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Neighborhood conditions for graphs and digraphs to be maximally edge-connected

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Abstract

Let G be a connected graph of order n, edge-connectivity λ and minimum degree δ. For a vertex v in G, the neighborhood $N(v)$ of v is defined as the set of all vertices adjacent to v and $d(v) = |N(v)|$ is the degree of v. A graph is called maximally edge-connected, if $\lambda = \delta$. In 1979, Goldsmith and Entringer gave the following neighborhood condition for graphs to be maximally edge-connected:
If

$$\sum_{x \in N(u)} d(x) \geq \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor^2 - \left\lfloor \frac{n}{2} \right\rfloor & \text{for all even } n \\ & \text{and for odd } n \leq 15, \\ \left\lfloor \frac{n}{2} \right\rfloor^2 - 7 & \text{for odd } n \geq 15, \end{cases}$$

for each vertex u of minimum degree, then $\lambda = \delta$.

In this article we show that the theorem of Goldsmith and Entringer remains valid for digraphs, and we give different improvements of this result. In addition, we present some new sufficient conditions for graphs and digraphs to be maximally edge-connected, depending on the neighborhood of an edge and vertex, respectively.

1 Terminology and introduction

We consider finite graphs and digraphs without loops and multiple edges. Let $V(D)$ be the vertex set of a graph or digraph D and $n = n(D) = |V(D)|$ its *order*. For a vertex $v \in V(D)$ of a digraph D, we denote the sets of *out-neighbors* and *in-neighbors* of v by $N^+(v)$ and $N^-(v)$, respectively. Furthermore, the *degree* $d(v)$ of a vertex v in a digraph D is defined as the minimum value of its *out-degree* $d^+(v) = |N^+(v)|$ and its *in-degree* $d^-(v) = |N^-(v)|$. We denote the minimum degree of a (di)graph D by $\delta = \delta(D)$. For two disjoint vertex sets X, Y of a digraph or graph let (X, Y) be the set of arcs or edges from X to Y. We denote the complete graph with p vertices by K_p. For each edge $e = ab$ in a graph G, let $\xi_G(e) = d(a) + d(b) - 2$ be the edge-degree of e, and let $\xi = \xi(G) = \min\{\xi_G(e) : e \in E(G)\}$ be the minimum edge-degree of G.

For other graph theory terminology we follow Chartrand and Lesniak [4].

An edge(arc)-cut of a (strongly) connected (di)graph D is a set of edges (arcs) whose removal disconnects D. The edge(arc)-connectivity $\lambda = \lambda(D)$ is defined as the minimum cardinality over all edge(arc)-cuts of D. A λ-cut is an edge(arc)-cut of cardinality λ. Since $\lambda \leq \delta$ for all (di)graphs, we call a (di)graph maximally edge(arc)-connected, if $\lambda = \delta$. In 1966, Chartrand [3] proved for graphs, that $\delta = \lambda$, if $\delta \geq \lfloor n/2 \rfloor$.

In 1979, Goldsmith and Entringer [7] showed that, if for each vertex u of minimum degree, the vertices in the neighborhood of u have sufficiently large degree sum, then the graph is maximally edge-connected.

Theorem 1.1 (Goldsmith, Entringer [7] 1979) *Let G be a connected graph of order $n \geq 2$, edge-connectivity λ and minimum degree δ. If*

$$\sum_{x \in N(u)} d(x) \geq \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor^2 - \left\lfloor \frac{n}{2} \right\rfloor & \text{for all even } n \\ & \text{and for odd } n \leq 15, \\ \left\lfloor \frac{n}{2} \right\rfloor^2 - 7 & \text{for odd } n \geq 15 \end{cases}$$

for each vertex u of minimum degree, then $\lambda = \delta$.

In this article we show that Theorem 1.1 remains valid for digraphs, and we give a generalization of this result. Inspired by Theorem 1.1, we present in the third section some new sufficient conditions for (di)graphs to be maximally edge(arc)-connected, depending on the neighborhood of an edge (arc) and vertex, respectively. These investigations lead to an extension of the following degree sequence condition by Dankelmann and Volkmann [5].

