

Study on n -Dimensional R-implications

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Abstract

R-implications are studied on $L_n(U)$ considering the conditions under which main properties are preserved, and their representability from U to $L_n(U)$ is also presented. Some results in the class of n -dimensional R-implications obtained from t-representable norms on $L_n(U)$ are discussed.

Keywords: n -dimensional fuzzy sets, fuzzy implications, n -dimensional R-implications.

This work focuses on mathematical description of related R-implications in the logical approach of n -DS mainly related to fuzzy implications. According to [11], the class of R-implications plays an important role in FL. In a broad sense, it is frequently applied to fuzzy control, analysis of vagueness in natural language and techniques of soft-computing as well as in the narrow sense, contributing to a branch of many valued logic enabling the investigation of deep logical questions [1].

1 Introduction

In [27], the notion of an n -dimensional fuzzy set (n -DS) on L_n -fuzzy set theory was introduced by Shang et. al. as a special class of L-fuzzy set theory, generalizing the theories underlying many other multivalued fuzzy logics: the interval-valued fuzzy set theory (IVFS) [26], the Atanassov's intuitionistic fuzzy set [2] (A-IFS) and its interval-valued approach [3]. In L_n -fuzzy set theory, the n -dimensional fuzzy sets membership values are n -tuples of real numbers in $U = [0, 1]$, ordered in increasing order, called n -dimensional intervals. In addition, even when the repetition of elements of n -tuples on the membership degrees is not considered, they can be defined as a (Typical) Hesitant Fuzzy Sets (HFS) [10, 28]. A historical and hierarchical analysis of this approach and other important extensions of the fuzzy set (FS) theory can be found in [12].

As the main idea, an n -DS considers several uncertainty levels in its membership functions, adding degrees of freedom making it possible to directly model uncertainties in computational systems based on FS [8]. Such uncertainties are frequently associated to many causes see, e.g. n -ary operators modelling imprecise parameters in time-varying systems or n distinct expert knowledge possibly obtained from questionnaires including uncertain words from natural language.

1.1 Main Contribution

As the main contribution, this paper introduces the definition of n -dimensional fuzzy R-implications (R- n -DI) in order to show that the main properties of R-implications on U can be preserved on $L_n(U)$. By considering projection functions and degenerate elements, the conjugation in the class of R-implications is presented. The ordering and neutrality properties are extended from U to $L_n(U)$. Our results use concepts and intrinsic properties as the identity and exchange principles from R-implications on U to R- n -DI on $L_n(U)$.

Aggregating functions, in particular t-norms along with fuzzy negations, are related to notions for n -dimensional intervals. Theoretical results from R-implications to their n -dimensional fuzzy approach are obtained. Focusing on the R-implication class, representable n -dimensional t-norms in conjunction with representable n -dimensional fuzzy negations and their interrelationship with the n -DSs are studied.

By using admissible order on $L_n(U)$ obtained by aggregation-sequences, Łukasiewicz R- n -DI and the minimum operator on $L_n(U)$, we are able to compare multiple alternatives, contributing to a decision making problem based on multiple attributes related to a selection of the best CIM software systems based on three decision maker evaluations.

1.2 Related Papers

Stretching the seminal studies of n -DS [27], main related papers exploring logical properties of corresponding fuzzy connectives in theoretical research (TR) areas providing support to applications in decision making problems (DMP) are summarized in Table 1.

Following the reported research, this paper studies the possibility of dealing with main properties n -dimensional R-implications on $L_n(U)$, exploring their application to solve DMP.

1.3 Outline of the Paper

This paper is organized as follows. In Preliminaries, we report the main characteristics of n -dimensional intervals and n -dimensional fuzzy negations are briefly discussed based on [8]. In Section 3, n -dimensional t-norms are studied including main properties, dual and conjugate constructions. In Section 4, the concepts and reasonable properties of n -dimensional fuzzy implications on $L_n(U)$ are also studied, as well as evidence on properties assuring their representability expressions. In Section 5, properties of R-implications are extended to n -dimensional fuzzy approach, main characteristics, duality and action of n -dimensional automorphisms. The conclusion section highlights main results and briefly comments on further work.

2 Preliminaries

In this section, we will briefly review some basic concepts of FL, concerned with the study of n -dimensional intervals, which can be found in [7, 9].

2.1 n -Dimensional Fuzzy Sets

Let $X \neq \emptyset$, $U = [0, 1]$ and $n \in \mathbb{N}^+ = \mathbb{N} - \{0\}$. By [27], an n -dimensional fuzzy set A over X is given as

$$A = \{(x, \mu_{A_1}(x), \dots, \mu_{A_n}(x)) : x \in X\},$$

when, for $i = 1, \dots, n$, the i -th membership degree of A denoted as $\mu_{A_i} : X \rightarrow U$ verifies the condition $\mu_{A_1}(x) \leq \dots \leq \mu_{A_n}(x)$.

In [7], the n -dimensional upper simplex, is given as

$$L_n(U) = \{\mathbf{x} = (x_1, \dots, x_n) \in U^n : x_1 \leq \dots \leq x_n\}, \quad (1)$$

and its elements are called n -dimensional intervals. For $i = 1, \dots, n$, the i -th projection of $L_n(U)$ is the function $\pi_i : L_n(U) \rightarrow U$ given by $\pi_i(x_1, \dots, x_n) = x_i$.

¹ n -dimensional Fuzzy Preference Relations

²Multi Expert Decision Making

³Multiple Attribute Group Decision Making Problem

A degenerate element $\mathbf{x} \in L_n(U)$ verifies the condition

$$\pi_i(\mathbf{x}) = \pi_j(\mathbf{x}), \text{ for each } i, j = 1, \dots, n, \quad (2)$$

and will be denoted by $/x/$, for $x \in U$.

Remark 1. By extending the \leq -order on U to higher dimensions, for $\mathbf{x}, \mathbf{y} \in L_n(U)$, it holds that:

$$\mathbf{x} \leq \mathbf{y} \text{ iff } \pi_i(\mathbf{x}) \leq \pi_i(\mathbf{y}) \text{ for each } i = 1, \dots, n. \quad (3)$$

Thus $(L_n(U), \leq)$ is a lattice. Additionally, for all $\mathbf{x}, \mathbf{y} \in L_n(U)$ the following relation is also considered

$$\mathbf{x} \preceq \mathbf{y} \Leftrightarrow \mathbf{x} = \mathbf{y}, \quad (4)$$

this is the same as saying that $\pi_n(\mathbf{x}) \leq \pi_1(\mathbf{y})$.

Moreover, it is related to partial orders on $L_n(U)$, one can easily observe that \preceq is more restrictive than \leq , meaning that $\mathbf{x} \preceq \mathbf{y} \Rightarrow \mathbf{x} \leq \mathbf{y}$.

By [17], $\mathcal{L}_n(U) = (L_n(U), \vee, \wedge, /0/, /1/)$ is a distributive complete lattice, which is continuous, with $/0/$ and $/1/$ being their bottom and top element, respectively. By [7], for all $\mathbf{x}, \mathbf{y} \in L_n(U)$, the supremum and infimum on $\mathcal{L}_n(U)$ is given as:

$$\mathbf{x} \vee \mathbf{y} = (\max(\pi_1(\mathbf{x}), \pi_1(\mathbf{y})), \dots, \max(\pi_n(\mathbf{x}), \pi_n(\mathbf{y}))) \quad (5)$$

$$\mathbf{x} \wedge \mathbf{y} = (\min(\pi_1(\mathbf{x}), \pi_1(\mathbf{y})), \dots, \min(\pi_n(\mathbf{x}), \pi_n(\mathbf{y}))). \quad (6)$$

Observe that $L_1(U) = U$ and $\mathcal{L}_2(U)$ reduces to the usual lattice of all the closed subintervals on U .

