Towards the mathematics of quantum field theory

(An advanced course)

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Contents

Introduction

This book is an applied pure mathematics textbook on quantum field theory. Its aim is to introduce mathematicians (and, in particular, graduate students) to the mathematical methods of theoretical and experimental quantum field theory, with an emphasis on

coordinate free presentations

of the mathematical objects in play, but also of the mathematical theories underlying those mathematical objects. The main objective of this presentation is to promote interaction between mathematicians and physicists, by supplying them a common and flexible language, hopefully for the good of both communities, even if the mathematical one is our primary target.

We will now give an elementary presentation, based on simple examples, of our general approach to the fulfillment of the above commitments. We refer to the book's heart for references and historical background on the notions introduced here, and to the end of this introduction for a (non-exhaustive) list of the author's original contributions.

1 Variational calculus and parametrized geometry

The notion of field is very general. From the mathematician's viewpoint, it means a function

$$x : M \longrightarrow C$$

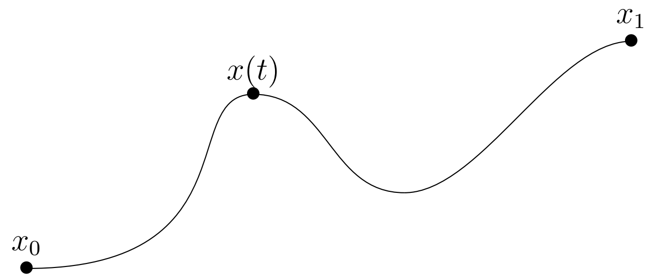
between two spaces, that represents the motion of a physical object in a configuration space C, with parameter space M. The notion of space must be taken here in a widely generalized sense, because even electrons have a mathematical formalization as functions between spaces of a non-classical kind, i.e., not modeled on subsets of the real affine space $\mathbb{R}^n$.

One may then base classical mechanics on variational calculus, following Lagrange and Hamilton. This is done by using Maupertuis' principle of least action, that says that the true physical trajectories of a system may be chosen among fields as those that minimize (or extremize) a given functional $S(x)$, called the action functional.

Let us first illustrate this very general notion by the simple example of newtonian mechanics in the plane. Let $M = [0, 1]$ be a time parameter interval, and $C = \mathbb{R}^2$ be the configuration space. In this case, a field is simply a function

$$x : [0, 1] \longrightarrow \mathbb{R}^2,$$

that represents the motion of a particle (i.e., a punctual object) in the plane.



The action functional of the particle in free motion is given by

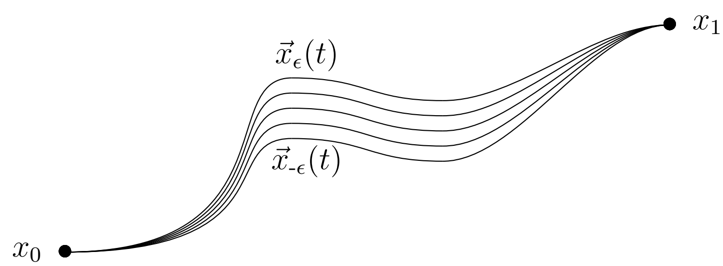
$$S(x) := \int_M \frac{1}{2} m \|\partial_t x(t)\|^2 dt,$$

where m is a real number encoding the mass of the particle. We will suppose given the additional datum of some starting and ending points $x_0, x_1 \in C$ for the trajectories, and of the corresponding speed vectors $\vec{v}_0$ and $\vec{v}_1$.

With this additional datum, one may compute the physical trajectories by computing the infinitesimal variation of the action functional S along a small path

$$\begin{array}{rccc} \vec{x}: &]-\epsilon, \epsilon[\times[0,1] & \to & \mathbb{R}^2 \\ & (\lambda, t) & \mapsto & \vec{x}_\lambda(t) \end{array}$$

in the space of fields at a field $x = \vec{x}_0$, with fixed starting and ending points,



by the formula

$$\frac{\delta S}{\delta \vec{x}}(x) := \partial_\lambda S(\vec{x}_\lambda)|_{\lambda=0}.$$

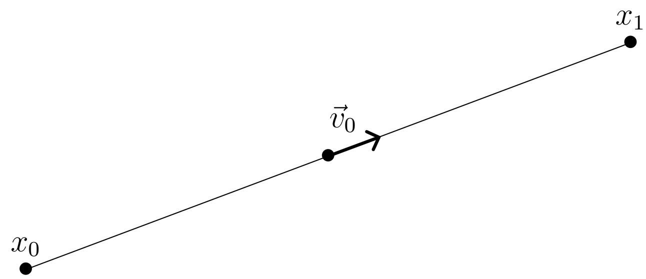
By applying an integration by part, one gets

$$\frac{\delta S}{\delta \vec{x}}(x) = \int_M \langle -m\partial_t^2 x, \partial_\lambda \vec{x} \rangle dt,$$

and this expression is zero for all variations $\vec{x}$ of x if an only if Newton's law for the motion of a free particle

$$m\partial_t^2 x = 0$$

is fulfilled. This equation may be solved using the given initial condition $(x_0, \vec{v}_0)$, or the given final condition $(x_1, \vec{v}_1)$. In both cases, the model tells us, with no surprise, that the free particle is simply moving on a straight line at constant speed, starting at the given point, with the given fixed speed.



We may identify the space of trajectories with the space of pairs $(x_0, \vec{v}_0)$, that are initial conditions for the corresponding differential equation. This space of pairs is the cotangent space

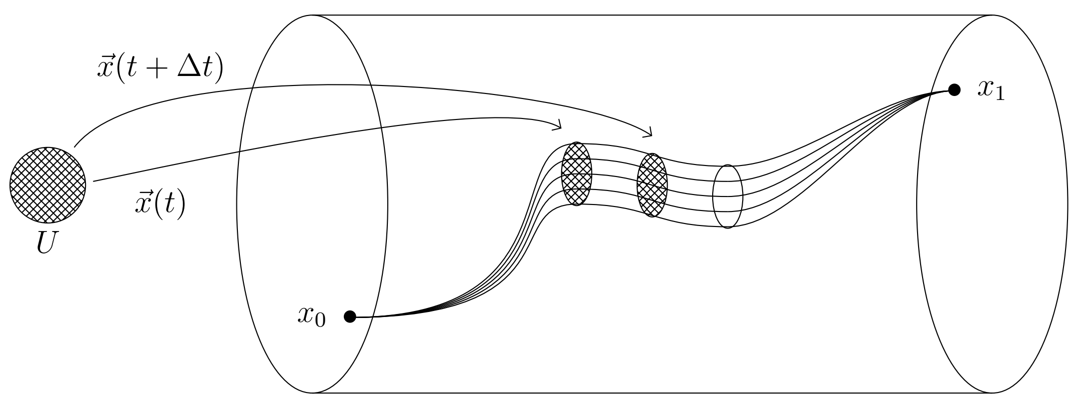
$$T^*C \cong \mathbb{R}^2 \times \mathbb{R}^2$$

of the configuration space.

Remark that the above computation of the equations of motion for the given action functional does not really involve the datum of a topology on the space H of fields with fixed starting and ending point. One only needs to know the value of the action $S : H \to \mathbb{R}$ on a family $\{\vec{x}_\lambda\}$ of fields parametrized by the small interval $]-\epsilon, \epsilon[$. One may even easily define the value of S on such a family, whenever λ is a point in an open subset U of $\mathbb{R}^n$ for some n. The awkward idea of replacing the simple parameter interval $U =]-\epsilon, \epsilon[$ by any such open will allow us to think of the space H of fields as a space of the same nature as the spaces M and C of parameters and configurations. For $U \subset \mathbb{R}^n$, let us denote $M(U)$, $C(U)$ and $\mathbb{R}(U)$ the sets of smooth maps from U to M, C and $\mathbb{R}$. One may think of these sets as sets of points of M, C and $\mathbb{R}$ parametrized by U. Let us also define the space of histories by setting

$$H(U) := \{x : U \times M \to C \text{ smooth, such that for all } u,\ x(u, 0) = x_0 \text{ and } x(u, 1) = x_1\}.$$

These are points of the space of fields parametrized by U, with given fixed starting and ending points. Here is an example with parameter $\lambda \in U = D(0, 1) \subset \mathbb{R}^2$ the open unit disc:



Remark that all these sets are compatible with smooth changes of parameters $V \to U$, because of the stability of smoothness by composition. One may then define a smooth functional

$$S : H \to \mathbb{R}$$

as the datum of a family of set functions

$$S_U : H(U) \to \mathbb{R}(U),$$

that are compatible with smooth changes of parameters. In modern mathematical language, M, C, H and $\mathbb{R}$ defined as above are functors from the category of smooth open subsets in real affine spaces to sets, and S is a natural transformation.

The main advantage of the above interpretation of variational calculus is that it allows us to work with a much more general notion of space, and these generalized kinds of spaces are necessary to understand the computations of quantum field theory in a completely geometrical viewpoint, generalizing what we explained above. One may call this geometry

parametrized geometry.

The combination of parametrized geometry with usual differential geometric tools, like manifolds, involves a refinement of the functorial notion of parametrized space, by the notion of sheaf, that allows to define a smooth manifold as the parametrized space obtained by pasting some basic open subset $U \subset \mathbb{R}^n$, seen as basic parametrized spaces, along smooth maps.

2 Motion with interactions and geometry of differential equations

We continue to consider the motion of a Newtonian particle in the plane, given by a map

$$x : M \to C,$$

with $M = [0, 1]$ and $C = \mathbb{R}^2$, but now suppose that it is not moving freely, but interacts with its environment. Since Newton, the mathematical formalization of this environment is given by a function

$$V : C \to \mathbb{R}$$

called the potential energy density. The associated force field is given by the gradient vector field

$$\vec{F} = -\overrightarrow{\mathrm{grad}}(V).$$

The action functional for this interacting particle is given by a functional

$$S(x) := \int_M L(x(t), \partial_t x(t))dt,$$

where the function

$$L(x, \partial_t x) := \frac{1}{2} m \|\partial_t x\|^2 + V(x)$$

is called the Lagrangian density. Functionals S of the above form are called local functionals. The associated equations of motion are again given by Newton's law

$$m\partial_t^2 x = \vec{F}(x).$$

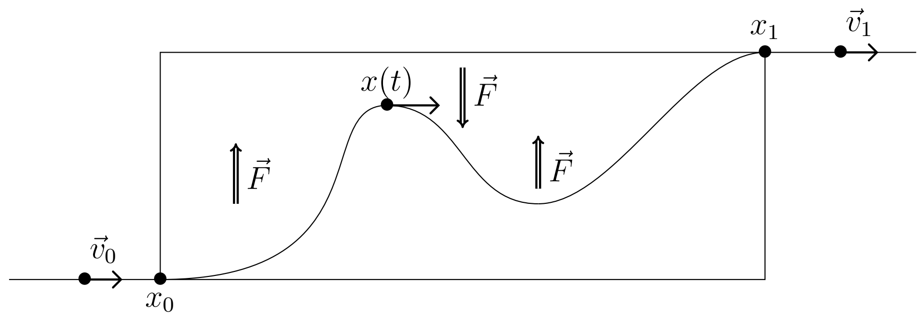
As before, we suppose that the initial and final conditions $(x_0, \vec{v}_0)$ and $(x_1, \vec{v}_1)$ are fixed. We may split the action functional $S = S_{free} + S_{int}$ into a sum of the free (kinetic) action functional

$$S_{free}(x) = \int_M \frac{1}{2} m \|\partial_t x(t)\|^2 dt,$$

whose equations of motion were already considered in the previous section, and the interaction functional

$$S_{int}(x) = \int_M V(x(t))dt.$$

We may then represent the motion of our particle as being free before and after the experiment, and being influenced by the potential within the experiment's spacetime box.



It will be convenient, along the way, to develop, following Lagrange, a more algebraic version of variational calculus, that corresponds to a differential calculus for local functionals.

In the above example, one simply considers the manifold $\text{Jet}(M, C)$ with coordinates (t, x_i) given by formal derivatives of the variables x in $C = \mathbb{R}^2$, indexed by integers $i \in \mathbb{N}$. One may see the Lagrangian density as a function $L : \text{Jet}(M, C) \to \mathbb{R}$ defined by

$$L(t, x_0, x_1) := \frac{1}{2} m \|x_1\|^2 + V(x_0).$$

The only additional datum needed to formulate variational calculus in purely algebraic terms is the differential relation between the family of formal coordinates x_i, given by

$$\partial_t x_i = x_{i+1}.$$

This universal differential relation is encoded in a connection ∇ on the bundle $\text{Jet}(M, C) \to M$, called the Cartan connection. The action of ∂_t on the jet bundle functions being given by the total derivative operator $\frac{d}{dt}$, the differential calculus underlying variational calculus is completely encoded by the above differential structure on the jet bundle.

There is also a parametrized geometry underlying this differential calculus, that is morally a geometry of formal solution spaces to non-linear partial differential equations, called $\mathcal{D}$-geometry. Its parameter spaces are essentially given by jet bundles $(\text{Jet}(U, U \times V), \nabla)$ of morphisms $\pi : U \times V \to U$ (for $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ open subsets), together with their Cartan connection. This geometry of differential equations is often used by physicists to study classical and quantum systems, and in particular their local symmetries and conservation laws.

3 Measurable densities and functorial analysis

We continue to study the motion of a particle $x : [0,1] \to \mathbb{R}^2$ in the plane, but will change the action functional S by replacing it with a parametrization-invariant version, given by the formula

$$S(x) := \int_M \frac{1}{2} m \|\partial_t x(t)\| dt$$

that involves the norm, i.e., the square root of the metric on $\mathbb{R}^2$. In the viewpoint of parametrized geometry, this expression makes sense when we replace the trajectory x by a smooth family

$$x : [0,1] \times U \to \mathbb{R}^2$$

parametrized by an arbitrary open set $U \subset \mathbb{R}^n$. However, the square root being a non-smooth function at 0, the function

$$\begin{array}{ccc} U & \longrightarrow & \mathbb{R} \\ \lambda & \longmapsto & S(x_\lambda) \end{array}$$

will not be smooth in the parameter in general, if the particle gets zero speed at some point of the trajectory. We thus have to restrict the domain of definition of the functional S to a parametrized subspace $D_S \subset H$, called its domain of definition.

More generally, we could take any Lebesgue integrable density $L : [0,1] \times T\mathbb{R}^2 \to \mathbb{R}$ and force the formula

$$S(x_\lambda) := \int_M L(t, x_\lambda(t), \partial_t x_\lambda(t)) dt$$

to define a smooth function of the auxiliary parameter $\lambda \in U$. The definition domain for this functional is easily computed using Lebesgue's domination criterion for the derivative of an integral with parameters. It is given by the subspace $D_S \subset H$ with parametrized points

$$D_S(U) := \left\{ x_\lambda | \text{ locally on } U, \exists g_i \in L^1([0,1]),\ |\partial_\lambda^i L(t, x_\lambda(t), \partial_t x_\lambda(t))| \leq g_i(t) \right\}.$$

The above discussion shows that it is necessary to generalize the notion of functional introduced in parametrized geometry, by authorizing partially defined functions, to get more flexibility and a better compatibility with the usual tools of functional analysis. This extension of functional analysis is called

functorial analysis.

4 Differential calculus and functional geometry

It was already remarked by Weil that one may formulate differential calculus in an algebraic way, by using algebras with nilpotent elements, like for example $\mathcal{C}^\infty(\mathbb{R})/(\epsilon^2)$, where $\epsilon = \mathrm{id} : \mathbb{R} \to \mathbb{R}$ is the standard coordinate function on $\mathbb{R}$. One may think of this algebra as the algebra of coordinate functions on an infinitesimal thickening $\vec{\bullet}$ of the point $\bullet$, defined by adding a universal tangent vector to it. Indeed, if M is a manifold, the choice of a tangent vector $\vec{v}$ to M at a point $m \in M$ is equivalent to the choice of a derivation

$$D : \mathcal{C}^\infty(M) \to \mathbb{R}$$

compatible with the evaluation algebra morphism

$$\begin{array}{rccc} \mathrm{ev}_m : & \mathcal{C}^\infty(M) & \to & \mathbb{R} \\ & f & \mapsto & f(m), \end{array}$$

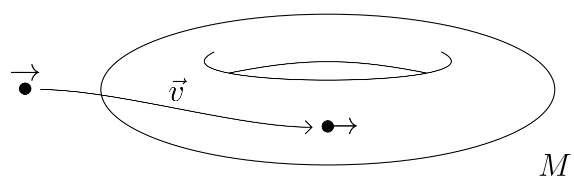
meaning that for all $f, g \in \mathcal{C}^\infty(M)$, one has

$$D(fg) = f(m)D(g) + D(f)g(m).$$

The datum of such a derivation is also equivalent to that of the algebra morphism

$$\begin{array}{rccc} h_D : & \mathcal{C}^\infty(M) & \to & \mathcal{C}^\infty(\mathbb{R})/(\epsilon^2) \\ & f & \mapsto & f(m) + \epsilon D(f). \end{array}$$

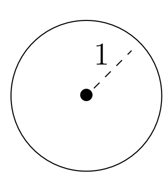
We may interpret this algebra morphism as a morphism $\vec{v} : \vec{\bullet} \to M$, i.e., as a point of M together with a tangent vector on it:



This last interpretation of tangent vectors admits a very general formulation, that works essentially without change in all the parametrized settings we discuss in this introduction.

To make this general formulation work, say, in the smooth setting, one first has to replace the parameters $U \subset \mathbb{R}^n$ by their algebras $\mathcal{C}^\infty(U)$ of functions, which makes no essential difference because of the equivalence between the datum of U and the datum of $\mathcal{C}^\infty(U)$.

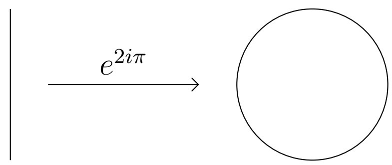
Then, one needs to allow more general algebras as parameters. For example, we would like to consider the nilpotent algebra $\mathcal{C}^\infty(\mathbb{R})/(\epsilon^2)$, as an admissible parameter. More generally, algebras of formal solutions to equations, like $\mathcal{C}^\infty(U)/(f_1, \ldots, f_n)$ may also be useful. Remark that using such algebras is a very natural thing to do also in classical differential geometry, when one is interested by varieties defined by equations. For example, the algebra of smooth functions on the real circle S^1



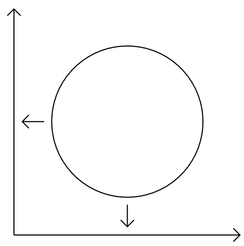
may be easily defined as the quotient algebra

$$\mathcal{C}^\infty(S^1) := \mathcal{C}^\infty(\mathbb{C})/(|z|^2 - 1) \cong \mathcal{C}^\infty(\mathbb{R}^2)/(x^2 + y^2 - 1).$$

One may use the exponential map $e^{2i\pi} : \mathbb{R} \to S^1$



or the implicit function theorem,



to define S^1 as a smooth manifold. It is however less comfortable to work with charts here, than to work directly with the above algebra. Indeed, $S^1 \subset \mathbb{C}$ is naturally defined by the closed algebraic condition $|z|^2 = 1$, and most of its properties may be actually derived from the immediate study of this equation. However, if we think of such quotients as usual real algebras, given by sets, together with compatible multiplication and addition operations, we run into the following trouble: the natural product map

$$\mathcal{C}^\infty(U) \otimes \mathcal{C}^\infty(V) \longrightarrow \mathcal{C}^\infty(U \times V)$$

is not an isomorphism, because some functions on the right may only be described as infinite converging series of tensors.

This problem may be overcomen by using a completed topological tensor product. However, this also has the drawback of introducing unnatural complications, that are furthermore uncompatible with generalizations to the other kinds of parametrized geometries discussed in this introduction. There is however a very elegant solution to this problem of completing the category of algebras of the form $\mathcal{C}^\infty(U)$, for $U \subset \mathbb{R}^n$, in a way that forces the product maps

$$\mathcal{C}^\infty(U) \otimes \mathcal{C}^\infty(V) \longrightarrow \mathcal{C}^\infty(U \times V)$$

to be isomorphisms. One does that by remarking that the functor

$$\underline{\mathcal{C}}^\infty(U) : \mathbb{R}^m \supset V \mapsto \mathcal{C}^\infty(U, V),$$

of smooth functions on U with values in open sets of arbitrary real affine spaces, commutes with transversal fiber products of open sets, i.e., fulfills

$$\underline{\mathcal{C}}^\infty(U, V \overset{t}{\underset{W}{\times}} V') \cong \underline{\mathcal{C}}^\infty(U, V) \underset{\underline{\mathcal{C}}^\infty(U,W)}{\times} \underline{\mathcal{C}}^\infty(U, V').$$

One then defines a more general smooth algebra to be a set-valued functor

$$\mathbb{R}^m \supset V \mapsto A(V)$$

fulfilling the above condition of commutation with transversal fiber products. This process of generalizing spaces by generalizing functions is called

functional geometry.

When conveniently combined with parametrized geometry, it furnishes a workable definition of vector field on a space of fields and more generally, of differential calculus on parametrized spaces.

5 Motion with symmetries and homotopical geometry

Before discussing further the quantization problem, we have to deal with systems that have some given gauge symmetries, that make the Cauchy problem for the equations of motion not well posed. To get a well defined notion of quantization for such a system, one needs a canonical way of breaking the symmetry, to get back a well posed Cauchy problem, with nice initial conditions. This method is usually called the gauge fixing procedure.

Here is how things look like in Yang-Mills gauge theory, that is at the heart of modern particle physics. Let M be a manifold (like the affine space $\mathbb{R}^2$ we used before) equipped with a metric, and let G be a compact Lie group with Lie algebra $\mathfrak{g}$, equipped with an invariant pairing. Let P be a principal G-bundle over M. It is given by a (locally trivial) fiber bundle $P \to M$ together with an action of the group bundle $G_M := G \times M$ that is simply transitive, giving locally an isomorphism $G_M \cong P$.

A principal G-connection on P is morally given by a G-equivariant parallel transport of sections of P along infinitesimally closed points in M. This may also be formalized as a G-invariant differential 1-form

$$A \in \Omega^1(P, \mathfrak{g})^G$$

on P, fulfilling an additional non-degeneracy condition. The corresponding action functional is then given by

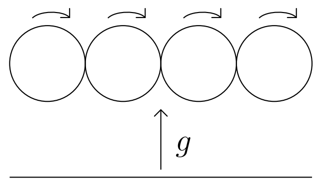
$$S(A) := \int_M \frac{1}{4}\langle F_A \wedge *F_A\rangle,$$

where

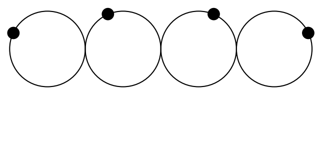
$$F_A := dA + [A \wedge A] \in \Omega^2(P, \mathfrak{g})^G$$

is the curvature of the connection and $*$ is the Hodge $*$-operator on differential forms associated to the given metric g on M. The solutions of the corresponding equations of motion on a compact manifold M contain flat connections.

Let us consider the situation $M = \mathbb{R}$ and P a principal S^1-bundle. The gauge symmetries are then given by functions $g : M \to S^1$



and a principal connection A on the trivial bundle $P = S^1 \times \mathbb{R}$ is given by a parallel transport along $\mathbb{R}$



for points in the circle fibers S^1 of P.

There is no reason to choose a particular principal bundle P, and one may also think of the action functional as a function $S(P, A)$ on isomorphism classes of principal bundles with connection. Homotopical geometry is a setting where one may naturally interpret the bundle P as a field, i.e., a function

$$P : M \longrightarrow C = BG,$$

where BG is the classifying space of principal G-bundles. Bundles with connections also have a classifying space BG_{conn}, so that pairs (P, A) composed of a principal bundle and a principal G-connection may also be seen as fields

$$(P, A) : M \longrightarrow C = BG_{conn},$$

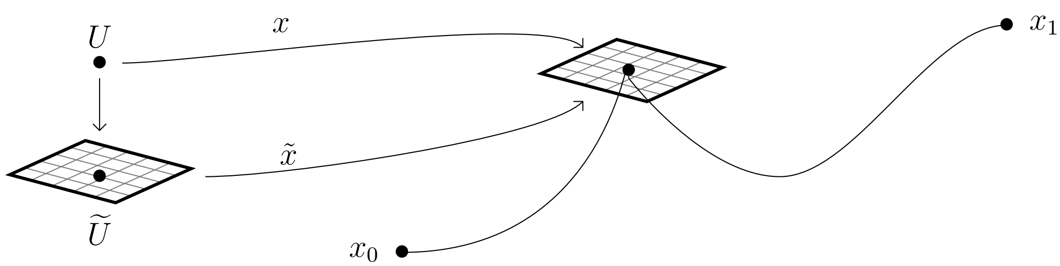
with values in this classifying space. It is not possible to define spaces like BG in the classical parametrized setting: one must enhance the sets of parametrized points to groupoids. Moreover, if we want to work with higher principal bundles (also called gerbes), associated to cohomology classes in $H^n(X, G)$ (where G is now suppose to be a commutative Lie group, like S^1 above), we need also to work with higher groupoids. This general enhancement of equivariant geometry under group actions is given by the setting of homotopical geometry. The space of fields of Yang-Mills theory is given by the space $\underline{\mathrm{Hom}}(M, BG_{conn})$ whose points parametrized by an open subset $U \subset \mathbb{R}^n$ are given by the groupoid of isomorphism classes of pairs (P, A) parametrized by U.

The advantage of formulating gauge theory in this setting is that it allows a clear treatment of gauge symmetries, that are manifest in the above definition of Yang-Mills theory. It also gives a more geometric intuition at the obstructions to quantizing a system with symmetries, called anomalies in physics, and (differential) group and Lie algebra cohomology in mathematics.

6 Deformation theory and derived geometry

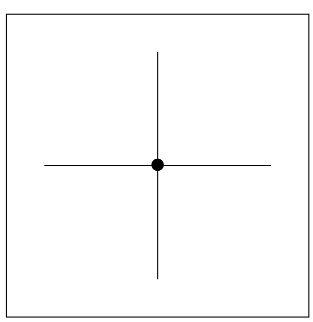
Many problems of physics can be formulated as deformation problems. As we already explained when we discussed functional geometry, differential calculus itself can be formulated in terms of infinitesimal thickenings and infinitesimal deformations of points, that correspond to tangent vectors. We have also seen that the geometrical formulation of theories with symmetries, like Yang-Mills theory, involve the use of an equivariant geometry, called homotopical geometry. The definition of differential invariants of equivariant spaces also involves the use of derived geometrical parameters. Perturbative quantization, as we will see, is also a deformation problem from classical field theory to quantum field theory.

Derived geometry is a parametrized geometry, where one replaces usual open parameters $U \subset \mathbb{R}^n$ by some homotopical generalizations $\widetilde{U} \to \mathbb{R}^n$, obtained by adding to the real coordinates $u \in U$ higher homotopical coordinates $\tilde{u} \in \widetilde{U}$. The coordinate algebra on $\widetilde{U}$ is a simplicial smooth algebra $\mathcal{C}^\infty(\tilde{U})$ such that $\pi_0(\mathcal{C}^\infty(\tilde{U})) = \mathcal{C}^\infty(U)$. These parameters are used to measure obstructions to deformations, i.e., non-smoothness properties.

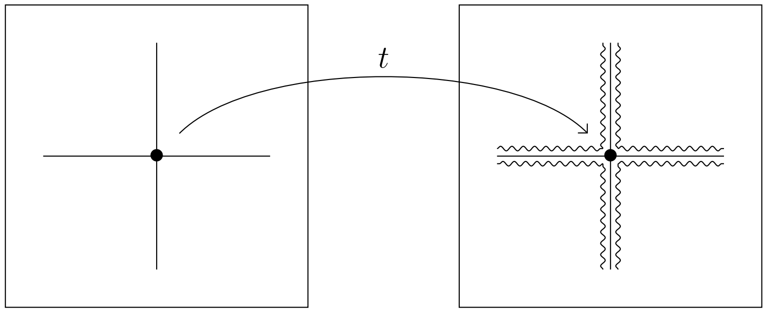


To be precise, coordinate functions on these homotopical parameter spaces are obtained by generalizing smooth algebras (used to formalize infinitesimal calculus in functional geometry). This generalization is obtained by considering functors $\mathbb{R}^m \supset U \mapsto A(U)$ on open subsets of arbitrary real affine spaces, with values in simplicial sets, and commuting (homotopically) with transversal fiber products of open sets.

To illustrate the general relation between deformation theory and derived geometry in simple terms, let us consider the singular cross, given by the union X of the two main axis in the plane,



with polynomial equation $xy = 0$ in $\mathbb{R}[x, y]$. This equation is singular at the origin. This singularity can be explained by the fact that the coordinate $\mathbb{R}$-algebra $\mathbb{R}[x, y]/(xy)$ on X is not smooth, because the universal point $\mathrm{id} : X \to X$ of X can not be deformed along the infinitesimal thickening $t : X \to T$, with coordinate algebra $\mathbb{R}[x, y]/(xy)^2$.



Such a deformation is defined as a factorization of the map $\mathrm{id} : X \to X$ in a sequence $X \xrightarrow{t} T \to X$, which would correspond to a factorization

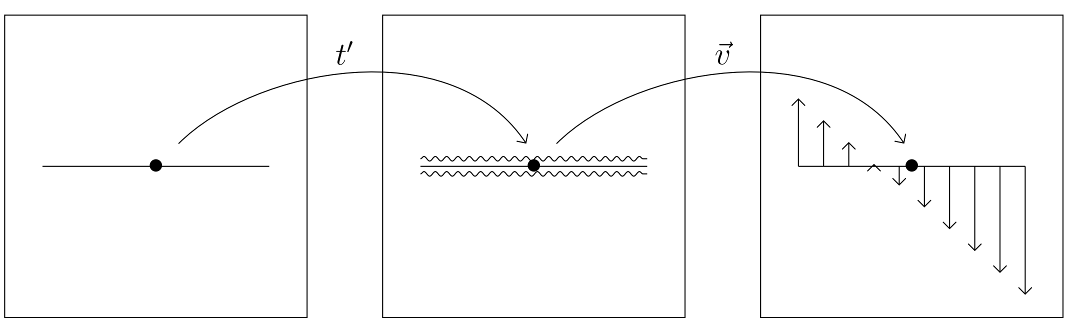
$$\mathbb{R}[x,y]/(xy) \longrightarrow \mathbb{R}[x,y]/(xy)^2 \xrightarrow{t^*} \mathbb{R}[x,y]/(xy) \quad (\text{composite} = \mathrm{id})$$

at the level of coordinate algebras. But such a factorization does not exist. The obstruction to this existence problem leaves in the cohomology group $\mathrm{Ext}^1(\mathbb{L}_X, \mathcal{I})$, where $\mathcal{I} = (xy)$ is the square zero ideal in $\mathbb{R}[x,y]/(xy)^2$, equipped with its canonical module structure over $\mathcal{O}_X = \mathbb{R}[x,y]/(xy)$, and $\mathbb{L}_X$ is a differential graded module over $\mathcal{O}_X$ called the cotangent complex.

In the case of the simpler equation $y = 0$ of one of the horizontal axis line L, we don't have this problem, because the choice of a deformation of the universal point of L along its order two thickening $t' : L \to T'$ is given by a factorization

$$\mathbb{R}[x,y]/(y) \longrightarrow \mathbb{R}[x,y]/(y)^2 \xrightarrow{(t')^*} \mathbb{R}[x,y]/(y) \quad (\text{composite} = \mathrm{id}),$$

that correspond to the choice of a global vector field on L (that we draw here with a rotation of 90 degrees),



given by a morphism $\vec{v} : x \mapsto x + \vec{v}(x).y$, with $\vec{v}(x) \in \mathbb{R}[x]$. The cotangent complex $\mathbb{L}_L$ of this smooth space is concentraded in degree zero, and equal there to the module of differential forms Ω^1_L. The space of deformations of the identity point $\mathrm{id} : L \to L$ along the thickening $L \to T'$ is then given by the space

$$\mathrm{Hom}_L(T', L) \cong \mathrm{Ext}^0(\mathbb{L}_L, \mathcal{O}_L)$$

of sections of the thickening map $t' : L \to T'$. So in the smooth case, one can interpret deformations of points in a given space in terms of degree zero cohomology of the cotangent complex.

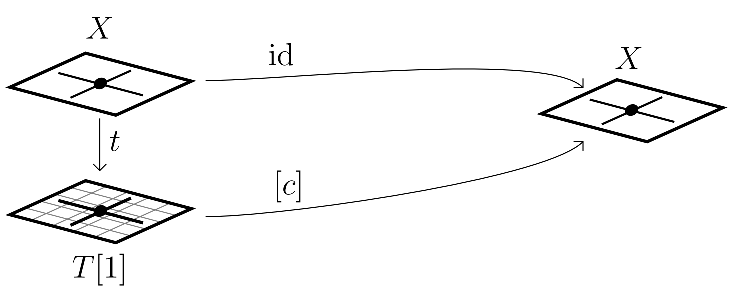
The aim of derived geometry is to also interpret the space of obstruction classes $[c] \in \mathrm{Ext}^1(\mathbb{L}_X, \mathcal{I})$ to the deformation of the singular cross X as a space of infinitesimal extensions

$$\mathrm{Hom}_L(T, X) \cong \mathrm{Ext}^1(\mathbb{L}_X, \mathcal{I}_X).$$

This can only be done by using an infinitesimal extension $t : X \to T[1] = X(\mathcal{I}[1])$ with coordinate ring the nilpotent extension

$$\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{I}[1]) := \mathbb{R}[x, y]/(xy) \oplus (xy)/(xy)^2[1],$$

with cohomological infinitesimal variable in a module of cohomological degree -1, and zero differential.

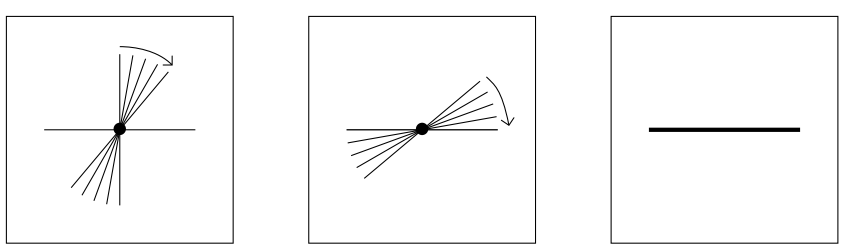


Another important motivation for working with derived geometrical spaces is given by the fact that many spaces considered in physics are given by non-transverse intersections of subspaces $L_1, L_2 \subset X$ in some given space X. For example, the space of trajectories for Yang-Mills theory is given by the intersection

$$\mathrm{Im}(dS) \cap H \subset T^*H$$

of the image of the differential of the action functional with the zero section inside the cotangent bundle T^*H. Morally, this intersection is non-transverse, and one would like to take a small generic perturbation of $\mathrm{Im}(dS)$ (meaning taking a kind of formal tubular neighborhood of this subspace inside T^*H) to get back a transverse intersection.

Let us illustrate this idea of small generic perturbation by explaining it in a simpler situation. Let us consider the intersection, in the plane, of the horizontal axis $y = 0$ and of the line $y = \lambda x$ of slope $\lambda \geq 0$.



If we vary $\lambda > 0$ the line turns around, but the intersection remains the origin, and the dimension of the intersection is zero there, because the coordinate ring on an intersection is given by the tensor product

$$\mathbb{R}[x,y]/(y, y-\lambda x) = \mathbb{R}[x,y]/(y) \underset{\mathbb{R}[x,y]}{\otimes} \mathbb{R}[x,y]/(y-\lambda x),$$

and is isomorphic to $\mathbb{R}$, i.e., to the coordinate algebra of functions on a point. We may also compute this dimension of the point by differential methods: its cotangent space is also of dimension zero. However, if λ tends to zero, the two lines get identified, and their intersection is not anymore transversal. The intersection is then a line, not a point, and the intersection dimension is 1. We thus have bad permanence properties for the intersection dimension in parametrized families. But the derived tensor product

$$\mathbb{R}[x,y]/(y) \overset{\mathbb{L}}{\underset{\mathbb{R}[x,y]}{\otimes}} \mathbb{R}[x,y]/(y)$$

may be thought of as a coordinate algebra for the derived intersection of the line with itself. This tensor product is computed by replacing one of the two sides of the tensor product by the differential graded $\mathbb{R}[x,y]$-algebra

$$\mathrm{Sym}_{dg-\mathbb{R}[x,y]}(\mathbb{R}[x,y][1] \xrightarrow{\cdot y} \mathbb{R}[x,y])$$

that is a cofibrant resolution of the algebra $\mathbb{R}[x,y]/(y)$ (we may omit the symmetric algebra operation, because the corresponding differential graded module is already canonically a differential graded algebra). This results in a differential graded ring over $\mathbb{R}[x,y]/(y)$ that has cohomology only in degrees 0 and 1. The cotangent complex of this space, i.e., its complex of differential forms, has cohomology of dimension 1 in degree zero, and 1 in degree 1. The Euler characteristic of its cohomology thus gives back a derived intersection of virtual dimension 0 at the origin. One may actually think of this derived intersection as a kind of generic perturbation of the non-transverse intersection, obtained by moving back one of the lines, to a line of non-zero slope.

So we will keep in mind that the optimal notion of formal generic perturbation, that has good permanence properties in families, and that makes morally all reasonable intersections transverse, is given by the derived intersection. In the flat Yang-Mills situation, this corresponds to a derived mapping space

$$\mathbb{R}\underline{\mathrm{Hom}}(M, BG_{flat})$$

encoding derived principal bundles with flat connections. As we said, its parameters are given by homotopical infinitesimal extensions $U \to \tilde{U}$ of open subsets $U \to \mathbb{R}^n$, that encode obstructions to deforming the field variables (P, A). One may show that this derived space is naturally equipped with a local symplectic structure, that makes it an analog of the space T^*C of initial conditions for the free particle mechanics, that has the advantage of keeping the gauge symmetries of the problem manifest.

The definition of a well-paused Cauchy problem for the equations of motion is usually given by physicists through the additional choice of a so-called Lagrangian subspace L in the (graded) derived critical space, called the gauge fixing.

Physicists also give another way to define a gauge fixed Cauchy problem, based on a generalization of Noether's theorem, that says roughly that the relations between equations of motions, called Noether identities, are in one to one correspondence with local symmetries of the system. This general method, called the Batalin-Vilkovisky formalism, essentially yields, in the Yang-Mills case, the same result as the above alluded to derived critical space of the action functional on the stacky space of gauge field histories (P, A).

7 Fermionic particles and super-geometry

One way of interpreting mathematically Dirac's quantization of the electron is to consider that this quantum particle has a classical counterpart whose time parameter is anticommuting with itself. Let us first describe mathematically how one may define such a strange object.

We have already roughly defined the notion of space parametrized by smooth open subsets of affine spaces. Since such an open subset $U \subset \mathbb{R}^n$ is determined, up to smooth isomorphism, by its algebra of real valued smooth functions $\mathcal{C}^\infty(U)$, we may define a more general notion of parametrized geometry by using more general types of algebras. Let us consider Grassmann algebras of the form $\mathcal{C}^\infty(U, \wedge^*\mathbb{R}^m)$ for $U \subset \mathbb{R}^n$ smooth. These associative algebras are not commutative but fulfill a super-commutation rule, given by

$$fg = (-1)^{|f|.|g|} gf,$$

where $|f|$ and $|g|$ denote the degree in the anticommuting variable. We will think of these algebras as coordinates $\mathcal{C}^\infty(U^{n|m})$ on some spaces $U^{n|m}$, that are parameters for a new type of geometry, called parametrized super-geometry.

This geometry generalizes parametrized smooth geometry in the following sense: we may extend the parametrized spaces $M = [0, 1]$, $C = \mathbb{R}^2$, and H to the category of super-opens $U^{n|m}$, by setting, for example

$$\mathbb{R}^2(U^{n|m}) = \mathbb{R}^{2|0}(U^{n|m}) := \mathcal{C}^\infty(U, \wedge^{2*}, \mathbb{R}^m)^2,$$

and more generally

$$M(U^{n|m}) := \mathrm{Hom}_{\mathrm{ALG}}(\mathcal{C}^\infty(M), \mathcal{C}^\infty(U^{n|m})).$$

We don't explain here why the above extensions are natural, but will illustrate these new spaces by describing the classical counterpart of the planar euclidean fermionic particle.

The parameter space for this particle is the odd line $M = \mathbb{R}^{0|1}$, with coordinate functions the polynomial algebra $\mathbb{R}[\theta]$ in an anticommuting parameter. So we consider the space of maps

$$x : \mathbb{R}^{0|1} \to C,$$

where $C = \mathbb{R}^2$ is the plane, as before. The natural inclusion $\{*\} \subset \mathbb{R}^{0|1}$ of the point $\{*\} = \mathbb{R}^{0|0}$ into the odd line induces a bijection of sets

$$\mathrm{Hom}(\mathbb{R}^{0|1}, C) \xrightarrow{\sim} \mathrm{Hom}(\{*\}, C) = C,$$

so that the fermionic particle looks like an inert object, because its trajectories do not depend on the parameter θ.

However, if we work with the parametrized super-space $\underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, C)$ of maps from $\mathbb{R}^{0|1}$ to C, whose points parametrized by $U^{n|m}$ are defined by

$$\underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, C)(U^{n|m}) := \mathrm{Hom}(\mathbb{R}^{0|1} \times U^{n|m}, C),$$

one may easily prove that this space is a non-trivial super-space, with algebra of coordinates the super-algebra

$$\Omega^*(C) = \mathcal{C}^\infty(\mathbb{R}^2, \wedge^*(\mathbb{R}^2)^*)$$

of differential forms on $C = \mathbb{R}^2$. This superspace is called the odd tangent bundle of C, and denoted $T[1]C$. The given fixed metric on C identifies it with the odd cotangent bundle of C, denoted $T^*[1]C$, whose algebra of coordinates is the algebra of multivector fields. We will also think of it as the space of initial conditions for the equations of motion.

8 Quantization of the free classical and fermionic particles

Let us now define a simplified notion of quantization for the two above described classical systems, that is very close to the one originally used by Dirac.

Recall that we have defined two kinds of spaces of trajectories in $C = \mathbb{R}^2$: the system of the classical particle is given by maps $x : [0, 1] \to C$, with space of initial conditions for the equations of motion given by T^*C, and the fermionic particle is given by maps $x : \mathbb{R}^{0|1} \to C$, with space of initial conditions for the equations of motion given by the odd cotangent bundle $T^*[1]C$. Both spaces are equipped with a symplectic structure, that allows us to give a static formulation of the mechanics of the system, called Hamiltonian mechanics.

A canonical quantization of a system with space of initial conditions P (chosen among T^*C and $T^*[1]C$) is given by a filtered algebra (A, F^*) (meaning an algebra A with a collection of imbricated subspaces $F^0 \subset \cdots F^n \cdots \subset A$ compatible with its product operation), such that there is an isomorphism of the associated graded algebra

$$\mathrm{gr}^F A := \oplus_{n \geq 0} F^{n+1}/F^n \xrightarrow{\sim} \mathcal{C}^\infty(P).$$

It is easy to see that the so-called Weyl algebra of differential operators

$$\mathcal{D}_C = \{\sum_n a_n(t)\partial_t,\ a_n \text{ smooth on C}\}$$

on $C = \mathbb{R}^2$, with its filtration by the degree in ∂_t, is a canonical quantization of the algebra of functions $\mathcal{C}_p^\infty(T^*C)$ on T^*C that are polynomial in the fiber coordinate. This algebra has a natural representation on the space $\mathcal{C}^\infty(C, \mathbb{R})$ of real valued functions on C.

The canonical quantization of the algebra $\mathcal{C}^\infty(T^*[1]C)$ is given by the Clifford algebra $\mathrm{Cliff}(T^*C, g)$, where g is (opposite to) the standard metric on $\mathbb{R}^2$. This algebra has a natural representation on the odd Hilbert space $H = L^2(C, S)$, where S is the spinorial representation of the Clifford algebra, seen as a super-vector space lying in odd degree.

One may refine the above notion of canonical quantization, to obtain also operators $(\Delta + m^2)$ and $(\not{D} + m)$ on the above super-Hilbert spaces, that give informations on the associated quantum equations of motion.

The models of modern particle physics are based on a second quantization procedure, that sees the above quantum operators as the classical equations of motion for some new field configurations, with new parameter space M' the original configuration space C of punctual mechanics, and new configuration spaces C' the supermanifolds $\mathbb{R}$ and S. The spaces of field configurations are thus given by parametrized spaces of maps

$$\varphi : M' \to \mathbb{R} \text{ and } \psi : M' \to S,$$

called bosonic and fermionic fields. These constructions show that it is very helpful to have a formalism for classical and quantum field theory that allows a uniform treatment of all these examples.

9 Quantizing interactions: the functional integral

We continue to discuss the basic example of the motion of a particle

$$x : M \to C,$$

with time parameter in $M = [0, 1]$, configuration space the affine plane $C = \mathbb{R}^2$, and interacting Lagrangian density

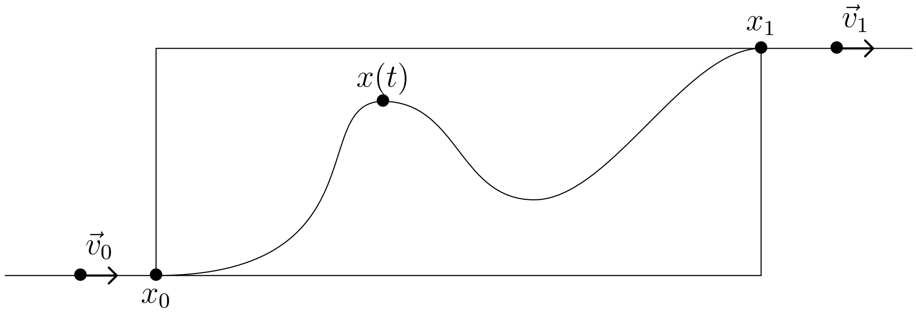
$$L(x, \partial_t x) := \frac{1}{2} m \|\partial_t x\|^2 + V(x).$$

Recall that the equations of motion are given by

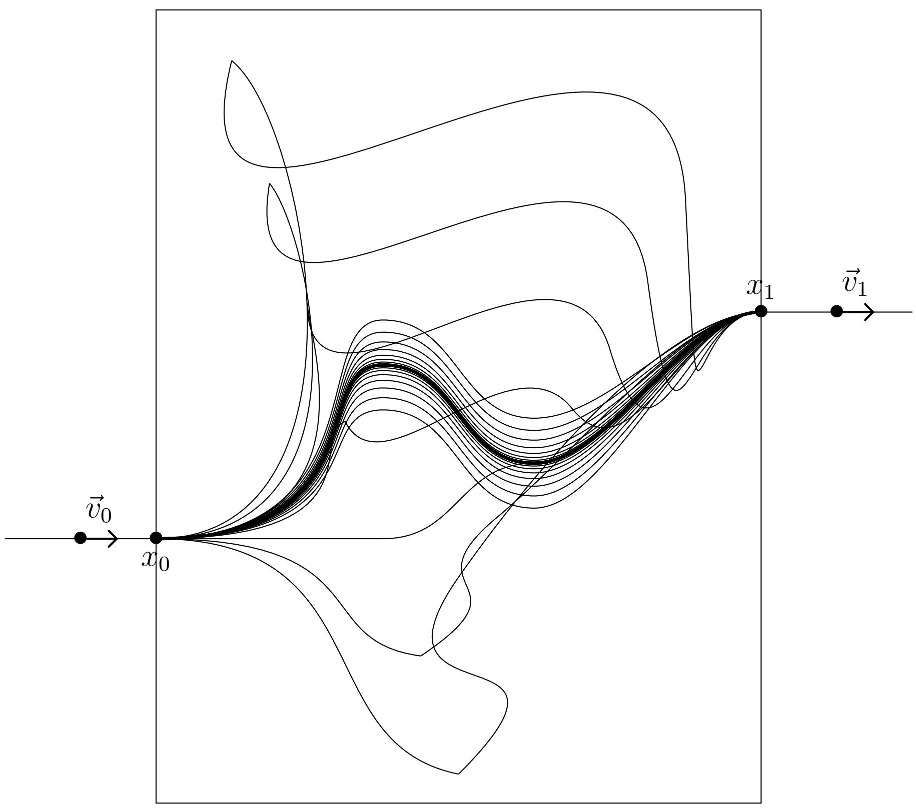
$$m\partial_t^2 x = -\overrightarrow{\mathrm{grad}}(V)(x).$$

The Cauchy theorem for differential equations tells us that the space of solutions to the above equation is, again, isomorphic to the cotangent space T^*C, that encodes initial conditions. One may very well quantize this space as above, but the associated quantum equations of motion are not as simple as the operator $\Delta + m^2$, that we previously used: one must take into account interactions. This is where the perturbative renormalization method comes in: one will treat the theory as depending of a small (or formal) coefficient λ, called the coupling constant, that one may put in front of the potential, to get a term $\lambda V(x)$ in the Lagrangian density.

One may give a very intuitive presentation of this perturbative quantization method by using Feynman's functional integral approach to quantization. Recall from above the shape of the classical motion of a particle in a Newtonian potential V.



Feynman's interpretation of a quantum process is that the quantum particle actually takes all possible paths between the incoming and the outgoing configurations, but that the probability to observe it on a given path concentrates around the classical solutions to the equations of motion.



Remark that the widely spread idea that Feynman's functional integral does not exist is false. What is true, is that this mathematical device is not an integral in the naive sense of functional analysis, meaning a continuous positive functional

$$\int_H : \mathcal{C}^0(H) \to \mathbb{R},$$

where $H \subset \mathrm{Hom}(M, C)$ is the topological space of field histories. This would not even make any sense in the case of electrons, that are fermionic particles, because of their anti-commuting coordinates.

In all known cases, though, one may use functorial analysis to describe a space $\mathcal{O}(H)$ of (possibly formal) functionals on H together with a well defined functional

$$\langle - \rangle_H : \mathcal{O}(H) \to \mathbb{R}[[\hbar]]$$

that formally fulfills properties similar to the average value of a normalized Gaussian integration measure on H of the form

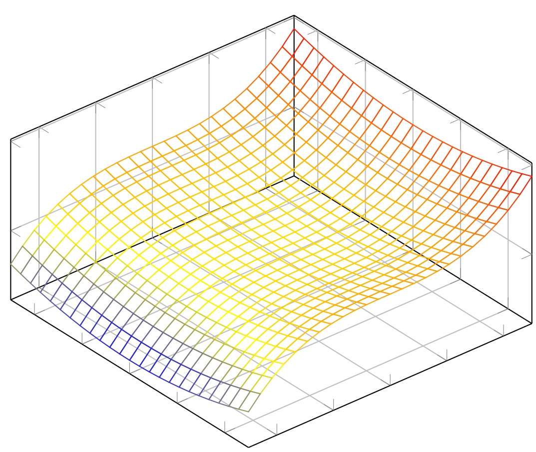
$$d\mu_S = \frac{e^{\frac{i}{\hbar}S(x)}d\mu(x)}{\int_H e^{\frac{i}{\hbar}S(x)}d\mu(x)}.$$

The definition of this integral is even harder to give if one works with the higher dimensional parameter space $M = [0,1] \times \mathbb{R}^3$ given by Lorentzian spacetime, and with configuration space, say $C = \mathbb{R}$. Field histories are then given by real valued functions

$$\varphi : M \to \mathbb{R},$$

and, M being non-compact, one is really willing to do a very big infinite dimensional analog of a Gaussian integral.

A very simple way to get intuition on the renormalization problem is to formulate, following Feynman's thesis, the problem of defining the functional integral as the limit of a family of finite dimensional Lebesgue integrals, by choosing a compact box $B \subset M$, that represents the laboratory, and a lattice $\Lambda \subset M$, that represents the grid of the experimental device. The intersection $M' = B \cap \Lambda \subset M$ is a finite set, so that the mean value $\langle - \rangle_H$ is well defined on the space of restricted field configurations $\varphi_{|M'} : M' \to \mathbb{R}$,



that identify with the finite dimensional real vector space $\mathbb{R}^{M'}$. The renormalization problem then comes from the fact that, if $f \in \mathcal{O}(H)$ is a reasonable functional with finite dimensional mean value

$$\langle f \rangle_H(B, \Lambda) := \langle f \rangle_{\mathbb{R}^{M'}},$$

the limit

$$\langle f\rangle_H := \lim_{\substack{B\to M\\ \Lambda\to M}} \langle f\rangle_H(B,\Lambda)$$

does not exist when the box size increases up to M and the lattice step tends to zero. The whole point of renormalization theory is to give a refined way of taking this limit, by eliminating, inductively on the degree of the formal power series involved, all the infinities that appear in it.

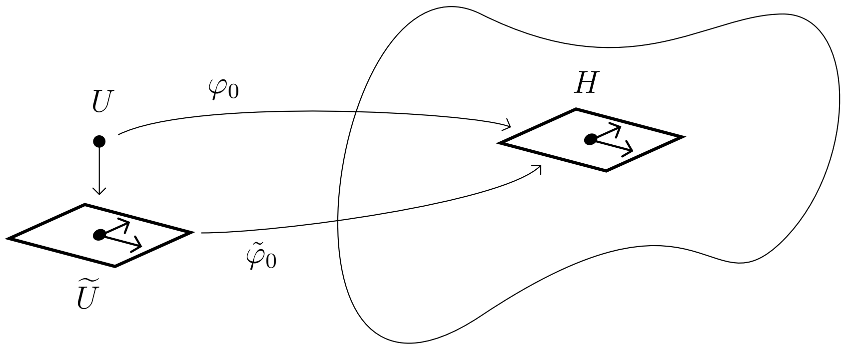
10 The geometry of perturbative quantization

The main advantage of formulating classical field theory in terms of parametrized geometry is that it allows to give a geometrical viewpoint of the various perturbative quantization methods that appear in physics, working directly on the space of field configurations. This way of formulating quantization, that is close to DeWitt's covariant field theory, has the advantage of allowing a general treatment of theories with symmetries.

The basic idea is to consider a given solution $\varphi_0 : M \to C$ of the equations of motion, called the background field, and to work with the formal completion $\widehat{\mathcal{O}}_{\varphi_0}(H)$ of the algebra of functionals on the space H of fields histories (one may also work with the derived critical space of the action functional here) around this background field. To get an idea of the type of algebra that one may obtain by this formal completion process, consider a vectorial field theory on a manifold M, with configuration space $C = E \to M$ a vector bundle and space of histories $H = \underline{\Gamma}(M,E)$ its space of sections. If φ_0 is the zero field, the formal completion $\widehat{\mathcal{O}}_{\varphi_0}(H)$ may be described explicitely by using distributional symmetric formal power series:

$$\widehat{\mathcal{O}}_{\varphi_0}(H) = \prod_{n\geq 0} \mathrm{Hom}_{\mathbb{R}}(\Gamma(M^n, E^{\boxtimes n}),\mathbb{R})^{S_n}.$$

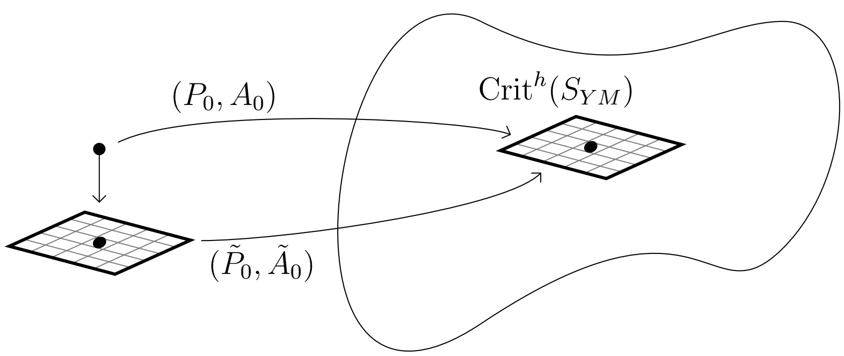
One may think of these formal power series as the algebra of functions on the restriction of the space $\underline{\Gamma}(M,E)$ to infinitesimal thickenings $\bullet \to \widetilde{T}$ of the point $\varphi_0 = 0$ in H, given by algebras of the form $\mathcal{C}^\infty(\mathbb{R}^n)/(x_1,\dots,x_n)^m$. Here is a drawing of the case $n = 2$ and $m = 2$.



In the case of a theory with symmetries, e.g., Yang-Mills theory, we take the formal completion

$$\widehat{\mathrm{Crit}^h(S_{YM})}_{(P_0,A_0)}$$

of the (flat part of the) derived critical space $\mathrm{Crit}^h(S_{YM}) = \mathbb{R}\mathrm{Hom}(M, BG_{flat})$ at a given background flat gauge field (P_0, A_0). This formal completion is defined as the restriction of the derived critical space to the category of nilpotent homotopical thickenings $\tilde{T}$ of the point (P_0, A_0).



The formal derived space is completely determined by its tangent (homotopical) local Lie algebra $\mathfrak{g}$, so that the formal completion of the functionals of the classical field theory is completely determined as the Chevalley-Eilenberg algebra

$$\mathcal{O}\left(\widehat{\mathrm{Crit}^h(S_{YM})}_{(P_0,A_0)}\right) = \mathrm{CE}^*(\mathfrak{g})$$

of this Lie algebra (formulated in the $\mathcal{D}$-geometrical setting, to make $\mathfrak{g}$ finite dimensional in the differential sense). There is a local (i.e., differentially structured) Poisson bracket on the derived critical space $\mathrm{Crit}^h(S_{YM})$, and this induces an invariant pairing on $\mathfrak{g}$, i.e., a Poisson structure on $\mathrm{CE}^*(\mathfrak{g})$.

One then formulates and solves various different (but essentially equivalent) renormalization problems to define a perturbative functional integral (i.e., a renormalized gaussian average)

$$\langle - \rangle_H : \widehat{\mathcal{O}}_{\varphi_0}(H) \to \mathbb{R}[[\hbar]].$$

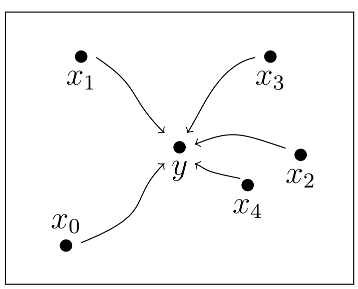
Parametrized geometry is also the natural setting to formalize a non-perturbative renormalization problem, in terms of non-formal functionals in $\mathcal{O}(H)$.

Finally, one may give a very algebraic formulation of the perturbative quantization problem, that is a quantum analog of Lagrange's algebraic formulation of variational calculus. This involves the use of the Ran space $\mathrm{Ran}(M)$ of configurations of points in the parameter space for trajectories, that gives a geometric way to encode formal power series of elements in $\mathcal{O}(H)$, once for all background fields $\varphi_0 : M \to C$. This space is also a convenient setting for the geometry of multi-jets, that give a tool for the simultaneous study of Taylor series of functions at different points.

This space may be defined as the parametrized space whose points with values in an open set $U \subset \mathbb{R}^n$ are given by

$$\mathrm{Ran}(M)(U) = \{[S] : S \times U \to M \text{ smooth, with } S \text{ a finite set}\}.$$

It is morally the union of all finite powers M^S of M along all diagonal maps (and in particular, along symmetries induced by bijections of S). One may also see this space as a useful way to encode notions that are compatible with the collapsing of families of points in M to one given point.



M

The corresponding parametrized geometry is given by the $\mathcal{D}$-geometry of factorization spaces on the Ran space. The quantization problem may be formalized in the following way: if X is a derived $\mathcal{D}$-space over M (for example, the derived critical space $\mathrm{Crit}^h(S)$ of a given action functional, given by the Euler-Lagrange system of partial differential equations), one may associate to it a derived $\mathcal{D}$-space $X^{(Ran)}$ (by a kind of homotopical multi-jet construction) over $\mathrm{Ran}(M)$, with an additional factorization property. Its algebra of functions is called a commutative factorization algebra. One may try to deform this commutative structure to a non-commutative factorization algebra. In most cases, it is not possible in the non-derived setting, as one can see on the example of quantum group deformations: there are no such deformations in a strict set theoretical context, because of the classical Heckman-Hilton argument of algebraic topology, that says that two compatible monoid operations on a set are equal and commutative. However, if one works with a homotopical version of factorization spaces, whose algebras of functions are commutative homotopical algebras, the quantization problem is better behaved. This imposes to see the Ran space as a derived space, whose points are parametrized by homotopical smooth algebras. The most natural thing to do is to work directly with the differential graded category of modules over the space $X^{(Ran)}$, that forms a factorization category, whose deformations are called categorical factorization quantizations. This gives an optimal setting to state a general quantization problem, giving a kind of quantum version of Lagrange's analytic formulation of mechanics. Remark that this formulation can't be obtained without the use of homotopical methods.

11 What's new in this book?

We now give a (non-exhaustive) list of the original tools, ideas and results, that we developed writing this book.

We propose here a reference book, that gives a coherent and complete mathematical toolbox for the coordinate-free treatment of classical and quantum field theories. Giving such an exhaustive presentation, made accessible to graduate students and researchers through a compact formalism, is an original contribution to the mathematical literature, even if there already exist good specialized references on some parts of the subject, cited in the bulk of our book.

Moreover, we have always tried, when possible, to give general methods, that apply to all of the numerous examples treated in this book. This allows us to avoid unworthy repetitions, that would have prevented us to treat so much material in one single book. The mathematician's approach of going from the general to the particular case also allows us to give optimal and precise hypothesis and results on theories that are usually treated in the physics literature through the study of particular examples.

We also describe a large class of examples of variational problems from classical, quantum and theoretical physics, directly in our language. Finally, we give an overview of various modern quantization and renormalization procedures, together with a mathematical formalization of them that is coherent with our general formalism.

We may coarsly say that the bundle composed of Chapters 1, 2, 3, 11, 12, 19, 21 and 23 (see also Section 8.11) contains substantial mathematical contributions of the author to the mathematical formalization of classical or quantum field theories. The other chapters contain results that were gathered in the mathematical and physical literature, and that were often in a form that didn't make them accessible to a graduate student in mathematics, so that the presentation that we give of them is at least pedagogically original. The cautious choice, and the concise, but exhaustive description of optimal mathematical tools for the presentation of our results, was also an important part of this work, that has clearly some pedagogical and bibliographical content.

Here is a more precise description of some of our main scientific contributions:

- We describe in Section 1.1 the main tools of higher category theory purely in an abstract setting, independent of the model chosen to describe higher categories. This allows us to shorten the usual presentation of these tools, and to concentrate on their applications. Some of the notions introduced there are not yet in the litterature.

- To give a conceptual presentation of the various structures used in this book, we introduced in Section 1.1 the general doctrinal techniques for higher categorical logic, and developed a systematical categorification process, called the doctrine machine (see Definition 8.11.2), that gives automatically the right definitions for both higher categorical structures, and algebraic and geometric structures, in their classical (see Chapter 2) and homotopical (see Chapter 9) fashions.

- In particular, our definition of monoidal and symmetric monoidal higher categories in Sections 1.2 and 1.3 is obtained by generalizing directly the categorical logical interpretation of monoids and commutative monoids. This approach is equivalent to the other ones (in particular, to Maclane and Segal's approaches), but looks much more natural. It also has the advantage of having a direct generalization to the higher categorification of other sketch-like theories (e.g., finitely presented or finite limit theories).

- The idea of parametrized and functional geometry (see Chapter 2), was already present in the literature, for example in the work of Grothendieck's and Souriau's schools, and in the synthetic geometry community, grounded by Lawvere's categorical approach to dynamics. However, it's systematic use for the formalization of the known physical approaches to quantum field theory is new. Our constructions are directly adapted to spaces with boundary and corners (see Example 2.2.8), contrary to the above references, that mostly work with closed manifolds. We also treat in the same setting super-symmetric field theories (see Section 2.3.3), and the geometry of non-algebraic partial differential equations (see Section 11.5).

- The proper generalization of classical functional analysis to parametrized and functional geometry, called functorial analysis (see Chapter 3), is also a new idea that is particularly fruitful to combine homotopical and super-manifold methods with more classical methods of analysis. There is not really another way to solve this important problem of the formalization of (non-perturbative) quantum field theory computations.

- The systematic definition of differential invariants through categorical methods, grounded by Quillen's work, was extended a bit in our work (see Section 1.5). This way, we don't have to repeat the definition of differential invariants in the various geometrical settings we use. The application of these methods to define jet spaces with boundaries and jet spaces in smooth super-geometry, seems to be new.

- We have introduced the notion of monoidally enriched differential geometry (see Section 2.3.3), to cover various types of functional geometries present in this book, like super-geometry, graded geometry and relatively algebraic $\mathcal{D}$-geometry.

- We have developed the new notion of smooth algebra with corners (see Example 2.2.8), that gives, through the associated differential and jet calculus, a natural setting for the study of non-topological higher dimensional quantum field theories with boundaries. This also opens the door to interesting research directions in differential geometry of higher and derived stacks with corners, and to their factorization quantizations (see Chapter 23).

- We have given a non-abelian cohomological interpretation of Cartan's formalism in general relativity (see Sections 6.4 and 13.4.2). This opens the possibility to study global properties of moduli stacks of Cartan geometries.

- We have formalized supersymmetric local differential calculus purely in terms of $\mathcal{D}$-modules (see Chapter 11), which gives a better take at its analogy with classical local differential calculus.

- Our formalization of $\mathcal{D}$-geometry over the Ran space (see Sections 11.5, 11.3, and 23.4) in smooth and analytic geometry is also new.

- Our formalization of the classical Batalin-Vilkovisky formalism and gauge fixing procedure (see Chapter 12), presented in a coordinate free language, that works also for supersymmetric theories, with optimal finiteness hypothesis, is also a new mathematical result. This part of the theory, that is the technical heart of our contributions, expands and improves on the two publications [Pau11a] and [Pau11b], by adding a new and precise geometric formalization of the gauge fixing procedure for local field theories.

- The mathematical formalization of non-perturbative quantum field theory (see Chapter 19) using functorial analysis and the Wilson-Wetterich non-perturbative renormalization group, compatible with our formalization of gauge theories, is also new.

- Our formalization of the Epstein-Glaser causal renormalization method (see Chapter 21), in terms of functional geometry and functorial analysis improves on the purely functional analytic approach used in the literature, by giving a clear meaning to the notion of space of graded or differential graded fields adapted to this method. This generalization is also adapted to the treatment of higher gauge theories, that were not treated before with this approach in the literature.

- Our general formalization of the categorical quantization problem for factorization spaces in Section 23.6 is also new.

- The use of the analytic derived Ran space, and of microlocal algebraic analysis in the geometric study of factorization algebras (See Remark 23.2.13) is also an original contribution, that opens the door to a causal treatment, à la Epstein-Glaser, of the quantization problem for factorization algebras on Lorentzian spacetimes, which is an interesting issue of the theory, already studied in the physics litterature in particular cases, for example by Hollands and Wald.

12 What's not in this book?

This books essentially treats the formalization of

quantitative aspects

of quantum field theories, with main aim to give to mathematicians a direct access to the standard physics texbooks in this subject. The quantum part of the book puts the focus on perturbative considerations, and we have partially skipped some important global aspects, like instantons, and Witten's various analytic generalizations of Morse theory.

A very interesting mathematical research direction that we have not treated at all in this book is the study of

qualitative aspects

of quantum field theories, that one may putatively call the theory of quantum dynamical systems.

The starting point for the theory of dynamical systems may be said to be Poincaré's paper on the three body problem: he transformed a mistake into interesting mathematical developments. The entry door into the theory of differential dynamical systems that is the closest to the spirit of our book would be Thom's transversality theory (grounded in [Tho56]), expanded to multijets by Mather [Mat70] (see also [Cha01] for a presentation of the subject and [GG73], Section II.4 for proofs). Remark that these transversality methods are based on codimension considerations for subspaces of infinite dimensional spaces of functions (called spaces of fields in our book) defined by partial differential equations. The better permanence properties of transversality in derived geometrical spaces may give new tools to mathematicians to refine their knowledge of classical transversality methods in dynamics.

The qualitative study of non-perturbative quantum field theory, through the Wilson-Wetterich non-perturbative (also sometimes called functional) renormalization group method, is another interesting open mathematical problem, that has, to the author's knowledge, not yet been studied so much by pure mathematicians.

Our precise mathematical formalization of the theory through the methods of functorial analysis, that complete, but are strictly compatible with the functional analytic ones, may be useful to the interested researchers.

13 Aims, means and specifications

The main problem that we had to face when we started this project, was the vast zoology of types of structures that appear in the mathematical formalization of quantum field theory. Here are the specifications that we used to arrive to a locally finitely presented book.

Our viewpoint of physics is very naive and reductionist: a physical theory is given by a family of experiments and a model, i.e.

a mathematical machinery,

that explains these experiments. The main constraint on such a theory is that one must be able to discuss it with collegues and students, so that they test it, correct it and improve it with later developments. We will mostly be interested by models, and refer to the physics litterature for the description of the corresponding experiments. Some of the models we present are theoretical extrapolations on usual physical models. They are either hoped to be tested at some point, or only useful to uncover and prove nice mathematical theorems. We don't care about these motivational aspects, leaving to the reader the opportunity to think about her own ones.

There are various ways to describe a mathematical machinery, and all of them are based on some common metamathematical (i.e. natural) language.

In the set theoretical language, one has the *axiom of choice* that gives a tool to prove

existence results.

In the categorical language (see Lawvere [Law05], [Law06] and Ehresmann [Ehr81]), one has *Yoneda's lemma*, that allows to prove general

unicity statements.

The combination of the categorical and set theoretical approach is morally necessary to be able to prove that a mathematical problem is *well posed* (in a sense generalizing Hadamard's notion of well posed partial differential system), meaning that its solution exists and is unique (in some sense). A special occurrence of this general philosophy is given by the following statement of the part of logics called model theory (grounded by Goedel's pioneering work): a non-trivial coherent theory (coherent means here, with good finiteness properties) admits, under the axiom of choice, at least a model. Once conveniently generalized to the setting of higher categorical and homotopical logic, this *doctrinal axiom of choice* can be used to guess nice formulations for the problems in play, that allow us to get existence results for their solutions. Of course, given a problem, defining precisely the coherence of its theory, and formulating it coherently is often much harder than solving it, but the general idea that Yoneda gives unicity and coherence (plus axiom of choice) gives existence was our useful guide to look for nice formulations of the problems we are interested in.

The rough idea of categorical mathematics is to define general mathematical objects and theories not as sets with additional structures, but as objects of a (possibly higher) category (in a metamathematical, i.e., linguistic sense), with a particular universal mapping property. Suppose given a notion of n-category for $n \geq 0$. All theories, types of theories and their models (also called semantics) that will be considered in this book can be classified, in a coordinate free fashion, by the following definition of a doctrine. The coordinate definition of a given theory is usually called its syntax. We will often describe theories using their syntax, but work with their semantics in a coordinate-free, higher categorical fashion.

Definition. Let $n \geq 1$ be an integer. A *doctrine* is an $(n+1)$-category $\mathcal{D}$. A *theory* of type $\mathcal{D}$ is an object $\mathcal{C}$ of $\mathcal{D}$. A *model* for a theory of type $\mathcal{D}$ in another one is an object

$$M : \mathcal{C}_1 \to \mathcal{C}_2$$

of the n-category $\underline{\mathrm{Mor}}(\mathcal{C}_1, \mathcal{C}_2)$.

Remark that the doctrines we use are usually given by $(n+1)$-categories of n-categories with additional *structures* and *properties*. One may define additional structures through a (possibly inductive) sketch-like construction (see Chapters 1 and 2), and additional properties through existence of representants for some functors. We decided to use the above general identification between higher category theory and categorical logic, to get the optimal pedagogical presentation, but the proper formulation of a doctrinal axiom of choice would involve a refined language, necessary to define the notion of coherence. This book being applications-oriented, we didn't want to burry its results under six feet of formalism[1].

The types of doctrines used in our presentation of quantum field theory are for example given by:

- the 2-category of categories with finite products (or more generally finite limits) with product preserving functors between them and natural transformations, whose theories are called algebraic theories. These will be used to formalize infinitesimals in smooth differential calculus.

- the 2-category of (nerves of) sites (categories with Grothendieck topology), whose models are called sheaves, or spaces. These will be used to formalize differential geometry on spaces of fields.

- the 2-category of monoidal (resp. symmetric monoidal, resp. multi-) categories with monoidal functors and monoidal natural transformations between them, whose theories are called PROs (resp. PROPs, resp. multicategories). These will be used to formalize fermionic differential calculus, local functional calculus and homotopical Poisson reduction of general gauge theories.

- the 2-category of Quillen's model categories with Quillen adjunctions between them. These will be used to formalize geometrically homotopical Poisson reduction of general gauge theories and higher gauge field theories.

- the ${}^{\infty}2$-category of ∞-categories, that are useful to formalize deformation quantization of gauge theories; the difficulty to pass from ${}^{\infty}1$- to ${}^{\infty}2$-categories are not bigger than those that have to be overcome to pass to ${}^{\infty}n$-categories, and the general theory is very useful for making devissages and classifying theories in the spirit of the Baez and Dolan's categorification program.

- We will also encounter n-doctrines related to topological quantum field theories.

- There is no good reason to restrict to doctrines for $n = 1$, because we may also be interested by studying the relations between doctrines, their geometries, and so on. These may only be studied in higher doctrines.

[1] Did we?

It is known that distilled axioms are pretty indigestible. We strongly suggest the reader follow the complicated alembics of Chapters 1 and 2 simultaneously with respective sections of more wholesome following chapters[2].

We don't aim at a complete account of all these theories, but we will present enough of each of them to make our description of physical models mathematically clear and consistent. There are many other ways to approach the mathematics of quantum field theory. All of them have advantages on those used here. They were chosen because i am more comfortable with them, and because i don't see another path to cover all i wanted to say in that book.

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[2]Exercice: find another occurence of this paragraph in the mathematical litterature.

[3]Workshop's archives: http://www.math.jussieu.fr/~fpaugam/GdT-physique/GdT-physique.html.

discussions about mathematics and physics and for his important involvement in the workshop's activities.

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[4]Workshop's archives: http://www.math.jussieu.fr/~fpaugam/documents/GdT-renormalisation-factorisation.html.

Part I

Mathematical preliminaries

Chapter 1

A categorical toolbox

The reader can skip this chapter at first reading and refer to it when needed. We give here a very short account of the tools from (higher) category theory that will be used along this book. Some of these are not so standard outside of the category theory community, so that we recall them for later reference in other chapters. Mac Lane's book [ML98], Grothendieck's seminar [AGV73] and Kashiwara-Schapira's book [KS06] can also be excellent complements, for classical categorical results. We refer to Chapter 8 for a quick presentation of the tools of higher category theory used in our book, with a documented reference list. More informations may be found in Baez and Dolan's groundbreaking article [BD98], in Simpson's textbook [Sim10], in Cheng and Lauda's introduction [CL04], in Lurie's reference books [Lur09c] and [Lur09d], and in Joyal's book [Joy11] (see also [Rez09], [Mal10] and [Ara12]).

In all this chapter, we work with a broad notion of *sets* (also called *classes*). This will lead to problems only when one works with limits indexed by very big classes. We will use standard techniques, when needed, to overcome this problem. The reader interested by technical details can refer to the theory of Grothendieck universes [AGV73].

1.1 Higher categories, doctrines and theories

In this section, we describe the basic constructions of higher category theory that we will use along this book, by generalizing quite directly usual categorical methods. The reader interested only by these may replace everywhere n by 1.

We first give a simple definition of a notion of n-category, following Simpson's book [Sim10]. One can find a finer axiomatic in [Sim01] and [BS11].

Definition 1.1.1. A theory of n-categories is supposed to have a notion of sum and product. A 0-category is a set. An *n-category* $\mathcal{A}$ is given by the following data:

1. (OB) a set $\mathrm{Ob}(\mathcal{A})$ of *objects*, compatible with sums and products of n-categories.

2. (MOR) for each pair (x, y) of objects, an $(n-1)$-category $\underline{\mathrm{Mor}}(x, y)$. One defines

$$\underline{\mathrm{Mor}}(\mathcal{A}) := \coprod_{x,y\in \mathrm{Ob}(\mathcal{A})} \underline{\mathrm{Mor}}(x, y)$$

and by induction $\underline{\mathrm{Mor}}^i(\mathcal{A}) := \underline{\mathrm{Mor}}(\underline{\mathrm{Mor}}(\dots(\mathcal{A})))$ for $0 \leq i \leq n$. One has $\underline{\mathrm{Mor}}^0(\mathcal{A}) := \mathcal{A}$ and $\underline{\mathrm{Mor}}^n(\mathcal{A})$ is a set. Denote $\mathrm{Mor}(\mathcal{A}) = \mathrm{Ob}(\underline{\mathrm{Mor}}(\mathcal{A}))$. By construction, one has source and target maps

$$s_i, t_i : \mathrm{Mor}^i(\mathcal{A}) \to \mathrm{Mor}^{i-1}(\mathcal{A})$$

that satisfy

$$s_i s_{i+1} = s_i t_{i+1},\ t_i s_{i+1} = t_i t_{i+1}.$$

3. (ID) for each $x \in \mathrm{Ob}(\mathcal{A})$, there should be a natural element $1_x \in \mathrm{Mor}(x, x)$. This gives morphisms
$$e_i : \mathrm{Mor}^i(\mathcal{A}) \to \mathrm{Mor}^{i+1}(\mathcal{A})$$
such that $s_{i+1}e_i(u) = u$ and $t_{i+1}e_i(u) = u$.

4. (EQUIV) on each set $\mathrm{Mor}^i(\mathcal{A})$, there is an equivalence relation $\sim$ compatible with the source and target maps. The induced equivalence relation on $\mathrm{Mor}^i(x, y)$ is also denoted $\sim$.

5. (COMP) for any $0 < i \leq n$ and any three $i-1$-morphisms u, v and w sharing the same source and target, there is a well-defined composition map

$$(\mathrm{Mor}^i(u, v)/\sim) \times (\mathrm{Mor}^i(v, w)/\sim) \to (\mathrm{Mor}^i(u, w)/\sim)$$

which is associative and has the classes of identity morphisms as left and right units.

6. (EQC) Equivalence and composition are compatible: for any $0 \leq i < n$ and $u, v \in \mathrm{Mor}^i(u, v)$, sharing the same source and target, then $u \sim v$ if and only if there exists $f \in \mathrm{Mor}^{i+1}(u, v)$ and $g \in \mathrm{Mor}^{i+1}(v, u)$ such that $f \circ g \sim 1_u$ and $g \circ f \sim 1_v$. This allows one to define the category $\tau_{\leq 1}\underline{\mathrm{Mor}}^i(u, v)$ with objects $\mathrm{Mor}^i(u, v)$ and morphisms between $w, z \in \mathrm{Mor}^i(u, v)$ given by equivalence classes in $\mathrm{Mor}^{i+1}(w, z)/\sim$.

An n-category is called *strict* if its composition laws all lift on morphisms (meaning that all compositions maps are well defined and strictly associative). We will also sometimes call an n-category *weak* if it is not strict.

We will give in Section 8.10 a sketch of proof of the following theorem.

Theorem 1.1.2. *In the ZFC axiomatic formulation of set theory, there exists a theory of n-categories, for all n. The class of n-categories forms an $(n+1)$-category n*CAT.

Proof. We refer to Simpson's book [Sim10], Rezk's paper [Rez09] and Ara's paper [Ara12] for three different (but essentially equivalent) constructions of a theory of n-categories. □

Simpson and Rezk actually define homotopical versions of higher categories, called $^\infty n$-categories. There is also a unicity result for these theories, due to Barwick and Schommer-Pries [BS11]. All the higher categorical definitions of this section are also adapted to this homotopical setting, without changes. The reader is invited to specialize to the case where $n = 1$ to get the usual notions of category theory.

The notion of n-category allows us to define a very general notion of type of structure (in the sense of categorical logic), that we call a doctrine.

Definition 1.1.3. A *doctrine* is an $(n+1)$-category $\mathcal{D}$. A *theory* of type $\mathcal{D}$ is an object $\mathcal{C}$ of $\mathcal{D}$. A *model* for a theory of type $\mathcal{D}$ in another one is an object

$$M : \mathcal{C}_1 \to \mathcal{C}_2$$

of the n-category $\underline{\mathrm{Mor}}_{\mathcal{D}}(\mathcal{C}_1, \mathcal{C}_2)$.

The above definition is an extension of Lawvere's terminology [Law06], that was used in a special case. In the literature, a theory (or more precisely its generating data) is also called a language and a syntax and the study of its models is called its semantics. The idea of categorical logics is to define theories using a language and a syntax, but to study their models through categorical methods, in order to have tools to compare theories and treat them in a coordinate-free and invariant (or, as physicists would say, covariant) way. Relating the above general notion to the explicit presentations of doctrines used by Lawvere would involve a general notion of *structures and properties* on n-categories (e.g., of n-monad in an n-category), that is out of the scope of these notes, but that will be clear in all the examples we will describe.

Definition 1.1.4. Let $\mathcal{D}$ be a doctrine. A *presentation* of $\mathcal{D}$ is the datum of a higher doctrine $\mathcal{E}$ and of two theories $\mathcal{C}_1$ and $\mathcal{C}_2$ of type $\mathcal{D}$ such that there is an equivalence of doctrines

$$\mathcal{D} \cong \underline{\mathrm{Mor}}_{\mathcal{E}}(\mathcal{C}_1, \mathcal{C}_2)$$

identifying theories of type $\mathcal{D}$ with models of the theory $\mathcal{C}_1$ in the theory $\mathcal{C}_2$.

We now define the notion of tensor product of theories, that is a very useful tool to define new theories from existing ones.

Definition 1.1.5. Let $\mathcal{D} \subset n\mathrm{C{\scriptsize AT}}$ be a doctrine of higher categories with additional structures. We say that $\mathcal{D}$ has *internal homomorphisms* if, for $\mathcal{C}_1$ and $\mathcal{C}_2$ two theories, the higher category $\underline{\mathrm{Mor}}(\mathcal{C}_1, \mathcal{C}_2)$ is naturally equipped with the structure of an object of $\mathcal{D}$. If $\mathcal{D}$ is a doctrine with internal homomorphisms, and $\mathcal{C}_1$ and $\mathcal{C}_2$ are two theories, their *Lawvere tensor product* is the theory $\mathcal{C}_1 \otimes \mathcal{C}_2$ such that for all theory $\mathcal{C}_3$, one has a natural equivalence

$$\underline{\mathrm{Mor}}(\mathcal{C}_1, \underline{\mathrm{Mor}}(\mathcal{C}_2, \mathcal{C}_3)) \cong \underline{\mathrm{Mor}}(\mathcal{C}_1 \otimes \mathcal{C}_2, \mathcal{C}_3).$$

One may also use ${}^{\infty}n$-categories to study the homotopy theory of doctrines and theories (structures up-to-homotopy), that plays an important role in physics. The corresponding objects will be called ∞-doctrines and ∞-theories. What we explain here is also adapted to the homotopical setting. We refer to Section 8.11 for more details on this.

Many of the doctrines that we will use are given by 2-categories very close to the 2-category Cat of categories, that we now describe concretely. A category $\mathcal{C}$ is a strict 1-category. More precisely:

Definition 1.1.6. A category $\mathcal{C}$ is given by the following data:

1. a set $\mathrm{Ob}(\mathcal{C})$ called the *objects* of $\mathcal{C}$,

2. for each pair of objects X, Y, a set $\mathrm{Hom}(X, Y)$ called the set of *morphisms*,

3. for each object X a morphism $\mathrm{id}_X \in \mathrm{Hom}(X, X)$ called the *identity morphism*,

4. for each triple of objects X, Y, Z, a composition law for morphisms

$$\circ : \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z).$$

One supposes moreover that this composition law is associative, i.e., $f \circ (g \circ h) = (f \circ g) \circ h$ and that the identity is a unit, i.e., $f \circ \mathrm{id} = f$ et $\mathrm{id} \circ f = f$.

Definition 1.1.7. A *functor* $F : \mathcal{C}_1 \to \mathcal{C}_2$ between two categories is given by the following data:

1. for each object X of $\mathcal{C}_1$, an object $F(X)$ of $\mathcal{C}_2$.

2. for each morphism $f : X \to Y$ of $\mathcal{C}_1$, a morphism $F(f) : F(X) \to F(Y)$ of $\mathcal{C}_2$.

One supposes moreover that F is compatible with identities and composition, i.e.,

$$F(\mathrm{id}) = \mathrm{id} \text{ and } F(f \circ g) = F(f) \circ F(g).$$

A *natural transformation* $\Phi : F \Rightarrow G$ between two functors $F, G : \mathcal{C}_1 \to \mathcal{C}_2$ is given by the datum, for every object X of $\mathcal{C}_1$, of a morphism $\Phi_X : F(X) \to G(X)$, such that for every morphism $f : X \to Y$ in $\mathcal{C}_1$, the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{\Phi_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\Phi_Y]{} & G(Y) \end{array}$$

Definition 1.1.8. The 2-category $\textsc{Cat}$ is the strict 2-category whose:

1. objects are categories.

2. morphisms are functors.

3. 2-morphisms are natural transformations.

One of the most important tool of category theory is the notion of adjunction. We first give a 2-categorical definition of this notion.

Definition 1.1.9. Let $\mathcal{E}$ be a 2-category. An *adjoint pair* in $\mathcal{E}$ is a pair

$$F : \mathcal{D} \leftrightarrows \mathcal{C} : G$$

of morphisms, together with a pair of morphisms

$$i : 1_{\mathcal{C}} \to F \circ G \text{ and } e : G \circ F \to 1_{\mathcal{D}},$$

fulfilling the triangular equalities

$$(Gi) \circ (eG) = 1_G \text{ and } (Fe) \circ (iF) = 1_F.$$

We now define the notion of adjunction between functors that is better adapted to homotopical generalizations. For $n = 1$, it is compatible with the above definition when $\mathcal{E} = \text{CAT}$ is the 2-category of categories.