Theorem 1.2 (Dankelmann, Volkmann [5] 1997) *Let D be a (di)graph of order n with edge(arc)-connectivity λ and degree sequence $d_1 \geq d_2 \geq ... \geq d_n = \delta$. If $\delta \geq \lfloor n/2 \rfloor$ or $\delta \leq \lfloor n/2 \rfloor - 1$ and*

$$\sum_{i=1}^{2k} d_{n+1-i} \geq kn - 3,$$

for some k with $2 \leq k \leq \delta$, then $\lambda = \delta$.

Further sufficient conditions for equality of edge(arc)-connectivity and minimum degree of a (di)graph were given by several authors, as for example: Balbuena and Carmona [1], Bollobás [2], Fàbrega and Fiol [6], Lesniak [9], Plesník [10], Plesník and Znám [11], and Xu [12].

2 Generalizations of the result by Goldsmith and Entringer

In the sequel we will use the following notation:
For each $v \in V(D)$ and $A \subseteq V(D)$, let $N_A^+(v) = N^+(v) \cap A$ and $N_A^-(v) = N^-(v) \cap A$.

Theorem 2.1 *Let D be a strongly connected digraph of order n with arc-connectivity λ and minimum degree $\delta \leq \lfloor n/2 \rfloor - 1$.*

a) If

$$\sum_{x \in N^+(u)} d(x) \geq \max\{\delta(\delta+1), d^+(u)(n-\delta-3)+\delta, d^+(u)(n-\delta-3)-n+4\delta+2\}$$

and

$$\sum_{x \in N^-(u)} d(x) \geq \max\{\delta(\delta+1), d^-(u)(n-\delta-3)+\delta, d^-(u)(n-\delta-3)-n+4\delta+2\}$$

for each vertex u of minimum degree, then $\lambda = \delta$.

b) If $\delta = d^+(u) = d^-(u)$,

$$\sum_{x \in N^+(u)} d(x) \geq \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor^2 - \left\lfloor \frac{n}{2} \right\rfloor & \text{for all even } n \\ & \text{and for odd } n \leq 15, \\ \left\lfloor \frac{n}{2} \right\rfloor^2 - 7 & \text{for odd } n \geq 15 \end{cases}$$

and

$$\sum_{x \in N^-(u)} d(x) \geq \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor^2 - \left\lfloor \frac{n}{2} \right\rfloor & \text{for all even } n \\ & \text{and for odd } n \leq 15, \\ \left\lfloor \frac{n}{2} \right\rfloor^2 - 7 & \text{for odd } n \geq 15 \end{cases}$$

for each vertex u of minimum degree, then $\lambda = \delta$.

c) Furthermore, if $\delta = \lfloor n/2 \rfloor - 1$,

$$\sum_{x \in N^+(u)} d(x) \geq \delta(\delta+1) = \lfloor n/2 \rfloor^2 - \lfloor n/2 \rfloor$$

and

$$\sum_{x \in N^-(u)} d(x) \geq \delta(\delta+1) = \lfloor n/2 \rfloor^2 - \lfloor n/2 \rfloor$$

for each vertex u of minimum degree, then $\lambda = \delta$.

Proof. Let S be an arbitrary λ-cut and let X and Y denote the vertex sets of the two components of $D - S$, such that $|(X,Y)| = \lambda$. We assume to the contrary, that $\lambda \leq \delta - 1$. If $|X| \leq \delta$ or $|Y| \leq \delta$, then $\lambda \geq \delta$, which implies $\lambda = \delta$, a contradiction. Now let $|X|, |Y| \geq \delta + 1$ and let u be an arbitrary vertex of minimum degree, without loss of generality $u \in X$. If $u \in Y$, then we work through the following proof by using the second parts of the conditions of the theorem. We define $X_0 \subset X$ as the set of vertices, which do not have an out-neighbor in Y. In an analog manner we define

$Y_0 \subset Y$ as the set of vertices without an in-neighbor in X. Clearly (or see Case 1 of the proof of Theorem 3.1), if $X_0 = \emptyset$, then $\lambda = \delta$. Thus, let in the following $X_0 \neq \emptyset$. If we assume that $|X| = \delta + 1$, then each vertex $v \in X_0$ satisfies $d^+(v) = \delta$, which implies that

$$\begin{aligned}\sum_{x\in N^+(v)} d(x) &\leq \sum_{x\in N^+(v)} d^+(x)\\ &\leq \delta(|X|-1)+|S|\\ &\leq \delta^2+\delta-1 \leq \delta(\delta+1)-1\\ &\leq \lfloor n/2\rfloor^2 - \lfloor n/2\rfloor - 1,\end{aligned}$$

a contradiction to a), b) and c). Analogously, $|Y| = \delta + 1$ contradicts a), b) and c).