2.2 Fuzzy Negations on $L_n(U)$

As conceived in [8], the notion of fuzzy negation was extended to $L_n(U)$ and main concepts reported below.

Definition 1. A function $\mathcal{N} : L_n(U) \rightarrow L_n(U)$ is an n -dimensional fuzzy negation (n -DN) if it satisfies:

$$\mathcal{N}1: \mathcal{N}(/0/) = /1/ \text{ and } \mathcal{N}(/1/) = /0/;$$

$$\mathcal{N}2: \text{ If } \mathbf{x} \leq \mathbf{y} \text{ then } \mathcal{N}(\mathbf{x}) \geq \mathcal{N}(\mathbf{y}).$$

In addition, if \mathcal{N} is an involutive function,

$$\mathcal{N}3: \mathcal{N}(\mathcal{N}(\mathbf{x})) = \mathbf{x};$$

then \mathcal{N} is a strong n -DN.

According to [8, Prop. 3.1], if N_1, \dots, N_n are fuzzy negations such that $N_1 \leq \dots \leq N_n$, then $\widetilde{N_1 \dots N_n} : L_n(U) \rightarrow L_n(U)$ is a n -DN given by

$$\widetilde{N_1 \dots N_n}(\mathbf{x}) = (N_1(\pi_n(\mathbf{x})), \dots, N_n(\pi_1(\mathbf{x}))). \quad (7)$$

And, when $N = N_1 = \dots = N_n$, $\widetilde{N_1 \dots N_n}$ is denoted as \widetilde{N} .

Example 1. Consider $N_{D1}, N_{D2} : U \rightarrow U$ as fuzzy negations respectively given as follows:

$$N_{D1}(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{otherwise;} \end{cases} \quad N_{D2}(x) = \begin{cases} 0, & \text{if } x = 1, \\ 1, & \text{otherwise;} \end{cases}$$

Technique	Contribution	Class
n -Dimensional Intervals and Fuzzy S-implications [31]	Study main properties characterizing the class of S-implications on $L_n(U)$	TR
Towards the study of main properties of n -Dimensional QL-implicators [32]	n-dimensional QL-implicators are studied considering duality and conjugation operators.	TR
Equilibrium Point of Representable Moore Continuous n -Dimensional Interval Fuzzy Negations [18]	Studies some conditions that guarantee the existence of equilibrium point in classes of representable (Moore continuous) n-dimensional interval fuzzy negations .	TR
Moore Continuous n -Dimensional Interval Fuzzy Negations [21]	Characterizing the notion of (continuous) n -dimensional interval Moore metric using the definitions of (continuous) Moore metric and n-dimensional interval fuzzy negations .	TR
n -Dimensional Fuzzy Negations [9]	Presenting n-representable fuzzy negations on $L_n(U)$, analyzing main classes such as continuous and monotone by part.	DMP
Natural n -dimensional fuzzy negations for n -dimensional t-norms and t-conorms [19]	Studying n-dimensional fuzzy negations , applying these studies mainly on natural n -dimensional fuzzy negations for n-dimensional triangular norms and triangular conorms .	TR
An algorithm for MCDM using n -DFS, admissible orders and OWA operators [15]	Introduces the concept of admissible order for n-DS presenting a construction method for those orders and studying OWA operators for aggregating tuples .	DMP
On n -dimensional strict fuzzy negations [20]	Investigate the class of representable n-dimensional strict fuzzy negations .	TR
A class of fuzzy multisets with a fixed number of memberships [8]	Define a generalization of Atanassov's operators for n -dimensional fuzzy values (called n -dimensional intervals).	DMP
Characterization Theorem for t-Representable n -Dimensional Triangular Norms [7]	Generalization of the notion of t-representability for n-dimensional t-norms and provide a characterization theorem for that class of n -dimensional t-norms.	TR
The n -dimensional fuzzy sets and Zadeh fuzzy sets based on the finite valued fuzzy sets [27]	Definition of cut set on n-dimensional fuzzy sets studying the decomposition and representation theorems of n -DS.	TR

Table 1: Distribution of papers based on n -dimensional fuzzy sets

and, $N_S, N_K, N_R : U \rightarrow U$ given as $N_S(x) = 1 - x$, $N_K(x) = 1 - \sqrt{x}$ and $N_R(x) = 1 - x^2$. It follows from Eq.(7) that:

(i) $\widetilde{N_{D1}}, \widetilde{N_R}, \widetilde{N_S}, \widetilde{N_K}, \widetilde{N_{D2}} : L_n(U) \rightarrow L_n(U)$ is a representable n -DN;

(ii) $\widetilde{N_{D1}}, \widetilde{N_S}, \widetilde{N_K}, \widetilde{N_R}, \widetilde{N_{D2}} : L_n(U) \rightarrow L_n(U)$ are the n -dimensional interval extensions of the above fuzzy negations.

Proposition 1. [9] Let \mathcal{N} be an n -DN. Then, a function $N_i : U \rightarrow U$ is a fuzzy negation defined by

$$N_i(x) = \pi_i(\mathcal{N}(x)), \forall i = 1, \dots, n, x \in U. \quad (8)$$

According to [9, Definition 29] and [23], an n -dimensional automorphism on $L(U)$ is reported below:

Definition 2. A function $\varphi : L_n(U) \rightarrow L_n(U)$ is an n -dimensional automorphism if φ is bijective and the following condition is satisfied

$$\mathbf{x} \leq \mathbf{y} \Leftrightarrow \varphi(\mathbf{x}) \leq \varphi(\mathbf{y}), \forall \mathbf{x}, \mathbf{y} \in L_n(U). \quad (9)$$

3 Triangular Norms on $L_n(U)$

By [9], a function $A : L_n(U)^k \rightarrow L_n(U)$ is an n -dimensional aggregation (n -DA) if the following con-

ditions are verified:

- A1. $A(/0/, \dots, /0/) = /0/$ and $A(/1/, \dots, /1/) = /1/$;
A2. $(\mathbf{x}_i \leq \mathbf{y}_i)_{i \in \{1..k\}} \Rightarrow A(\mathbf{x}_1, \dots, \mathbf{x}_k) \leq A(\mathbf{y}_1, \dots, \mathbf{y}_k)$,
 $\forall (\mathbf{x}_1, \dots, \mathbf{x}_k), (\mathbf{y}_1, \dots, \mathbf{y}_k) \in L_n(U)^k$.

Example 2. By Eq.(6), the minimum aggregation operator $\mathcal{F}_\wedge : L_n(U)^k \rightarrow L_n(U)$ is given as:

$$\mathcal{F}_\wedge(\mathbf{x}_1, \dots, \mathbf{x}_k) = (\wedge(\mathbf{x}_{i1}, \dots, \mathbf{x}_{ik}))_{i \in \{1, \dots, n\}} \quad (10)$$

when $\mathbf{x}_{ik} = \pi_i(\mathbf{x}_k)$.

According to [15], a linear order \sqsubseteq on $L_n(U)$ is called admissible if for all $\mathbf{x}, \mathbf{y} \in L_n(U)$ it satisfies: $\mathbf{x} \leq \mathbf{y} \Rightarrow \mathbf{x} \sqsubseteq \mathbf{y}$, meaning that \sqsubseteq refines \leq .

