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On properties of adjoint polynomials of graphs and their applications

Haixing Zhao   Ruying Liu

Department of Mathematics
Qinghai Normal University
Xining, Qinghai 810008
P.R. CHINA
h.x.zhao@eyou.com

Xueliang Li

Center for Combinatorics
Nankai University
Tianjin 300071
P.R. CHINA
x.li@eyou.com

Ligong Wang

Department of Mathematics
Northwestern Polytechnical University
Xi'an, Shaanxi 710072
P.R. CHINA

Abstract

For a graph G, we denote by $P(G,\lambda)$ the chromatic polynomial of G and by $h(G,x)$ the adjoint polynomial of G. A graph G is said to be *chromatically unique* if for any graph H, $P(H,\lambda) = P(G,\lambda)$ implies $H \cong G$. In this paper, we investigate some algebraic properties of the adjoint polynomials of some graphs. Using these properties, we obtain necessary and sufficient conditions for $K_n - E(\cup_{a,b}T_{1,a,b})$ and $\overline{(\cup_i C_{n_i}) \cup (\cup_i D_{m_j}) \cup (\cup_{a,b}T_{1,a,b})}$ to be chromatically unique if $G_i \in \{C_n, D_n, T_{1,a,b} | n \geq 5, 3 \leq a \leq 10, a \leq b\}$ and $h(P_m) \nmid h(G_i)$ for all $m \geq 2$. Moreover, many new chromatically unique graphs are given.

1 Introduction

All graphs considered here are finite and simple. For notations and terminology not defined here, we refer to [1]. For a graph G, let $V(G)$, $E(G)$, $p(G)$ and $q(G)$ denote the set of vertices, the set of edges, the number of vertices and the number of edges of G, respectively. The degree of a vertex v of G is denoted by $d_G(v)$, or simply by d_v. We denote by $\overline{G}$ the complement of G. Let G and H be two graphs. Here $G \cup H$ denotes the disjoint union of G and H, and mH denotes the disjoint union of m copies of H.

Let C_i (respectively, P_j) denote the cycle (respectively, the path) with i (respectively, j) vertices, where $i \geq 3$ (respectively, $j \geq 2$). We denote by D_k the graph obtained from C_3 and P_{k-2} by identifying a vertex of C_3 with an end-vertex of P_{k-2} and by T_{l_1,l_2,l_3} the tree with a vertex u of degree 3 such that $T_{l_1,l_2,l_3} - u = P_{l_1} \cup P_{l_2} \cup P_{l_3}$, where $k \geq 4$ and $l_i \geq 1$, $i = 1, 2, 3$. We denote by K_n the complete graph with n vertices. Let G be a subgraph of K_n. We denote by $K_n - E(G)$ the graph obtained from K_n by deleting all the edges of G.

For a positive integer r, a partition $\{A_1, A_2, \cdots, A_r\}$ of $V(G)$ is called an *r-independent partition* of a graph G if every A_i is a nonempty independent set of G. Let $m_r(G)$ denote the number of r-independent partitions of $V(G)$. Then the chromatic polynomial of G is $P(G, \lambda) = \sum_{r \geq 1} \alpha(G, r)(\lambda)_r$, where

$$(\lambda)_r = \lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - r + 1) \text{ for all } r \geq 1;$$

see [13] for more details. Two graphs G and H are called *chromatically equivalent*, denoted by $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph G is called *chromatically unique* (or simply *χ-unique*) if $H \cong G$ whenever $H \sim G$.

Definition 1.1. [11,12] Let G be a graph with p vertices. The polynomial

$$h(G, x) = \sum_{i=0}^{p-1} \alpha(\overline{G}, p - i)x^{p-i}$$

is called the *adjoint polynomial* of G. A graph G is called *adjointly unique* if $h(H, x) = h(G, x)$ implies that H is isomorphic to G.

From Definition 1.1, we have

Theorem 1.1. [3-5] *Any graph G is adjointly unique if and only if $\overline{G}$ is χ-unique.*

Definition 1.2. [11,12] Let G be a graph and $h(G, x) = x^{\alpha(G)}h_1(G, x)$, where $h_1(G, x)$ is a polynomial with a nonzero constant term. If $h_1(G, x)$ is irreducible over the rational number field, then G is called an *irreducible graph*.

The adjoint polynomial of G has many algebraic properties, such as the recursive relation, divisibility, reducibility over the rational number field, etc.; see [3–6] and

[10–12,14–16] for more details. These properties are very useful in the study of chromatic uniqueness of graphs. Many chromatically equivalent classes of graphs have been found by applying these properties, see [3–5] and [9–12]. In [6,9,14], Du, Liu, Li and Wang shown that if D_n and T_{l_1,l_2,l_3} are irreducible, then $\overline{\bigcup_{l_1,l_2,l_3} T_{l_1,l_2,l_3}}$ and $\overline{\bigcup_{j=1}^{t} D_{m_j}}$ are χ-unique.

The main goal of this paper is to study the algebraic properties of $h(P_n)$ and $h(T_{l_1,l_2,l_3})$. Using these properties, we investigate the chromaticity of $K_n - E(\cup_{a,b}T_{1,a,b})$ and $\overline{(\cup_i C_{n_i}) \cup (\cup_i D_{m_j}) \cup (\cup_{a,b}T_{1,a,b})}$, where $3 \leq a \leq 10$ and $a \leq b$. Moreover we obtain many new chromatically unique graphs.

For convenience, we sometimes simply denote $h(G,x)$ by $h(G)$ and $h_1(G,x)$ by $h_1(G)$. Let $f(x)$ and $g(x)$ be two polynomials in x. We denote by $(g(x), f(x))$ the greatest common factor of $g(x)$ and $f(x)$. Also $g(x)|f(x)$ (respectively., $g(x) \not| f(x)$) means that $g(x)$ divides $f(x)$ (respectively, $g(x)$ does not divide $f(x)$). We denote by $\partial f(x)$ the degree of $f(x)$. For any real number a, $\lfloor a \rfloor = \max\{x \mid x \leq a$ and x is an integer$\}$.

2 Preliminaries

Definition 2.1. [12] The *character* of a graph G is defined as follows:

$$R(G) = \begin{cases} 0, & \text{if } q(G) = 0, \\ b_2(G) - \binom{b_1(G) - 1}{2} + 1, & \text{if } q(G) > 0, \end{cases}$$

where $b_1(G)$ and $b_2(G)$ denote the second and the third coefficients of $h(G)$, respectively.

Lemma 2.1. [12] *Let G and H be two graphs. If $h(G,x) = h(H,x)$ or $h_1(G,x) = h_1(H,x)$, then $R(G) = R(H)$.*

Lemma 2.2. [12] *Let G be a graph with k components $G_1, G_2, \ldots, G_k$. Then*

(i) $h(G) = \prod_{i=1}^{k} h(G_i)$,

(ii) $R(G) = \sum_{i=1}^{k} R(G_i)$.