Definition 1.1.10. Let $\mathcal{C}$ and $\mathcal{D}$ be two n-categories. A *pair of adjoint functors* between $\mathcal{C}$ and $\mathcal{D}$ is a pair

$$F : \mathcal{D} \leftrightarrows \mathcal{C} : G$$

of functors together with a pair of morphisms of functors

$$i : 1_{\mathcal{C}} \to F \circ G \text{ and } e : G \circ F \to 1_{\mathcal{D}},$$

such that for all objects $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, the induced morphisms

$$\underline{\mathrm{Mor}}_{\mathcal{C}}(FY, X) \leftrightarrows \underline{\mathrm{Mor}}_{\mathcal{D}}(Y, GX)$$

are equivalences of $(n-1)$-categories inverse of each other.

Example 1.1.11. The pair

$$F : \text{GRP} \leftrightarrows \text{SETS} : G$$

given by the forgetful functor F and the free group functor G is an adjoint pair in CAT.

We now define the notion of left and right Kan extensions, that are useful higher categorical analogues of the notion of universal object in a category. These are useful to formulate limits and colimits by their universal properties, and to discuss localization of categories and derived functors, that are necessary tools for the formulation of homotopical and homological algebra.

Definition 1.1.12. Let $\mathcal{D}$ be an $(n+1)$-category for $n \geq 1$, $F : C \to D$ and $f : C \to C'$ be morphisms between two objects of $\mathcal{D}$. A (local) *right Kan extension* of F along f, if it exists, is a morphism

$$\mathrm{Ran}_f F = f_* F : C' \to D$$

equipped with a 2-morphism $a : \mathrm{Ran}_f F \circ f \Rightarrow F$ that induces, for all $G \in \mathrm{Mor}^1(C', D)$, an equivalence of $(n-1)$-categories

$$\begin{array}{ccc} \underline{\mathrm{Mor}}_{\underline{\mathrm{Mor}}(C',D)}(G, \mathrm{Ran}_f F) & \xrightarrow{\sim} & \underline{\mathrm{Mor}}_{\underline{\mathrm{Mor}}(C,D)}(f^* G, F) \\ c & \mapsto & a \star c \end{array},$$

where $f^* : \underline{\mathrm{Mor}}(C', D) \to \underline{\mathrm{Mor}}(C, D)$ denotes composition by f. Similarly, one defines a left Kan extension as a morphism

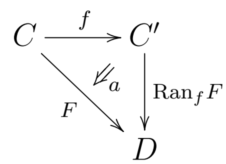
$$\mathrm{Lan}_f F = f_! F : C' \to D$$

equipped with a natural equivalence

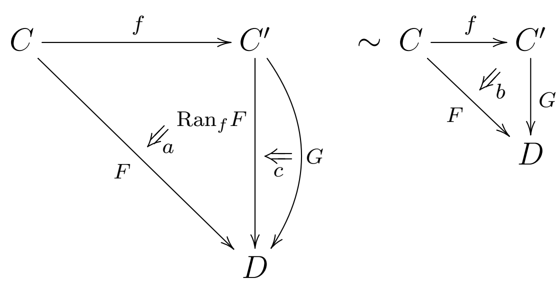
$$\underline{\mathrm{Mor}}_{\underline{\mathrm{Mor}}(C,D)}(F, f^*(-)) \cong \underline{\mathrm{Mor}}_{\underline{\mathrm{Mor}}(C',D)}(\mathrm{Lan}_f F, -).$$

Remark 1.1.13. The notion of Kan extension may also be defined in a weaker "almost cartesian closed" setting, where $\mathcal{D}$ is an n-category whose morphism $(n-1)$-categories $\underline{\mathrm{Mor}}_{\mathcal{D}}(C, D)$ naturally form n-categories. This will be useful to simplify the treatment of $^\infty 1$-Kan extensions, avoiding passing through the setting of $^\infty 2$-categories, by using the fact fact that functors between two $^\infty 1$-categories also form an $^\infty 1$-category. This method is used by Lurie in [Lur09d], in the setting of quasi-categories.

More concretely, a right Kan extension is a pair $(\mathrm{Ran}_f F, a)$ composed of a morphism $\mathrm{Ran}_f F : C' \to D$ and a 2-morphism $a : \mathrm{Ran}_f F \circ f \Rightarrow F$ giving a diagram



such that for every pair (G, b) composed of a morphism $G : C' \to D$ and of a 2-morphism $b : G \circ f \Rightarrow F$, there exists a 2-morphism $c : G \Rightarrow \mathrm{Ran}_f F$, such that the equivalence $b \sim a \circ c$, visualized by the following diagrams



is true up to a coherent system of higher isomorphisms.

We now define the notion of limits and colimits in n-categories, that will allow us to define interesting doctrines.

Definition 1.1.14. Let $\mathcal{D}$ be an $(n+1)$-category for $n \geq 1$, with a terminal object $\mathbb{1}$, defined by the fact that for all object $\mathcal{C}$ of $\mathcal{D}$, one has

$$\underline{\mathrm{Mor}}(\mathcal{C}, \mathbb{1}) \cong \mathbb{1}_{n\mathrm{CAT}},$$

where $\mathbb{1}_{n\mathrm{CAT}}$ is the n-category with only one k-morphism for all k. Let I and $\mathcal{C}$ be two objects of $\mathcal{D}$. A *diagram* in $\mathcal{C}$ is a morphism $F : I \to \mathcal{C}$. Let $f = 1 : I \to \mathbb{1}$ be the terminal morphism in $\mathcal{D}$. The *limit* of F is defined as the right Kan extension

$$\lim F := \mathrm{Ran}_1 F$$

and the *colimit* of F is defined as the left Kan extension

$$\mathrm{colim}\, F := \mathrm{Lan}_1 F.$$

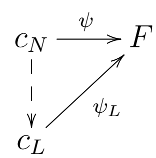
Here is a more concrete description of this notion, in the case where $\mathcal{D} = n\text{CAT}$ is the $(n+1)$-category of n-categories. Given a diagram in $\mathcal{C}$ and an object N of $\mathcal{C}$, we denote $c_N : I \to \mathcal{C}$ the constant functor given by $c_N(X) = N$. The functor $N \mapsto c_N$ identifies with the composition

$$1^* : \underline{\mathrm{Mor}}(\mathbb{1}, \mathcal{C}) \to \underline{\mathrm{Mor}}(I, \mathcal{C})$$

with the final morphism $1 : I \to \mathbb{1}$.

Definition 1.1.15. A *cone* from N to F (resp *cocone* from F to N) is a 2-morphism $\psi : c_N \to F$ (resp. $\psi : F \to c_N$).

A limit $L = \lim F$ for a functor $F : I \to \mathcal{C}$ is a universal cone, meaning a cone $\psi_L : c_L \to F$ such that for every other cone $\psi : c_N \to F$, there exists a factorization



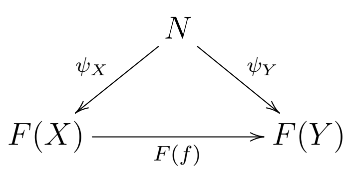
that is "unique up to higher equivalence", meaning that for all N, composition with ψ_L is an equivalence of $(n-2)$-categories

$$\psi_L^* : \underline{\mathrm{Mor}}_{\underline{\mathrm{Mor}}(I,\mathcal{C})}(c_N, c_L) \xrightarrow{\sim} \underline{\mathrm{Mor}}_{\underline{\mathrm{Mor}}(I,\mathcal{C})}(c_N, F).$$

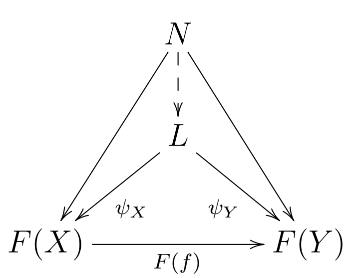
In the case $n = 1$ of usual categories, the above map is simply a bijection between sets, meaning that the above factorization is unique.

Inverting the sense of arrows, one identifies the notion of colimit $L = \operatorname{colim} F$ with that of universal cocone.

More concretely, in the case of categories, i.e., for $n = 1$, a cone is a family of morphisms $\psi_X : N \to F(X)$ indexed by objects in I such that for every morphism $f : X \to Y$ in I, the following diagram



commutes. A limit $L = \lim F$ for a functor $F : I \to \mathcal{C}$ is a cone such that for every cone ψ from N to F, there is a unique morphism $N \to L$ that makes the diagram



commute.

Example 1.1.16. Limits (resp. colimits) indexed by discrete categories identify with products (resp. coproducts).

We will call a limit or colimit in the $(n+1)$-category of n-categories *small* if it is indexed by an n-category whose $(n-1)$-truncation (given by identifying isomorphic $(n-1)$-morphisms) is an object of the n-category $(n-1)-\mathrm{CAT}$ of small $(n-1)$-categories, defined by induction from the category of small sets.

Proposition 1.1.17. *Let $\mathcal{C}$ be an $(n+1)$-category. For every object X of $\mathcal{C}$, the functor*

$$\underline{\mathrm{Mor}}(X,-):\mathcal{C}\to n\mathrm{CAT}$$

commutes with all small colimits.

Proof. This follows from the definition of limits and colimits. □

Definition 1.1.18. Let $\mathcal{C}$ be an n-category. Its Yoneda dual n-categories are defined as

$$\mathcal{C}^\wedge := \underline{\mathrm{Mor}}_{n\mathrm{CAT}}(\mathcal{C},(n-1)\mathrm{CAT}) \text{ and } \mathcal{C}^\vee := \underline{\mathrm{Mor}}_{n\mathrm{CAT}}(\mathcal{C}^{op},(n-1)\mathrm{CAT})$$

If $X\in\mathcal{C}$, we denote $h^X := \underline{\mathrm{Mor}}(X,-)\in\mathcal{C}^\wedge$ and $h_X := \underline{\mathrm{Mor}}(-,X)\in\mathcal{C}^\vee$ the associated functors.

The main interest of the Yoneda dual n-category is that it contains all small limits and colimits, so that it is very similar to the n-category $(n-1)\mathrm{CAT}$. One may think of it as a kind of limit or colimit completion of the given category.

Proposition 1.1.19. *Let $\mathcal{C}$ be an n-category. Then its Yoneda duals*

$$\mathcal{C}^\vee := \underline{\mathrm{Hom}}(\mathcal{C}^{op},\mathrm{SETS}) \text{ and } \mathcal{C}^\wedge := \underline{\mathrm{Hom}}(\mathcal{C},\mathrm{SETS})$$

have all limits and colimits. Moreover, the embedding $\mathcal{C}\to\mathcal{C}^\wedge$ preserves limits and the embedding $\mathcal{C}\to(\mathcal{C}^\vee)^{op}$ preserves colimits.

Proof. Follows from the definition of limits and colimits. □

Definition 1.1.20. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two n-categories. A functor $F:\mathcal{C}_1\to\mathcal{C}_2$ is *fully faithful* if for every pair (X,Y) of objects of $\mathcal{C}_1$, the corresponding morphism

$$\mathrm{Mor}(F):\underline{\mathrm{Mor}}_{\mathcal{C}_1}(X,Y)\to\underline{\mathrm{Mor}}_{\mathcal{C}_2}(F(X),F(Y))$$

is an equivalence of $(n-1)$-categories.

We now give a formulation of the higher Yoneda lemma, that is a reasonable hypothesis to impose on the formalism. The case $n=1$ is the usual Yoneda lemma.

Hypothesis 1 (Yoneda's lemma)**.** Let $\mathcal{C}$ be an n-category. The functors

$$\begin{array}{ccc}\mathcal{C}^{op} & \to & \mathcal{C}^\wedge\\ X & \mapsto & h^X\end{array} \text{ and } \begin{array}{ccc}\mathcal{C} & \to & \mathcal{C}^\vee\\ X & \mapsto & h_X\end{array}$$

are fully faithful, meaning that the morphisms of $(n-1)$-categories

$$\underline{\mathrm{Mor}}_{\mathcal{C}}(X,Y)\to\underline{\mathrm{Mor}}_{\mathcal{C}^\wedge}(h^Y,h^X) \text{ and } \underline{\mathrm{Mor}}_{\mathcal{C}}(X,Y)\to\underline{\mathrm{Mor}}_{\mathcal{C}^\vee}(h_X,h_Y)$$

are equivalences.

The first example of doctrine is given by the notion of n-algebraic theory (see Lawvere's thesis [Law04] for the case $n = 1$). The generalization to higher dimension is useful for the study of monoidal categories (see Sections 1.2 and 1.3).

Definition 1.1.21. The doctrine of *finite product n-categories* $n\mathrm{FPC{\scriptstyle AT}} \subset n\mathrm{C{\scriptstyle AT}}$ is the $(n+1)$-category:

1. whose objects are n-categories $\mathcal{T}$ with finite products (and in particular, a final object, given by the empty product).

2. whose morphisms are product preserving functors.

If $\mathcal{T} \in \mathrm{FPC{\scriptstyle AT}}$ is a theory, the set of generators of the monoid of isomorphism classes of its objects is called its set of *sorts*, and its morphisms are also called its *operations*. An *algebraic theory* is a theory $\mathcal{T} \in \mathrm{FPC{\scriptstyle AT}}$ with only one sort, i.e., whose objects are all of the form $x^{\times r}$ for a fixed object $x \in \mathcal{T}$ and $r \geq 0$.

Example 1.1.22. The category $\mathcal{T}_{\mathrm{G{\scriptstyle RP}}}$ oppposite to that of finitely generated free groups, with group morphisms between them is called the theory of groups. A model $G : \mathcal{T}_{\mathrm{G{\scriptstyle RP}}} \to \mathrm{S{\scriptstyle ETS}}$ is simply a group. The theory $\mathcal{T}_{\mathrm{G{\scriptstyle RP}}}$ is single-sorted. The theory of rings (given by the category opposite to that of free finitely generated rings) is also single sorted.

Example 1.1.23. More generally, given a set theoretic algebraic structure (rings, modules over them, etc...) that admits a free construction, the corresponding theory $\mathcal{T}$ is given by the opposite category to the category of free finitely generated objects, and models

$$M : \mathcal{T} \to \mathrm{S{\scriptstyle ETS}}$$

correspond exactly to the given type of structure on the set $M(\{1\})$. An interesting example is given by the algebraic theory $\mathrm{A{\scriptstyle FF}}_{\mathcal{C}^\infty}$ of smooth rings, whose objects are smooth affine spaces $\mathbb{R}^n$ for varying n and whose morphisms $f : \mathbb{R}^n \to \mathbb{R}^m$ are smooth functions. A model of this theory

$$A : \mathrm{A{\scriptstyle FF}}_{\mathcal{C}^\infty} \to \mathrm{S{\scriptstyle ETS}}$$

is called a smooth ring. These will be useful to formulate infinitesimal calculus on functional spaces.

Example 1.1.24. The 2-sorted theory $\mathcal{T}_{\mathrm{R{\scriptstyle INGS}},\mathrm{M{\scriptstyle OD}}}$ of commutative rings and modules, is the category with finite products opposite to the category whose objects are pairs (A, M) composed of a free finitely generated commutative ring and a free module on it, and whose morphisms are pairs $(f, \varphi) : (A, M) \to (B, N)$ composed of a ring morphisms $f : A \to B$ and a module morphisms $\varphi : M \to f^*N :=_A N$.

We now define the notion of higher sketch. Sketches were first introduced by Ehresmann in [Ehr82]. This is a natural generalization of the notion of algebraic theory. Many theories we will use are given by (higher) sketches. We refer to Barr-Wells [BW05] for an overview of sketches. Remark that the original definition of Ehresmann uses graphs with partial compositions, not categories. It is more elegant since it really gives a kind of generators and relations presentation of categories, but generalizing it to higher dimension is not that easy, so that we leave this charming exercice to the interested reader.

Definition 1.1.25. Let $(\mathfrak{L}, \mathfrak{C}) = (\{I\}, \{J\})$ be two classes of small n-categories, called categories of indices. The doctrine of $(\mathfrak{L}, \mathfrak{C})$-*sketches* is the $(n+1)$-category:

1. whose objects are triples $(\mathcal{T}, \mathcal{L}, \mathcal{C})$, called *sketches*, composed of n-categories $\mathcal{T}$, equipped with a class $\mathcal{L}$ of I-indexed cones for $I \in \mathfrak{L}$ and a class $\mathcal{C}$ of J-indexed cocones for $J \in \mathfrak{C}$.

2. whose morphisms are functors that respect the given classes of cones and cocones.

A sketch $(\mathcal{T}, \mathcal{C}, \mathcal{L})$ is called a *full sketch* if $\mathcal{C}$ and $\mathcal{L}$ are composed of colimit cocones and limit cones indexed by categories $I \in \mathfrak{L}$ and $J \in \mathfrak{C}$.

Remark 1.1.26. Remark that some doctrines may be described by sketch theories, with additional *properties*. For example, the theory of cartesian closed categories (resp. that of elementary topoi), is described by theories for the finite limit sketch, with the additional property of having an inner homomorphism object (resp. a power-object, analogous to the set of parts of a given set). The general notion of doctrine encompases these generalizations, and many others.

Example 1.1.27.

1. The doctrine FALG of algebraic theories identifies with the doctrine of $(\mathfrak{L}, \mathfrak{C})$-sketches, with $\mathfrak{L}$ the class of all finite discrete categories (whose only morphisms are identities) and $\mathfrak{C} = \emptyset$.

2. The doctrine FPALG of finitely presented algebraid theories is defined as the doctrine of $(\mathfrak{L}, \mathfrak{C})$-sketches with $\mathfrak{L}$ the class of all finite discrete categories and $\mathfrak{C}$ the class of finite categories. An example of a full sketch for this doctrine is given by the category of finitely presented algebras (coequalizers of two morphisms between free algebras) for a given algebraic theory $\mathcal{T}$. For example, finitely presented groups or finitely presented commutative unital $\mathbb{R}$-algebras are theories of this kind.

3. The doctrine FLALG of finite limit theories is defined by setting $\mathfrak{L}$ to be the class of finite categories and $\mathfrak{C} = \emptyset$. We will use the following example of finite limit theory in our formalization of geometry on spaces of fields. Let $\mathcal{T} = \mathrm{OPEN}_{\mathcal{C}^\infty}$ be the category of smooth open subsets $U \subset \mathbb{R}^n$ for varying n with smooth maps between them. Let $\mathcal{C} = \{.\}$ be the point (final object) and $\mathcal{L}$ be the class of transversal pullback diagrams. Recall that these are pullback diagrams

$$\begin{array}{ccc} U \times_V W & \longrightarrow & V \\ \downarrow & & \downarrow g \\ U & \xrightarrow{f} & W \end{array}$$

such that for all $x \in U$ and $y \in V$ with $f(x) = g(y)$, the images of tangent spaces are transversal, i.e., $D_x f(T_x U) \cup D_y g(T_y V)$ generate $T_{f(x)} W$. This transversality condition means that the fiber product, is a smooth open (because of the implicit function theorem). A model

$$A : (\mathrm{OPEN}_{\mathcal{C}^\infty}, \{.\}, - \underset{-}{\overset{t}{\times}} -) \to (\mathrm{SETS}, \{.\}, - \underset{-}{\times} -)$$

of this finite limit theory gives a smooth algebra

$$A_{|\mathrm{AFF}_{\mathcal{C}^\infty}} : \mathrm{AFF}_{\mathcal{C}^\infty} \to \mathrm{SETS}$$

so that one can think of it as a refinement of the notion of smooth algebra. We will see in Proposition 2.2.6 that both objects are actually equivalent.

1.2 Monoidal categories

We will start by defining the most general notion of monoidal category in a doctrinal fashion, and then give more concrete definitions of particular examples.

Definition 1.2.1. Let $\mathcal{D}$ be the doctrine given by the $(n+2)$-category of $(n+1)$-categories with finite products. Let $(\mathcal{T}_{\mathrm{MON}}, \times)$ be the finitary algebraic theory of monoids, given by the category with finite products opposite to that of finitely generated free monoids. The theory $\mathcal{E}_k$ is defined as the tensor product of $\mathcal{D}$-theories

$$\mathcal{E}_k := \mathcal{T}_{\mathrm{MON}}^{\otimes k}.$$

in the sense of definition 1.1.5. A model of the theory $\mathcal{E}_k$ with values in $(n\mathrm{CAT}, \times)$ is called a *k-tuply monoidal n-category*. A 1-monoidal n-category is also called a *monoidal n-category.*

Concretely, a monoidal category is simply a product preserving morphism

$$(\mathcal{C}, \otimes) : (\mathcal{T}_{\mathrm{MON}}, \times) \to (\mathrm{CAT}, \times)$$

with values in the (weak) 2-category of categories. It is thus the categorical analog of a monoid.

Remark that our definition of monoidal higher category is a bit different from the one usually used in the litterature, but both are equivalent, by Mac Lane's coherence theorem [ML98]. The following theorem was first stated by Baez and Dolan in [BD98]. A complete proof can be found in Lurie's book [Lur09c], Section 5.1.2.

Theorem 1.2.2. *A k-monoidal n-category is equivalent to an $(n+k)$-category with only one morphism in degrees smaller than k.*

The following result was explained by Baez and Dolan in [BD98] and is fully proved by Simpson in [Sim10], Chapter 23.

Theorem 1.2.3. *A k-tuply monoidal n-category for $k > n+1$ is equivalent to a $k+1$-tuply monoidal n-category.*

Remark that this doctrinal viewpoint looks quite abstract, but it has the advantage of having a direct homotopical generalization, contrary to the one where one writes explicitely all diagrams. It is moreover very easy to generalize this higher categorification method to other kinds of sketch like theories (e.g., finite product, finite limit or finite presentation theories). We now give Mac Lane's compact presentation, that is based on his coherence theorem. This theorem says roughly that fixing one associativity constraint for the three terms tensor products is enough to fix uniquely all others, if a coherence among these constraints, called the pentagonal axiom, is fulfilled. The next theorem may also be taken as a definition, if one wants to avoid the use of higher categories.

Theorem 1.2.4. *A* monoidal category *is given by a tuple*

$$(\mathcal{C},\otimes)=(\mathcal{C},\otimes,\mathbb{1},\mathrm{un}^r,\mathrm{un}^l,\mathrm{as})$$

composed of

a) a category $\mathcal{C}$,

b) a bifunctor $\otimes:\mathcal{C}\times\mathcal{C}\to\mathcal{C}$,

c) an object $\mathbb{1}$ of $\mathcal{C}$ called the unit object,

d) for each object A of $\mathcal{C}$, two unity isomorphisms

$$\mathrm{un}^r_A : A\otimes\mathbb{1}\xrightarrow{\sim} A \text{ and } \mathrm{un}^l_A:\mathbb{1}\otimes A\xrightarrow{\sim} A.$$

e) for each triple (A,B,C) of objects of $\mathcal{C}$, an associativity isomorphism

$$\mathrm{as}_{A,B,C}:A\otimes(B\otimes C)\xrightarrow{\sim}(A\otimes B)\otimes C,$$

that are supposed to make the following diagrams commutative:

i. pentagonal axiom for associativity isomorphisms:

$$\begin{array}{ccccc}
((A\otimes B)\otimes C)\otimes D & \xrightarrow{\mathrm{as}_{A,B,C}\otimes D} & (A\otimes(B\otimes C))\otimes D & \xrightarrow{\mathrm{as}_{A,B\otimes C,D}} & A\otimes((B\otimes C)\otimes D)\\
\downarrow{\scriptstyle \mathrm{as}_{A\otimes B,C,D}} & & & & \downarrow{\scriptstyle A\otimes\mathrm{as}_{B,C,D}}\\
(A\otimes B)\otimes(C\otimes D) & & \xrightarrow{\qquad\mathrm{as}_{A,B,C\otimes D}\qquad} & & A\otimes(B\otimes(C\otimes D))
\end{array}$$

ii. compatibility of unity and associativity isomorphisms:

$$\begin{array}{ccc}
(A\otimes\mathbb{1})\otimes B & \xrightarrow{\mathrm{as}_{A,\mathbb{1},B}} & A\otimes(\mathbb{1}\otimes B)\\
{\scriptstyle \mathrm{un}^l_A\otimes B}\searrow & & \swarrow{\scriptstyle A\otimes\mathrm{un}^r_B}\\
 & A\otimes B &
\end{array}$$

Proof. We refer to Lurie's book [Lur09c], Section 5.1.2 and Mac Lane's book [ML98], Section VII.2. □

Definition 1.2.5. A monoidal category $(\mathcal{C},\otimes)$ is called

1. *closed* if it has internal homomorphisms, i.e., if for every pair (B, C) of objects of $\mathcal{C}$, the functor
$$A \mapsto \mathrm{Hom}(A \otimes B, C)$$
is representable by an object $\underline{\mathrm{Hom}}(B, C)$ of $\mathcal{C}$.

2. *strict* if the associativity and unity transformations are equalities.

Example 1.2.6.
1. Any category with finite products is a monoidal category with monoidal structure given by the product.

2. The category $\mathrm{MOD}(K)$ of modules over a commutative unital ring K with its ordinary tensor product is a closed monoidal category. If one works with the standard construction of the tensor product as a quotient
$$M \otimes_K N := K^{(M \times N)} / \sim_{bil},$$
of the free module on the product by the bilinearity relations, the associativity isomorphisms are not equalities. Instead, they are given by the corresponding canonical isomorphisms, uniquely determined by the universal property of tensor products.

3. The category $(\mathrm{CAT}, \times)$ of categories with their product is monoidal.

4. The category $(\mathrm{Endof}(\mathcal{C}), \circ)$ of endofunctors $F : \mathcal{C} \to \mathcal{C}$ of a given category, with its monoidal structure given by composition, is a monoidal category.

Definition 1.2.7. Let $(\mathcal{C}, \otimes)$ be a monoidal category. A *monoid* in $\mathcal{C}$ is a triple $(A, \mu, 1)$ composed of

1. an object A of $\mathcal{C}$,

2. a multiplication morphism $\mu : A \otimes A \to A$,

3. a unit morphism $1 : \mathbb{1} \to A$,

such that for each object V of $\mathcal{C}$, the above maps fulfill the usual associativity and unit axiom with respect to the given associativity and unity isomorphisms in $\mathcal{C}$. A comonoid in $(\mathcal{C}, \otimes)$ is a monoid in the opposite category $(\mathcal{C}^{op}, \otimes)$. A (left) *module* over a monoid $(A, \mu, 1)$ in $\mathcal{C}$ is a pair (M, μ_M) composed of an object M of $\mathcal{C}$ and a morphism $\mu_M : A \otimes M \to M$ that is compatible with the multiplication μ on A. A comodule over a comonoid is a module over the corresponding monoid in the opposite category.

Example 1.2.8.
1. In the monoidal category $(\mathrm{SETS}, \times)$, a monoid is simply an ordinary monoid.

2. In the monoidal category $(\mathrm{MOD}(K), \otimes)$ for K a commutative unital ring, a monoid A is simply a (not necessarily commutative) K-algebra. A module over A in $(\mathrm{MOD}(K), \otimes)$ is then an ordinary module over A. A comonoid A is a coalgebra and a comodule over it is a comodule in the usual sense.

3. In the monoidal category $(\mathrm{Endof}(\mathcal{C}), \circ)$ of endofunctors of a category $\mathcal{C}$, a monoid is called a monad (or a triple) and a comonoid is called a comonad. If

$$F : \mathcal{D} \leftrightarrows \mathcal{C} : G$$

is an adjunction, the endofunctor $A = F \circ G : \mathcal{C} \to \mathcal{C}$ is equipped with natural unit $1 : \mathrm{Id}_{\mathcal{C}} \to A$ and multiplication morphisms $\mu : A \circ A \to A$ that makes it a monad. The functor $B = G \circ F : \mathcal{D} \to \mathcal{D}$ is equipped with natural counit $1 : B \to \mathrm{Id}_{\mathcal{C}}$ and comultiplication morphisms $\Delta : B \to B \circ B$ that makes it a comonad.

Definition 1.2.9. The doctrine MonCat is defined as the 2-category:

1. whose objects are monoidal categories,

2. whose morphisms are monoidal functors, i.e., functors $F : \mathcal{C}_1 \to \mathcal{C}_2$ equipped with natural isomorphisms

$$u : \mathbb{1}_{\mathcal{D}} \to F(\mathbb{1}_{\mathcal{C}}) \text{ and } \mu_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$$

that fulfill natural compatibility conditions with respect to the monoidal structures.

3. whose 2-morphisms are monoidal natural transformations, i.e., natural transformations $\Phi : F \Rightarrow G$ between monoidal functors compatible with the additional structure.

The monoidal analog of Lawvere's algebraic theories (see definition 1.1.21) is given by the following notion.

Definition 1.2.10. Theories for the doctrine MonCat given by strict monoidal categories whose objects are all of the form $x^{\otimes n}$ for a fixed object x and $n \geq 0$ are called *linear theories*.

Example 1.2.11. The simplest non-trivial linear theory is the strict monoidal category $(\Delta, +)$ of finite ordered sets (finite ordinals) with increasing maps between them and monoidal structure given by the sum of ordinals. A model

$$M : (\Delta, +) \to (\mathcal{C}, \otimes)$$

in a monoidal category is simply a monoid in $(\mathcal{C}, \otimes)$.

One can easily define a notion of multilinear theory as a linear theory with many generators. We refer to Leinster's book for a complete account of these theories [Lei04].

Definition 1.2.12. A *PRO* is a triple $(\mathcal{C}, \otimes, S)$ composed of a strict monoidal category $(\mathcal{C}, \otimes)$ and a set S such that:

1. the monoid of objects of $\mathcal{C}$ is isomorphic to the free monoid on S, and

2. for any elements $b_1, \ldots, b_m$ and $a_1, \ldots, a_n$ of S, the natural map

$$\coprod_{a_1^1,\ldots,a_n^{k_n}} \big(\mathrm{Hom}_{\mathcal{C}}(a_1^1 \otimes \ldots \otimes a_1^{k_1}, a_1) \times \cdots \times \mathrm{Hom}_{\mathcal{C}}(a_n^1 \otimes \ldots \otimes a_n^{k_n}, a_n)\big) \\ \to \mathrm{Hom}_{\mathcal{C}}(b_1 \otimes \ldots \otimes b_m, a_1 \otimes \ldots \otimes a_n)$$

   is a bijection, where the union is over all $n, k_1, \ldots, k_n \in \mathbb{N}$ and $a_i^j \in S$ such that there is an equality of formal sequences

$$(a_1^1, \ldots, a_1^{k_1}, \ldots, a_n^1, \ldots, a_n^{k_n}) = (b_1, \ldots, b_m).$$

A *morphism of PROs* is a pair (F, f) composed of a monoidal functor $F : (\mathcal{C}_1, \otimes) \to (\mathcal{C}_2, \otimes)$ and a map of sets $f : S_1 \to S_2$ such that the map on monoids of objects is the result of applying the free monoid functor to f.

On usually thinks of a PRO (also called a multicategory) as a category $\mathcal{C}$, equipped with families of multilinear operations $\mathrm{Hom}_{\mathcal{C}}(a_1, \ldots, a_n; b)$ with composition maps for multilinear operations that fulfill natural associativity and unit conditions.

Definition 1.2.13. A category $\mathcal{C}$ *enriched over a monoidal category* $(\mathcal{V}, \otimes)$ is given by the following data:

1. a class $\mathrm{Ob}(\mathcal{C})$ of objects.

2. for each pair (X, Y) of objects, an object $\mathrm{Hom}(X, Y)$ in $\mathcal{V}$.

3. for each object X, a morphism $\mathrm{id}_X : \mathbb{1} \to \mathrm{Hom}(X, X)$ in $\mathcal{V}$.

4. for each triple (X, Y, Z) of objects, a composition morphism

$$\mathrm{Hom}(X, Y) \otimes \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$$

with the usual associativity and unit conditions for composition (with respect to the given monoidal structure).

Example 1.2.14. 1. A monoidal category with internal homomorphisms is enriched over itself. In particular, the monoidal category $(\textsc{Mod}(K), \otimes)$ is enriched over itself.

2. A category enriched over the monoidal category $(\textsc{Cat}, \times)$ of categories is a strict 2-category. More generally, a category enriched over the monoidal category $(n\textsc{Cat}, \times)$ is a strict $(n + 1)$-category.

1.3 Symmetric monoidal categories

We present here a definition of symmetric monoidal higher categories that is equivalent to the one that is traditionally used (generalizing Segal's theory of Γ-spaces, as in [Lur09c]). The equivalence, due to Shulman, results from the fact that Γ is the free semi-cartesian symmetric monoidal theory, and that it's cartesian closure it the theory $\mathcal{T}_{\text{CMon}}$, that we use. The clear advantage of our approach is that it allows to define other kinds of categorified algebraic structures, based on other sketch theories (e.g., finite product theories, finite limit theories and finitely generated theories).

Definition 1.3.1. Let $\mathcal{D}$ be the doctrine given by the $(n+2)$-category of $(n+1)$-categories with finite products. Let $\mathcal{E}_\infty := (\mathcal{T}_{\text{CMon}}, \times)$ be the finitary algebraic theory of commutative unital monoids, given by the category with finite products opposite to that of finitely generated free commutative unital monoids. A *symmetric monoidal n-category* is a model for the theory $\mathcal{E}_\infty$ in the $(n+1)$-category $n\text{Cat}$.

We use general Definition 1.2.1 to define braided monoidal categories.

Definition 1.3.2. A *braided monoidal category* is a 2-tuply monoidal category.

The following result is very similar to what was explained in the setting of monoidal structures in Theorem 1.2.3.

Proposition 1.3.3. *There is an equivalence of $(n+1)$-categories between symmetric monoidal n-categories and k-tuply monoidal categories for $k > n+1$. In particular, a 3-tuply monoidal category is a symmetric monoidal category.*

Proof. See the proof of loc. cit. for references. □

We now give a theorem that may be used as a more concrete definition.

Theorem 1.3.4. *A* braided monoidal category *is a tuple*

$$(\mathcal{C}, \otimes) = (\mathcal{C}, \otimes, \mathbb{1}, \text{un}^r, \text{un}^l, \text{as}, \text{com})$$

such that $(\mathcal{C}, \otimes, \mathbb{1}, \text{un}^r, \text{un}^l, \text{as})$ is a monoidal category, equipped with the following additional datum:

f) for each pair (A, B) of objects of $\mathcal{C}$, a commutation isomorphism (also called a braiding)

$$\text{com}_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A,$$

that is supposed to make the following diagrams commutative:

iii. hexagonal axiom for compatibility between the commutation and the associativity isomorphisms:

$$\begin{array}{ccccccc} & & A \otimes (B \otimes C) & \xrightarrow{\text{as}} & (B \otimes C) \otimes A & & \\ & \overset{\text{com}}{\nearrow} & & & & \overset{\text{com}}{\searrow} & \\ (A \otimes B) \otimes C & & & & & & B \otimes (C \otimes A) \\ & \underset{\text{com}\otimes\text{id}}{\searrow} & & & & \underset{\text{id}\otimes\text{com}}{\swarrow} & \\ & & (B \otimes A) \otimes C & \xrightarrow[\text{as}]{} & B \otimes (A \otimes C) & & \end{array}$$

iv. compatibility between units and commutation:

$$\mathrm{un}^r \circ com = \mathrm{un}^l.$$

A symmetric monoidal category *is a braided monoidal category that further fulfils*

v. the idempotency of the commutation isomorphism:

$$\mathrm{com}_{A,B} \circ \mathrm{com}_{B,A} = \mathrm{id}_A.$$

Proof. We refer to Lurie's book [Lur09c], Section 5.1.2. □

We now introduce an important finiteness condition on an object of a monoidal category: that of having a monoidal dual.

Definition 1.3.5. Let $(\mathcal{C}, \otimes)$ be a monoidal category. A *dual pair* in $(\mathcal{C}, \otimes)$ is given by a pair $(A, A^\vee)$ of objects of $\mathcal{C}$ and two morphisms

$$i : \mathbb{1} \to A^\vee \otimes A \text{ and } e : A \otimes A^\vee \to \mathbb{1}$$

fulfilling the triangle equalities meaning that the composite maps

$$A \xrightarrow{A\otimes i} A \otimes A^\vee \otimes A \xrightarrow{e\otimes A} A$$

$$A^\vee \xrightarrow{i\otimes A^\vee} A^\vee \otimes A \otimes A^\vee \xrightarrow{A^\vee\otimes e} A^\vee$$

are the identities. One then calls $A^\vee$ a right dual of A and A a left dual of $A^\vee$. A monoidal category whose objects both have left and right duals is called *rigid*.

A dual pair can be seen as an adjoint pair in the sense of definition 1.1.9, in the "delooped" 2-category $B(\mathcal{C}, \otimes)$, with only one object, morphisms given by objects of $\mathcal{C}$ (whose composition is given by the tensor product) and 2-morphisms given by morphisms in $\mathcal{C}$ (with their usual composition).

Remark 1.3.6. Rigid braided monoidal categories have a well defined notion of trace for endomorphisms. Indeed, if $(A, A^\vee)$ is a dual pair with unit $i : \mathbb{1} \to A^\vee \otimes A$ and counit $e : A \otimes A^\vee \to \mathbb{1}$, one can define a trace map

$$\mathrm{Tr} : \mathrm{End}(A) \to \mathrm{End}(\mathbb{1})$$

by the composition

$$\mathrm{Tr}(f) := \mathbb{1} \xrightarrow{i} A^\vee \otimes A \xrightarrow{A^\vee\otimes f} A^\vee \otimes A \xrightarrow{\mathrm{com}} A \otimes A^\vee \xrightarrow{e} \mathbb{1}.$$

Example 1.3.7. 1. Any category with finite products is symmetric monoidal with $\otimes = \times$.

2. The category $(\textsc{Mod}(K), \otimes)$ of modules over a commutative unital ring is closed symmetric monoidal with monoidal structure given by the usual tensor product of modules and the canonical associativity, unit and commutativity constraints. These constraints are constructed by applying the universal property
$$\mathrm{Bil}(M, N; K) \cong \mathrm{Hom}(M \otimes N, K)$$
of the tensor product. The internal homomorphism objects $\underline{\mathrm{Hom}}(M, N)$ are given by the natural module structure on the set $\mathrm{Hom}_{\textsc{Mod}(K)}(M, N)$ of module morphisms.

3. The category $\textsc{Mod}_{fd}(K)$ of finite dimensional modules over a commutative field K is a closed and rigid symmetric monoidal category, i.e., it has duals and internal homomorphisms. The trace of an endomorphism $f : M \to M$ in the monoidal sense gives its usual trace in $\mathrm{End}(\mathbb{1}) = K$.

4. The category $\mathrm{Cob}(n)$ of oriented $(n-1)$-dimensional manifolds with morphisms given by oriented cobordisms $\partial N = M \coprod \bar{M}'$ between them (with N an n-dimensional manifold with boundary) is a rigid monoidal category, with duality given by the orientation reversing operation $M^\vee := \bar{M}$.

An important motivation for introducing the formalism of symmetric monoidal categories is given by the following example, that plays a crucial role in physics.

Example 1.3.8. The categories $\textsc{Mod}_g(K)$ and $\textsc{Mod}_s(K)$ of $\mathbb{Z}$-graded and $\mathbb{Z}/2$-graded modules over a commutative unital ring K are symmetric monoidal categories. Objects of $\textsc{Mod}_g(K)$ (resp. $\textsc{Mod}_s(K)$) are direct sums
$$V = \oplus_{k \in \mathbb{Z}} V^k \text{ (resp. } V = \oplus_{k \in \mathbb{Z}/2} V^k)$$
of K-modules and morphisms between them are linear maps respecting the grading. If $a \in V^k$ is a homogeneous element of a graded module V, we denote $\deg(a) := k$ its degree. The tensor product of two graded modules V and W is defined as the graded module given by the usual tensor product of the underlying modules, equipped with the grading
$$(V \otimes W)^k = \oplus_{i+j=k} V^i \otimes W^j.$$
The tensor product is associative with unit $\mathbb{1} = K$ in degree 0, and with the usual associativity isomorphisms of K-modules. The main difference with the tensor category $(\textsc{Mod}(K), \otimes)$ of usual modules is given by the non-trivial commutation isomorphisms
$$\mathrm{com}_{V,W} : V \otimes W \to W \otimes V$$
defined by extending by linearity the rule
$$v \otimes w \mapsto (-1)^{\deg(v)\deg(w)} w \otimes v.$$
One thus obtains symmetric monoidal categories $(\textsc{Mod}_g(K), \otimes)$ and $(\textsc{Mod}_s(K), \otimes)$ that are moreover closed, i.e., have internal homomorphisms. Once one has fixed the symmetric monoidal category $(\textsc{Mod}_g, \otimes, \mathrm{as}, \mathrm{un}, \mathrm{com})$, one gets the following canonical additional

constructions for free. The internal homomorphism object in $(\textsc{Mod}_g, \otimes)$ is given by the usual module

$$\underline{\mathrm{Hom}}(V, W) = \underline{\mathrm{Hom}}_{\textsc{Mod}(K)}(V, W)$$

of all linear maps, equipped with the grading

$$\underline{\mathrm{Hom}}_{\textsc{Mod}_g(K)}(V, W) = \oplus_{n\in\mathbb{Z}}\mathrm{Hom}^n(V, W)$$

where the degree n component $\mathrm{Hom}^n(V, W)$ is the set of all linear maps $f : V \to W$ such that $f(V^k) \subset W^{k+n}$. There is a natural evaluation morphism on the underlying modules

$$\mathrm{ev} : \underline{\mathrm{Hom}}_{\textsc{Mod}(K)}(M, N) \underset{\textsc{Mod}(K)}{\otimes} M \to N$$

and one only has to check that it is compatible with the grading to show that, for every $U \in \textsc{Mod}_g(K)$, there is a natural bijection

$$\mathrm{Hom}_{\textsc{Mod}_g(K)}(U, \underline{\mathrm{Hom}}(V, W)) \cong \mathrm{Hom}_{\textsc{Mod}_g(K)}(U \otimes V, W).$$

One can check that the tensor product of two internal homomorphisms $f : V \to W$ and $f' : V \to W'$ is given by the Koszul sign rule on homogeneous components:

$$(f \otimes g)(v \otimes w) = (-1)^{\deg(g)\deg(v)} f(v) \otimes g(w).$$

Definition 1.3.9. If $(\mathcal{C}, \otimes)$ is symmetric monoidal, a *commutative monoid* in $\mathcal{C}$ is a monoid $(A, \mu, 1)$ that fulfills the commutativity condition

$$\mu \circ \mathrm{com}_{A,A} = \mu.$$

We denote $\textsc{Alg}_{\mathcal{C}}$ (resp. $\textsc{CAlg}_{\mathcal{C}}$) the category of monoids (resp. commutative monoids) in $\mathcal{C}$. These will also be called *algebras* and *commutative algebras* in $\mathcal{C}$.

Example 1.3.10. 1. Commutative monoids in the symmetric monoidal category $(\textsc{Sets}, \times)$ are simply commutative monoids.

2. Commutative monoids in the symmetric monoidal category $(\textsc{Mod}(\mathbb{Z}), \otimes)$ of $\mathbb{Z}$-modules with their usual tensor product are ordinary commutative unital rings.

3. Commutative monoids in the symmetric monoidal category $(\textsc{Mod}_g(K), \otimes)$ of graded K-modules are called graded commutative K-algebras. Commutative algebras in $(\textsc{Mod}_s(K), \otimes)$ are called (commutative) super-algebras over K. One can find more on super-algebras in [DM99]. Recall that the commutativity condition for the multiplication of a super algebra uses the commutation isomorphism of the tensor category $(\textsc{Mod}_s(K), \otimes)$, so that it actually means a graded commutativity: (A, μ) is commutative if $\mu \circ \mathrm{com}_{A,A} = \mu$, which means

$$a.b = (-1)^{\deg(a)\deg(b)} b.a.$$

If $(\mathcal{C}, \otimes)$ is symmetric monoidal, A is a commutative monoid in $\mathcal{C}$, and M is a left A-module, one can put on M a right A-module structure $\mu^r : M \otimes A \to M$ defined by $\mu^r_M := \mu^l_M \circ \mathrm{com}_{M,A}$. We will implicitly use this right A-module structure in the forthcoming.

The left and right module structures are really different if the commutation isomorphism is non-trivial, e.g., in the category $\mathrm{Mod}_g(K)$ of graded modules. This is a key point of the formalism of symmetric monoidal categories: the sign nightmares are hidden in the monoidal structure, so that one can compute with generalized algebras as if they were just ordinary algebras.

Definition 1.3.11. Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category with small colimits and $V \in \mathcal{C}$. The *symmetric algebra* of V is the free commutative monoid in $\mathcal{C}$ defined by

$$\mathrm{Sym}(V) := \coprod_{n \geq 0} V^{\otimes n}/S_n.$$

If $F : \mathrm{CAlg}_{\mathcal{C}} \to \mathcal{C}$ is the forgetful functor, for every commutative monoid B in $\mathcal{C}$, one has a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(V, F(B)) \cong \mathrm{Hom}_{\mathrm{CAlg}_{\mathcal{C}}}(\mathrm{Sym}(V), B).$$

Definition 1.3.12. The doctrine SMonCat is defined as the 2-category:

1. whose objects are symmetric monoidal categories.
2. whose morphisms are monoidal functors that are compatible with the commutativity constraint.
3. whose 2-morphisms are monoidal natural transformations.

Example 1.3.13. The symmetric monoidal category $\mathrm{Cob}(n)$ of cobordisms, whose objects are oriented $(n-1)$-dimensional manifolds, morphisms are oriented cobordisms and monoidal structure is the disjoint union, can be thought as a theory for the doctrine SMonCat, called the theory of topological quantum field theories. Its models

$$M : (\mathrm{Cob}(n), \coprod) \to (\mathcal{C}, \otimes)$$

are called (Atiyah-type) topological quantum field theories. Its higher categorical extension $\mathrm{Cob}_{\infty_n}(n)$, as described by Baez-Dolan and refined by Lurie, is in some sense very simple because it is the free symmetric monoidal ${}^{\infty}n$-category with fully dualizable objects on one object (the point), as shown by Lurie in [Lur09a] (full dualizability is a higher categorical analog of monoidal dualizability).

The analog of Lawvere's algebraic theories (see definition 1.1.21) in the doctrine SMonCat is given by the notion of PROP. We refer to [Val04] and [Mer10] for a full account of this theory.

Definition 1.3.14. A *PROP* is a strict symmetric monoidal category whose objects are all of the form $x^{\otimes n}$ for a single object x and $n \geq 0$. A *wheeled PROP* is a rigid PROP, i.e., a strict symmetric monoidal category with duals generated by a single object x.

There is a nice combinatorial way to encode PROPs, given by the notion of properad, that clarifies the fact that PROPs are theories of multilinear operations with multiple input and multiple output. Among them, theories with simple outputs are called operads. We refer to [Val04] for a full account of properad theory and to [LV10] and [Fre09] for a full account of operad theory.

Let $(\mathcal{V}, \otimes)$ be a symmetric monoidal category with finite sums.

Definition 1.3.15. An $\mathbb{S}$-*bimodule* in $\mathcal{V}$ is a sum $\mathcal{M} = \coprod_{n,m\geq 0} \mathcal{M}(n,m)$ whose generic element $\mathcal{M}(n,m)$ is both a left $\mathbb{S}_n$-module and a right $\mathbb{S}_m$-module in $\mathcal{V}$. An $\mathbb{S}$-*module* is a sum $\mathcal{M} = \coprod_{n\geq 0} \mathcal{M}(n)$ whose generic element $\mathcal{M}(n)$ is a left $\mathbb{S}_n$-module.

Definition 1.3.16. A *properad* (resp. *wheeled properad*) $\mathcal{P}$ in $\mathcal{V}$ is a PROP (resp. a wheeled PROP) $(\mathcal{C}, \otimes)$ enriched over $(\mathcal{V}, \otimes)$. Its underlying $\mathbb{S}$-bimodule is

$$\mathcal{P}(n,m) := \underline{\mathrm{Hom}}(n,m).$$

An *operad* in $\mathcal{V}$ is a PROP $(\mathcal{C}, \otimes)$ enriched over $(\mathcal{V}, \otimes)$ with finite sums whose $\mathbb{S}$-bimodule is of the form

$$\mathcal{P}(n,m) = \oplus \lim_{n_1+\cdots+n_m=n} \underline{\mathrm{Hom}}(n_1, \mathbb{1}) \otimes \cdots \otimes \underline{\mathrm{Hom}}(n_m, \mathbb{1}).$$

We denote $\textsc{Properads}(\mathcal{V}, \otimes)$, $\textsc{WProperads}(\mathcal{V}, \otimes)$ and $\textsc{Operads}(\mathcal{V}, \otimes)$ the categories of properads, wheeled properads, and operads in $\mathcal{V}$.

Definition 1.3.17. An *algebra over a properad* (resp. *over an operad*) in $\mathcal{V}$ is a model

$$M : (\mathcal{P}, \otimes, \mathbb{1}) \to (\mathcal{V}, \otimes, \mathbb{1})$$

in the doctrine $\textsc{SMonCat}_{\mathcal{V}}$ of symmetric monoidal categories enriched over $\mathcal{V}$. More concretely,

1. if one defines the *endomorphism properad* of an object $M \in \mathcal{V}$ by

$$\mathcal{E}nd(M)(n,m) := \underline{\mathrm{Hom}}(M^{\otimes n}, M^{\otimes n}),$$

   an algebra over a properad $\mathcal{P}$ is the datum of a family of $\mathbb{S}_n \times \mathbb{S}_m$-equivariant morphisms

$$\mathcal{P}(n,m) \to \mathcal{E}nd(M)(n,m)$$

   that is compatible with tensor products and composition of morphisms.

2. if one defines the *endomorphism operad* of an object $M \in \mathcal{V}$ by

$$\mathcal{E}nd(M)(n) := \underline{\mathrm{Hom}}(M^{\otimes n}, M),$$

   an algebra over an operad $\mathcal{P}$ is the datum of a family of $\mathbb{S}_n$-equivariant morphisms

$$\mathcal{P}(n) \to \mathcal{E}nd(M)(n)$$

   that is compatible with tensor products and composition of morphisms.

Example 1.3.18.

- The operad Comu in $(\mathrm{V\small{ECT}}_k, \otimes)$ given by $\mathrm{Comu}(n) = k = \mathbb{1}_{\mathrm{V\small{ECT}}_k}$ is the commutative operad, whose algebras are commutative (unital) algebras in $\mathrm{V\small{ECT}}_k$. More generally, representations of Comu in a symmetric monoidal category $(\mathcal{C}, \otimes)$ are algebras (i.e., commutative monoids) in $(\mathcal{C}, \otimes)$.
- The operad Assu in $(\mathcal{C}, \otimes)$ is given by $\mathrm{Assu}(n) = \mathbb{1}[\mathbb{S}_n]$. Algebras over Assu are associative unital algebras (i.e., non-commutative monoids) in $(\mathcal{C}, \otimes)$. The morphism $\mathrm{Assu} \to \mathrm{Comu}$ sends $\sigma \in \mathbb{S}_n$ to 1.
- The operad Ass is given by $\mathrm{Ass}(0) = 0$ and $\mathrm{Ass}(n) = \mathbb{1}[\mathbb{S}_n]$ for $n \geq 1$. Its algebras are non-unital associative algebras.
- The operad $\mathrm{Lie} \subset \mathrm{Ass}$ is given by $\mathrm{Lie}(n) = \mathrm{Ind}_{\mathbb{Z}/n\mathbb{Z}}^{S_n}(V)$, where V is the representation of $\mathbb{Z}/n\mathbb{Z}$ given by its action on a primitive n-th root of unity. The operad LIE has as algebras Lie algebras (not easy to prove; see [LV10], Section 13.2 for more details and references).

One can also define operads by generators and relations, starting with $\mathbb{S}$-modules and using the free operad construction. There is also a purely combinatorial description of properads in terms of $\mathbb{S}$-bimodules. We only give this description for operads, referring to [Val04] for the case of properads.

Every $\mathbb{S}$-module M defines an endofunctor $\tilde{M} : \mathcal{V} \to \mathcal{V}$, called the Schur functor, by the formula

$$\tilde{M}(V) := \coprod_{n \geq 0} M(n) \otimes_{\mathbb{S}_n} V^{\otimes n}.$$

The composite of two $\mathbb{S}$-modules M and N is the $\mathbb{S}$-module

$$M \circ N = \oplus_{n \geq 0} M(n) \otimes_{\mathbb{1}[S_n]} N^{\otimes n}$$

and the unit $\mathbb{S}$-module is given by $M(0) = \mathbb{1}$.

Proposition 1.3.19. *The composition product gives $\mathbb{S}$-modules a monoidal category structure*

$$(\mathbb{S} - \mathrm{M\small{OD}}(\mathcal{V}), \circ, \mathbb{1}).$$

The Schur functor is a strict monoidal functor

$$\begin{array}{ccc} (\mathbb{S} - Mod(\mathcal{V}), \circ) & \to & (\mathrm{Endof}(\mathcal{V}), \circ) \\ M & \mapsto & \tilde{M} \end{array}$$

Remark that an operad naturally defines a monad on $\mathcal{V}$, because the Schur functor is strictly monoidal.

Proposition 1.3.20. *The functor*

$$\begin{array}{ccc} \mathrm{O\small{PERADS}}(\mathcal{V}, \otimes) & \to & \mathrm{M\small{ON}}(\mathbb{S} - \mathrm{M\small{OD}}, \circ, \mathbb{1}) \\ (\mathcal{P}, \otimes) & \mapsto & (\mathcal{P}(n))_{n \geq 0} \end{array}$$

from the category of operads to the category of monoids in the monoidal category of $\mathbb{S}$-modules has a left adjoint.

Using the above adjunction, we will also call operad a monoid in the monoidal category of $\mathbb{S}$-modules.

1.4 Grothendieck topologies

The category of sheaves for a given Grothendieck topology on a category $\mathcal{C}$ is essentially defined to improve on the category of presheaves (i.e., the Yoneda dual $\mathcal{C}^\vee$). Recall that the Yoneda embedding

$$\mathcal{C} \to \mathcal{C}^\vee := \underline{\mathrm{Hom}}(\mathcal{C}^{op}, \mathrm{SETS})$$

is fully faithful and sends all limits to limits. The problem is that it does not respect colimits. Consider the example of the category OPEN_X of open subsets of a given topological space X. If U and V are two open subsets, their union can be described as the colimit

$$U \cup V = U \coprod_{U \times_X V} V.$$

If $U \mapsto \underline{U} := \mathrm{Hom}(-, U)$ denotes the Yoneda embedding for $\mathcal{C} = \mathrm{OPEN}_X$, one usually has a map

$$\underline{U} \coprod_{\underline{U \times_X V}} \underline{V} \to \underline{U \cup V}$$

in the category $\mathcal{C}^\vee$, but this is not an isomorphism. If one works in the category $\mathrm{SH}(\mathrm{OPEN}_X, \tau)$ of sheaves on X, this becomes an isomorphism.

Definition 1.4.1. Let $\mathcal{C}$ be a category with finite limits. A *Grothendieck topology* τ on $\mathcal{C}$ is the datum of families of morphisms $\{f_i : U_i \to U\}$, called *covering families*, and denoted Cov_U, such that the following holds:

1. (Identity) The identity map $\{\mathrm{id} : U \to U\}$ belongs to Cov_U.

2. (Refinement) If $\{f_i : U_i \to U\}$ belongs to Cov_U and $\{g_{i,j} : U_{i,j} \to U_i\}$ belong to Cov_{U_i}, then the composed covering family $\{f_i \circ g_{i,j} : U_{i,j} \to U\}$ belongs to Cov_U.

3. (Base change) If $\{f_i : U_i \to U\}$ belongs to Cov_U and $f : V \to U$ is a morphism, then $\{f_i \times_U f : U_i \times_U V \to V\}$ belongs to Cov_V.

4. (Local nature) If $\{f_i : U_i \to U\}$ belongs to Cov_U and $\{f_j : V_j \to U\}$ is a small family of arbitrary morphisms such that $f_i \times_U f_j : U_i \times_U V_j \to U_i$ belongs to Cov_{U_i}, then $\{f_j\}$ belongs to Cov_U.

A category $\mathcal{C}$ equipped with a Grothendieck topology τ is called a *site*.

We remark that, for this definition to make sense one needs the fiber products that appear in it to exist in the given category. A more flexible and general definition in terms of sieves can be found in [KS06].

Suppose that we work on the category $\mathcal{C} = \mathrm{OPEN}_X$ of open subsets of a given topological space X, with inclusion morphisms and its ordinary topology. The base change axiom then says that a covering of $U \subset X$ induces a covering of its open subsets. The local character means that families of coverings of elements of a given covering induce a (refined) covering.

Definition 1.4.2. Let $(\mathcal{C}, \tau)$ be a category with Grothendieck topology. A functor $X : \mathcal{C}^{op} \to \text{SETS}$ is called a *sheaf* if the sequence

$$X(U) \longrightarrow \prod_i X(U_i) \rightrightarrows \prod_{i,j} X(U_i \times_U U_j)$$

is exact. The category of sheaves is denoted $\text{SH}(\mathcal{C}, \tau)$.

One can think of a sheaf in this sense as something analogous to the family of continuous functions on a topological space. A continuous function on an open set is uniquely defined by a family of continuous functions on the open subsets of a given covering, whose values are equal on their intersections.

Since sheaves are given by a limit preservation property, they can be seen as models of a sketch (see Definition 1.1.25) with underlying category $\mathcal{C}^{op}$.

From now on, one further supposes that the topology is sub-canonical, meaning that for all $U \in \mathcal{C}$, $\text{Hom}(-, U) : \mathcal{C}^{op} \to \text{SETS}$ is a sheaf for the given topology.

We now describe in full generality the problem solved by Grothendieck topologies.

Definition 1.4.3. Let $(\mathcal{C}, \tau)$ be a site. The *nerves of coverings* associated to τ in $\mathcal{C}$ are defined by

$$N(\{U_i \to U\}) = \{N_j = U_{i_1} \times_U \cdots \times_U U_{i_n}\} \to U.$$

Theorem 1.4.4. *Let $(\mathcal{C}, \tau)$ be a site with a sub-canonical topology. The Yoneda functor*

$$\mathcal{C} \to \text{SH}(\mathcal{C}, \tau)$$

is a fully faithful embedding that preserves general limits, and sends the nerves

$$N(\{U_i \to U\}) = \{N_j = U_{i_1} \times_U \cdots \times_U U_{i_n}\} \to U$$

of covering families to colimits.

This means that, contrary to the Yoneda embedding $\mathcal{C} \to \mathcal{C}^\vee$, the embedding of $\mathcal{C}$ in the category $\text{SH}(\mathcal{C}, \tau)$ preserves coverings. This will be essential for the definition of "varieties" modeled on objects in $\mathcal{C}$, by pasting of representable functors along open coverings.

1.5 Categorical infinitesimal calculus

In this section, we use Quillen's tangent category [Qui70] (see also the nlab website) to give a purely categorical definition of the main tools of differential calculus. The main advantage of this abstract approach is that it is very concise and allows us to deal at once with all the generalized geometric situations we will use.

Remark that one can generalize directly this approach to the higher categorical situation of doctrines and theories, by replacing everywhere the category $\mathcal{C}$ by an ${}^\infty n$-category, and the category of abelian groups in $\mathcal{C}/A$ by the stabilization $\text{Stab}(\mathcal{C}/A)$ (see Section 8.7). We refer to Section 9.6 for an example of a concrete application of this general idea to differential calculus in homotopical geometry. This can be useful for studying the deformation theory of higher categories with additional structures (that play an important role in homological mirror symmetry, for example; see [Kon95]).

In the following, we work with categories that do not necessarily admit finite limits.

Definition 1.5.1. Let $\mathcal{C}$ be a category with final object $\underline{0}$.

1. An *abelian group object* in $\mathcal{C}$ is a triple $(A, \mu, 0)$ composed of an object A of $\mathcal{C}$, a multiplication morphism $\mu : A \times A \to A$ and an identity morphism $0 : \underline{0} \to A$ that define a functor
$$\mathrm{Hom}(-, A) : \mathcal{C} \to \textsc{Grab}$$
from $\mathcal{C}$ to the category of abelian groups.

2. A *torsor over an abelian group object* $(A, \mu, 0)$ is a pair (B, ρ) composed of an object B of $\mathcal{C}$ and an action morphism $\rho : A \times B \to B$ that induce and action of $\mathrm{Hom}(-, A) : \mathcal{C} \to \textsc{Grab}$ on $\mathrm{Hom}(-, B) : \mathcal{C} \to \textsc{Sets}$ and such that the morphism
$$(\rho, \mathrm{id}) : A \times B \to B \times B$$
is an isomorphism.

The categorical definition of square zero extensions can be found on the nlab contributive website.

Definition 1.5.2. Let I be the category with two objects and a morphism between them. Let $\mathcal{C}$ be a category.

1. The *arrow category* $[I, \mathcal{C}]$ is the category whose objects are morphisms $B \to A$ in $\mathcal{C}$ and whose morphisms are given by commutative squares.

2. The *tangent category* $T\mathcal{C}$ is the category of abelian group objects in the arrow category $[I, \mathcal{C}]$, whose objects are triples
$$(B \to A, \mu : B \times_A B \to B, \underline{0} : A \to B),$$
and whose morphisms are commutative squares
$$\begin{array}{ccc} B & \longrightarrow & B' \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & A' \end{array}$$
such that the induced morphism $B \to f^*B' := A \times_{A'} B'$ is a morphism of abelian group objects over A. The fiber of $T\mathcal{C}$ at A is denoted $\textsc{Mod}(A)$ and called the category of modules over A.

3. A left adjoint to the domain functor
$$\begin{array}{rccc} \mathrm{dom} : & T\mathcal{C} & \to & \mathcal{C} \\ & [B \to A] & \mapsto & B \end{array}$$
is called a *cotangent functor* and denoted $\Omega^1 : \mathcal{C} \to T\mathcal{C}$.

Example 1.5.3. If $\mathcal{C} = \textsc{Rings}$ is the category of commutative unital rings, we will see in Theorem 2.3.1.3 that its tangent category $T\mathcal{C}$ is equivalent to the category $\textsc{Mod}$ of pairs (A, M), composed of a ring A and an A-module M. The corresponding arrow in $\mathcal{C}/A$ is given by the square zero extension

$$p : A \oplus M \to A.$$

In this setting, the value of the cotangent functor on $A \in \mathcal{C}$ is simply the module of Kähler forms, given by the square zero extension

$$A \oplus I/I^2 \cong (A \otimes A)/I^2 \to A,$$

with zero section induced by $p_1^* : A \to A \otimes A$, $a \mapsto a \otimes 1$.

Example 1.5.4. The category $\mathcal{C} = \textsc{Open}_{\mathcal{C}^\infty}^{op}$ does not have many pullbacks, meaning that it is hard to take amalgamed sums of smooth opens. This implies that for every $A \in \mathcal{C}$, there are not many abelian group objects $p : B \to A$ in $\mathcal{C}/A$. We will now show that the only such object is given by the identity map $\mathrm{id}_A : A \to A$. Indeed, suppose given an abelian cogroup object $p^* : V \to U$. This means that one has a section $0^* : U \to V$, so that $p^* : V \to U$ is injective. Moreover, it is open and closed because 0^* is a continuous section of $V \to U$. This means that V is a connected component of U. Denote $U = V \coprod W$. In this case, the amalgamed sum exists and is given by

$$U \coprod_V U \cong V \coprod W \coprod V.$$

The comultiplication $\mu^* : U \to U \coprod_V U$ has a section $\mathrm{id}_U \coprod_V 0^*$, so that it is an open and closed map, and sends U to a connected component of the above disjoint sum. This implies that $U = V$. We conclude that the tangent category $T\mathcal{C}$ is equal to $\mathcal{C}$, and that the cotangent functor $\Omega^1 : \mathcal{C} \to T\mathcal{C}$ is just the identity functor. This shows why the use of generalized algebras is necessary to define an interesting categorical differential calculus in smooth geometry.

Definition 1.5.5. Let $\mathcal{C}$ be a category. Let $M \to A$ be a module over A. A section $D : A \to M$ is called an M-valued *derivation*. We denote $\mathrm{Der}(A, M)$ the set of M-valued derivations.

By definition, if there exists a cotangent module Ω^1_A, one has

$$\mathrm{Der}(A, M) \cong \mathrm{Hom}_{\textsc{Mod}(A)}(\Omega^1_A, M).$$

We now refine the tangent category construction to define higher infinitesimal thickenings.

Definition 1.5.6. Let $\mathcal{C}$ be a category.

1. The *first thickening category* $\mathrm{Th}^1\mathcal{C}$ is the category of torsors

$$(C \to A, \rho : B \times_A C \to C)$$

in the arrow category $[I, \mathcal{C}]$ over objects $(B \to A, \mu, \underline{0})$ of $T\mathcal{C}$.

2. The *n-th thickening category* $\mathrm{Th}^n\mathcal{C}$ is the category of objects $D \to A$ in the arrow category $[I, \mathcal{C}]$ such that there exists a sequence

$$D = C_n \to C_{n-1} \to \cdots \to C_0 = A$$

with all $C_{i+1} \to C_i$ in $\mathrm{Th}^1\mathcal{C}$.

3. Given $A \in \mathcal{C}$, a left adjoint to the forgetful functor

$$\begin{array}{ccc} \mathrm{Th}^n\mathcal{C}/A & \to & \mathcal{C}/A \\ [B \to A] & \mapsto & [B \to A] \end{array}$$

is called an *infinitesimal thickening functor* and denoted $\mathrm{Th}^n : \mathcal{C}/A \to \mathrm{Th}^n\mathcal{C}/A$.

4. In particular, the functor

$$\mathrm{Jet}^n := \mathrm{Th}^n \circ \Delta^* : \mathcal{C} \to \mathrm{Th}^n\mathcal{C},$$

where $\Delta^*(A) := [A \coprod A \to A]$ is the codiagonal map, is called the *jet functor*.

Proposition 1.5.7. *Let $\mathcal{C}$ be a category with pullbacks and $A \in \mathcal{C}$. There is a natural isomorphism*

$$\mathrm{Jet}^1 A \xrightarrow{\sim} \Omega^1_A$$

in $\mathrm{Th}^1\mathcal{C}/A$.

Proof. By definition, $[\mathrm{Jet}^1 A \to A]$ is the infinitesimal thickening of the codiagonal map $[A \coprod A \to A]$. In particular, it has two retractions, denoted $d_0, d_1 : A \to \mathrm{Jet}^1 A$. One can consider d_0 as a trivialization of the torsor $\mathrm{Jet}^1 A \to A$ over an abelian group object, giving $\mathrm{Jet}^1 A \to A$ the structure of an abelian group object over A, equipped with a section $d_1 : A \to \mathrm{Jet}^1 A$. With this structure, it fulfils the universal property of the abelian group object $[\Omega^1_A \to A]$, so that both are isomorphic. □

Definition 1.5.8. Let (Legos, τ) be a site and $X \in \mathrm{Sh}(\mathrm{Legos}, \tau)$ be a space. suppose that there exists a thickening functor Th^k on $\mathrm{Sh}(\mathrm{Legos}, \tau)^{op}$. Let $C \to M$ be a morphism of spaces. The relative k-th jet space $\mathrm{Jet}^k(C/M) \to M$ is defined as the space

$$\mathrm{Jet}^k(C/M) := C \times_M \mathrm{Jet}^k(M).$$

We now define the notion of Grothendieck connection, that is a kind of infinitesimal pasting (i.e., descent) datum. Let (Legos, τ) be a site and $X \in \mathrm{Sh}(\mathrm{Legos}, \tau)$ be a space. suppose that there exists a first thickening functor Th^1 on $\mathrm{Sh}(\mathrm{Legos}, \tau)^{op}$. Denote $X^{(1)} := \mathrm{Th}^1(\Delta_{X^2})$ the first infinitesimal thickening of $\Delta_{X^2} : X \to X \times X$ and $X^{(1)}_3 := \mathrm{Th}^1(\Delta_{X^3})$ the first infinitesimal thickening of $\Delta_{X^3} : X \to X \times X \times X$. Let $p_1, p_2 : X^2 \to X$ be the two natural projections and let $p_{1,2}, p_{2,3}, p_{1,3} : X^3 \to X \times X$ the natural projections. We will also denote $p_1, p_2, p_3 : X^3 \to X$ the natural projections.

Definition 1.5.9. Let $C \to X$ be a space morphism. A *Grothendieck connection* on $C \to X$ is the datum of an isomorphism

$$A : p_1^* C_{|X^{(1)}} \xrightarrow{\sim} p_2^* C_{|X^{(1)}}$$

over $X^{(1)}$ that reduces to identity on X by pullback along $X \to X^{(1)}$. Given a Grothendieck connection on $C \to X$, one can pull it back to $X_3^{(1)}$ through the projections $p_{i,j}$, getting isomorphisms

$$A_{i,j} : p_i^* C_{|X_3^{(1)}} \xrightarrow{\sim} p_j^* C_{|X_3^{(1)}}.$$

The *curvature of the connection* $A : p_1^* C_{|X^{(1)}} \xrightarrow{\sim} p_2^* C_{|X^{(1)}}$ is the isomorphism

$$F_A := A_{1,3}^{-1} \circ A_{2,3} \circ A_{1,2} : p_1^* C_{|X_3^{(1)}} \xrightarrow{\sim} p_1^* C_{|X_3^{(1)}}.$$

An *integrable connection* is a connection whose curvature F_A is the identity morphism.

This notion of connection actually generalizes to any kind of geometrical object on X that has a natural notion of pull-back, for example a linear connection on a vector bundle, a G-equivariant connection on a G-bundle, etc...

Example 1.5.10. Let $X = \mathbb{R}$ and $C = \mathbb{R}^2$ equipped with their polynomial algebras of functions $\mathcal{O}_X = \mathbb{R}[x]$ and $\mathcal{O}_C = \mathbb{R}[x, u]$. We work in the category $\mathcal{C}$ of $\mathbb{R}$-algebras. One then has

$$\mathrm{Jet}^1 \mathcal{O}_X := \mathbb{R}[x, x_0]/(x - x_0)^2,$$

$$\mathcal{O}_{p_1^* C} = \mathbb{R}[x, x_0, u]/(x - x_0)^2,\ \mathcal{O}_{p_2^* C} = \mathbb{R}[x_0, x, u]/(x - x_0)^2$$

A connection on C must be identity on X, i.e., modulo $(x - x_0)$. The trivial Grothendieck connection on $C \to X$ is given by the isomorphism

$$\mathrm{id} : p_1^* C \xrightarrow{\sim} p_2^* C.$$

If $A \in (x - x_0)/(x - x_0)^2$ is an element (that represents a differential form on X), then $\epsilon : u \mapsto u + A$ is also a Grothendieck connection on C, whose inverse isomorphism is $u \mapsto u - A$.

Definition 1.5.11. Let $(\textsc{Legos}, \tau)$ be a site with amalgamated sums, $\mathcal{C}$ be the opposite category $\textsc{Legos}^{op}$, and $X \in \textsc{Sh}(\textsc{Legos}, \tau)$ be a sheaf. A morphism $U \to T$ in $\textsc{Legos}$ that is an object of $\mathrm{Th}^n(\mathcal{C})$ for some n is called an *infinitesimal thickening*. One says that X is

1. *formally smooth* (resp. *formally unramified*, resp. *formally étale*) if for every infinitesimal thickening $U \to T$, the natural map

$$X(T) \to X(U)$$

   is surjective (resp. injective, resp. bijective);

2. *locally finitely presented* if X commutes with directed limits;

3. *smooth* (resp. unramified, resp. étale) if it is locally finitely presented and formally smooth (resp. formally unramified, resp. formally étale).

Definition 1.5.12. Let (Legos, τ) be a site with amalgamated sums and $X \in \text{Sh}(\text{Legos}, \tau)$ be a sheaf. The *de Rham space* of X is the sheaf associated to the presheaf whose points with values in $T \in \text{Legos}$ are defined by

$$X_{DR}(T) := X(T)/\sim_{inf}$$

where $x \sim_{inf} y : T \to X$ if there exists an infinitesimal thickening $U \to T$ such that $x_{|U} = y_{|U}$. We denote $p_{DR} : X \to X_{DR}$ the natural projection map.

Remark 1.5.13. We may also define the de Rham space as the quotient of X by the action of the groupoid $\Pi_1^{inf}(X)$ of pairs of points $(x, y) \in X^2(T)$ that are infinitesimally close, i.e., such that there exists a thickening $S \to T$ with $x_{|U} = y_{|U}$. This is a subgroupoid of the groupoid $\text{Pairs}(X)$ whose arrows are pairs of points in X, with composition

$$(x, y) \circ (y, z) = (x, z).$$

Both groupoids are related to the fundamental groupoid $\Pi_1(X)$ of paths in X up to homotopy. An action of the groupoid $\text{Pairs}(X)$ on a bundle $E \to X$ is equivalent to a trivialization of E. An action of the groupoid $\Pi_1(X)$ on E is a parallel transport for the elements of the fibers of E along (homotopy classes of) paths in X. The groupoid $\Pi_1^{inf}(X)$ is intermediary between these two groupoids: an action of it on a bundle $E \to X$ is equivalent to a parallel transport for elements of the fibers of E along infinitesimal paths in X. More concretely, it is an integrable connection. The relation with the fundamental groupoid is given by the fact that a connection on a path is simply a differential equation, and that solving the corresponding cauchy problem allows to define a parallel transport along paths. If one has a parallel transport along paths, one also morally gets, by passing to the infinitesimal limit, a connection.

Remark 1.5.14. One may also define the de Rham space as the space $\pi_0(\Pi_\infty^{inf}(X))$ of connected components of the ∞-groupoid of infinitesimal simplices in X, whose degree n component is

$$\Pi_\infty^{inf}(X)_n(U) := \{(t, x_i) \in T(U) \times X^n(T),\ t : U \to T \text{ thickening },\ \forall i, j,\ x_{i|U} = x_{j|U}\}.$$

We refer to Section 9.3 for more details. If

$$X \to X^{(1)} \to X \times X$$

is the infinitesimal thickening of the diagonal to order 1, we may see it as a point in $\Pi_\infty^{inf}(X)_2(U)$ for $U = X$, because if we denote $T = X^{(1)}$, we have that the natural map $\gamma : T \to X^2$ may be interpreted as a pair $(x_1, x_2) \in X^2(T)$ of points of X with values in T, that are equal on U, the natural map $U \to T$ being a first order thickening. One interprets γ as an infinitesimal path between the points x_1 and x_2.

Proposition 1.5.15. *If a Grothendieck connection A on $B \to X$ is integrable, the isomorphism A can be extended to an action of $\Pi_\infty^{inf}(X)$ on B, so that it gives a space $B' \to X_{DR}$ whose pullback along $\pi : X \to X_{DR}$ is B.*

Definition 1.5.16. Let $(\textsc{Legos}, \tau)$ be a site with amalgamated sums and $X \in \textsc{Sh}(\textsc{Legos}, \tau)$ be a sheaf.

1. A *crystal of space* on X is a space morphism $Z \to X_{DR}$.
2. A crystal of space $Z \to X_{DR}$ is called a *crystal of varieties* over X if its pullback $Z \times_{X_{DR}} X \longrightarrow X$ is a variety in $\textsc{Var}(\textsc{Legos}, \tau)$.
3. A *crystal of module* $\mathcal{M}$ over X is a module over the space X_{DR}, i.e., an object $X_{DR} \to \mathcal{M}$ of the tangent category $T\left(\textsc{Sh}(\textsc{Legos}, \tau)^{op}\right)$.

The category of crystal of varieties (resp. crystal of modules) over X is denoted $\textsc{Var}_{cris}/X$ (resp. $\textsc{Mod}_{cris}(X)$).

Proposition 1.5.17. *Let $(\textsc{Legos}, \tau)$ be a site with amalgamated sums. Let X be a formally smooth space in $\textsc{Sh}(\textsc{Legos}, \tau)$. There is a natural adjunction*

$$p_{DR}^* : \textsc{Mod}_{cris}(X) \leftrightarrows \textsc{Mod}(X) : p_{DR,*}$$

between modules on X and crystals of modules on X. More generally, there is a natural adjunction

$$p_{DR}^* : \mathrm{Sp}/X_{DR} \leftrightarrows \mathrm{Sp}/X : p_{DR,*}.$$

One denotes $\mathrm{Jet}(C/X) \to X_{DR}$ the crystal of space over X given by $p_{DR,*}(C)$, and also (by extension), the corresponding variety $\mathrm{Jet}(C/X) = p_{DR}^*(p_{DR,*}C) \to X$.

Chapter 2

Parametrized and functional differential geometry

The aim of this chapter is to present flexible tools to do differential geometry on spaces of fields, i.e., on spaces of "functions" between generalized "spaces". These tools will be used all along this book.

Let us describe our general approach in the particular case of differential geometry. There are essentially two viewpoints in the modern approach to differential geometry: the *parametrized* and the *functional* viewpoint. Let $\text{OPEN}_{\mathcal{C}^\infty}$ be the category of open subsets $V \subset \mathbb{R}^n$ for varying n with smooth maps between them. Let $U \subset \mathbb{R}^n$ be such an open subset.

1. The parametrized approach is based on the study of the *functor of points*

$$\begin{array}{rccc} \underline{U}: & \text{OPEN}_{\mathcal{C}^\infty}^{op} & \to & \text{SETS} \\ & V & \mapsto & \text{Hom}(V, U) \end{array}$$

   of U parametrized by other geometric spaces V (e.g., a point, an interval, etc...);

2. The functional approach is based on the study of the *functor of functions*

$$\begin{array}{rccc} \mathcal{O}(U): & \text{OPEN}_{\mathcal{C}^\infty} & \to & \text{SETS} \\ & V & \mapsto & \text{Hom}(U, V) \end{array}$$

   on U with values in other geometric spaces V (e.g., the real or complex numbers).

Remark that these two approaches are equivalent if we vary U and V symmetrically. The main idea is the same in both situations: one generalizes spaces by using contravariant functors

$$X : \text{OPEN}_{\mathcal{C}^\infty}^{op} \to \text{SETS}$$

that commute with some prescribed limit cones (e.g., those given by nerves of coverings; this gives sheaves on Grothendieck sites), and one generalizes functions by using covariant functors

$$\mathcal{A} : \text{OPEN}_{\mathcal{C}^\infty} \to \text{SETS}$$

that commute with some prescribed limit cones (e.g., those given by finite products or transversal fiber products; this gives generalized algebras in the sense of Lawvere and Dubuc). Both these construction are naturally formalized in the setting of doctrines explained in Section 1.1, and in particular, using sketches (see Definition 1.1.25).

The parametrized approach has been systematically developed by Grothendieck and his school (see [Gro60] and [AGV73]) in algebraic geometry. In differential geometry, it is called the diffeological approach, and was developped by Chen [Che77], Souriau [Sou97] and Iglesias Zeimour [IZ99], with applications to classical mechanics (see [BH08] for a survey).

The functional approach is widely used in analysis, with deep roots in functional analytic methods. It has also been developed on more categorical and logical grounds after Lawvere's thesis [Law04]. A particular combination of the functional and parametrized approach in classical differential geometry is called synthetic differential geometry. It has been developed following Lawvere's ideas [Law79] by Moerdijk-Reyes in [MR91] and Kock [Koc06]. It has also been used in analytic geometry by Dubuc, Taubin and Zilber (see [DT83], [DZ94] and [Zil90]), in a way that is closer to our presentation.

Our aim is to have flexible tools to do differential geometry on spaces of fields (i.e., spaces of "functions" between "spaces", in a generalized sense). The parametrized approach gives a well behaved notion of differential form, and the functional approach gives a well behaved notion of vector field. We will use a combination of the parametrized and functional approach that is optimal for our purpose of studying with geometrical tools the mathematics of classical and quantum field theories.

We give here an axiomatic presentation that works in all the examples of geometric situations that will be used in this book.

2.1 Parametrized geometry

We first recall some results about sites from Section 1.4.

Let $(\textsc{Legos}, \tau)$ be a site (category with a Grothendieck topology) whose objects, called legos, will be the basic building blocks for our generalized spaces. Let $\textsc{Legos}^\vee := \underline{\mathrm{Hom}}(\textsc{Legos}^{op}, \textsc{Sets})$ be the Yoneda dual category. One has a fully faithful embedding

$$\begin{array}{ccc} \textsc{Legos} & \to & \textsc{Legos}^\vee \\ X & \mapsto & \underline{X} \end{array}$$

where

$$\underline{X} := \mathrm{Hom}(-, X) : \textsc{Legos}^{op} \to \textsc{Sets}$$

is the contravariant functor naturally associated to X. The Yoneda dual category has all limits (e.g. fiber products) and colimits (e.g. quotients) and also internal homomorphisms, so that it could be a nice category to do geometry. However, the natural embedding $\textsc{Legos} \to \textsc{Legos}^\vee$ does not preserve finite colimits given by pastings of open subspaces. For example, if $\textsc{Legos}$ is the category $\textsc{Open}_X$ of open subsets of a given topological space X, and one looks at a union

$$U = U_1 \cup U_2 = U_1 \coprod_{U_1 \cap U_2} U_2 \subset X$$

of open subsets of X, the natural morphism in $\text{LEGOS}^\vee$ given by

$$\underline{U_1} \coprod_{\underline{U_1} \times_{\underline{X}} \underline{U_2}} \underline{U_2} \to \underline{U} := \underline{U_1 \coprod_{U_1 \cap U_2} U_2}$$

is usually *not* an isomorphism. The notion of sheaf for a Grothendieck topology was introduced exactly to solve this problem: the above natural morphism, with colimits taken in the category $\text{SH}(\text{LEGOS}, \tau)$, is indeed an isomorphism.

We will thus work with generalized spaces given by sheaves on Grothendieck sites. We now define varieties as generalized spaces that can be obtained by pasting legos.

Definition 2.1.1. Let (LEGOS, τ) be a site and let $f : X \to Y$ be a morphism of sheaves on (LEGOS, τ).

- If X and Y are representable in LEGOS, one says that f is *open* if it is injective on points and it is the image of a morphism

  $$\coprod_i f_i : \coprod U_i \to Y$$

  for $\{U_i \xrightarrow{f_i} Y\}$ a covering family for τ.

- In general, one says that f is *open* if it is injective on points and it is universally open, meaning that, for every $x : \underline{U} \to Y$, with $\underline{U}$ representable, the fiber

  $$f \times_Y x : X \times_Y \underline{U} \to \underline{U}$$

  is isomorphic to the embedding $\underline{W} \subset \underline{U}$ of an open subset of the representable space $\underline{U}$.

- A *variety* is a space X that has an open covering by representable spaces, meaning that there exists a family $f_i : \underline{U_i} \to X$ of open morphisms with $U_i \in \text{LEGOS}$ such that the map

  $$\coprod_i f_i : \coprod_i \underline{U_i} \to X$$

  is an epimorphism of sheaves.

We denote $\text{VAR}(\text{LEGOS}, \tau) \subset \text{SH}(\text{LEGOS}, \tau)$ the category of varieties.

Definition 2.1.2. For (LEGOS, τ) the following sites, the corresponding varieties are called:

- $\mathcal{C}^k$*-manifolds* for $k \leq \infty$, if LEGOS is the category $\text{OPEN}_{\mathcal{C}^k}$ of open subsets of $\mathbb{R}^n$ for varying n with $\mathcal{C}^k$-maps between them, and τ is the usual topology;

- $\mathcal{C}^k$*-manifolds with corners* for $k \leq \infty$ (see Example 2.2.8 for a refined notion, that uses maps respecting the corner stratification), if LEGOS is the category $\text{OPEN}_{\mathcal{C}^k_+}$ of open subsets of $\mathbb{R}^n_+$ for varying n with smooth maps between them (recall that for $U \subset \mathbb{R}^n_+$, a map $f : U \to \mathbb{R}$ is $\mathcal{C}^k$ if there exists an open $\tilde{U} \subset \mathbb{R}^n$ with $U = \tilde{U} \cap \mathbb{R}^n_+$ and a $\mathcal{C}^k$-function $\tilde{f} : \tilde{U} \to \mathbb{R}$ such that $\tilde{f}_{|U} = f$), and τ is the usual topology;

- *real analytic manifolds* if LEGOS is the category $\text{OPEN}_{an,\mathbb{R}}$ of open subsets of $\mathbb{R}^n$ with real analytic maps between them and τ_{an} is the usual analytic topology generated by subsets of the form

$$U_{f,g} := \{x \in U;\ |f(x)| < |g(x)|\}$$

  for $f, g \in \mathcal{O}(U)$;

- *complex analytic manifolds* if LEGOS is the category $\text{STEIN}_{\mathbb{C}}$ of Stein open subsets U of $\mathbb{C}^n$ with analytic maps between them and τ_{an} the usual analytic topology defined as in the real analytic case;

- *schemes* if LEGOS is the category $\text{AFF}_{\mathbb{Z}} := \text{CALG}_{\mathbb{Z}}^{op}$ of affine schemes (opposite the category of commutative unital rings), equipped with the topology generated by the localization maps $\text{Spec}(A[f^{-1}]) \to \text{Spec}(A)$;

- *algebraic varieties* (schemes of finite type) if LEGOS is the category $\text{AFF}_{\mathbb{Z}}^{ft}$ dual to the category $\text{CALG}_{\mathbb{Z}}^{ft}$ of $\mathbb{Z}$-algebras of finite type (quotients $A = \mathbb{Z}[X_1, \dots, X_n]/I$ of some polynomial algebra with finitely many variables).

To be able to define differential invariants using a categorical setting for infinitesimal thickenings as in Section 1.5, one needs at least to have the following condition.

Definition 2.1.3. A Grothendieck site (LEGOS, τ) is called *differentially convenient* if LEGOS has finite coproducts and an initial object.

In the above examples, the only differentially convenient sites are the categories of schemes. Indeed, the pasting of two open subsets U and V of $\mathbb{R}^n$ (or $\mathbb{C}^n$) for $n > 0$ along a point can't be described by an open subset of $\mathbb{R}^m$ for any $m \geq 0$. An example of an infinitesimal thickening in the setting of schemes is given by the embedding

$$\text{Spec}(\mathbb{R}) \to \text{Spec}(\mathbb{R}[x]/(x^2))$$

of a point in its first infinitesimal neighborhood. Its importance for the algebraic formulation of differential calculus was emphasized by Weil. We will see later in this section how to improve on the above examples of sites to get infinitesimals in all the situations of definition 2.1.2.