By [15, Definition 5], let $\mathcal{A} = (A_1, \dots, A_k)$ be a sequence of n aggregation functions $A_i : U^n \rightarrow U$. For $\mathbf{x}, \mathbf{y} \in L_n(U)$, the following holds:

- $\mathbf{x} \sqsubset \mathbf{y}$ iff there exists $k \in \{1, \dots, n\}$ such that $A_j(\mathbf{x}) = A_j(\mathbf{y}), \forall j \in \{1, \dots, k-1\}$ and $A_k(\mathbf{x}) < A_k(\mathbf{y})$;
- $\mathbf{x} \sqsubseteq \mathbf{y}$ iff $x \sqsubset y$ or $x = y$.

In addition, let $\mathcal{A} = (A_1, \dots, A_n)$ be an aggregation-sequence of functions $A_i : U^n \rightarrow U$. Based on [15, Proposition 1], the order relation \sqsubseteq on $L_n(U)$ is ad-

missible iff for each $\mathbf{x}, \mathbf{y} \in L_n(U)$, the following holds:

$$A_i(\mathbf{x}) = A_i(\mathbf{y}), \forall i \in \{1, \dots, n\} \Leftrightarrow \mathbf{x} = \mathbf{y}.$$

Proposition 2. [15, Prop.2] Let $\mathcal{A} = (A_1, \dots, A_n)$ be an aggregation-sequence such that $A_i : U^n \rightarrow U$ is given by: $A_i(\mathbf{x}) = (\alpha_{i1}(\pi_1(\mathbf{x})), \dots, \alpha_{in}(\pi_n(\mathbf{x})))$, whenever $\alpha_{i1} + \dots + \alpha_{in} = 1$ and $0 \leq \alpha_{ij} \leq 1$, for $1 \leq i, j \leq n$. The $\sqsubseteq_{[A]}$ -order on $L_n(U)$ is admissible iff the corresponding matrix $[A] = (\alpha_{ij})_{n \times n}$ is regular.

Example 3. Consider the aggregation-sequence $A_1, A_2, A_3 : U^3 \rightarrow U$ where

$$A_1(\mathbf{x}) = 0.1x_1 + 0.5x_2 + 0.4x_3;$$

$$A_2(\mathbf{x}) = 0.3x_1 + 0.4x_2 + 0.3x_3;$$

$$A_3(\mathbf{x}) = 0.2x_1 + 0.4x_2 + 0.4x_3.$$

Under the conditions of Proposition 2, $[A] = (\alpha_{ij})_{3 \times 3}$ is a regular matrix. Therefore, $A_i(\mathbf{x}) = A_i(\mathbf{y}), \forall i \in \{1, 2, 3\} \Leftrightarrow \mathbf{x} = \mathbf{y}$ means that $\mathbf{x} \sqsubseteq_{[A]} \mathbf{y}$.

In [19], the notion of t-norms on U was extended to $L_n(U)$, and their main properties are reported below.

Definition 3. [8, Def.3.4] A function $\mathcal{T} : L_n(U)^2 \rightarrow L_n(U)$ is an n -dimensional t-norm (n -DT) if it is commutative, associative, monotonic w.r.t. the product order and has $/1/$ as its neutral element.

Let \mathcal{T} be n -DT. The natural n -DN \mathcal{T} is the function $\mathcal{N}_{\mathcal{T}} : L_n(U)^2 \rightarrow L_n(U)$ given as

$$\mathcal{N}_{\mathcal{T}}(\mathbf{x}) = \sup\{\mathbf{z} \in L_n(U) : \mathcal{T}(\mathbf{x}, \mathbf{z}) = /0/\} \quad (11)$$

According to [8], the conditions under which an n -DT on \mathcal{L}_n can be obtained from a finite subset of t-norm on U are reported as follows.

Theorem 1. [19, Theorem 3.3] If there exist t-norms T_1, \dots, T_n such that $T_1 \leq \dots \leq T_n$ then $\mathcal{T} : L_n(U)^2 \rightarrow L_n(U)$ is a t -representable n -DT defined by

$$\widetilde{T_1 \dots T_n}(\mathbf{x}, \mathbf{y}) = (T_1(\pi_1(\mathbf{x}), \pi_1(\mathbf{y})), \dots, T_n(\pi_n(\mathbf{x}), \pi_n(\mathbf{y}))).$$

By Theorem 1, a t -representable n -DT is expressed as

$$\widetilde{T_1 \dots T_n}(\mathbf{x}_1, \mathbf{x}_2) = (T_1(x_{11}, x_{21}), \dots, T_n(x_{1n}, x_{2n})). \quad (12)$$

In addition, if $T_1 = \dots = T_n = T$, $\widetilde{T_1 \dots T_n}$ in Eq. (12) is denoted by \widetilde{T} . See additional studies in [19].

Example 4. Considering the t-norms on U given as:

$$T_D(x, y) = \begin{cases} 0, & \text{if } x, y \in U, \\ \min(x, y), & \text{otherwise;} \end{cases} \quad T_P(x, y) = xy;$$

$$T_{LK}(x, y) = \max(x + y - 1, 0); \quad T_M(x, y) = \min(x, y).$$

1. By Eq.(11), the natural n -DN and its n -DT are given as $(\widetilde{T}_D, \widetilde{N}_{D2}), (\widetilde{T}_P, \widetilde{N}_S), (\widetilde{T}_{LK}, \widetilde{N}_S), (\widetilde{T}_M, \widetilde{N}_S)$.

2. By Eq.(12), $T_M, T_P, T_{LK}, T_{WB} : L_n(U)^2 \rightarrow L_n(U)$ is an example of t -representable n -DT.

Proposition 3. [8, Theorem 3.6] Let \mathcal{T} be a n -DT and φ be a n -DA. Then $\mathcal{T}^\varphi : L_n(U)^2 \rightarrow L_n(U)$ is an n -DT given as

$$\mathcal{T}^\varphi(\mathbf{x}, \mathbf{y}) = \varphi^{-1}(\mathcal{T}(\varphi(\mathbf{x}), \varphi(\mathbf{y}))). \quad (13)$$

4 Fuzzy Implications on $L_n(U)$

Studies on n -dimensional fuzzy implications on lattice $(L_n(U), \leq)$ were carried out, extending the preliminary studies on representability of fuzzy implications [14, 16] preserving their main properties. And, if $n = 2$, n -dimensional fuzzy implications can be seen as extensions of interval-valued fuzzy implications. Thus, their properties on U can also be investigated in an n -dimensional sense in $L_n(U)$.