Lemma 2.3. [12] *Let G be a graph with $e = uv \in E(G)$ and e does not belong to any triangle of G. Then*

$$h(G,x) = h(G - uv, x) + xh(G - \{u,v\}, x),$$

where $G - uv$ denotes the graph obtained from G by deleting the edge uv and $G - \{u,v\}$ denotes the graph obtained from G by deleting the vertices u and v together with their incident edges, respectively.

Lemma 2.4. [12] *Let G be a connected graph with n vertices. Then* *(i)* $R(G) \leq 1$, *and the equality holds if and only if* $G \cong P_n (n \geq 2)$ *or* $G \cong K_3$.

(ii) $R(G) = 0$ *if and only if* G *is one of the graphs* K_1, C_n, D_n *and* T_{l_1,l_2,l_3}, *where* $n \geq 4$ *and* $l_i \geq 1$, $i = 1, 2, 3$.

Lemma 2.5. [12] *(i) For* $n \geq 2$, $h(P_n) = \sum_{k \leq n} \binom{k}{n-k} x^k$;

(ii) For $n \geq 4, h(C_n) = \sum_{k \leq n} \frac{n}{k} \binom{k}{n-k} x^k$.

From Definition 1.2 and Lemma 2.5, we have

Lemma 2.6 *(i) For* $n \geq 2$, $\partial(h_1(P_n)) = \lfloor \frac{n}{2} \rfloor$ *and* $\alpha(P_n) = \lfloor \frac{n+1}{2} \rfloor$;

(ii) for $n \geq 3$, $h(P_n) = x(h(P_{n-1}) + h(P_{n-2}))$.

Lemma 2.7. [15] *For* $n \geq 2$, *we have:*

(i) $h(P_n) | h(P_m)$ *if and only if* $(n+1)|(m+1)$;

(ii) $h(P_n)$ *is irreducible if and only if* $n = 3$ *or* $n+1$ *is prime.*

Lemma 2.8 [14] *(i) For* $n \geq 4$, $h(T_{1,1,n-2}, x) = xh(C_n, x)$;

(ii) For $n \geq 4$, $h(T_{1,2,n-3}, x) = xh(D_n, x)$.

3 Some properties of adjoint polynomials of graphs

Lemma 3.1. *(i) For* $t \geq 1$ *and* $m \geq 4$, $h(T_{1,t,m}) = x[h(T_{1,t,m-1}) + h(T_{1,t,m-2})]$.

(ii) Let $n = |V(T_{1,t,m})| = m + t + 2$. *Then*

$$\partial h_1(T_{1,t,m}) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor, & \text{if } t \text{ and } m \text{ are even,} \\ \left\lfloor \frac{n-1}{2} \right\rfloor, & \text{otherwise.} \end{cases}$$

(iii) Let $n = |V(T_{1,t,m})| = m + t + 2$. *Then*

$$\alpha(T_{1,t,m}) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor, & \text{if } t \text{ and } m \text{ are even,} \\ \left\lfloor \frac{n+2}{2} \right\rfloor, & \text{otherwise.} \end{cases}$$

Proof. (i) From Lemma 2.3, it is obvious.

(ii) Choose the edge $e = uv \in E(T_{1,t,m})$ such that $d_u = 1$ and $d_v = 3$. By Lemma 2.3, $h(T_{1,t,m}) = x[h(P_{m+t+1}) + h(P_t)h(P_m)]$. From Lemma 2.6, we have $\partial(h_1(P_{m+t+1})) = \lfloor \frac{m+t+1}{2} \rfloor$ and $\partial(h_1(P_t)h_1(P_m)) = \lfloor \frac{m}{2} \rfloor + \lfloor \frac{t}{2} \rfloor$. Clearly, $\partial(h_1(P_{m+t+1})) \geq \partial(h_1(P_t)h_1(P_m))$. Noticing that $\partial(h(P_{m+t+1})) = \partial(h(P_t)h(P_m)) + 1$, we have

$$\partial(h_1(T_{1,t,m})) = \partial(h_1(P_{m+t+1})) + 1 \quad \text{for} \quad \partial(h_1(P_{m+t+1})) = \partial(h_1(P_t)h_1(P_m))$$

and

$$\partial(h_1(T_{1,t,m})) = \partial(h_1(P_{m+t+1})) \quad \text{for} \quad \partial(h_1(P_{m+t+1})) > \partial(h_1(P_t)h_1(P_m)).$$

It is not difficult to verify that $\partial(h_1(P_{m+t+1})) = \partial(h_1(P_t)h_1(P_m))$ only if m and t are even. So, (ii) of the lemma holds.

Clearly, (iii) follows from (ii) of the lemma. □

Lemma 3.2. *Let $\{g_i(x)\}_i$ $(i \geq 0)$ be a sequence of polynomials with integral coefficients and $g_n(x) = x(g_{n-1}(x) + g_{n-2}(x))$. Then*

(i) $g_n(x) = h(P_k)g_{n-k}(x) + xh(P_{k-1})g_{n-k-1}(x)$;

(ii) $h_1(P_n)|g_{n+1+i}(x)$ if and only if $h_1(P_n)|g_i(x)$, for any positive integers n and i.

Proof. (i) By induction on k. By $h(P_1) = x$ and $h(P_0) = 1$, we have

$$g_n(x) = h(P_1)g_{n-1}(x) + xh(P_0)g_{n-2}(x).$$

So, (i) of the lemma holds when $k = 1$. Suppose that it is true for $k \leq l-1$. From the recursive relation of $g_n(x)$, Lemma 2.6(ii) and the induction hypothesis, we have

$$\begin{aligned} g_n(x) &= x(g_{n-1}(x) + g_{n-2}(x)) \\ &= xh(P_{l-1})g_{n-l}(x) + x^2h(P_{l-2})g_{n-l-1}(x) + \\ &\qquad xh(P_{l-2})g_{n-l}(x) + x^2h(P_{l-3})g_{n-l-1}(x) \\ &= h(P_l)g_{n-l}(x) + xh(P_{l-1})g_{n-l-1}(x). \end{aligned}$$

(ii) From (i) of the lemma, for any integers n and i, it follows

$$g_{n+1+i}(x) = h(P_{n+1})g_i(x) + xh(P_n)g_{i-1}(x).$$

It is not difficult to see that $(h_1(P_n), h_1(P_{n+1})) = 1$ and $(h_1(P_n), x) = 1$ for $n \geq 2$. So, from the above equality we have $h_1(P_n)|g_{n+1+i}(x)$ if and only if $h_1(P_n)|g_i(x)$. □

Lemma 3.3. [10] *Let $1 \leq r_1 \leq r_2$ and $r_1 \leq s_1 \leq s_2$ such that $r_1+r_2 = s_1+s_2$. Then $h(P_{r_1})h(P_{r_2}) - h(P_{s_1})h(P_{s_2}) = (-1)^{r_1}x^{r_1+1}h(P_{s_1-r_1-1})h(P_{s_2-r_1-1})$, where $h(P_0) = 1$.*

Theorem 3.1. *For $k \geq 1$ and $t \geq 1$ such that $kt > 3$, we have that $h(P_{t-1})|h(T_{1,t,kt-3})$, $h(P_t)|h(T_{1,t,kt+k-1})$ and $h(P_{t+2})|h(T_{1,t,k(t+3)})$.*

Proof. Suppose that $g_0(x) = (-1)^t\dfrac{h(P_t)^2}{x^t}$,
$g_1(x) = (-1)^{t-1}\left\{h(P_t)h(P_{t-3}) + h(P_{t-1})^2\right\}/x^{t-2}$ and $g_n(x) = x[g_{n-1}(x) + g_{n-2}(x)]$. We have the following claim.