The most standard example of a generalized geometrical setting, with spaces defined as sheaves on a site, is given by the theory of diffeology, which was developed by J. W. Smith [Smi66] and K. T. Chen [Che77], and used in the physical setting by J.-M. Souriau to explain the geometric methods used by physicists to study variational problems (see [Sou97] and [IZ99] for references and historical background, and [BH08] for an overview).

Definition 2.1.4. Let $\text{OPEN}_{\mathcal{C}^\infty}$ be the category of open subsets of $\mathbb{R}^n$ for varying n with smooth maps between them and τ be the usual topology on open subsets. The category of *diffeologies* (also called *smooth spaces*) is the category $\text{SH}(\text{OPEN}_{\mathcal{C}^\infty}, \tau)$.

Definition 2.1.5. An *ordinary smooth manifold* is a topological space X and a family of open embeddings $f_i : U_i \to X$, with $U_i \subset \mathbb{R}^n$ open subsets, such that for every pair (U_i, U_j), the continuous map

$$f_{i|U_i \cap U_j}^{-1} \circ f_{j|U_i \cap U_j} : U_i \cap f_i^{-1}(f_j(U_j)) \to U_i \cap f_j^{-1}(f_i(U_j))$$

is a smooth map (i.e., a morphism in $\mathrm{OPEN}_{\mathcal{C}^\infty}$). The family $(f_i : U_i \to X)_{i \in I}$ is called an *atlas*. An atlas maximal for atlas inclusion is called a *maximal atlas*. A morphism of manifolds is a continuous map that induces a morphism of maximal atlases (in the obvious sense). We denote $\mathrm{VAR}_{\mathcal{C}^\infty}^{ordinary}$ the category of ordinary smooth manifolds.

We now check, on a simple example, that our notion of variety corresponds to the usual one.

Theorem 2.1.6. *The map*

$$\begin{array}{ccc} \mathrm{VAR}_{\mathcal{C}^\infty}^{ordinary} & \to & \mathrm{SH}(\mathrm{OPEN}_{\mathcal{C}^\infty}, \tau) \\ M & \mapsto & [U \mapsto \mathrm{Hom}_{\mathrm{VAR}_{\mathcal{C}^\infty}^{ordinary}}(U, M)] \end{array}$$

is a fully faithful embedding of smooth varieties in diffeologies and it induces an equivalence of categories

$$\mathrm{VAR}_{\mathcal{C}^\infty}^{ordinary} \xrightarrow{\sim} \mathrm{VAR}(\mathrm{OPEN}_{\mathcal{C}^\infty}, \tau).$$

Proof. There is a natural Grothendieck topology τ on the category $\mathrm{VAR}_{\mathcal{C}^\infty}^{ordinary}$ with covering families given by open morphisms $f_i : U_i \to M$ that cover M with $U_i \in \mathrm{LEGOS}$. We can thus use the natural fully faithful embedding

$$\mathrm{VAR}_{\mathcal{C}^\infty}^{ordinary} \to \mathrm{SH}(\mathrm{VAR}_{\mathcal{C}^\infty}^{ordinary}, \tau).$$

If $U_i \to M$ is a covering family, one defines its nerve system $N(\{U_i \to X\})$ to be the inductive system of all finite fiber products $N_j = U_{i_1} \times_M \cdots \times_M U_{i_n}$. Remark that all its objects are legos. The inductive limit $\varinjlim N_j$ is equal to M, by definition of our notion of variety. There is a natural functor

$$\mathrm{SH}(\mathrm{VAR}_{\mathcal{C}^\infty}^{ordinary}, \tau) \to \mathrm{SH}(\mathrm{OPEN}_{\mathcal{C}^\infty}, \tau)$$

sending X to its evaluation on legos. If $Y \in \mathrm{SH}(\mathrm{VAR}_{\mathcal{C}^\infty}^{ordinary}, \tau)$ is a sheaf, for every ordinary manifold $X = \varinjlim_j N_j$ given by the limit of the nerve of a covering, one has

$$Y(X) = \mathrm{Hom}_{\mathrm{SH}_{\mathrm{VAR}}}(X, Y) = \mathrm{Hom}_{\mathrm{SH}_{\mathrm{VAR}}}(\varinjlim_j N_j, Y) = \varprojlim_j \mathrm{Hom}_{\mathrm{VAR}}(N_j, Y) = \varprojlim_j Y(N_j),$$

so that $Y(X)$ is determined by the values $Y(N_j)$ of Y on some legos. Applying the same reasoning to morphisms, one concludes that the above functor is fully faithful. Remark that in both categories, varieties are defined as inductive limits of nerves for the same topology with same covering opens in $\mathrm{OPEN}_{\mathcal{C}^\infty}$. One thus gets an equivalence

$$\mathrm{VAR}(\mathrm{VAR}_{\mathcal{C}^\infty}^{ordinary}, \tau) \xrightarrow{\sim} \mathrm{VAR}(\mathrm{OPEN}_{\mathcal{C}^\infty}, \tau).$$

To prove the theorem, it remains to show that the natural functor

$$\mathrm{VAR}_{\mathcal{C}^\infty}^{ordinary} \to \mathrm{VAR}(\mathrm{VAR}_{\mathcal{C}^\infty}^{ordinary}, \tau)$$

is essentially surjective, since it is already fully faithful. Let X be a variety in $\mathrm{VAR}(\mathrm{VAR}_{\mathcal{C}^\infty}^{ordinary}, \tau)$. By hypothesis, we have a family $f_i : \underline{U_i} \hookrightarrow X$ of open embeddings of legos $U_i \in \mathrm{LEGOS}$ such that

$$\coprod_{i \in I} \underline{U_i} \to X$$

is a sheaf epimorphism. This means that $X = \varinjlim_i \underline{U_i}$ in the category of sheaves. By hypothesis, all maps $\underline{U_i} \to X$ are open, so that the maps of the nerve inductive system $N_j = N_j(\{U_i \to X\})$ of legos are open. Let $\tilde{X}$ be the topological space given by the colimit $\tilde{X} = \varinjlim N_j$ in the category TOP of topological spaces. One has open embeddings $U_i \subset X$ that are smoothly compatible, so that $\tilde{X}$ is a variety. In fact, $\tilde{X}$ is the inductive limit $\varinjlim N_j$ in the category $\mathrm{VAR}_{\mathcal{C}^\infty}^{ordinary}$. The main properties of sheaves for a Grothendieck topology, given in theorem 1.4.4, is that their colimits along nerves of coverings correspond to actual colimits in the original category. This shows that $\underline{\tilde{X}} \cong X$ and concludes the proof. □

We now recall, for comparison purposes, the notion of locally ringed space. Remark that it allows the introduction of infinitesimal thickenings, that can be used for a categorical formulation of differential calculus. We refer to [Gro60] for more details. The main drawbacks of this category are the following:

- it is algebraic in nature (i.e., germs are local rings), so that it is not always well adapted to the functorial statement of non-algebraic equations of analysis.
- it does not have a notion of internal homomorphisms, i.e., the locally ringed space of morphisms between locally ringed spaces does not exist in general.

We will see in Section 2.2 another more natural path to the functorial definition of differential invariants.

Definition 2.1.7. A *locally ringed space* consists of a pairs $(X, \mathcal{O}_X)$ composed of a topological space X and a sheaf of functions $\mathcal{O}_X$, with germs given by local rings. A *morphism of locally ringed spaces* is given by a pair

$$(f, f^\natural) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

with $f : X \to Y$ a continuous map and $f^\natural : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ a morphism of sheaves of rings (meaning for each open $U \subset Y$, morphisms $f^\natural : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$, compatible with restrictions) such that the corresponding morphism of germs is local (i.e. sends the maximal ideal of one ring into the maximal ideal of the other) at every point of X.

The category of locally ringed spaces contains all kinds of varieties, and has a special auto-duality property: their varieties are not more complicated than them.

Proposition 2.1.8. *Let* LRSPACES *be the category of locally ringed spaces. Equip* LRSPACES *with its standard topology τ given by open immersions compatible with the sheaves of functions.*

- *If* VAR $\subset$ LRSPACES *is a full subcategory with fiber products (but not necessarily stable by fiber products in* LRSPACES*) with a well defined restriction of the standard topology τ, the natural functor*

$$\text{VAR} \to \text{VAR}(\text{VAR}, \tau)$$

is fully faithful.

- *All categories of varieties described in definition 2.1.2 embed fully faithfully in the category of locally ringed spaces.*

Proof. The first statement follows directly from the local nature of locally ringed spaces. The second one can be checked locally, in which case it is clear because one has enough functions (the fact that germs are local is not easy to prove, though). □

We now define algebraic varieties relative to a symmetric monoidal category.

Definition 2.1.9. Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category and denote

$$\text{LEGOS}_{\mathcal{C}} := \text{ALG}_{\mathcal{C}}^{op}$$

the category dual to commutative $\mathcal{C}$-algebras. Let A be a commutative algebra in $(\mathcal{C}, \otimes)$. The functor $\underline{\text{Spec}}(A) \in \text{LEGOS}^{\vee}$ given by

$$\begin{array}{rccc} \underline{\text{Spec}}(A): & \text{LEGOS}_{\mathcal{C}}^{op} = \text{ALG}_{\mathcal{C}} & \to & \text{SETS} \\ & B & \mapsto & \text{Hom}(A, B) \end{array}$$

is called the *spectrum* of A.

It is possible, following Toen and Vaquié in [TV09b], to define a Zariski topology on LEGOS by generalizing a known characterization of Zariski opens in ordinary schemes (monomorphisms of spaces that are flat and finitely presented).

Definition 2.1.10. An algebra morphism $f : A \to B$ (or the corresponding morphism $\underline{\text{Spec}}(B) \to \underline{\text{Spec}}(A)$) is called a *Zariski open* if it is a flat and finitely presented monomorphism:

1. (monomorphism) for every algebra C, $\underline{\text{Spec}}(B)(C) \subset \underline{\text{Spec}}(A)(C)$,
2. (flat) the base change functor $- \otimes_A B : \text{MOD}(A) \to \text{MOD}(B)$ is left exact (i.e., commutes with finite limits), and
3. (finitely presented) if $\underline{\text{Spec}}_A(B)$ denotes $\underline{\text{Spec}}(B)$ restricted to A-algebras, $\underline{\text{Spec}}_A(B)$ commutes with filtered inductive limits.

A family of morphisms $\{f_i : \underline{\mathrm{Spec}}(A_i) \to \underline{\mathrm{Spec}}(A)\}_{i\in I}$ is called a *Zariski covering* if all its elements are Zariski open and moreover

4. there exists a finite subset $J \subset I$ such that the functor

$$\prod_{i\in J} - \otimes_A A_j : \mathrm{MOD}(A) \to \prod_{j\in J} \mathrm{MOD}(A_j)$$

   is conservative (i.e., preserves isomorphisms).

The *Zariski topology* on $\mathrm{LEGOS}_{\mathcal{C}}$ is the Grothendieck topology generated by Zariski coverings. The corresponding varieties and spaces are called *schemes* and *spaces relative to* $(\mathcal{C}, \otimes)$ and their categories are denoted $\mathrm{SCH}(\mathcal{C}, \otimes)$ and $\mathrm{SH}(\mathcal{C}, \otimes)$.

Example 2.1.11.

1. Usual schemes are schemes relative to the symmetric monoidal category $(\mathrm{MOD}(\mathbb{Z}), \otimes)$ of abelian groups. This result was proved by Grothendieck in EGA [Gro67].

2. Super and graded schemes are schemes relative to the symmetric monoidal categories $(\mathrm{MOD}_s(\mathbb{Z}), \otimes)$ and $(\mathrm{MOD}_g(\mathbb{Z}), \otimes)$.

3. If $(X, \mathcal{O})$ is a locally ringed space, schemes relative to the symmetric monoidal category $(\mathrm{QCoh}(X), \otimes)$ of quasi-coherent sheaves on X (cokernels of morphisms of arbitrary free modules) are called relative schemes on X.

We now explain how to use contravariant constructions on a site (LEGOS, τ) to define contravariant constructions (e.g. differential forms) on spaces in $\mathrm{SH}(\mathrm{LEGOS}, \tau)$. These contravariant constructions apply to all the geometric sites (LEGOS, τ) we will use in this book.

Definition 2.1.12. Suppose given a category $\mathcal{C}$ and a site (LEGOS, τ). Let

$$\Omega : \mathrm{LEGOS}^{op} \to \mathcal{C}$$

be a sheaf for the topology τ with values in $\mathcal{C}$, that we will call a *differential geometric construction* on LEGOS. Let $X \in \mathrm{SH}(\mathrm{LEGOS}, \tau)$ be a space, and let LEGOS/X denote the category of morphisms $x : \underline{U_x} \to X$ where $U_x \in \mathrm{LEGOS}$. The *differential geometric construction* $\Omega(X)$ *induced on* X is defined as the inverse limit in $\mathcal{C}$ (supposed to exist)

$$\Omega(X) := \varprojlim_{x\in \mathrm{LEGOS}/X} \Omega(U_x).$$

More generally, if $i : U \to X$ is an open morphism of spaces, one can define $\Omega(U)$. This gives (if it exists) a sheaf on X for the topology induced by τ

$$\Omega : \mathrm{SH}(\mathrm{LEGOS}, \tau)/X \to \mathcal{C}.$$

Example 2.1.13. Let $(\mathrm{Legos},\tau) = (\mathrm{Open}_{\mathcal{C}^\infty},\tau)$ be the usual topology on smooth legos and let

$$\Omega = \Omega^1 : \mathrm{Open}_{\mathcal{C}^\infty}^{op} \to \mathrm{Vect}_{\mathbb{R}}$$

be the sheaf of differential 1-forms. If $X : \mathrm{Open}_{\mathcal{C}^\infty} \to \mathrm{Sets}$ is a diffeology, a differential 1-form on X is the datum, for each $x : \underline{U} \to X$ of a differential form $[x^*\omega]$ on U, i.e., a section of $T^*U \cong U \times (\mathbb{R}^n)^* \to U$, such that if $f : x \to y$ is a morphism, i.e., a commutative diagram

$$\begin{array}{ccc} \underline{U_x} & \xrightarrow{x} & X \\ {\scriptstyle f}\downarrow & \nearrow_{y} & \\ \underline{U_y} & & \end{array},$$

one has

$$f^*[y^*\omega] = x^*\omega.$$

One can define similarly differential n-forms and the de Rham complex

$$(\Omega^*_X, d)$$

of the diffeology X. This example of course generalizes to any kind of spaces whose local models (i.e., Legos) have a well behaved (i.e., local and contravariant) notion of differential form. Remark that the de Rham complex of a space of maps $\underline{\mathrm{Hom}}(X,Y)$ between two smooth varieties has a nice description (up to quasi-isomorphism) in terms of iterated integrals, that is due to Chen [Che77] in the very particular case of the circle $X = S^1$, and much more recently to Ginot-Tradler-Zeinalian [GTZ09] in general. This description roughly says that there is a natural iterated integral quasi-isomorphism of differential graded algebras (see Section 8.8 for a definition of this notion)

$$\mathrm{it}_X : \mathrm{CH}_X(\Omega^*(M)) \longrightarrow \Omega^*(\underline{\mathrm{Hom}}(X,M))$$

that gives a concrete combinatorial description of the de Rham cohomology of the mapping space in terms of the higher Hochschild homology (whose definition is out of the scope of these notes) of differential forms on M.

Remark 2.1.14. One can go one step further by defining the notion of a locally trivial bundle on a space. Let Grpd be the 2-category of groupoids (i.e., categories with only invertible morphisms). The idea is to use the sheaf

$$\Omega = \mathrm{Bun} : \mathrm{Legos}^{op} \to \mathcal{C} = \mathrm{Grpd}$$

with value in the 2-category of groupoids. A bundle B over a diffeology X is the following datum:

- for each point $x : U \to X$, a bundle $[x^*B] \to X$,
- for each morphism $f : x \to y$ of points, an isomorphism

  $$i_f : f^*[y^*B] \xrightarrow{\sim} [x^*B]$$

  of bundles on U,

- for each commutative triangle

$$T = \begin{array}{ccc} x & \xrightarrow{g\circ f} & z \\ {\scriptstyle f}\downarrow & \nearrow_{g} & \\ y & & \end{array}$$

of points, an isomorphism

$$i_T : f^* i_g \xrightarrow{\sim} i_{g\circ f}$$

- and so on...

Of course, one can not continue to write down all the commutative diagrams in play by hand and there is a nice mathematical theory to treat properly this type of geometrical notion, called homotopical geometry. We will come back to this example in Chapter 9.

2.2 Functional geometry

Recall that if X and Y are sets, one defines the X-indexed product of Y by

$$\prod_X Y := \mathrm{Hom}_{\textsc{Sets}}(X, Y).$$

In particular, $\prod_\emptyset X$ is always equal to the one point set $\{.\} = \mathrm{Hom}(\emptyset, X)$. More generally, an empty product in a given category is a final object.

Up to now, we have considered spaces defined as sheaves X on some Grothendieck site $(\textsc{Legos}, \tau)$, meaning contravariant functors

$$X : \textsc{Legos}^{op} \to \textsc{Sets}$$

that respect nerves of coverings in τ. In particular, a given lego U gives a space

$$\underline{U} := \mathrm{Hom}(-, U)$$

also called the functor of points of U. The main drawbacks of this approach are the following:

1. Classical categories of legos (e.g., the category $\textsc{Open}_{\mathcal{C}^\infty}$ of smooth open subsets of $\mathbb{R}^n$) are usually not rich enough to contain infinitesimal thickenings, that play a central role in the categorical approach to differential calculus, presented in Section 1.5. This has to do with the fact that the types of function algebras that we have on legos are not general enough to allow the definition of infinitesimal thickenings, so that the corresponding differential invariants (e.g., the tangent category) are trivial. See Example 1.5.4 for a more precise explanation.

2. Given a smooth function $f : U \to \mathbb{R}$ on an open subset of $\mathbb{R}^n$, one would like the zero set space $Z(f) \subset \underline{U}$, defined by the formula

$$\begin{array}{rccc} Z(f): & \textsc{Open}^{op}_{\mathcal{C}^\infty} & \to & \textsc{Sets} \\ & V & \mapsto & Z(f)(U) := \{x : V \to U,\ f \circ x = 0\}, \end{array}$$

to be representable by a lego whose algebra of functions ought to be something like $\mathcal{C}^\infty(U)/(f)$.

The most natural way to solve the above problems by enriching a category of legos was discovered by Lawvere in [Law04]. One can simply embed it in the opposite category $\overline{\textsc{Legos}} = \textsc{Alg}^{op}_{\textsc{Legos}}$ of a category $\textsc{Alg}_{\textsc{Legos}}$ of covariant functors

$$A : \textsc{Legos} \to \textsc{Sets}$$

that respect some prescribed categorical structures (e.g., limits). Of course, one would like to have a natural Grothendieck topology $\bar{\tau}$ induced by τ and a fully faithful embedding of Grothendieck sites

$$(\textsc{Legos}, \tau) \to (\overline{\textsc{Legos}}, \bar{\tau}),$$

to get back the usual notion of varieties. We will use sketches (see Definition 1.1.25) to solve this problem of "adding infinitesimals and solution spaces to legos".

The following example is due to Lawvere's [Law79], and was used by Dubuc [Dub81] and Moerdijk-Reyes [MR91] to describe generalized smooth spaces.

Example 2.2.1. Here is an example of Lawvere's algebraic theory (see Definition 1.1.21) related to (generalized) differential geometry: For $k \leq \infty$, let $\textsc{Aff}_{\mathcal{C}^k}$ be the category with objects the affine spaces $\mathbb{R}^n$ for varying $n \geq 0$ and with morphisms the $\mathcal{C}^k$-maps between them. It is a full finite product sketch, i.e., a category with finite products. A model

$$A : \textsc{Aff}_{\mathcal{C}^k} \to \textsc{Sets}$$

for this theory (i.e., a product preserving functor) is called a (finitary) $\mathcal{C}^k$-*algebra*. We denote $\textsc{Alg}^{fin}_{\mathcal{C}^k}$ the category of $\mathcal{C}^k$-algebras. One can think of a $\mathcal{C}^k$-algebra A as a set $A(\mathbb{R})$, equipped with operations

$$[f] : A(\mathbb{R})^n \to A(\mathbb{R})$$

associated with smooth functions $f : \mathbb{R}^n \to \mathbb{R}$. In particular, $A(\mathbb{R})$ is equipped with an $\mathbb{R}$-algebra structure induced by the one on $\mathbb{R}$, since the operations

$$\begin{array}{rccc} + : & \mathbb{R} \times \mathbb{R} & \to & \mathbb{R} \\ \times : & \mathbb{R} \times \mathbb{R} & \to & \mathbb{R} \\ 0 : & \{.\} & \to & \mathbb{R} \\ 1 : & \{.\} & \to & \mathbb{R} \end{array}$$

are all $\mathcal{C}^k$-maps. If $\textsc{Open}_{\mathcal{C}^k}$ denotes the category of open subsets in $\mathbb{R}^n$ for $n \geq 0$ with $\mathcal{C}^k$-maps between them, there is a fully faithful embedding

$$\begin{array}{rccc} \mathcal{C}^k : & \textsc{Open}^{op}_{\mathcal{C}^k} & \to & \textsc{Alg}^{fin}_{\mathcal{C}^k} \\ & U & \mapsto & [\mathbb{R}^n \mapsto \mathrm{Hom}_{\mathcal{C}^k}(U, \mathbb{R}^n)] \end{array}$$

that sends transversal fiber products (i.e., the products that are well defined in this category) to pushouts. This category is the one usually used by people working in synthetic differential geometry (see e.g., Moerdijk-Reyes [MR91]), because it contains algebras of the form $\mathbb{R}[\epsilon]/(\epsilon^2)$, that play the role of infinitesimal thickenings. If we denote $\textsc{Open}_{\mathcal{C}^k_+}$ the category of open subset of $\mathbb{R}^n_+$ with smooth morphisms between them, there is also a fully faithful embedding

$$\mathcal{C}^k_+ : \textsc{Open}^{op}_{\mathcal{C}^k_+} \to \textsc{Alg}^{fin}_{\mathcal{C}^k}.$$

One can actually show that if $U \subset \mathbb{R}^n_+$ is an open subset, one has

$$\mathcal{C}^k_+(U) = \lim_{\tilde{U} \supset U} \mathcal{C}^k(\tilde{U})$$

where the limit is taken in $\textsc{Alg}^{fin}_{\mathcal{C}^k}$ and indexed by open subsets $\tilde{U} \subset \mathbb{R}^n$ such that $U \subset \tilde{U} \cap \mathbb{R}^n_+$ (see [MR91], Chapter 1). One can also use finer notions of morphisms between open subset of $\mathbb{R}^n_+$ with corners. This give a corresponding theory of $\mathcal{C}^k$-ring with corners (see Example 2.2.8 for a refined discussion).

A drawback of the theory of smooth rings described in the previous example is that it has no a priori topology and one has to fix a topology on the category of smooth rings (by using the Zariski topology on finitely generated rings, for example). We now give another way of defining a notion of generalized algebra that allows the direct definition of a Grothendieck topology on their opposite category. We are inspired here by Dubuc's work on synthetic analytic geometry, with Taubin and Zilber (see [DT83], [DZ94], [Zil90]).

Definition 2.2.2. A *differential geometric context* is a tuple

$$\mathbf{C} = (\textsc{Legos}, \tau, \mathfrak{L}, \mathfrak{C}, \mathcal{L}, \mathcal{C})$$

composed of a Grothendieck site $(\textsc{Legos}, \tau)$, two classes $\mathfrak{L}$ and $\mathfrak{C}$ of small categories, and an $(\mathfrak{L}, \mathfrak{C})$-sketch $(\textsc{Legos}, \mathcal{L}, \mathcal{C})$, that are compatible in the following sense: cones in $\mathcal{L}$ and cocones in $\mathcal{C}$ are stable by pullbacks along covering maps. A model

$$A : (\textsc{Legos}, \mathcal{L}, \mathcal{C}) \to \textsc{Sets}$$

is called a *generalized algebra* (in the given differential geometric context). If $U \in \textsc{Legos}$, we denote $\mathcal{O}(U)$ the corresponding generalized algebra of functions given by

$$\mathcal{O}(U)(V) := \mathrm{Hom}(U, V).$$

A generalized algebra is called *local* if it sends covering nerves to colimits. We denote $\textsc{Alg}_{\mathbf{C}}$ the category of generalized algebras.

Theorem 2.2.3. *Let* $\mathbf{C} = (\textsc{Legos}, \tau, \mathfrak{L}, \mathfrak{C}, \mathcal{L}, \mathcal{C})$ *be a differential geometric context. We denote* $\mathrm{Spec}(A)$ *an object of* $\overline{\textsc{Legos}} := \textsc{Alg}^{op}_{\mathbf{C}}$*. The families*

$$\{\mathrm{Spec}(A_j) \to \mathrm{Spec}(A)\}_{j \in J}$$

whose pullback along any map $\mathrm{Spec}(\mathcal{O}(U)) \to \mathrm{Spec}(A)$, *for* $U \in \textsc{Legos}$, *is induced from covering family* $\{U_j \to U\}_{j\in J}$ *in* τ, *form a Grothendieck topology* $\bar{\tau}$ *on* $\overline{\textsc{Legos}}$. *The natural functor*

$$(\textsc{Legos}, \tau) \to (\overline{\textsc{Legos}}, \bar{\tau})$$

is a fully faithful embedding of sites compatible with the topology and induces an embedding

$$\textsc{Var}(\textsc{Legos}, \tau) \to \textsc{Var}(\overline{\textsc{Legos}}, \bar{\tau})$$

of the corresponding categories of varieties.

We now illustrate the above construction by defining a natural category of generalized $\mathcal{C}^k$, complex and real analytic rings, equipped with a Grothendieck topology.

Example 2.2.4. Let $\mathfrak{L}$ be the classes of finite (possibly empty) categories and $\mathfrak{C} = \emptyset$. Recall that an empty limit is equivalent to a final object. The doctrine of $(\mathfrak{L}, \mathfrak{C})$-sketches is called the doctrine of finite limit sketches. Let Legos be one of the following categories:

1. open subsets $U \subset \mathbb{R}^n$ with $\mathcal{C}^k$-maps between them, for varying n, and some fixed $0 < k \leq \infty$,

2. open subsets $U \subset \mathbb{R}^n$ with real analytic maps between them,

3. open subsets $U \subset \mathbb{C}^n$ (one may also restrict to Stein subsets) with complex analytic maps between them.

Equip Legos with its usual topology, the class $\mathcal{L}$ of transversal (possibly empty) pullbacks. The corresponding algebras are respectively called $\mathcal{C}^k$-rings, real analytic rings and complex analytic rings. Let $(\overline{\textsc{Legos}}, \bar{\tau})$ be the corresponding category of generalized affine spaces. The natural functor

$$\textsc{Legos}^{op} \to \textsc{Sh}(\overline{\textsc{Legos}}, \bar{\tau})$$

is a fully faithful embedding that extends to the corresponding category of varieties. The categories of sheaves thus constructed are called the category of $\mathcal{C}^k$-spaces and generalized real and complex analytic spaces respectively. The main advantages of the category of generalized spaces on ordinary varieties are the following:

1. generalized algebras contain infinitesimal objects, like $\mathcal{C}^\infty(\mathbb{R})/(\epsilon^2)$ for example in the smooth case. This allows us to use the methods of categorical infinitesimal calculus (see Section 1.5) to define their differential invariants.

2. one can define the solution algebra $\mathcal{O}(U)/(f)$ for a given equation $f = 0$ with $f \in \mathcal{O}(U)$.

3. generalized spaces contain spaces of functions $\underline{\mathrm{Hom}}(X, Y)$ between two given spaces. This is important since spaces of functions, called spaces of fields by physicists, are omnipresent in physics.

Remark 2.2.5. There are various refinements of the notion of analytic ring and building blocks for synthetic analytic geometry. Their main interest is that they allow the treatment of local properties of analytic spaces, and not only of infinitesimal ones. There are based on Dubuc-Taubin-Zilber's notion of local analytic ring [Zil90] and locally analytically ringed space [DT83]. These analytic schemes based on local analytic rings have the advantage on usual analytic rings of making germs of analytic functions at a given point "finitely generated" in the categorical sense (see [DZ94]). One may also use Stein opens as building blocks for analytic rings.

A nice result is that finitary smooth algebras are equivalent to smooth algebras. This result is not true anymore in the holomorphic or algebraic case, and is deeply related to the fact that smooth manifolds have partition of unity, which implies that the Zariski and usual topology are identified in this smooth setting.

Proposition 2.2.6. *Let* $\textsc{Alg}_{\mathcal{C}^\infty}$ *be the category of smooth algebras, given by functors*

$$A : (\textsc{Open}_{\mathcal{C}^\infty}, -\overset{t}{\underset{-}{\times}}-) \to (\textsc{Sets}, -\underset{-}{\times}-)$$

that commute with transversal and empty fiber products. Let $\textsc{Alg}^{fin}_{\mathcal{C}^\infty}$ *be the category of finitary smooth algebras, given by functors*

$$A : (\textsc{Aff}_{\mathcal{C}^\infty}, \times) \to (\textsc{Sets}, \times)$$

on the category of smooth affine spaces $\mathbb{R}^n$ *commuting with (possibly empty) products. The natural functor*

$$\textsc{Alg}_{\mathcal{C}^\infty} \to \textsc{Alg}^{fin}_{\mathcal{C}^\infty}$$

is an equivalence.

Proof. This result follows from the two following facts:

1. if $A \in \textsc{Alg}^{fin}_{\mathcal{C}^\infty}$ is a finitary smooth algebra and $U \overset{t}{\underset{V}{\times}} W$ is a transversal fiber product, one has
$$A(U \overset{t}{\underset{V}{\times}} W) \cong A(U) \times_{A(V)} A(W),$$
where $A(U) := \mathrm{Hom}_{\textsc{Alg}^{fin}_{\mathcal{C}^\infty}}(\mathcal{O}(U), A)$. This allows us to construct a left inverse to the above natural functor.

2. the finitary smooth algebra of functions on an open subset $U \subset \mathbb{R}^n$ identifies naturally with the localization $\mathcal{C}^\infty(\mathbb{R}^n)[f^{-1}]$ of the algebra of functions on the affine space along a given smooth function $f : \mathbb{R}^n \to \mathbb{R}$.

□

We now describe various types of semianalytic spaces, that play an important role in algebraic analysis. We are inspired here by non-archimedean analytic geometry (see [Ber90] and [Hub96]), and by the theory of real semianalytic and subanalytic sets (see [Gab96], [Łoj93], [BM88]).

Example 2.2.7. Let LEGOS be one of the following categories:

1. relatively compact complex rational domains $U \subset \mathbb{C}^n = \mathbb{A}^n_{\mathbb{C}}$ defined by finite intersections of domains
$$U_{f,g} := \{x \in \mathbb{C}^n,\ |f(x)| < |g(x)|\},$$
for f and g analytic, with overconvergent functions (i.e., functions that are analytic on the closure $\bar{U}_{f,g}$).

2. relatively compact rational domains $U \subset \mathbb{C}^n$ stable by complex conjugation σ (i.e., relatively compact domains U in the real Berkovich affine space $\mathbb{A}^n_{\mathbb{R}} = \mathbb{C}^n/\sigma$), with overconvergent real analytic functions on them.

One may also possibly use only Stein semianalytic subsets, depending on the specifications for the geometry to be constructed. Equip LEGOS with the Grothendieck topology generated by usual coverings by rational domains, and with the class of limit cones given by transversal pullbacks. The corresponding algebras are called (overconvergent) complex and real semianalytic algebras and the corresponding varieties are called (overconvergent) semianalytic spaces. These can be equipped with their natural Grothendieck topology, induced by the usual one (and given by finite coverings). Remark that complex subanalytic subsets, defined as projections of relatively compact constructible domains (in the algebra generated by rational domains for complement, union and intersection), along affine projections $\mathbb{A}^n_X := X \times \mathbb{C}^n \to X$ identify naturally with Gabrielov real subanalytic subsets [1] of the complex space X. To get a similar result in the real semianalytic setting, one needs to add etale coverings (working with projections of etale constructible subsets). Remark also that the space of points of the semianalytic site on a given semianalytic space is something like a real or complex analytic version of the Zariski-Riemann space of a ring (see [HK94]). It may also be useful to work with spaces modeled on the Grothendieck site of subanalytic subsets, to relate our constructions to the methods of algebraic analysis, presented in Chapter 10.

One may also define a refined notion of smooth algebra with corners by the following process. We are inspired here by Joyce's notion of manifold with corners of any dimension (see [Joy09]), but give a different approach.

Example 2.2.8. An open subset with corners $U \subset (\mathbb{R}_+)^n$ may be stratified by the depth of its points. The depth of a point $u = (u_1, \dots, u_n)$ is the number of its components that are nonzero. There is a stratification

$$U = \coprod_{k=0}^{n} D^k(U)$$

where $D^k(U) := \{x \in U,\ \mathrm{depth}(x) = k\}$. Recall that the smooth structure on a subset of $(\mathbb{R}_+)^n$ is obtained by using smooth functions on an arbitrary $\tilde{U}$ around U, where $\tilde{U} \subset \mathbb{R}^n$ is open and $U = \tilde{U} \cap (\mathbb{R}_+)^n \subset \mathbb{R}^n$. We denote $\mathrm{OPEN}^{strat}_{\mathcal{C}^\infty}$ the category of stratified open subsets with corners, with smooth maps respecting the stratification, in the following

[1] Remark due to Gabrielov: use that $|f| = c$ is constructible and $\log |e^f| = \mathrm{Re}(f)$

sense: a point of depth k must be sent to a point of depth smaller or equal to k. With this condition there are non-trivial maps $[0,1] \to \mathbb{R}$, and any non-trivial map $\mathbb{R} \to [0,1]$ sends $\mathbb{R}$ into $]0,1[\subset [0,1]$. One may define the notion of transversal fiber product of stratified open, and use it to define the category $\mathrm{ALG}_{\mathcal{C}^\infty}^{strat}$ of stratified smooth algebras. The topology on stratified open subsets is induced by the usual topology on $\mathbb{R}^n$, using the fact that a smooth open subset of $\mathbb{R}^n_+$ correspond to germs of open subsets of $\mathbb{R}^n$ around $\mathbb{R}^n_+$. This gives a corresponding category of stratified smooth spaces. One may also use the formalism of Section 1.5 to define partial differential equations on stratified spaces. Now if $S : \underline{\Gamma}(M,C) \to \mathbb{R}$ is a local action functional on the space of sections of a bundle C on a compact stratified space M, then we may use the stratified structure to compute explicitely the variation of the action, through a stratified integration by part. The relation of this with the stratified jet formalism remains to be explored. Recall that in all this book, we chose to use the notion of histories (see our viewpoint of Lagrangian variational problems in Definition 7.1.1) exactly to kill the boundary terms of the integration by part, and avoid the above technicalities. This formalism could help to give a differential version of the notion of higher topological quantum field theory (see [Lur09a]).

2.3 Differential invariants

We now describe the concrete implementations of the tools of categorical infinitesimal calculus, presented in Section 1.5, in the differential geometric situations that will be in use in our description of physical systems.

2.3.1 Algebraic differential invariants

We essentially follow Lychagin [Lyc93] and Toen-Vezzosi [TV08] here.

Definition 2.3.1.1. A symmetric monoidal category $(\mathcal{V},\otimes)$ is called *pre-additive* if the initial and final object coincide (we denote it 0), the natural map

$$M \oplus N \to M \times N,$$

given by $(m,n) \mapsto (m,0)\oplus(0,n)$ is an isomorphism, and the monoidal structure commutes with finite direct sums.

From now on, we denote $(\mathcal{V},\otimes)$ a pre-additive symmetric monoidal category. Let $\mathrm{ALG}_{\mathcal{V}}$ be the category of commutative monoids in $\mathcal{V}$. These are the basic building blocks of (generalized) algebraic varieties, as explained in Definition 2.1.10.

Lemma 2.3.1.2. *Let $A \in \mathrm{ALG}_{\mathcal{V}}$ be a commutative monoid in $\mathcal{V}$ and $M \in \mathrm{MOD}(A)$ be a module (in the usual sense of symmetric monoidal categories). The direct sum $A \oplus M$ is naturally equipped with a commutative monoid structure such that the projection $A\oplus M \to A$ is a monoid morphism.*

Proof. The multiplication map on $A \oplus M$ is defined by the formula

$$\mu_{A\oplus M} : (A \oplus M) \otimes (A \oplus M) \cong A \otimes A \oplus A \otimes M \oplus M \otimes A \oplus M \otimes M \to A \oplus M$$

given by combining the four morphisms

$$\begin{array}{c} \mu \oplus 0 : A \otimes A \to A \oplus M, \\ 0 \oplus \mu_M^l : A \otimes M \to A \oplus M, \\ \mu_M^r \oplus U : M \otimes A \to A \oplus M, \text{ and} \\ 0 : M \otimes M \to M. \end{array} .$$

The unit $1 : \mathbb{1} \to A \oplus M$ is given by $1_A \oplus 0$. This makes $A \oplus M$ a commutative monoid in $\mathcal{V}$ such that the projection $A \oplus M \to A$ is a monoid morphism. □

Theorem 2.3.1.3. *Let* $\text{MOD}_{\mathcal{V}}$ *be the category of pairs* (A, M) *composed of a commutative monoid* A *and a module* M *over it, with morphisms given by pairs* $(f, f^\sharp) : (A, M) \to (B, N)$ *of a monoid morphism* $f : A \to B$ *and an* A*-module morphism* $f^\sharp : M \to f_*N$ *(where* f_*N *is* N *seen as an* A*-module).*

1. *the functor*
$$\begin{array}{ccc} \text{MOD}_{\mathcal{V}} & \to & T\text{ALG}_{\mathcal{V}} \\ (A, M) & \mapsto & [A \oplus M \to A] \end{array}$$
that sends a module over a monoid to the corresponding square zero extension is an equivalence.

2. *The category* $\text{Th}^n\text{ALG}_{\mathcal{V}}$ *is the subcategory of* $[I, \mathcal{C}]$ *whose objects are quotient morphisms* $A \to A/I$*, with kernel a nilpotent ideal* I *(i.e. nilpotent sub* A*-module of* A*) of order* $n + 1$ *(i.e., fulfilling* $I^{n+1} = 0$*).*

3. *The jet functor* $\text{Jet}^n : \mathcal{C} \to \text{Th}^n\mathcal{C}$ *is given by the corresponding ordinary jet algebra:*
$$\text{Jet}^n(A) = (A \otimes A)/I^{n+1}$$
where I *is the kernel of the multiplication map* $m : A \otimes A \to A$*.*

Remark that $\text{Jet}^n A$ is naturally equipped with two A-algebra structures denoted d_0 and d_1.

Definition 2.3.1.4. The A-module of *internal vector fields* is defined by

$$\Theta_A := \underline{\text{Hom}}_A(\Omega^1_A, A).$$

The A-module of *differential operators of order n* is defined by

$$\mathcal{D}^n_A := \underline{\text{Hom}}_A((\text{Jet}^n(A), d_1), A).$$

One defines a composition operation

$$\mathcal{D}^n_A \underset{\mathbb{1}}{\otimes} \mathcal{D}^n_A \to \mathcal{D}^n_A$$

by $D_1 \otimes D_2 \mapsto D_1 \circ D_2 := D_1 \circ d_0 \circ D_2 : \text{Jet}^n(A) \to A$. The A-module of *differential operators* is the colimit

$$\mathcal{D}_A := \text{colim}_n \mathcal{D}^n_A.$$

The intuition behind the definition of derivations in a our general situation is the following: a homomorphism $D : A \to M$ in $\mathcal{C}$ will be called a derivation if

$$D(fg) = D(f)g + fD(g).$$

To be mathematically correct, we need to compute derivations internally and using the tensor structure.

Proposition 2.3.1.5. *Let M be an A-module. The internal derivation object $\underline{\mathrm{Der}}(A, M)$ is the kernel of the inner Leibniz morphism (defined by adjunction of inner homomorphisms with the tensor product)*

$$\underline{\mathrm{Leibniz}} : \quad \underline{\mathrm{Hom}}(A, M) \rightrightarrows \underline{\mathrm{Hom}}(A \otimes A, M)$$

$$D \longmapsto \big(D \circ \mu,\ \mu^l_M \circ (\mathrm{id}_A \otimes D) + \mu^r_M \circ (D \otimes \mathrm{id}_A)\big)$$

In particular, on has $\Theta_A = \underline{\mathrm{Der}}(A, A)$.

Remark that for $\mathcal{C} = \textsc{Mod}_g(K)$ the symmetric monoidal category of graded modules, the Leibniz condition can be expressed as a graded Leibniz rule by definition of the right A-module structure on M above. More precisely, an internal morphism $D : A \to M$ is a graded derivation if it fulfills the graded Leibniz rule

$$D(ab) = D(a)b + (-1)^{\deg(D)\deg(f)} aD(b).$$

The restriction of this construction to the category $\textsc{Mod}_s(K)$ of super modules gives the notion of super-derivation.

We carefully inform the reader that if $A = \mathcal{C}^\infty(\mathbb{R})$ is the algebra of smooth function on $\mathbb{R}$, then in Ω^1_A, one has

$$d\exp(x) \neq \exp(x)dx.$$

This shows that contrary to the notion of derivation, that is a purely algebraic and depends only on the algebra of function, and not on the kind of geometry one is working with (smooth, analytic, algebraic, etc...), differential forms depend on the type of geometry in play. We refer to Section 2.3.2 for a solution to this problem.

Remark that if the category $\mathcal{C}$ is linear over a commutative ring K, Θ_A is endowed with a Lie bracket

$$[.,.] : \Theta_A \otimes \Theta_A \to \Theta_A$$

induced by the commutator of endomorphisms in $\underline{\mathrm{Hom}}(A, A)$. There is also a natural action

$$[.,.] : \Theta_A \otimes A \to A$$

and its is compatible with the above bracket and induces for each $\partial \in \Theta_A$ a derivation $\partial : A \to A$. All the above datum is called the A-Lie algebroid structure on Θ_A.

Recall from [BD04], 2.9.1 some basics on Lie algebroids.

Definition 2.3.1.6. A *Lie algebroid over* A is an A-module L equipped with a bracket

$$[.,.] : L \otimes L \to L \text{ and an } A\text{-linear anchor map } \tau : L \to \Theta_A$$

that fulfills for every $x, y, z \in L$ and $a \in A$:

1. Jacobi's identity: $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$,
2. anti-commutativity: $[x, y] = -[y, z]$,
3. compatibility of the anchor with the Lie bracket: $[x, ay] = \tau(x)(a).y + a[x, y]$.

An L-module is an A-module M with an action of L compatible with the L-action on A: for every $a \in A$, $m \in M$ and $x \in L$, one has

1. $x(am) = x(a)m + a(xm)$,
2. $(ax)m = a(xm)$ (left module structure) or $(ax)m = x(am)$ (left module structure).

The tangent Lie algebroid Θ_A is clearly the final Lie A-algebroid.

Proposition 2.3.1.7. *If A is smooth (meaning that Ω^1_A is a projective A-module of finite type), then $\mathcal{D}_A$ (resp. $\mathcal{D}_A^{op}$) is the enveloping algebra of the Lie-A-algebroid Θ_A of vector fields: if B is an associative A-algebra in $(\mathcal{C}, \otimes)$ equipped with an A-linear map $i : \Theta_A \to B$ (that induces the adjoint Θ_A-action on B) such that the inclusion $A \subset B$ is a morphism of left (resp. right) Θ_A-modules, there exists a unique morphism*

$$\mathcal{D}_A \to B$$
$$\text{(resp. } \mathcal{D}_A^{op} \to B\text{)}$$

extending $i : \Theta_A \to B$.

Definition 2.3.1.8. Let A be an algebra in $(\mathcal{C}, \otimes)$. A (left) $\mathcal{D}_A$*-module* is an object of M in $\mathcal{C}$, equipped with a left multiplication

$$\mu^l_{\mathcal{D}} : \mathcal{D}_A \otimes M \to M$$

that is compatible with the multiplication in $\mathcal{D}_A$.

Proposition 2.3.1.9. *Suppose that A is smooth, so that $\mathcal{D}_A$ is the enveloping algebra of the A-algebroid Θ_A of vector fields. The category of left $\mathcal{D}_A$-modules is equipped with a symmetric monoidal structure $(\text{Mod}(\mathcal{D}_A), \otimes)$ defined by*

$$\mathcal{M} \otimes \mathcal{N} := \mathcal{M} \otimes_A \mathcal{N}$$

with the $\mathcal{D}_A$-module structure induced by the action of derivations $\partial \in \Theta_A$ by

$$\partial(m \otimes n) = \partial(m) \otimes n + m \otimes \partial(n).$$

Proof. The action above is compatible with the action of functions on vector fields and with the Lie bracket of vector fields, so that it extends to an action of $\mathcal{D}_A$ since A is smooth by Proposition 2.3.1.7. □

Definition 2.3.1.10. A *differential complex* is a graded A-module M equipped with a $\mathcal{D}_A^{op}$-linear morphism $d : M \otimes_A \mathcal{D}_A \to M \otimes_A \mathcal{D}_A[1]$. Let $(\mathrm{Diffmod}_{dg}(A), \otimes)$ be the symmetric monoidal category of differential complexes. The *algebra of differential forms* on A is the free algebra in $(\mathrm{Diffmod}_{dg}(A), \otimes)$ on the de Rham differential $d : A \to \Omega^1_A$, given by the symmetric algebra

$$\Omega^*_A := \mathrm{Sym}_{(\mathrm{Diffmod}_{dg}(A),\otimes)}\left([\underset{0}{A} \overset{d}{\to} \underset{1}{\Omega^1_A}] \right).$$

We now recall the basic properties of differential forms.

Theorem 2.3.1.11. *The natural map*

$$\Theta_A \to \underline{\mathrm{Hom}}_A(\Omega^1_A, A)$$

extends to a morphism

$$i : \Theta_A \to \mathrm{Hom}_{\mathrm{Diffmod}_g(A)}(\Omega^*_A, \Omega^*_A[1])$$

called the interior product map (its images are odd derivations of Ω^*_A). The natural map

$$\Theta_A \to \underline{\mathrm{Hom}}(A, A)$$

extends to a morphism

$$L : \Theta_A \to \mathrm{Hom}_{\mathrm{MoD}_g(A)}(\Omega^*_A, \Omega^*_A),$$

called the Lie derivative. One has "Cartan's formula"

$$L_X\omega = i_X(d\omega) + d(i_X\omega).$$

Proof. One can consider the monoidal category $(\mathrm{Diffmod}_{tridg}(A), \otimes)$ composed of graded A-differential-modules M equipped with a triple of differential operators

$$\overset{L}{\circlearrowleft} M \underset{i}{\overset{d}{\rightleftarrows}} M[1] .$$

Recall that the Lie derivative is defined by $L_X(f) = X(f)$ on functions $f \in A$ and by $(L_X\omega)(Y) = \omega([X,Y])$ on 1-forms $\omega \in \Omega^1_A$. One then has, for every $X \in \Theta_A$, an isomorphism

$$(\Omega^*_A, d, i_X, L_X) \cong \mathrm{Sym}_{(\mathrm{Diffmod}_{tridg}(A),\otimes)}\left(\left[\overset{L_X}{\circlearrowleft}\underset{0}{A} \underset{i_X}{\overset{d}{\rightleftarrows}} \overset{L_X}{\circlearrowleft}\underset{1}{\Omega^1_A} \right] \right)$$

and the equality $i_X(df) = X(f)$ implies Cartan's formula by extension to the symmetric algebra. □

Example 2.3.1.12. In $(\mathcal{C}, \otimes) = (\text{Vect}_{\mathbb{R}}, \otimes)$, for $A = \mathbb{R}[X_1, \dots, X_n]$, the interior product of a $k+1$-form with a vector field is a k-form given by

$$i_X\omega(X_1, \dots, X_k) = \omega(X, X_1, \dots, X_k).$$

It can be computed explicitely by using that i_X is a derivation and that it is known on 1-forms. The Lie derivative can be computed explicitely using the Leibniz rule for the exterior differential

$$d(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^{\deg(\eta)}\omega \wedge d(\eta),$$

the Leibniz rule for the interior product

$$i_X(\omega \wedge \eta) = i_X(\omega) \wedge \eta + (-1)^{\deg(\eta)}\omega \wedge i_X(\eta),$$

and Cartan's formula

$$L_X\omega = i_X(d\omega) + d(i_X\omega).$$

We now describe the natural geometrical setting for the functor of point approach to algebraic partial differential equations.

Definition 2.3.1.13. Let A be an algebra in $(\mathcal{C}, \otimes)$. Algebras in the monoidal category

$$(\text{Mod}(\mathcal{D}_A), \otimes)$$

are called $\mathcal{D}_A$-*algebras*, and their category is denoted $\text{Alg}_{\mathcal{D}_A}$.

Recall from Beilinson and Drinfeld's book [BD04], Section 2.3.2, the existence of universal $\mathcal{D}$-algebras, called algebraic jet algebras. This way of looking at infinite jets shows that they are typically finite dimensional objects ($\mathcal{D}$-algebras of finite type), contrary to the preconceived idea of a large part of the mathematical community.

Proposition 2.3.1.14. *Let A be an algebra in $(\mathcal{C}, \otimes)$ and $\text{Alg}_{\mathcal{D}}$ be the category of commutative unital $\mathcal{D}_A$-algebras. The functor*

$$\text{Forget}_{\mathcal{D}} : \text{Alg}_{\mathcal{D}} \to \text{Alg}(A)$$

that sends a $\mathcal{D}_M$-algebra to the underlying A-algebra has a right adjoint called the algebraic jet algebra $\text{Jet}_{alg}(B)$.

Proof. The algebra $\text{Jet}_{alg}(B)$ is given by the quotient of the symmetric algebra

$$\text{Sym}^*(\mathcal{D}_A \otimes_A B)$$

in the symmetric monoidal category $(\text{Mod}(\mathcal{D}), \otimes)$, by the ideal generated by the elements

$$\partial(1 \otimes r_1.1 \otimes r_2 - 1 \otimes r_1r_2) \in \text{Sym}^2(\mathcal{D} \otimes_A B) + \mathcal{D} \otimes_A B$$

and

$$\partial(1 \otimes 1_B - 1) \in \mathcal{D} \otimes B + A,$$

$r_i \in B$, $\partial \in \mathcal{D}$, 1_B the unit of B. □

Example 2.3.1.15. If $A = \mathbb{R}[X]$ is the ring of polynomial functions on $\mathbb{R}$ and $B = \mathbb{R}[x, u]$ is the algebra of coordinates on the trivial algebraic bundle $\pi : C = \mathbb{R}^2 \to \mathbb{R} = M$, one gets as jet algebra

$$\mathrm{Jet}(B) = \mathbb{R}[x, u_0, \dots, u_n, \dots]$$

the infinite polynomial algebra with action of ∂_x given by

$$\partial_x u_i = u_{i+1}.$$

It corresponds to the algebraic jet bundle $\pi_\infty : \mathrm{Jet}(C) \to M$. A section of the bundle $\pi : C \to M$ is a polynomial $P \in \mathbb{R}[x]$ and the jet map

$$\Gamma(M, C) \to \Gamma(M, \mathrm{Jet}(C))$$

is given by sending $P(x) \in \Gamma(M, C) \cong \mathbb{R}[x]$ to its taylor expansion

$$j_\infty P(x) \in \Gamma(M, \mathrm{Jet}(C)) \cong \mathbb{R}[x]^{\{u_0, \dots, u_n, \dots\}},$$

given by

$$(j_\infty P)(x) = (\partial_x^n P(x))_{n \geq 0} = \sum_{n \geq 0} \frac{\partial_x^n P(x)}{n!}(X - x)^n.$$

Remark 2.3.1.16. One can apply directly the methods of algebraic differential calculus presented in this section to the symmetric monoidal category $(\mathrm{Mod}(\mathcal{O}_X), \otimes_{\mathcal{O}_X})$ of quasi-coherent modules over a variety $(X, \mathcal{O}_X)$ to define covariant differential invariants (vector fields, differential operators). However, algebraic methods don't give the right result in general for contravariant constructions (differential forms, jets). Indeed, one only gets this way Kähler differentials and jets, that are not enough for the study of smooth equations. This problem can be overcome at least in three ways:

1. one can do categorical infinitesimal calculus in a full subcategory of the category of locally ringed spaces.

2. one can restrict the functors of differential calculus (derivations, differential operators) to a category of geometric modules. This is the approach used by Nestruev [Nes03].

3. one can work with smooth algebras, as will be explained in Section 2.3.2; we will use this approach because it is optimal for our purposes.

2.3.2 Smooth differential invariants

Having studied the algebraic case in the previous section, we now show that the formalism of categorical infinitesimal calculus given in Section 1.5 gives back ordinary smooth differential calculus on smooth varieties. This is the purpose of the following theorem.

Theorem 2.3.2.1. *Let*

$$\begin{array}{rccc} \mathcal{C}^\infty(M): & \textsc{Aff}_{\mathcal{C}^\infty} & \to & \textsc{Sets} \\ & \mathbb{R}^n & \mapsto & \mathcal{C}^\infty(M, \mathbb{R}^n) \end{array}$$

be the smooth algebra of functions on a smooth variety M.

1. *There is a fully faithful monoidal embedding of the category of vector bundles on M in the category $\textsc{Mod}(\mathcal{C}^\infty(M))$ of modules over the corresponding smooth algebra.*

2. *The $\mathcal{C}^\infty(M)$-module $\Omega^1(M)$ of ordinary differential forms on M identifies with the module of differential forms Ω^1_X in the sense of definition 1.5.6.*

3. *The smooth algebra $\mathcal{C}^\infty(J^n M)$ of smooth functions on ordinary jet space for smooth functions on M identifies with the smooth algebra $\mathrm{Jet}^n(\mathcal{C}^\infty(M))$ in the sense of definition 1.5.6.*

4. *If $C \to M$ is a smooth bundle, the $\mathcal{C}^\infty(M)$-algebra $\mathcal{C}^\infty(\mathrm{Jet}^n(C/M))$ identifies with the smooth algebra of jets in the category of smooth $\mathcal{C}^\infty(M)$-algebras in the sense of definition 1.5.6.*

Proof. If $E \to M$ is a vector bundle, with zero section $\underline{0} : M \to E$, we may describe the smooth algebra $\mathcal{C}^\infty(M)$ as the quotient of $\mathcal{C}^\infty(E)$ by the class $\mathcal{I}$ of functions $f : E \to \mathbb{R}$ such that $f \circ \underline{0}$ is the zero function. The functor

$$\begin{array}{ccc} (\textsc{Vect}(M), \times_M, \otimes) & \to & (\textsc{Mod}(\mathcal{C}^\infty(M)), \oplus, \otimes) \\ [E \to M] & \mapsto & [\mathcal{C}^\infty(E)/\mathcal{I}^2 \to \mathcal{C}^\infty(M)] \end{array}$$

is fully faithful and bimonoidal. This means that it sends the fiber product $E \times_M E$ to the relative fiber product of smooth algebras, which corresponds to the sum of modules, and the tensor product of vector bundles to the pointed tensor product of smooth algebras. Differential forms in the sense of Definition 1.5.6 are given by the smooth algebra

$$\Omega^1_M := \mathcal{C}^\infty(M \times M)/\mathcal{I}^2_\Delta$$

over $\mathcal{C}^\infty(M)$ defined by the square of the diagonal ideal. The identification with usual differential forms may be done by choosing one of the projection to make this algebra a module, and by trivializing on a small open subset. The proof of the other results are of a similar nature and are left to the reader. □

Example 2.3.2.2. Let $M = \mathbb{R}$ and consider the trivial bundle $C = M \times \mathbb{R}$ on M. The k-th jet bundle for $C \to M$ is (non-canonically) identified with the space

$$\mathrm{Jet}^k(C/M) = M \times \mathbb{R}[T]/(T^{k+1}) \cong M \times \mathbb{R}^{k+1},$$

whose coordinates $(x, u_i)_{i=0,\dots,k}$ play the role of formal derivatives (i.e., coefficients of the taylor series) of the "functional variable" u_0. The canonical section $C \to \mathrm{Jet}^k(C/M)$ corresponds geometrically to the map

$$\begin{array}{rccc} j_k : & \Gamma(M, C) & \to & \Gamma(M, \mathrm{Jet}^k C) \\ & [x \mapsto f(x)] & \mapsto & \left[x \mapsto \sum_{i=0}^k \frac{\partial^i f}{\partial x^i}(x).\frac{(X-x)^i}{i!}\right], \end{array}$$

that sends a real valued function to its truncated Taylor series. The infinite jet space identifies with

$$\mathrm{Jet}(C/M) = M \times \mathbb{R}^{\mathbb{N}}.$$

Its functions given by smooth functions that depend on finitely many coordinates. It is naturally a crystal of smooth varieties over M (in the sense of Definition 1.5.16).

Example 2.3.2.3. More generally, if $M = \mathbb{R}^n$ and $C = M \times \mathbb{R}^m$, one gets the smooth algebra bundle

$$\mathrm{Jet}^k C = M \times \oplus_{i \leq k} \mathrm{Sym}^i(\mathbb{R}^n, \mathbb{R}^m),$$

with coordinates (x, u_α) for $x \in X$ and $u_\alpha \in \mathbb{R}^m$, α being a multi-index in $\mathbb{N}^n$ with $|\alpha| \leq k$. The jet map corresponds to higher dimensional Taylor series

$$\begin{array}{cccc} j_k : & \Gamma(M, C) & \to & \Gamma(M, \mathrm{Jet}^k C) \\ & [x \mapsto f(x)] & \mapsto & \left[x \mapsto \sum_{|\alpha| \leq k} (\partial_x^\alpha f)(x). \frac{(X-x)^\alpha}{\alpha!}\right], \end{array}$$

where multi-index notations are understood.

Remark that one can prove more generally that categories of spaces given by convenient subcategories of the category of locally ringed spaces can also be treated with the methods of Section 1.5. The reader used to this setting can thus just use the differential invariants he is used to work with.

2.3.3 Super and graded geometry

We now explain how to use our formalism of categorical infinitesimal calculus in the super and graded setting, to define differential invariants of supermanifolds, that play a fundamental role in the description of mathematical models of field theory (e.g. in quantum electrodynamics, to cite the most famous one). We refer to [Nis00] and [Sac09] for other approaches to synthetic super-geometry.

Definition 2.3.3.1. The category of *smooth open superspaces* is the category $\mathrm{OPEN}^s_{\mathcal{C}^\infty}$ whose objets are pairs $U^{n|m} = (U, \mathbb{R}^m)$, with $U \subset \mathbb{R}^n$ an open subset, and whose morphisms are given by the set

$$\mathrm{Hom}(U^{n|m}, V^{k|l}) = \mathrm{Hom}_{\mathrm{ALG}^s_{\mathbb{R}}}(\mathcal{C}^\infty(V, \wedge^* \mathbb{R}^l), \mathcal{C}^\infty(U, \wedge^* \mathbb{R}^m))$$

of morphisms of real superalgebras. The category of *smooth affine superspaces* is the full subcategory $\mathrm{AFF}^s_{\mathcal{C}^\infty} \subset \mathrm{OPEN}^s_{\mathcal{C}^\infty}$ given by pairs $\mathbb{R}^{n|m} = (\mathbb{R}^n, \mathbb{R}^m)$.

Definition 2.3.3.2. A *smooth super-diffeology* is a sheaf on the site $(\mathrm{OPEN}^s_{\mathcal{C}^\infty}, \tau)$ of smooth open superspaces. A variety relative to this site is called a *supermanifold*.

Proposition 2.3.3.3. *A morphism $f : U^{n|m} \to V^{k|l}$ of superspaces induces a smooth map $|f| : U \to V$ between the underlying open sets and a commutative diagram*

$$\begin{array}{ccc} U^{m|n} & \xrightarrow{f} & U^{k|l} \\ {\scriptstyle 0}\uparrow\downarrow{\scriptstyle p} & & {\scriptstyle 0}\uparrow\downarrow{\scriptstyle p} \\ U^{m|0} & \xrightarrow{|f|} & U^{k|0} \end{array}$$

The product of two smooth open superspaces $(U, \mathbb{R}^m)$ *and* $(V, \mathbb{R}^l)$ *is given by* $(U \times V, \mathbb{R}^{m+l})$.

Proof. Remark first that if $A = \mathcal{C}^\infty(U, \wedge^* \mathbb{R}^m)$, then the odd and even parts of A are

$$A^0 = \mathcal{C}^\infty(U, \wedge^{2*}\mathbb{R}^m) \text{ and } A^1 = \mathcal{C}^\infty(U, \wedge^{2*+1}\mathbb{R}^m).$$

The subset $A^0 \subset A$ is a subalgebra and the subset A^1 is a sub-A^0-module. If we consider the ideal $(A^1) \subset A$ generated by A^1, we get a quotient algebra

$$|A| = A/(A^1),$$

that is exactly isomorphic to $\mathcal{C}^\infty(U) = \mathcal{C}^\infty(U^{m|0})$. This construction is functorial in morphisms of open superspaces. This implies that a morphism $U^{n|m} \to V^{k|l}$ between open superspaces induces a diagram of superalgebras

$$\begin{array}{ccc} U^{m|n} & \xrightarrow{f} & U^{k|l} \\ {\scriptstyle 0}\uparrow & & \uparrow{\scriptstyle 0} \\ U^{m|0} & \xrightarrow{|f|} & U^{k|0} \end{array}$$

The product of usual open subsets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^k$ is $U \times V$, and we have

$$\mathcal{C}^\infty(U) \otimes \wedge^* \mathbb{R}^m \cong \mathcal{C}^\infty(U, \wedge^* \mathbb{R}^m)$$

so that $U^{m|0} \times \mathbb{R}^{0|m} \cong U^{n|m}$. We also have

$$\wedge^* \mathbb{R}^m \otimes \wedge^* \mathbb{R}^n \cong \wedge^* \mathbb{R}^{m+n}.$$

Combining these results, we get

$$U^{n|m} \times V^{k|l} \cong (U \times V)^{n+k|m+l}.$$

This implies in particular that there is a natural projection $U^{n|m} \to U^{n|0}$, given by the morphism $\mathcal{C}^\infty(U) \to \mathcal{C}^\infty(U, \wedge^* \mathbb{R}^m)$. This projection is functorial in $U^{n|m}$, meaning that there is a commutative diagram

$$\begin{array}{ccc} U^{m|n} & \xrightarrow{f} & U^{k|l} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ U^{m|0} & \xrightarrow{|f|} & U^{k|0} \end{array}$$

because $p \circ 0 = \mathrm{id}_{U^{m|0}}$ and $|f| = p \circ f \circ 0$. □

A diagram

$$\begin{array}{ccc} & & U^{m|n} \\ & & \downarrow \\ V^{k|l} & \longrightarrow & W^{r|s} \end{array}$$

is called transversal if the tangent superspace $TW^{r|s}$ is generated by $TU^{m|n} \cup TV^{k|l}$.

Definition 2.3.3.4. A *finitary smooth superalgebra* is a functor

$$A : (\text{AFF}^s_{\mathcal{C}^\infty}, \times) \to (\text{SETS}, \times)$$

that preserves finite products. A *smooth superalgebra* is a functor

$$A : (\text{OPEN}^s_{\mathcal{C}^\infty}, - \underset{-}{\overset{t}{\times}} -) \to (\text{SETS}, - \underset{-}{\times} -)$$

that sends finite transversal fiber products to fiber products. We denote $\text{ALG}^{s,fin}_{\mathcal{C}^\infty}$ and $\text{ALG}^s_{\mathcal{C}^\infty}$ the categories of finitary and general smooth superalgebras. If A is a given superalgebra, we denote $A^0 := A(\mathbb{R}^{1|0})$ and $A^1 := A(\mathbb{R}^{0|1})$.

Since $\mathbb{R}^{m,n} \cong (\mathbb{R}^{1|0})^m \times (\mathbb{R}^{0|1})^n$, if A is a given superalgebra, one has by definition

$$A(\mathbb{R}^{m|n}) \cong (A^0)^m \times (A^1)^n.$$

Definition 2.3.3.5. A *smooth superspace* is a sheaf on the site $(\text{ALG}^{s,op}_{\mathcal{C}^\infty}, \tau)$ of smooth open superspaces. A variety relative to this site is called a *super-variety* or *smooth superscheme.*

Proposition 2.3.3.6. *The forgetful functor*

$$\begin{array}{ccc} \text{ALG}^s_{\mathcal{C}^\infty} & \to & \text{ALG}^s_{\mathbb{R}} \\ A & \mapsto & A(\mathbb{R}^{1|1}) \end{array}$$

has a right adjoint geom *called the geometrization functor. Moreover, there is a well defined functor*

$$\begin{array}{ccc} \text{ALG}^s_{\mathcal{C}^\infty} & \to & \text{ALG}^s_{\mathcal{C}^\infty} \\ A & \mapsto & A^0 : [U^{m|n} \mapsto A(U^{m|0})] \end{array}$$

and for each A, functorial morphisms $0 : A^0 \to A$ and $p : A \to A^0$ of smooth superalgebras such that $p \circ 0 = \text{id}_{A^0}$.

Proof. If A is a real superalgebra, one defines its geometrization by

$$\text{geom}(A)(U^{n|m}) := \text{Hom}_{\text{ALG}^s_{\mathbb{R}}}(\mathcal{C}^\infty(U, \wedge^* \mathbb{R}^m), A).$$

This is a smooth superalgebra by construction. The map $A \mapsto A^0$ is functorial because of Proposition 2.3.3.3: a morphism $f : U^{m|n} \to V^{k|l}$ induces a natural map $|f| : U^{m|0} \to V^{k|0}$. The morphisms of functors $0 : A^0 \to A$ and $p : A \to A^0$ are induced by the maps $0 : U^{n|0} \to U^{n|m}$ and $p : U^{n|m} \to U^{0|n}$ (that are also functorial by loc. cit.). □

Proposition 2.3.3.7. *The maps $\tilde{A} : U \mapsto A(U^{n|1})$ and $\tilde{A}^0 : U \mapsto A(U^{n|0})$ induce a functor*

$$\begin{array}{cccc} \text{mod} : & \text{ALG}^s_{\mathcal{C}^\infty} & \to & \text{MOD}(\text{ALG}_{\mathcal{C}^\infty}) \\ & A & \mapsto & [\tilde{A} \to \tilde{A}^0] \end{array}$$

with values in smooth modules. This functor has a left adjoint, denoted $M \mapsto \text{Sym}(M[1])$.

Proof. Both functors $\tilde{A}$ and $\tilde{A}^0$ commute with transversal pullbacks and are thus smooth algebras. Remark that the underlying algebra of $\tilde{A}$ is the set $A(\mathbb{R}^{1|1}) = A^0 \times A^1$, with multiplication induced by the map

$$\begin{array}{rccc} \times : & \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} & \to & \mathbb{R}^{1|1}, \\ & (x_1, \theta_1), (x_2, \theta_2) & \mapsto & (x_1 x_2, x_1\theta_2 + x_2\theta_1). \end{array}$$

The zero morphism $0 : U^{n|0} \to U^{n|1}$, the projection $p : U^{n|1} \to U^{n|0}$, and the addition

$$+ : U^{n|1} \times_{U^{n|0}} U^{n|1} \cong U^{n|2} \longrightarrow U^{n|1}$$

given by sending (u, θ_1, θ_2) to $(u, \theta_1 + \theta_2)$ are all functorial in U, and make $U^{n|1}$ an abelian group object over U. This implies that $\tilde{A}$ is an abelian group object over $\tilde{A}^0$. In particular, its addition is given by

$$+ : (\tilde{A} \times_{\tilde{A}^0} \tilde{A})(U) \cong A(U^{n|1} \times_{U^{n|0}} U^{n|1}) \longrightarrow A(U^{n|0}) = \tilde{A}(U).$$

We have thus shown that $\mathrm{mod}(A)$ is indeed a smooth module. The left adjoint to mod must fulfil

$$\mathrm{Hom}_{\mathrm{ALG}^s_{\mathcal{C}^\infty}}(\mathrm{Sym}(M[1]), B) \cong \mathrm{Hom}_{\mathrm{MOD}_{\mathcal{C}^\infty}}(M, \mathrm{mod}(B)).$$

Concretely, if $M \to A$ is the structure map, it is given by

$$\mathrm{Sym}(M[1])(U^{n|m}) := \left(A \underset{\mathrm{geom}(A(\mathbb{R}))}{\otimes} \mathrm{geom}(\wedge^{2*}_{A(\mathbb{R})} M)\right)(U) \times (\wedge^{2*+1}_{A(\mathbb{R})} M)^m,$$

where $\mathrm{geom}(A(\mathbb{R})) \to A$ is the natural adjunction map and M also denotes its underlying $A(\mathbb{R})$-module. By construction, $\mathrm{Sym}(M[1])$ commutes with transversal fiber products. The adjunction morphism $\mathrm{Sym}(\mathrm{mod}(B)[1]) \to B$ is defined by the following: first, recall that $\mathrm{mod}(B) = [M \to A]$ is given by $M = B^1$ and $A = B^0$, and the desired map is given on the odd part by the multiplication map

$$\wedge^{2*+1}_{A(\mathbb{R})} M \to M = B^1.$$

There is also a natural multiplication morphism of real algebras

$$\wedge^{2*}_{A(\mathbb{R})} M \to B^0(\mathbb{R})$$

that corresponds to a morphism of smooth algebras

$$\mathrm{geom}(\wedge^{2*+1}_{A(\mathbb{R})} M) \to B^0.$$

Since B^0 is an A-algebra through the identity map $A = B^0$, we get by extension of scalars a morphism of smooth algebras

$$A \underset{\mathrm{geom}(A(\mathbb{R}))}{\otimes} \mathrm{geom}(\wedge^{2*}_{A(\mathbb{R})} M) \to B^0.$$

The other adjunction morphism $M \to \mathrm{mod}(\mathrm{Sym}(M[1])$ associated to a module $[M \to A]$ is given by the natural morphism $A \to \mathrm{Sym}(M[1])^0$ and the map $M \to \wedge^{2*+1}_{A(\mathbb{R})} M$, given more precisely by the map of sets

$$M(U) = A(U) \times M \longrightarrow \left(A \underset{\mathrm{geom}(A(\mathbb{R}))}{\otimes} \mathrm{geom}(\wedge^{2*}_{A(\mathbb{R})} M) \right)(U) \times (\wedge^{2*+1}_{A(\mathbb{R})} M).$$

□

Proposition 2.3.3.8. *Let M be a smooth manifold and Ω^1_M be the cotangent bundle of M. Let $\Omega^*_M := \mathrm{geom}(\Gamma(M, \wedge^* \Omega^1_M))$ be the smooth superalgebra of differential forms on M. There is a natural isomorphism*

$$\underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, M) \cong \underline{\mathrm{Spec}}(\Omega^*_M).$$

Proof. Remark that $\Omega^*_M = \mathrm{Sym}_{\mathcal{O}_M}(\Omega^1_M[1])$, where Ω^1_M is the smooth module of differential forms. Let A be a smooth superalgebra. One has

$$\begin{array}{rcc} \underline{\mathrm{Spec}}(\Omega^*_M)(A) & := & \mathrm{Hom}_{\mathrm{ALG}^s_{\mathcal{C}^\infty}}(\Omega^*_M, A) \\ & \cong & \mathrm{Hom}_{\mathrm{ALG}^s_{\mathcal{C}^\infty}}(\mathrm{Sym}_{\mathcal{O}_M}(\Omega^1_M[1]), A) \\ & \cong & \mathrm{Hom}_{\mathrm{MOD}_{\mathcal{C}^\infty}}(\Omega^1_M, \mathrm{mod}(A)) \\ & \cong & \mathrm{Hom}_{\mathrm{ALG}_{\mathcal{C}^\infty}}(\mathcal{O}_M, \tilde{A}). \end{array}$$

Now we may also compute the other side of the desired isomorphism. Remark that $\tilde{A} \cong (\mathcal{C}^\infty(\mathbb{R}^{0|1}) \otimes A)^0$. Indeed, we have by definition

$$(\mathcal{C}^\infty(\mathbb{R}^{0|1}) \otimes A)^0(U^{n|m}) := (\mathcal{C}^\infty(\mathbb{R}^{0|1}) \otimes A)(U^{n|0}) \cong A(U^{n|1}) =: \tilde{A}(U),$$

so that

$$\begin{array}{rcc} \underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, M)(A) & := & \mathrm{Hom}_{\mathrm{ALG}^s_{\mathcal{C}^\infty}}(\mathcal{O}_M, \mathcal{C}^\infty(\mathbb{R}^{0|1}) \otimes A), \\ & \cong & \mathrm{Hom}_{\mathrm{ALG}_{\mathcal{C}^\infty}}(\mathcal{O}_M, (\mathcal{C}^\infty(\mathbb{R}^{0|1}) \otimes A)^0), \\ & \cong & \mathrm{Hom}_{\mathrm{ALG}_{\mathcal{C}^\infty}}(\mathcal{O}_M, \tilde{A}), \end{array}$$

which concludes the proof. □

It is easy to generalize superspaces to spaces equipped with an additional linear structure (for example $\mathbb{Z}$-graded spaces). We explain shortly this generalization, that will be useful in our study of gauge theories.

Definition 2.3.3.9. Let (LEGOS, τ) be a site equipped with a class of transversal fiber products, and $\mathcal{C} = (\mathcal{C}, \mathcal{C}_0, \otimes)$ be a symmetric monoidal category $(\mathcal{C}, \otimes)$ equipped with a full subcategory $\mathcal{C}_0$ and a functor

$$\begin{array}{rccc} \mathcal{O}(-, \mathrm{Sym}(-)) : & \mathrm{LEGOS} \times \mathcal{C}_0 & \to & \mathrm{ALG}(\mathcal{C}, \otimes) \\ & (U, E) & \mapsto & \mathcal{O}(U, \mathrm{Sym}(E)), \end{array}$$

such that $\mathcal{O}(-, \mathrm{Sym}(E))$ is a sheaf for all E. The category $\mathrm{LEGOS}^{\mathcal{C}}$ of $\mathcal{C}$-enriched legos is the category of pairs (U, E) composed of a lego U, of an object E of $\mathcal{C}_0$, with morphisms $f : (U, E) \to (V, F)$ given by morphisms of $\mathcal{C}$-algebras

$$f^* : \mathcal{O}(V, \mathrm{Sym}(F)) \to \mathcal{O}(U, \mathrm{Sym}(E)).$$

We suppose now given a notion of transversal fiber product on the category $\mathrm{Legos}^{\mathcal{C}}$.

Definition 2.3.3.10. A $\mathcal{C}$-enriched algebra modeled on Legos is a set-valued functor on $\mathrm{Legos}^{\mathcal{C}}$ that commutes with transversal fiber products. We denote $\mathrm{Alg}_{\mathrm{Legos},\mathcal{C}}$ the category of $\mathcal{C}$-enriched algebras, equipped with the Grothendieck topology induced by τ.

All the sites Legos described in this chapter have a natural notion of transversal pullback. The above definition specializes to our definition of smooth superalgebras if we work with the monoidal category $(\mathrm{Vect}^s_{\mathbb{R}}, \otimes)$ of real super vector spaces, the subcategory $\mathrm{Vect}^{odd}_{\mathbb{R}}$ of odd vector spaces, and with the site of smooth open sets.

Definition 2.3.3.11. The category of smooth graded algebras is defined as the category

$$\mathrm{Alg}^g_{\mathcal{C}^\infty} := \mathrm{Alg}_{\mathcal{C},\mathrm{Open}_{\mathcal{C}^\infty}}$$

of $\mathcal{C}$-enriched smooth algebras, where $\mathcal{C} = (\mathrm{Vect}^g_{\mathbb{R}}, \mathrm{Vect}^{\pm}, \otimes)$ is the monoidal category of graded real vector spaces, equipped with the subcategory $\mathrm{Vect}^{\pm}$ of spaces with no degree zero part.

A fake function on a graded smooth space is a morphism

$$S : X \to \mathbb{A}^1$$

where $\mathbb{A}^1(A) = A$ is the space of all elements in the graded algebra A. A real valued function on a graded smooth space is a morphism

$$S : X \to \underline{\mathbb{R}}$$

where $\underline{\mathbb{R}}(A) := A^0$ is the space of degree zero elements in the graded algebra A.

Example 2.3.4. Let M be a variety and $\Omega^*(M)$ be its algebra of differential form, seen as a smooth super-algebra. Then a fake smooth function

$$f : \underline{\mathrm{Spec}}^s(\Omega^*(M)) \to \mathbb{A}^1$$

is simply a differential form ω in $\Omega^*(M)$, and a smooth function

$$f : \underline{\mathrm{Spec}}^s(\Omega^*(M)) \to \underline{\mathbb{R}}$$

is a differential form ω in $\Omega^{2*}(M)$. If now $\Omega^*(M)$ is considered as a smooth graded algebra, fake smooth functions don't change but a smooth function

$$f : \underline{\mathrm{Spec}}^g(\Omega^*(M)) \to \underline{\mathbb{R}}$$

is an ordinary function f on M (of degree 0).

We finish by giving the example that gives the main motivation to study super-spaces by their functors of points.

Example 2.3.5. Let $p : S \to M$ be a vector bundle of a smooth manifold M. The super-algebra

$$\mathcal{O}_{\Pi S} := \Gamma(M, \wedge^* S)$$

has a natural smooth super-algebra structure. Its spectrum is the corresponding geometric odd bundle

$$\Pi p : \Pi S := \underline{\mathrm{Spec}}(\mathcal{O}_{\Pi S}) \to M.$$

This bundle has no non-trivial sections, i.e., $\underline{\Gamma}(M, \Pi S) = \{0\}$, because a $\mathcal{C}^\infty(M)$-algebra morphism

$$\mathcal{O}_{\Pi S} = \Gamma(M, \wedge^* S) \to \mathcal{C}^\infty(M)$$

is necessarily trivial on the generators of the exterior algebra. Physicists, and among them, DeWitt in [DeW03], solve this problem by using points of $\underline{\Gamma}(M, \Pi S)$ with values in the free odd algebra on a countable number of generators

$$A = \Lambda_\infty := \widehat{\wedge^* \mathbb{R}^{(\mathbb{N})}}$$

(of course, his formulation is a bit different of ours, but the idea is the same). These are given by $\mathcal{C}^\infty(M)$-algebra morphisms

$$\mathcal{O}_{\Pi S} \to \mathcal{C}^\infty(M) \otimes A,$$

and since A has also odd components, one can find plenty of them. There is no reason to choose this coefficient algebra and not another one, so that the natural space of sections $\underline{\Gamma}(M, S)$ must really be thought of as the functor on super-algebras

$$A \mapsto \underline{\Gamma}(M, S)(A)$$

as we did in this whole chapter.

Chapter 3

Functorial analysis

In this chapter, we give a translation of the classical methods of the analysis of linear partial differential equations in the language used in this book, using functions with particular given definition domains. The advantage of this viewpoint of analysis is that it generalizes directly to the super or graded setting, that is strictly necessary to treat the basic models of particle physics. It is also better suited to the combination of the homotopical geometric methods of Chapter 9 with the usual methods of classical analysis. This combination is useful to better understand the conceptual meaning of gauge theoretical computations of Chapter 12.

We will describe in Chapters 10 and 11 a coordinate free approach to partial differential equations, very much in the spirit of the functor of points approach to resolution of equations. One needs both approaches to understand well the models of mathematical physics.

The analysis of partial differential equations can be based on the notion of well posed problem, that was introduced by Hadamard. Roughly speaking, a partial differential equation E with initial/boundary value f in a topological space of functions $\mathcal{F}$ is called well-posed if

1. there exists a solution to E around the given initial value,
2. this solution is unique,
3. the solution depends continuously on the initial value f.

The most general result on this problem is the Cauchy-Kowalewskaya theorem (see Theorem 10.6.5 for the formulation of a generalization of this statement adapted to the study of systems of partial differential equations on manifolds) that states that any reasonable real-analytic non-linear partial differential equation has a unique solution depending analytically on the given analytic initial condition. In the case of evolution equations, one also often asks for a persistence condition (the solution remains in the same functional space as the initial condition).

The main method of the analysis of partial differential equations is to construct a large space $\mathcal{F}$ of generalized functions (for example Schwartz distributions) in which the existence of a weak solution for a given type of equation is assured, and to find a subspace

$\mathcal{H}$ (for example a Sobolev space) in which the uniqueness is guaranteed. The existence and unicity can be proved by invoking a fixed point theorem. Existence can also be proved by a compacity criterion (by extracting a converging sub-sequence of a given sequence). Being in $\mathcal{H}$ is then often seen as a regularity condition on solutions, that can be proved by a priori estimates on the operator. In the case of non-linear equations, one can also use the local inversion theorem and perturbative methods, seeing the equation as a perturbation of a linear one by a non-linear term multiplied by a small parameter. We refer to Evans' textbook for a general overview of the analysis of non-linear equations [Eva98], and to Kriegl-Michor [KM97] for an approach to general non-linear analysis using convenient locally convex topological vector spaces (including the implicit function theorem) that is conceptually close to ours.

Remark that in this book, we didn't discuss at all the theory of qualitative results on partial differential equations, that started with the three body problem study of Poincaré, and that is now called the theory of dynamical systems. The approach that is the closest to the geometric theory of partial differential equations we discussed is Thom's transversality theorem for jets, generalized by Masur for multijets. One may also quote Gromov's generalization of Smalle's embedding theorem, called the h-principle, that says roughly, that in good cases, a given formal solution to a partial differential system can be deformed to a smooth one.

In this Chapter, we will mostly be interested by the existence of solutions. For this purpose, we introduce in Sections 3.1 and 3.2, abstract notions of functionals and distributions, that are more flexible than the usual ones, because they are also adapted to the non-linear setting. We will thus use them extensively.

3.1 Functionals and functional derivatives

To illustrate the general definitions of spaces we have given in Section 2.1, we will use them to define a flexible notion of functional and distribution on a function space. The idea of distribution is already at the heart of Dirac's work on quantum mechanics [Dir82].

We first recall from Bourbaki [Bou70] the definition of partially defined functions between sets.

Definition 3.1.1. A *span* $f : X \to Y$ between two sets is an arbitrary map $\Gamma_f \to X \times Y$. A *correspondence* $f : X \to Y$ between two sets is a span given by a subset $\Gamma_f \subset X \times Y$. A *partially defined function* $f : X \to Y$ between two sets is a correspondence $\Gamma_f \subset X \times Y$ whose projection on X is injective, with image denoted D_f and called the *domain of definition* of f. The set of correspondences (resp. partially defined functions) $f : X \to Y$ is denoted $\mathrm{Hom}_{\mathrm{COR}}(X, Y)$ (resp. $\mathrm{Hom}_{\mathrm{PAR}}(X, Y)$). The category of spans $f : X \to Y$ is denoted $\mathrm{Mor}^1_{\mathrm{SPAN}}(X, Y)$.

Proposition 3.1.2. *Sets with correspondences (resp. partially defined functions) between them form a category, denoted* COR *(resp.* PAR*). Sets with spans between them form a 2-category, denoted* SPAN.

Proof. For $f : X \to Y$ and $g : Y \to Z$ two spans given by $\Gamma_f \to X \times Y$ and $\Gamma_g \to Y \times Z$, we denote $g \circ f$ the span given by the fiber product $\Gamma_f \times_Y \Gamma_g \subset X \times Y \times Z \to X \times Z$. By construction, one has a natural isomorphisms

$$\Gamma_{(h\circ g)\circ f} \xrightarrow{i_{f,g,h}} \Gamma_{h\circ(g\circ f)},$$

and the universal property of fiber product imply that these isomorphisms fulfill the natural coherence condition to make SPAN a 2-category. For correspondences, the inclusion $R_f \subset X \times Y$ allows one to define composition as the projection in $X \times Z$ of the span composition $\Gamma_{f\circ g}$ of $f : X \to Y$ and $g : Y \to Z$. This reduces that 2-categorical structure to a usual category structure. □

We now define a general notion of functional, that is very close to the notion of functional used by physicists in quantum field theory books: they usually write down a formula without taking too much care of its definition domain.

Definition 3.1.3. Let (LEGOS, τ) be a site and X and Y in $\text{SH}(\text{LEGOS}, \tau)$ be two spaces. A *correspondence* (resp. *partially defined function*) between X and Y is is given by a functorial family

$$f_U : X(U) \to Y(U)$$

of partially defined functions (resp. correspondences) indexed by $U \in \text{LEGOS}$. The *definition domain* of a correspondence f is the sheaf $\underline{\mathrm{D}}_f$ associated to the presheaf $U \mapsto D_f(U)$. For $\mathcal{C} = \text{COR}, \text{PAR}$, the sheaf $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$ is the sheaf associated to the presheaf

$$U \mapsto \mathrm{Hom}_{\mathcal{C}}(X \times U, Y).$$

Sheaves with correspondences and partially defined functions between them form a categories, denoted respectively $\text{COR}(\text{LEGOS}, \tau)$ and $\text{PAR}(\text{LEGOS}, \tau)$.

We now define the notion of functional, that plays an important role in physics.

Definition 3.1.4. Let (LEGOS, τ) be a site, X, Y and Z in $\text{SH}(\text{LEGOS}, \tau)$ be three spaces. A partially defined function (resp. correspondence)

$$f : \underline{\mathrm{Hom}}(X, Y) \to Z$$

is called a *functional* (resp. *multivalued functional*) on the function space $\underline{\mathrm{Hom}}(X, Y)$ with values in Z.

The general geometrical setting presented in Chapter 2 for differential calculus on spaces of functions can be applied directly to functionals.