Now we assume $|X|, |Y| \geq \delta + 2$, which implies $|X|, |Y| \leq n - \delta - 2$ and $\delta \leq \lfloor n/2\rfloor - 2$. If $N_Y^+(u) = \emptyset$, then

$$\begin{aligned}\sum_{x\in N^+(u)} d(x) &\leq \sum_{x\in N_X^+(u)} d^+(x)\\ &\leq |N_X^+(u)|(|X|-1)+|S|\\ &\leq d^+(u)(n-\delta-3)+\delta-1,\end{aligned}$$

a contradiction to a). Furthermore, if in addition $d^+(u) = \delta$, then

$$d^+(u)(n-\delta-3)+\delta-1 = \delta(n-\delta-2)-1 \leq (\lfloor n/2\rfloor-2)(n-\lfloor n/2\rfloor)-1,$$

since the function $f(\delta) = \delta(n-\delta-2)-1$ increases for $\delta \leq \lfloor n/2\rfloor - 2$. Hence a contradiction to b).

If $N_Y^+(u) \neq \emptyset$, then $|N_Y^+(u)| \leq |N_X^+(u)|$, because otherwise there exists an edge-cut with $|S| - |N_Y^+(u)| + |N_X^+(u)| < |S|$ arcs, which is impossible. Furthermore, we observe

$$\begin{aligned}\sum_{x\in N^+(u)} d(x) &= \sum_{x\in N_X^+(u)} d(x) + \sum_{x\in N_Y^+(u)} d(x)\\ &\leq \sum_{x\in N_X^+(u)} d^+(x) + \sum_{x\in N_Y^+(u)} d^-(x)\\ &\leq |N_X^+(u)|(|X|-1)+|S|-|N_Y^+(u)|+|N_Y^+(u)|(|Y|-1)+|S|\\ &= |N_X^+(u)||X|+|N_Y^+(u)||Y|+2|S|-d^+(u)-|N_Y^+(u)|\\ &\leq |N_X^+(u)||X|+|N_Y^+(u)||Y|+2(\delta-1)-2d^+(u)+|N_X^+(u)|\\ &= |N_X^+(u)||X|+|N_Y^+(u)|(n-|X|)-2+2\delta-2d^+(u)+|N_X^+(u)|\\ &= |X|(|N_X^+(u)|-|N_Y^+(u)|)+n|N_Y^+(u)|-2+2\delta-2d^+(u)+|N_X^+(u)|\\ &\leq (n-\delta-2)(2|N_X^+(u)|-d^+(u))+n(d^+(u)-|N_X^+(u)|)\\ &\quad -2+2\delta-2d^+(u)+|N_X^+(u)|\\ &\leq |N_X^+(u)|(n-2\delta-3)+\delta d^+(u)-2+2\delta\\ &\leq (d^+(u)-1)(n-2\delta-3)+\delta d^+(u)-2+2\delta\\ &= d^+(u)(n-\delta-3)-n+4\delta+1,\end{aligned}$$

a contradiction to a) and b), since

$$\begin{aligned} d^+(u)(n-\delta-3)-n+4\delta+1 &= \delta(n-\delta+1)-n+1 \\ &\le (\lfloor n/2\rfloor-2)(n-\lfloor n/2\rfloor+3)-n+1 \end{aligned}$$

for $\delta \le \lfloor n/2\rfloor - 2$ and $d^+(u)=\delta$.

Since we have discussed all possible cases, the proof is complete. □

It is easy to see that Theorem 2.1 b) implies the result by Goldsmith and Entringer for graphs in Theorem 1.1.