Claim. For $n \geq t+3$, $g_n(x) = h(T_{1,t,n-t-2})$.
Proof of the claim: Noticing that $h(P_t)^2 = x(h(P_t)h(P_{t-2}) + h(P_t)h(P_{t-1}))$, from Lemmas 3.2 and 3.3, we can obtain by calculating that

$$\begin{aligned} g_{t+3}(x) &= h(P_{t+2})g_1(x) + xh(P_{t+1})g_0(x) \\ &= \frac{(-1)^{t-1}h(P_t)}{x^{t-2}}\left[h(P_{t-3})h(P_{t+2}) - h(P_{t-2})h(P_{t+1})\right] \\ &\qquad + \frac{(-1)^{t-1}h(P_{t-1})}{x^{t-2}}\left[h(P_{t-1})h(P_{t+2}) - h(P_t)h(P_{t+1})\right] \\ &= h(P_3)h(P_t) + x^3h(P_{t-1}). \end{aligned}$$

By Lemma 2.3, $h(T_{1,t,1}) = h(P_3)h(P_t) + x^3h(P_{t-1})$. Thus, $g_{t+3}(x) = h(T_{1,t,1})$.

Similarly, from Lemmas 2.3, 3.2 and 3.3, we can show that $g_{t+4}(x) = h(T_{1,t,2}) = h(P_4)h(P_t)+x^2h(P_2)h(P_{t-1})$. Using the recursive relation of $g_n(x)$, from (i) of Lemma 3.1, we know that for $n \geq t+3$, $g_n(x) = h(T_{1,t,n-t-2})$. This completes the proof of the claim.

Using the recursive relation of $g_n(x)$, from (i) of Lemma 3.2, we can obtain by calculating that $g_{t+2}(x) = \frac{g_{t+4}(x)-xg_{t+3}(x)}{x} = h(P_{t+2})$, $g_{t+1}(x) = \frac{g_{t+3}(x)-xg_{t+2}(x)}{x} = xh(P_t)$ and $g_{t-1}(x) = \frac{(x+1)g_{t+1}(x)-g_{t+2}(x)}{x} = xh(P_{t-1})$. Clearly, $h_1(P_{t-1})|g_{t-1}(x)$, $h_1(P_t)|g_{t+1}(x)$, $h_1(P_{t+2})|g_{t+2}(x)$. So, by (ii) of Lemma 3.2, $h_1(P_{t-1})|g_{kt+t-1}(x)$, $h_1(P_t)|g_{(t+1)k+t+1}(x)$ and $h_1(P_{t+2})|g_{(t+3)k+t+2}(x)$. Namely, $h_1(P_{t-1})|h(T_{1,t,kt-3})$, $h_1(P_t)|h(T_{1,t,kt+k-1})$ and $h_1(P_{t+2})|h(T_{1,t,k(t+3)})$. Thus, from (i) of Lemma 2.6 and (iii) of Lemma 3.1, it is not difficult to see that $h(P_{t-1})|h(T_{1,t,kt-3})$, $h(P_t)|h(T_{1,t,kt+k-1})$ and $h(P_{t+2})|h(T_{1,t,k(t+3)})$. This completes the proof of the theorem. □

Theorem 3.2. *For $l \geq 2$, $m \geq 1$ and $k \geq 1$, we have:*

(i) $h(P_l)|h(T_{1,1,m})$ if and only if $(l,m) \in \{(3,4k)\}$;

(ii) $h(P_l)|h(T_{1,2,m})$ if and only if $(l,m) \in \{(2,3k-1),(4,5k)\}$;

(iii) $h(P_l)|h(T_{1,3,m})$ if and only if $(l,m) \in \{(2,3k),(3,4k-1),(5,6k)\}$;

(iv) $h(P_l)|h(T_{1,4,m})$ if and only if $(l,m) \in \{(3,4k-3),(4,5k-1),(6,7k)\}$;

(v) $h(P_l)|h(T_{1,5,m})$ if and only if $(l,m) \in \{(2,3k-1),(3,4k),(4,5k-3),(5,6k-1),(7,8k)\}$;

(vi) $h(P_l)|h(T_{1,6,m})$ if and only if $(l,m) \in \{(2,3k),(5,6k-3),(6,7k-1),(8,9k)\}$.

Proof. Let $g_0(x) = (-1)^t\frac{h(P_t)^2}{x^t}$, $g_1(x) = (-1)^{t-1}\frac{h(P_t)h(P_{t-3})+h(P_{t-1})^2}{x^{t-2}}$ and $g_n(x) = x[g_{n-1}(x)+g_{n-2}(x)]$. From the proof of Theorem 3.1, one can see that if $n \geq t+3$, $g_n(x) = h(T_{1,t,n-t-2})$.

Without loss of generality, assume that $n = (l+1)k+i$, where $0 \leq i \leq l$. By Lemma 3.2, $h_1(P_l)|g_n(x)$ if and only if $h_1(P_l)|g_i(x)$ for $0 \leq i \leq l$. Note that $g_i(x) = h(T_{1,t,i-t-2})$ for $l \geq t+3$. From (i) of Lemma 2.6 and (ii) of Lemma 3.1, we have $\partial h_1(P_l) = \lfloor l/2 \rfloor$ and $\partial(g_i(x)) = \partial h_1(T_{1,t,i-t-2}) \leq \lfloor i/2 \rfloor \leq \lfloor l/2 \rfloor$. Thus, if $h_1(P_l)|h_1(T_{1,t,i-t-2})$, then $\partial(h_1(P_l)) = \partial(h_1(T_{1,t,i-t-2}))$. Moreover, it must hold that $h_1(P_l) = h_1(T_{1,t,i-t-2})$. So, by Lemma 2.1, $R(P_l) = R(T_{1,t,i-t-2})$, which contradicts the fact that $R(P_l) \neq R(T_{1,t,i-t-2})$. Therefore, we have that if $l \geq t+3$, then $h(P_l) \nmid h(T_{1,t,i-t-2})$. Thus, we only need to consider the cases of $l \leq t+2$.