Definition 4. [31, Def.7] A function $\mathcal{I} : L_n(U)^2 \rightarrow L_n(U)$ is a n -dimensional fuzzy implicator (n -DI) if \mathcal{I} meets the boundary conditions:

$$\mathcal{I}0(\mathbf{a}): \mathcal{I}(/1/, /1/) = \mathcal{I}(/0/, /1/) = \mathcal{I}(/0/, /0/) = /1/;$$

$$\mathcal{I}0(\mathbf{b}): \mathcal{I}(/1/, /0/) = /0/.$$

Other properties of an implicator \mathcal{I} are as follows:

$$\mathcal{I}1: \mathbf{x} \leq \mathbf{z} \rightarrow \mathcal{I}(\mathbf{x}, \mathbf{y}) \geq \mathcal{I}(\mathbf{z}, \mathbf{y});$$

$$\mathcal{I}2: \mathbf{y} \leq \mathbf{z} \rightarrow \mathcal{I}(\mathbf{x}, \mathbf{y}) \leq \mathcal{I}(\mathbf{x}, \mathbf{z});$$

$$\mathcal{I}3: \mathcal{I}(/1/, \mathbf{y}) = \mathbf{y};$$

$$\mathcal{I}4: \mathcal{I}(/x/, /x/) = /1/;$$

$$\mathcal{I}5: \mathcal{I}(\mathbf{x}, /0/) = \mathcal{N}(\mathbf{x});$$

$$\mathcal{I}6: \mathcal{I}(\mathbf{x}, \mathcal{I}(\mathbf{y}, \mathbf{z})) = \mathcal{I}(\mathbf{y}, \mathcal{I}(\mathbf{x}, \mathbf{z}));$$

$$\mathcal{I}7: \mathcal{I}(\mathbf{x}, \mathbf{y}) = /1/ \Leftrightarrow \mathbf{x} \leq \mathbf{y}.$$

Proposition 4. Let $\phi : L_n(U) \rightarrow L_n(U)$ be an n -DA and $\mathcal{I} : L_n(U)^2 \rightarrow L_n(U)$ be an n -DI. Properties from \mathcal{I}_0 to \mathcal{I}_7 are invariant under the conjugate-operator $\mathcal{I}^\phi : L_n(U)^2 \rightarrow L_n(U)$ given by

$$\mathcal{I}^\phi(\mathbf{x}, \mathbf{y}) = \phi^{-1}(\mathcal{I}(\phi(\mathbf{x}), \phi(\mathbf{y}))). \quad (14)$$

Proof. Let \mathcal{I} be an n -DI verifying properties from \mathcal{I}_0 to \mathcal{I}_7 . For $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{y} \in L_n(U)$ the following holds:

$\mathcal{I}0$: For $\mathbf{z} \in L_n(U)$, we have the boundary conditions:

$$\mathcal{I}^\phi(/0/, /0/) = \phi^{-1}(\mathcal{I}(\phi(/0/), \phi(/0/))) = \phi^{-1}(/1/) = /1/;$$

$$\mathcal{I}^\phi(/0/, /1/) = \phi^{-1}(\mathcal{I}(\phi(/0/), \phi(/1/))) = \phi^{-1}(/1/) = /1/;$$

$$\mathcal{I}^\phi(/1/, /1/) = \phi^{-1}(\mathcal{I}(\phi(/1/), \phi(/1/))) = \phi^{-1}(/1/) = /1/;$$

$$\mathcal{I}^\phi(/1/, /0/) = \phi^{-1}(\mathcal{I}(\phi(/1/), \phi(/0/))) = \phi^{-1}(/0/) = /0/.$$

$\mathcal{I}1$: Consider $\mathbf{x}_1 \leq \mathbf{x}_2$. By the monotonicity of bijection ϕ , we obtain the following expression:

$$\mathcal{I}^\phi(\mathbf{x}_1, \mathbf{y}) = \phi^{-1}(\mathcal{I}(\phi(\mathbf{x}_1), \phi(\mathbf{y}))) \leq \phi^{-1}(\mathcal{I}(\phi(\mathbf{x}_2), \phi(\mathbf{y}))) = \mathcal{I}^\phi(\mathbf{x}_2, \mathbf{y})$$

$\mathcal{I}2$: Analogous to $\mathcal{I}1$.

$$\mathcal{I}3: \mathcal{I}^\phi(/1/, \mathbf{y}) = \phi^{-1}(\mathcal{I}(/1/, \phi(\mathbf{y}))) = \phi^{-1}(/1/) = /1/.$$

$$\mathcal{I}4: \mathcal{I}^\phi(\mathbf{x}, \mathbf{x}) = \phi^{-1}(\mathcal{I}(\phi(\mathbf{x}), \phi(\mathbf{x}))) = \phi^{-1}(/1/) = /1/.$$

$$\mathcal{I}5: \mathcal{I}^\phi(\mathbf{x}, /0/) = \phi^{-1}(\mathcal{I}(\phi(\mathbf{x}), /0/)) = \phi^{-1}(/1/) = /1/.$$

$\mathcal{I}6$: Since \mathcal{I} satisfies exchange principle, we have that

$$\begin{aligned} \mathcal{I}^\phi(\mathbf{x}, \mathcal{I}^\phi(\mathbf{y}, \mathbf{z})) &= \mathcal{I}^\phi(\mathbf{x}, \phi^{-1}(\mathcal{I}(\phi(\mathbf{y}), \phi(\mathbf{z})))) \\ &= \phi^{-1}(\mathcal{I}(\phi(\mathbf{x}), \mathcal{I}(\phi(\mathbf{y}), \phi(\mathbf{z})))) \\ &= \phi^{-1}(\mathcal{I}(\phi(\mathbf{y}), \mathcal{I}(\phi(\mathbf{x}), \phi(\mathbf{z})))) \\ &= \mathcal{I}^\phi(\mathbf{y}, \phi^{-1}(\mathcal{I}(\phi(\mathbf{x}), \phi(\mathbf{z})))) = \mathcal{I}^\phi(\mathbf{y}, \mathcal{I}^\phi(\mathbf{x}, \mathbf{z})). \end{aligned}$$

$\mathcal{I}7$: Since \mathcal{I} verifies $\mathcal{I}7$, we obtain the next result:

$$\begin{aligned} \mathcal{I}^\phi(\mathbf{x}, \mathbf{y}) = /1/ &\Leftrightarrow \phi^{-1}(\mathcal{I}(\phi(\mathbf{x}), \phi(\mathbf{y}))) = /1/ \\ &\Leftrightarrow (\mathcal{I}(\phi(\mathbf{x}), \phi(\mathbf{y}))) = /1/ \Leftrightarrow \phi(\mathbf{x}) \leq \phi(\mathbf{y}) \Leftrightarrow \mathbf{x} \leq \mathbf{y}. \end{aligned}$$

Concluding, Proposition 4 holds. \square

An n -DI \mathcal{I} which also satisfies $\mathcal{I}1$ and $\mathcal{I}2$ is called an n -dimensional fuzzy implication or fuzzy implication on $L_n(U)$.

Proposition 5. [31, Prop 5] Let $I_1, \dots, I_n : U^2 \rightarrow U$ be functions such that $I_1 \leq \dots \leq I_n$. Then, for $\mathbf{x}, \mathbf{y} \in L_n(U)$, a function $\widetilde{I_1 \dots I_n} : L_n(U)^2 \rightarrow L_n(U)$ given by

$$\widetilde{I_1 \dots I_n}(\mathbf{x}, \mathbf{y}) = (I_1(\pi_n(\mathbf{x}), \pi_1(\mathbf{y})), \dots, I_n(\pi_1(\mathbf{x}), \pi_n(\mathbf{y}))), \quad (15)$$

is an n -DI iff I_1, \dots, I_n are also fuzzy implicators.

By Proposition 5, \mathcal{I} is called a representable n -DI if there exist fuzzy implications $I_1 \leq \dots \leq I_n$ such that $\mathcal{I} = \widetilde{I_1 \dots I_n}$. When $I_1 = \dots = I_n = I$, expression $\widetilde{I_1 \dots I_n}$ in (15) is denoted by \widetilde{I} .

