796276532375370649758118580	3395511574986949596095528303319615976
20	92377186384480708124691385084514	389661831861394309160671597470227141260
21	4752336614980811788918812100037742	45098341153890172933720046926712990564560

	8
9	5099310422090391496
10	13704298462588421435 90
11	3618388506356605498671 08
12	963302342508236129503911 06
13	2557746408606478354017970 4656
14	6790908638898662414354621367512
15	180515969765861305856326493177883 6
16	479129944343010546109088991715236416
17	12730041953682239419638166618864564918 4
18	338261027113403831688843698872293110943 84
19	8979236361265716993306719665508238908776976
20	238578435295941557134594213099867491745931970 8
21	6339120600650115539720299511429313412423603978 84

	9
10	18863224566732658126214 06
11	1141157848087251861218689584
12	69672655908024337765517832360 2
13	4243774862164278323209575329395 88
14	258314370052676252547508161792845076
15	1574086635850346996251978992102341065 84
16	95862402732775474032155409928239096103758
17	5838850475302068371124169003957330002794459 2
18	35566426067004899788863212091709859151400608530
19	2166250833202153403767902758527681948618872856355 6
20	1319487454709262230787670795267586713016390156945837 4
21	8036948764369763953790577856239599746267394595749873840

	10
11	37002212962949588531680755330 00
12	51862395090247737604326838473170 46
13	7247002682640559436615798235931339896
14	101173370732648706220563793591210657503 98
15	14150401583373465229901724791393777370134784
16	1977245182040265131005316074335174924232945755 2
17	27621450843840887218990061450105233143881959666882
18	386172254014899369107727829762011091927910112699344 36
19	53966440955564154159299163994384711501289328711565283828
20	753924101118201482711761411468109977416387709659435577238 64
21	10539868764350771070592736036466538171149090197348001349790235 6

	11
12	37876936551840320966610628618651925148
13	121402092153950231324585586130824397430012
14	389032793370109597048905957618960843454928780
15	1247957154818714231167692627613954525459553489760
16	4000530861217936937718467031082287221106775301679632
17	12827178200143748875916639761552763045881555115673871108
18	41131942214008030330396229388647009892233616816110396993384
19	131871104152434055394932389037544596658619102898388853091537684
20	422844677347323069341119623264468668637507205901326503250062515160
21	1355818225616814885111623888266548024949278051992385325116348813648320

	12
13	2057531167554923191716098476547452936303559816
14	15135966390015397603787004585272907255345527159826
15	111461063562302696604479927163003512700802805270795140
16	820304317785870781155659672999638245323589735358550492548
17	6037934332337557990078821479693027619443539596784120493446814
18	44446667782852638009444434205609953599124886134850863139214864548
19	327147367464816386218807204910644814643628999306225311427092475119290
20	2408109861468534305392091510249260537571381543439663189583852844291195092
21	17725519945130909412315332275988201185236166081323788360807670153637536027490

	13
14	586340285239094014146896232103084281534632727832540386
15	9908708792974770240927420863222030005257415264190005189536
16	167343843887831028257934754628053113502807342495676245892985168
17	2826509049388081395651627182698272385513687418304114896170224846228
18	47748124779018832213773531824034706450591100112466141784241000968040018
19	806496100989529194226920485221038700752447848810808290660704482447375395576
20	13622708035739214382522237886455936778948654505685659825672154887141735767565934
21	230112638507338218737269765760620404055812 142846094787590388143159532871066590837960

	14
15	881223459510018479646607531395623320992836698979648353862715840
16	34158205493452681748415170389835968705502115534688011109995471392658
17	1324253415891740528614894353981159482065258254770576413982811145192309350
18	51343641982241623486318855901521501851962060492744595750057504514833188113660
19	199044738080011524048101532200123942212544 8337164546859585990800470162358207773768
20	7716977357577546409792801558927854549023436 7361756780402055142486616664480293069529472
21	2991805079634744602832491153224153529820275330 659995136118650418909210279448637191405713434

	15
16	697883666554437769710801441076900479472532516683304445180451107164899 8208
17	6209608114925324059222407274300260124057817214489572128855378839495516465 22016
18	552570074947829042656562135942862697739518 598582085125908108023117523536690990834 96
19	491655207690209837589304736737453651266343 67 676849352328333891531765098990212668448781 48
20	437485948870599245126870446988818392851151342 07 101968726096803594210711195749166625143233486 8
21	389275575560581846209090896517576005395593089720 9 090887193434487083950490314440654870907751748912 0

	16
17	2910017073512907349849383645039697136825 87 4377964026759555932657103109752368734726 70
18	5943315010227131411523409369076195617028836 39 890139404216926814375537904947731495264789 68
19	1213695763106093182232996067143637978876333 5566 7325717080283457255792609997609862468637900 80638
20	247867984647872830857601051746001852313666799 37464 154401120226760303282366964956309886847612211 56594
21	506200700057101647692816487442795954471148266 81998431 307339023722150991254710094107341014315713745 5018868

	17
18	639343255174582741399937531141544444155498751663 431622332171486879663127581084968960387350821 10
19	2996560167135298048586007174465919700322680931 76493 4851819526436890040714145892896710694725416269 7688
20	140457039788995329757555792546126935495855635266962735 375328074312953618889157930841501803675393279146711 56
21	658344742211237385637395479024106151584708236132412 38258 404161822831190619127997283129502757206184358540788 17272

	18
19	73992025157277056655733241410727961339556717817562 2122 85401090078699073238672641754110867094212848478472 864
20	796004367488594225174930420344354485279452046766170950419 853316292054706828089629403975267861416867377195483458 28
21	856318530798812652729582379405764809870240715663678072173190 134947722598312974104028269752359145359210196004295650669 58

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(Received 6 Jan 2007; revised 13 June 2007)