Case 1. $t = 1$. Clearly, $l \leq 3$.

By calculating we have that $g_0(x) = -x$, $g_1(x) = x$, $g_2(x) = x^2$ and $g_3(x) = h(P_3)$. It is easy to verify that $h_1(P_l)|g_i(x)$ if and only if $l = i = 3$ for $2 \leq l \leq 3$ and $0 \leq i \leq 3$. By Lemma 3.2(ii), $h_1(P_3)|g_{4k+3}(x)$. Thus, $h_1(P_l)|h(T_{1,1,m})$ if and only if $l = 3$ and $m = 4k$, where $k \geq 1$. From (i) of Lemma 2.6 and (iii) of Lemma 3.1, we can obtain that if $m \geq 4$, then $h(P_l)|h(T_{1,1,m})$ if and only if $l = 3$ and $m = 4k$ for $k \geq 1$. This completes the proof of (i) of the theorem.

Case 2. $t = 2$. So, $l \leq 4$.

By calculating, it is easy to obtain that $g_0(x) = [h_1(P_2)]^2$, $g_1(x) = -x^2$, $g_2(x) = 2x^2 + x$, $g_3(x) = x^2 h(P_2)$ and $g_4(x) = x^2 h_1(P_4)$. One can see that $h_1(P_l)|g_i(x)$ if and only if $(l, i) \in \{(2, 0), (2, 3), (4, 4)\}$ for $2 \leq l \leq 4$ and $0 \leq i \leq 4$. From Lemma 3.2(ii), it is not difficult to see that $h_1(P_2)|g_{3k+3}$ and $h_1(P_4)|g_{5k+4}$. Hence, $h_1(P_l)|h(T_{1,2,m})$ if and only if $(l, m) \in \{(2, 3k - 1), (4, 5k)\}$. Similar to the proof of (i), we know that (ii) holds.

Case 3. $t = 3$. So, $l \leq 5$.

By calculating, we have that $g_0(x) = -[h_1(P_3)]^2$, $g_1(x) = x(x^2 + 3x + 3)$, $g_2(x) = -x^2 h_1(P_2)$, $g_3(x) = x^2(2x + 3)$, $g_4(x) = x^3 h_1(P_3)$ and $g_5(x) = x^3 h_1(P_5)$. One can verify that $h_1(P_l)|g_i(x)$ if and only if $(l, i) \in \{(3, 0), (2, 2), (3, 4), (5, 5)\}$ for $2 \leq l \leq 4$ and $0 \leq i \leq 5$. A completely similar proof of Case 1, we can show that (iii) holds.

Similarly, we can show that (iv), (v) and (vi) hold. Here we give the expression of $g_i(x)$ whereas the details of the proof are omitted.

When $t = 4$, $g_0(x) = [h_1(P_4)]^2$, $g_1(x) = -x(x^3 + 5x^2 + 7x + 1)$, $g_2(x) = x(x^3 + 4x^2 + 5x + 1)$, $g_3(x) = -x^3 h_1(P_3)$, $g_4(x) = x^2(2x^2 + 5x + 1)$, $g_5(x) = x^3 h_1(P_4)$ and $g_6(x) = x^3 h_1(P_6)$.

When $t = 5$, $g_0(x) = -[h_1(P_5)]^2$, $g_1(x) = x(x^4 + 7x^3 + 16x^2 + 13x + 4)$, $g_2(x) = -x^2(x^3 + 6x^2 + 11x + 5)$, $g_3(x) = x^2 h_1(P_2)h_1(P_3)$, $g_4(x) = -x^3 h_1(P_4)$, $g_5(x) = x^3(2x^2 + 7x + 4)$, $g_6(x) = x^4 h_1(P_5)$ and $g_7(x) = x^4 h_1(P_7)$.

When $t = 6$, $g_0(x) = [h_1(P_6)]^2$, $g_1(x) = -x(x^5 + 9x^4 + 29x^3 + 40x^2 + 22x + 2)$, $g_2(x) = xh_1(P_2)(x^4 + 7x^3 + 15x^2 + 9x + 1)$, $g_3(x) = -x(x^4 + 7x^3 + 16x^2 + 12x + 1)$, $g_4(x) = x^2(x^4+6x^3+12x^2+9x+1)$, $g_5(x) = -x^4 h_1(P_5)$, $g_6(x) = x^3(2x^3+9x^2+9x+1)$, $g_7(x) = -x^4 h_1(P_6)$ and $g_8(x) = x^4 h_1(P_8)$.

The proof of the theorem is complete. □

From Theorem 3.2, it is not difficult to see that for $1 \leq t \leq 6$ and $n \geq 2$, $h(P_n)|h(T_{1,t,m})$ if and only if $n + 1|t$, or $n + 1|t + 1$, or $n + 1|t + 3$. So, we propose the following problem.

Problem 3.1. For $n \geq 2$ and $m \geq t \geq 1$, find a necessary and sufficient condition of $h(P_n)|h(T_{1,t,m})$. In particular, is it true that $h(P_n)|h(T_{1,t,m})$ if and only if $n + 1|t$, or $n + 1|t + 1$, or $n + 1|t + 3$?

For a graph G, let $f(G, x)$ denote the characteristic polynomial of G. We denote by $\gamma(G)$ and $\beta(G)$ the maximum root of $f(G, x)$ and the minimum root of $h(G, x)$, respectively.

Lemma 3.4. *(i)* [14] *For a tree T, we have that $\beta(T) = -(\gamma(G))^2$;*

(ii) [2] *For T_{l_1,l_2,l_3}, we have that $\gamma(T_{l_1,l_2,l_3}) \leq \sqrt{2 + \sqrt{5}}$ if and only if l_1, l_2, l_3*

satisfy the following: $l_1 = 1$*, or* $l_1 = l_2 = 2$*, or* $l_1 = 2$ *and* $l_2 = l_3 = 3$.

From Lemma 3.4, the following lemma can be obtained.

Lemma 3.5. *For* T_{l_1,l_2,l_3}*,* $\beta(T_{l_1,l_2,l_3}) \geq -(2+\sqrt{5})$ *if and only if* l_1, l_2, l_3 *satisfy the following:* $l_1 = 1$*, or* $l_1 = l_2 = 2$*, or* $l_1 = 2$ *and* $l_2 = l_3 = 3$. □

We denote by A_n the graph obtained from C_{n-1} by adding a pendant edge. Let P_{a+b+3} be a path $x_1x_2\cdots x_{a+b+3}$. We denote by $A_{a,b}$ the graph obtained by adding pendant edges at x_{a+1} and x_{a+b+2} in P_{a+b+3}. In particular, $A_{1,n-2}$ is denoted simply by W_n. An *internal* x_1x_k*-path* of a graph G is a path $x_1x_2x_3\cdots x_k$ (possibly $x_1 = x_k$) of G such that d_{x_1} and d_{x_k} are at least 3 and $d_{x_2} = d_{x_3} = \cdots = d_{x_{k-1}} = 2$ (unless $k = 2$).