Remark 2. Let $\mathcal{I}(L_n(U))$ be the family of all n -DIs. For $\mathbf{x}, \mathbf{y} \in L_n(U)$, $I_1 \dots I_n \in \mathcal{I}(L_n(U))$ and $i = 1, \dots, n$, the following holds:

1. $\pi_i(\widetilde{I_1 \dots I_n}(\mathbf{x}, \mathbf{y})) = I_i(\pi_{n+1-i}(\mathbf{x}), \pi_i(\mathbf{y}))$;
2. $\pi_i(\widetilde{I_1 \dots I_n}(/x/, /y/)) = I_i(x, y)$;
3. $\pi_i(\widetilde{I}(/x/, /y/)) = I(x, y)$.

Let N be a strong fuzzy negation. According to [31, Proposition 7], the \widetilde{N} -dual of $\widetilde{I_1 \dots I_n} \in \mathcal{I}(L_n(U))$ is the function $\widetilde{I_1 \dots I_n}_{\widetilde{N}} : L_n(U)^2 \rightarrow L_n(U)$ given as

$$\widetilde{I_1 \dots I_n}_{\widetilde{N}}(\mathbf{x}, \mathbf{y}) = I_{n_N} \dots I_{1_N}(\mathbf{x}, \mathbf{y}). \quad (16)$$

Proposition 6. [31, Prop 8] An n -DI $\widetilde{I_1 \dots I_n} \in \mathcal{I}(L_n(U))$ is a n -dimensional fuzzy implication iff $I_1 \dots I_n$ are also fuzzy implications on U .

Other main properties of fuzzy implications on U are preserved by the representable n -DI on $L_n(U)$.

Proposition 7. Let $i \in \mathbb{N}_n$ and $k = 3, \dots, 7$. An n -DI $\widetilde{I_1 \dots I_n} : L_n(U)^2 \rightarrow L_n(U)$ w.r.t. a n -DN $N_1 \dots N_n$ verifies the property $\mathcal{I}k$ iff each FI $I_i : U^2 \rightarrow U$ w.r.t. to N_{n-i+1} , for $i = 1 \dots n$, verifies the corresponding property I_k .

Proof. (\Rightarrow) Firstly, let $I_1, \dots, I_n \in \mathcal{I}(L_1(U))$ fuzzy implications satisfying properties from $\mathcal{I}3$ to $\mathcal{I}7$. For $I_1 \dots I_n \in \mathcal{I}(L_n(U))$ given by Eq.(15) and $\mathbf{x}, \mathbf{y} \in L_n(U)$, the following holds:

$$\mathcal{I}3 : \widetilde{I_1 \dots I_n}(/1/, \mathbf{y}) = (I_1(1, y_1), \dots, I_n(1, y_n)) = (y_1, \dots, y_n) = \mathbf{y} \text{ (by Eq.(15) and } \mathcal{I}3)$$

$$\mathcal{I}4 : \widetilde{I_1 \dots I_n}(/x/, /x/) = (I_1(x, x), \dots, I_n(x, x)) = (1, \dots, 1) = /1/ \text{ (by Eq.(15) and } \mathcal{I}4)$$

$$\mathcal{I}5 : \widetilde{I_1 \dots I_n}(\mathbf{x}, /0/) = (I_1(x_n, 0), \dots, I_n(x_1, 0)) = (N_1(x_n), \dots, N_n(x_1)) = \mathcal{N}(\mathbf{x}) \text{ (by Eq.(15) and } \mathcal{I}5)$$

$$\begin{aligned} \mathcal{I}6 : \widetilde{I_1 \dots I_n}(\mathbf{x}, \widetilde{I_1 \dots I_n}(\mathbf{y}, \mathbf{z})) &= \widetilde{I_1 \dots I_n}(\mathbf{x}, (I_1(y_n, z_1), \dots, I_n(y_1, z_n))) \text{ (by Eq.(15))} \\ &= (I_1(x_n, I_1(y_n, z_1)), \dots, I_n(x_1, I_n(y_1, z_n))) \text{ (by Eq.(15))} \\ &= (I_1(y_n, I_1(x_n, z_1)), \dots, I_n(y_1, I_n(x_1, z_n))) \text{ (by } \mathcal{I}5) \\ &= \widetilde{I_1 \dots I_n}(\mathbf{y}, (I_1(x_n, z_1), \dots, I_n(x_1, z_n))) \text{ (by Eq.(15))} \\ &= \widetilde{I_1 \dots I_n}(\mathbf{y}, \widetilde{I_1 \dots I_n}(\mathbf{x}, \mathbf{z})) \text{ (by Eq.(15)).} \end{aligned}$$

$$\begin{aligned} \mathcal{I}7 : \widetilde{I_1 \dots I_n}(\mathbf{x}, \mathbf{y}) &= /1/ \\ &\Leftrightarrow (I_1(x_n, y_1), \dots, I_n(x_1, y_n)) = /1/ \text{ (by Eq.(15))} \\ &\Leftrightarrow I_1(x_n, y_1) = 1, \dots, I_n(x_1, y_n) = 1 \text{ (by } \mathcal{I}7) \\ &\Leftrightarrow x_1 \leq \dots \leq x_n \leq y_1 \leq \dots \leq y_n \Leftrightarrow \mathbf{x} \leq \mathbf{y}. \text{ (}\Leftarrow\text{)} \end{aligned}$$

Let $I_1, \dots, I_n \in \mathcal{I}(L_n(U))$ given by Eq.(15) and verify properties \mathcal{I}_k , for $k = 3, \dots, 7$. Based on projections π_i , for each $i = 1, \dots, n$, the following holds:

$\mathcal{I}3$: By $\mathcal{I}3$, $\widetilde{I_1 \dots I_n}(/1/, /y/) = /y/$ implies $I_i(1, y) = y$.

$\mathcal{I}4$: By $\mathcal{I}4$, $\widetilde{I_1 \dots I_n}(/x/, /x/) = /1/$ implies $I_i(x, x) = 1$.

$\mathcal{I}5$: By $\mathcal{I}5$, if $\widetilde{I_1 \dots I_n}(/x/, /0/) = \widetilde{N}(/x/)$ then it implies $I_i(x, 0) = N(x)$.

$\mathcal{I}6$: By $\mathcal{I}6$, $\widetilde{I_1 \dots I_n}(/x/, \widetilde{I_1 \dots I_n}(/y/, /z/)) = \widetilde{I_1 \dots I_n}(/y/, \widetilde{I_1 \dots I_n}(/x/, /z/))$. Then, it holds that $I_i(x, I_i(y, z)) = I_i(y, I_i(x, z))$.

$\mathcal{I}7$: By $\mathcal{I}7$, $\widetilde{I_1 \dots I_n}(/x/, /y/) = /1/ \Rightarrow /x/ \leq /y/$. Then, the following is verified: $I_i(x, y) = 1 \Rightarrow x \leq y$. \square

5 R-Implications on $\mathcal{L}_n(U)$

The definition and the main properties of R-implications extended from U to $L_n(U)$ are discussed below.

Definition 5. A function $\mathcal{I}_{\mathcal{T}} : L_n(U)^2 \rightarrow L_n(U)$ is called an n -dimensional R-implication (R- n -DI) if there exists n -DT $\mathcal{T} : L_n(U)^2 \rightarrow L_n(U)$ for $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L_n(U)$, such that

$$\mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y}) = \sup\{\mathbf{z} \in L_n(U) : \mathcal{T}(\mathbf{x}, \mathbf{z}) \leq \mathbf{y}\}. \quad (17)$$

The next proposition extends results from [5, Theorem 5.5].