Furthermore, the next theorem is a direct consequence of part a) and c) of Theorem 2.1.

Theorem 2.2 *Let G be a connected graph of order n with edge-connectivity λ and minimum degree $\delta \le \lfloor n/2\rfloor - 1$.*

If $\delta \le \lfloor n/2\rfloor - 2$ and

$$\begin{aligned} \sum_{x\in N(u)} d(x) &\ge \max\{\delta(\delta+1), \delta(n-\delta-2), \delta(n-\delta+1)-n+2\} \\ &= \delta(n-\delta-2)+\max\{0, 3\delta-n+2\} \end{aligned}$$

for each vertex u of minimum degree, then $\lambda=\delta$.

If $\delta = \lfloor n/2\rfloor - 1$ and

$$\sum_{x\in N(u)} d(x) \ge \delta(\delta+1) = \lfloor n/2\rfloor^2 - \lfloor n/2\rfloor$$

for each vertex u of minimum degree, then $\lambda=\delta$.

Theorem 2.2 is a generalization of Theorem 1.1. In order to see that Theorem 2.2 is an improvement, if n is even and $\delta \le \lfloor n/2\rfloor - 2$ or n is odd and $\delta \le \lfloor n/2\rfloor - 3$, we write it in another way.

Let $\delta = \lfloor n/2\rfloor - k$, where $k \in N$, and let $\nu = 1$, if n is odd, and $\nu = 0$, if n is even. Then

$$\begin{aligned} \delta(n-\delta-2) &= \left(\left\lfloor\frac{n}{2}\right\rfloor - k\right)\left(n-\left\lfloor\frac{n}{2}\right\rfloor + k - 2\right) \\ &= \left(\left\lfloor\frac{n}{2}\right\rfloor - k\right)\left(\left\lfloor\frac{n}{2}\right\rfloor + k + \nu\right) - 2\left(\left\lfloor\frac{n}{2}\right\rfloor - k\right) \\ &= \left\lfloor\frac{n}{2}\right\rfloor^2 - k^2 + \nu\left(\left\lfloor\frac{n}{2}\right\rfloor - k\right) - 2\left\lfloor\frac{n}{2}\right\rfloor + 2k \\ &= \left\lfloor\frac{n}{2}\right\rfloor^2 - \left\lfloor\frac{n}{2}\right\rfloor + 2k - k^2 - \left\lfloor\frac{n}{2}\right\rfloor + \nu\left(\left\lfloor\frac{n}{2}\right\rfloor - k\right) \end{aligned}$$

and thus

$$\delta(n-\delta-2) < \left\lfloor\frac{n}{2}\right\rfloor^2 - \left\lfloor\frac{n}{2}\right\rfloor$$

for all $k \geq 2$. If n is odd with $n \geq 15$, then

$$\delta(n-\delta-2) < \left\lfloor \frac{n}{2} \right\rfloor^2 - 7$$

for all $k \geq 2$.

Furthermore

$$\begin{aligned}
\delta(n-\delta-2) - n + 3\delta + 2 &= \left\lfloor \frac{n}{2} \right\rfloor^2 - \left\lfloor \frac{n}{2} \right\rfloor + 2k - k^2 - \left\lfloor \frac{n}{2} \right\rfloor \\
&\quad + \nu \left(\left\lfloor \frac{n}{2} \right\rfloor - k \right) - n + 3 \left(\left\lfloor \frac{n}{2} \right\rfloor - k \right) + 2 \\
&= \left\lfloor \frac{n}{2} \right\rfloor^2 - \left\lfloor \frac{n}{2} \right\rfloor - k - k^2 + 2 + \nu \left(\left\lfloor \frac{n}{2} \right\rfloor - k - 1 \right),
\end{aligned}$$

which implies

$$\delta(n-\delta-2) - n + 3\delta + 2 < \left\lfloor \frac{n}{2} \right\rfloor^2 - \left\lfloor \frac{n}{2} \right\rfloor$$

for all $k \geq 2$ and even n. If n is odd, $n \leq 15$ and $\delta \leq \lfloor n/2 \rfloor - 2$, then also

$$\delta(n-\delta-2) - n + 3\delta + 2 < \left\lfloor \frac{n}{2} \right\rfloor^2 - \left\lfloor \frac{n}{2} \right\rfloor,$$

if $k \geq 3$.