Lemma 3.6. [2] *(i) Let* G_{uv} *denote the graph obtained from* G *by introducing a new vertex on the edge* uv *of* G *. If* uv *is an edge on an internal path of* G *and* $G \not\cong W_n$*, then* $\gamma(G_{uv}) < \gamma(G)$*;*
(ii) If H *is a proper subgraph of* G*, then* $\gamma(H) < \gamma(G)$*;*
(iii) If $n \geq 2$*, then* $\gamma(W_n) = 2$.

From Lemmas 3.4 and 3.6, we have

Lemma 3.7. *Let* G *be a tree.*
(i) If uv *is an edge on an internal path of* G *and* $G \neq W_n$*, then* $\beta(G) < \beta(G_{uv})$*;*
(ii) If H *is a proper subgraph of* G*, then* $\beta(G) < \beta(H)$*;*
(iii) If $n \geq 2$*, then* $\beta(W_n) = -4$. □

Lemma 3.8. [14] *For any* $n \geq 2$*, we have:*
(i) $h(T_{1,n,n+3}) = h(P_{n+1})h(A_{n+3})$,
(ii) $h(T_{1,n,n}) = h(P_n)h(A_{n+2})$,
(iii) $h(T_{1,n,2n+5}) = h(C_{n+2})h(T_{1,n+1,n+2})$,
(iv) $h(T_{2,2,n}) = h(P_2)h(A_{n+3})$,
(v) $h(T_{2,3,3}) = x^3h(P_3)(x^3 + 6x^2 + 8x + 2)$,
(vi) $\beta(T_{1,n,n}) = \beta(T_{1,n-1,n+2})$ *and* $\beta(T_{1,n,n+1}) = \beta(T_{1,n-1,2n+3})$.

Theorem 3.3. *(i) For* $n \geq 2$ *and* $m \geq 6$*,*

$$\beta(T_{1,2,m+1}) < \beta(T_{1,2,m}) < \beta(T_{1,2,5}) < \beta(T_{1,1,n}) < \beta(T_{1,1,n-1}).$$

(ii) For $3 \leq l \leq 11$*,* $n \geq 3$ *and* $m \geq l+3$*,*

$$\beta(T_{1,l,m+1}) < \beta(T_{1,l,m}) < \beta(T_{1,l,l+2}) < \beta(T_{1,l-1,n}) < \beta(T_{1,l-1,n-1}).$$

(iii) For $T_1 \in \{T_{1,l_1,l_2} | 3 \leq l_1 \leq 10, l_1 \leq l_2\}$ *and* $T_2 \in \{T_{1,l_1,l_2} | 1 \leq l_1 \leq l_2\}$*, we have* $\beta(T_1) = \beta(T_2)$ *and* $T_1 \not\cong T_2$ *if and only if* $\beta(T_{1,n,n}) = \beta(T_{1,n-1,n+2})$ *and* $\beta(T_{1,n,n+1}) = \beta(T_{1,n-1,2n+3})$.

Proof. The proof of (i) and (ii): By Lemmas 2.2 and 2.3,

$$h(T_{1,l_1,l_2}) = xh(P_{l_1+l_2+1}) + xh(P_{l_1})h(P_{l_2})$$

and

$$h(A_{a,b}) \quad = \quad xh(T_{1,1,a+b+1}) + xh(P_a)h(T_{1,1,b}).$$

By calculating, we have $h(A_{1,1}) = x^7 + 6x^6 + 8x^5$. By Lemma 2.8, one can get that $h(A_{a,b}) = x^2h(C_{a+b+3}) + x^2h(C_{b+2})h(P_a)$ for $b \geq 2$. From Lemma 2.5, by calculating we obtain the coefficients of $h(T_{1,l_1,l_2})$ and $h(A_{a,b})$, given in Tables 1 and 2.

(l_1, l_2)	The coefficients of $h(T_{1,l_1,l_2})$: $b_0, b_1, b_2, b_3, \cdots$
$(2, 5)$	$1, 8, 20, 17, 4$
$(3, 5)$	$1, 9, 27, 31, 11$
$(4, 6)$	$1, 11, 44, 78, 59, 15, 1$
$(5, 7)$	$1, 13, 65, 157, 188, 102, 19$
$(6, 8)$	$1, 15, 90, 276, 458, 400, 164, 24, 1$
$(7, 9)$	$1, 17, 119, 443, 945, 1159, 776, 250, 29$
$(8, 10)$	$1, 19, 152, 666, 1741, 2773, 2636, 1402, 365, 35, 1$
$(9, 11)$	$1, 21, 189, 953, 2954, 5812, 7237, 5515, 2393, 515, 41$
$(10, 12)$	$1, 23, 230, 1312, 4708, 11054, 17120, 17216, 10787, 3899, 706, 48, 1$
$(11, 13)$	$1, 25, 275, 1751, 7143, 19517, 36274, 45644, 37982, 19958, 6111, 945, 55$

Table 1. The coefficients of $h(T_{1,l_1,l_2})$.

(a, b)	The coefficients of $h(A_{a,b})$: $b_0, b_1, b_2, b_3, \cdots$
$(1, 1)$	$1, 6, 8$
$(2, 7)$	$1, 13, 64, 148, 162, 75, 11$
$(3, 7)$	$1, 14, 76, 201, 266, 160, 31$
$(4, 9)$	$1, 17, 118, 430, 880, 1002, 589, 152, 13$
$(5, 11)$	$1, 20, 169, 785, 2184, 3718, 3795, 2177, 610, 58$
$(6, 13)$	$1, 23, 229, 1293, 4556, 10388, 15379, 14443, 8152, 2503, 351, 17$
$(7, 15)$	$1, 26, 298, 1981, 8455, 24225, 47328, 62764, 55198, 30744, 10003, 1636, 93$
$(8, 17)$	$1, 29, 376, 2876, 14421, 49819, 121296, 209304, 253878, 211718, 116689,$ $39840, 7574, 671, 21$
$(9, 19)$	$1, 32, 463, 4005, 23075, 93380, 272734, 581647, 906015, 1020680, 814606,$ $445093, 157785, 33292, 3585, 136$
$(10, 21)$	$1, 35, 559, 5395, 35119, 162981, 555750, 1414270, 2700775, 3860021,$ $4085950, 3142790, 1704795, 623400, 143448, 18620, 1140, 25$

Table 2. The coefficients of $h(A_{a,b})$.

Note: *For each $h(G)$ in Tables 1 and 2,* $h(G, x) = \sum_{i=0}^{p(G)} b_i x^{p(G)-i}$, *where* $p(T_{1,l_1,l_2}) = l_1 + l_2 + 2$ *and* $p(A_{a,b}) = a + b + 5$.

Using software Mathematica, we get the minimum roots of $h(T_{1,l_1,l_2})$ and $h(A_{a,b})$, given in Table 3.