Definition 3.1.5. Let $\mathbf{C} = (\text{LEGOS}, \tau, \cdots)$ be a differential geometric setting, with category of algebras ALG and category of spaces $\mathrm{Sp} := \text{SH}(\text{ALG}^{op}, \tau)$. Suppose given a free algebra $A[\epsilon]$ in one variable, and denote $\mathbb{D}_k$ the space

$$\mathbb{D}_k(A) := \underline{\mathrm{Spec}}(A[\epsilon]/(\epsilon^{k+1})).$$

The *k-th jet bundle of a space* X is the space

$$\mathrm{Jet}^k(X) := \underline{\mathrm{Hom}}(\mathbb{D}_k, X) \to \mathrm{Jet}^0(X) = X.$$

The *k-th derivative of a span* $f : X \to Y$ is the jet span

$$\mathrm{Jet}^k(f) : \mathrm{Jet}^k(X) \to \mathrm{Jet}^k(Y)$$

with graph $\mathrm{Jet}^k(\Gamma_f) \to \mathrm{Jet}^k(X) \times \mathrm{Jet}^k(Y)$. In particular, the k-th derivative of a partially defined function $f : X \to Y$ has definition domain $\mathrm{Jet}^k(\underline{\mathrm{D}}_f)$.

The jet space of a mapping space may be computed more explicitely.

3.2 Smooth functionals and distributions

Definition 3.2.1. Consider the site $(\mathrm{O}\textsc{pen}_{\mathcal{C}^\infty}, \tau)$ whose objects are open subsets $U \subset \mathbb{R}^k$ for varying k, with morphisms given by smooth maps between them, equipped with its ordinary topology τ. Let $\Omega \subset \mathbb{R}^n$ be an open subset and consider the space $\underline{\mathcal{C}}^\infty(\Omega, \mathbb{R}^m)$ of smooth $\mathbb{R}^m$-valued functions on Ω, whose U-points are given by the set of parametrized smooth functions

$$\underline{\mathcal{C}}^\infty(\Omega, \mathbb{R}^m)(U) := \mathrm{Hom}_{\mathcal{C}^\infty}(\Omega \times U, \mathbb{R}^m).$$

We denote $\underline{\mathcal{C}}^\infty(\Omega) := \underline{\mathcal{C}}^\infty(\Omega, \mathbb{R})$. Let $\underline{\mathbb{R}}$ be the *smooth affine line*, defined by $\underline{\mathbb{R}}(U) := \mathrm{Hom}_{\mathcal{C}^\infty}(U, \mathbb{R})$. A functional

$$f : \underline{\mathcal{C}}^\infty(\Omega, \mathbb{R}^m) \to \underline{\mathbb{R}}$$

is called a *smooth functional*.

Example 3.2.2. Let $\Omega \subset \mathbb{R}^n$ be an open subset. Let $L : \Omega \times \mathbb{R}[x_1, \dots, x_n] \to \mathbb{R}$ be a measurable function (called Lagrangian density) that depends on Ω but also on finitely many multi-indexed coordinates in Ω (formal partial derivatives of the dependent variable). The corresponding smooth (action) functional

$$\begin{array}{rccc} S: & \underline{\mathcal{C}}^\infty(\Omega) & \to & \underline{\mathbb{R}} \\ & \varphi(x,u) & \mapsto & \int_\Omega L(x, \varphi(x,u), \partial_\alpha \varphi(x,u)) dx \end{array}$$

has a definition domain given (for example) by Lebesgue's dominated derivation condition

$$\underline{\mathrm{D}}_S(U) = \left\{ \varphi(x,u) \,\middle|\, \begin{array}{c} \forall \beta, \text{ locally on } U, \text{ there exists } g \in L^1(\Omega) \text{ such that} \\ |\partial_\beta^u L(\varphi(x,u), \partial_\alpha \varphi(x,u))| \leq g(x) \end{array} \right\}.$$

The above example shows that the definition domain of a functional is usually very deeply related to the formula that defines the functional and that it is a hard analyst work to find a *common* definition domain for a large class of functionals. This work can be overcome in linear analysis but becomes really hard in non-linear analysis. General functionals give a natural setting to work with formulas without taking too much care of their definition domain at first sight, but the analyst's work needs to be done at some point to be sure that the definition domain is, at least, non-empty.

We now show that functionals in our sense are stable by the usual limit operation.

Proposition 3.2.3. *Let $\underline{\mathcal{C}}^\infty(\Omega)$ be the function space on an open subset $\Omega \subset \mathbb{R}^k$ and*

$$(f_n : \underline{\mathcal{C}}^\infty(\Omega) \to \underline{\mathbb{R}})_{n\geq 0}$$

be a family of smooth functionals. Then the limit

$$f = \lim_{n\to\infty} f_n : \underline{\mathcal{C}}^\infty(\Omega) \to \underline{\mathbb{R}}$$

is a well defined functional.

Proof. For $U \subset \mathbb{R}^n$ an open subset, we define

$$\underline{\mathrm{D}}_f(U) := \left\{ \varphi \in \underline{\mathcal{C}}^\infty(\Omega)(U) \,\middle|\, \begin{array}{c} \exists N,\ \forall n \geq N,\ \varphi \in \underline{\mathrm{D}}_{f_n}(U) \\ \text{and } \lim_{n\to\infty} f_n(\varphi) \text{ exists} \end{array} \right\}.$$

Then f is a well defined functional with (possibly empty) definition domain $\underline{\mathrm{D}}_f$. □

Proposition 3.2.4. *Let $\Omega \subset \mathbb{R}^n$ be an open subset. A smooth functional*

$$f : \underline{\mathcal{C}}^\infty(\Omega) \to \underline{\mathbb{R}}$$

is uniquely defined by the datum $(f_{\{.\}}, \underline{\mathrm{D}}_f)$ of the corresponding set theoretic (partially defined) function

$$f_{\{.\}} : \mathcal{C}^\infty(\Omega) \to \mathbb{R}$$

and the definition domain $\underline{\mathrm{D}}_f \subset \underline{\mathcal{C}}^\infty(\Omega)$.

Proof. Indeed, for every $u \in U$, there is a unique morphism $\{.\} \to U$ that sends the point to u. The commutativity of the diagram

$$\begin{array}{ccc} \underline{\mathcal{C}}^\infty(\Omega)(U) & \xrightarrow{f_U} & \underline{\mathbb{R}}(U) \\ \downarrow & & \downarrow \\ \underline{\mathcal{C}}^\infty(\Omega)(\{.\}) & \xrightarrow{f_{\{.\}}} & \underline{\mathbb{R}}(\{.\}) \end{array}$$

implies that if $\varphi : \Omega \times U \to \mathbb{R}$ is a point in $\underline{\mathrm{D}}_f(U)$, then one has

$$f_U(\varphi)(u) = f_{\{.\}}(\varphi(-, u))$$

in $\underline{\mathbb{R}}(U) := \mathcal{C}^\infty(U)$, which shows that the values of f are uniquely determined by the values of the ordinary function $f_{\{.\}} : \mathcal{C}^\infty(\Omega) \to \mathbb{R}$. The definition domain is contained in the set

$$\tilde{\underline{\mathrm{D}}}_f(U) = \{\varphi(x, u) | \forall u, \varphi(-, u) \in D_{f_{\{.\}}} \text{ and } u \mapsto f_{\{.\}}(\varphi(x, u)) \in \mathcal{C}^\infty(U)\}.$$

□

We now define a refined class of functionals, using the category of smooth algebras. This class is useful to make differential calculus on functionals a purely algebraic theory, with help of the tools of categorical infinitesimal calculus, described in Section 1.5.

Definition 3.2.5. Consider the site $(\mathrm{ALG}^{fp,op}_{\mathcal{C}^\infty}, \tau)$ whose objects are spectra $\underline{\mathrm{Spec}}(A)$ of finitely presented smooth algebras, equipped with their usual topology τ. For $\Omega \subset \mathbb{R}^n$ an open subset, we denote $\underline{\mathbf{C}}^\infty(\Omega, \mathbb{R}^m)$ the space of smooth $\mathbb{R}^m$-valued functions on Ω, whose A-points are given by the set of parametrized smooth functions

$$\underline{\mathbf{C}}^\infty(\Omega, \mathbb{R}^m)(A) := ((\mathcal{C}^\infty(\Omega) \otimes A)(\mathbb{R}))^n .$$

We denote $\underline{\mathbf{C}}^\infty(\Omega) := \underline{\mathbf{C}}^\infty(\Omega, \mathbb{R})$. Let $\underline{\mathbb{R}}$ be the *refined smooth affine line*, defined by $\underline{\mathbb{R}}(A) := A(\mathbb{R})$. A functional

$$\mathbf{f} : \underline{\mathbf{C}}^\infty(\Omega, \mathbb{R}^m) \to \underline{\mathbb{R}}$$

is called a *refined smooth functional*.

Proposition 3.2.6. *Smooth functionals and refined smooth functionals are related by two natural maps*

$$F : \mathrm{Hom}_{\mathrm{PAR}}(\underline{\mathbf{C}}^\infty(\Omega, \underline{\mathbb{R}}^m), \underline{\mathbb{R}}) \leftrightarrows \mathrm{Hom}_{\mathrm{PAR}}(\underline{\mathcal{C}}^\infty(\Omega, \mathbb{R}^m), \underline{\mathbb{R}}) : G.$$

Proof. It is enough to treat the case $m = 1$. Suppose given a refined smooth functional

$$\mathbf{f} : \underline{\mathbf{C}}^\infty(\Omega, \mathbb{R}) \to \underline{\mathbb{R}}.$$

One can evaluate it on the smooth algebras $A = \mathcal{C}^\infty(U)$, for any $U \subset \mathbb{R}^k$. For such an algebra, the natural isomorphism

$$\mathcal{C}^\infty(\Omega) \otimes \mathcal{C}^\infty(U) \cong \mathcal{C}^\infty(\Omega \times U)$$

of smooth algebras implies that one has the equalities

$$\underline{\mathbf{C}}^\infty(\Omega, \mathbb{R})(A) = \underline{\mathcal{C}}^\infty(\Omega, \mathbb{R})(U) \ \text{ and } \ \underline{\mathbb{R}}(A) = \underline{\mathbb{R}}(U).$$

This thus gives a map $\mathbf{f} \mapsto f$ from refined smooth functionals to usual smooth functionals. Now suppose given a smooth functional

$$f : \underline{\mathcal{C}}^\infty(\Omega, \mathbb{R}) \to \underline{\mathbb{R}}.$$

For a finitely presented smooth algebra A, and a presentation $p : \mathcal{C}^\infty(\mathbb{R}^k) \to A$, the refined functional $\mathbf{f} : \underline{\mathbf{C}}^\infty(\Omega, \mathbb{R}) \to \underline{\mathbb{R}}$ that we are seeking for should give a commutative diagram of partially defined functions

$$\begin{array}{ccc} \mathcal{C}^\infty(\Omega \times \mathbb{R}^k) & \xrightarrow{\mathbf{f}_{\mathbb{R}^k}} & \mathcal{C}^\infty(\mathbb{R}^k) , \\ \downarrow p & & \downarrow p \\ (\mathcal{C}^\infty(\Omega) \otimes A)(\mathbb{R}) & \xrightarrow{\mathbf{f}_A} & A(\mathbb{R}) \end{array}$$

with $\mathbf{f}_{\mathbb{R}^k} = f_{\mathbb{R}^k}$. Suppose first that $\mathbf{f}$ is a fully defined function. The two vertical maps induced by p being surjective, we have to define $\mathbf{f}_A$ by the formula

$$\mathbf{f}_A(x) = p(f_{\mathbb{R}^k}(\tilde{x})),$$

where $\tilde{x} \in p^{-1}(\{x\})$. Remark that any two presentations of A can be refined to another one through a commutative diagram

$$\begin{array}{ccc} \mathcal{C}^\infty(\mathbb{R}^{k_1+k_2}) & \longrightarrow & \mathcal{C}^\infty(\mathbb{R}^{k_1}) \\ \downarrow & & \downarrow \\ \mathcal{C}^\infty(\mathbb{R}^{k_2}) & \longrightarrow & A \end{array} ,$$

so that the above definition does not depend on the chosen presentation for A. Any morphism $h : A \to B$ between smooth algebras induces a commutative diagram

$$\begin{array}{ccc} \mathcal{C}^\infty(\mathbb{R}^{k_1+k_2}) & & \\ \downarrow & \searrow & \\ \mathcal{C}^\infty(\mathbb{R}^{k_1}) & & \mathcal{C}^\infty(\mathbb{R}^{k_2}) \\ \downarrow & & \downarrow \\ A & \xrightarrow{h} & B \end{array}$$

that gives a functoriality morphism $h : \mathbf{f}_A \to \mathbf{f}_B$. We thus have fully defined a refined function

$$\mathbf{f} : \underline{\mathbf{C}}^\infty(\Omega, \mathbb{R}) \to \underline{\mathbb{R}}.$$

If $\mathbf{f}$ is only partially defined, it domain of definition $\underline{\mathrm{D}}_{\mathbf{f}}$ can be computed from the definition domain $\underline{\mathrm{D}}_f$, and is given by the intersection

$$\underline{\mathrm{D}}_{\mathbf{f}}(A) := \bigcap_{p:\mathcal{C}^\infty(\mathbb{R}^k)\to A} p(\underline{\mathrm{D}}_f(\mathbb{R}^k)) \subset (\mathcal{C}^\infty(\Omega) \otimes A)(\mathbb{R})$$

indexed by presentations of A. □

Remark 3.2.7. The above result remains true, with essentially the same proof, if one works with finitely presented non-finitary smooth algebras, defined in Section 2.2, as covariant functors

$$A : \mathrm{O{\scriptsize PEN}}_{\mathcal{C}^\infty} \to \mathrm{S{\scriptsize ETS}}$$

that respect transversal products and final objects. The finite presentation condition on such an algebra A means that there exists $U \subset \mathbb{R}^k$ and a finitely generated ideal $I \subset \mathcal{C}^\infty(U)$ such that $A \cong \mathcal{C}^\infty(U)/I$ as non-finitary smooth algebra.

Definition 3.2.8. Let $\Omega \subset \mathbb{R}^n$ be an open subset. A *distribution* is a smooth $\underline{\mathbb{R}}$-linear functional

$$f : \underline{\mathcal{C}}^\infty(\Omega, \mathbb{R}^m) \to \underline{\mathbb{R}}$$

such that the definition domain of the underlying function $f_{\{.\}}$ contains $\mathcal{C}_c^\infty(\Omega, \mathbb{R}^m)$. It is called compactly supported if it has full definition domain $\underline{\mathrm{D}}_{f_{\{.\}}} = \mathcal{C}^\infty(\Omega, \mathbb{R}^m)$. We denote $\mathcal{C}^{-\infty}(\Omega, \mathbb{R}^m)$ the space of distributions and $\mathcal{C}_c^{-\infty}(\Omega, \mathbb{R}^m)$ the space of compactly supported distributions.

Example 3.2.9. Let $\Omega \subset \mathbb{R}^n$ be an open subset and $f : \Omega \to \mathbb{R}^m$ be a measurable function. Suppose given a non-degenerate symmetric pairing (e.g., the Euclidean or Lorentzian bilinear form)

$$\langle -, - \rangle : \mathbb{R}^m \otimes \mathbb{R}^m \to \mathbb{R}.$$

Then the functional

$$\begin{array}{rccc} f : & \underline{\mathcal{C}}^\infty(\Omega, \mathbb{R}^m) & \to & \underline{\mathbb{R}} \\ & \varphi(x,u) & \mapsto & \int_\Omega \langle f(x), \varphi(x,u) \rangle dx \end{array}$$

is a distribution with definition domain given (for example) by Lebesgue's dominated derivation condition

$$\underline{\mathrm{D}}_f(U) = \left\{ \varphi(x,u) \left| \begin{array}{c} \text{locally on } U, \text{ there exists } g(x) \in L^1(\Omega) \text{ such that} \\ \text{for all } u, \ |\langle f(x), \varphi(x,u) \rangle| \le g(x) \end{array} \right. \right\}.$$

Example 3.2.10. Let $\Omega \subset \mathbb{R}^n$ be an open subset and $x_0 \in \Omega$ be a fixed point. Then the functional

$$\begin{array}{rccc} \delta_{x_0} : & \underline{\mathcal{C}}^\infty(\Omega) & \to & \underline{\mathbb{R}} \\ & \varphi(x,u) & \mapsto & \varphi(x_0,u) \end{array}$$

is a compactly supported distribution with full definition domain $\underline{\mathcal{C}}^\infty(\Omega)$.

We now compare our approach to distributions and functionals to the usual approach that uses functional analytic methods and topological vector spaces. Our setting is much more flexible since we are completely free of choosing definition domains for the functionals we use, but the comparison is however useful to relate our work to what is usually done in mathematical physics books. We will also need these results to give our mathematical formalization of causal perturbative quantum field theories in Chapter 21.

Proposition 3.2.11. *A distribution $f : \underline{\mathcal{C}}^\infty(\Omega, \mathbb{R}^m) \to \underline{\mathbb{R}}$ is an ordinary distribution if its definition domain contains the subspace $\underline{\mathcal{C}}^\infty_p(\Omega, \mathbb{R}^m) \subset \underline{\mathcal{C}}^\infty_c(\Omega, \mathbb{R}^m)$ of properly supported functions, defined by*

$$\underline{\mathcal{C}}^\infty_p(\Omega, \mathbb{R}^m)(U) := \left\{ \varphi(x,u) \left| \begin{array}{c} \textit{locally on } U, \\ \textit{there exists a compact subset } K \subset \Omega \textit{ such that} \\ \textit{for all } u, \ \mathrm{supp}(\varphi(-,u)) \subset K \end{array} \right. \right\}$$

Proof. A distribution is an $\mathbb{R}$-linear continuous function on the topological vector space $\mathcal{C}^\infty_c(\Omega)$ equipped with the topology given by the family of seminorms

$$N_{K,k}(\varphi) := \sum_{|\alpha| \le k} \sup_{x \in K} |\partial_\alpha \varphi(x)|$$

indexed by compact subsets $K \subset \Omega$ and integers $k \in \mathbb{N}$. A properly supported function $\varphi(x,u)$ defines a smooth function between open subsets of locally convex vector spaces

$$\varphi : U \to \mathcal{C}^\infty_c(\Omega)$$

and its composition with a continuous linear (thus smooth) functional $f : \mathcal{C}^\infty_c(\Omega) \to \mathbb{R}$ defines a smooth function $f \circ \varphi : U \to \mathbb{R}$. This shows that if f is an ordinary distribution, then f gives a well defined natural transformation $f : \underline{\mathcal{C}}_p^\infty(\Omega, \mathbb{R}^m) \to \underline{\mathbb{R}}$. Conversely, if f is a distribution in our sense, it defines a linear function $f_{\{.\}} : \mathcal{C}_c^\infty(\Omega) \to \mathbb{R}$ that is automatically continuous because it sends smooth functions $\varphi : U \to \mathcal{C}_c^\infty(\Omega)$ to smooth functions $f(\varphi) : U \to \mathbb{R}$. Indeed, if $\varphi_n : \Omega \to \mathbb{R}$ is a sequence of smooth maps that converges in the topological vector space $\mathcal{C}_c^\infty(\Omega)$, one can find a smooth family $\psi : \mathbb{R}\times\Omega \to \mathbb{R}$ in $\underline{\mathcal{C}}_p^\infty(\Omega, \mathbb{R}^m)(\mathbb{R})$ and a family of points $u_n \in \mathbb{R}$ converging to 0 such that

$$\psi(-, u_n) = \varphi_n(-) \text{ and } \psi(-, 0) = \lim_n \varphi_n(-).$$

The functoriality of f in all inclusions $\{.\} \subset \mathbb{R}$ implies that

$$\lim_n f(\varphi_n) = \lim_n f(\psi(u_n)) = \lim_{t\to 0} f(\psi(t)) = f(\psi(0)) = f(\lim_n \varphi_n),$$

which finishes the proof. □

Definition 3.2.12. Let $E \to M$ be a vector bundle over a smooth variety. Let $\underline{\Gamma}_p(M^n, E^n) \subset \underline{\Gamma}_c(M^n, E^n)$ be the subspace given by properly supported sections. The space of *distributional polynomials* is defined by

$$\mathcal{O}^{poly}_{\Gamma(M,E)} := \oplus_{n\geq 0}\mathrm{Hom}_{\underline{\mathbb{R}}}(\underline{\Gamma}_p(M^n, E^n), \mathbb{R})^{S_n}.$$

The space of *distributional formal power series* is defined by

$$\mathcal{O}^{formal}_{\Gamma(M,E)} := \prod_{n\geq 0} \mathrm{Hom}_{\underline{\mathbb{R}}}(\underline{\Gamma}_p(M^n, E^n), \mathbb{R})^{S_n}.$$

Proposition 3.2.13. *The natural map*

$$\begin{array}{ccc} \mathcal{O}^{poly}_{\Gamma(M,E)} & \to & \mathcal{O}_{\Gamma(M,E)} := \mathrm{Hom}_{\mathrm{PAR}}(\underline{\Gamma}(M,E), \underline{\mathbb{R}}) \\ \sum_{n\geq 0} F_n & \mapsto & [s : M \to E] \mapsto \sum_{n\geq 0} F_n(s \times \cdots \times s) \end{array}$$

is injective and compatible with the natural algebra structures on both sides. Let $\mathcal{O}_0 \subset \mathcal{O}_{\underline{\Gamma}(M,E)}$ be the algebra of functionals on $\underline{\Gamma}(M,E)$ defined in $0 \in \Gamma(M,E)$, $\mathfrak{m}_0 \subset \mathcal{O}_0$ be the ideal of functions that annihilate at 0. Denote

$$\widehat{\mathcal{O}}_0 := \lim \mathcal{O}_0/\mathfrak{m}_0^n$$

the corresponding completion. The natural map

$$\mathcal{O}^{formal}_{\Gamma(M,E)} \to \widehat{\mathcal{O}}_0$$

is injective and compatible with the algebra structures on both sides.

Proof. The injectivity of the first map follows from the result of linear algebra that says that a multilinear function on a module over a ring of characteristic 0 is uniquely determined by the corresponding homogeneous functional (e.g., a bilinear form is uniquely determined by the corresponding quadratic form). To be more precise, we follow Douady [Dou66] by defining, for any function $h : X \to Y$ between two vector spaces,

$$\Delta_x(h)(y) := \frac{1}{2}(h(y + x) - h(y - x)).$$

If $f : X \to Y$ is a polynomial function of degree n associated to the multilinear map $\tilde{f} : X^{\otimes n} \to Y$, one has

$$\tilde{f}(x_1, \dots, x_n) = \frac{1}{n!}\Delta_{x_n} \circ \dots \circ \Delta_{x_1} f(0).$$

The algebra $\mathcal{O}^{formal}$ is obtained as a completion of $\mathcal{O}^{poly}$ with respect to the ideal $\mathfrak{n}_0 = \oplus_{n\geq 0}\mathrm{Hom}_{\underline{\mathbb{R}}}(\underline{\mathrm{D}}_n^{dist}, \mathbb{R})^{S_n}$ of multilinear functionals that annihilate at 0. One can thus check the injectivity at the finite level, in which case it follows from the definition of $\mathfrak{n}_0$ and $\mathfrak{m}_0$. □

We illustrate the above proposition by defining a notion of analytic function on a function space that is very similar to the one used by Douady to define Banach manifolds [Dou66].

Definition 3.2.14. Let $E \to M$ be a vector bundle. A functional $f : \underline{\Gamma}(M, E) \to \underline{\mathbb{R}}$ is called *distributionally analytic* if for every point $x \in \underline{\Gamma}(M, E)(U)$, there exists an open subspace $U_x \subset \underline{\Gamma}(M, E)$ that contains x and a formal functional

$$G = \sum_n G_n \in \mathcal{O}^{formal}_{\Gamma(U_x, E_{|U_x})}$$

such that f is the sum of the corresponding absolutely converging series:

$$f_{|U_x} = \sum_{n\geq 0} G_n.$$

To conclude this section, we hope that the reader is now convinced that our approach to functional calculus is compatible with the usual functional analytic one, that is based on topological vector spaces. We need however more flexibility, in particular for fermionic and graded calculus, so that we will stick all along to our methods.

3.3 Generalized weak solutions

Let $\Omega \subset \mathbb{R}^n$ be an open subset and denote $X = \underline{\mathcal{C}}^\infty(\Omega, \mathbb{R})$. Recall that the space of functionals $\mathcal{O}_X$ is the space $\underline{\mathrm{Hom}}_{\mathrm{PAR}}(X, \underline{\mathbb{R}})$ of partially defined functions

$$f : X \to \underline{\mathbb{R}}.$$

In this setting, one can easily define a notion of weak solution to a (not necessarily linear) problem.

Definition 3.3.1. The space

$$X^{**} := \underline{\mathrm{Hom}}_{\mathrm{PAR}}(\mathcal{O}_X, \underline{\mathbb{R}})$$

is called the *nonlinear bidual* of X. Let $P : X \to X$ be an operator (morphism of spaces). An element $F \in X^{**}$ is called a *weak solution* to the equation $\{P = 0\}$ if it fulfills the equation

$$F(f(P(\varphi))) = 0.$$

Example 3.3.2. Let $P(D) = \sum_\alpha a_\alpha \partial_\alpha$ be a linear partial differential operator on $\mathcal{C}^\infty(\Omega)$ with smooth coefficients $a_\alpha \in X$. A distributional weak solution f to $P(D)f = 0$ is a distribution $f : X \to \underline{\mathbb{R}}$ (as defined in Section 3.2) such that for all test function $\varphi \in X$, one has

$$\langle f, {}^t P(D)\varphi\rangle = 0,$$

where ${}^tP(D).\varphi := \sum_\alpha (-1)^\alpha \partial_\alpha(a_\alpha.\varphi)$ is the transpose operator for the standard integration pairing

$$\begin{array}{rccc} \int : & X \otimes_{\underline{\mathbb{R}}} X & \to & \underline{\mathbb{R}} \\ & (\varphi, \psi) & \mapsto & \int_\Omega \varphi(x)\psi(x)dx. \end{array}$$

One can interpret such a solution as a weak solution in our sense by using the integration pairing to define an embedding $\iota : X \hookrightarrow \mathcal{O}_X$ by

$$\langle \iota(\varphi), \psi\rangle := \int_\Omega \varphi(x)\psi(x)dx.$$

One can then identify ordinary distributions (i.e., linear functionals $f : X \to \underline{\mathbb{R}}$) with particular elements of X^{**} (i.e., functionals on $\mathcal{O}_X$) with definition domain $\iota(X) \subset \mathcal{O}_X$. This gives an embedding

$$\iota : \mathcal{C}^{-\infty}(\Omega) \hookrightarrow X^{**}.$$

One then checks that a weak solution that is $\underline{\mathbb{R}}$-linear with definition domain contained in $\iota(\mathcal{C}^{-\infty}(\Omega))$ is the same as a distributional weak solution. One can also use different bilinear pairings on X and this gives other notions of linear weak solution. The above construction also works with $\mathbb{R}^m$-valued distributions.

We now define the Fourier transform, that is a construction specific to the affine space $\Omega = \mathbb{R}^n$.

Definition 3.3.3. Let $\langle -, -\rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a given non-degenerate bilinear pairing. If $f : \mathbb{R}^n \to \mathbb{R}$ is an L^1 function, its *Fourier transform* is defined as the function

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-2i\pi\langle x,\xi\rangle} f(x)dx.$$

The *Fourier transform of a distribution* $f : \underline{\mathcal{C}}^\infty(\Omega) \to \underline{\mathbb{R}}$ is the distribution $\hat{f}$ given by the formula

$$\langle \hat{f}, \varphi\rangle = \langle f, \hat{\varphi}\rangle,$$

with definition domain given by

$$\underline{\mathrm{D}}_{\hat{f}}(U) = \{\varphi(x, u),\ e^{-2i\pi\langle x,\xi\rangle}\varphi(x) \in \underline{\mathrm{D}}_f(U)\}.$$

The main property of Fourier transform is that it replaces differential operators by multiplication operators.

Proposition 3.3.4. *If $P(\xi) = \sum_{|\alpha|\leq k} a_\alpha x^\alpha$ is a polynomial on $\mathbb{R}^n$ with complex coefficients, we define a differential operator*

$$P(D) = \sum_{|\alpha|\leq k} a_\alpha D^\alpha$$

with $D^\alpha = D_1^{\alpha_1}\dots D_n^{\alpha_n}$ *and* $D_i = \frac{\langle x_i, x_i\rangle}{2i\pi}\partial_{x_i}$. *We then have*

$$\langle \widehat{P(D)f}, \varphi\rangle = \langle P(\xi)\hat{f}, \varphi\rangle.$$

The space $\mathcal{C}_c^\infty(\Omega)$ is not stable by Fourier transform, meaning that the Fourier transform of a function with compact support need not be with compact support. (in quantum mechanical terms, it means that you can not measure at the same time the position and the impulsion of a given particle). We now define Schwartz type distributions, whose definition domain remains stable by the operation of derivation and Fourier transform.

Definition 3.3.5. The *standard Schwartz space* is the subspace $\underline{\mathcal{S}}(\mathbb{R}^n) \subset \underline{\mathcal{C}}^\infty(\mathbb{R}^n)$ defined by

$$\underline{\mathcal{S}}(\mathbb{R}^n)(U) := \left\{ \varphi(x,u) \;\middle|\; \begin{array}{c} \text{locally on } U, \text{ for all } \alpha, \beta, \\ \text{there exists } C_{\alpha,\beta,U} \text{ such that} \\ \forall u \in U,\ \|x^\alpha \partial_\beta \varphi(-,u)\|_\infty < C_{\alpha,\beta,U} \end{array} \right\}.$$

A distribution whose definition domain contains $\underline{\mathcal{S}}(\mathbb{R}^n)$ is called a *Schwartz distribution.* The space of Schwartz distributions is denoted $\mathcal{S}'(\mathbb{R}^n)$.

The space of Schwartz distributions is the continuous dual $\mathcal{S}'(\mathbb{R}^n)$ of the topological vector space $\mathcal{S}(\mathbb{R}^n) := \underline{\mathcal{S}}(\mathbb{R}^n)(\{.\})$ for the topology induced by the family of seminorms

$$N_k(\varphi) := \sum_{|\alpha|,|\beta|\le k} \sup_{x\in\mathbb{R}^n} |x^\beta \partial_\alpha(x)|.$$

It is stable by the Fourier transform. There are natural inclusions

$$\mathcal{C}_p^\infty(\mathbb{R}^n) \subset \underline{\mathcal{S}}(\mathbb{R}^n) \subset \underline{\mathcal{C}}^\infty(\mathbb{R}^n) \text{ and } \mathcal{C}_c^{-\infty}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{C}^{-\infty}(\mathbb{R}^n).$$

We finish this section by a general formulation of the Schwartz kernel theorem (see Schwartz's reference book [Sch66]). The properties of operators between functional spaces

$$P : \mathcal{C}_c^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n)$$

defined by a kernel function $K \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, i.e., by a formula of the form

$$P(f) = \int_{\mathbb{R}^n} K(x,y) f(x) dx,$$

can be read on the properties of its kernel. Schwartz's kernel theorem shows that essentially any reasonable operator is of this kind, if one uses distributions as kernels.

Theorem 3.3.6. *Let* $\Omega_1 \subset \mathbb{R}^n$ *and* $\Omega_2 \subset \mathbb{R}^m$ *be two open subsets. Let* $P : \mathcal{C}_c^\infty(\Omega_1) \to \mathcal{C}^{-\infty}(\Omega_2)$ *be a continuous operator. Then there exists a distribution* $K \in \mathcal{C}^{-\infty}(\Omega_1 \times \Omega_2)$ *such that*

$$\langle Pf, \varphi\rangle = \langle K, f \otimes \varphi\rangle.$$

More generally, let $P : \mathcal{C}_c^\infty(\Omega_1, \mathbb{R}^l) \to \mathcal{C}^{-\infty}(\Omega_2, \mathbb{R}^k)$ *be a continuous operator. Then there exists a distribution* $K \in \mathcal{C}^{-\infty}(\Omega_1 \times \Omega_2, (\mathbb{R}^l)^* \boxtimes \mathbb{R}^k)$ *such that*

$$\langle Pf, \varphi\rangle = \langle K, f \otimes \varphi\rangle.$$

Remark that there is a complex analytic approach to distributions due to Sato, called the theory of hyperfunctions. We refer to chapter 10 for an introduction to hyperfunction theory and other methods of linear algebraic analysis.

3.4 Spectral theory

The formalism of quantum mechanics (to be explained in Chapter 16), being mainly based on the spectral theory of operators on Hilbert spaces, we here recall the main lines of it, referring to von Neumann's book [vN96] for more details. This theory is a kind of generalization to infinite dimension of the theory of diagonalization of hermitian matrices.

Definition 3.4.1. A *Hilbert space* is a pair $\mathcal{H} = (\mathcal{H}, \langle ., . \rangle)$ composed of a topological vector space $\mathcal{H}$ and a sesquilinear pairing $\langle ., . \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that $\mathcal{H}$ is complete for the norm $\|f\| = \sqrt{\langle f, f \rangle}$.

The main objective of spectral theory is the study of the spectrum of an operator on a Hilbert space.

Definition 3.4.2. If $A : \mathcal{H} \to \mathcal{H}$ is an operator (i.e., a linear map), its *spectrum* $\mathrm{Sp}(A)$ is the subset of $\mathbb{C}$ defined by

$$\mathrm{Sp}(A) = \{\lambda \in \mathbb{C} | \, A - \lambda.\mathrm{Id} \text{ non invertible}\}.$$

Definition 3.4.3. If $A : \mathcal{H} \to \mathcal{H}$ is an operator, its *adjoint* A^* is defined by

$$\langle Af, g \rangle = \langle f, A^* g \rangle.$$

We now define various classes of operations on the Hilbert space $\mathcal{H}$.

Definition 3.4.4. Let $A : \mathcal{H} \to \mathcal{H}$ be a linear operator.

1. One says that A is *bounded* if there exists a constant $C \geq 0$ such that

   $$\|Ax\| \leq C\|x\|,$$

   and a minimal such constant C will be denoted $\|A\|$.

2. One says that A is *positive* (bounded) if $\mathrm{Sp}(A) \subset [0, +\infty[\subset \mathbb{C}$.

3. One says that A is *compact* if the image of the unit ball $B(0,1) = \{f \in \mathcal{H} | \, \|f\| \leq 1\}$ of $\mathcal{H}$ by A has compact closure, i.e.,

   $$\overline{A(B(0,1))} \text{ compact.}$$

4. One says that A is *self-adjoint* if $A^* = A$.

5. One says that A is *a projection* if A is bounded and

   $$A = A^* = A^2.$$

We denote $\mathcal{B}(\mathcal{H})$ (resp. $\mathcal{B}(\mathcal{H})^+$, resp. $\mathcal{K}(\mathcal{H})$, resp. $\mathrm{Proj}(\mathcal{B}(\mathcal{H}))$, resp. $\mathcal{B}^{sa}(\mathcal{H})$) the set of bounded (resp. positive, resp. compacts, resp. projection, resp. bounded self-adjoint) operators on $\mathcal{H}$. If $A, B \in \mathcal{B}(\mathcal{H})$ are two bounded operators, one says that A *is smaller than* B if $B - A$ is positive, i.e.,

$$A \leq B \Leftrightarrow B - A \in \mathcal{B}(\mathcal{H})^+.$$

Theorem 3.4.5. *Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded operator. One has an inclusion* $\mathrm{Sp}(A) \subset \mathbb{R}$ *is and only if A is self-adjoint. If $A \in \mathcal{K}(\mathcal{H})$ is compact then* $\mathrm{Sp}(A) \subset \mathbb{C}$ *is discrete.*

Proof. See Reed-Simon [RS80], Theorem VI.8. □

Definition 3.4.6. We denote Borel(Sp(A)) the Borel σ-algebra on Sp(A). A *spectral measure* for a bounded operator $A \in \mathcal{B}(\mathcal{H})$ is a multiplicative measure on the spectrum of A with values in the space of projectors. More precisely, this is a map

$$\begin{array}{rccc} E: & \mathrm{Borel}(\mathrm{Sp}(A)) & \to & \mathrm{Proj}(\mathcal{B}(\mathcal{H})) \\ & B & \mapsto & E(B) \end{array}$$

such that

1. $E(\emptyset) = 0$,
2. $E(\mathrm{Sp}(A)) = 1$,
3. E is sigma-additive, i.e.,
$$E\left(\coprod_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} E(B_i),$$
4. E is multiplicative, i.e., $E(B \cap C) = E(B).E(C)$.

The spectral theorem, that is a generalization to infinite dimension of the diagonalization theorem of symmetric or hermitian endomorphisms, can be formulated as follows.

Theorem 3.4.7. *Let $A \in \mathcal{B}(\mathcal{H})$ be a bounded self-adjoint operator. There exists a unique spectral measure*

$$E_A : \mathrm{Borel}(\mathrm{Sp}(A)) \to \mathrm{Proj}(\mathcal{B}(\mathcal{H}))$$

such that

$$A = \int_{\mathrm{Sp}(A)} \lambda dE_A(\lambda).$$

Proof. This theorem is also true for an unbounded self-adjoint operator (densely defined) and its proof can be found in Reed-Simon [RS80], Chapters VII.2 et VIII.3. □

The main interest of this theorem is that it gives a measurable functional calculus, that allows for example to evaluate a measurable positive function $f : \mathbb{R} \to \mathbb{R}_+$ on a given bounded self-adjoint operator A by

$$f(A) = \int_{\mathrm{Sp}(A)} f(\lambda) dE_A(\lambda).$$

This type of computation is very useful in quantum mechanics for the characteristic functions $\mathbb{1}_{[a,b]} : \mathbb{R} \to \mathbb{R}$, that allow to restrict to parts of the spectrum contained in an interval.

One also has the following theorem:

Theorem 3.4.8. *If two operators A and B of $\mathcal{B}(\mathcal{H})$ commute, there exists an operator $R \in \mathcal{B}(\mathcal{H})$ and two measurable functions $f, g : \mathbb{C} \to \mathbb{C}$ such that*

$$A = f(R) \text{ et } B = g(R).$$

The Stone theorem allows one to translate the study of unbounded self-adjoint operators to the study of one parameter groups of unitary operators. A proof can be found in [RS80], Chapter VIII.4.

Theorem 3.4.9 (Stone's theorem). *The datum of a strongly continuous one parameter family of unitary operators on $\mathcal{H}$, i.e., of a group morphism $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ fulfilling*

$$\text{if } \varphi \in \mathcal{H} \text{ and } t \to t_0 \text{ in } \mathbb{R}, \text{ then } U(t)\varphi \to U(t_0)\varphi,$$

is equivalent with the datum of an self-adjoint operator A, by the map $A \mapsto e^{itA}$.

3.5 Green functions and parametrices

Definition 3.5.1. Let $P(D) = \sum_\alpha a_\alpha \partial_\alpha$ be a partial differential operator on $\Omega \subset \mathbb{R}^n$.

- A *Green function* (also called a *fundamental solution*) for the operator $P(D)$ is a distribution f on Ω^2 such that

$$P(D)f(x, y) = \delta_x(y)$$

  whenever $x, y \in \Omega$.

- A *parametrix* for the operator $P(D)$ is a distribution f on Ω^2 such that

$$P(D)f(x, y) = \delta_x(y) + \omega(x, y)$$

  where ω is a smooth function on Ω^2.

A Green function on $\Omega = \mathbb{R}^n$ is called a fundamental solution because of the following result: if g is a function that can be convoluted with the fundamental solution f (for example, if g has compact support), one can find a solution to the inhomogeneous equation

$$P(D)h = g$$

by simply putting $h = f * g$ to be the convolution of the fundamental solution and g.

Example 3.5.2. In Euclidean field theory on flat space, linear trajectories are related to the solutions and eigenvalues of the Laplacian $\Delta = \sum_{i=1}^n \partial_{x_i}^2$. This operator corresponds to the polynomial $\|x\|^2 = \sum_{i=1}^n x_i^2$. Using the formula

$$\int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}_+} \int_{S^{n-1}} f(x.\omega) r^{n-1} dr d\omega,$$

that is always true for a positive function, one gets that the function $\frac{1}{\|x\|^2}$ is locally integrable if $n > 2$ so that it defines a distribution D. One can check that it is tempered and that its inverse Fourier transform f is a fundamental solution of the Laplacian.

Example 3.5.3. The first model of electromagnetic waves is given by the wave operator

$$\frac{1}{c^2}\partial_t^2 - \sum_{i=2}^{n} \partial_{x_i}^2.$$

Its solutions are quite different of those of the Laplacian. One first remarks that the corresponding polynomial $\frac{1}{c^2}t^2 - \sum_{i=2}^{n} x_i^2$ has real zeroes so that it cannot be naively inverted as a locally integrable function. The easiest way to solve the wave equation is to make the space Fourier-transform (on variable x) that transform it in ordinary differential operator

$$\frac{1}{c^2}\partial_t^2 - \|x\|^2.$$

One can also use the complex family of distributions parametrized by a real nonzero constant ϵ given by the locally integrable function

$$\frac{\mathbb{1}_X}{-\frac{1}{c^2}t^2 + \sum_{i=2}^{n} x_i^2 + i\epsilon}$$

where X is the light cone (where the denominator is positive for $\epsilon = 0$). The Fourier transform of the limit of this distribution for $\epsilon \to 0$ gives a fundamental solution of the wave equation called the Feynman propagator. We refer to Chapter 10 for more details on this complex analytic approach to distributions, called algebraic analysis.

3.6 Complex analytic regularizations

The Sato-Bernstein polynomial gives an elegant solution to the following problem of Gelfand: if P is a real polynomial in n variable with positive coefficients, show that the distribution

$$\langle P^s, f\rangle := \int_{\mathbb{R}^n} P(x)^s f(x)dx,$$

defined and holomorphic in s for $\mathrm{Re}(s) > 0$, can be meromorphically continued to the whole complex plane.

One can try to answer this for $P(x) = x^2$. An integration by parts gives

$$(2s+1)(2s+2)\langle x^{2s}, f\rangle = \int_{\mathbb{R}} x^{2(s+1)} f"(x)dx =: \langle \partial_x^2 x^{2(s+1)}, f\rangle.$$

This functional equation gives the desired meromorphic continuation to the complex plane with poles in the set $\{-1/2, -1, -3/2, -2, \dots\}$.

The generalization of this method to any real polynomial with positive coefficients can be done using a similar functional equation, that is given by the following theorem due to Sato (in the analytic setting) and Bernstein (in the algebraic setting).

Theorem 3.6.1. *If $f(x)$ is a no-zero polynomial in several variables with positive real coefficients, there is a non-zero polynomial $b(s)$ and a differential operator $P_x(s)$ with polynomial coefficients such that*

$$P_x(s).f(x)^{s+1} = b(s)f(x)^s.$$

Proof. A full and elementary proof of this theorem can be found in the memoir of Chadozeau and Mistretta [CM00]. One may also refer to the original paper of Bernstein [Ber71] and to Kashiwara's book [Kas03], Chapter 6. □

As an example, if $f(x) = x_1^2 + \cdots + x_n^2$, one has

$$\sum_{i=1}^{n} \partial_i^2 f(x)^{s+1} = 4(s+1)\left(s+\frac{n}{2}\right) f(x)^s$$

so the Bernstein-Sato polynomial is

$$b(s) = (s+1)\left(s+\frac{n}{2}\right).$$

This gives a method to compute the fundamental solution of the Laplacian.

More generally, if $f(x) = \sum_{i=1}^{n} a_i x_i^2$ is a quadratic form on $\mathbb{R}^n$, and $f^{-1}(D) = \sum_{i=1}^{n} \frac{1}{a_i}\partial_i^2$ is the second order differential operator corresponding to the inverse of the given quadratic form, one has

$$f^{-1}(D).f(x)^{s+1} = 4(s+1)\left(s+\frac{n}{2}\right) f(x)^s.$$

This gives a method to compute the fundamental solution of the wave equation.

Theorem 3.6.2 (Malgrange-Ehrenpreis)**.** *Let $P(D)$ be a differential operator in $\mathbb{R}^n$ with constant coefficients. There exists a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$P(D)f = \delta_0.$$

Proof. By using the Fourier transform

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n),$$

we reduce the problem to finding a distributional inverse to the polynomial $P \in \mathbb{C}[x_1, \ldots, x_n]$ that defines the differential operator $P(D)$. Indeed, the Fourier transformed equation of $P(D)f = \delta_0$ is

$$P(x).\mathcal{F}(f) = 1.$$

If the given polynomial has no real zeroes, its inverse is continuous on $\mathbb{R}^n$ and thus locally integrable. The corresponding distribution solves the problem. Otherwise, we need more efforts. Replacing P by $P.\bar{P}$ reduces the question to a polynomial P with positive coefficients. One can then apply the Malgrange-Ehrenpreis theorem 3.6.2 to P to get a non-zero polynomial $b(s)$ and a differential operator $P_x(s)$ with polynomial coefficients such that

$$P_x(s).f(x)^{s+1} = b(s)f(x)^s.$$

This gives meromorphic continuation of the Schwartz distribution $P(x)^s$. If we take the constant term of the Laurent series of this meromorphic function at -1, we get a distribution D that fulfills

$$P(x).D = 1,$$

so that the problem is solved. □

The Malgrange-Ehrenpreis theorem is very elegant but one can often find fundamental solutions more easily by other methods.

3.7 Heat kernel regularization

We refer to [BGV92] for a detailed study of heat kernels. The heat kernel $K_\tau(x,y)$ on a Riemannian manifold (M,g) is the solution of the Heat equation

$$\partial_\tau K_\tau(x,y) = \Delta_g K_\tau(x,y),$$

where Δ_g is the Laplace-Beltrami operator, subject to the boundary condition

$$\lim_{\tau\to 0} K_\tau(x,y) = \delta_x(y).$$

Using the functional calculus presented in Section 3.4, one can also write the corresponding integral operator $\varphi(y) \mapsto (T_\tau\varphi)(y) := \int K_\tau(x,y)\varphi(x)dx$ as

$$T_\tau = e^{\tau\Delta}.$$

One can use this to write down the heat kernel presentation of the green function P for the Laplacian operator with mass $\Delta_g + m^2$ as the integral

$$P(x,y) = \int_0^\infty e^{\tau m^2} K_\tau(x,y)d\tau.$$

The heat kernel regularization is simply given by cutting off the above integral

$$P_{\epsilon,L} = \int_\epsilon^L e^{-\tau m^2 K_\tau} d\tau.$$

It is a smooth function that approximates the Laplacian propagator.

3.8 Wave front sets of distributions and functionals

The wave front set is a refinement in the cotangent space of the singular support of a distribution (domain where a distribution is not smooth). It gives information on the propagation of singularities, and gives precise conditions for the existence of pullbacks (and in particular products) of distributions.

We refer to Hormander [Hör03], Chap. VIII for a concise presentation of the smooth version of wave front sets. A more analytic approach, including a complete overview of algebraic microlocal analysis is given in chapter 10.

The definition of the wave front set is based on the fact that one can analyse the regularity of a Schwartz function by studying the increasing properties of its Fourier transform.

Definition 3.8.1. Let M be a manifold and $u \in \mathcal{C}^{-\infty}(M)$ be a distribution on M. One says that u is *microlocally smooth near a given point* $(x_0, \xi_0) \in T^*M$ if there exists a smooth cutoff function $\varphi \in \mathcal{C}_c^\infty(M)$, with $\varphi(x_0) \neq 0$, such that the Fourier transform $\widehat{(\varphi u)}$

(in a given chart U of M containing the support of φ) is of rapid decrease at infinity in a small conic neighborhood of ξ_0, i.e, there exist $\epsilon > 0$ and C_ϵ such that

$$\forall N > 0,\ \forall \xi,\ \left\| \frac{\xi}{\|\xi\|} - \frac{\xi_0}{\|\xi_0\|} \right\| < \epsilon \Rightarrow\ \|\xi\|^N \widehat{(\varphi u)}(\xi) \leq C_\epsilon.$$

The *wave front set* of u is the set $\mathrm{WF}(u)$ of points of T^*M at which f is not microlocally smooth.

The projection of $\mathrm{WF}(u)$ to M is the singular support of u (points around which u is not equal to a smooth function).

Theorem 3.8.2. *Let $f : N \to M$ be a smooth map between smooth manifolds and u be a distribution on M. Suppose that*

$$\mathrm{WF}(u) \cap T_N^* M = \emptyset,$$

i.e., no point of $\mathrm{WF}(u)$ is normal to $df(TN)$. Then the pullback f^*u of u to a distribution on N can be defined such that $\mathrm{WF}(f^*u) \subset f^*\mathrm{WF}(u)$. This pullback is the unique way to extend continuously the pullback of functions to distributions subject to the above wavefront condition.

Proof. We only give a sketch of proof, refering to Hormander [Hör03], Chap VIII for more details. Let $u_i \to u$ be sequence of smooth functions converging to u. One defines the pullback of u_i by $f^*u_i := u_i \circ f$. The wave front set condition ensures that f^*u_i converges to a well defined distribution f^*u on N. □

Corollary 3.8.3. *There is a unique way to extend continuously the product of smooth functions to a product $(u, v) \mapsto u.v$ on distributions subject to the wave front condition*

$$\mathrm{WF}(u) \cap \mathrm{WF}(v)^o = \emptyset,$$

where $\mathrm{WF}(v)^o := \{(x, \xi) \in T^*M,\ \text{such that } (x, -\xi) \in \mathrm{WF}(v)\}$.

Proof. The product is defined as the pullback of the tensor product $u \otimes v \in \mathcal{C}^{-\infty}(M \times M)$ along the diagonal. The result then follows from the estimation of the wave front set of the tensor product

$$\begin{array}{c} \mathrm{WF}(u \otimes v) \\ \cap \\ (\mathrm{WF}(u) \times \mathrm{WF}(v)) \cup ((\mathrm{supp}\ u \times \{0\}) \times \mathrm{WF}(v)) \cup (\mathrm{WF}(u) \times (\mathrm{supp}\ v) \times \{0\}). \end{array}$$

□

Chapter 4

Linear groups

This chapter mostly contains very classical results, and is included for later references in the description of the physical examples of Chapter 13, 14 and 15, because these classical results are a bit disseminated in the mathematical litterature.

Symmetry considerations are nowadays an important tool to explain mathematically most of the theories of modern physics.

We start here by introducing linear algebraic and Lie groups through their functor of points, following a path between SGA 3 [DG62] and Waterhouse's book [Wat79]. Our presentation of Lie groups gives here a concrete illustration of the interest of the notion of smooth algebra for basic differential geometric constructions. We also use extensively the book of involutions [KMRT98] because algebras with involutions give the smoothest approach to the structural theory of algebraic groups over non algebraically closed fields (e.g., over $\mathbb{R}$).

4.1 Generalized algebraic and Lie groups

Definition 4.1.1. Let $\textsc{Alg}$ and C be two categories and suppose given a faithful functor

$$\mathrm{Forget} : C \to \textsc{Sets}$$

that makes us think of C as a category of sets with additional structures. A functor

$$F : \textsc{Alg} \to C$$

is called a *C-structured functor* on $\textsc{Alg}$. If $\textsc{Alg}$ is the category $\textsc{Alg}_K$ of commutative algebras over a field K, we call F an algebraic C-object functor.

Example 4.1.2. Let $\textsc{Alg} = \textsc{Alg}_{\mathbb{R}}$ be the category of real algebras and $C = \textsc{Ass}$ be the category of associative unital algebras. The functor

$$\begin{array}{cccc} \mathrm{M}_n : & \textsc{Alg}_{\mathbb{R}} & \to & \textsc{Ass} \\ & A & \mapsto & (\mathrm{M}_n(A), \times, +, \underline{0}, \mathrm{id}) \end{array}$$

makes M_n an algebraic associative algebra functor. The functor $\mathrm{GL}_n : \textsc{Alg}_{\mathbb{R}} \to \textsc{Grp}$ defined by

$$\mathrm{GL}_n(A) := \{(M, N) \in M_n(A) \times M_n(A) |\, MN = NM = I\}$$

makes GL_n an algebraic group functor. Both these functors are representable respectively by the algebras

$$A_{\mathrm{M}_n} := \mathbb{R}\left[\{M_{i,j}\}_{i,j=1,\dots,n}\right]$$

and

$$A_{\mathrm{GL}_n} := \mathbb{R}\left[\{M_{i,j}\}_{i,j=1,\dots,n}, \{N_{i,j}\}_{i,j=1,\dots,n}\right]/(MN = NM = I),$$

meaning that there are natural bijections

$$\mathrm{M}_n(A) \cong \mathrm{Hom}_{\textsc{Alg}_{\mathbb{R}}}(A_{\mathrm{M}_n}, A) \text{ and } \mathrm{GL}_n(A) \cong \mathrm{Hom}_{\textsc{Alg}_{\mathbb{R}}}(A_{\mathrm{GL}_n}, A).$$

Example 4.1.3. Let $\textsc{Alg} = \textsc{Open}_{\mathcal{C}^\infty}^{op}$ be opposite to the category $\textsc{Open}_{\mathcal{C}^\infty}$ of open subsets U of $\mathbb{R}^n$ for varying n with smooth maps between them. The functor

$$\begin{array}{rccc} \mathcal{C}^\infty : & \textsc{Open}_{\mathcal{C}^\infty}^{op} & \to & \textsc{Alg}_{\mathbb{R}} \\ & U & \mapsto & \mathcal{C}^\infty(U) \end{array}$$

allows us to define

$$\mathrm{M}_n^{sm} : \textsc{Open}_{\mathcal{C}^\infty}^{op} \to \textsc{Ass} \text{ and } \mathrm{GL}_n^{sm} : \textsc{Open}_{\mathcal{C}^\infty}^{op} \to \textsc{Grp}$$

called the smooth matrix algebra functor and the smooth general linear group functor (also called the general linear Lie group). One can also show that both these functors are representable respectively by $U_{\mathrm{M}_n} = \mathbb{R}^{n^2}$ and by the open subspace $U_{\mathrm{GL}_n} = \det^{-1}(\mathbb{R}^*) \subset U_{\mathrm{M}_n}$.

Example 4.1.4. The smooth special linear group $\mathrm{SL}_n^{sm} : \textsc{Open}_{\mathcal{C}^\infty}^{op} \to \textsc{Grp}$ defined by

$$\mathrm{SL}_n^{sm}(U) := \{g \in \mathrm{M}_n(U),\ \det(g) = 1\}$$

is not clearly representable because it corresponds to a closed subset of $\mathbb{R}^{n^2}$. A better setting to work with it is given by the category of (finitary) smooth algebras defined in Example 2.2.1. In this setting, the functor

$$\mathrm{SL}_n^{sm} : \textsc{Alg}_{\mathcal{C}^\infty}^{fin} \to \textsc{Grp}$$

defined by

$$\mathrm{SL}_n^{sm}(A) := \{g \in \mathrm{M}_n(A),\ \deg(g) = 1\}$$

is representable by the smooth algebra

$$\mathcal{C}^\infty(\mathrm{SL}_n) = \mathcal{C}^\infty(\mathrm{M}_n)/(\det(M) = 1).$$

Similarly, the functor

$$\mathrm{GL}_n^{sm} : \textsc{Alg}_{\mathcal{C}^\infty}^{fin} \to \textsc{Grp}$$

is representable by the smooth algebra

$$\mathcal{C}^\infty(\mathrm{GL}_n) = \mathcal{C}^\infty(\mathrm{M}_n^2)/(MN = NM = I).$$

The setting of smooth algebras thus gives a better analogy between algebraic and Lie groups.

We now give two examples of group functors that are not representable. This kind of group is of common use in physics, where it is called a (generalized) gauge group.

Example 4.1.5. Let $\Omega \subset \mathbb{R}^d$ be an open subset and consider the group functor

$$\begin{array}{rccc} \underline{\mathrm{Aut}}(\Omega): & \mathrm{OPEN}_{\mathcal{C}^\infty}^{op} & \to & \mathrm{GRP} \\ & U & \mapsto & \left\{ \begin{array}{c} \varphi : U \times \Omega \to \Omega \text{ smooth such that} \\ \varphi(u,-) : \Omega \to \Omega \text{ diffeomorphism } \forall u \in U \end{array} \right\} \end{array}$$

describing diffeomorphisms of Ω. It is not representable because of its infinite dimensionality. Such diffeomorphism groups play an important role as symmetry groups in generally covariant theories, like Einstein's general relativity.

Example 4.1.6. Let $G = \mathrm{GL}_n^{sm}$ be the general linear Lie group and $\Omega \subset \mathbb{R}^p$ be an open subset. The functor

$$\begin{array}{rccc} \underline{\mathrm{Hom}}(\Omega, G): & \mathrm{OPEN}_{\mathcal{C}^\infty}^{op} & \to & \mathrm{SETS} \\ & U & \mapsto & \{\varphi : U \times \Omega \to U_{\mathrm{GL}_n} \text{ smooth}\} \end{array}$$

is naturally equipped with a group functor structure induced by that of GL_n^{sm}. This kind of group play an important role as symmetry groups in Yang-Mills gauge theories, like Maxwell's electromagnetism.

To finish this section, we remark that the setting of smooth algebras allows us to give a completely algebraic treatment of linear Lie groups. We will thus only present the theory of algebraic matrix groups, since it is enough for our needs. We however finish this section by defining a notion of affine smooth group, that contains Lie groups but is a bit more flexible and is the perfect analog of affine group schemes in the smooth setting.

Definition 4.1.7. A *smooth affine group* is a representable functor

$$G : \mathrm{ALG}_{\mathcal{C}^\infty} \to \mathrm{GRP}$$

from the category of (finitary) smooth algebras (see Example 2.2.1) to the category of groups. More precisely, there exists a smooth algebra A_G and for any smooth algebra A, a natural isomorphism

$$G(A) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{ALG}_{\mathcal{C}^\infty}}(A_G, A).$$

A *Lie group* is a smooth affine group whose smooth algebra of functions is formally smooth and finitely presented.

Example 4.1.8. The universal covering $\widetilde{\mathrm{SL}_2}$ of SL_2 is a smooth affine group that may not be described as a subgroup of the general linear group on a given vector space. In particular, it is an example of a non-algebraic smooth affine group, all of whose faithful linear representations are infinite dimensional.

4.2 Affine group schemes

The reader who wants to have a partial and simple summary can use Waterhouse's book [Wat79]. For many more examples and a full classification of linear algebraic groups on non-algebraically closed fields (like $\mathbb{R}$ for example, that is useful for physics), we refer to the very complete book of involutions [KMRT98].

Definition 4.2.1. A *linear algebraic group* over a field K is an algebraic group functor that is representable by a quotient of a polynomial ring, i.e., it is a functor

$$G : \textsc{Alg}_K \to \textsc{Grp}$$

and a function algebra A_G such that for any algebra A, there is a natural bijection

$$G(A) \tilde{\to} \mathrm{Hom}_{\textsc{Alg}_K}(A_G, A).$$

One can show that any linear algebraic group G (with finitely generated algebra of functions A_G) is a Zariski closed subgroup of the group $\mathrm{GL}_{n,\mathbb{R}}$ for some n that we defined in the previous section by

$$\mathrm{GL}_n(A) := \{(M,N) \in M_n(A) \times M_n(A) \cong A^{n^2} \times A^{n^2} |\, MN = NM = I\}.$$

This finite dimensional representation is actually obtained as a sub-representation of the regular representation

$$\rho_{reg} : G \to \mathrm{GL}(A_G)$$

of G on its algebra of functions. For example, the additive group $\mathbb{G}_a$ whose points are given by

$$\mathbb{G}_a(A) = A$$

with its additive group structure is algebraic with function algebra $K[X]$ and it can be embedded in GL_2 by the morphism

$$\begin{array}{cccc} r: & \mathbb{G}_a & \to & \mathrm{GL}_2 \\ & a & \mapsto & \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right). \end{array}$$

Remark that if V is a real vector space (without a particular choice of a basis, that would break the symmetry, as physicists use to say), one can also define $\mathrm{GL}(V)$ by

$$\mathrm{GL}(V)(A) := \{(M,N) \in \mathrm{End}_A(V \otimes A)^2 |\, MN = NM = \mathrm{id}_V\}.$$

A representation of an algebraic group, also called a G-module, is a group morphism

$$G \to \mathrm{GL}(V)$$

where V is a finite dimensional vector space.

Definition 4.2.2. The *Lie algebra* of the group G is the space

$$\begin{array}{cccc} \mathrm{Lie}(G): & \textsc{Alg}_K & \to & \textsc{Grp} \\ & B & \mapsto & \{X \in \mathrm{Hom}_{\textsc{Alg}_K}(A_G, B[\epsilon]/(\epsilon^2)) | X \mod \epsilon = I\}. \end{array}$$

Remark 4.2.3. In the setting of smooth Lie groups, the same definition works by replacing $K[\epsilon]/(\epsilon^2)$ by the smooth algebra $\mathbb{C}^\infty(\mathbb{R})/(\epsilon^2)$ where ϵ is the standard coordinate function on $\mathbb{R}$.

One can think of elements of the Lie algebra as points $I + \epsilon X \in G(B[\epsilon]/(\epsilon^2))$ where X plays the role of a tangent vector to G at the identity point. There is a natural action of G on its Lie algebra called the *adjoint action*

$$\rho_{ad} : G \to \mathrm{GL}(\mathrm{Lie}(G))$$

defined by

$$g \mapsto \left[X \in G(R[\epsilon]/(\epsilon^2)) \mapsto gXg^{-1} \in G(R[\epsilon]/(\epsilon^2))\right].$$

The derivative of this action at identity gives a natural map

$$\begin{array}{rccc} \mathrm{ad}: & \mathrm{Lie}(G) & \to & \mathrm{End}_K(\mathrm{Lie}(G)) \\ & X & \mapsto & [Y \mapsto \mathrm{ad}(X)(Y)] \end{array}$$

called the *adjoint action* of the Lie algebra on itself. This allows to define the Lie bracket by

$$\begin{array}{rccc} [.,.]: & \mathrm{Lie}(G) \times \mathrm{Lie}(G) & \to & \mathrm{Lie}(G) \\ & (X,Y) & \mapsto & [X,Y] := \mathrm{ad}(X)(Y). \end{array}$$

The symmetric bilinear form

$$\begin{array}{rccc} \langle -,- \rangle: & \mathrm{Lie}(G) \times \mathrm{Lie}(G) & \to & \mathrm{Lie}(G) \\ & (X,Y) & \mapsto & \mathrm{Tr}(\mathrm{ad}(X)\mathrm{ad}(Y)) \end{array}$$

is called the *Killing form*. The group G is called *semisimple* if this form is non-degenerate. It is also equipped with a module structure over the ring space $\mathbb{A}$ given by the affine space $\mathbb{A}(B) = B$ for which the bracket is linear.

4.3 Algebras with involutions and algebraic groups

The study of algebraic groups on non-algebraically closed fields (for example, unitary groups over $\mathbb{R}$) is easier to do in the setting of algebras with involutions because one can do there at once a computation that works for all classical groups. For example, the computation of Lie algebras is done only once for all classical groups (orthogonal, symplectic, linear and unitary) in this section. We refer to the book of involutions [KMRT98] for a very complete account of this theory.

On non-algebraically closed fields, algebraic groups are defined using a generalization of matrix algebras called central simple algebras, and involutions on them.

Definition 4.3.1. Let K be a field. A *central simple algebra* A over K is a finite dimensional nonzero algebra A with center K which has no two-sided ideals except $\{0\}$ and A. An *involution* on (a product of) central simple algebra(s) A is a map $\sigma : A \to A$ such that for all $x, y \in A$,

1. $\sigma(x+y) = \sigma(x) + \sigma(y)$,
2. $\sigma(xy) = \sigma(y)\sigma(x)$,
3. $\sigma^2 = \mathrm{id}$.

The center K of A is stable by σ and $\sigma_{|K}$ is either an automorphism of order 2 or the identity. In the first case, K is of degree 2 over the fixed subfield L.

Theorem 4.3.2 (Wedderburn)**.** *Let A be a finite dimensional algebra over K. The following conditions are equivalent:*

1. *A is a central simple algebra over K.*
2. *There exists a finite extension K'/K on which A becomes a matrix algebra, i.e., such that*
$$A \otimes_K K' \cong \mathrm{M}_n(K).$$

Corollary 4.3.3. *Let A be a central simple algebra of dimension n^2 over K. Let*

$$i : A \to \mathrm{End}_K(A)$$

be the action of A on itself by left multiplication. The characteristic polynomial of $i(a)$ is the n-th power of a polynomial

$$P_{A,a} = X^n - \mathrm{Tr}_A(a)X^{n-1} + \cdots + (-1)^n \mathrm{Nm}_A(a)$$

of degree n called the reduced characteristic polynomial. The coefficients Tr_A and Nm_A are respectively called the reduced norm and reduced trace on A.

Before giving some examples of algebras with involutions, we define the corresponding algebraic groups.

Definition 4.3.4. Let (A, σ) be an algebra (that is a product of central simple algebras) with involution over (a product of fields) K.

1. The group $\mathrm{Isom}(A, \sigma)$ of *isometries* of (A, σ) is defined by
$$\mathrm{Isom}(A, \sigma)(R) = \{a \in A_R^\times |\, a.\sigma_R(a) = 1\}.$$
2. The group $\mathrm{Aut}(A, \sigma)$ of *automorphisms* of (A, σ) is defined by
$$\mathrm{Aut}(A, \sigma)(R) = \{\alpha \in \mathrm{Aut}_R(A_R) | \alpha \circ \sigma_R = \sigma_R \circ \alpha\}.$$
3. The group $\mathrm{Sim}(A, \sigma)$ of *similitudes* of (A, σ) is defined by
$$\mathrm{Sim}(A, \sigma)(R) = \{a \in A_R^\times | a.\sigma(a) \in K_R^\times\}.$$

The first example of algebra with involution is the one naturally associated to a non-degenerate bilinear form $b : V \times V \to K$ on an K-vector space. One defines an algebra with involution (A, σ_b) by setting $A = \mathrm{End}_K(V)$ and the involution σ_b defined by

$$b(x, f(y)) = b(\sigma_b(f)(x), y)$$

for $f \in A$ and $x, y \in V$. If the bilinear form b is symmetric, the corresponding groups are the classical groups

$$\mathrm{Iso}(A, \sigma) = \mathrm{O}(V, b),\ \mathrm{Sim}(A, \sigma) = \mathrm{GO}(V, b),\ \mathrm{Aut}(A, \sigma) = \mathrm{PGO}(V, b).$$

If the bilinear form b is antisymmetric, the corresponding groups are

$$\mathrm{Iso}(A, \sigma) = \mathrm{Sp}(V, b),\ \mathrm{Sim}(A, \sigma) = \mathrm{GSp}(V, b),\ \mathrm{Aut}(A, \sigma) = \mathrm{PGSp}(V, b).$$

Another instructive example is given by the algebra

$$A = \mathrm{End}_K(V) \times \mathrm{End}_K(V^\vee)$$

where V is a finite dimensional vector space over K, equipped with the involutions sending (a, b) to $(b^\vee, a^\vee)$. It is not a central simple algebra but a product of such over $K \times K$ and $\sigma_{|K}$ is the exchange involution. One then gets

$$\begin{array}{ccc} \mathrm{GL}(V) & \xrightarrow{\sim} & \mathrm{Iso}(A, \sigma) \\ m & \mapsto & (m, (m^{-1})^\vee) \end{array}$$

and

$$\mathrm{PGL}(V) \cong \mathrm{Aut}(A, \sigma).$$

If A is a central simple algebra over a field K with σ-invariant subfield $L \subset K$ of degree 2, one gets the unitary groups

$$\mathrm{Iso}(A, \sigma) = \mathrm{U}(V, b),\ \mathrm{Sim}(A, \sigma) = \mathrm{GU}(V, b),\ \mathrm{Aut}(A, \sigma) = \mathrm{PGU}(V, b).$$

It is then clear that by scalar extension to K, one gets an isomorphism

$$U(V, b)_K \cong \mathrm{GL}(V_K)$$

by the above construction of the general linear group. One also defines $\mathrm{SU}(V, q)$ as the kernel of the reduced norm map

$$\mathrm{Nm} : \mathrm{U}(V, b) \to K^\times.$$

We now compute at once all the classical Lie algebras.

Proposition 4.3.5. *Let (A, σ) be a central simple algebra with involution. We have*

$$\mathrm{Lie}(\mathrm{Sim}(A, \sigma)) = \{m \in A,\ m + \sigma(m) = 0\}.$$

Proof. If $1 + \epsilon m \in \mathrm{Sim}(A, \sigma)(K[\epsilon]/(\epsilon^2))$ is a generic element equal to 1 modulo ϵ, the equation

$$(1 + m\epsilon)(1 + \sigma(m)\epsilon) = 0$$

implies $a + \sigma(a) = 0$. □

The above proposition gives us the simultaneous computation of the Lie algebras of $\mathrm{U}(V, b)$, $\mathrm{O}(V, b)$ and $\mathrm{Sp}(V, b)$.

4.4 Clifford algebras and spinors

This section is treated in details in the book of involutions [KMRT98], in Deligne's notes on spinors [Del99] and in Chevalley's book [Che97]. The physical motivation for the study of the Clifford algebra is the necessity to define a square root of the Laplacian (or of the d'Alembertian) operator, as is nicely explained in Penrose's book [Pen05], Section 24.6.

4.4.1 Clifford algebras

Let (V, q) be a symmetric bilinear space over a field K.

Definition 4.4.1.1. The *Clifford algebra* $\mathrm{Cliff}(V, q)$ on the bilinear space (V, q) is the universal associative and unitary algebra that contains V and in which q defines the multiplication of vectors in V. More precisely, it fulfills the universal property

$$\mathrm{Hom}_{\mathrm{ALG}_K}(\mathrm{Cliff}(V, q), B) \cong \{j \in \mathrm{Hom}_{\mathrm{VECT}_K}(V, B) | j(v).j(w) + j(w).j(v) = q(v, w).1_B\}$$

for every associative unitary K-algebra B.

One defines explicitely $\mathrm{Cliff}(V, q)$ as the quotient of the tensor algebra $T(V)$ by the bilateral ideal generated by expression of the form

$$m \otimes n + n \otimes m - q(m, n).1 \text{ with } m, n \in V.$$

Remark that the degree filtration on the tensor algebra $T(V)$ induces a natural filtration F on the Clifford algebra whose graded algebra

$$\mathrm{gr}^F(\mathrm{Cliff}(V, q)) = \wedge^* V$$

is canonically isomorphic to the exterior algebra. One can extend this isomorphism to a linear isomorphism

$$\wedge^* V \xrightarrow{\sim} \mathrm{Cliff}(V, q)$$

by showing that the linear maps $f_k : \wedge^k V \to \mathrm{Cliff}(V, q)$ defined by

$$f_k(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}\ (\sigma) v_{\sigma(1)} \dots v_{\sigma(k)}$$

induces the above isomorphism of graded algebras.

Consequently, one has $\dim_K(\mathrm{Cliff}(V, q)) = 2^{\dim_K V}$. The automorphism $\alpha : V \to V$ given by $m \mapsto -m$ induces an automorphism $\alpha : \mathrm{Cliff}(V) \to \mathrm{Cliff}(V)$ such that $\alpha^2 = \mathrm{id}$. One thus gets a decomposition

$$\mathrm{Cliff}(V, q) = \mathrm{Cliff}^0(V, q) \oplus \mathrm{Cliff}^1(V, q),$$

giving $\mathrm{Cliff}(V, q)$ a super-algebra structure.

Proposition 4.4.1.2. *Let (V, q) be an even dimensional quadratic space. The Clifford algebra $\mathrm{Cliff}(V, q)$ is a central simple algebra and $\mathrm{Cliff}^0(V, q)$ is either central simple over a quadratic extension of K or a product of two central simple algebras over K.*

Proof. The general structure theorem for even Clifford algebras can be found in the book of involution [KMRT98], page 88. We describe here explicitely the case of hyperbolic quadratic spaces. The general case follows from the two following facts:

1. every even dimensional quadratic space over an algebraically closed field is an orthogonal sum of a hyperbolic quadratic space and of a non-degenerate quadratic space.
2. if $(W, q) = (V_1, q_1) \perp (V_2, q_2)$ is an orthogonal sum, the natural morphism

$$C(V_1, q_2) \otimes_g C(V_2, q_2) \to C(W, q)$$

is an isomorphism of graded algebras (where the tensor product is the graded algebra tensor product).

A quadratic space (V, q) with V of dimension $2n$ is called hyperbolic if it is isomorphic to a hyperbolic quadratic space of the form $(\mathbb{H}(U), q_h)$, with U a subspace of dimension n, U^* its linear dual and $\mathbb{H}(U) = U \oplus U^*$ its hyperbolic quadratic space equipped with the bilinear form $q_h(v \oplus \omega) = \omega(v)$. Denote

$$S := \wedge^* U.$$

For $u + \varphi \in \mathbb{H}(U)$, let ℓ_u be the left exterior multiplication by u on S and d_φ be the unique derivation on S extending φ, given by

$$d_\varphi(x_1 \wedge \cdots \wedge x_r) = \sum_{i=1}^{r} (-1)^{i+1} x_1 \wedge \dots \hat{x_i} \wedge x_r \varphi(x_i).$$

One shows that the map

$$\begin{array}{ccc} \mathbb{H}(U) & \to & \mathrm{End}(\wedge^* U) \\ \varphi + u & \mapsto & \ell_u + d_\varphi \end{array}$$

extends to an isomorphism

$$\mathrm{Cliff}(V, q) \xrightarrow{\sim} \mathrm{End}(S).$$

This isomorphism is actually an isomorphism

$$\mathrm{Cliff}(V, q) \xrightarrow{\sim} \underline{\mathrm{End}}(S)$$

of super-algebras. Moreover, if we denote

$$S^+ := \wedge^{2*} U \text{ and } S^- := \wedge^{2*+1} U$$

the even and odd parts of S, the restriction of this isomorphism to $\mathrm{Cliff}^0(V, q)$ induces an isomorphism

$$\mathrm{Cliff}^0(V, q) \xrightarrow{\sim} \mathrm{End}_K(S^+) \times \mathrm{End}_K(S^-)$$

which can be seen as the isomorphism

$$\mathrm{Cliff}^0(V, q) \xrightarrow{\sim} \underline{\mathrm{End}}^0(S)$$

induced by the above isomorphism on even parts of the algebras in play. □

Remark 4.4.1.3. The fact that $\mathrm{gr}^F \mathrm{Cliff}(V,q) \cong \wedge^* V$ can be interpreted by saying that the Clifford algebra is the canonical quantization of the exterior algebra. Indeed, the Weyl algebra (of polynomial differential operators) also has a filtration whose graded algebra is the polynomial algebra, and this is interpreted by physicists as the fact that the Weyl algebra is the canonical quantization of the polynomial algebra. More precisely, if V is a vector space, and ω is a symplectic 2-form on V (for example, the fiber of a cotangent bundle with its canonical 2-form), one can identify the algebra of algebraic differential operators on V with the algebra that fulfills the universal property

$$\mathrm{Hom}_{\mathrm{ALG}_{\mathbb{R}}}(\mathcal{D}_V, B) \cong \{j \in \mathrm{Hom}_{\mathrm{VECT}_{\mathbb{R}}}(V, B) | j(v).j(w) - j(w).j(v) = \omega(v,w).1_B\}.$$

In this case, the graded algebra

$$\mathrm{gr}^F \mathcal{D}_V \cong \mathrm{Sym}^*(V)$$

is identified with the polynomial algebra $\mathrm{Sym}^*(V)$. The above isomorphism can be extended to a linear isomorphism

$$\mathcal{D}_V \to \mathrm{Sym}^*(V)$$

by using the symmetrization formula

$$a_1 \dots a_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} a_{\sigma(1)} \dots a_{\sigma(n)}.$$

The corresponding product on $\mathrm{Sym}^*(V)$ is called the Moyal product. A combination of these two results can be given in the setting of Clifford super-algebras with respect to complex valued super-quadratic forms. The introduction of Clifford algebras in the definition of the Dirac operator and in fermionic field theory is thus, in this sense, equivalent to the use of classical anti-commuting coordinates on fermionic bundles.

Remark 4.4.1.4. The above analogy between differential operators and elements of the Clifford algebra can be used in geometry, as is explained in the paper of Getzler [Get83] on the super proof of the Atiyah-Singer index theorem. Let M be a differential manifold. One can think of differential forms $\Omega^*(M)$ as functions on the super-bundle $T[1]M$, and the Clifford algebra of $TM \oplus T^*M$, being isomorphic to $\mathrm{End}(\wedge^* T^*M)$, gives operators on the state space $\Omega^*(M)$. This situation is analogous to the Weyl quantization of the symplectic manifold (T^*M, ω) by the action of differential operators $\mathcal{D}(M)$ on $L^2(M)$.

4.4.2 Spin group and spinorial representations

Definition 4.4.2.1. The *Clifford group* of (V,q) is the group

$$\Gamma(V,q) := \{c \in \mathrm{Cliff}(V,q)^\times | cv\alpha(c)^{-1} \in v \text{ for all } v \in V\}.$$

One has by definition a natural action (by conjugation) of this group on V that induces a morphism

$$\Gamma(V,q) \to \mathrm{O}(V,q).$$

The special Clifford group of (V,q) is the subgroup $\Gamma^+(V,q)$ of even elements in $\Gamma(V,q)$ given by

$$\Gamma^+(V,q)=\Gamma(V,q)\cap \mathrm{Cliff}^0(V,q).$$

The tensor algebra $T(V)$ is naturally equipped with an anti-automorphism given by $m_1\otimes\cdots\otimes m_k\mapsto m_k\otimes\cdots\otimes m_1$ on the homogeneous elements of degree k. Since the defining ideal for the Clifford algebra is stable by this anti-automorphism (because q is symmetric), one gets an anti-automorphism of $\mathrm{Cliff}(V,q)$ called the transposition and denoted $c\mapsto^t c$.

Definition 4.4.2.2. The *spinor group* is the subgroup of $\Gamma^+(V,q)$ defined by

$$\mathrm{Spin}(V,q)=\{c\in\Gamma^+(V,q)|\,^tcc=1\}.$$

One has a natural action (by multiplication) of $\mathrm{Spin}(V,q)$ on the real vector space $\mathrm{Cliff}(V,q)$ that commutes with the idempotent $\alpha:\mathrm{Cliff}(V,q)\to\mathrm{Cliff}(V,q)$ and thus decomposes in two representations $\mathrm{Cliff}^0(V,q)$ and $\mathrm{Cliff}^1(V,q)$.

Definition 4.4.2.3. Let (V,q) be an even dimensional quadratic space. A rational representation S of $\mathrm{Spin}(V,q)$ will be called

1. a *rational spinorial representation* if it is a $\mathrm{Cliff}(V,q)$-irreducible submodule of the action of $\mathrm{Cliff}(V,q)$ on itself.

2. a *rational semi-spinorial representation* if it is a $\mathrm{Cliff}^0(V,q)$-irreducible submodule of the action of $\mathrm{Cliff}^0(V,q)$ on itself.

If K is algebraically closed and $(V,q)=(\mathbb{H}(U),q_h)$ is the hyperbolic quadratic space, one has $\mathrm{Cliff}(V,q)\cong\mathrm{End}(\wedge^*U)$, so that the spinorial representation is simply $\wedge^*U$ and the semi-spinorial representations are $S^+=\wedge^{2*}U$ and $S^-=\wedge^{2*+1}U$.

If (V,q) is the Lorentzian space, given by $q(t,x)=-c^2t^2+x_1^2+x_2^2+x_3^2$ on Minkowski space $V=\mathbb{R}^{3,1}$, the algebra $\mathrm{Cliff}^0(V,q)$ is isomorphic to the algebra $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}M_{2,\mathbb{C}}$ of complex matrices viewed as a real algebra. The group $\mathrm{Spin}(3,1):=\mathrm{Spin}(V,q)$ then identifies to the group $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{SL}_{2,\mathbb{C}}$ of complex matrices of determinant 1 seen as a real algebraic group. There is a natural representation of $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}M_{2,\mathbb{C}}$ on the first column $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{C}^2$ of $\mathrm{Cliff}^0(V,q)\cong\mathrm{Res}_{\mathbb{C}/\mathbb{R}}M_{2,\mathbb{C}}$ and this is the representation of $\mathrm{Spin}(3,1)$ that we will call the real spinor representation. Its extension to $\mathbb{C}$ decomposes in two representations isomorphic to $\mathbb{C}^2$ that are exchanged by complex conjugation. These two representations are called the semi-spinorial representations and correspond in physics to the electron and the anti-electron.

For a more concrete and motivated approach to these matters, the reader is advised to read Section 24.6 of Penrose's book [Pen05].

4.4.3 Pairings of spinorial representations

Let (V,q) be an even dimensional quadratic space. Remark that the natural action of the vector space $V\subset\mathrm{Cliff}(V,q)$ on a rational spinorial representation S of $\mathrm{Cliff}(V,q)$ induces a morphism of $\mathrm{Spin}(V,q)$-representations

$$V \otimes S \to S.$$

This pairing will be useful to describe the Dirac operator on the spinor bundle.

If (V,q) is a quadratic form over an algebraically closed field, one has two semi-spinorial representations S^+ and S^- of $\mathrm{Spin}(V,q)$ and the above morphism induces two morphisms of $\mathrm{Spin}(V,q)$ representations

$$V \otimes S^{\pm} \to S^{\mp}$$

which are also useful to define Dirac operators between semi-spinorial bundles. More general semi-spinorial representations have to be studied on a case by case basis (see [Del99], part I, for a more complete description).

We now define the two natural pairings that are necessary to the description of the Dirac Lagrangian in Section 14.2.

Proposition 4.4.3.1. *Let (V,q) be an even dimensional quadratic space over an algebraically closed field. The spinorial representation S is naturally equipped with three pairings*

$$\epsilon : S \otimes S \to K,$$

and

$$\Gamma : S \otimes S \to V,\ \tilde{\Gamma} : S \otimes S \to V.$$

Proof. The form $\epsilon : S \otimes S \to K$ is the non-degenerate bilinear form on S for which

$$\epsilon(vs,t) = \epsilon(s,vt)$$

for all $v \in V$. It is the bilinear form whose involution on

$$\underline{\mathrm{End}}(S) \cong \mathrm{Cliff}(V,q)$$

is the standard anti-involution of $\mathrm{Cliff}(V,q)$. The Clifford multiplication map

$$c : V \otimes S \to S$$

induces a morphism $c : S^\vee \otimes S \to V^\vee$, that gives a morphism

$$\Gamma := q^{-1} \circ c \circ (\epsilon^{-1} \otimes \mathrm{id}_S) : S \otimes S \to V.$$

One the defines $\tilde{\Gamma}$ by

$$\tilde{\Gamma} := \Gamma \circ (\epsilon \otimes \epsilon) : S^\vee \otimes S^\vee \to V.$$

□

We now give the real Minkowski version of the above proposition, which is useful to define the super Poincaré group, and whose proof can be found in [Del99], theorem 6.1.

Proposition 4.4.3.2. *Let (V,q) be a quadratic space over $\mathbb{R}$ of signature $(1,n-1)$. Let S be an irreducible real spinorial representation of $\mathrm{Spin}(V,q)$. The commutant Z of $S_{\mathbb{R}}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.*

1. *Up to a real factor, there exists a unique symmetric morphism* $\Gamma : S \otimes S \to V$. *It is invariant under the group* Z^1 *of elements of norm* 1 *in* Z.

2. *For* $v \in V$, *if* $Q(v) > 0$, *the form* $(s,t) \mapsto q(v, \Gamma(s,t))$ *on* S *is positive or negative definite.*

4.5 General structure of linear algebraic groups

We refer to the Grothendieck-Demazure seminar [DG62] for the general theory of root systems in linear algebraic groups and for the proof of the classification theorem for reductive groups.

Definition 4.5.1. An algebraic group is called:

1. *reductive* if the category of its representations is semi-simple, i.e., every exact sequence
$$0 \to U \to V \to W \to 0$$
of G-module has an equivariant splitting, i.e., induces a direct sum decomposition
$$V \cong W \oplus U.$$

2. *unipotent* if it is a successive extension of additive groups.

3. *a torus* if after extending it to a (here the) finite extension $\mathbb{C}$ of $\mathbb{R}$, by $A_{G_\mathbb{C}} := A_G \otimes_\mathbb{R} \mathbb{C}$, it becomes isomorphic to GL_1^n (one says the torus splits on $\mathbb{C}$).

Let T be a torus. Its character group is the functor

$$\begin{array}{rccc} X^*(T): & \textsc{Alg}_\mathbb{R} & \to & \textsc{Grab} \\ & B & \mapsto & \mathrm{Hom}(T_B, \mathrm{GL}_{1,B}) \end{array}$$

with values in the category of Grab of abelian groups and its cocharacter group is the functor

$$\begin{array}{rccc} X_*(T): & \textsc{Rings} & \to & \textsc{Grab} \\ & B & \mapsto & \mathrm{Hom}(\mathrm{GL}_{1,B}, T_B). \end{array}$$

One has a perfect pairing
$$\langle .,. \rangle : X_*(T) \times X^*(T) \to \underline{\mathbb{Z}}$$
given by $(r,s) \mapsto r \circ s$ with $\underline{\mathbb{Z}}$ the functor that associates to an algebra $A = \prod A_i$ product of simple algebras A_i the group $\mathbb{Z}^i$.

Lemma 4.5.2. *If* $T = \mathrm{GL}_1^n$, *a representation* $T \to \mathrm{GL}(V)$ *decomposes in a sum of characters:* $V \cong \oplus \chi_i$, *with* $\chi_i \in X^*(T)$.

Proposition 4.5.3. *Let* G *be a reductive group. The family of sub-tori* $T \subset G$ *has maximal elements that one calls maximal tori of* G.

The first structure theorem of linear algebraic groups gives a “dévissage” of general group in unipotent and reductive groups.

Theorem 4.5.4. *Let P be a linear algebraic group. There exists a biggest normal unipotent subgroup R_uP of P called the unipotent radical of P and the quotient P/R_uP is reductive.*

To give a more detailed study of the structure of reductive groups, one introduce new invariants called the roots.