Proposition 8. If \mathcal{T} is an n -DT then $\mathcal{I}_{\mathcal{T}} \in \mathcal{I}(L_n(U))$. Moreover, it verifies $\mathcal{I}0, \mathcal{I}1, \mathcal{I}2, \mathcal{I}3$ and $\mathcal{I}4$. In addi-

tion, it also verifies \mathcal{I}_5 , meaning that its natural negation $\mathcal{N}_{\mathcal{T}}$ coincides with the $\mathcal{N}_{\mathcal{T}}$ given in Eq.(11).

Proof. Let \mathcal{T} be an n -DT and $\mathcal{I}_{\mathcal{T}}$ be a function defined by Eq.(17). Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L_n(U)$.

\mathcal{I}_0 : The boundary conditions hold:

$$\begin{aligned} \mathcal{I}_{\mathcal{T}}(/1/, /1/) &= \sup\{\mathbf{z} \in L_n(U) : \mathcal{T}(/1/, \mathbf{z}) = \mathbf{z} \leq /1/\} = /1/; \\ \mathcal{I}_{\mathcal{T}}(/0/, /1/) &= \sup\{\mathbf{z} \in L_n(U) : \mathcal{T}(/0/, \mathbf{z}) = \mathbf{z} \leq /1/\} = /1/; \\ \mathcal{I}_{\mathcal{T}}(/0/, /0/) &= \sup\{\mathbf{z} \in L_n(U) : \mathcal{T}(/0/, \mathbf{z}) = \mathbf{z} \leq /0/\} = /1/; \\ \mathcal{I}_{\mathcal{T}}(/1/, /0/) &= \sup\{\mathbf{z} \in L_n(U) : \mathcal{T}(/1/, \mathbf{z}) = \mathbf{z} \leq /0/\} = /0/. \end{aligned}$$

\mathcal{I}_1 : Let $\mathbf{x}_1, \mathbf{x}_2 \in L_n(U)$. Based on monotonicity of \mathcal{T} , when $\mathbf{x}_1 \leq \mathbf{x}_2$, taking $\mathbf{z} \in L_n(U)$ such that $\mathcal{T}(\mathbf{x}_2, \mathbf{z}) \leq \mathbf{y}$ we should have that $\mathcal{T}(\mathbf{x}_1, \mathbf{z}) \leq \mathbf{y}$. Then, we obtain the inclusion:

$$\{\mathbf{z} \in L_n(U) : \mathcal{T}(\mathbf{x}_1, \mathbf{z}) \leq \mathbf{y}\} \supset \{\mathbf{z} \in L_n(U) : \mathcal{T}(\mathbf{x}_2, \mathbf{z}) \leq \mathbf{y}\}.$$

Then, $\mathcal{I}_{\mathcal{T}}(\mathbf{x}_1, \mathbf{y}) = \sup\{\mathbf{z} \in L_n(U) : \mathcal{T}(\mathbf{x}_1, \mathbf{z}) \leq \mathbf{y}\}$ and so, $\mathcal{I}_{\mathcal{T}}(\mathbf{x}_1, \mathbf{y}) \geq \sup\{\mathbf{z} \in L_n(U) : \mathcal{T}(\mathbf{x}_2, \mathbf{z}) \leq \mathbf{y}\}$. Therefore, $\mathcal{I}_{\mathcal{T}}(\mathbf{x}_1, \mathbf{y}) \geq \mathcal{I}_{\mathcal{T}}(\mathbf{x}_2, \mathbf{y})$.

\mathcal{I}_2 : Let $\mathbf{y}_1, \mathbf{y}_2 \in L_n(U)$, are arbitrarily fixed and $\mathbf{y}_1 \leq \mathbf{y}_2$. Analogous to \mathcal{I}_1 .

\mathcal{I}_3 : $\mathcal{I}_{\mathcal{T}}(/1/, \mathbf{y}) = \sup\{\mathbf{z} \in L_n(U) : \mathcal{T}(/1/, \mathbf{z}) = \mathbf{z} \leq \mathbf{y}\} = \mathbf{y}$.

\mathcal{I}_4 : $\mathcal{I}_{\mathcal{T}}(/x/, /x/) = \sup\{\mathbf{z} \in L_n(U) : \mathcal{T}(/x/, \mathbf{z}) \leq /x/\} = /1/$.

\mathcal{I}_5 : $\mathcal{I}_{\mathcal{T}}(\mathbf{x}, /0/) = \sup\{\mathbf{z} \in L_n(U) : \mathcal{T}(\mathbf{x}, \mathbf{z}) \leq /0/\} = \mathcal{N}_{\mathcal{T}}(\mathbf{x})$, by Eq.(11).

Concluding, Proposition 8 is verified. \square

The next proposition extends results from [4, Prop.2.5.10].

Proposition 9. *If $\mathcal{I}_{\mathcal{T}} : L_n(U)^2 \rightarrow L_n(U)$ is an R - n -DI generated from an n -DT $\mathcal{T} : L_n(U)^2 \rightarrow L_n(U)$ then its ϕ -conjugate $\mathcal{I}_{\mathcal{T}}^{\phi} : L_n(U)^2 \rightarrow L_n(U)$ is also an R - n -DI generated from a ϕ -conjugate n -DT $\mathcal{T}^{\phi} : L_n(U)^2 \rightarrow L_n(U)$, and the following is verified:*

$$\mathcal{I}_{\mathcal{T}}^{\phi}(\mathbf{x}, \mathbf{y}) = \mathcal{I}_{\mathcal{T}^{\phi}}(\mathbf{x}, \mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in L_n(U). \quad (18)$$

Proof. From Prop. 3, $\mathcal{T}^{\phi} : L_n(U)^2 \rightarrow L_n(U)$ is an n -DT implying that $\mathcal{I}_{\mathcal{T}^{\phi}} : L_n(U)^2 \rightarrow L_n(U)$ is an R - n -DI. Based on the continuity of bijection ϕ , the following holds:

$$\begin{aligned} \mathcal{I}_{\mathcal{T}}^{\phi}(\mathbf{x}, \mathbf{y}) &= \phi^{-1}(\mathcal{I}_{\mathcal{T}}(\phi(\mathbf{x}), \phi(\mathbf{y}))) \\ &= \phi^{-1}(\sup\{\mathbf{z} \in L_n(U) : \mathcal{T}(\phi(\mathbf{x}), \mathbf{z}) \leq \phi(\mathbf{y})\}) \\ &= \sup\{\phi^{-1}(\mathbf{z}) \in L_n(U) : \phi^{-1}(\mathcal{T}(\phi(\mathbf{x}), \mathbf{z})) \leq \mathbf{y}\} \\ &= \sup\{\mathbf{z} \in L_n(U) : \phi^{-1}(\mathcal{T}(\phi(\mathbf{x}), \phi(\mathbf{z}))) \leq \mathbf{y}\} \\ &= \sup\{\mathbf{z} \in L_n(U) : \mathcal{T}^{\phi}(\mathbf{x}, \mathbf{z}) \leq \phi(\mathbf{y})\} = \mathcal{I}_{\mathcal{T}^{\phi}}(\mathbf{x}, \mathbf{y}) \end{aligned}$$