(l_1, l_2)	$\beta(T_{1,l_1,l_2})$	(a, b)	$\beta(A_{a,b})$
$(2, 5)$	-4.0000	$(1, 1)$	-4.00000
$(3, 5)$	-4.09529	$(2, 7)$	-4.09529
$(4, 6)$	-4.16035	$(3, 7)$	-4.15875
$(5, 7)$	-4.19353	$(4, 9)$	-4.18970
$(6, 8)$	-4.21145	$(5, 11)$	-4.20829
$(7, 9)$	-4.22153	$(6, 13)$	-4.21937
$(8, 10)$	-4.22736	$(7, 15)$	-4.22597
$(9, 11)$	-4.23080	$(8, 17)$	-4.22993
$(10, 12)$	-4.23286	$(9, 19)$	-4.23232
$(11, 13)$	-4.23411	$(10, 21)$	-4.23378

Table 3. The minimum roots of $h(T_{1,l_1,l_2})$ and $h(A_{a,b})$.

By Lemma 3.7, we have

$$\beta(A_{a,b}) < \beta(A_{a,b+1}) < \beta(A_{a,b+2}) < \cdots < \beta(A_{a,b+k}) \; for \; k \geq 3 \tag{1}$$

and

$$\beta(A_{a,b}) < \beta(T_{1,a,b+2}). \tag{2}$$

From Table 3, one sees that $\beta(T_{1,2,5}) = \beta(A_{1,1})$ and $\beta(T_{1,3,5}) = \beta(A_{2,7})$, and $\beta(T_{1,l+1,l+3}) < \beta(A_{l,2l+1})$ for $3 \leq l \leq 10$. So, by (1), (2) and Lemma 3.7, we have:

(a) for $l = 1$, $m \geq 6$ and $n \geq 2$, $\beta(T_{1,2,m+1}) < \beta(T_{1,2,m}) < \beta(T_{1,2,5}) = \beta(A_{1,1}) = \beta(W_n) < \beta(T_{1,1,n}) < \beta(T_{1,1,n-1})$,

(b) for $l = 2$, $m \geq 6$ and $n \geq 2$, $\beta(T_{1,3,m+1}) < \beta(T_{1,3,m}) < \beta(T_{1,3,5}) = \beta(A_{2,7}) < \beta(A_{2,n+6}) < \beta(T_{1,2,n}) < \beta(T_{1,2,n-1})$,

(c) for $3 \leq l \leq 10$, $m \geq l + 4$ and $n \geq 2$, $\beta(T_{1,l+1,m+1}) < \beta(T_{1,l+1,m}) < \beta(T_{1,l+1,l+3}) < \beta(A_{l,2l+1}) < \beta(A_{l,n+2l}) < \beta(T_{1,l,n}) < \beta(T_{1,l,n-1})$.

Thus, from (a), (b) and (c), we know that (i) and (ii) of the theorem hold.

The proof of (iii). By (i) and (ii) of the theorem and (ii) of Lemma 3.7, we have:

(d) for $m \geq 6$ and $n \geq 2$, $\beta(T_{1,3,m+1}) < \beta(T_{1,3,m}) < \beta(T_{1,3,5}) < \beta(T_{1,3,4}) < \beta(T_{1,3,3}) = \beta(T_{1,2,5}) < \beta(T_{1,1,n}) < \beta(T_{1,1,n-1})$,

(e) for $m \geq 6$ and $n \geq 12$, $\beta(T_{1,3,m+1}) < \beta(T_{1,3,m}) < \beta(T_{1,3,5}) < \beta(T_{1,2,n}) < \beta(T_{1,2,n-1}) < \beta(T_{1,2,10}) < \beta(T_{1,2,9}) = \beta(T_{1,3,4}) < \beta(T_{1,2,8}) < \beta(T_{1,2,7}) < \beta(T_{1,2,6}) < \beta(T_{1,3,3}) = \beta(T_{1,2,5}) < \beta(T_{1,2,4}) < \beta(T_{1,2,3}) < \beta(T_{1,2,2})$,

(f) for $3 \leq l \leq 10$, $m \geq l+4$ and $n \geq 2l+8$, $\beta(T_{1,l+1,m+1}) < \beta(T_{1,l+1,m}) < \beta(T_{1,l+1,l+3}) < \beta(T_{1,l,n}) < \beta(T_{1,l,n-1}) < \beta(T_{1,l,2l+6}) < \beta(T_{1,l+1,l+2}) = \beta(T_{1,l,2l+5}) < \beta(T_{1,l,2l+4}) < \cdots < \beta(T_{1,l,l+5}) < \beta(T_{1,l,l+4}) < \beta(T_{1,l+1,l+1}) = \beta(T_{1,l,l+3}) < \beta(T_{1,l,l+2}) < \beta(T_{1,l,l+1}) < \beta(T_{1,l,l})$,

(g) for $l \geq 11$, $m \geq l+1$ and $n \geq 2$, $\beta(T_{1,l+1,m}) < \beta(T_{1,l+1,l+1}) \leq \beta(T_{1,12,12}) = \beta(T_{1,11,13}) < \beta(T_{1,10,n})$ by (vi) of Lemma 3.8.

So, from (d), (e), (f) and (g), it is not difficult to see that (iii) holds.

The proof of the theorem is complete. □

From the theorem, we propose the following.

Problem 3.2. Is it true that $\beta(T_{1,l,l+2}) < \beta(T_{1,l-1,n})$ for all $l \geq 3$ and $n \geq 1$.

4 Some chromatically unique graphs

Lemma 4.1. *Let $f_i(x)$ be function with integral coefficients. If $h_1(P_m) \not| f_i(x)$ for $m \geq 2$ and $i = 1, 2, \cdots, k$, then there is no $n \geq 2$ such that $h_1(P_n) | \prod_{i=1}^{k} f_i(x)$.*

Proof. Suppose that there is an $n \geq 2$ such that $h_1(P_n) | \prod_{i=1}^{k} f_i(x)$. Clearly, $n+1 \geq 3$. So, there is an n_1 such that $n+1 = (n_1+1)n_2$ with $n_1+1 = 4$ or n_1+1 prime. From Lemma 2.7, $h_1(P_3)$ and $h_1(P_{n_1})$ are irreducible and $h_1(P_3)|h_1(P_n)$ or $h_1(P_{n_1})|h_1(P_n)$. Thus, $h_1(P_3)| \prod_{i=1}^{k} f_i(x)$ or $h_1(P_{n_1})| \prod_{i=1}^{k} f_i(x)$, which implies that there is an i such that $h_1(P_3)|f_i(x)$ or $h_1(P_{n_1})|f_i(x)$. This contradicts the condition of the theorem. □

Lemma 4.2. [16] *For $j \geq 9$ and $n \geq 4$, $\beta(D_{j+1}) < \beta(D_j) < -4 < \beta(C_n) < \beta(C_{n-1})$.*

Lemma 4.3. [4,6] *For $j \geq 5$, $\cup_j C_j$ is adjointly unique.*

Theorem 4.1. *Let $n_i \geq 5$ and $m_j \geq 9$ for each i and j, and let $3 \leq l_1 \leq 10$ and $l_1 \leq l_2$. Let $G = (\cup_i C_{n_i}) \cup (\cup_j D_{m_j}) \cup (\cup_{l_1,l_2} T_{1,l_1,l_2})$. If $h(P_n) \not| h(C_{n_i})$, $h(P_n) \not| h(D_{m_j})$ and $h(P_n) \not| h(T_{1,l_1,l_2})$ for all $n \geq 2$, then $\overline{G}$ is χ-unique if and only if $l_2 \neq 2l_1+3$ and $(l_1, l_2) \neq (n_i - 1, n_i)$ for all i.*

Proof. From Theorem 1.1, we only need to consider the necessary and sufficient conditions for G to be adjointly unique.