Definition 4.5.5. Let G be a reductive group and T be a maximal torus of G. The *roots* of the pair (G, T) are the (non-trivial) weights of T in the adjoint representation of G on its Lie algebra. More precisely, they form a subspace of the space $X^*(T)$ of characters whose points are

$$R^* = \{\chi \in X^*(T) |\, \rho_{ad}(t)(X) = \chi(t).X\}.$$

The *coroots* $R_* \subset X_*(T)$ are the cocharacters of T that are dual to the roots with respect to the given perfect pairing between characters and cocharacters. The quadruple $\Phi(G, T) = (X_*(T), R_*, X^*(T), R^*, \langle ., .\rangle)$ is called the *root system* of the pair (G, T).

The main theorem of the classification theory of reductive group is that the root system determines uniquely the group and that every root system comes from a reductive group (whose maximal torus splits). Actually, the root system gives a system of generators and relations for the points of the give algebraic group. More precisely, every root correspond to a morphism

$$x_r : \mathbb{G}_a \to G$$

that is given by exponentiating the corresponding element of the Lie algebra, so that it fulfills

$$tx_r(a)t^{-1} = x_r(r(t)a)$$

and the group is generated by the images of these morphisms and by its maximal torus, the relations between them being given by the definition of a root. Actually, for any root $r \in R$, there is a homomorphism $\varphi_r : \mathrm{SL}_2 \to G$ such that

$$\varphi_r \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) = x_r(a) \text{ and } \varphi_r \left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right) = x_{-r}(a),$$

the image of the diagonal matrices being given by the image of the dual coroot to the given root.

We don't want to define the abstract notion of root system, that would be a necessary step to explain that every abstract root system is the root system of a given reductive group. We just state the unicity result.

Theorem 4.5.6. *The functor $(G, T) \mapsto \Phi(G, T)$ from reductive groups to root systems is conservative: if two groups have the same root system, they are isomorphic.*

Remark that this theorem is often formalized in the setting of split reductive groups but the methods of SGA 3 [DG62] allow us to state it in general, if we work with spaces of characters as sheaves for the etale topology on the base field, i.e., as Galois modules.

4.6 Representation theory of reductive groups

Because representation theory is very important in physics, and to be complete, we also recall the classification of representations of reductive groups (in characteristic 0). This can be found in the book of Jantzen [Jan87], part II.

A Borel subgroup B of G is a maximal closed and connected solvable subgroup of G. For example, the standard Borel subgroup of GL_n is the group of upper triangular matrices. If B is a Borel subgroup that contains the given maximal torus $T \subset G$, the set of roots whose morphism $x_r : \mathbb{G}_a \to G$ have image in G are called positive roots and denoted $R^+ \subset R$. One defines an order on $X^*(T)$ by saying that

$$\lambda \leq \mu \Leftrightarrow \mu - \lambda \in \sum_{\alpha \in R^+} \mathbb{N}\alpha.$$

Any G-module V decomposes in weight spaces (representations of T, i.e., sums of characters) as

$$V = \oplus_{\lambda \in X^*(T)} V_\lambda.$$

For the given order on $X^*(T)$, every irreducible representation of G as a highest non-trivial weight called the highest weight of V. We define the set of dominant weights by

$$X^*(T)_{dom} := \{\lambda \in X^*(T),\ \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in R^+\}.$$

If λ is a dominant weight, we define the G-module $L(\lambda)$ by

$$L(\lambda) = \mathrm{ind}_B^G(\lambda) := (\lambda \otimes A_G)^B$$

where A_G denotes the algebra of functions on G.

The main theorem of classification of irreducible representations for reductive groups is the following.

Theorem 4.6.1. *Suppose that G is reductive and splitted, i.e., $T \cong \mathbb{G}_m^n$. The map*

$$\begin{array}{cccc} L: & X^*(T)_{dom} & \to & \textsc{RepIrr}/\sim \\ & \lambda & \mapsto & L(\lambda) \end{array}$$

is a bijection between the set of dominant weight and the set of isomorphism classes of irreducible representations of G.

If G is not splitted, i.e., if T is a twisted torus, one has to work a bit more to get the analogous theorem, that is due to Tits, and which can be found in the book of involutions [KMRT98], in the section on Tits algebras. Remark that many groups in physics are not splitted, and this is why their representation theory is sometimes tricky to handle.

4.7 Structure and representations of SL_2

As explained above, SL_2 is the main building block for any other algebraic groups. It is defined by

$$\mathrm{SL}_2 = \{M \in \mathrm{GL}_2, \det(M) = 1\}.$$

Its maximal torus is

$$T = \left\{\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}\right\} \cong \mathrm{GL}_1.$$

The space of characters of T is identified with $\mathbb{Z}$ by $t \mapsto t^n$. The Lie algebra of SL_2 is given by

$$\mathrm{Lie}(\mathrm{SL}_2)(A) = \{I + \epsilon M \in \mathrm{SL}_2(A[\epsilon]/(\epsilon^2)),\ \det(I + \epsilon M) = 1\}.$$

If $M = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$, the determinant condition means $(1+\epsilon a).(1+\epsilon d) = 1 + \epsilon(a+d) = 1$, so that $\mathrm{Tr}(M) = 0$. We thus get

$$\mathrm{Lie}(\mathrm{SL}_2) = \{M \in M_2,\ \mathrm{Tr}(M) = 0\} = \left\{\begin{pmatrix} a & b \\ c & -a\end{pmatrix}\right\}.$$

The adjoint action of $t \in T$ on $\mathrm{Lie}(\mathrm{SL}_2)$ is given by

$$\left\{\begin{pmatrix} a & b \\ c & -a\end{pmatrix}\right\} \mapsto \left\{\begin{pmatrix} a & t^2 b \\ t^{-2}c & -a\end{pmatrix}\right\}$$

so that the roots are $r_2 : t \mapsto t^2$ and $r_{-2} : t \mapsto t^{-2}$. They correspond to the root morphisms

$$r_2, r_{-2} : \mathbb{G}_a \to \mathrm{SL}_2$$

given by

$$a \mapsto \begin{pmatrix} 1 & a \\ 0 & 1\end{pmatrix} \text{ and } a \mapsto \begin{pmatrix} 1 & 0 \\ a & 1\end{pmatrix}.$$

Remark that SL_2 is generated by T and the images of the root morphisms.

One shows that the dominant weights of SL_2 are given by $\mathbb{Z}_{>0} \subset \mathbb{Z} = X^*(T)$ and the corresponding irreducible representations are given by

$$V_n := \mathrm{Sym}^n(V)$$

where V is the standard representation of SL_2.

For GL_2, the classification of representations is similar, and gives a family of irreducible representations

$$V_{n,m} := \mathrm{Sym}^n(V) \otimes (\det)^{\otimes m}$$

where det is the determinant representation and (n, m) is a pair of integers with $n > 0$ and $m \in \mathbb{Z}$.

4.8 Structure and representations of SU_2

Recall that SU_2 is defined by

$$\mathrm{SU}_2 = \{M \in \mathrm{Res}_{\mathbb{C}/\mathbb{R}} M_{2,\mathbb{C}}, {}^t\bar{M}.M = M^t\bar{M} = 1\}.$$

Its Lie algebra can be computed by using

$$\mathfrak{su}_2 = \mathrm{Lie}(\mathrm{SU}_2) = \{M \in \mathrm{SU}_2(\mathbb{R}[\epsilon]/(\epsilon^2)),\ M = \mathrm{id} \mod \epsilon\}.$$

Indeed, if $I + \epsilon M$ is in $\mathfrak{su}_2$, it fulfills

$${}^t\bar{I}I + \epsilon[{}^t\bar{M}I +^t \bar{I}M] = 1$$

so that $\mathfrak{su}_2$ is given by

$$\mathfrak{su}_2 = \{M \in \mathrm{Res}_{\mathbb{C}/\mathbb{R}} M_{2,\mathbb{C}},\ {}^t\bar{M} + M = 0\}.$$

Actually, if one defines the quaternion algebra by

$$\mathbb{H} := \{M \in \mathrm{Res}_{\mathbb{C}/\mathbb{R}} M_{2,\mathbb{C}}, \exists \lambda \in \mathbb{R},\ {}^t\bar{M} + M = \lambda.\mathrm{id}\},$$

one can show that $\mathbb{H}$ is a non-commutative field with group of invertibles

$$U_2 := \mathbb{H}^\times.$$

The Lie algebra of U_2 is $\mathbb{H}$ itself. There is a natural norm map $\mathrm{Nm} : \mathbb{H} \to \mathbb{R}$ sending M to $\mathrm{Nm}(M) :=^t \bar{M}.M$ and one can identify SU_2 with the multiplicative kernel of

$$\mathrm{Nm} : U_2 = \mathbb{H}^\times \to \mathbb{R}^\times.$$

One has $\mathbb{H}_\mathbb{C} := \mathbb{H} \otimes_\mathbb{R} \mathbb{C} \cong \mathrm{M}_{2,\mathbb{C}}$, so that the special unitary group is a twisted version of SL_2, meaning that

$$\mathrm{SU}_2 \otimes_\mathbb{R} \mathbb{C} \cong \mathrm{SL}_{2,\mathbb{C}}.$$

This implies that their structure and representation theory are essentially equivalent. The maximal torus of SU_2 is isomorphic to SU_1, that is to the kernel of the norm map

$$\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_1 \to \mathrm{GL}_{1,\mathbb{R}}$$

that sends z to $z\bar{z}$. It is also a twisted version of the maximal torus GL_1 of SL_2.

The representations of SU_2 are given by taking irreducible representations of $\mathrm{SL}_{2,\mathbb{C}}$, i.e., the representations $V_n = \mathrm{Sym}^n(V)$ for V the standard representation, making their scalar restriction to $\mathbb{R}$, for example $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} V_n$ and then taking irreducible sub-representations of these for the natural action of $\mathrm{SU}(2)$. More concretely, they are simply given by the representations $V_{n,\mathbb{R}} = \mathrm{Sym}^n_\mathbb{R}(V_\mathbb{R})$, where $V_\mathbb{R}$ is the standard representation of $\mathrm{SU}(2)$ on $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{C}^2$.

For example, the representation $V_2 = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{C}\langle X^2, XY, Y^2\rangle$ of real dimension 6 has a natural real sub-representation of dimension 3 that corresponds to the morphism

$$\mathrm{SU}_2 \to \mathrm{SO}_3.$$

4.9 Structure of $\mathrm{SO}(V,q)$

Let (V,q) be a quadratic space over $\mathbb{R}$. The special orthogonal group $\mathrm{SO}(V,q)$ is the subgroup of $\mathrm{GL}(V)$ defined by

$$\mathrm{SO}(V,q) = \{f \in \mathrm{GL}(V),\ q(f(v),f(w)) = q(v,w)\}.$$

Its lie algebra is given by

$$\mathfrak{so}(V,q) = \{m \in \mathrm{End}(V),\ q(v,m(w)) + q(m(v),w) = 0\}.$$

Indeed, if $\mathrm{id} + \epsilon m \in \mathrm{SO}(V,q)(\mathbb{R}[\epsilon]/(\epsilon^2))$ is a generic element that reduces to id modulo ϵ, the equation

$$q((\mathrm{id}+\epsilon m)(v),(\mathrm{id}+\epsilon m)(w)) = q(v,w)$$

gives

$$\epsilon[q(v,m(w)) + q(m(v),w)] = 0.$$

There is a canonical identification

$$\begin{array}{ccc} \wedge^2 V & \xrightarrow{\sim} & \mathfrak{so}(V,q) \\ v \otimes w & \mapsto & [x \mapsto q(w,x).v - q(v,x).w]. \end{array}$$

This identification can be better understood by using the natural isomorphism $\mathfrak{spin}(V,q) \xrightarrow{\sim} \mathfrak{so}(V,q)$, the natural embedding

$$\mathfrak{spin}(V,q) \to \mathrm{Cliff}(V,q)$$

and the linear isomorphism

$$\wedge^* V \xrightarrow{\sim} \mathrm{Cliff}(V,q)$$

given in degree 2 by

$$\begin{array}{ccc} \wedge^2 V & \to & \mathrm{Cliff}(V,q) \\ x \wedge y & \mapsto & \frac{1}{2}(xy - yx). \end{array}$$

Chapter 5

Hopf algebras

We now make a small excursion in the setting of Hopf algebras, that is directly related to the notion of algebraic group, but that will also be useful to understand conceptually the combinatorial structures that appear in quantum field theory. Quantum group deformations are also interesting illustrations of the idea of categorical factorization quantization presented in Section 23.6. We refer to [KRT97], [Kas95] and [Maj95] for an introduction to the modern theory of Hopf algebras and quantum groups.

5.1 Definitions and examples

If G is an affine algebraic group over K, its algebra of functions is a commutative unital algebra, equipped with a coproduct $\Delta : A \to A \otimes A$, a counit $\epsilon : A \to K$ and an antipode $S : A \to A$ that are all algebra morphisms (given respectively by multiplication, unit and inverse in the corresponding algebraic group). The notion of Hopf algebra may be seen as a generalization of this situation, whose conceptual roots may be (a bit arbitrarily) taken in deformation theory. Indeed, the category of representations $\rho : G \to \mathrm{GL}(V)$ of an algebraic group G with Hopf algebra A is a symmetric monoidal category

$$(\textsc{Rep}(A), \otimes) := (\textsc{Rep}(G), \otimes),$$

that we may see as a 3-tuply monoidal category (see Section 1.3). One may then deform this category over K to a 1-tuply or 2-tuply monoidal category over $K[[\hbar]]$. This may be done by deforming A to a $K[[\hbar]]$-bialgebra whose category of modules is monoidal or braided monoidal.

Definition 5.1.1. Let $(\mathcal{C}, \otimes)$ be a monoidal category. A *bialgebra* is an object A of $\mathcal{C}$, equipped with a monoid structure $(\mu : A \otimes A \to A, e : \mathbb{1} \to A)$ and a comonoid structure $(\Delta : A \to A \otimes A, \epsilon : A \to \mathbb{1})$ that are compatible, meaning that Δ and ϵ are monoid morphisms. A bialgebra is called *commutative* (resp. *cocommutative*) if the underlying monoid (resp. comonoid) is commutative (resp. cocommutative).

The interest of the notion of bialgebra is that one may make the tensor product of two modules V and W over a given bialgebra A by putting on the tensor product $V \otimes W$, that is a module over $A \otimes A$, the module structure over A given by the coproduct $\Delta : A \to A \otimes A$. This construction gives the following important result.

Lemma 5.1.2. *The monoidal structure (associativity condition) on $(\mathcal{C}, \otimes)$ induces a natural monoidal category structure on the category* $\textsc{Mod}(A)$ *of modules over a bialgebra* A.

One may avantageously generalize the notion of bialgebra by weakening the above result.

Definition 5.1.3. A *quasi-bialgebra* is an object A of $\mathcal{C}$ equipped with a monoid structure $(\mu : A \otimes A \to A, e : \mathbb{1} \to 1)$ and a pair of monoid morphisms $(\Delta : A \to A \otimes A, \epsilon : A \to \mathbb{1})$, and with the additional datum of a monoidal structure (associativity, and left and right unit constraints) on the category $\textsc{Mod}(A)$ with its usual tensor functor induced by that of $\mathcal{C}$ and by Δ and ϵ. A *braided quasi-bialgebra* is a quasi-bialgebra equipped with a braiding of the corresponding monoidal category $(\textsc{Mod}(A), \otimes)$.

Definition 5.1.4. Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category with internal homomorphisms. Let (A, Δ, ϵ) be a comonoid and $(B, m, 1)$ be a monoid in $\mathcal{C}$. The internal homomorphism object $\underline{\mathrm{Hom}}(A, B)$ has a structure of a monoid, called the *convolution monoid*, and whose product

$$* : \underline{\mathrm{Hom}}(A, B) \otimes \underline{\mathrm{Hom}}(A, B) \to \underline{\mathrm{Hom}}(A, B),$$

may be defined (on the sets $\mathrm{Hom}(C, \underline{\mathrm{Hom}}(A, B))$, functorially in C), by the composition

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{f \otimes g} B \otimes B \xrightarrow{m} B.$$

Definition 5.1.5. A *Hopf algebra* is a bialgebra A whose identity endomorphism id_A has a two sided inverse S for convolution in $(\underline{\mathrm{End}}(A), *)$, called its *antipode*.

We now present two examples of Hopf algebras that play a fundamental role in perturbative causal quantum field theory of Chapter 21.

Example 5.1.6. 1. If V is a K-vector space, it corresponds to an algebraic additive group scheme $(\mathbb{V}, +)$, whose algebra of functions is the polynomial algebra $A = \mathrm{Sym}(V, \mathbb{R})$ given by

$$\mathrm{Sym}(V, \mathbb{R}) := \oplus_n \mathrm{Hom}_{\mathbb{R}}(V^{\otimes n}, \mathbb{R})^{S_n}.$$

The coproduct is then given by

$$\Delta(a(v_1, \ldots, v_n)) := a(v_1 + u_1, \ldots, v_n + u_n).$$

If V is finite dimensional, we may also dualize and defined the coproduct on V^* by $\Delta(a) = 1 \otimes a + a \otimes 1$ (extended to $S(V^*)$ by algebra morphism) and the counit by $\epsilon(a) = 0$.

2. Similarly, the above coproduct induced by addition on V extends to the associative algebra of multilinear functions $B = T(V, \mathbb{R})$ given by

$$T(V, \mathbb{R}) := \oplus_n \mathrm{Hom}_{\mathbb{R}}(V^{\otimes n}, \mathbb{R}).$$

   In the finite dimensional case, it may also be obtained as above.

Example 5.1.7. If (C, Δ, ϵ) is a coalgebra, the algebra $\mathrm{Sym}(C)$ may be equipped with a bialgebra structure Δ by setting $\Delta(c) = \Delta(c)$ for $c \in C$, and extending this formula by algebra morphism to $\mathrm{Sym}(C)$. Remark that $\mathrm{Sym}(C)$ is a comodule algebra on the coalgebra C, meaning that the natural map

$$\mathrm{Sym}(C) \to C \otimes \mathrm{Sym}(C)$$

defined by extending comultiplication on C to an algebra morphism is compatible with comultiplication on C. For example, if we take $C = \mathrm{Sym}(V)$, we get a coaction

$$\mathrm{Sym}(\mathrm{Sym}(V)) \to \mathrm{Sym}(V) \otimes \mathrm{Sym}(\mathrm{Sym}(V)).$$

This coaction plays an important role in the Hopf algebraic interpretation of the deformation quantization of free fields, to be described in Section 21.4.

Example 5.1.8. The example that has stimulated a great deal of the recent work on Hopf algebras is given by the envelopping Hopf algebra $U(\mathfrak{g})$ of a (semisimple) Lie algebra, and the quasi-bialgebras that correspond to deformations of its category of representations as a monoidal (or braided monoidal) category, sexily named quantum groups by Drinfeld. Remark that if $\mathfrak{g}$ is the Lie algebra of an algebraic group G with Hopf algebra of functions A, one may identify it with invariant differential operators on G, and this gives a duality

$$\begin{array}{rccc} \langle -, - \rangle : & U(\mathfrak{g}) \times A & \to & K \\ & (P, f) & \mapsto & (Pf)(e) \end{array}$$

that also identifies $U(\mathfrak{g})$ with an algebra of distributions on G supported at the unit e. This duality becomes perfect only when one passes to the formal completion $\hat{A}$ of the algebra A at e. Deforming the coproduct on $U(\mathfrak{g})$ is thus equivalent to deforming the product on the algebra $\hat{A}$ of formal functions on G around its unit. If $\mathfrak{g} = V$ is a vector space with its trivial commutative Lie algebra structure, then $U(\mathfrak{g})$ is the symmetric algebra $\mathrm{Sym}(V)$, and A is the Hopf algebra $\mathrm{Sym}(V, \mathbb{R})$ of polynomial functions on V. The completion $\hat{A}$ is given by

$$\hat{A} = \widehat{\mathrm{Sym}}(V, \mathbb{R}) := \prod_{n \geq 0} \mathrm{Hom}_{\mathbb{R}}(V^{\otimes n}, \mathbb{R})^{S_n}.$$

Deforming the coproduct on $\mathrm{Sym}(V)$ is thus equivalent to deforming the product on $\widehat{\mathrm{Sym}}(V, \mathbb{R})$, which is what is done when one defines $*$-products in perturbative causal quantum field theory (see Chapter 21).

We now give for completeness reasons the definition of quantum groups.

Definition 5.1.9. Let K be a field. A *quantum enveloping algebra* for a Lie algebra $\mathfrak{g}$ is given by a noncommutative and noncocommutative braided quasi-bialgebra A over $K[[\hbar]]$ with underlying module $U(\mathfrak{g})[[\hbar]]$, and that gives back the envelopping bialgebra structure $U(\mathfrak{g})$ modulo $\hbar$.

An important result of the theory of quantum groups is given by the following existence result, whose proof may be found in Drinfeld's ICM address [Dri87].

Theorem 5.1.10. *If $\mathfrak{g}$ is a semisimple Lie algebra, then there exists a quantum enveloping algebra $U_\hbar(\mathfrak{g})$ for $\mathfrak{g}$.*

Definition 5.1.11. Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category that is additive. Let A be a bialgebra in $\mathcal{C}$. An element $a \in A$ is called

1. *primitive* if
$$\Delta(a) = a \otimes 1 + 1 \otimes a,$$
i.e., if a is in the equalizer
$$\mathrm{Prim}(A) := \mathrm{Ker}((\Delta, \mathrm{id} \otimes e + e \otimes \mathrm{id}) : A \to A \otimes A).$$

2. *group like* if
$$\Delta(a) = a \otimes a,$$
i.e., if a is in the equalizer
$$\mathrm{Grp}(A) := \mathrm{Ker}((\Delta, (\mathrm{id} \otimes e).(e \otimes \mathrm{id})) : A \to A \otimes A).$$

5.2 Twists and deformations

The deformation theory of Hopf algebras may be studied by using a cohomology theory for them, similar to the Hochschild cohomology of associative algebra. A geometrically intuitive way to look at this cohomology theory is to think of it as the tangent space to the moduli space of monoidal (resp. braided monoidal) dg-categories at the point corresponding to the category $(\mathrm{MOD}(A), \otimes)$ of modules over the bialgebra that one wants to deform. This situation is similar to the deformation theory of associative algebras, where Hochschild cohomology complex may be interpreted as the tangent space to the moduli space of dg-categories (without monoidal structure) with Morita isomorphisms (bimodules) between them, at the point corresponding to the dg-category $\mathrm{MOD}(A)$ of modules over the given associative algebra to be deformed.

We now give a concrete description of the deformations of Hopf algebras that we will use to define $*$-products in quantum field theory. This may be found in [Maj95], Section 2.2. We adapt his definition to the general setting of symmetric monoidal categories.

Definition 5.2.1. Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category and A be a bialgebra in $\mathcal{C}$. For $\psi : A^{\otimes n} \to \mathbb{1}$ a linear functional with value in the unit object, we denote $\cdot_i$ the multiplication in i-th place for $0 \leq i \leq n$, $\psi \circ \cdot_0 = \epsilon \otimes \psi$ and $\psi \circ \cdot_{n+1} = \psi \circ \epsilon$. An

n-cochain on A is a linear functional $\psi : A^{\otimes n} \to \mathbb{1}$, invertible in the convolution algebra $\underline{\mathrm{Hom}}(A^{\otimes n}, \mathbb{1})$ (defined using the coalgebra structure on A, and the algebra structure on $\mathbb{1}$). The *coboundary operator* is defined by

$$\partial\psi := \prod_{i=0}^{n+1} \psi^{(-1)^i} \circ \cdot_i.$$

We know from Calaque and Etingof's survey [CE04] that

- 2-cocycles classify monoidal structures on the forgetful functor

$$(\mathrm{CoMod}(A), \otimes) \to (\mathcal{C}, \otimes)$$

  from comodules on A to $\mathcal{C}$, that deform the trivial monoidal structure;

- 3-cocycles classify the associativity conditions that one may put on $\mathrm{CoMod}(A)$, without changing the tensor product, by deforming the usual associativity condition on $\otimes$.

We now use Sweedler's notations for the coproduct

$$\Delta(a) = \sum a_{(1)} \otimes a_{(2)}.$$

When using these notations in monoidal categories, one must carefully insert the necessary associativity and commutativity isomorphisms in all formulas. We will keep this important fact implicit in our presentation. This allows us to interpret consistently the classical formulas in the general setting of symmetric monoidal categories, and in particular for super, graded, or differential graded modules.

Definition 5.2.2. Let (A, m, Δ) be a bialgebra and $\psi : A \otimes A \to \mathbb{1}$ be a unital 2-cocycle on A. The twisted product $m_\psi : A \otimes A \to A$ is defined by the composition

$$m_\psi(a, b) = \sum \psi(a_{(1)} \otimes b_{(1)}) a_{(2)} b_{(2)} \psi^{-1}(a_{(3)} \otimes b_{(3)}).$$

Theorem 5.2.3. *A 2-cocycle ψ on A induces an isomorphism of monoidal categories*

$$F_\psi : (\mathrm{CoMod}(A), \otimes) \to (\mathrm{CoMod}(A_\psi), \otimes_\psi)$$

by the natural isomorphism

$$\begin{array}{rccc} \sigma_\psi : & F_\psi(V) \otimes_\psi F_\psi(W) & \to & F_\psi(V \otimes W) \\ & v \otimes_\psi w & \mapsto & \psi(v_{(1)} \otimes w_{(1)}) v_{(2)} \otimes w_{(2)}. \end{array}$$

In particular, if (B, m) is an algebra in $(\mathrm{CoMod}(A), \otimes)$, called a *comodule algebra*, the above equivalence sends it to a twisted comodule algebra (B_ψ, m_ψ), with multiplication given by

$$a \cdot_\psi b = \sum \psi(a_{(1)} \otimes b_{(1)}) a_{(2)} b_{(2)}.$$

More specifically, one may deform the algebra A as a comodule over the bialgebra A.

One may think of a comodule algebra as an object generalizing the notion of homogeneous space X under an algebraic group G. Twisting such an object corresponds to a kind of deformation quantization for X, as module over G. We will be mainly interested by the situation where $G = \mathbb{V}$ is the additive algebraic group of a module V, and $X = \mathbb{V}$ is the associated affine space. These deformations play an important role in the causal approach to perturbative quantum field theory, described in Chapter 21. We now describe it concretely.

Example 5.2.4. Let V be a real vector space and $\psi : V \otimes V \to \mathbb{R}$ be a bilinear pairing on V, that we see as a bilinear pairing with values in the vector space $\hbar\mathbb{R} \cong \mathbb{R}$. Extend it to a unique 2-cocycle ψ on the symmetric bialgebra $(S(V)[[\hbar]], \Delta, \mu)$ by the conditions

$$\begin{array}{rcl} \psi(ab, c) & = & \sum \psi(a, c_{(1)})\psi(b, c_{(2)}), \\ \psi(a, bc) & = & \sum \psi(a_{(1)}, b)\psi(a_{(2)}, c). \end{array}$$

It is known that all 2-cocycles on $S(V)$ are of this form, and that a cocycle is a boundary if and only if it is symmetric. Recall that the symmetric bialgebra is simply the Hopf algebra of functions on the algebraic group $(\mathbb{V}, +)$ given by the space V with its addition. The twisting by a bilinear pairing corresponds to defining a deformation of the affine space $\mathbb{V}$, seen as a homogeneous space (comodule algebra) under the group $(\mathbb{V}, +)$. This gives an associative product on $S(V)[[\hbar]]$ given by

$$a \overset{\hbar}{*} b = \sum \psi(a_{(1)}, b_{(1)})a_{(2)}b_{(2)},$$

that we may call the *Moyal product*, if the bilinear form ψ is symplectic.

Example 5.2.5. One may refine Example 5.2.4 by the following. Let V be a vector space and $C = S(V)$ be the associated cofree coalgebra. By Example 5.1.7, one gets on $S(C)$ a comodule algebra structure under C, i.e., a morphism

$$\Delta : S(C) \to C \otimes S(C)$$

that is compatible with comultiplication on C. If $\psi : V \otimes V \to \mathbb{R}$ is a bilinear pairing on V, that one extends to a 2-cocycle on $C = S(V)$ as in loc. cit., one gets a twisted product by the formula

$$a \overset{\hbar}{*} b = \sum \psi(a_{(1)}, b_{(1)})a_{(2)}b_{(2)},$$

where $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$ means that $a \in S(C)$, $a_{(1)} \in C$ and $a_{(2)}$ in $S(C)$.

Chapter 6

Connections and curvature

The mathematical formulation of particle physics involve various notions of connections on bundles. The easiest way to relate them is to use the very general notion of Grothendieck connection, that was described in categorical terms in Chapter 1.5. In each concrete cases, the notion of connection is easier to handle, and we give for it a simpler description.

6.1 Koszul connections

Definition 6.1.1. Let $F \to X$ be a vector bundle on a smooth manifold. Let $A = \mathcal{C}^\infty(X)$ and M be the A-module $\Gamma(X, F)$ of sections of F. A *Koszul connection* on F is an $\mathbb{R}$-linear map

$$\nabla : M \to M \otimes_A \Omega^1_A$$

that fulfills Leibniz rule

$$\nabla(f.s) = f.\nabla(s) + df \otimes s.$$

Definition 6.1.2. Let X be a manifold with function algebra A. Let $F \to X$ be a vector bundle and let $\nabla : \mathcal{F} \to \mathcal{F} \otimes_A \Omega^1_A$ be a Koszul connection. The *curvature* of ∇ is the composition

$$C(\nabla) := \nabla_2 \circ \nabla \in \mathrm{End}_A(\mathcal{F}) \otimes_A \Omega^2_A,$$

where ∇_i is defined by

$$\nabla_i := \mathrm{id}_{\mathcal{F}} \otimes d + (-1)^i \nabla \wedge \mathrm{id}_{\Omega^1_A} : \mathcal{F} \otimes_A \Omega^i_A \to \mathcal{F} \otimes_A \Omega^{i+1}_A.$$

Remark that the A-linearity of the curvature is not clear a priori.

Proposition 6.1.3. *Let $F \to X$ be a vector bundle on a smooth manifold. The data of a linear Grothendieck connection and of a Koszul connection on F are equivalent.*

Proof. We refer to [BO78] for more details on the link between connections and infinitesimal neighborhoods. Just recall that the exterior differential is a map $d : A \to \Omega^1_A \subset \mathrm{Jet}^1 A$. Denote $\mathcal{F} = \Gamma(X, F)$ the space of sections of F. If $\tau : \mathcal{F} \otimes_A \mathrm{Jet}^1 A \to \mathcal{F} \otimes_A \mathrm{Jet}^1 A$ is a Grothendieck connection, and $d_{1,\mathcal{F}} := d \otimes \mathrm{id}_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \otimes \mathrm{Jet}^1 A$, then the morphism $\theta = \tau \circ d_{1,\mathcal{F}} : \mathcal{F} \to \mathcal{F} \otimes \mathrm{Jet}^1 A$ allows us to construct a Koszul connection by

$\nabla(f) = \theta(f) - f \otimes 1$. Indeed, one has $\nabla(f) = \mathcal{F} \otimes \Omega^1$ because τ is identity on X. The association $\tau \mapsto \nabla$ is grounded on various universal properties that makes it unique. □

6.2 Ehresmann connections

We refer to the book [KMS93], 17.1, for a differential geometric approach to Ehresmann connections. We will mostly be interested by the more algebraic approach presented in the book of Krasilshchik and Verbovetsky [KV98], Section 5.2.

Definition 6.2.1. Let $p : B \to M$ be a surjective submersion of varieties. An *Ehresmann connection* on B is the datum of a section a of the natural projection

$$\mathrm{Jet}^1 B \to B.$$

Proposition 6.2.2. *Let $p : B \to X$ be a surjective submersion of varieties. Then the following data on B are equivalent:*

1. *An Ehresmann connection a,*
2. *A connection one form $v \in \Omega^1(B, TB) = \mathrm{End}(TB)$ such that $v^2 = v$ and the image of v is VB,*
3. *a section $a \in \Gamma_B(TB, VB)$ of the canonical exact sequence*

$$0 \to VB \to TB \to \pi^* TX \to 0$$

of bundles on B.

Proof. We first show that the two last data are equivalent. If $a : TB \to VB$ is a section of the natural map $VB \to TB$, the projection $v : TB \to TB$ on VB along the kernel of a gives a connection one form. Since its image is VB, the data of a and v are equivalent. Now we use the Krasilshchik-Verbovetsky approach to connections in [KV98], Section 5.2. Remark that the section $a : TB \to VB$ induces a natural morphism of functors on the category of admissible $\mathcal{O}_B$-modules (chosen to be the category of all modules in the case of algebraic varieties and the category of geometric modules for smooth varieties)

$$\nabla_a : \mathrm{Der}(\mathcal{O}_X, -) \to \mathrm{Der}(\mathcal{O}_B, -).$$

This corresponds on the representing objects to a morphism

$$\nabla_a : \mathcal{O}_B \to \mathcal{O}_B \otimes_{\mathcal{O}_X} \Omega^1_{B/X},$$

which can be extended to

$$s_a^* := \mathrm{id} + \nabla_a : \mathcal{O}_B \to \mathcal{O}_B \oplus \mathcal{O}_B \otimes_{\mathcal{O}_X} \Omega^1_{B/X} =: \mathcal{O}_{\mathrm{Jet}^1 B},$$

and then to a Grothendieck connection

$$\epsilon : \mathcal{O}_{\mathrm{Jet}^1 B} \xrightarrow{\sim} \mathcal{O}_{\mathrm{Jet}^1 B}.$$

Composing with the canonical projection $\mathcal{O}_{\mathrm{Jet}^1 B} \to \mathcal{O}_B$ gives a section $s : B \to \mathrm{Jet}^1 B$. □

Definition 6.2.3. The *curvature* of an Ehresmann connection is given by

$$R = \frac{1}{2}[v, v]$$

where $[.,.]$ denotes the Frölicher-Nijenhuis bracket of $v \in \Omega^1(E, TE)$ with itself. Thus $R \in \Omega^2(E, TE)$ is defined by

$$R(X, Y) = v([(\mathrm{id} - v)X, (\mathrm{id} - v)Y)]).$$

Proposition 6.2.4. *Let $p : B \to X$ be a bundle given by a morphism of varieties. The following data are equivalent:*

1. *An Ehresmann connection on B,*

2. *A Grothendieck connection on B.*

Proof. Denote $\mathcal{A} = \mathcal{C}^\infty(X)$ and $\mathcal{B} = \mathcal{C}^\infty(B)$. Suppose given a Grothendieck connection

$$p_1^* B \xrightarrow{\sim} p_2^* B.$$

In this case, it is equivalent to a map

$$\mathcal{B} \otimes_{\mathcal{A}} \mathrm{Jet}^1 \mathcal{A} \xrightarrow{\sim} \mathcal{B} \otimes_{\mathcal{A}} \mathrm{Jet}^1 \mathcal{A}$$

that induces identity on $\mathcal{A}$, and that we can compose with the projection $\mathcal{B} \otimes_{\mathcal{A}} \mathrm{Jet}^1 \mathcal{A} \to \mathcal{B}$ to get a map

$$\mathcal{B} \otimes_{\mathcal{A}} \mathrm{Jet}^1 \mathcal{A} \to \mathcal{B}$$

that is also a map

$$\mathrm{Jet}^1 \mathcal{B} \to \mathcal{B}$$

and gives a section of the natural projection $\mathrm{Jet}^1 B \to B$. These two data are actually equivalent because one can get back that Grothendieck connection by tensoring with $\mathcal{B}$ over $\mathcal{A}$. □

Corollary 6.2.5. *If $p : F \to X$ is a vector bundle on a smooth manifold, the datum of a linear Ehresmann connection on F and of a Koszul connection on F are equivalent.*

6.3 Principal connections

We use here the definition of Giraud [Gir71], Chapter III, 1.4, since it is adapted to general spaces.

Definition 6.3.1. Let G be a lie group and $P \to M$ be a space morphism equipped with an action $m : G \times_M P \to P$ of G. One says that (P, m) is a *principal homogeneous space* over M under G (also called a *torsor* under G over M) if

1. $P \to M$ is an epimorphism, i.e., there exists a covering family $\{U_i \to M\}$ such that $P_M(U_i)$ are non-empty (i.e., the bundle has locally a section on a covering of M),

2. the natural morphism

$$\begin{array}{ccc} G \times P & \to & P \times P \\ (g, p) & \mapsto & (p, gp) \end{array}$$

   is an isomorphism (i.e., the action of G on P is simply transitive).

Definition 6.3.2. A *principal G-connection* on P is a G-equivariant connection on $P \to M$.

Proposition 6.3.3. *The following are equivalent:*

1. *A principal G-connection on $p : P \to M$.*

2. *An equivariant $\mathfrak{g}$-valued differential form A on P, i.e., $A \in \Omega^1(P, \mathfrak{g})^G$ such that the diagram*

$$\begin{array}{ccc} \mathfrak{g} = T_{1_G} G & \xrightarrow{\mathrm{id}_{\mathfrak{g}}} & \mathfrak{g} \\ {\scriptstyle T_{1_G} m} \downarrow & \nearrow_{A} & \\ \Theta_P & & \end{array}$$

   commutes where $m : G \times P \to P$ is the action map.

Proof. The equivalence between an equivariant Ehresmann connection $A \in \Omega^1(TP, TP)^G$ and a differential form in $\Omega^1(P, \mathfrak{g})$ follows from the fact the derivative of the action map $m : G \times_M P \to P$ with respect to the G-variable at identity $e \in G$ defines a bundle map

$$i = D_e m : \mathfrak{g}_P := \mathfrak{g} \times P \to VP,$$

called a parallelization, between the vertical tangent bundle to P, defined by

$$0 \to VP \to TP \xrightarrow{Dp} p^*TM \to 0$$

and the trivial linear bundle with fiber the Lie algebra $\mathfrak{g}$ of G, that is an isomorphism. The fact that $i : \mathfrak{g}_P \to TP$ is valued in VP follows from the fact that G acts vertically on P. $\square$

Proposition 6.3.4. *The curvature of a principal G-connection A is identified with the form*

$$F = dA + [A \wedge A]$$

in $\Omega^2(P, \mathfrak{g})$ where the bracket exterior product is given by

$$[\omega \otimes h \wedge \nu \otimes k] = \omega \wedge \nu \otimes [h, k].$$

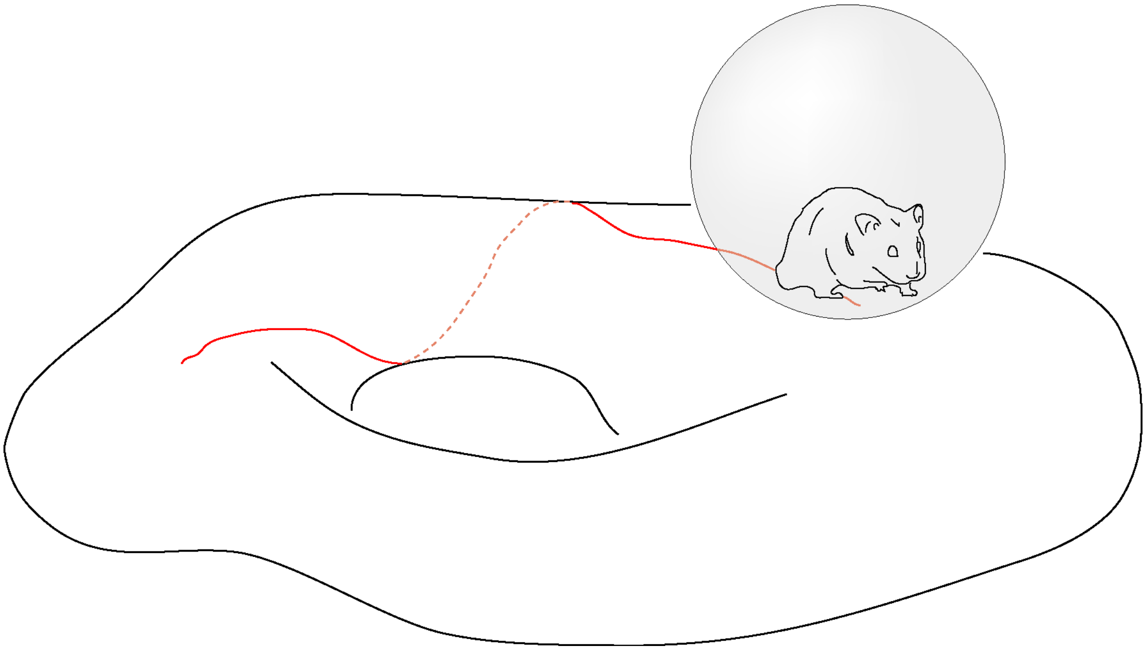
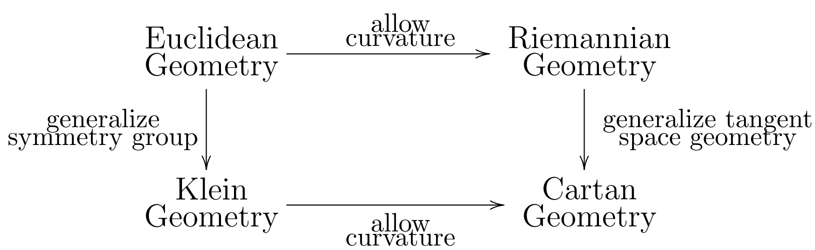


Figure 6.1: Cartan geometry and the hamster ball.

6.4 Cartan connections and moving frames

We refer to the excellent survey of Wise [Wis06] for a more complete description. Sharpe [Sha97] and Wise explain neatly the idea of Cartan geometry in a commutative diagram:



One can also say that a Cartan geometry is given by a space whose geometry is given by pasting infinitesimally some classical geometric spaces of the form G/H for $H \subset G$ two Lie groups. This can be nicely explained by the example of a sphere pasted infinitesimally on a space through tangent spaces, or, as in Wise's article [Wis06], by a Hamster ball moving on a given space as in figure 6.1.

Definition 6.4.1. Let M be a manifold, $H \subset G$ be two groups. A *Cartan connection* on M is the data of

1. a principal G-bundle Q on M,

2. a principal G-connection A on Q,

3. a section $s : M \to E$ of the associated bundle $E = Q \times_G G/H$ with fibers G/H,

such that the pullback

$$e = s^*A \circ ds : TM \to VE,$$

called the moving frame (vielbein), for $A : TE \to VE$ the associated connection, is a linear isomorphism of bundles.

The role of the section s here is to "break the G/H symmetry". It is equivalent to the choice of a principal H-sub-bundle $P \subset Q$.

One may interpret the datum (s, Q, A) of a Cartan connection as point of the stack (see Section 9.3 for a precise definition)

$$(s, Q, A) \in \underline{\mathrm{Hom}}(M, [G\backslash(G/H)]_{conn})$$

fulfilling the additional non-triviality condition.

In the cases of interest for this section, E is a linear bundle and the section s is simply the zero section that breaks the translation symmetry (action of $G/H = V$ on the sections of the vector bundle)

The first examples of Cartan connections are given by Klein geometries, i.e., by homogeneous spaces $E = G/H$ over the point space $M = \{.\}$. The corresponding Cartan connection is given by

1. the trivial principal G-bundle $Q = G$ on M,
2. the trivial G-connection A on Q,
3. a section $s : M \to E$ of the associated bundle $E = Q \times_G G/H = G/H \to M$, i.e., a point $x = s(-) \in G/H$.

The pull-back $e = s^*A : TM = M \times \{0\} \to VE = M \times \{0\}$ is an isomorphism.

Three examples of Klein geometries that are useful in physics are given by $H = \mathrm{SO}(n-1, 1)$ and

$$G = \begin{cases} \mathrm{SO}(n, 1) & \text{(de Sitter)} \\ \mathbb{R}^{n-1,1} \rtimes \mathrm{SO}(n-1, 1) & \text{(Minkowski)} \\ \mathrm{SO}(n-1, 2) & \text{(anti de Sitter)} \end{cases}$$

The de Sitter and anti de Sitter geometries are useful to study cosmological models with non-zero cosmological constant. Cartan geometries modeled on these Klein geometries are useful in general relativity. For example, the Minkowski Cartan geometry without torsion corresponds exactly to pseudo-Riemannian manifolds, that are the basic objects of general relativity.

Remark that the G-connection A on the principal G-bundle Q is equivalent to an equivariant $\mathfrak{g}$-valued differential form

$$A : TQ \to \mathfrak{g},$$

and its restriction to $P \subset Q$ gives an H-equivariant differential form

$$A : TP \to \mathfrak{g}.$$

This is the original notion of Cartan connection form.

Definition 6.4.2. The *curvature* of a Cartan connection is the restriction of the curvature of the corresponding Ehresman connection on the principal G-bundle Q to the principal H-bundle P. It is given by the formula

$$F_A := dA + \frac{1}{2}[A \wedge A] \in \Omega^2(P, \mathfrak{g}).$$

The torsion of the Cartan connection is given by the composition of F_A with the projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$.

Suppose that one can decompose $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$ in an H-equivariant way (the Cartan geometry is called reductive). The Cartan connection form thus can be decomposed in

$$A = \omega + e$$

for $\omega \in \Omega^1(P, \mathfrak{h})$ and $e \in \Omega^1(P, \mathfrak{g}/\mathfrak{h})$. In this particular case, e can also be seen as

$$e \in \Omega^1(M, \underline{\mathfrak{g}/\mathfrak{h}})$$

for $\underline{\mathfrak{g}/\mathfrak{h}}$ the H-bundle associated to $\mathfrak{g}/\mathfrak{h}$. This form is called the vielbein by Cartan. By definition of the Cartan connection, it gives an isomorphism

$$e : TM \to \underline{\mathfrak{g}/\mathfrak{h}}.$$

Chapter 7

Lagrangian and Hamiltonian systems

In this chapter, we describe Lagrange and Hamilton's geometric viewpoint of mechanics. We will mainly use Lagrangian methods in this book because their covariance is automatic in relativistic field theories and they allow a more conceptual treatment of symmetries. We thus only give a short account of the Hamiltonian formalism.

We will present in Chapters 13, 14 and 15 a very large family of examples from classical, quantum and theoretical physics, formulated directly in the language of the following section.

7.1 Lagrangian mechanics

The main interest of classical field theory is the notion of Lagrangian variational problem. We introduce here a very simple and general such notion, based on the parametrized approach to spaces, described in Chapter 2 (see also Chapter 9). We also use here the tools of functorial analysis from Chapter 3. We refer to Chapter 12 for a differential-algebraic treatment of symmetries in Lagrangian variational calculus, whose starting point is conceptually close to Lagrange's and Noether's original ideas on the subject.

Definition 7.1.1. A *Lagrangian variational problem* is made of the following data:

1. A space M called the parameter space for trajectories,
2. A space C called the configuration space for trajectories,
3. A morphism $\pi : C \to M$ (often supposed to be surjective),
4. A subspace $H \subset \Gamma(M, C)$ of the space of sections of π

$$\Gamma(M, C) := \{x : M \to C,\ \pi \circ x = \mathrm{id}\},$$

   called the space of histories,

5. A functional (partial function with a domain of definition) $S : H \to R$ (where R is a space in rings that is often the real line $\mathbb{R}$, the complex line $\mathbb{C}$, or $\mathbb{R}[[\hbar]]$) called the *action functional*.

An *observable* is a functional $f : H \to R$.

The description of classical physical systems can often be based on the following.

Principle 1 (Least action principle)**.** The space T of physical trajectories of a given classical variational problem is the subspace T of the space of histories H given by critical points of the action S, i.e.,

$$T = \{x \in H,\ d_x S = 0\}.$$

The advantage of the Lagrangian approach to classical mechanics is that it allows to give a clear informal intuition on the notion of quantum trajectory. This viewpoint was introduced by Feynman in his thesis, and is at the basis of most of the modern developments of quantum field theory.

Principle 2 (Functional integral quantization)**.** The space of quantum trajectories is the space H of all histories. The probability of observing a given history $x : M \to C$ is proportional to $e^{\frac{i}{\hbar}S(x)}$. More precisely, if $f : H \to R$ is an observable, its probability value is given by the absolute value of the complex Gaussian mean value [1]

$$\langle f\rangle_S := \frac{\int_H f(x)e^{\frac{i}{\hbar}S(x)}[dx]}{\int_H e^{\frac{i}{\hbar}S(x)}[dx]}.$$

Remark 7.1.2. The spaces of parameters and configurations that appear in the above definition may be manifolds, but one should also allow them to be more general types of spaces, like manifolds with corners, stacks or derived stacks (see Chapter 9 for an introduction). In any case, one needs some finiteness or geometricity conditions (see Section 9.5 for the general notion of a geometric stack) on both of them (essentially, an analog of integration on M, and an analog of duality on C) to be able to define interesting variational problems.

One can apply this general approach to the case of so-called *local action functionals* (see Chapters 11 and 12 for a detailed study of these objects), that are given by expressions of the form

$$S(x) = \int_M L(x(m), \partial_\alpha x(m))dm$$

for $L : \mathrm{Jet}(C) \to \mathbb{R}$ a function called the Lagrangian density, that depends on variables x_0 (functional variables) in the fiber of $\pi : C \to M$ and of finitely many formal additional variables x_α that play the role of formal derivatives of x_0. This will give a very nice description of the space T of physical trajectories for the classical system, as a solution space for a (usually nonlinear) partial differential equation called the Euler-Lagrange system.

Example 7.1.3. Consider the variational problem of classical Newtonian mechanics for a particle moving in a potential: one has $M = [0, 1]$, $C = [0, 1] \times \mathbb{R}^3$, $\Gamma(M, C) \cong \mathcal{C}^\infty([0, 1], \mathbb{R}^3)$ and for $(x_0, x_1, \vec{v}_0, \vec{v}_1)$ fixed in $(\mathbb{R}^3)^4$, the space of histories is given by

$$H := \{x \in \Gamma(M, C),\ x(0) = x_0,\ x(1) = x_1,\ \partial_t x(0) = \vec{v}_0,\ \partial_t x(1) = \vec{v}_1\}.$$

[1]The meaning of this expression has to be clearly specified mathematically, which is the central difficulty of this approach, that will be treated in Chapters 19, 20 and 21.

One equips $\mathbb{R}^3$ with its standard scalar product $\langle ., .\rangle$. The action functional is given by

$$S(x) = \int_M \frac{1}{2} m\langle \partial_t x(t), \partial_t x(t)\rangle - V(x(t))dt$$

where $V : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function, called the potential. If $\vec{h} : \mathbb{R} \to \mathbb{R}^3$ is a function tangent to H at some point x, one shows that $h(0) = h(1) = 0$. Defining

$$d_x S(\vec{h}) := \lim_{\epsilon \to 0} \frac{S(x + \epsilon h) - S(x)}{\epsilon},$$

one gets

$$d_x S(\vec{h}) = \int_0^1 \langle m\partial_t x, \partial_t \vec{h}\rangle - \langle d_x V(x), \vec{h}\rangle dt$$

and by integrating by parts using that $h(0) = h(1) = 0$, finally,

$$d_x S(\vec{h}) = \int_0^1 \langle -m\partial_t^2 x - d_x V(x), \vec{h}\rangle dt$$

The space of physical trajectories is thus the space of maps $x : \mathbb{R} \to \mathbb{R}^3$ such that

$$m.\partial_t^2 x = -V'(x).$$

This is the standard law of Newtonian mechanics. For example, if $V = 0$, the physical trajectories are those with constant speed, which corresponds to galilean inertial bodies.

Remark 7.1.4. The knowledge of the functional integral is not strictly necessary to make the functional integral quantization. One only needs to have a function

$$e^{\frac{i}{\hbar}S} : H \to S^1.$$

This viewpoint is used in the formalization of higher gauge theories, described in Section 15.5.

7.2 Symplectic and Poisson manifolds

Definition 7.2.1. The algebra of *multi-vector fields* on a given manifold P is the anti-symmetric algebra $\wedge^*\Theta_P$ on the $\mathcal{O}_P$-module of vector fields.

Proposition 7.2.2. *There exists a unique extension*

$$[., .]_{NS} : \wedge^*\Theta_P \otimes \wedge^*\Theta_P \to \wedge^*\Theta_P$$

of the Lie bracket of vector fields, that is moreover a bigraded derivation. More precisely, this bracket, called the Schouten-Nijenhuis bracket, fulfills:

1. *The graded Jacobi identity:*

$$[X, [Y, Z]] = [[X, Y], Z] + (-1)^{(|X|-1)(|Y|-1)}[Y, [X, Z]];$$

2. *The graded anti-commutativity:*

$$[X,Y] = (-1)^{(|X|-1)(|Y|-1)}[Y,X];$$

3. *The bigraded derivation condition:*

$$[X, Y \wedge Z] = [X,Y] \wedge Z + (-1)^{(|X|-1)(|Y|-1)}.$$

Definition 7.2.3. A *symplectic manifold* is a pair (P,ω) of a manifold P and a non-degenerate closed 2-form $\omega \in \Omega^2(P)$:

$$d\omega = 0, \quad \omega^\flat : TP \xrightarrow{\sim} T^*P.$$

A *Poisson manifold* is a pair (P,π) of a manifold P and a bivector $\pi \in \wedge^2\Theta_P$ whose Schouten-Nijenhuis bracket with itself

$$[\pi,\pi]_{SN} = 0$$

is zero. A *symplectomorphism*

$$f : (P_1,\omega_1) \to (P_2,\omega_2)$$

is a smooth map such that $f^*\omega_2 = \omega_1$. A *Poisson morphism*

$$f : (P_1,\pi_1) \to (P_2,\pi_2)$$

is a smooth map such that $f_*\pi_1$ makes sense and is equal to π_2.

Proposition 7.2.4. *The datum of a symplectic manifold is equivalent to the datum of a Poisson structure that is non-degenerate, i.e., such that*

$$\pi^\natural : T^*P \to TP$$

is an isomorphism.

Proof. The inverse $\pi^\natural$ of $\omega^\flat : TP \to T^*P$ gives the Poisson bivector. One checks that the nullity of the bracket is equivalent to the fact that ω is closed. □

Definition 7.2.5. If (P,π) is a Poisson manifold, we define its *Poisson bracket* by

$$\begin{array}{cccc} \{.,.\}_\pi : & \mathcal{O}_M \times \mathcal{O}_M & \to & \mathcal{O}_M \\ & (f,g) & \mapsto & \langle \pi, df \wedge dg\rangle. \end{array}$$

The main example of a symplectic manifold is given by the cotangent bundle $P = T^*X$ of a given manifold. If we define the Legendre 1-form by

$$\theta(v) = \mathrm{ev}(Dp \circ v)$$

where

- $Dp : T(T^*X) \to p^*TX := TX \times_X T^*X$ is the differential of $p : T^*X \to X$,
- ev : $TX \times_X T^*X \to \mathbb{R}_M$ is the natural evaluation duality and,
- and $v \in \Theta_{T^*X} = \Gamma(T^*X, T(T^*X))$ is a vector field.

The symplectic 2-form on P is defined by

$$\omega = -d\theta.$$

On $P = T^*\mathbb{R}^n$, with coordinates (q, p), the Liouville form is given by

$$\theta = pdq = \sum_i p_i dq_i$$

and the symplectic form by

$$\omega = dq \wedge dp = \sum_i dq_i \wedge dp_i.$$

The Poisson bracket is then given by

$$\{f, g\} = \sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}.$$

7.3 Dynamics of a Hamiltonian system

Definition 7.3.1. A *Hamiltonian system* is a tuple $(P, \pi, I, \mathrm{Hist}, H)$ composed of

1. a Poisson manifold (P, π) called the *phase space*, whose functions are called observables,
2. an interval $I \subset \mathbb{R}$ called the time *parameter space for trajectories*,
3. a subspace $\mathrm{Hist} \subset \mathrm{Hom}(I, P)$ called the *space of histories*,
4. a function $H : P \to \mathbb{R}$ called the *Hamiltonian*.

If $x : I \to P$ is a history in the phase space and $f \in \mathcal{O}_P$ is an observable, *Hamilton's equations* for x along f are given by

$$\frac{\partial(f \circ x)}{\partial t} = \{H, f\} \circ x.$$

The Hamiltonian function H defines a *Hamiltonian vector field*

$$X_H := \{H, .\} : \mathcal{O}_P \to \mathcal{O}_P.$$

On $T^*\mathbb{R}^n$, the Hamiltonian vector field is given by

$$X_H = \sum_i \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \sum_i \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}.$$

If the Poisson structure π is non-degenerate, i.e., if (P,π) comes from a symplectic manifold (P,ω), the Hamilton equation on a history $x : I \to P$ are given by

$$x^*(i_{X_H}\omega) = x^*(dH).$$

On $T^*\mathbb{R}^n$, this corresponds to

$$\begin{cases} \frac{\partial q_x}{\partial t} &= \frac{\partial H}{\partial p}, \\ \frac{\partial p_x}{\partial t} &= -\frac{\partial H}{\partial q}. \end{cases}$$

Theorem 7.3.2. *(Symplectic Noether theorem) Given a Hamiltonian system (P,π,H), a function f on P is constant along the trajectories of the Hamiltonian vector field if and only if the Hamiltonian is constant under the Hamiltonian vector field induced by f.*

Proof. Follows from the equality

$$X_f(H) = \{f,H\} = -\{H,f\} = -X_H(f).$$

□

7.4 Lagrangian manifolds and canonical transformations

We now introduce the notion of canonical transformation, with views towards applications to the gauge fixing procedure for gauge theories, in Section 12.5.

Definition 7.4.1. A *coisotropic submanifold* of a Poisson manifold (P,π) is a submanifold $L \subset P$ such that the ideal of functions on P that annihilate on L is stable by the Poisson bracket. An *isotropic submanifold* of a symplectic manifold (P,ω) is a submanifold L such that the symplectic form restricts to zero on L. A *Lagrangian submanifold* in a symplectic manifold is a submanifold that is both isotropic and coisotropic.

Example 7.4.2.
1. Let $P = \mathbb{R}^{2n}$ be the flat symplectic space, with coordinates (x,x^*) and symplectic form $dx \wedge dx^*$, whose matrix in the basis (x,x^*) given by

   $$dx \wedge dx^* := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

   Then the submanifolds L with coordinates only x or only x^* are Lagrangian. Submanifolds of dimension 1 are isotropic and submanifolds of codimension 1 are coisotropic.

2. The graph of a Poisson morphism $f : (P_1,\pi_1) \to (P_2,\pi_2)$ is a coisotropic submanifold in $(P_1 \times P_2, \pi_1 - \pi_2)$.

3. If $S : X \to \mathbb{R}$ is a smooth function, the image of its differential $dS : X \to T^*X$ is a Lagrangian submanifold.

Definition 7.4.3. Let (P, π) be a Poisson manifold. A *canonical transformation* is a Poisson isomorphism

$$f : (P, \pi) \to (P, \pi).$$

A canonical transformation of a symplectic manifold (P, ω) is a canonical transformation of the associated Poisson manifold. A canonical transformation f is called *Hamiltonian* if there exists a function $H : P \to \mathbb{R}$ such that f is the value at 1 of the flow of canonical transformations

$$\sigma_{X_H} : \mathbb{R} \times P \to P$$

associated to the Hamiltonian vector field $X_H := \{H, -\}$.

The following example will play a fundamental role in the general gauge fixing procedure for gauge theories, presented in Section 12.5.

Example 7.4.4. Let X be a smooth manifold and $\pi : P = T^*X \to X$ be the natural projection. Let $\psi : X \to \mathbb{R}$ be a smooth function, with differential $d\psi : X \to T^*X$. Then the map

$$\mathrm{id}_P + \pi \circ d\psi : P \to P$$

is a Hamiltonian canonical transformation with Hamiltonian ψ, called a vertical hamiltonian canonical transformation on T^*X. This function is also sometimes called the gauge fixing function.

7.5 Relation with Lagrangian variational problems

The main interest of the Hamilton formalism is that it translates the problem of solving the Euler-Lagrange equation of a Lagrangian variational problem, i.e., a partial differential equation, in the problem of solving an ordinary differential equation: the Hamilton equation. It is however hard to formalize without coordinates for a general Lagrangian variational problem. We refer to Vitagliano's paper [Vit09] for a jet space formulation of the relation between Lagrangian mechanics and multi-symplectic Hamiltonian mechanics, giving in this section only a coordinate depend description of the Legendre transform.

Let $(\pi : C \to M, S)$ be a Lagrangian variational problem in which

$$\pi : C = I \times X \to I = M$$

with $I \subset \mathbb{R}$ an interval and X a manifold. We suppose that the action functional $S : \underline{\Gamma}(M, C) \to \mathbb{R}$ is local and given by

$$S(x) = \int_I L(t, x, \partial_t x) dt,$$

with $L : \mathrm{Jet}^1 C = I \times TX \to \mathbb{R}$ a Lagrangian density. We also suppose that we have a trivialization of TX and T^*X given respectively by the coordinates (x, x_1) and (q, p). The Legendre transformation is given by the map

$$\begin{array}{rccc} \mathbb{F}L: & TX & \to & T^*X \\ & (x, x_1) & \mapsto & (q,p) := (x, \frac{\partial L}{\partial x_1}). \end{array}$$

If this map is an isomorphism, one defines the Hamiltonian by

$$H(q,p,t) = \langle p, x_1 \circ (\mathbb{F}L)^{-1}(q,p)\rangle - L(t, (\mathbb{F}L)^{-1}(q,p)).$$

In simplified notation, this gives

$$H(q,p,t) := pq_1(q,p) - L(t, q, q_1(q,p)).$$

The Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial x_1^n}\right) - \frac{\partial L}{\partial x^n} = 0,$$

can be written more explicitely

$$(\partial_t^2 x^{n'}) \frac{\partial^2 L}{\partial x_1^{n'} \partial x_1^n}(t, x, \partial_t x) = -(\partial_t x^{n'}) \frac{\partial^2 L}{\partial x^{n'} \partial x_1^n}(t, x, \partial_t x) + \frac{\partial L}{\partial x^n}(t, x, \partial_t x).$$

If we suppose the non-degeneracy condition that the matrix

$$\left(\frac{\partial^2 L}{\partial x_1^{n'} \partial x_1^n}(t, x, \partial_t x)\right)_{n,n'}$$

is always invertible, we can compute $\partial_t^2 x^{n'}$ from all the other variables. This then gives a translation of the Euler-Lagrange equation through the Legendre transform to the Hamilton equations

$$\begin{cases} \frac{\partial q}{\partial t} & = & \frac{\partial H}{\partial p}, \\ \frac{\partial p}{\partial t} & = & -\frac{\partial H}{\partial q}. \end{cases}$$

For example, if we start with the Lagrangian

$$L(t, x, x_1) = \frac{1}{2} m (x_1)^2 - V(x)$$

of Newtonian mechanics, we get the momentum variable $p = \frac{\partial L}{\partial x_1} = m x_1$, so that $x_1 = \frac{p}{m}$ and

$$H(t,q,p) = \sum p_i x_{i,1} - L(t, x, x_1) = \frac{p^2}{2m} + V(q).$$

The Hamilton equations are given by

$$\begin{cases} \frac{\partial q}{\partial t} & = & \frac{\partial H}{\partial p} = \frac{p}{m}, \\ \frac{\partial p}{\partial t} & = & -\frac{\partial H}{\partial q} = -V'(q) \end{cases}$$

and one recognize in them the usual equations of Newtonian mechanics.

7.6 Hamilton-Jacobi equations

The main interest of Hamilton-Jacobi equations is that they replace the problem of solving the second order Euler-Lagrange equations by a simpler, first order equation, whose formal solution space (in the sense of algebraic analysis, see Chapter 11) is a classical finite dimensional sub-variety of the Euler-Lagrange equation, that one can study with simple geometric tools. Vitagliano has actually expanded on this simple idea by defining an Hamilton-Jacobi problem for a general partial differential equation as a finite dimensional sub-equation (that gives a method to construct explicit solutions of a given complicated system).

We first give here a coordinate description, basing on Arnold [Arn99], p253-255. We then give a coordinate free presentation, basing on Vitagliano [Vit11] and Carinena et al. [CGM+06].

Definition 7.6.1. Let $\pi : C = I \times X \to I = M$ be a bundle of classical mechanics as in the last section and let $S : \underline{\Gamma}(M, C) \to \mathbb{R}$ be a local action functional of the form $S(x) = \int_I L(t, x, \partial_t x)dt$. Let $(t_0, x_0) \in C$ be a fixed point. The *Hamilton-Jacobi* action function

$$S_{hj} : C \to \mathbb{R}$$

is defined as the integral

$$S_{hj,(t_0,x_0)}(t, x) = \int_I L(t, x_{ext}, \partial_t x_{ext})dt$$

where $x_{ext} : M \to C$ is the extremal trajectory starting at (t_0, x_0) and ending at (x, t).

For this definition to make sense, one has to suppose that the mapping $(x_{0,1}, t) \mapsto (x, t)$, given by solving the equations of motion with initial condition $x(0) = x_0$ and $\partial_t x(0) = x_{0,1}$, is non-degenerate. This can be shown to be possible in a small neighborhood of the initial condition. We thus stick to this case.

Theorem 7.6.2. *The differential of the Hamilton-Jacobi action function (for a fixed initial point) is equal to*

$$dS_{hj} = pdq - Hdt$$

where $p = \frac{\partial L}{\partial q_1}$ *and* $H = pq_1 - L$ *are defined with help of the terminal velocity* $q_1 := \partial_t x_{ext}(t, x)$ *of the extremal trajectory.*

Corollary 7.6.3. *The action function satisfies the Hamilton-Jacobi equation*

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial x}, x, t\right) = 0.$$

We now give a coordinate free presentation of the Hamilton-Jacobi formalism, basing ourselves on Vitagliano [Vit11] and Carinena et al. [CGM+06]. As before, we denote $\pi : C = I \times X \to I = M$ the bundle of classical mechanics and $\mathrm{Jet}^1 C = TX \times I$ the corresponding jet space. Let $\mathrm{Jet}^\dagger C := T^*X \times I$ be its dual bundle. One has a natural isomorphism

$$\mathrm{Jet}^1 C \times_C \mathrm{Jet}^\dagger C \cong I \times TX \times_X T^*X.$$

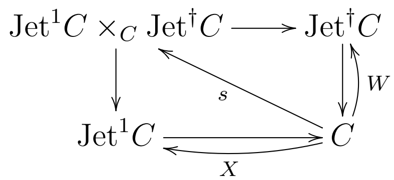
Definition 7.6.4. Let $L : \mathrm{Jet}^1 C \to \mathbb{R}$ be a Lagrangian density. Its Legendre correspondence

$$\mathrm{Leg}_L : \mathrm{Jet}^1 C \to \mathrm{Jet}^\dagger C,$$

is defined as the subspace $\Gamma_{\mathrm{Leg}_L} \subset \mathrm{Jet}^1 C \times_C \mathrm{Jet}^\dagger C$ given by

$$\Gamma_{\mathrm{Leg}_L} := \left\{ \left(t, x, x_1, \frac{\partial L}{\partial x_1} \right) \right\} \subset \mathrm{Jet}^1 C \times_C \mathrm{Jet}^\dagger C.$$

One says that a section $s \in \Gamma(C, \mathrm{Jet}^1 C \times_C \mathrm{Jet}^\dagger C)$, given by a pair of sections (X, W) of the form



is a solution to the *generalized Hamilton-Jacobi equation* if

$$i_X s^* \omega = s^* dE$$

and $\mathrm{im}(s) \subset \Gamma_{\mathrm{Leg}}$, where $E(t, x, x_1, p) = p x_1 - L(t, x, x_1)$ is the Hamiltonian function and ω is the canonical symplectic form on T^*X.

The following theorem was proved by Carinena et al. [CGM+06].

Theorem 7.6.5. *The two following conditions are equivalent:*

1. $s = (X, W)$ *is a solution to the generalized Hamilton-Jacobi equation,*
2. *every integral curve of* X *is a solution of the Euler-Lagrange equations.*

If further the Legendre transform is invertible, then $X = \mathrm{Leg}^{-1} \circ W$ *and the combination of the generalized Hamilton-Jacobi equation with the equation*

$$s^* \omega = 0$$

is locally equivalent to the condition that $W = dS$ *and* S *is a solution to the classical Hamilton-Jacobi equation.*

This formulation of the Hamilton-Jacobi formalism has been extended to higher field theories by Vitagliano (see [Vit10] and [Vit11]).

7.7 Poisson reduction

We refer to the book [Ber01] for an introduction and to [GS90] for a more complete account of the theory in the symplectic setting. For the Poisson setting, we refer to [But07], Part A, 1, 6.1. We use also the article of Kostant-Sternberg [KS87] and Marsden-Weinstein [MW74].

Definition 7.7.1. Let G be a Lie group. A *Hamiltonian G-space* is a tuple $(M, \pi, \varphi, \delta)$ composed of

1. A Poisson manifold (M, π),

2. A G-action $\varphi : G \times M \to M$ that is canonical, i.e., respects the Poisson bracket,

3. a linear map $\delta : \mathfrak{g} \to \mathcal{O}_M$ from the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ to the Poisson algebra of functions on M such that
$$\{\delta(\xi), .\} = \xi_M(-)$$
where $\mathfrak{g} \to \Theta_M, \xi \mapsto \xi_M$ is the infinitesimal action.

The *moment map* of a Hamiltonian G-space is defined as the G-equivariant map

$$\begin{array}{cccc} J: & M & \to & \mathfrak{g}^* \\ & m & \mapsto & [\xi \mapsto \delta(\xi)(m)]. \end{array}$$

Following Marsden and Weinstein, one says that $\mu \in \mathfrak{g}^*$ is a weakly regular value of J if

1. $J^{-1}(\mu)$ is a sub-manifold of M and

2. for every $m \in J^{-1}(\mu)$, the inclusion $T_m(J^{-1}(\mu)) \subset \mathrm{Ker}(D_m J)$ is an equality.

The main theorem of Mardsen-Weinstein's symplectic reduction is the following.

Theorem 7.7.2. *Let $(M, \pi, \varphi, \delta)$ be a Hamiltonian G-space (that is supposed to be symplectic). Suppose that $\mu \in \mathfrak{g}^*$ is a regular value for the moment map J and that the isotropy group G_μ of μ acts freely and properly on the constraint surface $C_\mu := J^{-1}(\mu)$. Then the manifold*
$$C_\mu / G_\mu$$
is equipped with a natural symplectic form.

An algebraic Poisson version of this construction can be given by the following.

Proposition 7.7.3. *Let $(M, \pi, \varphi, \delta)$ be a Hamiltonian G-space. Let $\mu \in \mathfrak{g}^*$ be an element and suppose that the ideal $\mathcal{I}_C$ of the constraint subspace $J^{-1}(\mu) \subset M$ is closed by the Poisson bracket (one talks of first class constraints). Then the algebra*

$$(\mathcal{O}_M / \mathcal{I}_C)^{\mathcal{I}_C}$$

of $\mathcal{I}_C$-invariant functions (such that $\{\mathcal{I}_C, f\} = 0$) in the quotient algebra is a Poisson algebra.

Remark that the above Poisson reduction identifies with the Mardsen-Weinstein reduction of the above theorem with its hypothesis.

7.8 The finite dimensional BRST formalism

The presentation of this section is based on Kostant-Sternberg [KS87]. The BRST formalism gives a simplified Hamiltonian analog of the Batalin-Vilkovisky homotopical Poisson reduction of gauge theories, that is treated in Chapter 12.

Let $(M, \pi, \varphi, \delta)$ be a Hamiltonian G-space and suppose that $0 \in \mathfrak{g}^*$ is a regular value of the moment map $J : M \to \mathfrak{g}^*$. This implies that the ideal $\mathcal{I}$ of the subspace $C := J^{-1}(0)$ is generated by the image of δ.

We consider δ as a dg-module

$$\mathcal{C} := [\underset{-1}{\mathfrak{g} \otimes \mathcal{O}_M} \xrightarrow{\delta} \underset{0}{\mathcal{O}_M}]$$

and define the Koszul resolution as the symmetric algebra

$$\mathcal{K} := \mathrm{Sym}_{\mathcal{O}_M - dg}(\mathcal{C}).$$

As a graded algebra, we have $\mathcal{K} \cong \oplus \wedge^* \mathfrak{g} \otimes \mathcal{O}_M$, and the differential is given by defining $\delta(1 \otimes f) = 0$ and $\delta(\xi \otimes 1) = 1 \otimes \delta(\xi)$. One then has that

$$H^0(\mathcal{K}, \delta) = \mathcal{O}_M / \mathcal{I}_C = \mathcal{O}_C.$$

We now consider $\mathcal{K}$ as a module over $\mathfrak{g}$ and define

$$d : \mathcal{K} \to \mathfrak{g}^* \otimes \mathcal{K} = \mathrm{Hom}(\mathfrak{g}, \mathcal{K})$$

by

$$(dk)(\xi) = \xi k,\ \xi \in \mathfrak{g},\ k \in \mathcal{K}.$$

It is a morphism of modules over the dg-algebra $\mathcal{K}$. One extends this differential to the algebra

$$\mathcal{A}^{\bullet,\bullet} := \mathrm{Sym}_{\mathcal{K}-dg}\left([\underset{0}{K} \xrightarrow{d} \underset{1}{\mathfrak{g}^* \otimes K}] \right).$$

This bidifferential bigraded algebra is called the BRST algebra. One gets as zero cohomology

$$H^0(\mathcal{A}^{\bullet,\bullet}, d) = \mathcal{K}^{\mathfrak{g}}$$

for the differential d the space of $\mathfrak{g}$-invariants in $\mathcal{K}$ and the cohomology

$$H^0(H^0(\mathcal{A}^{\bullet,\bullet}, \delta), d) = \mathcal{O}_C^{\mathfrak{g}}$$

is the space of functions on the Poisson reduction on the given Hamiltonian G-space.

The total complex $(\mathrm{Tot}(\mathcal{A}^{\bullet,\bullet}), D)$ is called the BRST complex. One can make the hypothesis that its zero cohomology is equal to the above computed space $\mathcal{O}_C^{\mathfrak{g}}$ of functions on the Poisson reduction.

Now remark that one has a canonical isomorphism

$$\wedge^* \mathfrak{g} \otimes \wedge^* \mathfrak{g}^* \cong \wedge^*(\mathfrak{g} \oplus \mathfrak{g}^*)$$

that induces a split scalar product on $\wedge^*(\mathfrak{g} \oplus \mathfrak{g}^*)$ given by the evaluation of linear forms. Let $C(\mathfrak{g} \oplus \mathfrak{g}^*)$ be the corresponding Clifford super-algebra.

Lemma 7.8.1. *The super-commutator in* $C(\mathfrak{g}\oplus\mathfrak{g}^*)$ *induces a super-Poisson structure on* $\wedge^*(\mathfrak{g}\oplus\mathfrak{g}^*)$.

Proof. Let c_i and c_j be representative elements in the Clifford algebra for some elements in the exterior algebra of respective degrees i and j. The class of the super-commutator $[c_i, c_j]$ in the degree $i+j$ exterior power depend only of the classes of c_i and c_j. This operation fulfills the axioms of a super Poisson algebra. □

The main theorem of homological perturbation theory is the following.

Theorem 7.8.2. *Under all the hypothesis given above, there exists an odd element* $\Theta \in \mathcal{A}^{\bullet,\bullet}$*, called the BRST generator, such that the Poisson bracket by* Θ *is precisely the BRST differential* D*, i.e.,*

$$\{\Theta, .\} = D$$

on the total complex $\mathrm{Tot}(\mathcal{A}^{\bullet,\bullet})$. *Since* $D^2 = 0$*, this element fulfills the so called classical master equation*

$$\{\Theta, \Theta\} = 0.$$

Remark that in the case of a group action on a point $X = \{\bullet\}$, the BRST generator $\Theta \in \wedge^3(\mathfrak{g}\oplus\mathfrak{g}^*)$ is simply given by the Lie bracket on $\mathfrak{g}$

$$\Theta = [.,.] \in \mathfrak{g}^* \wedge \mathfrak{g}^* \wedge \mathfrak{g}$$

and the BRST cohomology identifies with the Lie algebra cohomology.

Chapter 8

Homotopical algebra

One may think of homotopical algebra as a tool to compute and study systematically obstructions to the resolution of (not necessarily linear) problems. Since most of the problems that occur in physics and mathematics carry obstructions (see the introduction of this book, and Section 9.1 for an intuitive account of the main examples that we will encounter), one needs tools to study these and give an elegant presentation of the physicists' ideas (that often invented some of these techniques for their own safety, as one can see in the example of the BRST formalism of Section 7.8).

In this book, we chose to base our presentation on the setting of doctrines and of higher categorical logic (see Section 1.1), generalizing Lawvere and Ehresmann's categorical approach to mathematics. This setting carries in itself the methods of obstruction theory, because all the obstructions we want to study may be defined, at some point, as some kinds of (higher) Kan extensions of models of theories in some given doctrine.

In this chapter, we want to present the main tools that allow to compute explicitely these abstractly defined obstructions, by giving a concrete flavor to the idea of general obstruction theory. Along the way, we will also explain the various approaches that allow to base higher category theory on usual category theory, described in a set theoretical fashion. These results are useful to have a precise definition of what higher categories are, but they also allow to relate the classical axiom of choice of set theory with existence problems in doctrinal mathematics, i.e., with the doctrinal axiom of choice.

In the next chapter, we will also explain how this concrete approach to obstruction theory may be used to give a geometrical presentation of part of it, called homotopical geometry. The use of these geometrical methods gives fruitful intuitions to better understand the physicists' constructions, particularly for gauge theories.

The whole treatment of homotopical algebraic methods would require a book with a number of five hundred pages volumes strictly bigger than six (see Hovey's "model categories" [Hov99], Toen-Vezzosi's "Homotopical algebraic geometry I and II" [TV02], [TV08] and Lurie's "Higher topos theory" [Lur09d], "Higher algebra" [Lur09c] and "Derived algebraic geometry" [Lur09b] to have an idea of the scope). We refer to the above literature (and other introductory references along the way) for the interested reader.

8.1 Localizations and derived functors

The main objects of homotopical algebra are localizations of categories. Given a category $\mathcal{C}$ and a multiplicative class of morphisms, the localization $\mathcal{C}[W^{-1}]$ is the universal category in which all morphisms of W become isomorphisms. This universal property should be taken in a 2-categorical sense to be enough flexible. Recall that the symmetric monoidal category

$$(\text{Cat}, \times)$$

of categories with their cartesian product have an internal homomorphism object given for $\mathcal{C}$, $\mathcal{D}$ two categories by the category $\underline{\text{Mor}}(\mathcal{C}, \mathcal{D})$ of functors $F : \mathcal{C} \to \mathcal{D}$, with morphisms given by natural transformations.

We first give a definition of localization of categories adapted to n-categories and ${}^{\infty}n$-categories. In applications, we will often use ${}^{\infty}1$-localizations of usual categories, so that this level of generality is necessary.

Definition 8.1.1. Let $\mathcal{C}$ be an n-category and W be a multiplicative class of morphisms in $\mathcal{C}$. If $\mathcal{D}$ is an n-category, we denote $\underline{\text{Mor}}_{W^{-1}}(\mathcal{C}, \mathcal{D})$ the n-category of functors $F : \mathcal{C} \to \mathcal{D}$ that send morphisms in W to isomorphisms. A *localization* of $\mathcal{C}$ with respect to W is the datum of a pair $(\mathcal{C}[W^{-1}], L)$ composed of an n-category $\mathcal{C}[W^{-1}]$ and a functor $L : \mathcal{C} \to \mathcal{C}[W^{-1}]$ such that for every n-category $\mathcal{D}$, the functor

$$\begin{array}{ccc} \underline{\text{Mor}}(\mathcal{C}[W^{-1}], \mathcal{D}) & \to & \underline{\text{Mor}}_{W^{-1}}(\mathcal{C}, \mathcal{D}) \\ G & \mapsto & G \circ L \end{array}$$

is an equivalence of n-categories.

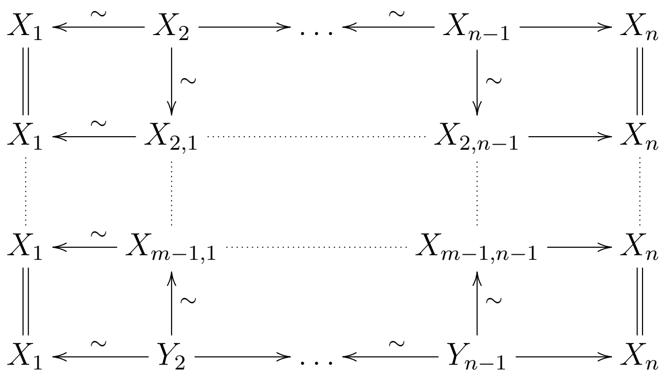
We now describe the existence result in the case of usual categories. Higher categories will be treated later.

Theorem 8.1.2. *Localizations of categories exist and are unique up to a unique equivalence of categories.*

The proof of the above theorem, that may be found in [GZ67], is "easy" and constructive: objects of the localized category are given by sequences

$$X_1 \xleftarrow{\sim} X_2 \longrightarrow \dots \xleftarrow{\sim} X_{n-1} \longrightarrow X_n$$

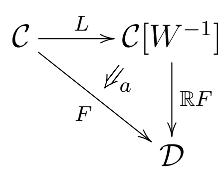
where left arrows are in the class W. Two such strings are equivalent if, after forcing them to the same length, they can be connected by a commutative diagram with vertical arrows in W of the form



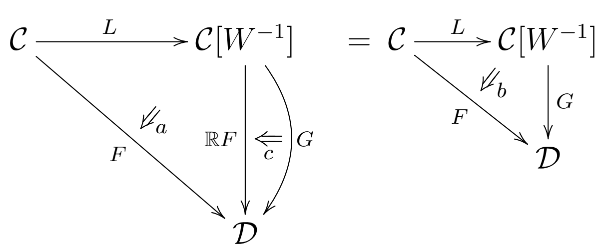
From the construction, it is not clear that the class $\mathrm{Hom}_{\mathcal{C}[W^{-1}]}(X, Y)$ of morphisms between two given objects of the localized category form a set. This is not so problematic because we allowed ourselves to work with categories where homomorphisms form classes. However, a more critical problem with the above construction is that one can't compute anything with it because its morphisms are not given in a concrete and explicit way.

We now define left and right derived functors, that are particular examples of left and right Kan extensions (see Definition 1.1.12). We refer to [KM08] for a complete treatment of derived functors using 2-categorical methods.

Definition 8.1.3. Let $\mathcal{C}$ be a category, W be a multiplicative set of morphisms in $\mathcal{C}$ and $L : \mathcal{C} \to \mathcal{C}[W^{-1}]$ be the corresponding localization. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. A *right derived functor for F* is a pair $(\mathbb{R}F, a)$, composed of a functor $\mathbb{R}F : \mathcal{C}[W^{-1}] \to \mathcal{D}$ and a natural transformation $a : \mathbb{R}F \circ L \Rightarrow F$ giving a diagram



such that for every pair (G, b) composed of a functor $G : \mathcal{C}[W^{-1}] \to \mathcal{D}$ and of a natural transformation $b : G \circ L \Rightarrow F$, there exists a unique natural transformation $c : G \Rightarrow \mathbb{R}F$ such that the equality $b = a \circ c$, visualized by the following diagrams



is true. A *total right derived functor* is a pair $(\mathbb{R}F, a)$ such that for all functors $H : N \to N'$, the composed pair $(H \circ \mathbb{R}F, H * a)$ is a right derived functor for $H \circ F$. One defines a *left derived functor for F* by the same property with the reversed direction for 2-arrows.

Remark that Definition 1.1.12 also works without change in the $^{\infty}2$-category of $^{\infty}1$-categories, giving a notion of $^{\infty}1$-derived functor. It may also be formulated in the $^{\infty}1$-category of $^{\infty}1$-categories, using that fact that functors between them also form an $^{\infty}1$-category (see Remark 1.1.13 and Corollary 8.6.8).

As said above, one can't really compute with localized categories because their morphism sets are combinatorially very big (and sometimes not even sets). The model category setting gives a method to replace objects of the localization with simpler objects called fibrant or cofibrant resolutions, so that morphisms in the localization are given by length two zigzags, and derived functors become explicitely computable. It is usually a tedious work to show that a given category fulfill the model category axioms, but in the examples of interest for physics, this work has already been done by topologists and algebraists, so that we can use them as black boxes to make explicit computations with physical theories.

8.2 Model categories

The setting of model categories was first defined by Quillen [Qui67] to make homotopy theory functorial and symmetric, and to treat with unified methods:

- classical homotopy theory, which involves a model category structure on the category of topological spaces, and
- homological algebra (derived categories), which involves a model category structure on the category of complexes of modules over a ring.

For short presentations, we refer to Toen's various introductory articles on his webpage, and in particular the course [Toe] and to Keller's notes [Kel06], 4.1. For more details, we use Hovey [Hov99] and Dwyer-Spalinski [DS95].

The aim of the theory of model categories is to give a workable notion of localization of a category $\mathcal{C}$ with respect to a given multiplicative class W of morphisms, whose elements will be denoted by arrows with tilde $\xrightarrow{\sim}$.

The general definition of model categories can be found in [Hov99], 1.1.3 (see also Quillen's original book [Qui67], and [Cis10] for weakened axioms).

Definition 8.2.1. A *model category* is a complete and cocomplete category $\mathcal{C}$ (i.e., a category that has all small limits and colimits) together with the following data

- three distinguished classes $(W, \mathrm{Fib}, \mathrm{Cof})$ of maps in $\mathcal{C}$, (whose elements are respectively called *weak equivalences*, *fibrations* and *cofibrations*, and respectively denoted $\xrightarrow{\sim}$, $\twoheadrightarrow$ and $\rightarrowtail$);
- There are two functorial factorizations $f \mapsto (p, i')$, $f \mapsto (p', i)$ (i.e., $f = p \circ i'$ and $f = p' \circ i$, functorialy in f),

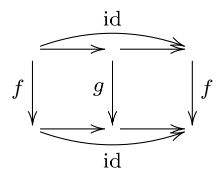
subject to the following axioms

1. (2 out of 3) If (f, g) are composable arrows (i.e. $f \circ g$ exists), then all f, g and $f \circ g$ are in W if any two of them are;

2. (Left and right lifting properties) Calling maps in $W \cap \mathrm{Fib}$ (resp. $W \cap \mathrm{Cof}$) *trivial fibrations* (resp. *trivial cofibrations*), in each commutative square solid diagram of the following two types

   the dotted arrow exists so as to make the two triangles commutative.