If n is odd and $n \geq 15$, then $\delta(n-\delta-2) - n + 3\delta + 2 < \left\lfloor \frac{n}{2} \right\rfloor^2 - 7$ for all $k \geq 3$.

The following remark is summing up the observations above.

Remark 2.1 *Let G be a graph of order n and minimum degree δ. It is*

$$\delta(n-\delta-2) + \max\{0, 3\delta - n + 2\} < \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor^2 - \left\lfloor \frac{n}{2} \right\rfloor & \text{for all even } n, \\ & \text{and } \delta \leq \lfloor n/2 \rfloor - 2. \\ \left\lfloor \frac{n}{2} \right\rfloor^2 - \left\lfloor \frac{n}{2} \right\rfloor & \text{for all odd } n \leq 15 \\ & \text{and } \delta \leq \lfloor n/2 \rfloor - 3. \\ \left\lfloor \frac{n}{2} \right\rfloor^2 - 7 & \text{for odd } n \geq 15 \\ & \text{and } \delta \leq \lfloor n/2 \rfloor - 3. \end{cases}$$

3 Neighborhood of an edge

Definition 3.1 *a) Let D be a digraph and let $e = uv$ be an arbitrary arc in D. We define*

$$N^+(e) = (N^+(u) \cup N^+(v)) \setminus \{u, v\} \text{ and } N^-(e) = (N^-(u) \cup N^-(v)) \setminus \{u, v\}.$$

Furthermore we define $\xi_N^+ = \xi_N^+(D) = \min\{|N^+(e)| : e$ is an arc in $D\}$, $\xi_N^- = \xi_N^-(D) = \min\{|N^-(e)| : e$ is an arc in $D\}$ and

$$\xi_N = \xi_N(D) = \min\{\xi_N^+, \xi_N^-\}.$$

We call ξ_N the minimum restricted arc-degree of D.

b) Let G be a graph and let $e = uv$ be an arbitrary edge in G. We define

$$N(e) = (N(u) \cup N(v)) \setminus \{u, v\} \text{ and } \xi_N = \xi_N(G) = \min\{|N(e)| : e \text{ is an edge in } G\}.$$

We call ξ_N the minimum restricted edge-degree of G.

Theorem 3.1 *Let D be a strongly connected digraph of order n with arc-connectivity λ and minimum degree δ. If*

$$\sum_{x \in N^+(e)} d^+(x) \geq |N^+(e)| \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) + \delta,$$

for each arc $e = uv$ with $d^+(u), d^+(v) \leq \lfloor n/2 \rfloor - 1$, and if

$$\sum_{x \in N^-(e)} d^-(x) \geq |N^-(e)| \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) + \delta,$$

for each arc $e = uv$ with $d^-(u), d^-(v) \leq \lfloor n/2 \rfloor - 1$, then $\lambda = \delta$.

Proof. Let S be an arbitrary λ-cut and let X and Y denote the vertex sets of the two components from $G - S$ such that $|(X, Y)| = \lambda$. In the following we assume that $|X| \leq \lfloor n/2 \rfloor$, because if $|X| \geq \lfloor n/2 \rfloor + 1$, then $|Y| \leq \lfloor n/2 \rfloor$ and we can work through the following proof with Y instead of X and by using the second part of the condition of the theorem. If $|X| \leq 1$, then we are done. Now let $|X| \geq 2$ and let $X_1 \subseteq X$ be the set of vertices incident with at least one arc of (X, Y) and $X_0 = X \setminus X_1$.

Case 1: Let $X_0 = \emptyset$.

Thus, each vertex in X is incident to at least one arc in (X, Y). Let x be an arbitrary vertex in X, then we obtain

$$\begin{aligned} \delta &\leq d^+(x) = |N^+(x)| = |N^+(x) \cap X| + |N^+(x) \cap Y| \\ &\leq \sum_{v \in (N^+(x) \cap X)} |N^+(v) \cap Y| + |N^+(x) \cap Y| \leq |(X, Y)|, \end{aligned}$$

and thus $\lambda = \delta$.