Let H be a graph such that $h(H) = h(G)$. Suppose that $H = \cup_i H_i$ and each H_i is connected. By Lemmas 2.2 and 2.4,

$$\prod_i h(H_i) = \prod_i h(C_{n_i}) \prod_j h(D_{m_j}) \prod_{l_1,l_2} h(T_{1,l_1,l_2}) \tag{3}$$

and

$$\sum_i R(H_i) = \sum_i R(C_{n_i}) + \sum_j R(D_{m_j}) + \sum_{l_1,l_2} R(T_{1,l_1,l_2}) = 0. \tag{4}$$

As $h(P_n) \not| h(H)$ for $n \geq 2$, it is obvious from Lemma 4.1 that $h(P_n) \not| h(H_i)$ for each i and $n \geq 2$. Thus, from (4) and Lemmas 2.2 and 2.4 and $h_1(P_4) = h_1(K_3)$, we have $R(H_i) = 0$ for each component H_i in H. Recalling that $h(P_n) \not| h(H_i)$ for each H_i and $n \geq 2$, by Lemmas 2.4 and 3.8, we have

$$H_i \in \{C_n, D_m, T_{a,b,c}, K_1 \mid n \geq 4, m \geq 4, 1 \leq a \leq b \leq c\} \tag{5}$$

and

$$H_i \notin \{T_{1,a,a}, T_{1,b,b+3}, T_{2,2,c}, T_{2,3,3} \mid a \geq 2, b \geq 1, c \geq 2\}. \tag{6}$$

By Lemma 2.8, $\beta(C_n) = \beta(T_{1,1,n-2})$ and $\beta(D_n) = \beta(T_{1,2,n-3})$ for $n \geq 4$. Therefore, by Lemma 3.5, $\beta(G) > -(2+\sqrt{5})$. So, $\beta(H) = \beta(G) > -(2+\sqrt{5})$. From Lemma 3.5 and (5) and (6), we have

$$H_i \in \{C_n, D_m, T_{1,b,c}, K_1 \mid n \geq 4, m \geq 4, 1 \leq b \leq c, b \neq c, c \neq b+3\}. \tag{7}$$

We construct a graph H' from H by replacing each component $T_{1,a,2a+5}$ by two components C_{a+2} and $T_{1,a+1,a+2}$ until none of the components is isomorphic to $T_{1,a,2a+5}$, where $a \geq 2$. Without loss of generality, let $H' = \bigcup_i H'_i$. From (3) and (7), we can easily get that

$$\prod_{i=1} h(H'_i) = \prod_i h(C_{n_i}) \prod_j h(D_{m_j}) \prod_{l_1,l_2} h(T_{1,l_1,l_2}), \tag{8}$$

$$H'_i \in \{C_n, D_m, T_{1,b,c}, K_1 \mid n \geq 4, m \geq 4, 1 \leq b \leq c, b \neq c, c \neq b+3, c \neq 2b+5\}. \tag{9}$$

In the following, we shall consider the minimum roots of the two sides of (8), namely, $\beta(H')$ and $\beta(G)$. Assume that T_{1,s_1,s_2} is a component of G and $\beta(G) = \beta(T_{1,s_1,s_2})$. Clearly, $3 \leq s_1 \leq 10$ and $s_2 \geq s_1$. From (8), we see that H' must have a component (say H'_1) such that $\beta(H'_1) = \beta(T_{1,s_1,s_2})$. As $\beta(C_n) = \beta(T_{1,1,n-2})$ and $\beta(D_n) = \beta(T_{1,2,n-3})$ for $n \geq 4$, we know by Theorem 3.3(iii) and (9) that $H'_1 \in \{T_{1,s_1,s_2}, T_{1,a,a+1}\}$. Suppose that $H'_1 \cong T_{1,a,a+1}$ and $T_{1,a,a+1} \ncong T_{1,s_1,s_2}$. From Theorem 3.3(iii) and $\beta(T_{1,a,a+1}) = \beta(T_{1,s_1,s_2})$, we have $T_{1,a-1,2a+3} \cong T_{1,s_1,s_2}$, which contradicts the fact that $s_2 \neq 2s_1 + 5$. Thus, $H'_1 \cong T_{1,s_1,s_2}$. Eliminating a factor $h(T_{1,s_1,s_2})$ from the two sides of (8), we arrive at

$$\prod_{i=2} h(H'_i) = \prod_i h(C_{n_i}) \prod_j h(D_{m_j}) \prod_{l_1,l_2} h(T_{1,l_1,l_2})/h(T_{1,s_1,s_2}). \tag{10}$$

From (10), we can obtain the following fact by repeating the above argument.

Fact 1. For each component T_{1,l_1,l_2} of G, there must be a component H'_i of H' such that $H'_i \cong T_{1,l_1,l_2}$.

Eliminating the factor $\prod_{l_1,l_2} h(T_{1,l_1,l_2})$ of $h(G)$ from the two sides of (8), it follows immediately that

$$\prod_{i=1} h(H''_i) = \prod_i h(C_{n_i}) \prod_j h(D_{m_j}) \tag{11}$$

and

$$H_i'' \in \{C_n, D_m, T_{1,b,c}, K_1 \mid n \geq 4, m \geq 4, 1 \leq b \leq c, b \neq c, c \neq b+3, c \neq 2b+5\}. \quad (12)$$

Since $p((\cup_i C_{n_i}) \cup (\cup_j D_{m_j})) = q((\cup_i C_{n_i}) \cup (\cup_j D_{m_j}))$, we have $p(\cup_i H_i'') = q(\cup_i H_i'')$. So, from (12), we have

$$H_i'' \in \{C_n, D_m \mid n \geq 4, m \geq 4\}. \quad (13)$$

From assumptions and Lemma 4.2, we have $\beta(D_{m_j}) < -4 < \beta(C_n)$ for $m_j \geq 9$. Similar to the argument of (8), from Lemma 4.2, we can get the following fact by comparing the minimum roots of the two sides of equation (11) .