3. (Retracts) If a morphism f is a retract of a morphism g, meaning that there is a commutative diagram



   and f is a weak equivalence (resp. a fibration, resp. a cofibration), then g is a weak equivalence (resp. a fibration, resp. a cofibration);

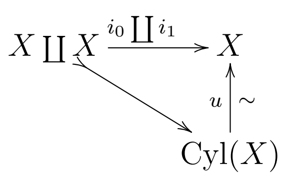
4. (Factorizations) For the functorial factorizations $f \mapsto (p, i')$, $f \mapsto (p', i)$, p is a fibration, i' a trivial cofibration, p' is a trivial fibration and i a cofibration.

The *homotopy category* of a model category $\mathcal{C}$ is defined as the localization

$$h(\mathcal{C}) := \mathcal{C}[W^{-1}].$$

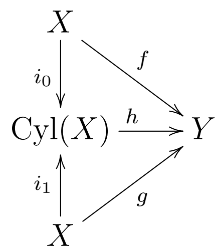
Remark that these axioms are self-dual, meaning that the opposite category is also a model category (with exchanged fibrations and cofibrations), and that the two classes Fib and Cof are determined by the datum of one of them and the class W of weak equivalences (by the lifting axiom). We now define the notion of homotopy of maps.

Definition 8.2.2. Let $X \in \mathcal{C}$, a *cylinder* for X is a factorization



of the canonical map $X \coprod X \to X$ into a cofibration followed by a weak equivalence. A cylinder for X in the opposite category is called a *path object* or a *cocylinder*. If

$f, g : X \to Y$ are maps, a *left homotopy* is a map $h : \mathrm{Cyl}(X) \to Y$ such that the following diagram commutes



A *right homotopy* is a left homotopy in the opposite category. Left and right homotopies are denoted $h : f \sim_l g$ and $h : f \sim_r g$.

The factorization axiom of model categories ensures the existence of functorial cylinders and cocylinders.

Example 8.2.3. In the category Top of (Hausdorff and compactly generated) topological spaces, one defines the classical cylinder of X by $\mathrm{Cyl}_c(X) = X \times [0,1]$. It is then clear that the injections $i_0, i_1 : X \to \mathrm{Cyl}_c(X)$ are closed and the projection $u : \mathrm{Cyl}_c(X) \to X$ is a homotopy equivalence because $[0,1]$ can be contracted to a point. One can use this classical cylinder to define in an ad-hoc way the notion of (left) homotopy of maps between topological spaces. Two maps $f, g : X \to Y$ are then called homotopic if there is a map $h : X \times [0,1] \to Y$ such that $h(0) = f$ and $h(1) = g$. This is the usual notion of homotopy. One then defines the homotopy groups of a pointed pointed topological space $(X, *)$ as the homotopy classes of pointed continuous maps

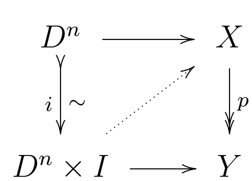
$$\pi_i(X) := \mathrm{Hom}_{\mathrm{Top}}((S^i, *), (X, *))/ \sim_l$$

from the pointed spheres to $(X, *)$. These can be equipped with a group structure by identifying S^n with the quotient of the hypercube $[0,1]^n$ by the relation that identifies its boundary with a point and setting the composition to be the map induced by

$$(f * g)(x_1, \ldots, x_n) = \begin{cases} f(2x_1, x_2, ...x_n) \text{ for } x_1 \in [0, 1/2] \text{ and} \\ (f * g)(x_1, x_2, \ldots, x_n) = g(2x_1 - 1, x_2, \ldots, x_n) \text{ for } x_1 \in [1/2, 1] \end{cases}$$

on the quotient space. The main example of model category is given by the category Top of topological spaces, where

- weak equivalences are given by weak homotopy equivalences (i.e., maps inducing isomorphisms on all the π_i's, for any choice of base-point),
- fibrations (also called Serre fibrations) $p : X \to Y$ have the right lifting property with respect to the maps $D^n \times \{0\} \to D^n \times I$ for all discs D^n:



- cofibrations $i : X \to Y$ have the left lifting property with respect to fibrations (these can also be described as retracts of transfinite compositions of pushouts of the generating cofibrations $S^{n-1} \to D^n$).

In this model category, the classical cylinder $\mathrm{Cyl}_c(X)$ is a cylinder for X if X is cofibrant. The mapping space $\mathrm{Cocyl}(X) := \underline{\mathrm{Hom}}([0,1], X)$, equipped with the topology generated by the subsets $V(K,U)$ indexed by opens U in X and compacts K in $[0,1]$, and defined by

$$V(K,U) = \{f : [0,1] \to X,\ f(K) \subset U\},$$

is a path (or cocylinder) for X if X is fibrant.

Recall that for $\mathcal{C}$ a model category, the associated homotopy category is the localization $h(\mathcal{C}) = \mathcal{C}[W^{-1}]$. It thus doesn't depend on fibrations and cofibrations. Since $\mathcal{C}$ is complete and cocomplete, it contains an initial object $\emptyset$ and a final object $*$. An object x is called fibrant (resp. cofibrant) if the natural map $x \to *$ (resp. $\emptyset \to x$) is a fibration (resp. a cofibration). The existence of functorial factorizations applied to these maps imply the existence of two functors $R, Q : \mathcal{C} \to \mathcal{C}$ together with natural transformations $\mathrm{id} \to R, Q$ that are object-wise natural equivalences. The functor R is called the fibrant replacement functor and Q the cofibrant replacement. They are valued respectively in the subcategory $\mathcal{C}_f$ and $\mathcal{C}_c$ of fibrant and cofibrant objects. Denote $\mathcal{C}_{cf}$ the subcategory of objects that are fibrant and cofibrant.

The main structure theorem of model category theory is the following (see [Hov99], Theorem 1.2.10).

Theorem 8.2.4. *Let $\mathcal{C}$ be a model category.*

1. *(Simple description of objects) The natural inclusion $\mathcal{C}_{cf} \to \mathcal{C}$ induces an equivalence of categories*
$$h(\mathcal{C}_{cf}) \to h(\mathcal{C})$$
whose quasi-inverse is induced by RQ.

2. *(Simple description of morphisms) The homotopy category $h(\mathcal{C}_{cf})$ is the quotient category of $\mathcal{C}_{cf}$ by the (left or right) homotopy relation $\sim$, i.e.,*
$$\mathrm{Hom}_{h(\mathcal{C}_{cf})}(x,y) \cong \mathrm{Hom}_{\mathcal{C}_{cf}}(x,y)/\sim .$$

3. *There is a natural isomorphism*
$$\mathrm{Hom}_{h(\mathcal{C})}(x,y) \cong \mathrm{Hom}_{\mathcal{C}}(QX, RY)/\sim .$$

This gives a nice description of maps in the homotopy category $h(\mathcal{C})$ as homotopy classes of maps in $\mathcal{C}$:

$$[x,y] := \mathrm{Hom}_{h(\mathcal{C})}(x,y) \cong [RQx, RQy] \cong \mathrm{Hom}_{\mathcal{C}}(RQx, RQy)/\sim \cong \mathrm{Hom}_{\mathcal{C}}(Qx, Ry)/\sim .$$

We now introduce the notion of homotopy limits (see Toen's lecture notes [Toe] for an introduction and Cisinski [Cis10] for an optimal formalization). These give tools to describe limits (defined in Section 1.1) in ∞-categories, by restriction to the case of higher groupoids through Yoneda's lemma.

Definition 8.2.5. Let I be a small category, $\mathcal{C}$ be a category, equipped with a class W of morphisms. Let $\mathcal{C}^I := \underline{\mathrm{Mor}}_{\mathrm{CAT}}(I, \mathcal{C})$ be the category of I-diagrams in $\mathcal{C}$ and W_I be the class of diagrams that are objectwise in W. Let $L : \mathcal{C} \to \mathcal{C}[W^{-1}]$ and $L_I : \mathcal{C}^I \to \mathcal{C}^I[W_I^{-1}]$ be the localizations and $c : (\mathcal{C}, W) \to (\mathcal{C}^I, W_I)$ be the constant diagram functor. A *homotopy limit functor* (resp. *homotopy colimit functor*) is a right (resp. left) adjoint to the localized constant functor

$$\bar{c} : \mathcal{C}[W^{-1}] \to \mathcal{C}^I[W_I^{-1}].$$

The main theorem on classical homotopy limits can be found in the book [DHKS04] or in [Cis10].

Theorem 8.2.6. *If $(\mathcal{C}, W)$ is a model category, then small homotopy limits exist in $\mathcal{C}$.*

8.3 Quillen functors and derived functors

One then wants to study functors between homotopy categories induced by morphisms of model categories. These are called Quillen functors. We refer to Dwyer-Spalinski [DS95] for the following approach to derived functors by their 2-universal property, that improves a bit on Hovey's presentation in [Hov99].

Definition 8.3.1. Let $\mathcal{C}$ and $\mathcal{D}$ be two model categories, and

$$F : \mathcal{C} \leftrightarrows \mathcal{D} : G$$

be a pair of adjoint functors.

1. If F preserves cofibrations and G preserves fibrations, one calls the pair (F, G) a *Quillen adjunction*, F a *left Quillen functor* and G a *right Quillen functor*. If (F, G) is a Quillen adjunction, the natural functor

$$\mathcal{C}_c \xrightarrow{F} \mathcal{D} \xrightarrow{L_{\mathcal{D}}} h(\mathcal{D})$$

$$(\text{resp. } \mathcal{D}_f \xrightarrow{G} \mathcal{C} \xrightarrow{L_{\mathcal{C}}} h(\mathcal{C})\ )$$

sends local equivalences in $\mathcal{C}_c$ (resp. local equivalences in $\mathcal{D}_f$) to isomorphisms, and defines an adjoint pair

$$\mathbb{L}F : h(\mathcal{C}) \cong h(\mathcal{C}_c) \leftrightarrows h(\mathcal{D}_f) \cong h(\mathcal{D}) : \mathbb{R}G.$$

This adjoint pair between $h(\mathcal{C})$ and $h(\mathcal{D})$ can be explicitely computed by setting

$$\mathbb{L}F = L_{\mathcal{D}} \circ F \circ Q \text{ and } \mathbb{L}G = L_{\mathcal{C}} \circ G \circ R.$$

2. Suppose in addition that for each cofibrant object A of $\mathcal{C}$ and fibrant object X of $\mathcal{D}$, a map $f : A \to G(X)$ is a weak equivalence in $\mathcal{C}$ if and only if its adjoint $f^\sharp : F(A) \to X$ is a weak equivalence in $\mathcal{D}$, then the pair (F, G) is called a *Quillen equivalence* and $\mathbb{L}F$ and $\mathbb{R}G$ are inverse equivalences of categories.

The main property of derived functors is that they are compatible with composition and identity. This follows easily from the above definition and the 2-universal property of localizations.

Theorem 8.3.2. *For every model categories $\mathcal{C}$, there is a natural isomorphism $\epsilon : \mathbb{L}(\mathrm{id}_{\mathcal{C}}) \to \mathrm{id}_{h(\mathcal{C})}$. For every pair of left Quillen functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ between model categories, there is a natural isomorphism $m_{GF} : \mathbb{L}G \circ \mathbb{L}F \to \mathbb{L}(G \circ F)$.*

The above compatibility isomorphism moreover fulfills additional associativity and coherence conditions that can be found in [Hov99], theorem 1.3.7. One gets the right derived version of this theorem by passing to the opposite category.

8.4 Bousfield's homotopical localizations

An important operation in homotopical sheaf theory is the notion of Bousfield localization of a model category, that solves the following problem: given a model category $\mathcal{C}$ and a subset S of maps in $\mathcal{C}$, find another model category $L_S\mathcal{C}$ in which the maps in S are weak equivalences. The localization functor must be a left Quillen functor

$$f : \mathcal{C} \to L_S\mathcal{C}$$

that is universal with respect to left Quillen functors $f : \mathcal{C} \to \mathcal{D}$ that send images of morphisms in S in $h(\mathcal{C})$ to isomorphisms in $h(\mathcal{D})$.

Let $\mathcal{C}[W^{-1}]$ be the ∞-localization of $\mathcal{C}$ (also called the Hammock localization, to be described in Theorem 8.6.10). This is an $^{\infty}1$-category, and in particular, its morphism spaces are ∞-groupoids (that we will model on simplicial sets in Section 8.5).

An object x of $\mathcal{C}$ is called S-local if it sees maps in S as weak equivalences, i.e., if it is fibrant and for any $y \to y'$ in S, the induced map

$$\underline{\mathrm{Mor}}_{\mathcal{C}[W^{-1}]}(y', x) \to \underline{\mathrm{Mor}}_{\mathcal{C}[W^{-1}]}(y, x)$$

is an equivalence of ∞-groupoids. A map $f : x \to x'$ is called an S-local equivalence if it is seen as an equivalence by any S-local map, i.e., if for any S-local object y in $\mathcal{C}$, the induced map

$$\underline{\mathrm{Mor}}_{\mathcal{C}}(x', y) \to \underline{\mathrm{Mor}}_{\mathcal{C}}(x, y)$$

is an equivalence of ∞-groupoids.

Theorem 8.4.1. *Let $\mathcal{C}$ be a standard model category. Then the following classes of maps in $\mathcal{C}$*

- $W_S :=$ *S-local equivalence;*
- $\mathrm{Cof}_S :=$ *cofibrations in $\mathcal{C}$;*

are part of a model structure on $\mathcal{C}$, denoted by $L_S\mathcal{C}$ and the identity functor induces a left Quillen functor

$$\mathcal{C} \to L_S\mathcal{C}$$

that is universal among left Quillen functors whose derived functor send morphisms in S to isomorphisms.

8.5 Simplicial sets and higher groupoids

We refer to the book [GJ99] for an introduction to simplicial sets.

The notion of simplicial set allows to give a purely combinatorial description of the homotopy category of the category TOP of topological spaces. It is also a fundamental tool in higher category theory, because it is the universally accepted model for the theory of ∞-groupoids.

Let Δ be the category whose objects are finite ordered sets $[n] = [0, \dots, n-1]$ and whose morphisms are nondecreasing maps.

Definition 8.5.1. A *simplicial set* is a contravariant functor

$$X : \Delta^{op} \to \text{SETS}.$$

We denote SSETS the category of simplicial sets.

To give examples of simplicial sets, one can use the following description of the category Δ by generators and relations.

Proposition 8.5.2. *The morphisms*

$$\begin{array}{llll} d^i : [n-1] \to [n] & 0 \le i \le n & \text{(cofaces)} \\ s^j : [n+1] \to [n] & 0 \le j \le n & \text{(codegeneracies)} \end{array}$$

given by

$$d^i([0, \dots, n-1]) = [0, \dots, i-1, i+1, \dots, n]$$

and

$$s^j([0, \dots, n+1]) = [0, \dots, j, j, \dots, n]$$

fulfill the so-called cosimplicial identities

$$\begin{cases} d^j d^i & = & d^i d^{j-1} & \text{if } i < j \\ s^j d^i & = & d^i s^{j-1} & \text{if } i < j \\ s^j d^j & = & 1 = s^j d^{j+1} & \\ s^j d^i & = & d^{i-1} s^j & \text{if } i > j+1 \\ s^j s^i & = & s^i s^{j+1} & \text{if } i \le j \end{cases}$$

and give a set of generators and relations for the category Δ.

Define the n simplex to be $\Delta^n := \text{Hom}(-, [n])$. Yoneda's lemma imply that for every simplicial set X, one has

$$\text{Hom}(\Delta^n, X) \cong X_n.$$

The boundary $\partial \Delta^n$ is defined by

$$(\partial \Delta^n)_m := \{f : [m] \to [n],\ \text{im}(f) \neq [n]\}$$

and the k-th horn $\Lambda^n_k \subset \partial \Delta^n$ by

$$(\Lambda^n_k)_m := \{f : [m] \to [n],\ k \notin \text{im}(f)\}.$$

The category SSETS has all limits and colimits (defined component-wise) and also internal homomorphisms defined by

$$\underline{\mathrm{Hom}}(X, Y) : [n] \mapsto \mathrm{Hom}_{\Delta_n}(X \times \Delta_n, Y \times \Delta_n) = \mathrm{Hom}(X \times \Delta^n, Y).$$

The geometric realization $|\Delta^n|$ is defined to be

$$|\Delta^n| := \{(t_0, \ldots, t_n) \in [0,1]^{n+1}, \sum t_i = 1\}.$$

It defines a covariant functor $|.| : \Delta \to \mathrm{TOP}$ with maps $\theta_* : |\Delta^n| \to |\Delta^m|$ for $\theta : [n] \to [m]$ given by

$$\theta_*(t_0, \ldots, t_m) = (s_0, \ldots, s_n)$$

where

$$s_i = \begin{cases} 0 & \text{if} \quad \theta^{-1}(i) = \emptyset \\ \sum_{j \in \theta^{-1}(i)} t_j & \text{if} \quad \theta^{-1}(i) \neq \emptyset. \end{cases}$$

The geometric realization of a general simplicial set X is the colimit

$$|X| = \mathrm{colim}_{\,\Delta^n \to X} |\Delta^n|$$

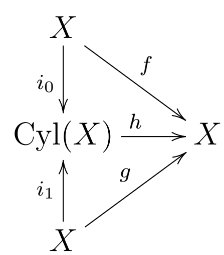
indexed by the category of maps $\Delta^n \to X$ for varying n. The ∞-groupoid (also called the singular simplex) of a given topological space Y is the simplicial set

$$\Pi_\infty(Y) : [n] \mapsto \mathrm{Hom}_{\mathrm{TOP}}(|\Delta^n|, Y).$$

The geometric realization and ∞-groupoid functors are adjoint meaning that

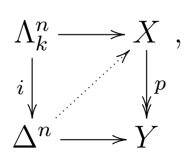
$$\mathrm{Hom}_{\mathrm{TOP}}(|X|, Y) \cong \mathrm{Hom}_{\mathrm{SSETS}}(X, \Pi_\infty(Y)).$$

Definition 8.5.3. The *simplicial cylinder* of a given simplicial set is defined as $\mathrm{Cyl}(X) := X \times \Delta^1$. Let $f, g : X \to Y$ be morphisms of simplicial sets. A *homotopy* between f and g is a factorization



Fibrations of simplicial sets are defined as maps that have the right lifting property with respect to all the standard inclusions $\Lambda^n_k \subset \Delta^n$, $n > 0$.

Definition 8.5.4. A morphism $p : X \to Y$ of simplicial sets is called a *fibration* (or a *Kan fibration*) if in every commutative square



the dotted arrow exists to as to make the two triangles commutative. A simplicial set X is called *fibrant* (or a *Kan complex*) if the projection map $X \to \{*\}$ is a fibration.

Let X be a simplicial set. Its set of connected components is defined as the quotient

$$\pi_0(X) := X_0/\sim$$

by the equivalence relation $\sim$ generated by the relation $\sim_1$, called "being connected by a path", i.e., $x_0 \sim_1 x_1$ if there exists $\gamma : \Delta^1 \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

Lemma 8.5.5. *The functor* $\pi_0 : \mathrm{SS}\textsc{ets} \to \textsc{Sets}$ *commutes with direct products.*

Proof. This follows from the fact that $\pi_0(X)$ identifies naturally to the set of connected components $\pi_0(|X|)$ of the geometric realization, and geometric realization commutes with products. □

The above lemma implies that simplicial sets may be equipped with a new category structure $\mathrm{SS}\textsc{ets}^{\pi_0}$, whose morphisms are given by elements in $\pi_0(\underline{\mathrm{Hom}}(X,Y))$, and whose composition is induced by the internal composition map

$$\underline{\mathrm{Hom}}(X,Y) \times \underline{\mathrm{Hom}}(Y,Z) \to \underline{\mathrm{Hom}}(X,Z).$$

Definition 8.5.6. A morphism $f : X \to Y$ of simplicial sets is called a *weak equivalence* if for all fibrant object Z, the natural map

$$\pi_0(\underline{\mathrm{Hom}}(f,Z)) : \pi_0(\underline{\mathrm{Hom}}(Y,Z)) \to \pi_0(\underline{\mathrm{Hom}}(X,Z))$$

is bijective.

The following theorem is due to Quillen [Qui67].

Theorem 8.5.7. *The category of simplicial sets is equipped with a model category structure such that*

1. *weak equivalences are weak equivalences of simplicial sets,*
2. *fibrations are Kan fibrations,*
3. *cofibrations are injections.*

Definition 8.5.8. The above defined model structure is called the model structure of higher groupoids, and denoted ${}^{\infty}\textsc{Grpd}$.

Let X be a fibrant simplicial set and $x : \Delta^0 \to X$ be a base point of X. For $n \geq 1$, define the homotopy group $\pi_n(X,x)$ as the set of homotopy classes of maps $f : \Delta^n \to X$ that fit into a diagram

$$\begin{array}{ccc} \Delta^n & \xrightarrow{f} & X \\ \uparrow & & \uparrow \\ \partial\Delta^n & \longrightarrow & \Delta^0 \end{array}.$$

The geometric realization of such maps correspond to maps

$$f : S^n \to |X|.$$

Theorem 8.5.9. *Weak equivalences of simplicial sets may be defined as morphisms that induce isomorphisms on homotopy groups, for any choice of base point $x : \Delta^0 \to X$.*

Theorem 8.5.10. *The geometric realization and the ∞-groupoid functors induce a Quillen adjunction*

$$|-| : {}^{\infty}\text{GRPD} \leftrightarrows \text{TOP} : \Pi_\infty$$

whose derived version is an equivalence of homotopy categories

$$\mathbb{L}|-| : h({}^{\infty}\text{GRPD}) \overset{\sim}{\leftrightarrows} h(\text{TOP}) : \mathbb{R}\Pi_\infty.$$

8.6 ∞-categories

In this section, we will often use the basic notions of higher category theory described in Section 1.1. In particular, we suppose known the abstract notions of Kan extensions and limits given in Definitions 1.1.12 and 1.1.14, with the adaptation given in Remark 1.1.13.

The simplicial approach to higher category theory finds its roots and expands on the Dwyer-Kan simplicial localization methods [DK80]. Recall from Section 8.5 that there is a natural model structure ${}^{\infty}$GRPD on the category of simplicial sets whose equivalences are weak homotopy equivalences. We now describe another model structure ${}^{\infty}$CAT whose equivalences are categorical equivalences. This leads to Joyal's quasi-categories. This gives a nice model for a theory of ${}^{\infty}$1-categories (see Section 8.10), that we will simply call ∞-categories. We refer to [Joy11], [Lur09d] and [Lur09c] for nice accounts of these structures and of their use in geometry and algebra. These are the simplest generalizations of nerves of categories that also allow to encode higher groupoids.

In this section, we give an elementary presentation of ∞-categories, that is easier to work with than the abstract one, whose definition will be sketched in Section 8.10. It is useful to present both approaches because the less elementary method of loc. cit. makes the generalization of usual categorical concepts more simple and natural than the present one.

Definition 8.6.1. Let $\mathcal{C}$ be a category. The *nerve* of $\mathcal{C}$ is the simplicial set $N(\mathcal{C})$ whose n-simplices are

$$N(\mathcal{C})_n := \text{Hom}_{\text{CAT}}([n], \mathcal{C}),$$

where $[n]$ is the linearly ordered set $\{0, \dots, n\}$. More concretely,

1. $N(\mathcal{C})_0$ is the set of objects of $\mathcal{C}$,
2. $N(\mathcal{C})_1$ is the set of morphisms in $\mathcal{C}$, and
3. $N(\mathcal{C})_n$ is the set of families

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n$$

of composable arrows.

To define the simplicial structure of $N(\mathcal{C})$, one only needs to define maps

$$\begin{array}{lll} d_i : N(\mathcal{C})_n \to N(\mathcal{C})_{n-1}, & \text{for } 0 \leq i \leq n & \text{(faces)} \\ s_j : N(\mathcal{C})_n \to N(\mathcal{C})_{n+1}, & \text{for } 0 \leq j \leq n & \text{(degeneracies)} \end{array}$$

satisfying the simplicial identities, described in Section 8.5. The faces d_i are given by composition

$$x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i-1}} x_{i-1} \xrightarrow{f_{i+1}\circ f_i} x_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_n} x_n$$

of two consecutive arrows and the degeneracies s_i by insertion of identities

$$x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_i} x_i \xrightarrow{\mathrm{id}_{x_i}} x_i \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_n} x_n.$$

Let $a_i : [1] \to [k]$ be the morphisms in the simplicial category Δ given by $a^i(0) = i$ and $a^i(1) = i+1$ for $0 \leq i \leq k-1$ and let a_i be the corresponding maps in Δ^{op}. Denote X the simplicial set $N(\mathcal{C})(a_i)$. The map

$$X(a_i) = X_k \to X_1$$

simply sends a family of k composable arrows to the i-th one. Their fiber product give the so-called *Segal maps*

$$\varphi_k : X_k \to X_1 \underset{X_0}{\times} \cdots \underset{X_0}{\times} X_1,$$

and by definition of X_k, these maps are bijections.

Proposition 8.6.2. *Let* Cat *be the category of small categories. The nerve functor*

$$\begin{array}{rccc} N : & \mathrm{Cat} & \to & \mathrm{SSets} \\ & C & \mapsto & [n] \mapsto \mathrm{Hom}_{\mathrm{Cat}}([n], C) \end{array}$$

is a fully faithful embedding, with essential image the subcategory of simplicial sets whose Segal maps are bijections. Moreover, it has a left adjoint τ_1, called the fundamental category *functor.*

Proof. The full faithfulness is clear from the above description of the nerve. The existence of an adjoint to the nerve functor follows formally from the completeness of the categories in play (see [GZ67]). We only describe it explicitely. Let X be a simplicial set. Two elements f and g of X_1 are called composable if there exists $h \in X_2$ such that $h_{[0,1]} = f$ and $h_{[1,2]} = g$. Their composition is then given by $h_{[0,2]}$. We put on X_1 the smallest equivalence relation generated by this composition, that is stable by composition. We define $\tau_1(X)$ as the category whose objects are elements in X_0, whose morphisms are families $(f_1, \ldots, f_n)$ of elements in X_1 that are composable, quotiented by the composition equivalence relation. □

Definition 8.6.3. A *quasi-fibration* of simplicial sets is a morphism $p : X \to Y$ such that every commutative square

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ \Delta^n & \longrightarrow & Y \end{array},$$

with $0 < k < n$ can be completed by a dotted arrow so as to make the two triangles commutative. A *categorical equivalence* of simplicial sets is a morphism $f : X \to Y$ that induces an equivalence of categories

$$\tau_1(f) : \tau_1(X) \to \tau_1(Y).$$

A *quasi-category* is a simplicial set X such that the morphism $X \to \{*\}$ is a quasi-fibration. More precisely, it is a simplicial set such that for $0 < k < n$, there exists a dotted arrow rendering the diagram

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

commutative. An ∞-*groupoid* is a quasi-category such that the dotted arrow in the above diagram exists of $0 \leq k \leq n$.

Remark that an ∞-groupoid is exactly the same as a fibrant simplicial set for the model structure $^\infty$GRPD of Section 8.6 (i.e., a homotopy type), and a quasi-category such that the dotted arrow rendering the diagram

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

commutative is unique is simply a nerve.

We state a useful characterization of quasi-categories, whose proof can be found in [Joy11].

Lemma 8.6.4. *A simplicial set $\mathcal{C}$ is a quasi-category if and only if the projection*

$$\underline{\mathrm{Hom}}_{\mathrm{SSets}}(\Delta^2, \mathcal{C}) \longrightarrow \underline{\mathrm{Hom}}_{\mathrm{SSets}}(\Lambda^2_1, \mathcal{C})$$

is a trivial fibration.

We now give an intrinsic formulation of the fact that the collection of ∞-categories form an ∞-category.

Theorem 8.6.5. *There exists a model structure on* SSETS, *denoted* $^\infty$CAT, *whose fibrations are quasi-fibrations, whose cofibrations are monomorphisms, and whose weak equivalences are categorical equivalences. Its fibrant objects are quasi-categories.*

Proof. We refer to [Joy11], and [Lur09d] for a complete proof. □

Theorem 8.6.5 may be improved, by saying that the collection of ∞-categories form an $^\infty$2-category (with non-invertible 2-morphisms). We will explain this shortly in Section 8.10. This is mandatory to our treatment of Kan extensions, limits, and colimits in Section 1.1, exactly as in the classical categorical setting, because these definitions use $^\infty$2-categorical universal properties. However, one may avoid passing to this 2-dimensional setting by using only the above results. This viewpoint was systematically taken care of by Joyal [Joy11] and Lurie [Lur09d].

Definition 8.6.6. An ∞-category is an object of the homotopy category of ${}^{\infty}\text{CAT}$.

The following result shows that quasi-categories themselves form a quasi-category.

Proposition 8.6.7. *If X is a simplicial set and $\mathcal{C}$ is a quasi-category, the simplicial set*

$$\underline{\text{Mor}}_{{}^{\infty}\text{CAT}}(X, \mathcal{C}) := \underline{\text{Hom}}_{\text{SSETS}}(X, \mathcal{C})$$

is a quasi-category. Moreover, if $F : X \to Y$ or $G : \mathcal{C} \to \mathcal{D}$ are categorical equivalences, then

$$\underline{\text{Mor}}(F, \mathcal{C}) : \underline{\text{Mor}}_{{}^{\infty}\text{CAT}}(X, \mathcal{C}) \to \underline{\text{Mor}}_{{}^{\infty}\text{CAT}}(Y, C)$$

and

$$\underline{\text{Mor}}(X, G) : \underline{\text{Mor}}_{{}^{\infty}\text{CAT}}(X, \mathcal{C}) \to \underline{\text{Mor}}_{{}^{\infty}\text{CAT}}(X, \mathcal{D})$$

are categorical equivalences.

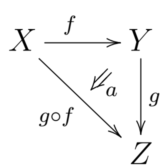
Corollary 8.6.8. *The model category ${}^{\infty}\text{CAT}$ is cartesian closed. The collection of ∞-categories form an ∞-category ${}^{\infty}\text{CAT}$.*

Proof. This is proved in [Joy11] and [Lur09d]. The quasi-category ${}^{\infty}\text{CAT}$ is naturally associated, through a simplicial nerve construction, to the simplicial category whose simplicial sets of morphisms are given by invertible ∞-functors, i.e., by a maximal Kan complex in the quasi-category $\underline{\text{Mor}}_{{}^{\infty}\text{CAT}}(\mathcal{C}, \mathcal{D})$. □

Definition 8.6.9. If $\mathcal{C}$ is a quasi-category representing an ∞-category, one calls

1. elements in $\mathcal{C}_0$ its *objects*,
2. elements in $\mathcal{C}_1$ its *morphisms*,
3. elements in $\mathcal{C}_2$ its *compositions*.

If $\mathcal{C}$ is a quasi-category, one may interpret elements in $\mathcal{C}_2$ as triangular 2-morphisms



in the corresponding ∞-category, encoding compositions. Remark that the arrow $g \circ f$ is only unique up to equivalence, and that the 2-morphism a is invertible only up to a 3-morphism, that may be drawn as a tetrahedron.

The following theorem is completely central and at the origin of the simplicial approach to higher category theory. It is essentially due to Dwyer and Kan [DK80].

Theorem 8.6.10. *Let $\mathcal{C}$ be an ∞-category and W be a class of morphisms in $\mathcal{C}$. There exists a localized ∞-category, i.e., a functor $L : \mathcal{C} \to \mathcal{C}[W^{-1}]$ such that for every ∞-category $\mathcal{D}$, the functor*

$$\begin{array}{ccc} \underline{\text{Mor}}(\mathcal{C}[W^{-1}], \mathcal{D}) & \to & \underline{\text{Mor}}_{W^{-1}}(\mathcal{C}, \mathcal{D}) \\ G & \mapsto & G \circ L \end{array}$$

is an equivalence of ∞-categories.

Theorem 8.6.11. *If $F : (\mathcal{C}, W_{\mathcal{C}}) \leftrightarrows (\mathcal{D}, W_{\mathcal{D}}) : G$ is a Quillen adjunction between model categories. There is an ∞-adjunction (in the sense of Definition 1.1.10)*

$$\mathbb{L}F : \mathcal{C}[W_{\mathcal{C}}^{-1}] \leftrightarrows \mathcal{D}[W_{\mathcal{D}}^{-1}] : \mathbb{R}G$$

between the associated ∞-localizations.

Proposition 8.6.12. *The identity functors*

$$^{\infty}\text{GRPD} \leftrightarrows {}^{\infty}\text{CAT}$$

induce a Quillen adjunction between the two given model categories on SSETS*. The corresponding adjunction on homotopy categories is also denoted*

$$i : h(^{\infty}\text{GRPD}) \leftrightarrows h(^{\infty}\text{CAT}) : \tau_0,$$

and the functor i is a fully faithful embedding.

Definition 8.6.13. Let $\mathcal{C}$ be a quasi-category, and x and y be two objects of $\mathcal{C}$. The simplicial set of morphisms from x to y is defined by the fiber product

$$\begin{array}{ccc} \underline{\text{Mor}}_{\mathcal{C}}(x,y) & \longrightarrow & \underline{\text{Hom}}_{\text{SSETS}}(\Delta^1, \mathcal{C}) \\ \downarrow & & \downarrow (s,t) \\ \Delta_0 & \xrightarrow{(x,y)} & \mathcal{C} \times \mathcal{C} \end{array}$$

We now show that ∞-categories have the nice property of being weakly enriched over ∞-groupoids, in the sense of the following proposition. This implies that they are a good candidate for a theory of $^{\infty}1$-categories.

Proposition 8.6.14. *The simplicial set $\underline{\text{Mor}}_{\mathcal{C}}(x,y)$ is a Kan complex. There exists a morphism of simplicial sets*

$$\circ : \underline{\text{Mor}}_{\mathcal{C}}(x,y) \times \underline{\text{Mor}}_{\mathcal{C}}(y,z) \to \underline{\text{Mor}}_{\mathcal{C}}(x,z)$$

*that gives an associative composition law in the category $^{\infty}$*GRPD*, unique up to isomorphism.*

Proof. From Lemma 8.6.4, we know that the natural map

$$\underline{\text{Hom}}_{\text{SSETS}}(\Delta^2, \mathcal{C}) \longrightarrow \underline{\text{Hom}}_{\text{SSETS}}(\Lambda^2_1, \mathcal{C})$$

is a trivial fibration, so that it has a section s. One has a natural cartesian square

$$\begin{array}{ccc} \underline{\text{Hom}}_{\text{SSETS}}(\Lambda^2_1, \mathcal{C}) & \longrightarrow & \underline{\text{Hom}}_{\text{SSETS}}(\Delta^1, \mathcal{C}) \\ \downarrow & & \downarrow t \\ \underline{\text{Hom}}_{\text{SSETS}}(\Delta^1, \mathcal{C}) & \xrightarrow{s} & \underline{\text{Hom}}_{\text{SSETS}}(\Delta^0, \mathcal{C}) = \mathcal{C} \end{array},$$

so that the map $\underline{\mathrm{Hom}}(d_1, \mathcal{C}) : \underline{\mathrm{Hom}}(\Delta^2, \mathcal{C}) \to \underline{\mathrm{Hom}}(\Delta^1, \mathcal{C})$ induces, by composition with s a composition law

$$\circ : \underline{\mathrm{Hom}}_{\mathrm{SSets}}(\Delta^1, \mathcal{C}) \underset{\mathcal{C}}{\times} \underline{\mathrm{Hom}}_{\mathrm{SSets}}(\Delta^1, \mathcal{C}) \to \underline{\mathrm{Hom}}_{\mathrm{SSets}}(\Delta^1, \mathcal{C}).$$

This map reduces to a well defined composition map

$$\circ : \underline{\mathrm{Mor}}_{\mathcal{C}}(x, y) \times \underline{\mathrm{Mor}}_{\mathcal{C}}(y, z) \to \underline{\mathrm{Mor}}_{\mathcal{C}}(x, z).$$

By construction, this law is unique up to isomorphism and associative in ${}^{\infty}\mathrm{Grpd}$. □

We now define the notion of limits in ∞-categories. One may give a more explicit presentation in the setting of quasi-categories, but this would take us too far (see [Lur09d] and [Joy11]).

Definition 8.6.15. Let $\mathcal{C}$ and I be two ∞-categories. A limit functor (resp. colimit functor) is a right (resp. left) adjoint to the constant ∞-functor

$$\mathcal{C} \cong \underline{\mathrm{Mor}}_{\infty\mathrm{Cat}}(*, \mathcal{C}) \to \underline{\mathrm{Mor}}_{\infty\mathrm{Cat}}(I, \mathcal{C}).$$

The only limits that we really need to compute explicitely are given by limits of higher groupoids, because we will always use Yoneda's lemma to reduce computations of limits in ∞-categories to computations of limits for their morphism spaces.

We now use the nerve construction to give an explicit description of homotopy limits and homotopy colimits of simplicial sets (see [GJ99], [BK72] and [Lur09d], Appendix A). The fact that every simplicial set is cofibrant implies that homotopy colimits have a simpler description than homotopy limits.

Definition 8.6.16. Let $f : I \to \mathrm{SSets}$ be a diagram of simplicial sets indexed by a small category. For each n, denote Ef_n the so-called *translation category*, with objects (i, x), with i an object of I and $x \in f(i)_n$, and morphisms $\alpha : (i, x) \to (j, y)$ given by morphisms $\alpha : i \to j$ of I such that $f(\alpha)_n(x) = y$. The *homotopy colimit* of f is the diagonal simplicial set of the bisimplicial set

$$\mathrm{hocolim} f = [n \mapsto N(Ef_n)],$$

obtained by composing with the diagonal functor $\Delta^{op} \to \Delta^{op} \times \Delta^{op}$. The *homotopy limit* of a diagram $f : I^{op} \to \mathrm{SSets}$ *whose values are Kan complexes* is defined dualy to colimits, as diagonals of nerve of translation categories of the diagram $f : I \to \mathrm{SSets}^{op}$.

Concretely, one has

$$(\mathrm{hocolim} f)_{n,m} = \coprod_{i_0 \to i_1 \to \cdots \to i_m} f(i_0)_n.$$

We refer to loc. cit. for a proof of the following proposition.

Proposition 8.6.17. *The above defined homotopy colimits and limits of simplicial sets are indeed the homotopy limits and colimits for the model structure ${}^{\infty}\mathrm{Grpd}$, and also the ∞-limits and ∞-colimits in the ∞-category ${}^{\infty}\mathrm{Grpd}$.*

We now give various examples of concrete computations of homotopy limits in the ∞-category

$$^{\infty}\mathrm{GRPD} = \mathrm{SSETS}[W_{eq}^{-1}]$$

obtained by localizing simplicial sets by weak homotopy equivalences (remark that we have the same notation of the quasi-category and its homotopy category; its meaning will be clear from the situation). These examples will play an important role in applications.

Example 8.6.18. 1. Let G be a group, seen as a category with invertible arrows, and let BG be the associated nerve. Then one has a homotopy cartesian diagram

$$\begin{array}{ccc} G & \longrightarrow & \{*\} \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & BG \end{array}.$$

2. Let $(X,*)$ be a pointed topological space and $\Omega X := \underline{\mathrm{Hom}}((S^1,*),(X,*))$ be the topological space of continuous pointed loops in X. Then one has the homotopy cartesian diagram

$$\begin{array}{ccc} \Omega X & \longrightarrow & \{*\} \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & X \end{array}.$$

3. Let X be a topological space and $LX := \underline{\mathrm{Hom}}(S^1, X)$ be the topological space of (unpointed) loops in X. Then one has the homotopy cartesian diagram

$$\begin{array}{ccc} LX & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle\Delta} \\ X & \xrightarrow{\Delta} & X\times X \end{array}$$

in TOP (meaning that applying Π_∞ : TOP $\to$ $^{\infty}$GRPD to it gives a homotopy cartesian diagram).

4. More generally, if $Y, Z \subset X$ are two closed subset of a topological space X, and

$$L_{Y\to Z}X \subset \underline{\mathrm{Hom}}([0,1], X)$$

is the subspace of paths starting on Y and ending on Z, then one has the homotopy cartesian diagram

$$\begin{array}{ccc} L_{Y\to Z}X & \xrightarrow{\mathrm{ev}_0} & Z \\ \downarrow{\scriptstyle \mathrm{ev}_1} & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

5. Even more generally, if $f : Y \to X$ and $g : Z \to X$ are continuous maps of topological spaces, then one has a decomposition $f = p \circ i$ of the form

$$Y \xrightarrow{i} Y \times_X \underline{\mathrm{Hom}}(\Delta_1, X) \xrightarrow{p} X,$$

with p a fibration and i a trivial cofibration. The homotopy fiber product of f and g is then given by the fiber product

$$Y \times_X^h Z = (Y \times_X \underline{\mathrm{Hom}}(\Delta_1, X)) \times_X Z,$$

that is the relative space of path $L_{f \to g}X$, given by triples (y, γ, z) with $y \in Y$ and $z \in Z$, and γ a path in X from $f(y)$ to $g(z)$. This construction also applies directly to a cartesian diagram of fibrant simplicial sets, i.e., of ∞-groupoids.

8.7 Stable ∞-categories

We refer to Lurie [Lur09c], Chapter 1, for the content of this section. The reader may also refer to Quillen [Qui67] for a more classical treatment of similar matters, in terms of pointed model categories.

Definition 8.7.1. Let $\mathcal{C}$ be an ∞-category. A *zero object* in $\mathcal{C}$ is an object that is both initial and final. An ∞-category is called *pointed* if it has a zero object.

The existence of a zero object implies the existence of a zero morphisms

$$0 : X \to Y$$

between every two objects in the homotopy category $h(\mathcal{C}) := \tau_1(\mathcal{C})$.

Definition 8.7.2. Let $\mathcal{C}$ be a pointed ∞-category. A *triangle* in $\mathcal{C}$ is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Z \end{array}$$

where 0 is a zero object of $\mathcal{C}$. A triangle in $\mathcal{C}$ is called a *fiber sequence* if it is a pullback square, and a *cofiber sequence* if it is a pushout square. If it is a fiber sequence, it is called a *fiber* for the morphisms $g : Y \to Z$. If it is a cofiber sequence, it is called a *cofiber* for the morphism f.

Definition 8.7.3. Let $\mathcal{C}$ be a pointed ∞-category. A *loop space object* is a fiber ΩX for the morphism $0 \to X$ and a *suspension object* ΣX is a cofiber for the morphism $0 \to X$.

Definition 8.7.4. An ∞-category $\mathcal{C}$ is called *stable* if it satisfies the following conditions:

1. There exists a zero object $0 \in \mathcal{C}$,

2. Every morphisms in $\mathcal{C}$ admits a fiber and a cofiber,

3. A triangle in $\mathcal{C}$ is a fiber sequence if and only if it is a cofiber sequence.

In a stable ∞-category, the loop space and suspension are autoequivalences

$$\Omega : \mathcal{C} \to \mathcal{C} \text{ and } \Sigma : \mathcal{C} \to \mathcal{C}$$

inverses of each other. Moreover, the homotopy category $h(\mathcal{C}) := \tau_1(\mathcal{C})$ is additive.

Theorem 8.7.5. *Let $\mathcal{C}$ be a stable ∞-category, let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Z \end{array}$$

be a fibration sequence, and let A be any object in $h(\mathcal{C})$. There is a long exact homotopy sequence

$$\cdots \to \mathrm{Hom}(A, \Omega^{n+1}Z) \to \mathrm{Hom}(A, \Omega^n X) \to \mathrm{Hom}(A, \Omega^n Y) \to \mathrm{Hom}(A, \Omega^n Z) \to \cdots$$

Definition 8.7.6. Let $\mathcal{C}$ be an ∞-category with finite limits. The *stabilization* of $\mathcal{C}$ is the limit

$$\mathrm{Stab}(\mathcal{C}) := \lim(\cdots C \xrightarrow{\Omega} C)$$

in the ∞-category of ∞-categories.

Definition 8.7.7. A *pointed model category* is a model category whose initial and final object coincide.

If (M, W) is a pointed model category, its ∞-localization $\mathcal{C} = M[W^{-1}]$ is a pointed ∞-category.

Example 8.7.8. The category TOP_* of pointed (Hausdorff and compactly generated) topological spaces is an example of a pointed model category. In this setting, the suspension is given by the derived functor of the functor

$$X \mapsto \Sigma(X) := X \wedge S^1$$

of smash product with S^1 (product in the category TOP_* of pointed spaces) and the loop is given by the pointed mapping space

$$X \mapsto \Omega(X) := \underline{\mathrm{Hom}}_{\mathrm{TOP}_*}(S^1, X).$$

The stabilization of the associated ∞-category TOP_* is the ∞-category Sp of spectra.

Example 8.7.9. The category $\mathrm{MOD}_{dg}(A)$ of dg-modules on a given ring (see Section 8.8) is another example of a pointed model category. In this setting, the suspension is given by the shift functor

$$X \mapsto X[1]$$

and the loop space is given by the functor

$$X \mapsto X[-1].$$

The cofiber of a map $f : M \to N$ of complexes is given by the graded module

$$C(f) = M[1] \oplus N$$

equipped with the differential

$$d_{C(f)} = \begin{pmatrix} d_{A[1]} & 0 \\ f[1] & d_B \end{pmatrix}$$

acting on matrix vectors and the inclusion $N \to C(f)$ and projection $C(f) \to M[1]$ are just the natural ones. If $f : M_0 \to N_0$ is concentrated in degree 0, then $C(f)$ is just given by f itself with M_0 in degree -1, so that $H^0(C(f))$ is the cokernel of f and $H^{-1}(C(f))$ is its kernel. If $f : M_0 \to N_0$ is injective, i.e., a cofibration for the injective model structure, one gets an exact sequence

$$0 \to M_0 \xrightarrow{f} N_0 \to H^0(C_f) \to 0.$$

8.8 Derived categories and derived functors

Let A be an associative unital ring. Let $(\mathrm{MOD}_{dg}(A), \otimes)$ be the monoidal category of graded (left) A-modules, equipped with a linear map $d : C \to C$ of degree -1 such that $d^2 = 0$, with graded morphisms that commute with d and tensor product $V \otimes W$ of graded vector spaces, endowed with the differential $d : d_V \otimes \mathrm{id}_W + \mathrm{id}_V \otimes d_W$ (tensor product of graded maps, i.e., with a graded Leibniz rule), and also the same anti-commutative commutativity constraint.

Remark that this category is pointed, meaning that its initial and final objects are identified (the zero dg-module). It is moreover additive, meaning that morphisms are abelian groups, and even an abelian category. We would like to localize this category with respect to the class W of quasi-isomorphisms, (i.e., morphisms that induce isomorphisms of cohomology space $H^* := \mathrm{Ker}(d)/\mathrm{Im}(d)$). This is usually done in textbooks by the use of the formalism of derived and triangulated categories. We provide here directly the homotopical description of this construction.

The projective model structure on the category $\mathrm{MOD}_{dg}(A)$ is defined by saying that:

- weak equivalences are quasi-isomorphisms,
- fibrations are degree-wise surjections, and
- cofibrations are maps with the left lifting property with respect to trivial fibrations.

The injective model structure on the category $\mathrm{Mod}_{dg}(A)$ is defined by saying that:

- weak equivalences are quasi-isomorphisms,
- cofibrations are degree-wise injections, and
- fibrations are maps with the left lifting property with respect trivial cofibrations.

Remark that it is hard to prove that these really define model category structures on $\mathrm{Mod}_{dg}(A)$ (see [Hov99]). Recall that a module P (resp. I) over A is called projective (resp. injective) if for all surjective (resp. injective) module morphism $M \to N$, the morphism

$$\mathrm{Hom}_{\mathrm{Mod}(A)}(P, M) \to \mathrm{Hom}_{\mathrm{Mod}(A)}(P, N)$$
$$(\text{resp. } \mathrm{Hom}_{\mathrm{Mod}(A)}(M, I) \to \mathrm{Hom}_{\mathrm{Mod}(A)}(N, I))$$

is surjective.

One can show that any cofibrant object in the projective model structure has projective components, and that having this property is enough to be cofibrant for bounded below dg-modules. Using this one can show that two morphisms $f, g : P \to Q$ between cofibrant dg-modules are homotopic if and only if there is an A-linear morphism $h : X \to Y$ homogeneous of degree -1 (called a chain homotopy) such that

$$f - g = d \circ h + h \circ d.$$

This can be seen by remarking that given a complex X, a path object for X is given by the chain complex

$$\mathrm{Cocyl}(X)_n = X_n \oplus X_n \oplus X_{n+1}$$

with the differential

$$\partial(x, y, z) = (\partial x, \partial y, -\partial z + x - y).$$

One also defines the cylinder of X as the complex defines by

$$\mathrm{Cyl}(X)_n := X_n \oplus X_{n-1} \oplus X_n$$

equipped with the differential

$$\partial(x, y, z) := (\partial x - y, -\partial x, x + \partial y).$$

It can be used to show that the left homotopy in injective model structure is given by a chain homotopy.

Definition 8.8.1. The homotopical category $h(\mathrm{Mod}_{dg}(A))$ is called the *derived category* of A and denoted $D(A)$.

Remark that in this case, a dg-module is cofibrant and fibrant if and only if it is fibrant. Applying theorem 8.2.4, one can compute $D(A)$ by taking the quotient category of the category of cofibrant dg-modules by the chain homotopy equivalence relation $\sim_{ch}$, i.e., the natural functor

$$\mathrm{Mod}_{dg,c}(A)/\sim_{ch} \longrightarrow h(\mathrm{Mod}_{dg}(A))$$

is an equivalence.

By definition of derived functors, if

$$F : \mathrm{Mod}(A) \leftrightarrows \mathrm{Mod}(B) : G$$

is an adjunction of additive functors such that F is left exact (respects injections) and G is right exact (respect surjections), one can extend it to a Quillen adjunction

$$F : \mathrm{Mod}_{dg,inj}(A) \leftrightarrows \mathrm{Mod}_{dg,proj}(B) : G$$

and this defines a derived adjunction

$$\mathbb{L}F : D(A) \leftrightarrows D(B) : \mathbb{L}G.$$

One can compute this adjunction by setting $\mathbb{L}F(M) := F(Q(M))$ and $\mathbb{L}G(M) := G(R(M))$ where R and Q are respective fibrant and cofibrant replacements of M. This gives back the usual derived functors of homological algebra, defined in [Har70].

For example, let A and B be two rings and M be an (A, B) bimodule. The pair

$$- \otimes_A M : \mathrm{Mod}_{dg,proj}(A) \leftrightarrows \mathrm{Mod}_{dg,inj}(B) : \mathrm{Hom}_B(M, -)$$

is a Quillen adjunction that corresponds to a pair of adjoint derived functors

$$- \overset{\mathbb{L}}{\otimes}_A M : D(A) \leftrightarrows D(B) : \mathbb{R}\mathrm{Hom}_B(M, -).$$

Similarly, the pair (we carefully inform the reader that one works with the opposite category on the right side, to get covariant functors)

$$\mathrm{Hom}_A(-, M) : \mathrm{Mod}_{dg,proj}(A) \leftrightarrows (\mathrm{Mod}_{dg,proj}(B))^{op} : \mathrm{Hom}_B(-, M)$$

is a Quillen adjunction that corresponds to a pair of adjoint derived functors

$$\mathbb{R}\mathrm{Hom}_A(-, M) : D(A) \leftrightarrows D(B)^{op} : \mathbb{R}\mathrm{Hom}_B(-, M).$$

The two above construction give back all the usual constructions of derived functors on bounded derived categories.

If $\mathrm{Mod}_{dg,+}$ denotes the subcategory of bounded below dg-modules, on can define a (different from the standard) cofibrant replacement functor $Q : \mathrm{Mod}_{dg,+}(A) \to \mathrm{Mod}_{dg}(A)$ by

$$Q(M) = \mathrm{Tot}(L(M)),$$

where $L(M)$ is the free resolution of M given by setting $L(M)_0 = A^{(M)}$, $f_0 : L(M)_0 \to M$ the canonical morphism and

$$f_{i+1} : L(M)_{i+1} := A^{(\mathrm{Ker}(f_i))} \to L(M)_i,$$

and $\mathrm{Tot}(L(M))_k = \sum_{i+j=k} L(M)_{i,j}$ is the total complex associated to the bicomplex $L(M)$ (with one of the differential given by M and the other by the free resolution degree).

Remark that the category $\mathrm{Mod}_{dg,+}(A)$ of positively graded dg-modules can be equipped with a model structure (see [DS95], Theorem 7.2) by defining

- weak equivalences to be quasi-isomorphisms,
- cofibrations to be monomorphisms with component-wise projective cokernel,
- fibrations to be epimorphisms component-wise in strictly positive degree.

The dual result is that the category $\textsc{Mod}_{dg,-}(A)$ of negatively graded dg-modules can be equipped with a model structure by defining

- weak equivalences are quasi-isomorphisms,
- fibrations are epimorphisms with component-wise injective kernel,
- cofibrations are component-wise monomorphisms in strictly negative degree.

Of course, one has Quillen adjunctions

$$i : \textsc{Mod}_{dg,+}(A) \leftrightarrows \textsc{Mod}_{dg}(A) : \tau_{\geq 0}$$

and

$$i : \textsc{Mod}_{dg,-}(A) \leftrightarrows \textsc{Mod}_{dg}(A) : \tau_{\leq 0}$$

given by the inclusion and truncation functors. These allow to define derived functors on bounded complexes, and to compute them by the usual method of projective and injective resolutions.

Bounded differential graded module may also be studied with simplicial methods. Simplicial A-modules are defined as contravariant functors $M : \Delta^{op} \to \textsc{Mod}(A)$.

Proposition 8.8.2. *There is an equivalence of categories*

$$N : \textsc{Mod}_s(A) \leftrightarrows \textsc{Mod}_{dg,+}(A) : \Gamma$$

between the category $\textsc{Mod}_s(A)$ of simplicial modules (which is equipped with the model structure induced by the one given on the underlying simplicial sets) and the category of positively graded complexes, which preserves the model structures.

Proof. The dg-module $N(M)$ associated to a simplicial module M has graded parts

$$N(M)_n = \cap_{i=1}^n \mathrm{Ker}(\partial_n^i)$$

and is equipped with the differential

$$d_n(a) = \partial_n^0$$

where ∂_n^i is induced by the non-decreasing map $[0, \ldots, \hat{i}, \ldots, n+1] \to [0, \ldots, n]$. The simplicial set associated to a given dg-module C is the family of sets

$$\Gamma(C)_n := \oplus_{n \twoheadrightarrow k} C_k$$

with naturally defined simplicial structure, for which we refer the reader to [GJ99], Chapter III. □

This gives a way to introduce simplicial methods in homological algebra.

Definition 8.8.3. Let K be a commutative unital ring. A differential graded algebra is an algebra in the symmetric monoidal category $\text{MOD}_{dg}(K)$, i.e., a pair (A, d) composed of a graded algebra A and a derivation $d : A \to A$, i.e., a morphism of K-module $A \to A$ of degree -1 and such that

$$d(ab) = d(a).b + (-1)^{\deg(a)}a.d(b)$$

and $d \circ d = 0$.

Theorem 8.8.4. *The model category structure on* $\text{MOD}_{dg}(K)$ *induces a model category structure on the category* $\text{ALG}_{dg,K}$ *of differential graded algebra.*

8.9 Derived operations on sheaves

We refer to Hartshorne's book [Har77], Chapter II.1 and III.1, for a short review of basic notions on abelian sheaves and to Kashiwara and Schapira's book [KS06] for a more complete account. The extension of the above methods to derived categories of Grothendieck abelian categories (for example, categories of abelian sheaves on spaces) can be found in [CD09].

Recall that a sheaf on a category $X = (\text{LEGOS}, \tau)$ with Grothendieck topology is a contravariant functor

$$X : \text{LEGOS}^{op} \to \text{SETS}$$

that fulfills the sheaf condition given in definition 1.4.1. More precisely, the sheaf condition is that for every covering family $\{f_i : U_i \to U\}$, the sequence

$$X(U) \longrightarrow \prod_i X(U_i) \rightrightarrows \prod_{i,j} X(U_i \times_U U_j)$$

is exact.

We first translate the sheaf condition in terms of nerves of coverings.

Definition 8.9.1. Let $\mathcal{U} := \{f_i : U_i \to U\}$ be a covering family for a Grothendieck topology. The *nerve* $N(\mathcal{U})$ is the simplicial object of LEGOS with n-vertices

$$N(\mathcal{U})_n := \coprod_{i_1,\dots,i_n} U_{i_1} \times_U \cdots \times_U U_{i_n},$$

and faces and degenerations given by restrictions and inclusions.

Proposition 8.9.2. *A contravariant functor* $X : \text{LEGOS}^{op} \to \text{SETS}$ *is a sheaf if and only if it commutes to colimits along nerves of coverings, i.e., one has*

$$\lim_{[n]\in\Delta} X(N(\mathcal{U})_n) \cong X(\operatorname{colim}_{[n]\in\Delta} N(\mathcal{U})_n) \cong X(U)$$

for every covering $\mathcal{U}$ *of* U*.*

An abelian sheaf on X is a sheaf with values in the category GRAB of abelian groups. We denote SHAB(X) the category of abelian sheaves on X.

Theorem 8.9.3. *(Beke) The category of complexes of abelian sheaves has a so called injective model category structure defined by setting*

- *weak equivalences to be quasi-isomorphisms, and*
- *cofibrations to be monomorphisms (injective in each degree).*

The projective model structure is harder to define because it involves a cohomological descent condition. We first define it in a simple but non-explicit way (explained to the author by D.-C. Cisinski) and then explain shortly the description of fibrations. We refer to [CD09] for more details on this result.

Theorem 8.9.4. *The category of complexes of abelian sheaves has a so called projective model category structure defined by setting*

- *weak equivalences to be quasi-isomorphisms, and*
- *cofibrations to have the left lifting property with respect to trivial fibrations, which are defined ad-hoc as morphisms $f : K \to L$ that are object-wise surjective in each degree (i.e., $f_{n,X} : K(X)_n \to L(X)_n$ is surjective for every n and X) and such that* $\mathrm{Ker}(f_X)$ *is acyclic).*
- *fibrations to have the right lifting property with respect to trivial cofibrations.*

We now give, just for the culture, the description of fibrations in the projective model structure on complexes of abelian sheaves.

Definition 8.9.5. Let $X \in$ LEGOS. A *hyper-cover* of X is a simplicial lego $U : \Delta^{op} \to$ LEGOS equipped with a morphism $p : U \to X$ of simplicial legos such that each U_n is a coproduct of representables, and $U \to X$ is a local acyclic fibration (locally for the given topology on LEGOS, it is an object-wise acyclic fibration of simplicial sets).

Theorem 8.9.6. *The fibrations in the projective model category structure on the category of complexes of abelian sheaves are given by epimorphisms of complexes (surjective in each degree), whose kernel K has the cohomological descent property, meaning that for every object X in* LEGOS *and every hyper-covering $U \to X$, the natural morphism*

$$K(X) \to \mathrm{Tot}(N(K(U)))$$

is a quasi-isomorphism.

Another way to define the left bounded derived category of sheaves for the projective model structure is given by making a Bousfield homotopical localization (see Section 8.4) of the category of left bounded complexes of abelian presheaves (or equivalently, simplicial presheaves), with its standard model structure, by the morphisms of complexes of presheaves that induce isomorphisms on the cohomology sheaves.

Definition 8.9.7. Let $f : X \to Y$ be a morphism of topological spaces. The *direct image* of an abelian sheaf $\mathcal{F}$ on X is defined by

$$f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U))$$

and the *inverse image* is defined as the sheaf associated to the presheaf

$$f^{-1}\mathcal{F}(U) := \lim_{f(U)\subset V} \mathcal{F}(V).$$

Theorem 8.9.8. *The direct and inverse image functors form a Quillen adjunction between categories of complexes of abelian sheaves*

$$f^* : C\textsc{Shab}(X)_{proj} \leftrightarrows C\textsc{Shab}(Y)_{inj} : f_*$$

that induce an adjoint pair of derived functors

$$\mathbb{L}f^* : D(\textsc{Shab}(X)) \leftrightarrows D(\textsc{Shab}(Y)) : \mathbb{R}f_*.$$

The above constructions can be extended to sheaves of modules over ringed spaces. A ringed space is a pair $(X, \mathcal{O}_X)$ of a space and of a sheaf of commutative rings, and a morphism of ringed spaces

$$f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

is a pair of a continuous map $f : X \to Y$ and a morphism of sheaves of rings $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$.

Definition 8.9.9. The *inverse image functor* $f^* : \textsc{Mod}(\mathcal{O}_Y) \to \textsc{Mod}(\mathcal{O}_X)$ is defined by

$$f^*\mathcal{F} := f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

Theorem 8.9.10. *The model category structures on the category of complexes of abelian sheaves induce model category structures on the category of complexes of $\mathcal{O}_X$-modules on a given ringed space $(X, \mathcal{O}_X)$. The direct and inverse image functors form a Quillen adjunction*

$$f^* : C\textsc{Mod}(\mathcal{O}_X)_{proj} \leftrightarrows C\textsc{Mod}(\mathcal{O}_Y)_{inj} : f^*.$$

8.10 Higher categories

We present here one model for the definition of the notion of higher category, that was developped essentially by Rezk [Rez09]. We base our presentation on the work of Ara (see [Ara10] and [Ara12], where this theory is compared to higher quasi-categories, and presented in the language of model categories; we had to adapt a bit these references to our doctrinal viewpoint, based on $^\infty1$-categories).

The approach to higher categories using an inductive construction and cartesian closed model category structures originates in the work of Simpson and Tamsamani (see [Tam95], and [Sim10] for historical background and more references) and of Joyal [Joy07]. There are various other models for higher categories. Comparison results may be found in

Bergner's survey [Ber06] and in Toen's article [Toe04] for the one-dimensional case, and in [BR12] and [Ara12] in higher dimensions. An axiomatic for characterization of the higher dimensional case may be found in Barwick and Schommer-Pries' paper [BS11].

We refer to Definition 1.1.1 for an informal definition of the general notion of higher category.

Definition 8.10.1. The *globular category* is the category generated by the graph

$$D_0 \underset{s}{\overset{t}{\leftleftarrows}} D_1 \underset{s}{\overset{t}{\leftleftarrows}} D_2 \underset{s}{\overset{t}{\leftleftarrows}} D_3 \cdots,$$

with relations given by

$$ss = st,\ ts = tt.$$

Definition 8.10.2. A *dimension array* T is an array of natural integers $(i_k, j_k) \in \mathbb{N}_{>0}^{n+1} \times \mathbb{N}^n$ of the form

$$T = \begin{pmatrix} i_1 & & i_2 & & i_3 & \cdots & i_{n+1} \\ & j_1 & & j_2 & \cdots & j_n & \end{pmatrix},$$

with $i_k > j_k$ and $i_{k+1} > j_k$ for all $0 \leq k \leq n$. The dimension of T is the maximal integer $\dim(T) = \max_k(i_k)$ that appears in T.

To each dimension array T, one associates a diagram in $\mathbb{G}$ given by composing iteratively the source and target maps

$$[T] = \begin{array}{ccccccccc} D_{i_1} & & D_{i_2} & & D_{i_3} & & \cdots & & D_{i_{n+1}} \\ & {\scriptstyle s}\searrow \ \swarrow{\scriptstyle t} & & {\scriptstyle s}\searrow \ \swarrow{\scriptstyle t} & & & & \swarrow{\scriptstyle t} & \\ & D_{j_1} & & D_{j_2} & & \cdots & D_{j_n} & & \end{array}.$$

We choose here a different convention of the usual one, by denoting Θ_0 the opposite to Joyal's category [Joy07].

Definition 8.10.3. The category Θ_0 is the universal category under $\mathbb{G}$ with all limits corresponding to the dimension arrays, i.e., the sketch associated to the category $\mathbb{G}$ and the family of limit diagrams $[T]$.

Definition 8.10.4. A *globular set* is a covariant functor

$$X : \mathbb{G} \to \text{SETS}.$$

The category of globular sets is denoted $\widehat{\mathbb{G}}$.

The natural fully faithful embedding

$$\mathbb{G} \hookrightarrow \widehat{\mathbb{G}}^{op}$$

extends to a fully faithful embedding

$$\Theta_0 \hookrightarrow \widehat{\mathbb{G}}^{op}.$$

To make a clear difference between the notions of:

- higher categories with invertible higher morphisms of arbitrary degrees, that we called ∞-categories, and
- higher category with non-invertible higher morphisms of arbitrary degrees,

we will call the later *ω-categories.*

Definition 8.10.5. A *strict* 0*-category* is a set. For $n \geq 1$, a *strict* n*-category* is a category enriched (in the sense of Definition 1.2.13) over the cartesian monoidal category of strict $(n-1)$-categories. The category of *strict ω-categories* is the limit

$$^{\omega}\mathrm{C{\small AT}}_{str} := \lim_{n} n\mathrm{C{\small AT}}_{str}$$

of the categories of strict n-categories.

There is a forgetful contravariant functor

$$^{\omega}\mathrm{C{\small AT}}_{str} \longrightarrow \widehat{\mathbb{G}}^{op}$$

that sends $\mathcal{C}$ to the globular set $\{\mathrm{Mor}^i(\mathcal{C})\}_{i\geq 0}$ of morphisms in $\mathcal{C}$.

Proposition 8.10.6. *The above forgetful functor commutes with arbitrary limits and thus has a right adjoint, denoted F_∞ and called the free strict ∞-category functor.*

We now define (the opposite category to) Joyal's category.

Definition 8.10.7. *Joyal's category* Θ is the full subcategory of $(\mathrm{C{\small AT}}^{\infty}_{str})^{op}$ whose objects are dimension arrays and whose morphisms are given by

$$\mathrm{Hom}_\Theta(T_1, T_2) := \mathrm{Hom}_{\mathrm{C{\small AT}}^{\infty}_{str}}(F_\infty([T_2]), F_\infty([T_1])).$$

The subcategory with objects given by arrays of dimension smaller than n is denoted Θ_n.

There is a sequence of covariant embeddings

$$\mathbb{G} \to \Theta_0 \to \Theta.$$

Each of these functors are obtained by extending the given categories: the first one by addition of limits to dimension array diagrams, and the other by addition of composition and unit morphisms between diagrams, like for example

$$m : D_1 \times_{D_0} D_1 \to D_1 \text{ or } 1 : D_i \to D_{i+1}.$$

The basic idea is that a strict ω-category is a functor

$$\mathcal{C} : \Theta \to \mathrm{S{\small ETS}}$$

that respects the colimits associated to array diagrams, so that $(\Theta, \{[T]\})$ is the limit sketch of higher categories. One may naively extend this by considering $^{\infty}\omega$-categories, that may be thought as ∞-functors

$$\mathcal{C} : \Theta \to {}^{\infty}\mathrm{G{\small RPD}}$$

that send array diagrams to limits. This have internal homomorphisms in the ${}^\infty 1$-categorical sense, meaning that if $\mathcal{C}_1$ and $\mathcal{C}_2$ is an ${}^\infty\omega$-category, there is a naturally associated ${}^\infty\omega$-category $\underline{\mathrm{Mor}}(\mathcal{C}_1, \mathcal{C}_2)$. We now give a more precise definition.

For T and array, seen as the limit of a diagram $[T]$ in Θ, we denote I_T the limit of $[T]$ in $\widehat{\Theta}^{op}$. There is a natural *higher Segal morphism*

$$T \to I_T.$$

For example, in $\Theta_0 = \Delta^{op}$, the array of lenght k corresponds to the objects Δ_k and the above morphism is the Segal morphism

$$\Delta_k \to I_k := \Delta_1 \underset{\Delta_0}{\times} \Delta_1 \cdots \underset{\Delta_0}{\times} \Delta_1.$$

Asking that a functor $\mathcal{C} : \Theta \to \text{Sets}$ commute with limits of array diagrams is equivalent to asking that its extension to presheaves sends the Segal morphisms $T \to I_T$ to bijections.

Let J_1 be the free groupoid on two objects 0 and 1, depicted (we omit the identity morphisms) as

$$J_1 := \ 0 \rightleftarrows 1$$

There is a natural functor

$$J_1 \to \Delta_0 = \{*\},$$

that corresponds to a morphism $\Delta_0 \to J_1$ in $\widehat{\Theta}^{op}$. For $i \geq 1$, we denote J_{i+1} the strict $(i+1)$-category with objects 0 and 1, $\mathrm{Mor}(1,0) = \emptyset$ and only non-trivial morphism i-category $\mathrm{Mor}(0,1) := J_i$. Explicitely, one has

$$J_2 := \ 0 \underset{}{\overset{}{\rightrightarrows}} 1 \quad (\Downarrow\Uparrow)$$

and

$$J_3 := \ 0 \ \Downarrow \ \underset{\Leftarrow}{\Rightarrow} \ \Downarrow \ 1\,.$$

There is a natural morphism $J_{i+1} \to D_i$ that sends the only two non-trivial i-arrows to the only non-trivial i-arrow. It corresponds to a morphism $D_i \to J_{i+1}$ in $\widehat{\Theta}^{op}$.

We now define ${}^\infty\omega$-categories as ∞-functors that send the above morphisms to equivalences. This corresponds to sending the corresponding limit diagrams to ∞-limit diagrams.

Definition 8.10.8. An ${}^\infty\omega$*-category* is an ∞-functor

$$\mathcal{C} : \Theta \to {}^\infty\text{Grpd}$$

that sends the Segal morphisms $T \to I_T$ and the morphisms $D_i \to J_{i+1}$ for $i \geq 0$ to equivalences. An ${}^\infty n$*-category* is an ∞-functor

$$\mathcal{C} : \Theta_n \to {}^\infty\text{Grpd}$$

fulfilling the same condition for all arrays of dimension smaller than n and for $i \geq n$.

The main theorem of Rezk [Rez09] (that is similar in spirit to Simpson's main result [Sim10]) can be formulated in the ∞-categorical language by the following.

Theorem 8.10.9. *The ∞-category of ${}^{\infty}n$-categories is cartesian closed.*

8.11 Theories up-to-homotopy and the doctrine machine

We now define a general notion of homotopical doctrine and homotopical theory, basing ourselves on the theory of homotopical higher categories discussed in Section 8.10. The main interest of this notion is that it gives a wide generalization of the theory of operadic structures up-to-homotopy, that is necessary to describe all the examples presented in this book, in terms of their universal properties. The notion of categorification was given an important impetus in Baez and Dolan's paper [BD98]. Our approach through sketches is new, even if it is already widely applied in particular examples.

The main motivation for studying the homotopy theory of theories is to understand how algebraic or geometric structures get transfered along homotopies. These homotopy transfer operations are often used in mathematical physics, particularly in the context of deformation theory. They give a conceptual understanding of some aspects of perturbative expansions (see for example Mnev-Merkulov's work in [Mer10] on the quantum BV formalism, and Tamarkin's formulation [Tam98] of the deformation quantization of Poisson manifolds).

Here is a standard example: consider the monoidal model category $(\mathrm{Mod}_{dg}(A), \otimes, W_{qis})$, and suppose given a chain homotopy

$$h \circlearrowright X \underset{g}{\overset{f}{\rightleftarrows}} Y \circlearrowleft k$$

between two dg-modules X and Y over a ring A. One has

$$fg - \mathrm{id}_Y = \partial(k) := d_Y k + k d_Y \;\text{ and }\; gf - \mathrm{id}_X = \partial(h) := d_X h + h d_X,$$

where partial denotes the differential on graded internal endomorphisms of X and Y induced by the graded morphisms d_X and d_Y. Suppose given on Y an associative (differential graded) multiplication

$$m : Y \otimes Y \to Y.$$

Then

$$m_2 := g \circ m \circ (f \times f) : X \otimes X \to X$$

is an operation on X, but it is not anymore associative in general. However, the obstruction for it being associative (associator) is homotopic to zero, so that there is a ternary operation (the homotopy)

$$m_3 : X^{\otimes 3} \to X$$

such that

$$m_2(a, m_2(b,c)) - m_2(m_2(a,b),c) = \partial(m_3)(a,b,c),$$

where ∂ is the differential induced on the internal homomorphisms $\underline{\mathrm{Hom}}(X^{\otimes 3}, X)$ by the formula

$$\partial(f) = d_X \circ f - (-1)^{\deg(f)} f \circ d_{X^{\otimes 3}}.$$

One can combine m_2 and m_3 to get new relations that are also true up to a higher homotopy $m_4 : X^{\otimes 4} \to X$, and so on... The combinatorics that appear here are a bit intricate, and the homotopy theory of operads is here to help us to deal with that problem.

We refer to Loday and Vallette's book [LV10], and Fresse's book [Fre09], for the standard algebraic approach to operads, that give a way to treat linear and monoidal theories, and the corresponding algebra structures, up-to-homotopy. An ∞-categorical presentation of the theory of operads up to homotopy may also be found in [Lur09c]. The main results in the model category approach may also be found in Berger and Moerdijk [BM03]. An important aspect of this theory, that we will not describe here, is its explicit computational aspect, based on Koszul duality results, that give small resolutions of the theories in play (see [LV10], Chapter 11).

We now define a setting, based on higher categories that allows us to define very general types of higher theories up-to-homotopy, and work with them essentially as with usual theories, presented in Section 1.1. These have a build-in abstract notion of homotopy transfer.

Definition 8.11.1. An ∞-doctrine is an ${}^{\infty}(n+1)$-category $\mathcal{D}$ of ${}^{\infty}n$-categories with additional structures and properties. A theory of type $\mathcal{D}$ is an object $\mathcal{C}$ of $\mathcal{D}$. A model for a theory of type $\mathcal{D}$ in another one is an object

$$M : \mathcal{C}_1 \to \mathcal{C}_2$$

of the ${}^{\infty}n$-category $\underline{\mathrm{Mor}}(\mathcal{C}_1, \mathcal{C}_2)$.

Of course, every doctrine $\mathcal{D}$ gives an ${}^{\infty}(n+1)$-doctrine. For example, the doctrine of symmetric monoidal categories is contained in the doctrine of symmetric monoidal ∞-categories. As an example, a dg-PROP (symmetric monoidal category enriched in dg-modules) can be seen as a particular kind of symmetric monoidal ∞-category, and a cofibrant replacement of it for the natural model structure will encode the corresponding structure up-to-homotopy.

Important examples of theories are given by classical Lawvere (finite product) or Ehresmann (sketches) theories, embedded in the ${}^{\infty}1$ setting. They give notions of algebraic and other objects up-to-homotopy, like ∞-Lie algebras, or ∞-associative algebras.

We now sum-up the general idea of categorification of theories, by defining a natural categorification technique, called the *doctrine machine*, that allows us to define homotopical versions of any of the theories we might be interested in, if it is described by an Ehresmann sketch in the sense of definition 1.1.25. This very general categorification method was already used to define ${}^{\infty}\omega$-categories, and will also be used in Chapter 9 to build homotopical analogs of geometrical spaces. One may extend easily this method to other kinds of universal structures on categories, like inner homomorphisms, to be able to treat cartesian closed categories. It is also possible to apply inductively the following construction process, as was already done for example in Sections 1.2 and 1.3 for monoidal theories. These generalizations are all contained in the general notion of doctrine, so that we stick to the setting of sketches.

Definition 8.11.2 (The doctrine machine)**.** Let $(\mathfrak{L},\mathfrak{C}) = (\{I\},\{J\})$ be two classes of categories, called categories of indices and let $\mathcal{T} = (\mathcal{T},\mathcal{L},\mathcal{C})$ be an $(\mathfrak{L},\mathfrak{C})$-sketch, in the sense of Definition 1.1.25, i.e., a category $\mathcal{T}$ with given families of I and J indexed diagrams in $\mathcal{T}$ for all $I \in \mathfrak{L}$ and $J \in \mathfrak{C}$. The ${}^{\infty}n$-*categorification* of this theory is simply the theory $\mathcal{T}$, seen as an ${}^{\infty}n$-categorical sketch. The ${}^{\infty}(n+1)$-category of its models

$$A : (\mathcal{T},\mathcal{L},\mathcal{C}) \longrightarrow ({}^{\infty}n\mathrm{C}\scriptstyle{\mathrm{AT}}\displaystyle, \lim_{\mathfrak{L}}, \mathrm{colim}_{\mathfrak{C}})$$

with values in the ${}^{\infty}(n+1)$-$(\mathfrak{L},\mathfrak{C})$-sketch of ${}^{\infty}n$-categories is denoted $\mathrm{A}\scriptstyle{\mathrm{LG}}\displaystyle_{{}^{\infty}n}(\mathcal{T})$ and called the ${}^{\infty}(n+1)$-category of ${}^{\infty}n$-$\mathcal{T}$-algebras.

One can of course apply the same idea to more general types of structures, and this complication sometimes simplifies the study of the corresponding theories. For example, Lurie [Lur09a] have shown that the theory of topological quantum field theories up to homotopy, given by the symmetric monoidal ${}^{\infty}n$-category of cobordisms, is the free symmetric monoidal ${}^{\infty}n$-category with strong duality on one object, i.e., the free n-monoidal theory with strong duality.

We now define more concretely ∞-Lie algebras, because they will play an important role in the quantum part of this book.

Definition 8.11.3. Let K be a base field. Let $(\mathcal{T}_{\mathrm{LIE}},\otimes)$ be the linear theory of Lie algebras, whose underlying monoidal category is opposite to that of free finitely generated Lie algebras. We see this theory as a linear symmetric monoidal theory. If $(\mathcal{C},\otimes)$ is a symmetric monoidal linear ∞-category, a Lie algebra in $(\mathcal{C},\otimes)$ is a monoidal ∞-functor

$$L : (\mathcal{T}_{\mathrm{LIE}},\otimes) \longrightarrow (\mathcal{C},\otimes).$$

One may describe Lie algebras in the ∞-category of differential graded modules by a more concrete "tree style" linear theory, called the L_∞ operad. We only define here the notion of L_∞-algebra.

Definition 8.11.4. A *curved* L_∞*-algebra* $\mathfrak{g}$ over a ring A of characteristic zero is the datum of

1. a positively graded flat module $\mathfrak{g}$,
2. a square zero differential $D : \vee^*\mathfrak{g} \to \vee^*\mathfrak{g}$ of degree -1 on the free graded cocommutative coalgebra on $\mathfrak{g}$.

The graded parts $\{l_i\}_{i\geq 0}$ of the associated canonical (cofreeness) map

$$l := \sum l_i : \vee^* \to \mathfrak{g}$$

are called the *higher brackets* of the L_∞-algebra. If $l_0 = 0$, one calls $\mathfrak{g}$ an L_∞*-algebra.*

Chapter 9

A glimpse at homotopical geometry

Homotopical geometry started with the work of Serre [Ser70] (on local derived intersection multiplicities), Illusie [Ill71] (on the deformation theory of singular spaces), and Quillen [Qui70] (on the deformation theory of algebras) to cite some of the many grounders. It has gotten a new impetus with Grothendieck's long manuscript [Gro13], Simpson-Hirschowitz' paper [SH98], and the work of the homotopy theory community (see e.g., Jardine's work [Jar87] on simplicial presheaves).

The algebraic version of this theory is now fully developped, with the foundational works given by Toen-Vezzosi's "Homotopical algebraic geometry I and II" [TV02], [TV08], and Lurie's "Higher topos theory" [Lur09d], "Higher algebra" [Lur09c] and "Derived algebraic geometry" [Lur09b].

The non-algebraic setting is currently in development (see Spivak's thesis [Spi08] and Lurie's work [Lur09b]), with inspirations coming from Lawvere's thesis [Law04], the synthetic differential geometry community (see e.g., Moerdijk-Reyes' book [MR91]), and Dubuc and Zilber's work [DZ94], that we have used also in the classical setting, to make the full presentation of homotopical geometric spaces clear and consistent.

We give here, basing ourselves on the doctrinal approach to categorical logic, explained in Chapter 1.1, a systematic construction method for homotopical spaces, that is more general than the ones present in the literature, and that is robust and general enough to be adapted to the predictable future developments of the subject. This presentation is also strictly compatible with the pedagogical spirit of our approach, based on the higher Yoneda lemma.

9.1 Motivations

In physics, for example, in gauge theory, one often has to study spaces defined by (say Euler-Lagrange) equations that are not smooth, because they are given by non transverse intersections (in gauge theory, this is related to the so-called Noether relations, to be treated in Chapter 12). This non smoothness means that one does not have a system of local coordinates on the solution space. For example, consider a functional $S : X \to \mathbb{R}$

on a given space. Its space of critical points may be described by the intersection

$$T = \mathrm{im}(dS) \cap T_0^* X \subset T^* X$$

of the image of the differential and of the zero section in the cotangent bundle. For gauge theories, this intersection is not transverse, and one has to compute the *derived* (or *homotopical*) intersection

$$T^h = \mathrm{im}(dS) \overset{h}{\cap} T_0^* X$$

to get a better space, with nice local coordinates. To illustrate this general problem of transversality on a simple example, consider two lines D_1 and D_2 in the algebraic affine space $\mathbb{R}^2$, defined by the equations $x = 0$ and $y = 0$. Their intersection is given by the origin $(0,0) \in \mathbb{R}^2$ whose coordinate ring may be computed as the tensor product

$$\mathcal{O}(D_1 \cap D_2) = \mathcal{O}(D_1 \times_{\mathbb{R}^2} D_2) = \mathcal{O}(D_1) \otimes_{\mathcal{O}(\mathbb{R}^2)} \mathcal{O}(D_2)$$

that gives explicitely

$$\mathcal{O}(D_1 \cap D_2) = A/(x) \otimes_A A/(y) \cong A/(x,y) \cong \mathbb{R},$$

where $A = \mathbb{R}[x,y]$. This space is of dimension zero and the intersection is transverse. Now consider the intersection of D_1 with itself. It is a non transverse intersection whose usual points are given by D_1, that is of dimension 1. We would like to force the transversality of this intersection to get a space of dimension 0. This is easily done by defining the derived intersection to be the space $D_1 \cap^h D_1$ with function algebra

$$\mathcal{O}(D^1 \cap^h D_1) := \mathcal{O}(D_1) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}(\mathbb{R}^2)} \mathcal{O}(D_1)$$

given by the derived tensor product. This may be computed, in this case of a linear equation, by taking the projective resolution of $A/(x)$ as an A-module

$$0 \to A \xrightarrow{.x} A \to A/(x) \to 0.$$

The derived tensor product (of A-modules) is then equal to the complex of modules over $A/(x)$ given by

$$[A \xrightarrow{.x} A] \otimes_A A/(x) = [A/(x) \xrightarrow{0} A/(x)].$$

It is thus the zero map with cohomology spaces $H^0 = A/(x)$ and $H^1 = A/(x)$. If we define the derived dimension of the derived intersection as the Euler characteristic of this complex, we get

$$\dim(D_1 \cap^h D_1) = \dim(H^0) - \dim(H^1) = 1 - 1 = 0,$$

so that the derived intersection is transverse, at least in terms of dimension. This idea of improving on intersection theory by using cohomological methods was already present in Serre's work [Ser70] and in Illusie [Ill71] and Quillen [Qui70]. The case of spaces of trajectories is computed in a similar way, but one has to pass to the setting of $\mathcal{D}$-geometry. This will be described in Chapter 12, where we describe general gauge theories.

The above example is also related to the problem of defining a nice deformation theory and differential calculus on a non smooth space. To illustrate this, take a smooth variety (for example the affine plane $\mathbb{A}^2 := \mathrm{Spec}(\mathbb{R}[x, y])$) and equation in it whose solution space M is not smooth (for example $xy = 0$). Then one cannot do a reasonable differential calculus directly on M. However, one can replace the quotient ring $A = \mathbb{R}[x, y]/(xy)$ by a dg-algebra A' that is a cofibrant resolution of A as an $\mathbb{R}[x, y]$-algebra, and do differential calculus on this new dg-algebra. Morally, we may think of $\mathrm{Spec}(A')$ in $\mathbb{A}^2$ as a kind of homotopical infinitesimal tubular neighborhood of M in $\mathbb{A}^2$. This will give a much better behaved non-smooth calculus. For example, the space of differential forms on A' gives the full cotangent complex of A.

To formalize this, one has to define a differential calculus "up to homotopy". This can be done in the general setting of *model* categories of algebras (see Toen-Vezzosi's seminal work [TV08] and Hovey's book [Hov99]), which also encompasses a good notion of higher stack, necessary to formalize correctly quotients and moduli spaces in covariant gauge theory. We will only give here a sketch of this theory, whose application to physics will certainly be very important, particularly gauge theory, where physicists independently discovered similar mathematical structures in the BRST-BV formalism.

We may also motivate the introduction of homotopical methods in physics by the fact that many of the physicists spaces of fields are equivariant objects. For example, in Yang-Mills theory, one works with a variable principal G-bundle P over a given spacetime M, that one may formalize properly in equivariant geometry (the theory of ∞-stacks) as a map

$$P : M \to BG$$

where BG is the smooth classifying space for principal G-bundles. It is very convenient to think of principal bundles in this way because this gives improved geometric tools to work on them. The proper setting for differential calculus on spaces like BG is given by derived geometry, that has been alluded to above.

Another good reason for using methods of homotopical geometry is the fact the quantization problem itself is in some sense a deformation problem: one wants to pass from a commutative algebra, say over $\mathbb{R}$, to an associative (or more generally a factorization) algebra, say over $\mathbb{R}[[\hbar]]$, by an infinitesimal deformation process (see Costello and Gwilliam's work [CG10]). The modern methods of deformation theory are deeply rooted in homotopical geometry, as one can see in the expository work of Lurie [Lur09e] of his formalization of the derived deformation theory program of Deligne, Drinfeld and Kontsevich. The conceptual understanding of the mathematical problem solved by the quantization and renormalization process thus uses homotopical methods.

9.2 Homotopical spaces

The main idea of homotopical geometry is to generalize the functorial approach to spaces by the following systematic process, that is an almost direct ∞-categorical generalization of the functorial approach to geometry, presented in Chapter 2. This categorification, that is very much in the doctrinal spirit of Section 1.1, is obtained by replacing the category

SETS of usual sets (i.e., 0-categories) by the ∞-category $^\infty$GRPD of higher groupoids (i.e., $^\infty$0-categories).