Case 2: Let $X_0 \neq \emptyset$.

Subcase 2.1: There exists no arc in X_0.

If v is an arbitrary vertex in X_0, then each out-neighbor neighbor of v lies in X_1. Hence we deduce that

$$\lambda = |(X,Y)| \geq |X_1| \geq |N^+(v)| \geq \delta,$$

and thus $\lambda = \delta$.

Subcase 2.2: There exists at least one arc in X_0.

Let $e = uv$ be an arbitrary arc in X_0. By the definition of the sets X and X_0, we have $d^+(u), d^+(v) \leq |X| - 1 \leq \lfloor n/2 \rfloor - 1$. Using $|N^+(x) \cap X| \leq \lfloor n/2 \rfloor - 1$ for each vertex $x \in X$ and the hypothesis for $N^+(e)$, we conclude that

$$\begin{aligned}
\lambda = |(X,Y)| &\geq \sum_{x \in N^+(e)} d^+(x) - \sum_{x \in N^+(e)} |N^+(x) \cap X| \\
&\geq |N^+(e)| \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) + \delta - |N^+(e)| \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \\
&\geq \delta,
\end{aligned}$$

and thus $\lambda = \delta$.

Since we have discussed all possible cases, the proof is complete. □

Corollary 3.1 *Let G be a connected graph of order n with edge-connectivity λ and minimum degree δ. If*

$$\sum_{x \in N(e)} d(x) \geq |N(e)| \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) + \delta,$$

for each edge $e = uv$ with $d(u), d(v) \leq \lfloor n/2 \rfloor - 1$ then $\lambda = \delta$.

Corollary 3.2 *Let D be a strongly connected digraph of order n with minimum degree δ and arc-connectivity λ. If*

$$\max\{d^+(u), d^+(v)\} \geq \left\lfloor \frac{n}{2} \right\rfloor \text{ and } \max\{d^-(u), d^-(v)\} \geq \left\lfloor \frac{n}{2} \right\rfloor$$

for each arc uv, then $\lambda = \delta$.

Corollary 3.3 (Hellwig, Volkmann [8] 2005) *Let G be a connected graph of order n with minimum degree δ and edge-connectivity λ. If for each edge e there exists at least one vertex v incident with e such that*

$$d(v) \geq \left\lfloor \frac{n}{2} \right\rfloor,$$

then $\lambda = \delta$.

Corollary 3.4 *Let D be a (strongly) connected (di)graph of order n with restricted minimum edge(arc)-degree ξ_N, edge(arc)-connectivity λ and degree sequence $d_1 \geq d_2 \geq \dots \geq d_n = \delta$. If*

$$\sum_{i=1}^{\xi_N} d_{n+1-i} \geq \xi_N \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) + \delta,$$

then $\lambda = \delta$.

Inspired by this corollary we present an extension of Theorem 1.2. In order to prove this extension we use the following lemma.

Lemma 3.1 *Let D be a strongly connected digraph with minimum degree δ, arc-connectivity λ and minimum restricted arc-degree ξ_N. If $\lambda < \delta$, then there exist two disjoint vertex sets X and Y with $X \cup Y = V(D)$, $|(X,Y)| = \lambda$ and $|X|, |Y| \geq \xi_N + 2$.*

Proof. Let X and Y be two disjoint vertex sets with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$. We define $X_0 \subset X$ as the set of vertices, which do not have an out-neighbor in Y. If there does not exist an arc with both endpoints in X_0, then $\lambda = \delta$, a contradiction (cf. the proof of Theorem 3.1). Hence let $e = uv$ be an arbitrary arc in X_0. Since $N^+(e) \subseteq X$, $u, v \notin N^+(e)$ and $|N^+(e)| \geq \xi_N$, we obtain the desired result that

$$|X| \geq \xi_N + 2.$$

Similarly one can show that $|Y| \geq \xi_N + 2$. □

As a direct consequence of Lemma 3.1 we obtain the following generalization of Chartrand's [3] classical result that $\lambda = \delta$ when $\delta \geq \lfloor n/2 \rfloor$.