Fact 2. For each component D_{m_j} of G, there must be one component H_i' such that $H_i' \cong D_{m_j}$ in H'.

Eliminating the factor $\prod_j h(D_{m_j})$ of $h(G)$ from the two sides of (11), it follows that

$$\prod_i h(H_i''') = \prod_i h(C_{n_i}), \ \ H_i''' \in \{C_n, D_m | n \geq 4, m \geq 4\}. \quad (14)$$

The following fact is obtained from (14) and assumptions and Lemma 4.3.

Fact 3. $\cup_i H_i''' \cong \cup_i C_{n_i}$.

From Facts 1, 2 and 3, it is clear that $H' \cong G$. Suppose that H has at least one component $T_{1,a,2a+5}$. Obviously, H' must contain the components $T_{1,a+1,a+2}$ and C_{a+2}. Recalling that $H' \cong G$, we have G must contain the components $T_{1,a+1,a+2}$ and C_{a+2}. This contradicts to the condition of the theorem. So, H does not contain the component $T_{1,a,2a+5}$. Therefore, $H \cong H' \cong G$. This completes the proof of the sufficient condition of the theorem.

From Lemma 3.8(iii), the necessity of the theorem is obvious.

This completes the proof of the theorem. □

From Lemma 2.8 and Theorem 3.2, we have that $h(P_n)|h(C_m)$ if and only if $n = 3$ and $m = 4k+2$, and $h(P_n)|h(D_m)$ if and only if $n = 2$ and $m = 3k+2$ or $n = 4$ and $m = 5k+3$, where $k \geq 1$. So, from Theorems 3.2 and 4.1, we have

Corollary 4.1. *Let* $G_i \in \{C_i | i \geq 5, i \not\equiv 2(mod 4)\} \cup \{D_j | j \geq 9, j \not\equiv 2(mod 3), j \not\equiv 3(mod 5)\} \cup \{T_{1,l_1,l_2} | 3 \leq l_1 \leq 6, l_1 \leq l_2, l_1 \neq l_2, l_1 \neq l_2+1, l_2 \neq 2l_1+5\}$ *and* $(l_1, l_2) \notin \{(3, 3k), (3, 4k-1), (4, 4k+1), (4, 5k-1), (4, 7k), (5, 3k+2), (5, 4k+4), (5, 5k+2), (6, 3k+3), (6, 7k-1) | k \geq 1\}$. *Then* $\overline{\cup_i G_i}$ *is* χ*-unique.* □

Theorem 4.2. *Let* $3 \leq l_1 \leq 10$, $l_1 \leq l_2$. *If* $h(P_m) \nmid h(T_{1,l_1,l_2})$ *for any* $m \geq 2$, *then* $K_n - E(\cup_{l_1,l_2} T_{1,l_1,l_2})$ *is* χ*-unique if and only if* $l_2 \neq 2l_1+5$, *where* $n \geq \sum_{l_1,l_2} |V(T_{1,l_1,l_2})|$.

Proof. Obviously, $\overline{K_n - E(\cup_{l_1,l_2} T_{1,l_1,l_2})} = rK_1 \cup (\cup_{l_1,l_2} T_{1,l_1,l_2})$, where $r = n - \sum_{l_1,l_2} |V(T_{1,l_1,l_2})|$. Let $G = rK_1 \cup (\cup_{l_1,l_2} T_{1,l_1,l_2})$. From Theorem 1.1, we only consider the necessary and sufficient conditions for G to be adjointly unique.

Let H be a graph such that $h(H) = h(G)$. Suppose that $H = \cup_i H_i$, where each H_i is connected. By Lemmas 2.2 and 2.4, we have

$$\prod_i h(H_i) = x^r \prod_{l_1,l_2} h(T_{1,l_1,l_2}) \qquad (15)$$

and

$$\sum_i R(H_i) = \sum_{l_1,l_2} R(T_{1,l_1,l_2}) = 0. \qquad (16)$$

Similar to the proof of Theorem 4.1, by assumptions and Lemmas 3.7 and 4.1, we have

$$H_i \in \{C_n, D_m, T_{1,b,c}, K_1 \mid n \geq 4, m \geq 4, 1 \leq b \leq c, b \neq c, c \neq b+3\}. \qquad (17)$$

We construct a graph H' from H by replacing each component $T_{1,a,2a+5}$ by two components C_{a+2} and $T_{1,a+1,a+2}$ until none of the components is isomorphic to $T_{1,a,2a+5}$, where $a \geq 2$. Without loss of generality, let $H' = \cup_i H'_i$. By (15) and (17), we obtain that

$$\prod h(H'_i) = x^r \prod_{l_1,l_2} h(T_{1,l_1,l_2}) \qquad (18)$$

and

$$H'_i \in \{C_n, D_m, T_{1,b,c}, K_1 \mid n \geq 4, m \geq 4, 1 \leq b \leq c, b \neq c, c \neq b+3, c \neq 2b+5\}. \qquad (19)$$

Similar to the proof of Theorem 4.1, by comparing the minimum roots of the two sides of (18) we have

Fact 4. For each component T_{1,l_1,l_2} of G, there must be a component H'_i of H' such that $H'_i \cong T_{1,l_1,l_2}$.

Eliminating the factor $\prod_{l_1,l_2} h(T_{1,l_1,l_2})$ of $h(G)$ from the two sides of (18), it follows immediately that

$$\prod h(H''_i) = x^r. \qquad (20)$$

From (19) and (20) and Fact 4, we have

Fact 5. *H' only has r isolated vertices and $H'_i \in \{T_{1,b,c}, K_1 \mid 1 \leq b \leq c, b \neq c, c \neq b+3, c \neq 2b+5\}$.*

By Facts 4 and 5, $H' \cong G$. Assume that H has at least one component $T_{1,a,2a+5}$. Then H' must contain a component C_{a+2}. This contradicts Fact 5. So, $H \cong H' \cong G$. The proof of sufficiency of the conditions of the theorem is complete.

From (iii) of Lemma 3.8, the necessity of the theorem is obvious.

This completes the proof of the theorem. □

Corollary 4.2. *Let $G_i \in \{T_{1,l_1,l_2} | 3 \leq l_1 \leq 6, l_1 \leq l_2, l_1 \neq l_2, l_1 \neq l_2+1, l_2 \neq 2l_1+5\}$ and $(l_1, l_2) \notin \{(3,3k), (3,4k-1), (4,4k+1), (4,5k-1), (4,7k), (5,3k+2), (5,4k+4), (5,5k+2), (6,3k+3), (6,7k-1) | k \geq 1\}$. Then $K_n - E(\cup_i G_i)$ is χ-unique, where*

$n \geq \sum_{l_1,l_2} |V(T_{1,l_1,l_2})|.$ □

It is not difficult to see that many results in [6,9,10,12,14] are special cases of our Corollaries 4.1 and 4.2.

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