Therefore, Proposition 9 is verified. \square

Corollary 1. *Let $\phi : L_n(U) \rightarrow L_n(U)$ be an n -DA and $\mathcal{I}_{\mathcal{T}} : L_n(U)^2 \rightarrow L_n(U)$ be an R - n -DI. Properties $\mathcal{I}_3, \mathcal{I}_5$ and \mathcal{I}_7 are invariant under the conjugate-operator $\mathcal{I}_{\mathcal{T}}^{\phi} : L_n(U)^2 \rightarrow L_n(U)$.*

Proof. It follows from Propositions 4, 8 and 9. \square

Example 5. *Let $I_{LK} : U^2 \rightarrow U$ be the Lukasiewicz fuzzy implication, $I_{LK}(x, y) = \max(1, 1 - x + y)$. The function $\mathcal{I}_{LK} : L_n^2(U) \rightarrow L_n(U)$ given as $\mathcal{I}_{LK}(\mathbf{x}, \mathbf{y}) = (\bigwedge(/1/, /1/ - \mathbf{x} + \mathbf{y}))$ is an R - n -DI obtained by taking \mathcal{T} in Eq.(17) as $\widetilde{\mathcal{T}_{LK}}$, see Example 4. In addition, we have that $\mathcal{I}_{LK} = I_{LK}$.*

6 Application on DMP

This section extends the application described in [29, Example 1] based on CIM (Computer-Integrated Manufacturing) software plays, from HFS to n -DS.

The triangle product $\triangleleft : L_n(U)^2 \rightarrow L_n(U)$ is given as $\triangleleft \equiv \mathcal{F}_{\wedge} \circ \widetilde{I_{LK}}$, taking $\widetilde{I_{LK}} : L_n(U)^2 \rightarrow L_n(U)$ as the Lukasiewicz R - n -DI in Example 5 and $\mathcal{F}_{\wedge} : L_n(U)^4 \rightarrow L_n(U)$ as the minimum operator given in Eq.(10).

Taking $k, j \in \{1, \dots, n_2\}$ and $i \in \{1, \dots, n_1\}$, the action of \triangleleft -operator can be given by $n_1 \times n_2$ -matrix whose elements $\mathbf{z}_{k,j} = \triangleleft(\mathbf{x}_{ki}, \mathbf{x}_{ji})$, are given as follows:

$$\mathbf{z}_{k,i} = \begin{cases} \left(\mathcal{F}_{\wedge} \circ \widetilde{I_{LK}}(\mathbf{x}_{ki}, \mathbf{x}_{li})_{(i=1\dots 4)} \right)_{(k,j=1\dots 7)}, & \text{if } k \neq j \\ (1.0, 1.0, 1.0), & \text{otherwise.} \end{cases} \quad (19)$$

It enables us to compare multiple alternatives in order to solve the following DMP. To help the user in the selection of seven kinds of CIM software systems filled in nowadays market, a data processing company aims to clarify differences of such systems [13]. The evaluations expressed by n -DS are shown in Table 2.

Let $A = \{A_1, A_2, \dots, A_7\}$ ($n_2 = 7$) be the set of CIM software alternatives and X be the set of 4 attributes related to functionality (a_1), usability (a_2), portability (a_3) and maturity (a_4) ($n_1 = 4$).

Selecting opinions of 3 decision makers (DM) to provide their evaluations with values between 0 and 1 for all alternative A_i w.r.t. each attribute are expressed as 3-dimensional intervals \mathbf{x}_{ij} and contained in the matrix $[D]_{7 \times 4} = (\mathbf{x}_{ki})_{k=1\dots 7, i=1\dots 4}$. Applying Eq.(19), the resulting matrix $L_{7 \times 7} = (\mathbf{z}_{kj})_{k,j=1\dots 7}$ is given in Table 3.

Consider 1^{st} and 2^{nd} lines (alternatives \mathbf{A}_1 and \mathbf{A}_2) in Table 2. For a component $\mathbf{z}_{21} = \mathcal{F}(\mathbf{y}_i)_{i=1\dots 4}$, $\mathbf{y}_i = \widetilde{I_{LK}}(\mathbf{x}_{2i}, \mathbf{x}_{1i}) = (\bigwedge(1, 1 - \mathbf{x}_{2i} + \mathbf{x}_{1i}))_{i=1\dots 4}$, it holds that:

$$\begin{aligned} \mathbf{y}_1 &= \widetilde{I}_{LK}(\mathbf{x}_{21}, \mathbf{x}_{11}) = \widetilde{I}_{LK}((0.85, 0.85, 0.9), (0.8, 0.85, 0.95)) \\ &= (\wedge(1, 1-0.9+0.8), \wedge(1, 1-0.8+0.85), \wedge(1, 1-0.95+0.95)) \\ &= (0.9, 1.0, 1.0) \end{aligned}$$

$$\begin{aligned} \mathbf{y}_2 &= \widetilde{I}_{LK}(\mathbf{x}_{22}, \mathbf{x}_{12}) = \widetilde{I}_{LK}((0.6, 0.7, 0.8), (0.7, 0.75, 0.8)) \\ &= (\wedge(1, 1-0.8+0.7), \wedge(1, 1-0.7+0.75), \wedge(1, 1-0.6+0.8)) \\ &= (0.9, 1.0, 1.0) \end{aligned}$$

$$\begin{aligned} \mathbf{y}_3 &= \widetilde{I}_{LK}(\mathbf{x}_{23}, \mathbf{x}_{13}) = \widetilde{I}_{LK}((0.2, 0.2, 0.2), (0.65, 0.65, 0.8)) \\ &= (\wedge(1, 1-0.2+0.65), \wedge(1, 1-0.2+0.65), \wedge(1, 1-0.2+0.8)) \\ &= (1.0, 1.0, 1.0) \end{aligned}$$

$$\begin{aligned} \mathbf{y}_4 &= \widetilde{I}_{LK}(\mathbf{x}_{24}, \mathbf{x}_{14}) = \widetilde{I}_{LK}((0.15, 0.15, 0.15), (0.3, 0.3, 0.35)) \\ &= (\wedge(1, 1-0.15+0.3), \wedge(1, 1-0.15+0.3), \wedge(1, 1-0.15+0.35)) \\ &= (1.0, 1.0, 1.0). \end{aligned}$$

So, the result component in Table 3 (shown in bold row-2; column-1) is a 3-dimensional interval \mathbf{z}_{21} given as:

$$\begin{aligned} \mathbf{z}_{21} &= \triangleleft(\mathbf{x}_{2i}, \mathbf{x}_{1i}) = \mathcal{F}_{\wedge} \circ \widetilde{I}_{LK}(\mathbf{x}_{2i}, \mathbf{x}_{1i})_{(i=1\dots 4)} \\ &= (\wedge(0.95, 1.0, 1.0, 1.0), \wedge(1.0, 1.0, 1.0, 1.0), \wedge(1.0, 1.0, 1.0, 1.0)) \\ &= (0.90, 1.0, 1.0) \end{aligned}$$

By observing, the value related to the i -th component of $\mathbf{z}_{kl} \in L_3(U)$ results from action of \triangleleft -operator over data provided by evaluations from the i -th DM.

Moreover, considering the results from Proposition 2 and taking the aggregation-sequence given in Example 3, we can compare the 3-dimensional intervals in Table 3 using the admissible $\sqsubseteq_{[A]}$ -order.