Corollary 3.5 *Let D be a (strongly) connected (di)graph of order n with minimum restricted edge(arc)-degree ξ_N and edge(arc)-connectivity λ. If $\xi_N \geq \lfloor n/2 \rfloor - 1$, then $\lambda = \delta$.*

It is easy to see that the following theorem generalizes Theorem 1.2, since $\xi_N + 2 > \delta$.

Theorem 3.2 *Let D be a digraph of order n with degree sequence $d_1 \geq d_2 \geq \dots \geq d_n = \delta$ and arc-connectivity λ. If*

$$\sum_{i=1}^{2k} d_{n+1-i} \geq kn - 3$$

for some k with $2 \leq k \leq \xi_N + 2$, then $\lambda = \delta$.

Proof. Suppose to the contrary that $\lambda < \delta$. Then there exist by Lemma 3.1 disjoint vertex sets X and Y with $X \cup Y = V(D)$, $|(X,Y)| = \lambda$ and $|X|, |Y| \geq \xi_N + 2$. Now let $S \subseteq X$, $T \subseteq Y$ be two vertex sets of cardinality k with $2 \leq k \leq \xi_N + 2$. Furthermore, choose S and T such that the number of arcs of (X,Y) incident with the vertices in S and T is minimum.

If $\xi_N + 2 - (\delta - 1) + 1 \leq k \leq \xi_N + 2$, then we conclude that

$$\begin{aligned}\sum_{x \in S} d(x) &\leq \sum_{x \in S} d^+(x) \leq k(|X| - 1) + \delta - 1 - (\xi_N + 2 - k) \\ &\leq k(|X| - 1) + \delta - 1 - (\delta + 1 - k) \\ &= k|X| - 2.\end{aligned}$$

Similarly, we have

$$\sum_{x \in T} d(x) \leq k|Y| - 2.$$

If $2 \leq k \leq \xi_N + 2 - (\delta - 1)$, then we conclude that

$$\sum_{x \in S} d(x) \leq \sum_{x \in S} d^+(x) \leq k(|X| - 1) = k|X| - k \leq k|X| - 2$$

and analogously

$$\sum_{x \in T} d(x) \leq k|Y| - 2.$$

Thus we obtain in both cases that

$$\sum_{i=1}^{2k} d_{n+1-i} \leq \sum_{x \in S \cup T} d(x) \leq k(|X| + |Y|) - 4 = kn - 4,$$

which yields a contradiction to the hypothesis. □

Corollary 3.6 *Let G be a graph of order n with degree sequence $d_1 \geq d_2 \geq \ldots \geq d_n = \delta$ and edge-connectivity λ. If*

$$\sum_{i=1}^{2k} d_{n+1-i} \geq kn - 3$$

for some k with $2 \leq k \leq \xi_N + 2$, then $\lambda = \delta$.

The following theorem can be proved in an analog manner to the proof of Theorem 3.1.

Theorem 3.3 *Let D be a strongly connected digraph of order n with arc-connectivity λ and minimum degree δ. If*

$$\sum_{x \in N^+(v)} d^+(x) \geq d^+(v) \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) + \delta,$$

for each vertex v with $d^+(v) \leq \lfloor n/2 \rfloor - 1$, and if

$$\sum_{x \in N^-(v)} d^-(x) \geq d^-(v) \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) + \delta,$$

for each vertex v with $d^-(v) \leq \lfloor n/2 \rfloor - 1$, then $\lambda = \delta$.

Corollary 3.7 *Let G be a connected graph of order n with edge-connectivity λ and minimum degree δ. If*

$$\sum_{x\in N(v)} d(x) \geq d(v)\left(\left\lfloor \frac{n}{2}\right\rfloor - 1\right) + \delta,$$

for each vertex v with $d(v) \leq \lfloor n/2\rfloor - 1$, then $\lambda = \delta$.

Corollary 3.8 *Let G be a connected graph of order n with edge-connectivity λ and minimum degree δ. If*

$$\sum_{x\in N(v)} d(x) \geq \left\lfloor \frac{n}{2}\right\rfloor^2 - 2\left\lfloor \frac{n}{2}\right\rfloor + 1 + \delta,$$

for each vertex v, then $\lambda = \delta$.