1. Define an ∞-category ALG_h of homotopical algebras (functors of functions) as the ∞-category of functors

$$A : (\text{LEGOS}, \mathcal{C}) \to ({}^\infty\text{GRPD}, \{\text{all limits}\})$$

   commuting with particular limits (e.g., finite products or transversal pullbacks), where $^\infty$GRPD is the ∞-category of ∞-groupoids, and LEGOS is a category of building blocs with a Grothendieck topology τ.

2. Extend the Grothendieck topology τ to a homotopical analog of Grothendieck topology τ_h on $\text{LEGOS}_h := \text{ALG}_h^{op}$, by defining the class of morphisms

$$Y \to X$$

   of simplicial presheaves on LEGOS^h that are hypercoverings (see Definition 9.2.3), as those whose pullbacks along all points $U \to X$ of X with values in $U \in$ LEGOS are hypercoverings for the original topology τ. Denote τ_h the corresponding class of cocones in LEGOS_h.

3. Define homotopical functors of points (derived spaces) as ∞-functors

$$X : (\text{LEGOS}_h^{op}, \tau_h) \to ({}^\infty\text{GRPD}, \{\text{all limits}\})$$

   sending hypercovering cocones to limits.

The analogous construction with SETS-valued functor describes the category of sheaves of sets on LEGOS as a localization of the category of presheaves by local equivalences.

We now give a concrete presentation of the above general methods in the setting of Quillen's model categories. This gives, for example, a finer take at the computational aspects of mapping spaces between stacks. The reader only interested by formal properties of higher and derived stacks may skip the rest of this section.

The homotopical analog of a Grothendieck topology τ is the following.

Definition 9.2.1. Let (LEGOS, W) be a model category with homotopy fiber products. A *model topology* τ on LEGOS is the data, for every lego U, of covering families $\{f_i : U_i \to U\}_{i\in I}$ of morphisms in h(LEGOS), fulfilling:

1. (Homotopy base change) For every morphism $f : V \to U$ in h(LEGOS) and every covering family $\{f_i : U_i \to U\}$ of U, $f \times_U^h f_i : V \times_U^h V_i \to V$ is a covering family.

2. (Local character) If $\{f_i : U_i \to U\}$ is a covering family and $\{f_{i,j} : U_{i,j} \to U_i\}$ are covering families, then $\{f_i \circ f_{i,j} : U_{i,j} \to U\}$ is a covering family.

3. (Isomorphisms) If $f : U \to V$ is an isomorphism in h(LEGOS), it is a covering family.

A triple $(\textsc{Legos}, W, \tau)$ composed of a model category $(\textsc{Legos}, W)$ and a model topology τ is called a *model site*.

We now define the notion of hypercovering, that replaces the classical notion of (nerve of a) covering in the setting of homotopical geometry.

Definition 9.2.2. Let $\Delta_{\leq n} \subset \Delta^{op}$ be the full subcategory on the objects [1], [2],...,[n]. The truncation functor

$$\tau_n : \mathrm{S}\textsc{Sets} = \underline{\mathrm{Mor}}_{\textsc{Cat}}(\Delta^{op}, \textsc{Sets}) \to \underline{\mathrm{Mor}}_{\textsc{Cat}}(\Delta_{\leq n}, \textsc{Sets}) =: \mathrm{S}\textsc{Sets}_{\leq n}$$

has a left adjoint

$$\mathrm{sk}_n : \mathrm{S}\textsc{Sets}_{\leq n} \to \mathrm{S}\textsc{Sets},$$

called the *n-skeleton*, and a right adjoint

$$\mathrm{cosk}_n : \mathrm{S}\textsc{Sets}_{\leq n} \to \mathrm{S}\textsc{Sets}$$

called the *n-coskeleton*.

Definition 9.2.3. Let $(\textsc{Legos}, \tau)$ be a site. A morphism $X \to Y$ of simplicial presheaves $X, Y \in \textsc{Pr}_{\mathrm{S}\textsc{Sets}}(\textsc{Legos})$ is called an *hypercovering for the topology τ* if for all n, the canonical morphism

$$Y_n \to (\mathrm{cosk}_{n-1} Y)_n \underset{(\mathrm{cosk}_{n-1} X)_n}{\times} X_n$$

is a local epimorphism for τ.

We can now give a sketch of the definition of the category of general homotopical spaces. We refer to Toen-Vezzosi [TV08] for more precise definitions.

Definition 9.2.4. Let $(\textsc{Legos}, \tau, W)$ be a model category equipped with a model topology. Let (C, W_C) be a model category. For $x \in \textsc{Legos}$ and $A \in C$, one defines a functor

$$\begin{array}{cccc} A \otimes h_x : & \textsc{Legos}^{op} & \to & C \\ & y & \mapsto & \coprod_{\mathrm{Hom}(y,x)} A. \end{array}$$

- The model category
$$(\textsc{Pr}_{(C,W_C)}(\textsc{Legos}, W), W_{global})$$
of C-valued *prestacks* on $\textsc{Legos}$ is the category of functors
$$X : \textsc{Legos}^{op} \to C,$$
equipped with the model structure obtained by left Bousfield localization of the object-wise model structure (induced by the model structure W_C on C) by the weak equivalences $A \otimes h_f$ for $f \in W$ and A in a family of objects that generates C by homotopical colimits.

- The model category
$$(\mathrm{SH}_{(C,W_C)}(\mathrm{LEGOS}, W, \tau), W_{local})$$
of C-valued *stacks* on LEGOS is the left Bousfield localization of $\mathrm{PR}_{(C,W_C)}(\mathrm{LEGOS}, W, \tau)$ by the class of homotopy τ-hypercoverings.

Definition 9.2.5. Let (LEGOS, τ) be a site, equipped with the trivial model structure whose weak equivalences are the isomorphisms, and $(C, W_C) = (\mathrm{SSETS}, W_{eq})$ be the model category of simplicial sets. The ∞-category of *higher stacks* is the ∞-localization

$$\mathrm{STACKS}_\infty(\mathrm{LEGOS}, \tau) := \mathrm{SH}_{(\mathrm{SSETS}, W_{eq})}(\mathrm{LEGOS}, W_{triv}, \tau)[W_{local}^{-1}].$$

Definition 9.2.6. Let (LEGOS, τ) be a site, equipped with a family $\mathcal{C}$ of limit cones (e.g. transversal pullbacks), and ALG_h be the model category of covariant functors

$$A : \mathrm{LEGOS} \to \mathrm{SSETS}$$

that commute homotopically with limits in $\mathcal{C}$, and that are quotients of algebras of the form $\mathcal{O}(U)$. Equip ALG_h with the Grothendieck topology τ_h induced by τ, and suppose that $(\mathrm{ALG}_h^{op}, \tau_h)$ is a model site. Let $(C, W_C) = (\mathrm{SSETS}, W_{eq})$ be the model category of simplicial sets. The ∞-category of *derived higher stacks* is the ∞-localization

$$\mathrm{DSTACKS}_\infty(\mathrm{LEGOS}, W, \tau) := \mathrm{SH}_{(\mathrm{SSETS}, W_{eq})}(\mathrm{LEGOS}_h, W, \tau_h)[W_{local}^{-1}].$$

One can also work with spectra valued algebras, and this gives a notion of unbounded derived stacks. If LEGOS is the category of affine schemes over $\mathbb{Q}$ (i.e., opposite to $\mathbb{Q}$-algebras), one can also replace simplicial algebras by differential graded algebras in the above construction, since both model categories are Quillen equivalent.

One can also work with unbounded differential graded algebras (or with ring spectra, in the simplicial language). This also gives a notion of (unbounded) derived stacks.

9.3 Non abelian cohomology

The main interest of derived and homotopical geometry is that it gives a setting, called non abelian cohomology, that allows one to treat geometrically the problems of obstruction theory. These ideas of using homotopy theory in geometry were developed by many people, starting with Quillen [Qui70] and Illusie [Ill71]. The idea of non-abelian cohomology was already present in Grothendieck's long letter to Quillen [Gro13]. It was also developped by Simpson and Hirschowitz [SH98]. A systematic treatment is now available in the work of Toen-Vezzosi (see [TV02] and [TV08]), and Lurie (see [Lur09d] and [Lur09b]). We refer to Sati-Schreiber-Stasheff [SSS09] and to Schreiber's book [Sch11] for nice applications of this formalism in the description of higher gauge theories.

Let (LEGOS, τ) be a site with a family $\mathcal{C}$ of limit cones. In this section, we will denote STACKS_∞ the ∞-category of ∞-stacks on the site $(\mathrm{LEGOS}, \mathcal{C})$, and $\mathrm{DSTACKS}_\infty$ the ∞-category of derived ∞-stacks (given by simplicial sheaves on the model site of simplicial generalized algebras $A : \mathrm{LEGOS} \to \mathrm{SSETS}$, given by functors that send hypercovering cocones to homotopy limits). Objects in STACKS_∞ will simply be called stacks, and objects in $\mathrm{DSTACKS}_\infty$ will be called derived stacks.

Definition 9.3.1. Let X and G be two stacks or derived stacks. One defines:

1. *the non-abelian G-cohomology space on X* as the ∞-groupoid $\underline{\mathrm{Mor}}(X, G)$ of morphisms from X to G;

2. *non-abelian G-valued cocycles on X* as objects of $\underline{\mathrm{Mor}}(X, G)$;

3. *non-abelian coboundaries* between these cocycles as morphisms in $\underline{\mathrm{Mor}}(X, G)$;

4. *non-abelian cohomology of X with coefficients in G* as the set
$$H(X, G) := \pi_0(\underline{\mathrm{Mor}}(X, G))$$
of connected components of $\underline{\mathrm{Mor}}(X, G)$.

Example 9.3.2. If A is a sheaf of abelian groups, and $A[-n]$ is the corresponding complex concentrated in degree n, the Dold-Kan correspondence (see Proposition 8.8.2) gives a stack $K(A, n)$ such that
$$H^n(X, A) = H(X, K(A, n)).$$
This shows that non-abelian cohomology generalizes usual cohomology of abelian sheaves.

Definition 9.3.3. The *pointed loop space* ΩA of a pointed object $\{*\} \to A$ is defined as the homotopy pullback
$$\begin{array}{ccc} \Omega A & \longrightarrow & \{*\} \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & A \end{array}.$$
A *delooping* $\{*\} \to BA$ of $\{*\} \to A$ is a pointed object such that
$$A \cong \Omega BA.$$
A pointed object that has a delooping is called an *∞-group*. If A is an ∞-group and $X \to BA$ is a non-abelian BA-valued cocycle, the homotopy fiber product
$$\begin{array}{ccc} P & \longrightarrow & \{*\} \\ \downarrow & & \downarrow \\ X & \longrightarrow & BA \end{array}$$
is called the corresponding *principal ∞-bundle.*

Remark that a section $s : X \to P$ of a principal ∞-bundle is equivalent to a morphism $s : X \to \{*\}$ over BA, i.e., a homotopy commutative diagram
$$\begin{array}{ccc} & & \{*\} \\ & \overset{s}{\nearrow} & \downarrow \\ X & \longrightarrow & BA \end{array}.$$

Example 9.3.4. Abelian sheaves of groups have the particular property of being deloopable arbitrarily many times. If A is a sheaf of abelian groups, one has

$$K(A, n) \cong B^n A.$$

The principal ∞-bundles associated to cocycles $X \to K(A, n)$ are called n-gerbes. Their equivalence classes correspond exactly to the cohomology set $H^n(X, A)$.

Example 9.3.5. Suppose we work in the algebraic, differential geometric, or analytic setting. A semi-simple group G is only once deloopable, with delooping the moduli space BG of principal G-bundles. In general, if G is an ∞-group and A is an arbitrarily deloopable ∞-stack, one thinks of

$$H^n(G, A) := \pi_0(\underline{\mathrm{Mor}}(BG, B^n A))$$

as the group cohomology of G with values in A.

Example 9.3.6. As before, suppose we work in the algebraic, differential geometric, or analytic setting. Let G be a semisimple group and $G \to \mathrm{GL}(V)$ be a representation of G. The push-forward

$$\begin{array}{ccc} V \times G & \longrightarrow & V \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & [V/G] \end{array}$$

defines a fiber bundle $[V/G] \to BG$ with fiber V over the point. This bundle has an n-delooping

$$K_{BG}(V, n) \to BG$$

relative to BG for all n, with fiber $K(V, n)$ over the point. If $P : X \to BG$ is a principal G-bundle on X, the bundle $V^P := P \times_G [V/G]$ associated to the representation V is given by the homotopy pullback

$$\begin{array}{ccc} V^P & \longrightarrow & [V/G] \\ \downarrow & & \downarrow \\ X & \longrightarrow & BG \end{array}.$$

A morphism $s : X \to [V/G]$ of stacks is the same thing as a section $s : X \to V^P$ of the corresponding principal bundle $P_s : X \xrightarrow{s} [V/G] \to BG$. More generally, the bundle $K(V^P, n)$ is defined as the pullback

$$\begin{array}{ccc} K(V^P, n) & \longrightarrow & K_{BG}(V, n) \\ \downarrow & & \downarrow \\ X & \longrightarrow & BG \end{array}.$$

A morphism $s : X \to K_{BG}(V, n)$ is the same thing as a cohomology class $s \in H^n(X, V^P)$.

Example 9.3.7. Let X be a smooth variety, seen as a (non-derived) stack modeled on algebras in the given geometrical setting. The infinitesimal ∞-groupoid of X is the stack associated to the prestack whose degree n component is

$$\Pi_\infty^{inf}(X)_n(T) := \{x \in X^n(T),\ \exists U \to T^n \text{ thickening },\ \forall i,j,\ x_{i|U} = x_{j|U}\}.$$

It is equipped with its natural simplicial structure induced by its inclusion in the simplicial space $n \mapsto X^n$. Let G be an ∞-group. The stack of principal G-bundles with flat connection on X is given by the stack

$$\mathrm{BunConn}_G(X) := \underline{\mathrm{Hom}}(\Pi_\infty^{inf}(X), BG).$$

This space plays an important role in higher gauge theory. If G is a semisimple group (in the algebraic, smooth or analytic context), one can give a simplified description of the above space as

$$\mathrm{BunConn}_G(X) \cong \underline{\mathrm{Hom}}(\Pi_1^{DR}X, BG),$$

where $\Pi_1^{DR}X := \tau_1(\Pi_\infty^{inf}(X))$ is the groupoid associated to the infinitesimal groupoid. This is a subgroupoid of the groupoid $\mathrm{Pairs}(X)$ of pairs of points in X, whose maps to BG are identify with flat G-equivariant Grothendieck connections on principal G-bundles.

Example 9.3.8. If (X,ω) is a smooth symplectic manifold and L_1 and L_2 are two Lagrangian submanifolds, their derived intersection gives the stack whose points with values in a simplicial smooth algebra A are

$$(L_1 \cap^h L_2)(A) = \{f : [0,1] \to |X(A)|,\ f(0) \in [L_1(A)|,\ f(1) \in |L_2(A)|\},$$

where $|-| : \mathrm{SSets} \to \mathrm{Top}$ is the geometric realization functor. This derived stack is actually representable by a simplicial smooth algebra, i.e., it may be described as the derived spectrum

$$L_1 \cap^d L_2 = \mathbb{R}\underline{\mathrm{Spec}}\left(\mathcal{C}^\infty(L_1) \overset{\mathbb{L}}{\underset{\mathcal{C}^\infty(X)}{\otimes}} \mathcal{C}^\infty(L_2)\right).$$

The above applies in particular to the case of a cotangent space $X = T^*M$ and the Lagrangian submanifolds are given by the zero section $L_1 = T_0^*M$ and the image $L_2 = \mathrm{im}(dS)$ of the differential of a given function $S : M \to \mathbb{R}$. The corresponding intersections are the usual, homotopical and derived critical points of S. One will usually use the stack of derived critical points to get a well behaved differential calculus.

Definition 9.3.9. The *free loop space* LX of a derived stack X is the homotopy pullback

$$\begin{array}{ccc} LX & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle\Delta} \\ X & \underset{\Delta}{\longrightarrow} & X \times X \end{array} .$$

The simplicial algebra of functions on LX is called the *Hochschild cohomology* of X.

Remark that this definition really gives the right Hochshild cohomology in algebraic, smooth and analytic geometry, on a uniform ground.

9.4 Differential cohomology

Differential cohomology is a differential version of non-abelian cohomology, that is used to formalize gauge theories on higher bundles with connections. We will give examples of such theories in Section 15.5. We refer to Sati-Schreiber-Stasheff [SSS09] for a short introduction to non-abelian cohomology and its application in physics. We base our presentation on Schreiber's book [Sch11], that gives a systematic and elegant treatment of higher gauge theories in the setting of cohesive homotopical geometry.

The general setting is given by an ∞-topos (i.e., ∞-category of sheaves on a homotopical site, see [Lur09d] for a thorought study and various characterizations)

$$\mathrm{H} = \mathrm{Sh}_{\infty\mathrm{Grpd}}(C, \tau)$$

whose global section adjunction

$$\mathrm{Disc} : {}^{\infty}\mathrm{Grpd} \leftrightarrows \mathrm{H} : \Gamma$$

is equipped with additional adjunctions

$$\Pi : \mathrm{H} \leftrightarrows {}^{\infty}\mathrm{Grpd} : \mathrm{Disc} \text{ and } \Gamma : \mathrm{H} \leftrightarrows {}^{\infty}\mathrm{Grpd} : \mathrm{coDisc},$$

called the cohesive ∞-groupoid and the codiscrete functor, fulfilling additional conditions. We will work with the cohesive ∞-topos

$$\mathrm{H} = \mathrm{Stacks}_{\infty}(\mathrm{Open}_{\mathcal{C}^{\infty}}, \tau)$$

of smooth ∞-stacks, but all constructions of this section generalize to other cohesive situations. We denote, as usual, $\underline{\mathrm{Mor}}(X, Y)$ the ∞-groupoid of morphisms between two smooth ∞-stacks.

Definition 9.4.1. Let $[c] \in H(B, C) = \pi_0(\underline{\mathrm{Mor}}(B, C))$ be a non-abelian cohomology class, called a *characteristic class*, and denote A its homotopy fiber

$$\begin{array}{ccc} A & \longrightarrow & \{*\} \\ \downarrow & & \downarrow \\ B & \xrightarrow{c} & C \end{array}.$$

The *c-twisted A-valued non-abelian cohomology space of X* is defined as the homotopy pullback

$$\begin{array}{ccc} \underline{\mathrm{Mor}}_{c-tw}(X, A) & \xrightarrow{tw} & H(X, C) \\ \downarrow & & \downarrow s \\ \underline{\mathrm{Mor}}(X, B) & \xrightarrow{\theta_*} & \underline{\mathrm{Mor}}(X, C) \end{array},$$

where $s : H(X, C) := \pi_0(\underline{\mathrm{Mor}}(X, C)) \to \underline{\mathrm{Mor}}(X, C)$ is a given choice of section.

We now define the notion of higher flat connection and flat differential cohomology.

Definition 9.4.2. Let X be an object of H. The *smooth ∞-groupoid of X* is the smooth singular space

$$\Pi(X) := \mathrm{LConst}(\Pi_\infty(|X|))$$

where $\Pi_\infty(|X|)$ is the singular ∞-groupoid of the geometric realization $|X|$ of X and LConst is the locally constant ∞-groupoid stack associated to a given ∞-groupoid.

We refer to Schreiber [Sch11] for the proof of the following result.

Theorem 9.4.3. *The smooth ∞-groupoid functor $\Pi : \mathrm{H} \to \mathrm{H}$ has a right adjoint denoted*

$$\flat : \mathrm{H} \to \mathrm{H}.$$

For A an object of H*, we denote $\flat_{dR}A := \{*\} \times^h_A \flat A$. If G is an ∞-group, the defining fibration sequence of $\flat_{dR}G$ extends to*

$$\flat_{dR}G \to G \to \flat G \xrightarrow{curv} \flat_{dR}BG,$$

and induces in particular a morphism

$$\theta : G \to \flat_{dR}BG$$

called the Mauer-Cartan form *on G.*

Definition 9.4.4. The *flat cohomology* of X with coefficients in A is defined by

$$H_{flat}(X, A) := H(\Pi(X), A) = H(X, \flat A).$$

The *de Rham cohomology* of X with coefficients in A is defined by

$$H_{dR}(X, A) := H(X, \flat_{dR}A).$$

The corresponding spaces are denoted $\underline{\mathrm{Mor}}_{flat}(X, A)$ and $\underline{\mathrm{Mor}}_{dR}(X, A)$. The *differential cohomology space* of X with coefficients in an ∞-group A is the θ-twisted cohomology, where $\theta : A \to \flat_{dR}BA$ is the Mauer-Cartan form. More concretely, it is the homotopy fiber product

$$\begin{array}{ccc} \underline{\mathrm{Mor}}_{diff}(X, A) & \longrightarrow & H_{dR}(X, BA) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Mor}}(X, A) & \xrightarrow{\theta_*} & \underline{\mathrm{Mor}}_{dR}(X, BA) \end{array} ,$$

where the right arrow is given by a choice of representative for each class in π_0. The *differential cohomology* of X with coefficients in an n-deloopable ∞-group A is

$$H^n_{diff}(X, A) := \pi_0(\underline{\mathrm{Mor}}_{diff}(X, B^nA)).$$

Example 9.4.5. If A is a Lie group, the cohomology

$$H^1(X,A) := H(X,BA)$$

is the set of isomorphism classes of smooth principal A-bundles on X. The flat cohomology set

$$H^1_{flat}(X,A) := H_{flat}(X,BA)$$

is the set of isomorphism classes of smooth principal A-bundles on X equipped with a flat connection. If $A = U(1)$, the differential cohomology set

$$H^n_{diff}(X,A) := H_{diff}(X,B^nA)$$

is the Deligne cohomology of X, usually defined as the homotopy pullback of integral cohomology and the complex of differential forms along real cohomology.

Definition 9.4.6. For $X \in \mathrm{H}$, the space $\underline{\mathrm{Mor}}_{conn}(X,BG)$ of ∞-*connections* is defined as the homotopy pullback

$$\begin{array}{ccc} \underline{\mathrm{Mor}}_{conn}(X,BG) & \longrightarrow & \prod_{c_i\in H(BG,B^{n_i}A),i\geq 1} \underline{\mathrm{Mor}}_{diff}(X,B^{n_i}A) \\ \downarrow \eta & & \downarrow \\ \underline{\mathrm{Mor}}(X,BG) & \longrightarrow & \prod_{c_i\in H(BG,B^{n_i}A),i\geq 1} \underline{\mathrm{Mor}}(X,B^{n_i}A) \end{array}$$

One defines

- an ∞-connection as a cocycle $\nabla \in \mathrm{H}_{conn}(X,BG) := \pi_0(\underline{\mathrm{Mor}}_{conn}(X,BG))$,
- the underlying ∞-bundle as $P := \eta(\nabla)$,
- a gauge transformation as a morphism in $\underline{\mathrm{Mor}}_{conn}(X,BG)$,
- the refined Chern-Weil homomorphism as the homomorphism

  $$c : H_{conn}(BG,A) \to H^n_{diff}(X,A)$$

  associated to $c \in H^n(BG,A)$.

We will see explicit examples of higher gauge field theories, defined by applying the present formalism of non-abelian cohomology, in Chapter 15.

Another way to formulate the above Chern-Weil theory is given by the following procedure, that may be found in [Sch11], Section 2.9.

Denote

$$\Omega^1_{cl}(-,A) := \flat_{dR}BA,$$

and choose recursively for each $n \in \mathbb{N}$, a morphism

$$\Omega^{n+1}_{cl}(-,A) \to B\Omega^n_{cl}(-,A)$$

out of a 0-truncated abelian group object (i.e., a sheaf of abelian groups).

Definition 9.4.7. For any $n \in \mathbb{N}$, the *differential refinement B^nA_{conn} of the classifying space of n-bundles* is the ∞-pullback

$$\begin{array}{ccc} B^nA_{conn} & \longrightarrow & \Omega^{n+1}_{cl}(-,A)\,. \\ \downarrow & & \downarrow \\ B^nA & \xrightarrow{curv} & \flat_{dR}B^{n+1}A \end{array}$$

For X a manifold, denote

$$H^n_{conn}(X,A) := \pi_0(\underline{\mathrm{Mor}}(X, B^nA_{conn})).$$

Definition 9.4.8. Let $c : BG \to B^nA$ be a smooth characteristic map and B^nA_{conn} be a differential refinement. A differential refinement of BG along c is an object BG_{conn} that fits into a factorization

$$\begin{array}{ccc} \flat BG & \xrightarrow{\flat c} & \flat B^nA \\ \downarrow & & \downarrow \\ BG_{conn} & \xrightarrow{\hat{c}} & B^nA_{conn} \\ \downarrow & & \downarrow \\ BG & \xrightarrow{c} & B^nA \end{array}$$

of the natural above vertical rectangle diagram.

If X is a manifold and $\hat{c} : BG_{conn} \to B^nA_{conn}$ is a differentially refined characteristic class, one may define the differentially refined Chern-Weil homomorphism

$$\hat{c} : \underline{\mathrm{Hom}}(M, BG_{conn}) \longrightarrow \underline{\mathrm{Hom}}(M, B^nA_{conn})$$

from the moduli space of G-bundles with connection to the moduli space of A-n-gerbes with connection on M.

9.5 Geometric stacks

We now give a short account of the notion of geometric higher stacks, that is originally due to Simpson and Hirschowitz [SH98]. We refer to [TV08], Section 1.3 for a complete treatment of geometric stacks, and to [LMB00] for the classical theory of algebraic stacks (i.e., geometric 1-stacks). We also refer to [Pri11] for a concrete description of geometric stacks in terms of simplicial diagrams of building blocs (that are affine schemes, in his case, but may be any reasonable kinds of generalized algebras in a geometric context). We prefer the approach through general stacks because it is also adapted to infinite dimensional spaces of fields (i.e., spaces of functions). The relation between geometricity and our considerations is that this is the most general finiteness hypothesis that one may put on both the space M of parameters and the configuration space C of a classical physical system.

Definition 9.5.1. Let STACKS be the ∞-category of stacks (resp. derived stacks) on a site (LEGOS, τ) (resp. homotopical site $(\overline{\text{LEGOS}}, \tau)$). Let $\mathbb{P}$ be a class of morphisms in the given site that is local with respect to the given topology. We define geometric stacks with respect to the class $\mathbb{P}$ by the following inductive process for $n \geq 0$:

1. A stack is (-1)-geometric if it is representable.

2. A morphism $f : X \to Y$ of stacks is (-1)-representable if for any representable stack Z and any morphism $Z \to Y$, the fiber product $Y \times_Z X$ is geometric.

3. A morphism $f : X \to Y$ is in (-1)-$\mathbb{P}$ if it is (-1)-representable and the pullback of f along a point $Z \to Y$ with values in a representable stack is a morphism in $\mathbb{P}$.

4. An n-atlas on a stack X is a family of morphisms $\{U_i \to X\}_{i \in I}$ such that

   (a) each U_i is representable.

   (b) each morphism $U_i \to X$ is in $(n-1)$-$\mathbb{P}$.

   (c) the total morphism
   $$\coprod_{i \in I} U_i \to X$$
   is an epimorphism.

5. A stack X is n-geometric if its diagonal morphism $X \to X \times X$ is $(n-1)$-representable and X admits an n-atlas.

6. A morphism of stacks $X \to Y$ is n-representable if for any representable stack Z and any morphism $Z \to Y$, the pullback $X \times_Y Z$ is n-geometric.

7. A morphism of stacks $X \to Y$ is in n-$\mathbb{P}$ if it is n-representable and for any morphism $Z \to Y$, there exists an n-atlas $\{U_i\}$ of $X \times_Y Z$, such that each composite morphism $U_i \to X$ is in $\mathbb{P}$.

Definition 9.5.2. In algebraic, differential or analytic geometry, a stack that is geometric with respect to the class of smooth morphisms is called an Artin stack, and a stack that is geometric with respect to the class of etale morphisms is called a Deligne-Mumford stack.

Example 9.5.3. Most of the stacks used in non-abelian cohomology and gauge theory are geometric. At least, we will always suppose that the parameter stack M and the configuration stack C for trajectories are both geometric in a convenient sense.

1. If G is a linear Lie group acting on a manifold M, the stacky quotient $[M/G]$ is a geometric Artin stack. In particular, $BG := [*/G]$ is a geometric stack.

2. More generally, if A is a commutative Lie group, then the stack $B^n A$ is also geometric.

3. Lurie's version of Artin's representability criterion (see [TV08], Appendix and [Lur09b]) implies that if M is a proper smooth scheme and G is a linear algebraic group, then the mapping stack

$$\underline{\mathrm{Hom}}(M, BG)$$

is also geometric. This may not remain true in the smooth or analytic setting.

9.6 Homotopical infinitesimal calculus

We now describe shortly differential calculus relative to a given ∞-category $\mathcal{C}$, explaining the necessary modifications that need to be done to adapt the methods of Section 1.5 to the ∞-categorical setting. This presentation of differential invariants is originally due to Quillen [Qui70] (see also Illusie [Ill71]). We refer to Lurie [Lur09c] and Toen-Vezzosi [TV08] for thorough treatments of the cotangent complex.

Definition 9.6.1. Let $\mathcal{C}$ be an ∞-category with finite limits. The tangent fibration $T\mathcal{C} \to [I, \mathcal{C}]$ to an ∞-category is the fiberwise stabilization of its codomain fibration

$$\mathrm{codom} : [I, \mathcal{C}] \to [\{0\}, \mathcal{C}] \cong \mathcal{C}.$$

More precisely, it is a categorical fibration $T\mathcal{C} \to \mathcal{C}$ (in the sense of, say, the model category ${}^{\infty}\mathrm{CAT}$) universal with the property that for each $A \in \mathcal{C}$, its fiber $T\mathcal{C}_A$ over A is the stabilization of the overcategory $\mathcal{C}/A$. If $A \in \mathcal{C}$, we denote $\mathrm{MOD}(A) := T\mathcal{C}_A$.

Definition 9.6.2. A *cotangent complex functor* for an ∞-category $\mathcal{C}$ is an adjoint

$$\mathrm{dom} : T\mathcal{C} \to [\{0\}, \mathcal{C}] \cong \mathcal{C} : \Omega^1$$

to the domain map on the tangent ∞-category.

Definition 9.6.3. A *square zero extension* in $\mathcal{C}$ is a morphism $A' \to A$ that is a torsor under a module $M \in T\mathcal{C}_A$. More precisely, if BM is a *delooping* of M, fulfilling $\Omega BM \cong M$, this torsor structure is given by a morphism

$$A' \to BM$$

in the overcategory $\mathcal{C}/A$.

Using the above definition of square zero extension, one may extend what was done in Section 1.5 to the ∞-categorical setting. This gives, in particular, a notion of thickenings and jets functors, and of crystals of module and spaces. We carefully inform the reader that working with non-flat connections in the derived setting is a bit involved, since the curvature must be seen as an additional structure on a given connection, not as a property.

We finish this section by an explicit definition of the de Rham space of a stack, generalized directly from the algebraic one studied by [GR11], and adapted to the various geometrical settings we will use in this book.

Definition 9.6.4. Let $X : \mathrm{ALG}_h^{op} \to {}^{\infty}\mathrm{GRPD}$ be a derived higher stack. The *de Rham stack* of X is defined by
$$X_{DR}(A) := X(\mathrm{disc}(\pi_0(A)_{red})),$$
where

- the adjunction
$$\pi_0 : \mathrm{ALG}_h \leftrightarrows \mathrm{ALG} : \mathrm{disc}$$
between the ∞-category ALG_h of homotopical algebras and the category ALG of usual algebras is induced by the usual adjunction
$$\pi_0 : {}^{\infty}\mathrm{GRPD} \leftrightarrows \mathrm{SETS} : \mathrm{disc};$$
- the functor $\mathrm{red} : \mathrm{ALG} \to \mathrm{ALG}$ sends an algebra A to the final object (that may be seen as a functor, if it doesn't exist) B in the category of nilpotent morphisms $A \to B$ (see Definition 1.5.6), that correspond to classical infinitesimal thickenings $\underline{\mathrm{Spec}}(B) = U \hookrightarrow T = \underline{\mathrm{Spec}}(A)$.

9.7 Derived symplectic structures

We now give a short sketch of the notion of closed form on a derived stack, and of derived symplectic form, that can be found in [PTVV11] (see also [BN10] and [TV09a]). This gives a direct generalization of the usual notions, that is quite well adapted to a global and canonical study of gauge theories. A stricking difference between the classical and the derived setting is the following: the fact that a differential form is closed is not a property but an additional structure, in general.

From now on, we work in the ∞-category $\mathrm{DSTACKS}_{\infty}(\mathrm{LEGOS}, \tau)$ of derived stack on one of the classical sites (algebraic, smooth or analytic geometry). We have already seen in Proposition 2.3.3.8 that one may see differential forms on a manifold M as functions on the superspace of maps $\underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, M)$. We will now present a refined version of this, in the setting of derived geometry, replacing $\mathbb{R}^{0|1}$ by the circle stack S^1.

Definition 9.7.1. The *circle stack* S^1 is the delooping $B\mathbb{Z}$ of the constant group stack $\mathbb{Z}$. The *derived loop space* is the stack
$$LX := \mathbb{R}\underline{\mathrm{Hom}}(S^1, X).$$
The *unipotent loop space* is the stack
$$L^u X := \mathbb{R}\underline{\mathrm{Hom}}(B\mathbb{G}_a, X).$$
The *formal loop space* $\hat{L}X$ is the formal completion of LX at the constant loop $X \subset LX$.

The natural morphism $S^1 = B\mathbb{Z} \to B\mathbb{G}_a$ induced by the inclusion $\mathbb{Z} \to \mathbb{G}_a$ induces a morphism
$$L^u X := \mathbb{R}\underline{\mathrm{Hom}}(B\mathbb{G}_a, X) \to LX.$$

In the affine case, i.e., $X = \underline{\mathrm{Spec}}(A)$, the above map is an equivalence. This means that the group stack $\underline{\mathrm{Aut}}(B\mathbb{G}_a)$ acts on LX. The more general case may be treated by using the following theorem of Ben-Zvi and Nadler [BN10].

Theorem 9.7.2. *Formal loops are unipotent, i.e., the natural map $\hat{L}X \to LX$ factors in*

$$\hat{L}X \to L^u X \to LX.$$

In particular, $\hat{L}X$ inherits an action of $\underline{\mathrm{Aut}}(B\mathbb{G}_a)$.

We can form the quotient stack

$$p : [\hat{L}X/\underline{\mathrm{Aut}}(B\mathbb{G}_a)] \to B\underline{\mathrm{Aut}}(B\mathbb{G}_a).$$

Definition 9.7.3. The *de Rham complex* $\mathrm{DR}(X)$ of X is the module $p_*\mathcal{O}_{[\hat{L}X/\underline{\mathrm{Aut}}(B\mathbb{G}_a)]}$ over $B\underline{\mathrm{Aut}}(B\mathbb{G}_a)$.

Concretely, the group stack $\underline{\mathrm{Aut}}(B\mathbb{G}_a)$ is isomorphic to the semi-direct product $\mathbb{G}_m \rtimes B\mathbb{G}_a$, and the de Rham algebra is the module over its functions with underlying complex the derived exterior product

$$\mathrm{DR}(X) = \oplus_p(\wedge^p\Omega^1_X)[p],$$

and action of the functions on $\underline{\mathrm{Aut}}(B\mathbb{G}_a)$ given by the de Rham differential, for the $B\mathbb{G}_a$ part, and the grading, for the $\mathbb{G}_m$ part. This looks like a complicated formulation, but it is necessary to treat canonically the derived case.

More generally, one may also formalize analogs of $\mathcal{D}$-modules, and quasi-coherent sheaves with non-flat connections on derived Artin stacks using the loop space formalism, as in [BN10]. We only give here a short sketch of these notions. We denote Aff Toen's affinization functor (see [Toe00]), that sends a derived stack X to the universal morphism $X \to \mathrm{Aff}(X)$ from X to an affine stack. We have $\mathrm{Aff}(S^1) = \mathrm{Aff}(B\mathbb{Z}) = B\mathbb{G}_a$. Let $S^1 \to S^3 \to S^2$ be the Hopf fibration and consider its rotation

$$\begin{array}{ccc} \Omega S^3 & \longrightarrow & \Omega S^2 \\ \downarrow & & \downarrow \\ \{e\} & \longrightarrow & S^1 \end{array}.$$

Definition 9.7.4. Let X be a derived Artin stack. A *graded de Rham module over X* is a quasi-coherent sheaf on the derived stack $[\hat{L}X/\mathrm{Aff}(S^1) \rtimes \mathbb{G}_m]$. A *curved graded de Rham module over X* is a quasi-coherent sheaf on the derived stack $[\hat{L}X/\mathrm{Aff}(\Omega S^2) \rtimes \mathbb{G}_m]$. The *curvature* of a curved graded de Rham module $\mathcal{F}$ is the associated module over the derived stack $[\hat{L}X/\mathrm{Aff}(\Omega S^3) \rtimes \mathbb{G}_m]$.

We refer to [BN10] for a proof of the following theorem and of other related results.

Theorem 9.7.5. *Let X be a usual scheme. There are natural functors from $\mathcal{D}$-modules to graded de Rham modules over X, and from quasi-coherent sheaves with connections to curved graded de Rham modules.*

We now define the cyclic complex as the complex of $B\mathbb{G}_a$-equivariant (i.e., S^1-equivariant) de Rham forms on X.

Definition 9.7.6. The *negative cyclic complex* of X is the direct image

$$NC(X) := q_*(\mathrm{DR}(X)),$$

where $q : B\underline{\mathrm{Aut}}(B\mathbb{G}_a) \to B\mathbb{G}_m$ is the natural projection.

There is a natural morphism

$$NC(X) \to \mathrm{DR}(X),$$

that induces a morphism

$$NC(X)(p) \longrightarrow \wedge^p \Omega^1_X[p]$$

for all $p \geq 0$.

For E a complex of modules, we denote $|E|$ the simplicial set associated to the truncation $\tau_{\leq 0}E$.

Definition 9.7.7. The simplicial set

$$\mathcal{A}^p(X, n) := | \wedge^p \Omega^1_X[n]|$$

is called the *space of p-forms of degree n* on the derived stack X. The simplicial set

$$\mathcal{A}^{p,cl}(X, n) := |NC(X)[n-p](p)|$$

is called the *space of closed p-forms of degree n* on the derived stack X. The homotopy fiber of the natural morphism

$$\mathcal{A}^{p,cl}(X, n) \to \mathcal{A}^p(X, n)$$

at a given form ω is called the *space of keys* of ω.

Remark that keys of ω are exactly the structures needed to close the form ω.

Definition 9.7.8. A degree n derived symplectic form on a derived stack X is a closed 2-form $\omega \in \mathcal{A}^{2,cl}(X, n)$ of degree n such that the associated morphism

$$\omega : \mathbb{T}_X \longrightarrow \Omega^1_X[n]$$

is an isomorphism.

We now quote the important structure theorems of [PTVV11], that is valid in the algebraic setting, and also probably in the analytic setting.

Theorem 9.7.9. *Let X be a derived Artin stack endowed with an $\mathcal{O}$-orientation of dimension d, and (F, ω) be a derived Artin stack with an n-shifted symplectic structure ω. Then the derived mapping stack $\underline{\mathrm{Hom}}(X, F)$ carries a natural $(n-d)$-shifted symplectic structure.*

We now present an example that gives a clear relation between Yang-Mills theory and differential calculus on derived stacks. Let G be a smooth affine group with Lie algebra $\mathfrak{g}$. An invariant pairing on $\mathfrak{g}$ induces a closed 2-form on BG. The above theorem allows in particular the construction of a shifted $2(1-d)$ symplectic form on the derived stack

$$\mathrm{B\scriptstyle{UN}}\mathrm{Conn}^G_{flat}(M) := \underline{\mathrm{Hom}}(M_{DR}, BG)$$

of principal G-bundles with flat connection, from a fundamental class $[M] \in H^{2d}_{DR}(M, \mathcal{O})$.

We now discuss derived isotropic and Lagrangian structures.

Definition 9.7.10. Let (F, ω) be an Artin stack equipped with an n-shifted symplectic form ω. An *isotropic structure* on a morphism $f : X \to F$ of derived Artin stacks is a path h between 0 and the closed 2-form $f^*\omega$ in the space $\mathcal{A}^{2,cl}(X, n)$. An isotropic structure is called a *Lagrangian structure* if the associated morphism

$$\Theta_h : \mathbb{T}_{X/F} \to \mathbb{L}_X[n-1]$$

is a quasi-isomorphism of perfect complexes.

Theorem 9.7.11. *Derived Lagrangian intersections in an n-shifted derived Artin stack are equipped with an $(n-1)$-shifted symplectic form.*

9.8 Deformation theory and formal geometry

The basis for this section is called the derived deformation theory program, and is essentially due to Deligne, Drinfeld and Kontsevich. We base our presentation on Lurie's ICM talk [Lur09e] (see also [Lur11]), on Hinich's [Hin01] and [Hin99], and on Quillen's article [Qui69]. The theory given here is fully developped over a field, and is in active development over other bases by Toen-Vezzosi and Gaitsgory-Rozenblyum. There is no reasonable doubt about the fact that a version of it will pass through to this more general setting.

Let k be a field. A spectral k-algebra is an algebra in the stable monoidal ∞-category $\mathrm{Sp}(k)$ of spectra over k. We could equivalently work with differential graded algebras, for k of characteristic zero. Let $\mathrm{Art}_k = \mathrm{ALG}^{sm}_k$ be the category of spectral artinian algebras over k, also called small augmented spectral k-algebras, given by pairs $(A, \epsilon : A \to k)$ of a nilpotent small spectral k-algebra A and a morphism ϵ such that if $\mathfrak{m}_A = \mathrm{Ker}(\epsilon)$ is the corresponding prime ideal, one has that $k = \pi_0(A/\mathfrak{m}_A)$. Recall that a k-module spectra is small if all its homotopy groups are finite dimensional and if they vanish in negative degrees and in big enough degrees. This category is in the jet category of the category of spectral k-algebras, in the sense of Section 1.5.

We also describe the coalgebra approach to deformation theory (see [Hin01]). Denote COALG^{sm}_k the category of small spectral coalgebras as the category of unital spectral coalgebras $(A, \epsilon : k \to A)$ obtained as linear duals of small spectral k-algebras.

By definition, duality gives an equivalence

$$\mathrm{Hom}_k(-, k) : \mathrm{ALG}^{sm}_k \leftrightarrows \mathrm{COALG}^{sm}_k : \mathrm{Hom}_k(-, k),$$

however, in infinite dimensional situations, it is preferable to work directly with small coalgebras. This will be the case when we pass to the chiral setting.

Remark that given a derived moduli functor

$$X : \text{ALG}(\text{Sp})_k \to \text{SSETS}$$

from spectral k-algebras to simplicial sets, and a point $x \in X(k)$, one defines its formal completion $\hat{X}_x$ by sending a small spectral algebra $A \in \text{Art}_k$ to the fiber of the map $X(A) \to X(k)$. The properties of this functor are axiomatized by the following definition.

Definition 9.8.1. A formal deformation problem is a functor

$$X : \text{Art}_k \to \text{SSETS}$$

that fulfils a kind of homotopical devissage condition, that guaranties its pro-representability, and that is

1. The space $X(k)$ is contratible,
2. If $A \to B$ and $A' \to B$ are maps between small spectral algebras with induce surjections on π_0, then X commutes homotopically to the fiber product $A \times_B A'$.

If X is a deformation problem, one may associate to it its tangent spectrum TX, given by the suspension compatible family of spaces $\{X(k \oplus k[n])\}_{k \geq 0}$.

The basic idea of deformation theory is that Artinian algebras break in peaces of the form $k \oplus k[n]$, and that the devissage axiom allows us to find back a formal deformation problem from its values on these simpler peaces, that are encoded by the tangent spectrum, so that the datum of the later is essentially equivalent to the knowledge of the deformation problem.

Derived Koszul duality is a natural ∞-adjunction

$$\text{Prim}[-1] : \text{COALG} \leftrightarrows \text{LIE} : \text{CE}$$

between commutative unital coalgebras and Lie algebras given by taking the primitive elements and the Chevalley-Eilenberg complex. If A is a commutative counital coalgebra, we denote $\bar{A} := \text{Ker}(\epsilon)$ the kernel of its counit.

Definition 9.8.2. Let $\mathfrak{g}$ be a spectral Lie algebra and A be a commutative coalgebra. The set of Mauer-Cartan elements for the pair is defined as

$$\text{MC}(A, \mathfrak{g}) := \text{Hom}(A, \text{CE}(\mathfrak{g})) \cong \text{Hom}(\text{Prim}[-1](A), \mathfrak{g}).$$

To every commutative coalgebra B, one associates a formal deformation problem by

$$F_B : A \mapsto \text{Hom}(A^*, B),$$

where the linear dual $A^* = \text{Hom}_k(A, k)$ of an artinian algebra is equipped with its canonical coalgebra structure. One can characterize deformation problems that are corepresentable.

Theorem 9.8.3. *The functor*

$$T : \textsc{Lie}_k \to \textsc{Def}_k$$

from Lie algebras to deformation problems that sends a Lie algebra $\mathfrak{g}$ *to*

$$F_{\mathfrak{g}}(A) := \mathrm{MC}(A^*, \mathfrak{g})$$

is fully faithful. It has a quasi-inverse on its image given by the tangent map functor

$$F \mapsto TF[-1] := \{F(k \oplus k[i+1])\}_{i \geq 0},$$

where the spectral Lie algebra structure comes from the canonical identification $TF[-1] \cong T\Omega F$ *and the infinitesimal composition of loops in* F*.*

Chapter 10

Algebraic analysis of linear partial differential equations

In this chapter, we present the natural coordinate free approach to systems of linear partial differential equations, in the setting of $\mathcal{D}$-modules. These tools will be use in Chapters 11 and 12 for the algebraic study of nonlinear partial differential equations and gauge theories, and in Chapter 23 for the study of the quantization of factorization algebras.

Linear algebraic analysis was founded by Mikio Sato in the sixties, with his groundbreaking work on hyperfunctions and microfunctions. The theory of $\mathcal{D}$-modules was then developped by Kashiwara and others. We refer to [Sch07], [And07] and [Sch09a] for historical background on these methods. In this chapter, we describe Kashiwara and Schapira's version of linear algebraic analysis, using subanalytic sheaves, that give a convenient generalization of Sato's ideas, allowing:

- a short presentation of microlocalization techniques, that give a conceptual and general solution to the problem of multiplication (or more generally pullback) of distributional solutions of partial differential systems on manifolds, and
- the treatment of functions with growth conditions, that give a precise link between the methods of algebraic analysis introduced by Sato, and the classical methods of the analysis of partial differential equations, as they are presented by Hormander in his reference book [Hör03].

We remark that algebraic analysis is usually used in the setting of real analytic geometry, meaning that one works with differential operators with real analytic coefficients, even if the solutions considered may have quite wild singularities. However, most of the tools and results can be adapted to the smooth setting, as was done by Hormander and others (see [Hör07]) for microlocal analysis. The main result that does not pass through is the Cauchy-Kowalewskaya-Kashiwara theorem about the Cauchy problem. Many of the physical models we will describe have indeed an underlying real analytic flavor.

This chapter is based on the articles [Sch10], [Sch12], [KS99] and [Pre07]. We refer to [KS90], [KSIW06] and [KS01] for a complete account of this theory and to [KKK86] and [SKK73] for a more classical presentation and many examples of explicit applications.

10.1 $\mathcal{D}$-modules and linear partial differential equations

We refer to Schneiders' review [Sch94] and Kashiwara's book [Kas03] for an introduction to $\mathcal{D}$-modules.

Let M be a smooth (real or complex) analytic manifold of dimension n and $\mathcal{D}$ be the algebra of differential operators on M. Recall that locally on M, one can write an operator $P \in \mathcal{D}$ as a finite sum

$$P = \sum_{\alpha} a_\alpha \partial^\alpha$$

with $a_\alpha \in \mathcal{O}_M$,

$$\partial = (\partial_1, \dots, \partial_n) : \mathcal{O}_M \to \mathcal{O}_M^n$$

the universal derivation and α some multi-indices.

To write down the equation $Pf = 0$ with f in an $\mathcal{O}_M$-module $\mathcal{S}$, one needs to define the universal derivation $\partial : \mathcal{S} \to \mathcal{S}^n$. This is equivalent to giving $\mathcal{S}$ the structure of a $\mathcal{D}$-module. The solution space of the equation with values in $\mathcal{S}$ is then given by

$$\mathcal{S}ol_P(\mathcal{S}) := \{f \in \mathcal{S},\ Pf = 0\}.$$

Remark that

$$\mathcal{S}ol_P : \mathrm{Mod}(\mathcal{D}) \to \mathrm{Vect}_{\mathbb{R}_M}$$

is a functor that one can think of as representing the space of solutions of P. Denote $\mathcal{M}_P$ the cokernel of the $\mathcal{D}$-linear map

$$\mathcal{D} \xrightarrow{.P} \mathcal{D}$$

given by right multiplication by P. Applying the functor $\mathcal{H}om_{\mathrm{Mod}(\mathcal{D})}(-, \mathcal{S})$ to the exact sequence

$$\mathcal{D} \xrightarrow{.P} \mathcal{D} \longrightarrow \mathcal{M}_P \to 0,$$

we get the exact sequence

$$0 \to \mathcal{H}om_{\mathrm{Mod}(\mathcal{D})}(\mathcal{M}_P, \mathcal{S}) \to \mathcal{S} \xrightarrow{P.} \mathcal{S},$$

which gives a natural isomorphism

$$\mathcal{S}ol_P(\mathcal{S}) = \mathcal{H}om_{\mathrm{Mod}(\mathcal{D})}(\mathcal{M}_P, \mathcal{S}).$$

This means that the $\mathcal{D}$-module $\mathcal{M}_P$ represents the solution space of P, so that the category of $\mathcal{D}$-modules is a convenient setting for the functor of point approach to linear partial differential equations.

Remark that it is even better to consider the derived solution space

$$\mathbb{R}\mathcal{S}ol_P(\mathcal{S}) := \mathbb{R}\mathcal{H}om_{\mathrm{Mod}(\mathcal{D})}(\mathcal{M}_P, \mathcal{S})$$

because it encodes also information on the inhomogeneous equation

$$Pf = g.$$

Indeed, applying $\mathcal{H}om_{\mathcal{D}}(-,\mathcal{S})$ to the exact sequences

$$0 \to \mathcal{I}_P \to \mathcal{D} \to \mathcal{M}_P \to 0$$

$$0 \to \mathcal{N}_P \to \mathcal{D} \to \mathcal{I}_P \to 0$$

where $\mathcal{I}_P$ is the image of P and $\mathcal{N}_P$ is its kernel, one gets the exact sequences

$$0 \to \mathcal{H}om_{\mathcal{D}}(\mathcal{M}_P,\mathcal{S}) \to \mathcal{S} \to \mathcal{H}om_{\mathcal{D}}(\mathcal{I}_P,\mathcal{S}) \to \mathcal{E}xt^1_{\mathcal{D}}(\mathcal{M}_P,\mathcal{S}) \to 0$$

$$0 \to \mathcal{H}om_{\mathcal{D}}(\mathcal{I}_P,\mathcal{S}) \to \mathcal{S} \to \mathcal{H}om_{\mathcal{D}}(\mathcal{N}_P,\mathcal{S}) \to \mathcal{E}xt^1_{\mathcal{D}}(\mathcal{I}_P,\mathcal{S}) \to 0$$

If $Pf = g$, then $QPf = 0$ for $Q \in \mathcal{D}$ implies $Qg = 0$. The second exact sequence implies that this system, called the algebraic compatibility condition for the inhomogeneous equation $Pf = g$ is represented by the $\mathcal{D}$-module $\mathcal{I}_P$, because

$$\mathcal{H}om_{\mathcal{D}}(\mathcal{I}_P,\mathcal{S}) = \{g \in \mathcal{S},\ Q.g = 0,\ \forall Q \in \mathcal{N}_P\}.$$

The first exact sequence shows that $\mathcal{E}xt^1_{\mathcal{D}}(\mathcal{M}_P,\mathcal{S})$ are classes of vectors $f \in \mathcal{S}$ satisfying the algebraic compatibility conditions modulo those for which the system is truly compatible. Moreover, for $k \geq 1$, one has

$$\mathcal{E}xt^k_{\mathcal{D}}(\mathcal{I}_P,\mathcal{S}) \cong \mathcal{E}xt^{k+1}_{\mathcal{D}}(\mathcal{M}_P,\mathcal{S})$$

so that all the $\mathcal{E}xt^k_{\mathcal{D}}(\mathcal{M}_P,\mathcal{S})$ give interesting information about the differential operator P.

Recall that the sub-algebra $\mathcal{D}$ of $\mathrm{End}_{\mathbb{R}}(\mathcal{O})$, is generated by the left multiplication by functions in $\mathcal{O}_M$ and by the derivation induced by vector fields in Θ_M. There is a natural right action of $\mathcal{D}$ on the $\mathcal{O}$-module Ω^n_M by

$$\omega.\partial = -L_\partial \omega$$

with L_∂ the Lie derivative.

There is a tensor product in the category $\textsc{Mod}(\mathcal{D})$ given by

$$\mathcal{M} \otimes \mathcal{N} := \mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}.$$

The $\mathcal{D}$-module structure on the tensor product is given on vector fields $\partial \in \Theta_M$ by Leibniz's rule

$$\partial(m \otimes n) = (\partial m) \otimes n + m \otimes (\partial n).$$

There is also an internal homomorphism object $\mathcal{H}om(\mathcal{M},\mathcal{N})$ given by the $\mathcal{O}$-module $\mathcal{H}om_{\mathcal{O}}(\mathcal{M},\mathcal{N})$ equipped with the action of derivations $\partial \in \Theta_M$ by

$$\partial(f)(m) = \partial(f(m)) - f(\partial m).$$

An important system is given by the $\mathcal{D}$-module of functions $\mathcal{O}$, that can be presented by the de Rham complex

$$\mathcal{D} \otimes \Theta_M \to \mathcal{D} \to \mathcal{O} \to 0,$$

meaning that $\mathcal{O}$, as a $\mathcal{D}$-module, is the quotient of $\mathcal{D}$ by the sub-$\mathcal{D}$-module generated by vector fields. The family of generators ∂_i of the kernel of $\mathcal{D} \to \mathcal{O}$ form a regular sequence, i.e., for every $k = 1,\dots,n$, ∂_k is not a zero divisor in $\mathcal{D}/(\partial_1,\dots,\partial_{k-1})$ (where $\partial_{-1} = 0$ by convention). This implies (see Lang [Lan93], XXI §4 for more details on Koszul resolutions) the following:

Proposition 10.1.1. *The natural map*

$$\mathrm{Sym}_{(\mathrm{Mod}_{dg}(\mathcal{D}),\otimes)}([\mathcal{D}\otimes\Theta_M\to\mathcal{D}])\longrightarrow\mathcal{O}$$

is a quasi-isomorphism of dg-$\mathcal{D}$-modules. The left hand side gives a free resolution of $\mathcal{O}$ as a $\mathcal{D}$-module called the universal Spencer complex.

Proof. This is the classical Koszul resolution of the regular ideal Θ_M in $\mathcal{D}$. For more details, see [Kas03], Proposition 1.6. □

Proposition 10.1.2. *The functor*

$$\mathcal{M}\mapsto\Omega^n_M\otimes_{\mathcal{O}}\mathcal{M}$$

induces an equivalence of categories between the categories $\mathrm{Mod}(\mathcal{D})$ *and* $\mathrm{Mod}(\mathcal{D}^{op})$ *of left and right $\mathcal{D}$-modules whose quasi-inverse is*

$$\mathcal{N}\mapsto\mathcal{H}om_{\mathcal{O}_M}(\Omega^n_M,\mathcal{N}).$$

The monoidal structure induced on $\mathrm{Mod}(\mathcal{D}^{op})$ *by this equivalence is denoted $\otimes^!$.*

Proof. The two functors are quasi-inverse of each other because Ω^n_M is a locally free $\mathcal{O}$-module of rank 1. A short computation shows that this equivalence is compatible with the $\mathcal{D}$-module structures. □

Definition 10.1.3. Let $\mathcal{S}$ be a right $\mathcal{D}$-module. The *de Rham functor* with values in $\mathcal{S}$ is the functor

$$\mathrm{DR}_{\mathcal{S}}:\mathrm{Mod}(\mathcal{D})\to\mathrm{Vect}_{\mathbb{R}_M}$$

that sends a left $\mathcal{D}$-module to

$$\mathrm{DR}_{\mathcal{S}}(\mathcal{M}):=\mathcal{S}\overset{\mathbb{L}}{\otimes}_{\mathcal{D}}\mathcal{M}.$$

The *de Rham functor* with values in $\mathcal{S}=\Omega^n_M$ is denoted DR and simply called the de Rham functor. One also denotes $\mathrm{DR}^r_{\mathcal{S}}(\mathcal{M})=\mathcal{M}\overset{\mathbb{L}}{\otimes}_{\mathcal{D}}\mathcal{S}$ if $\mathcal{S}$ is a fixed left $\mathcal{D}$-module and $\mathcal{M}$ is a varying right $\mathcal{D}$-module, and $\mathrm{DR}^r:=\mathrm{DR}^r_{\mathcal{O}}$.

Proposition 10.1.4. *The natural map*

$$\begin{array}{ccc}\Omega^n_M\otimes_{\mathcal{O}}\mathcal{D} & \to & \Omega^n_M\\ \omega\otimes Q & \mapsto & \omega(Q)\end{array}$$

extends to a $\mathcal{D}^{op}$-linear quasi-isomorphism

$$\Omega^*_M\otimes_{\mathcal{O}}\mathcal{D}[n]\overset{\sim}{\to}\Omega^n_M.$$

Proof. This follows from the fact that the above map is induced by tensoring the Spencer complex by Ω^n_M, and by the internal product isomorphism

$$\begin{array}{ccc}\mathrm{Sym}_{\mathrm{Mod}_{dg}(\mathcal{D})}([\mathcal{D}\underset{-1}{\otimes}\Theta_M\to\mathcal{D}\underset{0}{\otimes}\mathcal{O}])\otimes\Omega^n_M & \longrightarrow & (\Omega^*_M\otimes\mathcal{D}[n],d)\\ X\otimes\omega & \longmapsto & i_X\omega.\end{array}$$

□

We will see that in the super setting, this proposition can be taken as a definition of the right $\mathcal{D}$-modules of volume forms, called Berezinians.

The $\mathcal{D}$-modules we will use are usually not $\mathcal{O}$-coherent but only $\mathcal{D}$-coherent. The right duality to be used in the monoidal category $(\text{Mod}(\mathcal{D}), \otimes)$ to get a biduality statement for coherent $\mathcal{D}$-modules is thus not the internal duality $\underline{\text{Hom}}_{\mathcal{O}}(\mathcal{M}, \mathcal{O})$ but the derived dual $\mathcal{D}^{op}$-module

$$\mathbb{D}(\mathcal{M}) := \mathbb{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{D}).$$

The non-derived dual works well for projective $\mathcal{D}$-modules, but most of the $\mathcal{D}$-modules used in field theory are only coherent, so that one often uses the derived duality operation. We now describe the relation (based on biduality) between the de Rham and duality functors.

Proposition 10.1.5. *Let $\mathcal{S}$ be a coherent $\mathcal{D}^{op}$-module and $\mathcal{M}$ be a coherent $\mathcal{D}$-module. There is a natural quasi-isomorphism*

$$\mathbb{R}\text{Sol}_{\mathbb{D}(\mathcal{M})}(\mathcal{S}) := \mathbb{R}\mathcal{H}om_{\mathcal{D}^{op}}(\mathbb{D}(\mathcal{M}), \mathcal{S}) \cong \text{DR}_{\mathcal{S}}(\mathcal{M}),$$

where $\mathbb{D}(\mathcal{M}) := \mathbb{R}\text{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{D})$ is the $\mathcal{D}^{op}$-module dual of $\mathcal{M}$.

Proof. This follows from the biduality isomorphism

$$\mathcal{M} \cong \mathbb{R}\mathcal{H}om_{\mathcal{D}^{op}}(\mathbb{D}(\mathcal{M}), \mathcal{D}).$$

□

The use of $\mathcal{D}$-duality will be problematic in the study of covariant operations (like Lie bracket on local vector fields). We will come back to this in Section 11.4.

Definition 10.1.6. Let $\mathcal{M}$ be a coherent $\mathcal{D}$-module. A *coherent filtration* of $\mathcal{M}$ is a sequence $\{F^m(\mathcal{M})\}_{m\geq 0}$ of subsheaves of $\mathcal{M}$ such that

1. $\mathcal{M} = \bigcup_m F^m(\mathcal{M})$,
2. $F^m(\mathcal{D})F^l(\mathcal{M}) \subset F^{m+l}(\mathcal{M})$,
3. $\oplus_m F^m(\mathcal{M})$ is locally finitely generated as a $\oplus_m F^m(\mathcal{D})$-module.

The *characteristic ideal* of a filtered $\mathcal{D}$-module $(\mathcal{M}, F^\bullet\mathcal{M})$ is the annihilator $\mathcal{I}_{F^\bullet\mathcal{M}}$ of the $\mathcal{O}_{T^*M}$-module $\mathcal{O}_{T^*M} \otimes_{\pi^{-1}\text{gr}(\mathcal{D})} \text{gr}^F(\mathcal{M})$.

Proposition 10.1.7. *Let $\mathcal{M}$ be a coherent $\mathcal{D}$-module. There exists locally a coherent filtration $F^\bullet\mathcal{M}$ on $\mathcal{D}$. Moreover, the radical ideal*

$$\mathcal{I}_{\mathcal{M}} = \text{rad}(\mathcal{I}_{F^\bullet\mathcal{M}})$$

is independent of the local choice of a coherent filtration.

Proof. This follows from the local finiteness condition on $\oplus_m F^m(\mathcal{D})$. See [Kas03], Section 2.2 for more details. □

Definition 10.1.8. Let $\mathcal{M}$ be a coherent $\mathcal{D}$-module. The ideal $\mathcal{I}_{\mathcal{M}}$ defined in Proposition 10.1.7 is called the characteristic ideal of $\mathcal{M}$ and the corresponding variety $\text{char}(\mathcal{M})$ is called the characteristic variety of $\mathcal{M}$.

10.2 Berezinians and operations on super-$\mathcal{D}$-modules

We refer to Penkov's article [Pen83] for a complete study of the Berezinian in the $\mathcal{D}$-module setting and to Deligne-Morgan's lectures [DM99] and Manin's book [Man97] for more details on supermanifolds.

Let M be a supermanifold of dimension $n|m$ and denote Ω^1_M the $\mathcal{O}_M$-module of differential forms on M and Ω^*_M the super-$\mathcal{O}_M$-module of higher differential forms on M, defined as the exterior (i.e., odd symmetric) power

$$\Omega^*_M := \wedge^*\Omega^1_M := \mathrm{Sym}_{\mathrm{Mod}(\mathcal{O}_M)}\Omega^1_M[1].$$

Remark that Ω^*_M is strickly speaking a $\mathbb{Z}/2$-bigraded $\mathbb{R}$-module, but we can see it as a $\mathbb{Z}/2$-graded module because its diagonal $\mathbb{Z}/2$-grading identifies with $\mathrm{Sym}_{\mathrm{Mod}(\mathcal{O}_M)}T\Omega^1_M$, where $T : \mathrm{Mod}(\mathcal{O}_M) \to \mathrm{Mod}(\mathcal{O}_M)$ is the grading exchange. Thus from now on, we consider Ω^*_M as a mere $\mathbb{Z}/2$-graded module.

The super version of Proposition 10.1.4 can be taken as a definition of the Berezinian, as a complex of $\mathcal{D}$-modules, up to quasi-isomorphism.

Definition 10.2.1. The *Berezinian* of M is defined in the derived category of $\mathcal{D}_M$-modules by the formula

$$\mathrm{Ber}_M := \Omega^*_M \otimes_{\mathcal{O}} \mathcal{D}[n].$$

The *complex of integral forms* $I_{*,M}$ is defined by

$$I_{*,M} := \mathbb{R}\mathrm{Hom}_{\mathcal{D}}(\mathrm{Ber}_M, \mathrm{Ber}_M).$$

The following proposition (see [Pen83], 1.6.3) gives a description of the Berezinian as a $\mathcal{D}$-module.

Proposition 10.2.2. *The Berezinian complex is concentraded in degree* 0*, and equal there to*

$$\mathrm{Ber}_M := \mathcal{E}xt^n_{\mathcal{D}}(\mathcal{O}, \mathcal{D}).$$

It is moreover projective of rank 1 *over* $\mathcal{O}$*.*

Proof. This follows from the fact that

$$\wedge^*\Theta_M \otimes_{\mathcal{O}} \mathcal{D}[-n] \to \mathcal{O}$$

is a projective resolution such that

$$\mathrm{Ber}_M := \Omega^*_M \otimes_{\mathcal{O}} \mathcal{D}[n] = \mathbb{R}\mathcal{H}om_{\mathcal{D}}(\wedge^*\Theta_M \otimes_{\mathcal{O}} \mathcal{D}[-n], \mathcal{D})$$

and this de Rham complex is exact (Koszul resolution of a regular module) except in degree zero where it is equal to

$$\mathcal{E}xt^n_{\mathcal{D}}(\mathcal{O}, \mathcal{D}).$$

□

Proposition 10.2.3. *Suppose M is a supermanifold of dimension $m|n$. The Berezinian is a locally free $\mathcal{O}$-module of rank 1 on M with generator denoted $D(dx_1, \ldots, dx_m, d\theta_1, \ldots, d\theta_n)$. It $f : M \to M$ is an isomorphism of super-varieties (change of coordinate) with local tangent map $D_x f$ described by the even matrix*

$$D_x f = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

acting on the real vector space

$$T_x M = (T_x M)^0 \oplus (T_x M)^1,$$

the action of $D_x f$ on $D(dx_1, \ldots, dx_m)$ is given by the Berezin determinant

$$\mathrm{Ber}(D_x f) := \det(A - BD^{-1}C)\det(D)^{-1}.$$

Proof. This is a classical result (see [DM99] or [Man97]). □

In the super-setting, the equivalence of left and right $\mathcal{D}$-modules is given by the functor

$$\mathcal{M} \mapsto \mathcal{M} \otimes_{\mathcal{O}} \mathrm{Ber}_M$$

that twists by the Berezinian right $\mathcal{D}$-module, which can be computed by using the definition

$$\mathrm{Ber}_M := \Omega^*_M \otimes_{\mathcal{O}} \mathcal{D}[n]$$

and passing to degree 0 cohomology.

A more explicit description of the complex of integral forms (up to quasi-isomorphism) is given by

$$I_{*,M} := \mathbb{R}\mathcal{H}om_{\mathcal{D}}(\mathrm{Ber}_M, \mathrm{Ber}_M) \cong \mathcal{H}om_{\mathcal{D}}(\Omega^*_M \otimes_{\mathcal{O}} \mathcal{D}[n], \mathrm{Ber}_M)$$

so that we get

$$I_{*,M} \cong \mathcal{H}om_{\mathcal{O}}(\Omega^*_M[n], \mathrm{Ber}_M) \cong \mathcal{H}om_{\mathcal{O}}(\Omega^*_M[n], \mathcal{O}) \otimes_{\mathcal{O}} \mathrm{Ber}_M$$

and in particular $I_{n,M} \cong \mathrm{Ber}_M$.

Remark that Proposition 10.1.4 shows that if M is a non-super manifold, then Ber_M is quasi-isomorphic with Ω^n_M, and this implies that

$$I_{*,M} \cong \mathcal{H}om_{\mathcal{O}}(\Omega^*_M[n], \mathcal{O}) \otimes_{\mathcal{O}} \mathrm{Ber}_M \cong \wedge^* \Theta_M \otimes_{\mathcal{O}} \Omega^n_M[-n] \xrightarrow{i} \Omega^*_M,$$

where i is the internal product homomorphism. This implies the isomorphism

$$I_{*,M} \cong \Omega^*_M,$$

so that in the purely even case, integral forms essentially identify with ordinary differential forms.

The main use of the module of Berezinians is given by its usefulness in the definition of integration on supermanifolds. We refer to Manin [Man97], Chapter 4 for the following proposition.

Proposition 10.2.4. *Let M be a supermanifold, with underlying manifold $|M|$ and orientation sheaf* $\mathrm{or}_{|M|}$. *There is a natural integration map*

$$\int_M [dt^1 \dots dt^n d\theta^1 \dots d\theta^q] : \Gamma_c(M, \mathrm{Ber}_M \otimes \mathrm{or}_{|M|}) \to \mathbb{R}$$

given in a local chart (i.e., an open subset $U \subset \mathbb{R}^{n|m}$) for $g = \sum_I g_I \theta^I \in \mathcal{O}$ by

$$\int_U [dt^1 \dots dt^n d\theta^1 \dots d\theta^q] g := \int_{|U|} g_{1,\dots,1}(t) d^n t.$$

We finish by describing the inverse and direct image functors in the supergeometric setting, following the presentation of Penkov in [Pen83].

Let $g : X \to Y$ be a morphism of supermanifolds. Recall that for $\mathcal{F}$ a sheaf of $\mathcal{O}_Y$-modules on Y, we denote $g^{-1}\mathcal{F}$ the sheaf on X defined by

$$g^{-1}\mathcal{F}(U) := \lim_{g(U) \subset V} \mathcal{F}(V).$$

The $(\mathcal{D}_X, g^{-1}\mathcal{D}_Y)$ module of relative inverse differential operators is defined as

$$\mathcal{D}_{X \to Y} := \mathcal{O}_X \otimes_{g^{-1}\mathcal{O}_Y} g^{-1}\mathcal{D}_Y.$$

The $(g^{-1}\mathcal{D}_Y, \mathcal{D}_X)$ module of relative direct differential operators is defined as

$$\mathcal{D}_{X \leftarrow Y} := \mathrm{Ber}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X \to Y} \otimes_{g^{-1}\mathcal{O}_Y} (\mathrm{Ber}_Y^*).$$

Definition 10.2.5. The inverse image functor of $\mathcal{D}$-modules is defined by

$$g_{\mathcal{D}}^*(-) := \mathcal{D}_{X \to Y} \overset{\mathbb{L}}{\otimes}_{g^{-1}\mathcal{O}_Y} g^{-1}(-) : D(\mathcal{D}_Y) \to D(\mathcal{D}_X).$$

If $g : X \hookrightarrow Y$ is a locally closed embedding, the direct image functor is defined by

$$g_*^{\mathcal{D}}(-) := \mathcal{D}_{Y \leftarrow X} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} (-) : D(\mathcal{D}_X) \to D(\mathcal{D}_Y).$$

More generally, for any morphism $g : X \to Y$, one defines the right adjoint to $g_{\mathcal{D}}^*$ by the formula

$$g_*^{\mathcal{D}}(-) := \mathbb{R}f_*(- \overset{\mathbb{L}}{\otimes}_{\mathcal{D}} \mathcal{D}_{X \to Y}).$$

It may also be convenient to approach operations on $\mathcal{D}$-modules using spans (generalized correspondences) between varieties, like in [FG11].

10.3 Subanalytic sheaves

We now introduce the main technical tool of the modern presentation of linear algebraic analysis, that allows a smooth treatment of microlocalization and of generalized functions with growth conditions. We refer to [KS01] and [Pre07] for a detailed treatment of these matters.

Recall from example 2.2.7 the basics of semianalytic geometry.

Definition 10.3.1. A *rational domain* in the affine space $\mathbb{C}^n$ is a finite intersection of domains of the form

$$U_{f,g} := \{x \in \mathbb{C}^n,\ |f(x)| < |g(x)|\}$$

for $f, g \in \mathcal{O}(\mathbb{C}^n)$. We denote $\textsc{Legos}_{san}$ the category of relatively compact rational domains in $\mathbb{C}^n$ with holomorphic maps between them with its locally finite Grothendieck topology. A *semianalytic algebra* is a functor

$$A : \textsc{Legos}_{san} \to \textsc{Sets}$$

that commutes with transversal fiber products. It is called finitely generated if it is a quotient $\mathcal{O}(U)/(f_1, \ldots, f_n)$ where $\mathcal{O}(U) := \mathrm{Hom}(U, -)$ is the semianalytic algebra associated to a given lego U and $f_i \in \mathcal{O}(U)(\mathbb{C})$. We denote $(\overline{\textsc{Legos}}_{san}, \tau)$ the site of semianalytic affine schemes. A *semianalytic variety* is a sheaf on this site that is locally representable.

We now recall the basic results on subanalytic subsets. We give here a purely analytic presentation of subanalytic subsets, using only inequalities on norms of functions.

Definition 10.3.2. Let X be a complex semianalytic variety. A subset $V \subset X$ is called subanalytic if it is locally given by a projection along

$$\mathbb{A}^n_X = X \times \mathbb{C}^n \to X$$

of a subset in the algebra generated by relatively compact rational domains by union, intersection and complement. If M is a real semianalytic variety, a subset $V \subset X$ is called subanalytic if it is locally given by the real trace of a subanalytic subset in the complexification $M_{\mathbb{C}}$.

We refer to Gabrielov for the proof of the following theorem [Gab96]. The main difficulty is to pass to the complement.

Theorem 10.3.3. *Subanalytic subsets are stable by union, intersection, complement and projection along proper maps.*

Definition 10.3.4. Let X be a real or complex analytic space. Let X_{sa} be the site with objects the open subanalytic subsets of X and coverings given by usual coverings $U = \cup_{V \in S} V$, subject to the additional local finiteness hypothesis that for every compact subset K of X, there exists a finite subset $S_0 \subset S$ such that $K \cap U = K \cap (\cup_{V \in S_0} V)$.

There is a natural map of sites

$$\rho : X \to X_{sa},$$

and associated to this map, there are two pairs (ρ^{-1}, ρ_*) and $(\rho_!, \rho^{-1})$ of adjoint functors

$$\textsc{Mod}(k_X) \underset{\rho_!}{\overset{\rho_*}{\rightleftarrows}} \textsc{Mod}(k_{X_{sa}}) \quad (\text{with } \rho^{-1} : \textsc{Mod}(k_{X_{sa}}) \to \textsc{Mod}(k_X)).$$

We are now able to define constructible sheaves. (see [KS90], Chapter VIII for a detailed presentation).

Definition 10.3.5. Let k be a field and X be a complex analytic space. An object in $D^b(k_{X_{sa}})$ is called constructible if it has locally constant cohomology. An object $\mathcal{F}$ in $D^b(k_X)$ is called (semi-analytically) k-constructible if its direct image $\rho_*\mathcal{F}$ in $D^b(k_{X_{sa}})$ is constructible.

If Z is a locally closed subset of X, we denote

$$\begin{array}{rccc} \Gamma_Z: & \mathrm{Mod}(k_{X_{sa}}) & \to & \mathrm{Mod}(k_{X_{sa}}) \\ & \mathcal{F} & \mapsto & \mathcal{H}om(\rho_* k_Z, \mathcal{F}), \end{array}$$

and

$$\begin{array}{rccc} (-)_Z: & \mathrm{Mod}(k_{X_{sa}}) & \to & \mathrm{Mod}(k_{X_{sa}}) \\ & \mathcal{F} & \mapsto & \mathcal{F} \otimes \rho_* k_Z. \end{array}$$

Let X and Y be two real analytic manifolds and $f : X \to Y$ be a real analytic map. There is a natural adjoint pair

$$f_* : \mathrm{Mod}(k_{X_{sa}}) \leftrightarrows \mathrm{Mod}(k_{Y_{sa}}) : f^{-1},$$

and there is also a proper direct image functor defined by

$$\begin{array}{rccc} f_{!!}: & \mathrm{Mod}(k_{X_{sa}}) & \to & \mathrm{Mod}(k_{Y_{sa}}) \\ & \mathcal{F} & \mapsto & \operatorname{colim}_U f_* \mathcal{F}_U, \end{array}$$

where U ranges through the family of relatively compact open subanalytic subsets of X.

Let $\mathcal{R}$ be a sheaf of $k_{X_{sa}}$-algebras.

Theorem 10.3.6. *If $f : X \to Y$ is a real analytic map between real analytic manifolds, the functors f^{-1}, f_* and $f_{!!}$ extend to functors*

$$\begin{array}{rccc} f^{-1}: & \mathrm{Mod}(\mathcal{R}) & \to & \mathrm{Mod}(f^{-1}\mathcal{R}), \\ f_*: & \mathrm{Mod}(f^{-1}\mathcal{R}) & \to & \mathrm{Mod}(\mathcal{R}), \\ f_{!!}: & \mathrm{Mod}(f^{-1}\mathcal{R}) & \to & \mathrm{Mod}(\mathcal{R}). \end{array}$$

Moreover, the derived functor $\mathbb{R}f_{!!} : D^+(f^{-1}\mathcal{R}) \to D^+(\mathcal{R})$ admits a right adjoint denoted

$$f^! : D^+(\mathcal{R}) \to D^+(f^{-1}\mathcal{R}).$$

We now describe the main examples of subanalytic sheaves, given by functions with growth conditions. Let X be a real analytic manifold, $K_X = \mathbb{C}_X$, $\mathcal{C}^\infty_X$ be the sheaf of complex valued smooth functions and $\mathcal{D}_X$ be the ring of linear differential operators on X.

Definition 10.3.7. One says that a function $f \in \mathcal{C}^\infty(U)$ has *polynomial growth* at a point p of X if, for a local coordinate system $(x_1, \dots, x_n)$ around p, there exists a compact neighborhood K of p and a positive integer N such that

$$\sup_{x \in K \cap U} (\operatorname{dist}(x, X \backslash U))^N |f(x)| < \infty.$$

One says that f is *tempered at p* if all of its derivatives are of polynomial growth at p. One says that f is *tempered* if it is tempered at every point p of U. We denote $\mathcal{C}^{\infty,t}(U)$ the vector space of tempered functions.

We refer to [KS01] for a proof of the following theorem.

Theorem 10.3.8. *The mapping* $U \mapsto \mathcal{C}^{\infty,t}(U)$ *is a sheaf on* X_{sa}.

We now define Dolbeault complexes with growth conditions. These will be the main tools for the definition of generalized functions.

Definition 10.3.9. Let M be a real analytic manifold with complexification X and denote by $X^{\mathbb{R}} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} X$ the underlying real manifold, identified with the diagonal of $X^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong X \times \overline{X}$. The subanalytic sheaf of *tempered holomorphic functions* is the Dolbeault complex of X with coefficient in $\mathcal{C}_X^{\infty,t}$, defined by

$$\mathcal{O}_X^t := \mathbb{R}\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \mathcal{C}_X^{\infty,t}).$$

10.4 Sheaf theoretic generalized functions

We refer to Schapira's surveys [Sch10] and [Sch12] for this section. Let M be a real analytic manifold and X be a complexification of M. We denote $D^b(\mathbb{C}_X)$ the bounded derived category of sheaves of $\mathbb{C}$-vector spaces and $D'_X(-) := \mathbb{R}\mathcal{H}om_{\mathbb{C}_X}(-, \mathbb{C}_X)$ its duality functor. The constant sheaf on X with support on M is denoted $\mathbb{C}_M$.

The intuitive idea behind Sato's approach to generalized functions is to define them as sums of boundary values on M of holomorphic functions on $X \backslash M$. More precisely:

Definition 10.4.1. The sheaf of *hyperfunctions* on M is defined as the cohomology with support

$$\mathcal{B}_M := \mathbb{R}\mathcal{H}om_{\mathbb{C}_X}(D'_X(\mathbb{C}_M), \mathcal{O}_X).$$

Remark first that the sheaf $\mathcal{B}_M$ on X is a naturally equipped with a right $\mathcal{D}_X$-module structure. Using the Poincaré-Verdier duality isomorphism (see [KS90] for a proof of this statement) $D'_X(\mathbb{C}_M) \cong \mathrm{or}_M[-n]$, one shows that $\mathcal{B}_M$ is concentraded in degree zero and

$$\mathcal{B}_M \cong \mathcal{H}^n_M(X) \otimes \mathrm{or}_M$$

where or_M is the orientation sheaf on M.

Now consider a subanalytic open subset Ω of X and denote $\overline{\Omega}$ its closure. Assume that $M \subset \overline{\Omega}$ and $D'_X(\mathbb{C}_{\overline{\Omega}}) \cong \mathbb{C}_\Omega$ (this is true if $\Omega \subset X$ is locally homeomorphic to the inclusion of an open convex subset into $\mathbb{C}^n$). The morphism $\mathbb{C}_{\overline{\Omega}} \to \mathbb{C}_M$ defines by duality the morphism $D'_X(\mathbb{C}_M) \to D'_X(\mathbb{C}_{\overline{\Omega}}) \cong \mathbb{C}_\Omega$. Applying the functor $\mathbb{R}\mathrm{Hom}_{\mathbb{C}_X}(-, \mathcal{O}_X)$, one gets the boundary value morphism

$$b : \mathcal{O}(\Omega) \to \mathcal{B}(M).$$

The source and the target of this map can both be written

$$\mathbb{R}\mathrm{Hom}_{\mathbb{C}_X}(F, \mathcal{O}_X)$$

for F an $\mathbb{R}$-constructible sheaf on X (see Definition 10.3.5). We thus think of sheaves of this form as space of generalized analytic functions.

To illustrate the above general procedure, consider the case $M = \mathbb{R} \subset \mathbb{C} = X$ with $\Omega = \mathbb{C}\backslash\mathbb{R}$. One then has $\mathcal{B}_M \cong \mathcal{O}(\mathbb{C}\backslash\mathbb{R})/\mathcal{O}(\mathbb{C})$, the natural inclusion $\mathcal{O}(\mathbb{C}) \to \mathcal{B}(M)$ is given by $f \mapsto (f, 0)$ and the boundary value map is simply the projection

$$b : \mathcal{O}(\mathbb{C}\backslash\mathbb{R}) \to \mathcal{O}(\mathbb{C}\backslash\mathbb{R})/\mathcal{O}(\mathbb{C}).$$

For example, Dirac's δ function and Heaviside's step function are defined as the boundary values

$$\delta := b\left(\frac{1}{2\pi i z}, \frac{1}{2\pi i z}\right) \text{ and } H := b\left(\frac{1}{2\pi i}\log z, \frac{1}{2\pi i}\log z - 1\right)$$

where $\log(1) = 0$. Contrary to the usual approach of analysis, boundary values are not computed as limits, but as equivalence classes. The δ function may also be written more intuitively as

$$\delta(x) = \frac{1}{2i\pi}\left(\frac{1}{x - i0} - \frac{1}{x + i0}\right).$$

The main drawback of the above sheaf theoretic construction of spaces of generalized functions is that it prevents us to use functions with growth conditions, like tempered functions, because they don't form sheaves. One can overcome this problem by working with subanalytic sheaves.

Definition 10.4.2. The subanalytic sheaf of *hyperfunctions* is defined by

$$\mathcal{B}_M = \mathbb{R}\mathcal{H}om_{\mathbb{C}_{X_{sa}}}(D'\mathbb{C}_{M_{sa}}, \mathcal{O}_X).$$

The subanalytic sheaf of *tempered distributions* is defined by the formula

$$\mathcal{D}b^t_M := \mathbb{R}\mathcal{H}om_{\mathbb{C}_{X_{sa}}}(D'\mathbb{C}_{M_{sa}}, \mathcal{O}^t_X).$$

The subanalytic sheaf of *distributions* is defined by

$$\mathcal{D}b_M := \rho_*\rho^{-1}\mathcal{D}b^t_M.$$

We will now use similar constructions of cohomology with support, to define differential operators. Recall first that the integral kernel of the identity map on functions on $X = \mathbb{C}$ is given by

$$K(\mathrm{id}) = \delta(x, y) := \frac{1}{2i\pi}\frac{dy}{(x - y)}.$$

Indeed, Cauchy's formula tells us that

$$\mathrm{id}(f(x)) = \int_{\mathbb{C}} \delta(x, y)f(y) = \frac{1}{2i\pi}\int_{\mathbb{C}} \frac{f(y)}{(x - y)}dy.$$

One gets the kernel of the differential operator ∂_x by derivating the above equality, to yeald

$$K(\partial_x) = \frac{1}{2i\pi}\frac{dy}{(x - y)^2}.$$

We now give a definition of infinite order differential operators, by seeing differential operators as integral operators for some kernels on X^2 supported on the diagonal X. This definition can be directly microlocalized, to get the notion of microlocal operators (see Definition 10.5.7).

Definition 10.4.3. The ring of *differential operators of infinite order* is defined as

$$\mathcal{D}_X^{\mathbb{R}} := \mathbb{R}\mathcal{H}om\left(\mathbb{C}_\Delta, \mathcal{O}_{X\times X}^{(0,d_X)}\right),$$

where $\mathcal{O}_{X\times X}^{(0,d_X)} := \mathcal{O}_{X\times X} \overset{\mathbb{L}}{\otimes}_{p_2^{-1}\mathcal{O}_X} p_2^{-1}\Omega_X$. The ring of *differential operators* is defined as

$$\mathcal{D}_X := \mathbb{R}\mathcal{H}om\left(\mathbb{C}_\Delta, \mathcal{O}_{X\times X}^{t,(0,d_X)}\right),$$

where $\mathcal{O}_{X\times X}^{t,(0,d_X)} := \mathcal{O}_{X\times X}^{t} \overset{\mathbb{L}}{\otimes}_{p_2^{-1}\mathcal{O}_X} p_2^{-1}\Omega_X$.