See the comparison results in the following, where $\sqsubseteq_{[A]} \equiv \sqsubseteq$ and $\supseteq_{[A]} \equiv \supseteq$ are used by reducing notation:

$$\begin{array}{cccccccc} \mathbf{Z}_{21} \supseteq \mathbf{Z}_{12} & \mathbf{Z}_{12} \supseteq \mathbf{Z}_{21} & \mathbf{Z}_{13} \supseteq \mathbf{Z}_{31} & \mathbf{Z}_{14} \supseteq \mathbf{Z}_{41} & \mathbf{Z}_{15} \supseteq \mathbf{Z}_{51} & \mathbf{Z}_{16} \supseteq \mathbf{Z}_{61} & \mathbf{Z}_{17} \supseteq \mathbf{Z}_{71} \\ \mathbf{Z}_{31} \supseteq \mathbf{Z}_{13} & \mathbf{Z}_{32} \supseteq \mathbf{Z}_{23} & \mathbf{Z}_{23} \supseteq \mathbf{Z}_{32} & \mathbf{Z}_{24} \supseteq \mathbf{Z}_{42} & \mathbf{Z}_{25} \supseteq \mathbf{Z}_{52} & \mathbf{Z}_{26} \supseteq \mathbf{Z}_{62} & \mathbf{Z}_{27} \supseteq \mathbf{Z}_{72} \\ \mathbf{Z}_{41} \supseteq \mathbf{Z}_{14} & \mathbf{Z}_{42} \supseteq \mathbf{Z}_{24} & \mathbf{Z}_{43} \supseteq \mathbf{Z}_{34} & \mathbf{Z}_{34} \supseteq \mathbf{Z}_{43} & \mathbf{Z}_{35} \supseteq \mathbf{Z}_{53} & \mathbf{Z}_{36} \supseteq \mathbf{Z}_{63} & \mathbf{Z}_{37} \supseteq \mathbf{Z}_{73} \\ \mathbf{Z}_{51} \supseteq \mathbf{Z}_{15} & \mathbf{Z}_{52} \supseteq \mathbf{Z}_{25} & \mathbf{Z}_{53} \supseteq \mathbf{Z}_{35} & \mathbf{Z}_{54} \supseteq \mathbf{Z}_{45} & \mathbf{Z}_{45} \supseteq \mathbf{Z}_{54} & \mathbf{Z}_{46} \supseteq \mathbf{Z}_{64} & \mathbf{Z}_{47} \supseteq \mathbf{Z}_{74} \\ \mathbf{Z}_{61} \supseteq \mathbf{Z}_{16} & \mathbf{Z}_{62} \supseteq \mathbf{Z}_{26} & \mathbf{Z}_{63} \supseteq \mathbf{Z}_{36} & \mathbf{Z}_{64} \supseteq \mathbf{Z}_{46} & \mathbf{Z}_{65} \supseteq \mathbf{Z}_{56} & \mathbf{Z}_{56} \supseteq \mathbf{Z}_{65} & \mathbf{Z}_{57} \supseteq \mathbf{Z}_{75} \\ \mathbf{Z}_{71} \supseteq \mathbf{Z}_{17} & \mathbf{Z}_{72} \supseteq \mathbf{Z}_{27} & \mathbf{Z}_{73} \supseteq \mathbf{Z}_{37} & \mathbf{Z}_{74} \supseteq \mathbf{Z}_{47} & \mathbf{Z}_{75} \supseteq \mathbf{Z}_{57} & \mathbf{Z}_{76} \supseteq \mathbf{Z}_{67} & \mathbf{Z}_{67} \supseteq \mathbf{Z}_{76} \end{array}$$

Since $\mathbf{z}_{i1} \supseteq \mathbf{z}_{1i}$, for $i \in \{1, \dots, 7\}$, then \mathbf{A}_1 is the superior CIM software alternative by comparing it with other alternatives. The same analysis can be performed to other alternatives.

7 Conclusion

This work discusses n -dimensional R-implications, considering ϕ -conjugation under R-implications from U to $L_n(U)$. As main contribution, properties characterizing the class of R-implications on $L_n(U)$ are studied. An illustration on solving a DMP applied to a CIM software is discussed.

In sequence, this study considers the discussion of such extension of fuzzy connectives on $L_n(U)$ related to other special classes of fuzzy implications: Dishkant-

and Yager-implications [6, 24, 25, 30] also considers the (T,N)-implications [22]. Since inherent ordering related to n -dimensional intervals, further work considers admissible linear orders contributing with solutions for DMP on multi-attributes and multi-specialists.

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<i>D</i> -matrix	x_1	x_2	x_3	x_4
A1	(0.80, 0.85, 0.95)	(0.70, 0.75, 0.80)	(0.65, 0.65, 0.80)	(0.30, 0.30, 0.35)
A2	(0.85, 0.85, 0.90)	(0.60, 0.70, 0.80)	(0.20, 0.20, 0.20)	(0.15, 0.15, 0.15)
A3	(0.20, 0.30, 0.40)	(0.40, 0.40, 0.50)	(0.90, 0.90, 1.00)	(0.45, 0.50, 0.65)
A4	(0.80, 0.95, 1.00)	(0.10, 0.15, 0.20)	(0.20, 0.20, 0.30)	(0.60, 0.70, 0.80)
A5	(0.35, 0.40, 0.50)	(0.70, 0.90, 1.00)	(0.40, 0.40, 0.40)	(0.20, 0.30, 0.35)
A6	(0.50, 0.60, 0.70)	(0.80, 0.80, 0.90)	(0.40, 0.40, 0.60)	(0.10, 0.10, 0.20)
A7	(0.80, 0.80, 1.00)	(0.15, 0.20, 0.35)	(0.10, 0.10, 0.20)	(0.70, 0.70, 0.85)

Table 2: n -dimensional interval information

	A1	A2	A3	A4	A5	A6	A7
A1	(1.00,1.00,1.00)	(0.40,0.55,0.55)	(0.25,0.45,0.60)	(0.30,0.40,0.50)	(0.40,0.55,0.70)	(0.55,0.75,0.90)	(0.30,0.45,0.55)
A2	(0.90,1.00,1.00)	(1.00,1.00,1.00)	(0.30,0.45,0.55)	(0.30,0.45,0.60)	(0.45,0.55,0.65)	(0.60,0.75,0.85)	(0.35,0.50,0.75)
A3	(0.65,0.80,0.90)	(0.20,0.30,0.30)	(1.00,1.00,1.00)	(0.20,0.30,0.40)	(0.40,0.50,0.50)	(0.40,0.50,0.70)	(0.10,0.20,0.30)
A4	(0.50,0.60,0.75)	(0.35,0.45,0.55)	(0.20,0.35,0.60)	(1.00,1.00,1.00)	(0.35,0.45,0.70)	(0.30,0.40,0.60)	(0.80,0.90,1.00)
A5	(0.70,0.85,1.00)	(0.60,0.80,1.00)	(0.40,0.50,0.80)	(0.10,0.25,0.50)	(1.00,1.00,1.00)	(0.75,0.80,1.00)	(0.15,0.30,0.65)
A6	(0.80,0.95,1.00)	(0.60,0.80,0.80)	(0.50,0.60,0.70)	(0.20,0.35,0.40)	(0.65,0.80,1.00)	(1.00,1.00,1.00)	(0.25,0.40,0.55)
A7	(0.45,0.60,0.65)	(0.30,0.45,0.45)	(0.20,0.50,0.60)	(0.75,0.95,1.00)	(0.35,0.60,0.70)	(0.25,0.40,0.50)	(1.00,1.00,1.00)

Table 3: Action of \triangleleft -operator in n -dimensional intervals

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