4 Examples

The following example shows that Corollary 3.7 is independent of Theorem 1.1. Furthermore, it shows that Theorem 2.2 is an improvement of Theorem 1.1, if $n \geq 15$ and $\delta \leq \lfloor n/2\rfloor - 2$, respectively $\delta \leq \lfloor n/2\rfloor - 3$, if n is odd.

Example 4.1 *Let H be the complete graph K_{n-p-1} with vertex set $V(H) = \{x_1, x_2, \dots, x_{n-p-1}\}$. If n is even, then $n \geq 2p+4, n \geq 15, p \geq 2$ and if n is odd, then $n \geq 2p+7, n \geq 15, p \geq 2$. Let $V(H)$ together with the additional vertices $y, y_1, y_2, ..., y_p$ be the vertex set of the graph G. Apart from the edges in H, there exist the edges yy_i and y_ix_j for all $i = 1, 2, ..., p, j = 1, 2, ..., n-p-2$. Then $n(G) = n$ and $\delta(G) = p$. The vertex y is the only vertex in $V(G)$ of minimum degree, and the only vertex with degree less or equal $\lfloor n(G)/2\rfloor - 1$. We observe that*

$$\begin{aligned}\sum_{x\in N(y)} d(x) &= \delta(G)(n - \delta(G) - 1) \\ &= \delta(G)(n - \delta(G) - 2) + \delta(G) \\ &\geq \delta(G)(n(G) - \delta(G) - 2) + \max\{0, 3\delta(G) - n(G) + 2\}\end{aligned}$$

and

$$\begin{aligned}\sum_{x\in N(y)} d(x) &= \delta(G)(n(G) - \delta(G) - 1) \\ &\geq \delta(G)(\lfloor n(G)/2\rfloor) \\ &= d(y)\left(\lfloor n(G)/2\rfloor - 1\right) + \delta(G).\end{aligned}$$

Thus Corollary 3.7 and Theorem 2.2 imply that $\lambda(G) = \delta(G)$. But, since

$$\begin{aligned}\sum_{x\in N(y)} d(x) &= \delta(G)(n(G) - \delta(G) - 1) \\ &\leq (\lfloor n(G)/2\rfloor - 2)(\lfloor n(G)/2\rfloor + 1) \\ &< \lfloor n(G)/2\rfloor^2 - \lfloor n(G)/2\rfloor,\end{aligned}$$

if n is even, Theorem 1.1 does not show that $\lambda(G) = \delta(G)$ in this case. Furthermore, it is

$$\begin{aligned}\sum_{x\in N(y)} d(x) &= \delta(G)(n(G)-\delta(G)-1)\\ &\leq (\lfloor n(G)/2\rfloor - 3)(\lfloor n(G)/2\rfloor + 3)\\ &< \lfloor n(G)/2\rfloor^2 - 7,\end{aligned}$$

if n is odd with $\delta(G) = p \leq \lfloor n/2\rfloor - 3$, and thus Theorem 1.1 does not show that $\lambda(G) = \delta(G)$ in this case.

The following example shows that Corollary 3.6 is best possible in the sense that $\sum_{i=1}^{2k} d_{n+1-i} \geq kn - 4$ for some $2 \leq k \leq \xi_N + 2$ does not guarantee $\lambda = \delta$.

Example 4.2 *Let H_1 and H_2 be two disjoint copies of the complete graph K_p with $p \geq 3$. We define G as the disjoint union of H_1 and H_2. Furthermore there exist $p-2$ vertex disjoint edges with one endpoint in $V(H_1)$ and one in $V(H_2)$. Then $n(G) = 2p$, $\delta(G) = p-1$, $\lambda(G) \leq p-2 < \delta(G)$, $\xi_N(G) = p-2$ and*

$$\sum_{i=1}^{2k} d_{n+1-i} = \sum_{i=1}^{2(\xi_N(G)+2)} d_{n+1-i} = (2p)p - 4 = kn - 4$$

for $k = \xi_N(G) + 2$.

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