10.5 Microlocalization

We will define microlocalization as the application of a subanalytic sheaf kernel attached to a particular 1-form, as in [KSIW06]. Convolution of subanalytic sheaves kernels is defined in the following way. Consider a diagram

$$\begin{array}{ccccc} & & X\times Y\times Z & & \\ & \swarrow^{q_{12}} & \downarrow^{q_{13}} & \searrow^{q_{23}} & \\ X\times Y & & X\times Z & & Y\times Z \end{array}$$

of manifolds seen as subanalytic spaces. The composition of a kernel $\mathcal{F} \in D^b(K_{X\times Y})$ and $\mathcal{G} \in D^b(K_{Y\times Z})$ is defined by

$$\mathcal{F} \circ \mathcal{G} := \mathbb{R}q_{13!!}(q_{12}^{-1}\mathcal{F} \otimes q_{23}^{-1}\mathcal{G}).$$

We denote $\pi_X : T^*X \to X$ the natural projection. For a closed submanifold Z of X, denote $T_Z X$ its normal bundle, defined by the exact sequence

$$0 \to TZ \to TX \times_X Z \to T_Z X \to 0$$

and $T_Z X$ its conormal bundle, defined by the exact sequence

$$0 \to T_Z^* X \to T^*X \times_X Z \to T^*Z \to 0.$$

In particular, $T_X^* X$ denotes the zero section of T^*X. To a differentiable map $f : X \to Y$, one associates the diagram

$$T^*X \xleftarrow{f_d} T^*Y \times_Y X \xrightarrow{f_\pi} T^*Y.$$

The deformation $\widetilde{X}_Z$ to the normal bundle of Z in X is defined (see [KS90], Chap. IV) as the relative affine scheme over X given by the spectrum

$$\widetilde{X}_Z := \mathrm{Spec}_{\mathcal{O}_X}\left(\bigoplus_{i\in\mathbb{Z}} z^{-i}\mathcal{I}_Z^i\right),$$

where $\mathcal{I}_Z$ denotes the ideal of Z in $\mathcal{O}_X$ and its negative powers are defined to be $\mathcal{O}_X$. We will also denote $\widetilde{X}_Z$ the corresponding manifold. The natural projection $p : \widetilde{X}_Z \to \mathbb{A}^1$ has fiber the normal bundle $t^{-1}(0) \cong T_Z X$ and $t^{-1}(\mathbb{G}_m) \cong X \times \mathbb{G}_m$, where $\mathbb{G}_m = \mathbb{A}^1 \backslash \{0\}$. One has the following commutative diagram with cartesian squares:

$$\begin{array}{ccccc}
\{0\} & \longrightarrow & \mathbb{R} & \longleftarrow & \{t \in \mathbb{R},\ t > 0\} \\
\uparrow & & \uparrow{\scriptstyle t} & & \uparrow \\
T_Z X & \xrightarrow{s} & \widetilde{X}_Z & \xleftarrow{j} & \Omega \\
\downarrow{\scriptstyle \tau_Z} & & {\scriptstyle p}\downarrow & \swarrow{\scriptstyle \tilde{p}} & \\
Z & \xrightarrow{i} & X & &
\end{array}$$

Note that $p : \widetilde{X}_Z \to X$ is not smooth but its relative dualizing complex is $\omega_{\widetilde{X}_Z/X} = K_{\widetilde{X}_Z}[1]$.

Definition 10.5.1. Let S be a subset of X. The *normal cone to S along Z* is the set

$$C_Z(S) := T_M X \cap \overline{\tilde{p}^{-1}(S)}.$$

If $X = T^*Y$ is already the cotangent bundle of a manifold Y, we use the canonical symplectic form on T^*Y to identify T^*X with TX and consider $C_Z(S)$ as a closed subset of $T^*_M X$.

Now let M be a real analytic manifold. The deformation to the normal bundle of the diagonal

$$\Delta_{T^*M} : Z = T^*M \to T^*M \times T^*M = X$$

gives a diagram

$$\begin{array}{ccccccccc}
P & \longrightarrow & TT^*M & \xrightarrow{\sim} & T_{\Delta_{T^*M}}(T^*M \times T^*M) & \xrightarrow{s} & \widetilde{T^*M \times T^*M}_{\Delta_{T^*M}} & \xleftarrow{j} & \Omega \\
 & & & & {\scriptstyle \tau_{T^*M}}\downarrow & & \downarrow{\scriptstyle p} & \swarrow{\scriptstyle \tilde{p}} & \\
 & & & & T^*M & \xrightarrow[\Delta_{T^*M}]{} & T^*M \times T^*M & &
\end{array}$$

where $P \subset TT^*M$ is defined by

$$P = \{(x, \xi; v_x, v_\xi),\ \langle v_x, \xi \rangle = 0\}.$$

Definition 10.5.2. The *microlocalization kernel* $\mathcal{K}_{T^*M} \in D^b(K_{T^*M \times T^*M_{sa}})$ is defined by

$$\mathcal{K}_{T^*M} := \mathbb{R}p_{!!}\left(K_{\overline{\Omega}} \otimes \rho_!(K_P)\right) \otimes \rho_!\left(\omega^{\otimes -1}_{\Delta_{T^*M}/T^*M \times T^*M}\right)$$

The *microlocalization functor* is defined by

$$\begin{array}{rccc}
\mu : & D^b(K_{M_{sa}}) & \to & D^b(K_{T^*M_{sa}}) \\
 & \mathcal{F} & \mapsto & \mathcal{K}_{T^*M} \circ \pi_M^{-1}(\mathcal{F}),
\end{array}$$

where the convolution is taken over the triple product $M \times T^*M \times \{.\}$ as a subanalytic space.

The microlocalization functor allows straigtforward definitions of many important tools of algebraic analysis. Remark that our definition on the subanalytic site differs in general from the usual one (see for example [Pre07], Section 5.2).

Definition 10.5.3. For $\mathcal{F}, \mathcal{G} \in D^b(K_{M_{sa}})$, the *microlocal homomorphisms* between $\mathcal{F}$ and $\mathcal{G}$ are defined by

$$\mu hom(\mathcal{F}, \mathcal{G}) := \mathbb{R}\mathcal{H}om(\mu(\mathcal{F}), \mu(\mathcal{G})).$$

For $\mathcal{F} \in D^b(K_M)$, the *microsupport* of $\mathcal{F}$ is defined by

$$\mathrm{SS}(\mathcal{F}) := \mathrm{supp}(\mu(\mathcal{F})).$$

If $\mathcal{F}, \mathcal{G} \in D^b(K_M)$, we will also denote

$$\mu hom(\mathcal{F}, \mathcal{G}) := \rho^{-1}\mathbb{R}\mathcal{H}om(\mu(\rho_*\mathcal{F}), \mu(\rho_*\mathcal{G})).$$

By definition, for $\mathcal{F}, \mathcal{G} \in D^b(K_M)$, one has

$$\mathrm{supp}(\mu hom(\mathcal{F}, \mathcal{G})) \subset \mathrm{SS}(\mathcal{F}) \cap \mathrm{SS}(\mathcal{G}).$$

Example 10.5.4. If $Z \subset M$ is a closed submanifold, one has

$$\mathrm{SS}(\mathbb{C}_Z) = T^*_Z M.$$

We give a simpler characterization of the microsupport, that is its original definition (see [KS90]). Roughly speaking, $\mathcal{F}$ has no cohomology supported by the closed half spaces whose conormals do not belong to its microsupport.

Proposition 10.5.5. *For $\mathcal{F} \in D^b(K_M)$ and $U \subset T^*M$, one has $U \cap \mathrm{SS}(\mathcal{F})$ if and only if for any $x_0 \in M$ and any real $\mathcal{C}^\infty$-function $\varphi : X \to \mathbb{R}$ such that $\varphi(x_0) = 0$ and $d\varphi(x_0) \in U$, one has*

$$(\mathbb{R}\Gamma_{\varphi \geq 0}(\mathcal{F}))_{x_0} = 0.$$

Proof. We know from [KS90], Corollary 6.1.3, that, if SS' denotes the above characterization of the microsupport,

$$\mathrm{SS}'(\mathcal{F}) = \mathrm{supp}(\mu hom(\mathcal{F}, \mathcal{F})).$$

The result then follows from the sequence of equalities

$$SS'(\mathcal{F}) = \mathrm{supp}(\mu hom(\mathcal{F}, \mathcal{F})) = \mathrm{supp}(\mathbb{R}\mathcal{H}om(\mu(\mathcal{F}), \mu(\mathcal{F})) = \mathrm{supp}(\mu(\mathcal{F})) = \mathrm{SS}(\mathcal{F}).$$

□

The main property of the microsupport is the following.

Theorem 10.5.6. *For $\mathcal{F} \in D^b(K_M)$, the microsupport $\mathrm{SS}(\mathcal{F}) \subset T^*M$ is a closed involutive subset, i.e., contains its own orthogonal for the standard symplectic form.*

Proof. See [KS90], Theorem 6.5.4. □

We now define, in analogy with the notion of infinite order differential operator, presented in Definition 10.4.3, various sheaves of microlocal functions and microlocal operators. The main use of microlocal operators is to define inverses for differential operators in some subspaces of the cotangent bundle.

Definition 10.5.7. Let X be a complex analytic manifold. Denote $\dot{T}^*X := T^*X \backslash T^*_X X$ and $\gamma : \dot{T}^*X \to P_X := \dot{T}^*X/\mathbb{C}^*$ be the canonical projection. The ring of *microlocal operators* is defined as

$$\mathcal{E}_X^{\mathbb{R}} := \mu hom\left(\mathbb{C}_\Delta, \mathcal{O}_{X\times X}^{(0,d_X)}\right),$$

where $\mathcal{O}_{X\times X}^{(0,d_X)} := \mathcal{O}_{X\times X} \overset{\mathbb{L}}{\otimes}_{p_2^{-1}\mathcal{O}_X} p_2^{-1}\Omega_X$. The ring of *tempered microlocal operators* is defined as

$$\mathcal{E}_X^{\mathbb{R},f} := \mu hom\left(\mathbb{C}_\Delta, \mathcal{O}_{X\times X}^{t,(0,d_X)}\right),$$

where $\mathcal{O}_{X\times X}^{t,(0,d_X)} := \mathcal{O}^t_{X\times X} \overset{\mathbb{L}}{\otimes}_{p_2^{-1}\mathcal{O}_X} p_2^{-1}\Omega_X$. The ring $\mathcal{E}_X$ of *microdifferential operators* is the corresponding conic sheaf, defined as

$$\mathcal{E}_X := \gamma^{-1}\mathbb{R}\gamma_*\mathcal{E}_X^{\mathbb{R},f}.$$

The ring $\mathcal{E}_X$ is supported on $T^*X \cong T^*_\Delta(X\times X)$, and one has a natural isomorphism

$$\mathrm{sp} : \mathcal{D}_X \xrightarrow{\sim} (\mathcal{E}_X)_{|T^*_\Delta\Delta}$$

where $X \cong T^*_\Delta\Delta \subset T^*_\Delta(X\times X)$ is the zero section.

There is a natural morphism $\pi^{-1}\mathcal{D}_X \to \mathcal{E}_X$. Moreover, the composition of microdifferential operators can be computed locally in an easy way. When X is affine and U is open in T^*X, a microdifferential operator $P \in \mathcal{E}_X(U)$ is described by its "total symbol"

$$\sigma_{tot}(P)(x,\xi) = \sum_{-\infty<j\leq m} p_j(x,\xi),\ m\in\mathbb{Z},\ p_j \in \Gamma(U, \mathcal{O}_{T^*X}(j)),$$

with the condition:

$$\begin{cases} \text{for any compact subset } K \text{ of } U, \text{ there exists a positive constant} \\ C_K \text{ such that } \sup_K |p_j| \leq C_K^{-j}(-j)! \text{ for all } j<0, \end{cases}$$

(where $\mathcal{O}_{T^*X}(j) \subset \mathcal{O}_{T^*X}$ denotes functions on T^*X that are homogeneous of degree j in the fiber variable). The total symbol of the product is given by the Leibniz rule: if Q is another operator, one has

$$\sigma_{tot}(P\circ Q) = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\partial_\xi^\alpha \sigma_{tot}(P)\partial_x^\alpha\sigma_{tot}(Q).$$

Definition 10.5.8. If $\mathcal{M}$ is a coherent $\mathcal{E}_X$-module, its *characteristic variety* $\mathrm{char}(\mathcal{M})$ is defined to be the support of $\mathcal{M}$.

Proposition 10.5.9. *If $\mathcal{M}$ is a $\mathcal{D}$-module and $\mathcal{M}' = \mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{M}$ is its microlocalization, the characteristic variety of $\mathcal{M}$ equals the characteristic variety of $\mathcal{M}'$, i.e.,*

$$\mathrm{char}(\mathcal{M}) = \mathrm{supp}(\mathcal{M}').$$

Proof. See [Kas03], Theorem 7.27. □

Microfunctions are a microlocal version of hyperfunctions. They are naturally equipped with a right $\mathcal{E}_X$-module structure.

Definition 10.5.10. Let M be a real analytic manifold and X be a complexification of M. The sheaf of *microfunctions* on M is defined by

$$\mathcal{C}_M := \mu hom(D'_X(\mathbb{C}_M), \mathcal{O}_X).$$

The sheaf of *tempered microfunctions* on M is defined by

$$\mathcal{C}^t_M := \mu hom(D'_X(\mathbb{C}_M), \mathcal{O}^t_X).$$

There is a natural isomorphism

$$\mathrm{sp} : \mathcal{B}_M \xrightarrow{\sim} (\mathcal{C}_M)_{|T^*_M M},$$

called specialization, where $M \cong T^*_M M \subset T^*_M X$ is the zero section. Since $\mathcal{C}_M$ is a conic sheaf, this also induces a morphism

$$\mathrm{sp} : \mathcal{B}_M(M) \longrightarrow \mathcal{C}_M(T^*_M X)$$

on global sections.

Definition 10.5.11. Let $u \in \mathcal{B}_M$ be an hyperfunction. The support of $\mathrm{sp}(u)$ in $T^*_M X$ is called the analytic wave front set of u and denoted $\mathrm{WF}(u)$.

Proposition 10.5.12. *Let Ω be a subanalytic open subset of X, $\overline{\Omega}$ its closure and suppose that $D'_X(\mathbb{C}_\Omega) \cong \mathbb{C}_{\overline{\Omega}}$ and $M \subset \overline{\Omega}$. Let $\varphi \in \mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{O})(\Omega)$ be a solution to a $\mathcal{D}_X$-module $\mathcal{M}$. Then one has*

$$\mathrm{WF}(b(\varphi)) \subset T^*_M X \cap \mathrm{SS}(\mathbb{C}_\Omega) \cap \mathrm{char}(\mathcal{M}).$$

Proof. The morphism $\mathbb{C}_{\overline{\Omega}} \to \mathbb{C}_M$ defines by duality a morphism $D'_X(\mathbb{C}_M) \to D'_X(\mathbb{C}_{\overline{\Omega}}) \cong \mathbb{C}_\Omega$. Applying $\mu hom(-, \mathcal{O}_X)$, one gets that the boundary value morphism

$$b : \mathcal{O}(\Omega) \to \mathcal{C}(T^*M) \cong \mathcal{B}(M)$$

factors through $\mu hom(\mathbb{C}_\Omega, \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X))$, and one concludes by using the fact that $\mathrm{SS}(\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) \subset \mathrm{char}(\mathcal{M})$. □

Proposition 10.5.13. *Let N be a submanifold of M and u be a hyperfunction on M. Suppose that*

$$\mathrm{WF}(u) \cap T^*_N M = \emptyset.$$

Then there is a well defined restriction $u_{|N}$ of u to N. In particular, if $u_1 \in \mathcal{B}(M)$ and $u_2 \in \mathcal{B}(M)$ are two hyperfunctions, the pullback $u_1.u_2$ of the tensor product $u_1 \otimes u_2 \in \mathcal{B}(M \times M)$ to the diagonal $M \subset M \times M$ is well defined whenever

$$\mathrm{WF}(u) \cap \mathrm{WF}(v)^o = \emptyset,$$

where $\mathrm{WF}(v)^o = \{(x, -\xi) |\ (x, \xi) \in \mathrm{WF}(v)\}$.

Proof. See [KKK86], Theorem 3.1.3 and Theorem 3.1.5. □

Microfunctions were designed to analyse the directions of the propagation of singularities of hyperfunctions in the cotangent bundle. We denote $\dot{\pi} : \dot{T}^*_M X \to M$ the conormal bundle without its zero section. The following proposition shows that microfunctions outside of the zero section describe exactly the singular part of hyperfunctions, i.e., hyperfunctions up to (non-singular) analytic functions.

Proposition 10.5.14. *There is a natural exact sequence*

$$0 \to \mathcal{A}_M \to \mathcal{B}_M \to \dot{\pi}_* \mathcal{C}_M \to 0$$

of sheaves that is also exact at the level of global sections.

Proof. See [KS90], Proposition 11.5.2. □

10.6 Some key results of linear algebraic analysis

We refer to Schapira's surveys [Sch10] and [Sch12] for refined introductions to the general methods and results of linear algebraic analysis. Let M be a real analytic manifold with complexification X. Recall that if $\mathcal{M}$ is a $\mathcal{D}$-module and $\mathcal{S}$ is another $\mathcal{D}$-module, we denote

$$\mathbb{R}\mathrm{Sol}_{\mathcal{M}}(\mathcal{S}) := \mathbb{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{S})$$

the space of (derived) solutions of $\mathcal{M}$ with values in $\mathcal{S}$.

Theorem 10.6.1 (Propagation of singularities)**.** *Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module. Then one has an inclusion*

$$\mathrm{SS}(\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_M)) \subset C_{T^*_M X}(\mathrm{char}(\mathcal{M}))$$

of closed subsets of T^*T^*X. In particular, one has

$$\mathrm{supp}(\mathbb{R}\mathcal{H}om_{\mathcal{E}}(\mathcal{M}, \mathcal{C}_M)) \subset \mathrm{char}(\mathcal{M}) \cap T^*_M X.$$

Proof. See [KS90], Proposition 11.5.4. □

Definition 10.6.2. A coherent $\mathcal{D}_X$-module is called *elliptic* if

$$\mathrm{char}(\mathcal{M}) \cap T^*_M X \subset T^*_X X.$$

Theorem 10.6.3 (Elliptic regularity)**.** *Let $\mathcal{M}$ be an elliptic $\mathcal{D}$-module. The natural inclusion of analytic solutions in hyperfunction solutions is an isomorphism:*

$$\mathbb{R}\mathrm{Sol}_{\mathcal{M}}(\mathcal{A}_M) \xrightarrow{\sim} \mathbb{R}\mathrm{Sol}_{\mathcal{M}}(\mathcal{B}_M).$$

Proof. This follows from Theorem 10.6.1, by applying the functor $\mathbb{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, -)$ to the exact sequence

$$0 \to \mathcal{A}_M \to \mathcal{B}_M \to \dot{\pi}_* \mathcal{C}_M \to 0$$

of Proposition 10.5.14. □

Definition 10.6.4. Let $f : X \to Y$ be a morphism of complex manifolds. Consider the associated diagram

$$T^*X \xleftarrow{f_d} T^*Y \times_Y X \xrightarrow{f_\pi} T^*Y.$$

Let $\mathcal{N}$ be a $\mathcal{D}_Y$-module. One says that f is *non characteristic* for $\mathcal{N}$ if

$$f_\pi^{-1}\mathrm{char}(\mathcal{N}) \cap T^*_X Y \subset X \times_Y T^*_Y Y.$$

We now give Kashiwara's generalization of the classical Cauchy-Kowalewskaya theorem, that asserts essentially that, given some nice initial data, one can solve globally analytic partial differential equations.

Theorem 10.6.5 (Cauchy-Kowalewskaya-Kashiwara)**.** *Let $f : X \to Y$ be a morphism of complex manifolds and $\mathcal{N}$ be a $\mathcal{D}_Y$-module. If f is non characteristic for $\mathcal{N}$, the $\mathcal{D}_X$-module $f^*_{\mathcal{D}}(\mathcal{N})$ is coherent and there is a natural isomorphism*

$$f^{-1}\mathbb{R}\mathrm{Sol}_{\mathcal{N}}(\mathcal{O}_Y) \xrightarrow{\sim} \mathbb{R}\mathrm{Sol}_{f^*_{\mathcal{D}}\mathcal{N}}(\mathcal{O}_X).$$

Proof. This follows by devissage from the classical solution of the holomorphic Cauchy problem. We refer to [KS90], Theorem 11.3.5 for a proof. □

A related result gives a relation between the characteristic variety of a $\mathcal{D}$-module and its holomorphic solution space (see [KS90], Theorem 11.3.3).

Theorem 10.6.6 (Kashiwara-Schapira)**.** *Let X be a complex analytic manifold and $\mathcal{M}$ be a coherent $\mathcal{D}$-module. There is a natural bijection*

$$\mathrm{char}(\mathcal{M}) = \mathrm{SS}(\mathbb{R}\mathrm{Sol}_{\mathcal{M}}(\mathcal{O}_X)).$$

A very important structural result for microlocal analysis is that coherent $\mathcal{E}$-modules have a very simple form at generic points of their characteristic variety. The full statement of this theorem is too long for our presentation and we refer to [KKK86] for more precision. The proof is based on the extension of local canonical (i.e. symplectic) changes of variables in the cotangent bundle to transformations on microdifferential operators, called quantized contact transformation.

We will now state a simplified formulation of the regular Riemann-Hilbert correspondence between derived categories of regular holonomic $\mathcal{D}$-modules and $\mathbb{C}$-constructible sheaves. We start by recollecting some basic facts on holonomic $\mathcal{D}$-modules.

Definition 10.6.7. Let X be a complex analytic manifold. A $\mathcal{D}$-module $\mathcal{M} \in D^b(\mathcal{D}_X)$ is called *holonomic* if $\mathrm{char}(\mathcal{M})$ is a Lagrangian subspace of T^*X. The category of holonomic $\mathcal{D}$-modules is denoted $D^b_{hol}(\mathcal{D}_X)$.

The category of holonomic $\mathcal{D}$-modules gives the simplest category containing the $\mathcal{D}$-module $\mathcal{O}$ of functions and stable by the Grothendieck six operations. It plays an important role in the study of some topological quantum field theories described in Sections 23.2 and 23.5. We refer to Ayoub [Ayo07] for a general formalization of the four operations, called crossed functors, and to [FHM02] for a presentation of the Grothendieck-Verdier duality formalism of [Har70] in the language of adjunctions between symmetric monoidal categories. We also recommend [KS90] as a general reference on Verdier duality for sheaves.

Theorem 10.6.8. *Let $f : X \to Y$ be a morphism of smooth analytic manifolds. The adjoint pair $(f^*, f_*) := (f^*_{\mathcal{D}}, f^{\mathcal{D}}_*)$ of functors described in Definition 10.2.5, and the duality functor $\mathbb{D}$, can be completed with an adjoint pair $(f_!, f^!)$ of proper supported inverse and direct image, to give the Grothendieck six operations*

$$(f^*, f_*, f^!, f_!, \mathbb{D}, \boxtimes)$$

on the derived category of bounded complexes of holonomic $\mathcal{D}$-modules.

Definition 10.6.9. Let X be a complex manifold and $Z \subset X$ be a closed submanifold. We denote $\mathcal{O}_X(*Z)$ the subsheaf of the sheaf $\mathcal{K}_X$ of meromorphic functions on X given by sections that have poles on Z. We denote $\mathcal{D}(*Z) = \mathcal{D}_X(*Z)$ the ring of differential operators on X with coefficients in $\mathcal{O}(*Z) = \mathcal{O}_X(*Z)$. A $\mathcal{D}(*Z)$-module $(\mathcal{E}, \nabla)$ on X that is locally free of finite rank on $\mathcal{O}(*Z)$ is called a meromorphic connection on X with poles on Z.

We first recalling the standard dévissage of holonomic $\mathcal{D}$-modules (see Kashiwara [Kas03]) in meromorphic differential equations.

Proposition 10.6.10. *For X a manifold and $\mathcal{M}$ a holonomic $\mathcal{D}$-module on X, there exists a stratification $Z_0 = \emptyset \subset Z_1 \subset \dots Z_n = X$ of X by closed analytic subsets such that for all $i = 1, \dots, n$, the restriction of $\mathcal{M}$ on Z_i is a meromorphic connection on Z_i with poles on Z_{i-1}.*

Definition 10.6.11. Let X be a manifold and Z be a closed submanifold. A meromorphic connection $(\mathcal{E}, \nabla)$ on X with poles on Z is called *regular singular* if the natural restriction morphism

$$\mathbb{R}\mathrm{Sol}_{\mathcal{E}}(\mathcal{O}_X(*Z)) \to \mathbb{R}\mathrm{Sol}_{\mathcal{E}}(\mathcal{O}_{X \setminus Z})$$

is an isomorphism.

Definition 10.6.12. A complex of sheaves $F \in D^b(\mathbb{C}_X)$ is called *constructible* if it is the derived solution sheaf

$$F = \mathbb{R}\mathrm{Sol}_{\mathcal{M}}(\mathcal{O}_X) := \mathbb{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{O}_X)$$

of a holonomic $\mathcal{D}$-module $\mathcal{M}$. The category of constructible sheaves is denoted $D^b_{\mathbb{C}-c}(\mathbb{C}_X)$. A $\mathcal{D}$-module is called *regular holonomic* if it is of the form

$$\mathcal{M} = \mathbb{R}\mathcal{H}om_{\mathbb{C}_X}(F, \mathcal{O}^t_X)$$

where F is construtible and $\mathcal{O}^t_X$ is the subanalytic sheaf of tempered holomorphic functions (see Definition 10.3.9). The category of regular holonomic $\mathcal{D}$-modules is denoted $D^b_{reghol}(\mathcal{D}_X)$.

Theorem 10.6.13 (Riemann-Hilbert correspondence)**.** *The pair of functors*

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}}(-, \mathcal{O}_X) : D^b_{reghol}(\mathcal{D}_X)^{op} \leftrightarrows D^b_{\mathbb{C}-c}(\mathbb{C}_X) : \mathbb{R}\mathcal{H}om_{\mathbb{C}_X}(-, \mathcal{O}^t_X)$$

is an equivalence of categories. Moreover, one has the following concrete characterizations of both categories:

1. *a sheaf F is constructible if and only if it is locally constant of finite rank on an analytic stratification, i.e., there exist a finite sequence $Z_0 = \emptyset \subset Z_1 \subset \dots Z_n = X$ of closed analytic subsets such that $F_{|Z_{i+1}\setminus Z_i}$ is locally constant.*

2. *a holonomic $\mathcal{D}$-module $\mathcal{M}$ is regular if and only if one of the following equivalent properties is fulfilled:*

   (a) *there exists locally a coherent filtration $F^\bullet\mathcal{M}$ whose characteristic ideal $\mathcal{I}_{F^\bullet\mathcal{M}}$ is radical (see Definition 10.1.6), i.e., such that the radical ideal $\mathcal{I}_{\mathcal{M}}$ defining the characteristic variety* $\mathrm{char}(\mathcal{M})$ *annihilates* $\mathrm{gr}^F(\mathcal{M})$.

   (b) *there exists a dévissage of $\mathcal{M}$ into meromorphic connections on a stratification $Z_0 = \emptyset \subset Z_1 \subset \dots Z_n = X$ by closed analytic subsets, such that for $i = 1, \dots, n$, the restriction of $\mathcal{M}$ to Z_i is a regular singular meromorphic connection with poles on Z_{i-1}.*

Remark for the applications to diffeographism groups in renormalization a la Connes-Kreimer of Chapter 18, one needs a more general result, called the irregular Riemann-Hilbert correspondence: a classification of all holonomic $\mathcal{D}$-modules by constructible sheaves with additional combinatorial data. This general result is only known in lower dimensional situations (see [Sab09] for a description of the kind of structures necessary to prove instances of the irregular Riemann-Hilbert correspondence), but the results of Mochizuki [Moc08] and Kedlaya [Ked10] (that give a fine devissage for general holonomic $\mathcal{D}$-modules) give interesting hints to the definition of a convenient category of irregular (ramified) constructible coefficients, that will give an irregular Riemann-Hilbert correspondence (see also [**?**]).

Chapter 11

Algebraic analysis of non-linear partial differential equations

This chapter introduces the necessary tools for a differential algebraic coordinate-free study of non-linear partial differential equations in general, and Lagrangian mechanics (see Section 7.1 for an introduction compatible with our general approach) in particular.

The algebraic study of non-linear partial differential equation was initiated by E. Cartan. This section expands on the article [Pau11a]. We use the viewpoint of functorial analysis, developped in Chapter 3, meaning that functionals are defined as partially defined functions between spaces in the sense of Chapter 2. We use Beilinson and Drinfeld's functorial approach [BD04] to non-linear algebraic partial differential equations (with non-algebraic coefficients), and we relate this approach to ours by extending it to non-algebraic partial differential equations. We are also inspired by the work of Nestruev's diffiety school, and in particular Vinogradov [Vin01] and Krasilshchik and Verbovetsky [KV98].

11.1 Differential algebras and non-linear partial differential equations

In this section, we will use systematically the language of differential calculus in symmetric monoidal categories, and the functor of points approach to spaces of fields, described in Chapters 2 and 3. We first restrict our presentation to relatively algebraic partial differential equations (with non-algebraic coefficients), but our results also apply to smooth partial differential equations, using the formalism of Section 1.5. This extension of our approach to general partial differential equations will be described in Section 11.5, but it is pedagogically helpful to first consider the relatively algebraic situation.

We specialize the situation to the symmetric monoidal category

$$(\textsc{Mod}(\mathcal{D}_M), \otimes_{\mathcal{O}_M})$$

of left $\mathcal{D}_M$-module on a given (super-)manifold M. Recall that there is an equivalence

$$(\textsc{Mod}(\mathcal{D}_M), \otimes) \to (\textsc{Mod}(\mathcal{D}_M^{op}), \otimes^!)$$

given by tensoring with the $(\mathcal{D}, \mathcal{D}^{op})$-modules Ber_M and $\mathrm{Ber}_M^{-1} = \mathcal{H}om_{\mathcal{O}}(\mathrm{Ber}_M, \mathcal{O})$. The unit objects for the two monoidal structures are $\mathcal{O}$ and Ber_M respectively. If $\mathcal{M}$ is a $\mathcal{D}$-module (resp. a $\mathcal{D}^{op}$-module), we denote $\mathcal{M}^r := \mathcal{M} \otimes \mathrm{Ber}_M$ (resp. $\mathcal{M}^\ell := \mathcal{M} \otimes \mathrm{Ber}_M^{-1}$) the corresponding $\mathcal{D}^{op}$-module (resp. $\mathcal{D}$-module).

Recall that if $P \in \mathbb{Z}[X]$ is a polynomial, one can study the solution space

$$\underline{\mathrm{Sol}}_{P=0}(A) = \{x \in A,\ P(x) = 0\}$$

of P with values in any commutative unital ring. Indeed, in any such ring, one has a sum, a multiplication, a zero and a unit that fulfill the necessary compatibilities to be able to write down the polynomial. One can thus think of the mathematical object given by the category of commutative unital rings as solving the mathematical problem of giving a natural setting for a coordinate free study of polynomial equations. This solution space is representable, meaning that there is a functorial isomorphism

$$\underline{\mathrm{Sol}}_{P=0}(-) \cong \mathrm{Hom}_{\mathrm{RINGS}_{cu}}(\mathbb{Z}[X]/(P), -).$$

This shows that the solution space of an equation essentially determine the equation itself. Remark that the polynomial P lives in the free algebra $\mathbb{Z}[X]$ on the given variable that was used to write it.

Suppose now given the bundle $\pi_1 : C = \mathbb{R} \times \mathbb{R} \to \mathbb{R} = M$ of smooth manifolds. We would like to study an algebraic non-linear partial differential equation

$$F(t, \partial_t^i x) = 0$$

that applies to sections $x \in \Gamma(M, C)$, that are functions $x : \mathbb{R} \to \mathbb{R}$. It is given by a polynomial $F(t, x_i) \in \mathbb{R}[t, \{x_i\}_{i \geq 0}]$. The solution space of such an equation can be studied with values in any $\mathcal{O}$-algebra $\mathcal{A}$ equipped with an action of the differentiation ∂_t (that fulfills a Leibniz rule for multiplication), the basic example being given by the algebra $\mathrm{Jet}(\mathcal{O}_C) := \mathbb{R}[t, \{x_i\}_{i \geq 0}]$ above with the action $\partial_t x_i = x_{i+1}$. The solution space of the given partial differential equation is then given by the functor

$$\underline{\mathrm{Sol}}_{\mathcal{D},F=0}(\mathcal{A}) := \{x \in \mathcal{A},\ F(t, \partial_t^i x) = 0\}$$

defined on all $\mathcal{O}_C$-algebras equipped with an action of ∂_t. To be more precise, we define the category of $\mathcal{D}$-algebras, that solves the mathematical problem of finding a natural setting for a coordinate free study of polynomial non-linear partial differential equations with smooth super-function coefficients.

Definition 11.1.1. Let M be a supermanifold. A *$\mathcal{D}_M$-algebra* is an algebra $\mathcal{A}$ in the monoidal category of $\mathcal{D}_M$-modules. More precisely, it is an $\mathcal{O}_M$-algebra equipped with an action

$$\Theta_M \otimes \mathcal{A} \to \mathcal{A}$$

of vector fields on M such that the product in $\mathcal{A}$ fulfills Leibniz's rule

$$\partial(fg) = \partial(f)g + f\partial(g).$$

Recall from Chapter 2.3 that one can extend the jet functor to the category of smooth $\mathcal{D}$-algebras (and even to smooth super-algebras), to extend the forthcoming results to the study of non-polynomial smooth partial differential equations. The forgetful functor

$$\text{Forget} : \text{ALG}_{\mathcal{D}} \to \text{ALG}_{\mathcal{O}}$$

has an adjoint (free $\mathcal{D}$-algebra on a given $\mathcal{O}$-algebra)

$$\text{Jet} : \text{ALG}_{\mathcal{O}} \to \text{ALG}_{\mathcal{D}}$$

called the (infinite) jet functor. It fulfills the universal property that for every $\mathcal{D}$-algebra $\mathcal{B}$, the natural map

$$\text{Hom}_{\text{ALG}_{\mathcal{O}}}(\mathcal{O}_C, \mathcal{B}) \cong \text{Hom}_{\text{ALG}_{\mathcal{D}}}(\text{Jet}(\mathcal{O}_C), \mathcal{B})$$

induced by the natural map $\mathcal{O}_C \to \text{Jet}(\mathcal{O}_C)$ is a bijection.

Using the jet functor, one can show that the solution space of the non-linear partial differential equation

$$F(t, \partial_t^i x) = 0$$

of the above example is representable, meaning that there is a natural isomorphism of functors on $\mathcal{D}$-algebras

$$\underline{\text{Sol}}_{\mathcal{D}, F=0}(-) \cong \text{Hom}_{\text{ALG}_{\mathcal{D}}}(\text{Jet}(\mathcal{O}_C)/(F), -)$$

where (F) denotes the $\mathcal{D}$-ideal generated by F. This shows that the jet functor plays the role of the polynomial algebra in the differential algebraic setting. If $\pi : C \to M$ is a bundle, we define

$$\text{Jet}(C) := \underline{\text{Spec}}(\text{Jet}(\mathcal{O}_C)).$$

One can summarize the above discussion by the following array:

Equation	Polynomial	Partial differential
Formula	$P(x) = 0$	$F(t, \partial^\alpha x) = 0$
Naive variable	$x \in \mathbb{R}$	$x \in \text{Hom}(\mathbb{R}, \mathbb{R})$
Algebraic structure	commutative unitary ring A	$\mathcal{D}_M$-algebra A
Free structure	$P \in \mathbb{R}[x]$	$F \in \text{Jet}(\mathcal{O}_C)$
Solution space	$\{x \in A, P(x) = 0\}$	$\{x \in A, F(t, \partial^\alpha x) = 0\}$

Example 11.1.2. If $\pi : C = \mathbb{R}^{n+m} \to \mathbb{R}^n = M$ is a trivial bundle of dimension $m + n$ over M of dimension n, with algebra of coordinates $\mathcal{O}_C := \mathbb{R}[\underline{t}, \underline{x}]$ for $\underline{t} = \{t^i\}_{i=1,\dots,n}$ and $\underline{x} = \{x^j\}_{j=1,\dots,m}$ given in multi-index notation, its jet algebra is

$$\text{Jet}(\mathcal{O}_C) := \mathbb{R}[\underline{t}, \underline{x}_\alpha]$$

where $\alpha \in \mathbb{N}^m$ is a multi-index representing the derivation index. The $\mathcal{D}$-module structure is given by making $\frac{\partial}{\partial t^i}$ act through the total derivative

$$D_i := \frac{\partial}{\partial t^i} + \sum_{\alpha, k} x^k_{i\alpha} \frac{\partial}{\partial x^k_\alpha}$$

where $i\alpha$ denotes the multi-index α increased by one at the i-th coordinate. For example, if $\pi : C = \mathbb{R} \times \mathbb{R} \to \mathbb{R} = M$, one gets

$$D_1 = \frac{\partial}{\partial t} + x_1 \frac{\partial}{\partial x} + x_2 \frac{\partial}{\partial x_1} + \dots .$$

Definition 11.1.3. Let $\pi : C \to M$ be a bundle. A *partial differential equation* on the space $\Gamma(M, C)$ of sections of π is given by a quotient $\mathcal{D}_M$-algebra

$$p : \mathrm{Jet}(\mathcal{O}_C) \twoheadrightarrow \mathcal{A}$$

of the jet algebra of the $\mathcal{O}_M$-algebra $\mathcal{O}_C$. Its *local solution space* is the $\mathcal{D}$-space whose points with values in $\mathrm{Jet}(\mathcal{O}_C)$-$\mathcal{D}$-algebras $\mathcal{B}$ are given by

$$\underline{\mathrm{Sol}}_{\mathcal{D},(\mathcal{A},p)} := \{x \in \mathcal{B} |\ f(x) = 0,\ \forall f \in \mathrm{Ker}(p)\}$$

The *non-local solution space* of the partial differential equation $(\mathcal{A}, p)$ is the subspace of $\underline{\Gamma}(M, C)$ given by

$$\underline{\mathrm{Sol}}_{(\mathcal{A},p)} := \{x \in \underline{\Gamma}(M, C) |\ (j_\infty x)^* L = 0 \text{ for all } L \in \mathrm{Ker}(p)\}$$

where $(j_\infty x)^* : \mathrm{Jet}(\mathcal{O}_C) \to \mathcal{O}_M$ is (dual to) the Jet of x. Equivalently, $x \in \underline{\mathrm{Sol}}_{(\mathcal{A},p)}$ if and only if there is a natural factorization

$$\begin{array}{ccc} \mathrm{Jet}(\mathcal{O}_C) & \xrightarrow{(j_\infty x)^*} & \mathcal{O}_M \\ & {\scriptstyle p}\searrow & \uparrow \\ & & \mathcal{A} \end{array}$$

of the jet of x through p.

11.2 Local functionals and local differential forms

The natural functional invariant associated to a given $\mathcal{D}$-algebra $\mathcal{A}$ is given by the de Rham complex

$$\mathrm{DR}(\mathcal{A}) := (I_{*,M} \otimes_{\mathcal{O}} \mathcal{D}[n]) \overset{\mathbb{L}}{\otimes}_{\mathcal{D}} \mathcal{A}$$

of its underlying $\mathcal{D}$-module with coefficient in the universal complex of integral forms $I_{*,M} \otimes_{\mathcal{O}} \mathcal{D}[n]$, and its cohomology $h^*(\mathrm{DR}(\mathcal{A}))$. We will denote

$$h(\mathcal{A}) := h^0(\mathrm{DR}(\mathcal{A})) = \mathrm{Ber}_M \otimes_{\mathcal{D}} \mathcal{A}$$

where Ber_M here denotes the Berezinian object (and not only the complex concentrated in degree 0). If M is a non-super manifold, one gets

$$\mathrm{DR}(\mathcal{A}) = \Omega^n_M \overset{\mathbb{L}}{\otimes}_{\mathcal{D}} \mathcal{A} \qquad \text{and} \qquad h(\mathcal{A}) = \Omega^n_M \otimes_{\mathcal{D}} \mathcal{A}.$$

The de Rham cohomology is given by the cohomology of the complex

$$\mathrm{DR}(\mathcal{A}) = I_{*,M}[n] \otimes_{\mathcal{O}_M} \mathcal{A},$$

which gives

$$\mathrm{DR}(\mathcal{A}) = \wedge^* \Omega^1_M[n] \otimes_{\mathcal{O}_M} \mathcal{A}$$

in the non super case.

If $\mathcal{A}$ is a jet algebra of section of a bundle $\pi : C \to M$ with basis a classical manifold, the de Rham complex identifies with a sub-complex of the usual de Rham complex of $\wedge^*\Omega^1_{\mathcal{A}/\mathbb{R}}$ of $\mathcal{A}$ viewed as an ordinary ring. One can think of classes in $h^*(\mathrm{DR}(\mathcal{A}))$ as defining a special class of (partially defined) functionals on the space $\underline{\Gamma}(M, C)$, by integration along singular homology cycles with compact support.

Definition 11.2.1. Let M be a supermanifold of dimension $p|q$. For every smooth simplex Δ_n, we denote $\Delta_{n|q}$ the super-simplex obtained by adjoining q odd variables to Δ_n. The *singular homology of M with compact support* is defined as the homology $H_{*,c}(M)$ of the simplicial set

$$\mathrm{Hom}(\Delta_{\bullet|q}, M)$$

of super-simplices with compact support condition on the body and non-degeneracy condition on odd variables.

Recall that a functional $f \in \mathrm{Hom}(\underline{\Gamma}(M, C), \mathbb{A}^1)$ on a space of fields denotes in general, by definition, only a partially defined function (with a well-chosen domain of definition).

Proposition 11.2.2. *Let $\pi : C \to M$ be a bundle and $\mathcal{A}$ be the $\mathcal{D}_M$-algebra $\mathrm{Jet}(\mathcal{O}_C)$. There is a natural integration pairing*

$$\begin{array}{ccc} H_{*,c}(M) \times h^{*-n}(\mathrm{DR}(\mathcal{A})) & \to & \mathrm{Hom}(\underline{\Gamma}(M, C), \mathbb{A}^1) \\ (\Sigma, \omega) & \mapsto & [x \mapsto \int_\Sigma (j_\infty x)^* \omega] \end{array}$$

where $j_\infty x : M \to \mathrm{Jet}(C)$ is the taylor series of a given section x. If $p : \mathrm{Jet}(\mathcal{O}_C) \to \mathcal{A}$ is a given partial differential equation (such that $\mathcal{A}$ is $\mathcal{D}$-smooth) on $\Gamma(M, C)$ one also gets an integration pairing

$$\begin{array}{ccc} H_{*,c}(M) \times h^{*-n}(\mathrm{DR}(\mathcal{A})) & \to & \mathrm{Hom}(\underline{\mathrm{Sol}}_{(\mathcal{A},p)}, \mathbb{A}^1) \\ (\Sigma, \omega) & \mapsto & S_{\Sigma,\omega} : [x(t, u) \mapsto \int_\Sigma (j_\infty x)^* \omega]. \end{array}$$

Proof. Remark that the values of the above pairing are given by partially defined functions, with a domain of definition given by Lebesgue's domination condition to make $t \mapsto \int_\Sigma (j_\infty x_t)^* \omega$ a smooth function of t if x_t is a parametrized trajectory. The only point to check is that the integral is independent of the chosen cohomology class. This follows from the fact that the integral of a total divergence on a closed subspace is zero, by Stokes' formula (the super case follows from the classical one). □

Definition 11.2.3. A functional $S_{\Sigma,\omega} : \underline{\Gamma}(M,C) \to \mathbb{A}^1$ or $S_{\Sigma,\omega} : \underline{\mathrm{Sol}}_{(\mathcal{A},p)} \to \mathbb{A}^1$ obtained by the above constructed pairing is called a *quasi-local functional.* We denote

$$\mathcal{O}^{qloc} \subset \mathcal{O} := \underline{\mathrm{Hom}}(\underline{\Gamma}(M,C), \mathbb{A}^1)$$

the space of quasi-local functionals. A *local functional* is a quasi-local functional that comes from a cohomology class in $h(\mathcal{A}) = h^0(\mathrm{DR}(\mathcal{A}))$.

Remark that for $k \leq n$, the classes in $h^{*-k}(\mathrm{DR}(\mathcal{A}))$ are usually called (higher) conservation laws for the partial differential equation $p : \mathrm{Jet}(\mathcal{O}_C) \to \mathcal{A}$.

If $\mathcal{A}$ is a $\mathcal{D}$-algebra, i.e., an algebra in $(\mathrm{Mod}(\mathcal{D}), \otimes_{\mathcal{O}})$, one defines, as usual, an $\mathcal{A}$-module of differential forms as an $\mathcal{A}$-module $\Omega^1_{\mathcal{A}}$ in the monoidal category $(\mathrm{Mod}(\mathcal{D}), \otimes)$, equipped with a ($\mathcal{D}$-linear) derivation $d : \mathcal{A} \to \Omega^1_{\mathcal{A}}$ such for every $\mathcal{A}$-module $\mathcal{M}$ in $(\mathrm{Mod}(\mathcal{D}), \otimes)$, the natural map

$$\mathrm{Hom}_{\mathrm{Mod}(\mathcal{A})}(\Omega^1_{\mathcal{A}}, \mathcal{M}) \to \mathrm{Der}_{\mathrm{Mod}(\mathcal{A})}(\mathcal{A}, \mathcal{M})$$

given by $f \mapsto f \circ d$ is a bijection.

Remark that the natural $\mathcal{O}$-linear map

$$\Omega^1_{\mathcal{A}/\mathcal{O}} \to \Omega^1_{\mathcal{A}}$$

is an isomorphism of $\mathcal{O}$-modules. The $\mathcal{D}$-module structure on $\Omega^1_{\mathcal{A}}$ can be seen as an Ehresman connection, i.e., a section of the natural projection

$$\Omega^1_{\mathcal{A}/\mathbb{R}} \to \Omega^1_{\mathcal{A}/M}.$$

Example 11.2.4. In the case of the jet space algebra $\mathcal{A} = \mathrm{Jet}(\mathcal{O}_C)$ for $C = \mathbb{R}^{n+m} \to \mathbb{R}^n = M$, a basis of $\Omega^1_{\mathcal{A}/M}$ compatible with this section is given by the Cartan forms

$$\theta^i_\alpha = dx^i_\alpha - \sum_{j=1}^n x^i_{j\alpha} dt^j.$$

The de Rham differential $d : \mathcal{A} \to \Omega^1_{\mathcal{A}}$ in the $\mathcal{D}$-algebra setting and its de Rham cohomology (often denoted d^V in the literature), can then be computed by expressing the usual de Rham differential $d : \mathcal{A} \to \Omega^1_{\mathcal{A}/M}$ in the basis of Cartan forms.

As explained in Section 10.1, the right notion of finiteness and duality in the monoidal category of $\mathcal{D}$-modules is not the $\mathcal{O}$-finite presentation and duality but the $\mathcal{D}$-finite presentation and duality. This extends to the category of $\mathcal{A}$-modules in $(\mathrm{Mod}(\mathcal{D}), \otimes)$. The following notion of smoothness differs from the usual one over a ring, because we impose the $\mathcal{A}[\mathcal{D}]$-finite presentation, where $\mathcal{A}[\mathcal{D}] := \mathcal{A} \otimes_{\mathcal{O}} \mathcal{D}$, to have good duality properties. This is however equivalent to the usual categorical finite presentation (in terms of commutation to directed colimits).

Definition 11.2.5. The $\mathcal{D}$-algebra $\mathcal{A}$ is called $\mathcal{D}$-*smooth* if $\Omega^1_{\mathcal{A}}$ is a projective $\mathcal{A}$-module of finite $\mathcal{A}[\mathcal{D}]$-presentation in the category of $\mathcal{D}$-modules, and $\mathcal{A}$ is a (geometrically) finitely presented $\mathcal{D}$-algebra, meaning that there exists a finitely presented $\mathcal{O}$-module $\mathcal{M}$, a $\mathcal{D}$-ideal $\mathcal{I} \subset \mathrm{Jet}(\mathrm{Sym}_{\mathcal{O}}(\mathcal{M}))$, and a surjective morphism

$$\mathrm{Jet}(\mathrm{Sym}_{\mathcal{O}}(\mathcal{M}))/\mathcal{I} \twoheadrightarrow \mathcal{A}.$$

Proposition 11.2.6. *If $\mathcal{A} = \mathrm{Jet}(\mathcal{O}_C)$ for $\pi : C \to M$ a smooth map of varieties, then $\mathcal{A}$ is $\mathcal{D}$-smooth and the $\mathcal{A}[\mathcal{D}]$-module $\Omega^1_{\mathcal{A}}$ is isomorphic to*

$$\Omega^1_{\mathcal{A}} \cong \Omega^1_{C/M} \otimes_{\mathcal{O}_C} \mathcal{A}[\mathcal{D}].$$

In particular, if $\pi : C = \mathbb{R} \times M \to M$ is the trivial bundle with fiber coordinate u, one gets the free $\mathcal{A}[\mathcal{D}]$-module of rank one

$$\Omega^1_{\mathcal{A}} \cong \mathcal{A}[\mathcal{D}]^{(\{du\})}$$

generated by the form du.

Remark 11.2.7. One may identify local functionals on $\underline{\Gamma}(M, C)$ with cohomology classes in $h(\mathcal{A})$, where $\mathcal{A} := \mathrm{Jet}(\mathcal{O}_C)$ is the jet algebra, and

$$h(\mathcal{A}) = h^0(\mathrm{DR}(\mathcal{A})) := h^0(\mathrm{Ber}_M \overset{\mathbb{L}}{\otimes}_{\mathcal{D}} \mathcal{A}).$$

Up to now, we have been using this degree convention, because we were working mostly with left $\mathcal{D}$-modules, and $\mathbb{R}$-valued action funtionals. We may also see this from the right $\mathcal{D}$-module viewpoint as

$$h(\mathcal{A}) = h^n(\mathcal{A}^r \otimes_{\mathcal{D}} \mathcal{O}),$$

where $\mathcal{A}^r := \mathrm{Ber}_M \otimes_{\mathcal{O}} \mathcal{A}$ is the right $\mathcal{D}$-algebra associated to $\mathcal{A}$. In this viewpoint, it is desirable to think of volume forms in Ber_M as living in cohomological degree n, where n is the dimension of the supermanifold in play, so that $\mathcal{A}^r$ is in degree n and its cohomology also. It is possible to think of a local action functional $S \in h(\mathcal{A})$ as a space morphism

$$\underline{\Gamma}(M, C) \to \underline{\Gamma}(M, \mathrm{Ber}_M),$$

whose value can be integrated on compact subspaces of M, to get a real number. If not integrated, a density must be thought as living in cohomological degree n. This is what will explain the degree shifts that one finds in computations on the Ran space.

11.3 $\mathcal{D}$-modules on the Ran space and operations

We refer to Beilinson-Drinfeld's book [BD04] and to Francis-Gaitsgory's article [FG11] for a systematic study of $\mathcal{D}$-modules on the Ran space. We will give here a short presentation of local operations (also called $*$-operations by Beilinson and Drinfeld), and chiral operations, since these are at the basis of the deformation quantization approach to euclidean quantum field theory. Local operations are very important because they allow to manipulate algebraically covariant objects like local vector fields or local Poisson brackets. The main idea of Beilinson and Drinfeld's approach to geometry of $\mathcal{D}$-spaces is that functions and differential forms usually multiply by using ordinary tensor product of $\mathcal{D}$-modules, but that local vector fields and local differential operators multiply by using local operations. This gives the complete toolbox to do differential geometry on $\mathcal{D}$-spaces in a way that is very similar to ordinary differential geometry.

One defines the ∞-category (we may also use dg-categories here) of $\mathcal{D}$-modules on the Ran space as the ∞-limit (or the homotopy limit of dg-categories)

$$\mathcal{D}(\mathrm{Ran}(M)) := \lim \mathcal{D}(M^I)$$

of the categories $\mathcal{D}(M^I)$ of $\mathcal{D}$-modules on finite powers of M, indexed by the category of finite sets I with surjective maps between them. We refer to Section 11.5 for a refined geometric explanation of this definition.

By definition, a $\mathcal{D}$-module on the Ran space is a collection of $\mathcal{D}$-modules $\mathcal{M}^I$ on the finite powers M^I with homotopy equivalences

$$\Delta(\pi)(\mathcal{M}^I) \cong \mathcal{M}^J$$

for every surjection $\pi : I \twoheadrightarrow J$, with $\Delta(\pi) : M^J \to M^I$ the associated diagonal map.

The canonical projection functor

$$\Delta(\pi)^! : \mathcal{D}(\mathrm{Ran}(M)) \to \mathcal{D}(M^I)$$

has a left adjoint

$$\Delta(\pi)_* : \mathcal{D}(M^I) \to \mathcal{D}(\mathrm{Ran}(M)).$$

that may be described explicitely by the following construction (see [Roz10] for a more precise description): Let $\mathcal{M}$ be a $\mathcal{D}$-module on M^I and denote $\{\mathcal{M}^I\} := \Delta_*\mathcal{M}$ the corresponding $\mathcal{D}$-module on the Ran space. Denote $Z_{I,J} \subset M^{I \cup J}$ the (singular) subvariety given by the union of all diagonals corresponding to diagrams

$$I \twoheadrightarrow K \twoheadleftarrow J$$

with K an arbitrary finite set. Then we have (inverse and direct images are defined through the de Rham space construction for $Z_{I,J}$)

$$\mathcal{M}^J \cong (p_J)_*(p_I)^!\mathcal{M}.$$

In the case where $\mathcal{M}$ lives on M, there is a unique diagram

$$I = \{1\} \twoheadrightarrow K = \{1\} \twoheadleftarrow J : \pi$$

that corresponds to the natural projection $J \to \{1\}$, $Z_{I,J} \subset M \times M^J$ is the main diagonal and

$$\mathcal{M}^J \cong \Delta(\pi)_*\mathcal{M}$$

is the $\mathcal{D}$-module supported on the main diagonal with value $\mathcal{M}$.

There are two natural monoidal structures on $\mathcal{D}(\mathrm{Ran}(M))$. The local tensor product is given by

$$\mathcal{M} \otimes^* \mathcal{N} := \mathrm{union}_*(\mathcal{M} \boxtimes \mathcal{N}),$$

where

$$\mathrm{union} : \mathrm{Ran}(M) \times \mathrm{Ran}(M) \to \mathrm{Ran}(M)$$

is given by union of finite sets. The chiral tensor product is given by

$$\mathcal{M} \otimes^{ch} \mathcal{N} := \mathrm{union}_* \circ j_* \circ j^*(\mathcal{M} \boxtimes \mathcal{N}),$$

where j is the open embedding of the locus

$$(\mathrm{Ran}(M) \times \mathrm{Ran}(M))_{disj} \subset \mathrm{Ran}(M) \times \mathrm{Ran}(M)$$

corresponding to pairs of finite subsets of M that are disjoint.

We now describe the relation between the two descriptions of local operations given in the litterature (see [BD04] and [FG11]).

Proposition 11.3.1. *Let $\{\mathcal{M}_1, \dots, \mathcal{M}_n, \mathcal{N}\}$ be $\mathcal{D}$-modules on M, seen as $\mathcal{D}$-modules on the Ran space supported on the main diagonal. There is a natural isomorphism*

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}(\mathrm{Ran}(M))}(\mathcal{M}_1 \otimes^* \cdots \otimes^* \mathcal{M}_n, \mathcal{N}) \cong \mathbb{R}\mathcal{H}om_{\mathcal{D}_{M^n}}(\mathcal{M}_1 \boxtimes \cdots \boxtimes \mathcal{M}_n, \Delta_* \mathcal{N}).$$

Another way to explain the interest of $*$-operations, is that they induce ordinary operations in de Rham cohomology. Since de Rham cohomology is the main tool of local functional calculus (it gives an algebraic presentation of local functions, differential forms and vector fields on the space $\underline{\Gamma}(M, C)$ of trajectories of a given field theory), we will make a systematic use of these operations. We refer to Beilinson-Drinfeld [BD04] and [FG11] for a proof of the following result, that roughtly says that the de Rham functor is a monoidal functor. This means, in simpler terms, that local, i.e., $*$-operations can be used to define usual operations on quasi-local functionals.

Proposition 11.3.2. *The de Rham functor is an ∞-monoidal functor*

$$\mathrm{DR} : (\mathcal{D}(\mathrm{Ran}(M)), \otimes^*) \to (\mathrm{Sh}(\mathrm{Ran}(M)), \otimes^!).$$

In particular, the central de Rham cohomology functor $h : \mathrm{Mod}(\mathcal{D}_M^{op}) \to \mathrm{Mod}(\mathbb{R}_M)$ given by $h(\mathcal{M}) := h^0(\mathrm{DR}^r(\mathcal{M})) := \mathcal{M} \otimes_{\mathcal{D}} \mathcal{O}$ induces a natural map

$$h : \mathrm{Hom}_{\mathcal{D}(\mathrm{Ran}(M))}(\otimes^*_{i \in I} \mathcal{L}_i, \mathcal{M}) \to \mathrm{Hom}(\otimes_i h(\mathcal{L}_i), h(\mathcal{M}))$$

from $*$-operations to multilinear operations.

11.4 Local vector fields and their local bracket

We now explain, on the concrete example of local functional calculus, the usefulness of the general theory of local operations to do differential calculus on $\mathcal{D}$-spaces. The following construction is based on the notion of local dual for a module $\mathcal{M}$ over a $\mathcal{D}$-algebra, which is defined as the dual for the monoidal structure $\otimes^*$, i.e., by the isomorphism of functors

$$\mathrm{Hom}_{\mathcal{A}}(-, (\mathcal{M}^\circ)^\ell) \cong \mathrm{Hom}_{\mathcal{D}(\mathrm{Ran}(M))}(- \otimes^* \mathcal{M}, \mathcal{A}).$$

Concretely, if $\mathcal{M}$ is a projective $\mathcal{A}[\mathcal{D}]$-module of finite rank, one has

$$\mathcal{M}^\circ \cong \mathcal{H}om_{\mathcal{A}[\mathcal{D}]}(\mathcal{M}, \mathcal{A}[\mathcal{D}]).$$

We now define the notion of local vector fields.

Definition 11.4.1. Let $\mathcal{A}$ be a smooth $\mathcal{D}$-algebra. The $\mathcal{A}^r[\mathcal{D}^{op}]$-module of *local vector fields* is defined by

$$\Theta_{\mathcal{A}} = (\Omega^1_{\mathcal{A}})^\circ := \mathcal{H}om_{\mathcal{A}[\mathcal{D}]}(\Omega^1_{\mathcal{A}}, \mathcal{A}[\mathcal{D}]),$$

where $\mathcal{A}^r[\mathcal{D}^{op}] := \mathcal{A}^r \otimes_{\mathrm{Ber}_M} \mathcal{D}^{op}$ acts on the right though the isomorphism

$$\mathcal{A}^r[\mathcal{D}^{op}] \cong (\mathcal{A} \otimes_{\mathcal{O}} \mathrm{Ber}_M) \otimes_{\mathrm{Ber}_M} \mathcal{D}^{op} \cong \mathcal{A}[\mathcal{D}^{op}].$$

Remark now that in ordinary differential geometry, one way to define vector fields on a manifold M is to take the $\mathcal{O}_M$-dual

$$\Theta_M := \mathcal{H}om_{\mathcal{O}_M}(\Omega^1_M, \mathcal{O}_M)$$

of the module of differential forms. The Lie bracket

$$[.,.] : \Theta_M \otimes \Theta_M \to \Theta_M$$

of two vector fields X and Y can then be defined from the universal derivation $d : \mathcal{O}_M \to \Omega^1_M$, as the only vector field $[X, Y]$ on M such that for every function $f \in \mathcal{O}_M$, one has the equality of derivations

$$[X, Y].f = X.i_Y(df) - Y.i_X(df).$$

In the case of a $\mathcal{D}$-algebra $\mathcal{A}$, this construction does not work directly because the duality used to define local vector fields is not the $\mathcal{A}$-linear duality (because it doesn't have good finiteness properties) but the $\mathcal{A}[\mathcal{D}]$-linear duality. This explains why the Lie bracket of local vector fields and their action on $\mathcal{A}$ are new kinds of operations of the form

$$[.,.] : \Theta_{\mathcal{A}} \boxtimes \Theta_{\mathcal{A}} \to \Delta_*\Theta_{\mathcal{A}}$$

and

$$L : \Theta_{\mathcal{A}} \boxtimes \mathcal{A} \to \Delta_*\mathcal{A}$$

where $\Delta : M \to M \times M$ is the diagonal map and the box product is defined by

$$\mathcal{M} \boxtimes \mathcal{N} := p_1^*\mathcal{M} \otimes p_2^*\mathcal{N}$$

for $p_1, p_2 : M \times M \to M$ the two projections. One way to understand these construction is by looking at the natural injection

$$\Theta_{\mathcal{A}} \hookrightarrow \mathcal{H}om_{\mathcal{D}}(\mathcal{A}, \mathcal{A}[\mathcal{D}])$$

given by sending $X : \Omega^1_{\mathcal{A}} \to \mathcal{A}[\mathcal{D}]$ to $X \circ d : \mathcal{A} \to \mathcal{A}[\mathcal{D}]$. The theory of $\mathcal{D}$-modules tells us that the datum of this map is equivalent to the datum of a $\mathcal{D}^{op}_{M\times M}$-linear map

$$L : \Theta_{\mathcal{A}} \boxtimes \mathcal{A}^r \to \Delta_*\mathcal{A}^r.$$

Similarly, the above formula

$$[X, Y].f = X.i_Y(df) - Y.i_X(df)$$

of ordinary differential geometry makes sense in local computations only if we see $\Theta_{\mathcal{A}}$ as contained in $\mathcal{H}om_{\mathcal{D}}(\mathcal{A}, \mathcal{A}[\mathcal{D}])$ (and not in $\mathcal{H}om_{\mathcal{D}}(\mathcal{A}, \mathcal{A})$, contrary to what is usually done), so that we must think of the bracket as a morphism of sheaves

$$\Theta_{\mathcal{A}} \to \mathcal{H}om_{\mathcal{D}^{op}}(\Theta_{\mathcal{A}}, \Theta_{\mathcal{A}} \otimes \mathcal{D}^{op})^r.$$

This is better formalized by a morphism of $\mathcal{D}^{op}_{M\times M}$-modules

$$[.,.] : \Theta_{\mathcal{A}} \boxtimes \Theta_{\mathcal{A}} \to \Delta_*\Theta_{\mathcal{A}}$$

as above. Another way to understand these local operations is to make an analogy with multilinear operations on $\mathcal{O}_M$-modules. Indeed, if $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are three quasi-coherent $\mathcal{O}_M$-modules, one has a natural adjunction isomorphism

$$\mathcal{H}om_{\mathcal{O}_M}(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \cong \mathcal{H}om_{\mathcal{O}_M}(\Delta^*(\mathcal{F} \boxtimes \mathcal{G}), \mathcal{H}) \cong \mathcal{H}om_{\mathcal{O}_{M\times M}}(\mathcal{F} \boxtimes \mathcal{G}, \Delta_*\mathcal{H}),$$

and local operations are given by a $\mathcal{D}$-linear version of the right part of the above equality. It is better to work with this expression because of finiteness properties of the $\mathcal{D}$-modules in play.

11.5 Non-algebraic partial differential equations

We now use the general approach to differential calculus, given in Section 1.5, to describe a formulation of the jet space formalism that also applies to nonlinear partial differential equations on general types of varieties (e.g., smooth, analytic, super or graded manifolds). For example, this formalism is adapted to the study of a partial differential equation of the form

$$F(x, \varphi(x), \partial_\alpha \varphi(x)) = 0$$

for $F \in \mathcal{C}^\infty(\mathrm{Jet}(C))$ a smooth function on the jet space of a bundle $C \to M$ over a smooth manifold M. The formalism of $\mathcal{D}$-algebras described before is only adapted to the study of differential equations of this form, where F is a function that is polynomial in the higher jet coordinates, but the generalization is not that hard in our viewpoint, so that we present it shortly.

Recall from Section 1.5 that, if M is a space (meaning a sheaf on a differentially convenient site, see Definition 2.1.3), one may define its de Rham space $\pi : M \to M_{DR}$ as the quotient space

$$M_{DR} := \pi_0(\Pi^{inf}_\infty(M))$$

of M by the (free) action of the ∞-groupoid $\Pi^{inf}_\infty(M)$ of its infinitesimal paths.

The datum of an action of $\Pi^{inf}_\infty(M)$ on a space $Z \to M$ is the same as the datum of a space $Z_{DR} \to M_{DR}$ such that

$$Z \cong \pi^* Z_{DR} := Z_{DR} \times_{M_{DR}} M \to M,$$

which is the same as a flat Grothendieck connection on Z relatively to the morphism $Z \to M$.

In particular, a $\mathcal{D}$-module on a space M may be defined as a module over M_{DR}, which is the same as a space thickening

$$M_{DR} \to Z$$

that is equipped with an abelian cogroup object structure.

There is an adjunction

$$\pi^* : \text{SPACES}/M_{DR} \leftrightarrows \text{SPACES}/M : \pi_* =: \text{Jet}_{DR}$$

between spaces over M_{DR} and spaces over M, and we denote

$$\text{Jet} := \pi^* \circ \text{Jet}_{DR} : \text{SPACES}/M \to \text{SPACES}/M.$$

By definition, the jet space $\text{Jet}(C) \to M$ of a space $C \to M$ is equipped with a flat connection, given by the corresponding morphism $\text{Jet}_{DR}(C) \to M_{DR}$.

There is a natural inclusion

$$\underline{\Gamma}(M_{DR}, \text{Jet}_{DR}(C)) \hookrightarrow \underline{\Gamma}(M, \text{Jet}(C))$$

that identifies horizontal sections of the jet bundle, and an isomorphism

$$j_\infty : \underline{\Gamma}(M, C) \xrightarrow{\sim} \underline{\Gamma}(M_{DR}, \text{Jet}(C)_{DR})$$

that sends a section to its infinite jet, that is horizontal for the canonical connection on $\text{Jet}(C)$.

These considerations also work when C is a derived stack, like BG for G a Lie group.

Definition 11.5.1. The Ran space is the space

$$\text{Ran}(M) := \text{colim}_{\,I \twoheadrightarrow J} M^I$$

given by the colimit along diagonals $\Delta(\pi) : M^J \hookrightarrow M^I$ associated to surjections $\pi : I \twoheadrightarrow J$ of finite sets.

One may define a $\mathcal{D}$-module on the Ran space as an object $\mathcal{M}$ in the tangent category

$$\text{MOD}(\mathcal{D}_{\text{Ran}(M)}) := T(\text{Ran}(M)_{DR})$$

to the de Rham space of the Ran space. Concretely, it is given by a family of $\mathcal{D}$-modules on powers M^I of M that are strictly compatible with pullbacks along the various diagonal embeddings. One defines similarly a $\mathcal{D}$-space over the Ran space as a morphism

$$Z \to \text{Ran}(M)_{DR},$$

that is identified with a family of $\mathcal{D}$-spaces $Z^I \to M^I_{DR}$, compatible with pullbacks along diagonals.

Remark that the quantization problem for higher dimensional gauge theories (to be describe in Chapter 23), does not work well in this non-homotopical setting, essentially because quantizations are not strict geometric objects, but homotopical in nature (i.e., strict quantizations do not exist in higher dimension in general, as one can see in the example of quantum groups, described in Chapter 5; this may be intuitively explained by a Heckman-Hilton type argument that says that two commuting monoid structures on a set are equal and commutative). Remark also that the Batalin-Vilkovisky formalism for homotopical Poisson reduction of gauge theories (see Chapter 12) naturally gives a derived Poisson $\mathcal{D}$-space as output, so that it is very natural, also from the physicist's viewpoint, to quantize in the derived setting.

These remarks imply that it is profitable to see the Ran space as an ∞-colimit

$$\mathbb{R}\mathrm{Ran}(M) := \mathrm{colim}_{I \twoheadrightarrow J} M^I$$

of derived stacks, and to work with the stable (tangent) ∞-category

$$\mathcal{D}(\mathrm{Ran}(M)) := \mathrm{M{\scriptstyle OD}}(\mathbb{R}\mathrm{Ran}(M)_{DR})$$

of differential modules over it.

This category identifies canonically with the ∞-limit

$$\mathcal{D}(\mathrm{Ran}(M)) \cong \lim_{I \twoheadrightarrow J} \mathcal{D}(M^I).$$

We will simply denote $\mathrm{Ran}(M)$ the derived Ran space $\mathbb{R}\mathrm{Ran}(M)$, since both spaces are isomorphic as derived spaces.

In this setting, one also defines the $\otimes^*$ and $\otimes^{ch}$ monoidal structures on $\mathcal{D}(\mathrm{Ran}(M))$ as before. It is sometimes convenient to avoid working with algebras, and to use directly spaces over $\mathrm{Ran}(M)_{DR}$, to work with general partial differential equations and the corresponding $\mathcal{D}$-spaces. This geometric approach will be used in Chapter 23 to describe the general quantization problem for $\mathcal{D}$-spaces.

Chapter 12

Gauge theories and their homotopical Poisson reduction

In this chapter, we define general gauge theories and study their classical aspects. These may also be called *local variational problems*, because they fit with the general Lagrangian formalism of Section 7.1, but have the additional property of having as action functional

$$S : H \to R$$

a local functional (in the sense of Definition 11.2.3). The corresponding equations of motion are given by an Euler-Lagrange partial differential equation, whose study will be done in the setting of non-linear algebraic analysis, presented in Chapter 11.

This chapter is based and expands (by adding the gauge fixing procedure) on the author's precise and coordinate-free mathematical formalization in [Pau11b] of the theory of homotopical Poisson reduction of gauge theories. The main interest of our approach on the previous ones is that it is fully expressed in terms of functors of points. This allows us to always clearly state the nature of the various spaces and functions in play in the gauge fixing (and later in the functional integral) procedure. We also always give the most general finiteness hypothesis in which the formalism can be applied, which differs from the usual approach of the physical litterature, based on explicit examples, but also very often, on finite dimensional toy models.

We are inspired, when discussing the Batalin-Vilkovisky formalism, by a huge physical literature, starting with Peierls [Pei52] and DeWitt [DeW03] for the covariant approach to quantum field theory, and with [HT92] and [FH90] as general references for the BV formalism. More specifically, we also use Stasheff's work [Sta97] and [Sta98] as homotopical inspiration, and [FLS02], [Bar10] and [CF01] for explicit computations. We refer to the author's work loc. cit. for a more complete list of references on this subject.

12.1 A finite dimensional toy model

In this section, we will do some new kind of differential geometry on spaces of the form $X = \underline{\mathrm{Spec}}_{\mathcal{D}}(\mathcal{A})$ given by spectra of $\mathcal{D}$-algebras, that encode solution spaces of non-linear partial differential equations in a coordinate free fashion (to be explained in the next section).

Before starting this general description, that is entailed of technicalities, we present a finite dimensional analog, that can be used as a reference to better understand the constructions done in the setting of $\mathcal{D}$-spaces.

Let H be a finite dimensional smooth manifold (analogous to the space of histories $H \subset \underline{\Gamma}(M, C)$ of a given Lagrangian variational problem) and $S : H \to \mathbb{R}$ be a smooth function (analogous to an action functional on the space of histories). Let $d : \mathcal{O}_H \to \Omega^1_H$ be the de Rham differential and

$$\Theta_H := \mathcal{H}om_{\mathcal{O}_H}(\Omega^1_H, \mathcal{O}_H).$$

There is a natural biduality isomorphism

$$\Omega^1_H \cong \mathcal{H}om_{\mathcal{O}_H}(\Omega^1_H, \mathcal{O}_H).$$

Let $i_{dS} : \Theta_H \to \mathcal{O}_H$ be given by the insertion of vector fields in the differential $dS \in \Omega^1_H$ of the given function $S : H \to \mathbb{R}$.

The claim is that there is a natural homotopical Poisson structure on the space T of critical points of $S : H \to \mathbb{R}$, defined by

$$T = \{x \in H,\ d_x S = 0\}.$$

Define the algebra of functions $\mathcal{O}_T$ as the quotient of $\mathcal{O}_H$ by the ideal $\mathcal{I}_S$ generated by the equations $i_{dS}(\vec{v}) = 0$ for all $\vec{v} \in \Theta_H$. Remark that $\mathcal{I}_S$ is the image of the insertion map $i_{dS} : \Theta_H \to \mathcal{O}_H$ and is thus locally finitely generated by the image of the basis vector fields $\vec{x}_i := \frac{\partial}{\partial x_i}$ that correspond to the local coordinates x_i on H. Now let $\mathcal{N}_S$ be the kernel of i_{dS}. It describes the relations between the generating equations $i_{dS}(\vec{x}_i) = \frac{\partial S}{\partial x_i}$ of $\mathcal{I}_S$.

The differential graded $\mathcal{O}_H$-algebra

$$\mathcal{O}_P := \mathrm{Sym}_{dg}([\underset{-1}{\Theta_H} \xrightarrow{i_{dS}} \underset{0}{\mathcal{O}_H}])$$

is isomorphic, as a graded algebra, to the algebra of multi-vectors

$$\wedge^*_{\mathcal{O}_H} \Theta_H.$$

This graded algebra is equipped with an odd, so called Schouten bracket, given by extending the Lie derivative

$$L : \Theta_H \otimes \mathcal{O}_H \to \mathcal{O}_H$$

and Lie bracket

$$[.,.] : \Theta_H \otimes \Theta_H \to \Theta_H$$

by Leibniz's rule.

Proposition 12.1.1. *The Schouten bracket is compatible with the insertion map i_{dS} and makes $\mathcal{O}_P$ a differential graded odd Poisson algebra. The Lie bracket on Θ_H induces a Lie bracket on $\mathcal{N}_S$.*

Now, we will compute, following Tate [Tat57], a cofibrant resolution

$$\mathcal{B} \xrightarrow{\sim} \mathcal{O}_H/\mathcal{I}_S =: \mathcal{O}_T$$

of the algebra of functions on the critical space, as an $\mathcal{O}_H$-algebra. This proceeds by adding inductively to the algebra $\mathcal{O}_P$ higher degree generators to annihilate its cohomology. More precisely, if $H^1(\mathcal{O}_P, i_{dS}) \neq 0$, we choose a submodule

$$\mathfrak{g}_S^0 \subset \mathrm{Ker}(i_{dS}) \subset \Theta_H$$

such that

$$H^1\left(\mathrm{Sym}_{dg-\mathcal{O}_H}([\underset{-2}{\mathfrak{g}_S^0} \to \underset{-1}{\Theta_H} \to \underset{0}{\mathcal{O}_H}])\right) = 0.$$

And we apply the same procedure by adding a chosen $\mathfrak{g}_S^1$ to kill the H^2 of the algebra

$$\mathrm{Sym}_{dg-\mathcal{O}_H}([\underset{-2}{\mathfrak{g}_S^0} \to \underset{-1}{\Theta_H} \to \underset{0}{\mathcal{O}_H}]),$$

and so on... The graded module $\mathfrak{g}_S$ is the finite dimensional analog of the space of gauge (and higher gauge) symmetries. Suppose that $\mathfrak{g}_S$ is bounded with projective components of finite rank.

Definition 12.1.2. The *finite dimensional BV algebra* associated to $S : H \to \mathbb{R}$ is the bigraded $\mathcal{O}_H$-algebra

$$\mathcal{O}_{BV} := \mathrm{Sym}_{bigrad}\left(\left[\begin{array}{ccccc} \mathfrak{g}_S[2] & \oplus & \Theta_H[1] & \oplus & \mathcal{O}_H \\ & & & & \oplus \\ & & & & {}^t\mathfrak{g}_S^*[-1] \end{array}\right]\right),$$

where ${}^t\mathfrak{g}_S^*$ is the $\mathcal{O}_H$-dual of the graded module $\mathfrak{g}_S$ transposed to become a vertical ascending graded module.

The main theorem of the Batalin-Vilkovisky formalism, that is the aim of this section, is the following:

Theorem 12.1.3. *There exists a non-trivial extension*

$$S_{cm} = S_0 + \sum_{i \geq 1} S_i \in \mathcal{O}_{BV}$$

of the classical function S that fulfills the classical master equation

$$\{S_{cm}, S_{cm}\} = 0.$$

The differential $D = \{S_{cm}, .\}$ gives $\mathcal{O}_{BV}$ the structure of a differential graded odd Poisson algebra.

The corresponding derived space $\mathbb{R}\underline{\mathrm{Spec}}(\mathcal{O}_{BV}, D)$ can be thought as a kind of derived Poisson reduction

$$\mathbb{R}\underline{\mathrm{Spec}}(\mathcal{O}_{BV}, D) \cong \mathbb{R}\underline{\mathrm{Spec}}(\mathcal{O}_H/\mathcal{I}_H)\underset{\mathbb{R}}{/}\mathcal{N}_S$$

that corresponds to taking the quotient of the homotopical critical space of S (cofibrant replacement of $\mathcal{O}_T = \mathcal{O}_H/\mathcal{I}_H)$) by the foliation induced by the Noether relations $\mathcal{N}_S$.

Example 12.1.4. To be even more explicit, let us treat a simple example of the above construction. Let $H = \mathbb{R}^2$ be equipped with the polynomial function algebra $\mathcal{O}_H = \mathbb{R}[x, y]$. The differential one forms on H are given by the free $\mathcal{O}_H$-modules

$$\Omega^1_H = \mathbb{R}[x, y]^{(dx,dy)}.$$

Let $S \in \mathcal{O}_H$ be the function $F(x, y) = \frac{x^2}{2}$. One then has $dS = xdx$. The module Θ_H of vector fields is the free module

$$\Theta_H = \mathbb{R}[x, y]^{\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)}$$

and the insertion map is given by the $\mathbb{R}[x, y]$-module morphism

$$\begin{array}{cccc} i_{dS}: & \Theta_H & \to & \mathcal{O}_H \\ & \vec{v} & \mapsto & \langle dS, \vec{v}\rangle, \end{array}$$

and, in particular, $i_{dS}\left(\frac{\partial}{\partial x}\right) = x$, and $i_{dS}\left(\frac{\partial}{\partial y}\right) = 0$. The image of the insertion map is given by the ideal $\mathcal{I}_S = (x)$ in $\mathcal{O}_H = \mathbb{R}[x, y]$. The kernel $\mathcal{N}_S$ of the insertion map is the free submodule

$$\mathcal{N}_S = \mathfrak{g}_S = \mathbb{R}[x, y]^{\left(\frac{\partial}{\partial y}\right)}$$

of Θ_H. We apply the above inductive construction of a Koszul-Tate resolution to get a graded module $\mathfrak{g}_S$. The corresponding cofibrant resolution is given by a quasi-isomorphism

$$\mathcal{B}_S \xrightarrow{\sim} \mathcal{O}_H/\mathcal{I}_S.$$

The obtained algebra $\mathcal{B}$ is called the Koszul-Tate resolution of $\mathcal{O}_H/\mathcal{I}_S$. Now the bigraded BV algebra $\mathcal{O}_{BV}$ is given by

$$\mathcal{O}_{BV} := \mathrm{Sym}_{bigrad}\left(\left[\begin{array}{ccccc} \mathfrak{g}_S[2] & \oplus & \Theta_H[1] & \oplus & \mathcal{O}_H \\ & & & & \oplus \\ & & & & {}^t\mathfrak{g}_S^*[-1] \end{array}\right]\right).$$

The graded version is the $\mathcal{O}_H$-algebra

$$\mathcal{O}_{BV} = \mathrm{Sym}^*(\mathfrak{g}_S) \otimes \wedge^*\Theta_H \otimes \wedge^*\mathfrak{g}_S^*$$

on $4 = 1 + 2 + 1$ variables. The BV formalism gives a way to combine the Koszul-Tate differential with the Chevalley-Eilenberg differential by constructing an $S_{cm} \in \mathcal{O}_{BV}$ such that some components of the bracket $\{S_{cm}, .\}$ induce both differentials on the corresponding generators of the BV algebra.

The aim of this section is to generalize the above construction to local variational problems, where $H \subset \underline{\Gamma}(M, C)$ is a space of histories (subspace of the space of sections of a bundle $\pi : C \to M$) and $S : H \to \mathbb{R}$ is given by the integration of a Lagrangian density. The main difficulties that we will encounter and overcome trickily in this generalization are that:

1. The $\mathcal{D}$-module $\mathcal{D} \otimes_{\mathcal{O}} \mathcal{D}$ is not $\mathcal{D}$-coherent, so that a projective resolution of $\mathcal{N}_S$ will not be dualizable in practical cases. This will impose us to use finer resolutions of the algebra $\mathcal{O}_T$ of functions on the critical space.

2. Taking the bracket between two *densities* of vector fields on the space H of histories is not an $\mathcal{O}_M$-bilinear operation but a new kind of operation, called a locally bilinear operation and described in Section 11.4.

3. The ring $\mathcal{D}$ is not commutative. This will be overcome by using the equivalence between $\mathcal{D}$-modules and $\mathcal{D}^{op}$-modules given by tensoring by Ber_M.

Out of the above technical points, the rest of the constructions of this section are completely parallel to what we did on the finite dimensional toy model.

12.2 General gauge theories

Proposition 12.2.1. *Let M be a supermanifold and $\mathcal{A}$ be a smooth $\mathcal{D}$-algebra. There is a natural isomorphism, called the local interior product*

$$\begin{array}{rccc} i: & h(\Omega^1_{\mathcal{A}}) := \mathrm{Ber}_M \otimes_{\mathcal{D}} \Omega^1_{\mathcal{A}} & \longrightarrow & \mathcal{H}om_{\mathcal{A}[\mathcal{D}]}(\Theta^\ell_{\mathcal{A}}, \mathcal{A}) \\ & \omega & \longmapsto & [X \mapsto i_X \omega]. \end{array}$$

Proof. By definition, one has

$$\Theta_{\mathcal{A}} := \mathcal{H}om_{\mathcal{A}[\mathcal{D}]}(\Omega^1_{\mathcal{A}}, \mathcal{A}[\mathcal{D}])$$

and since $\mathcal{A}$ is $\mathcal{D}$-smooth, the biduality map

$$\Omega^1_{\mathcal{A}} \to \mathcal{H}om_{\mathcal{A}^r[\mathcal{D}^{op}]}(\Theta_{\mathcal{A}}, \mathcal{A}^r[\mathcal{D}^{op}])$$

is an isomorphism. Tensoring this map with Ber_M over $\mathcal{D}$ gives the desired result. □

Definition 12.2.2. If $\mathcal{A}$ is a smooth $\mathcal{D}$-algebra and $\omega \in h(\Omega^1_{\mathcal{A}})$, the $\mathcal{A}[\mathcal{D}]$ linear map

$$i_\omega : \Theta^\ell_{\mathcal{A}} \to \mathcal{A}$$

is called the *insertion map*. Its kernel $\mathcal{N}_\omega$ is called the $\mathcal{A}[\mathcal{D}]$-module of *Noether identities* and its image $\mathcal{I}_\omega$ is called the *Euler-Lagrange ideal*.

If $\mathcal{A} = \mathrm{Jet}(\mathcal{O}_C)$ for $\pi : C \to M$ a bundle and $\omega = dS$, the Euler-Lagrange ideal $\mathcal{I}_{dS}$ is locally generated as an $\mathcal{A}[\mathcal{D}]$-module by the image of the local basis of vector fields in

$$\Theta^\ell_{\mathcal{A}} \cong \mathcal{A}[\mathcal{D}] \otimes_{\mathcal{O}_C} \Theta_{C/M}.$$

If M is of dimension n and the relative dimension of C over M is m, this gives n equations (indexed by $i = 1, \ldots, n$, one for each generator of $\Theta_{C/M}$) given in local coordinates by

$$\sum_\alpha (-1)^{|\alpha|} D_\alpha \left(\frac{\partial L}{\partial x_{i,\alpha}} \right) \circ (j_\infty x)(t) = 0,$$

where $S = [Ld^n t] \in h(\mathcal{A})$ is the local description of the Lagrangian density.

We now define the notion of local variational problem with nice histories. This type of variational problem can be studied completely by only using geometry of $\mathcal{D}$-spaces. This gives powerful finiteness and biduality results that are necessary to study conceptually general gauge theories.

Definition 12.2.3. Let $\pi : C \to M$, $H \subset \underline{\Gamma}(M, C)$ and $S : H \to \mathbb{R}$ be a Lagrangian variational problem (in the sense of Definition 7.1.1), and suppose that S is a local functional, i.e., if $\mathcal{A} = \mathrm{Jet}(\mathcal{O}_C)$, there exists $[L\omega] \in h(\mathcal{A}) := \mathrm{Ber}_M \otimes_{\mathcal{D}} \mathcal{A}$ and $\Sigma \in H_{c,n}(M)$ such that $S = S_{\Sigma, L\omega}$. The variational problem is called a *local variational problem with nice histories* if the space of critical points $T = \{x \in H,\ d_x S = 0\}$ identifies with the space $\underline{\mathrm{Sol}}(\mathcal{A}/\mathcal{I}_{dS})$ of solutions to the Euler-Lagrange equation.

The notion of variational problem with nice histories can be explained in simple terms by looking at the following simple example. The point is to define H by adding boundary conditions to elements in $\underline{\Gamma}(M, C)$, so that the boundary terms of the integration by part, that we do to compute the variation $d_x S$ of the action, vanish.

Remark that one may refine the above notions, improving on the use of histories to kill boundary terms, by the use of manifolds with corners, basing on the formalism described in Example 2.2.8 and the general differential calculus formalism of Section 1.5.

Example 12.2.4. Let $\pi : C = \mathbb{R}^3 \times [0,1] \to [0,1] = M$, $\mathcal{A} = \mathbb{R}[t, x_0, x_1, \ldots]$ be the corresponding $\mathcal{D}_M$-algebra with action of ∂_t given by $\partial_t x_i = x_{i+1}$, and $S = \frac{1}{2} m (x_1)^2 dt \in h(\mathcal{A})$ be the local action functional for the variational problem of Newtonian mechanics for a free particle in $\mathbb{R}^3$. The differential of $S : \underline{\Gamma}(M, C) \to \underline{\mathbb{R}}$ at $u : U \to \underline{\Gamma}(M, C)$ along the vector field $\vec{u} \in \Theta_U$ is given by integrating by part

$$\langle d_x S, \vec{u} \rangle = \int_M \langle -m \partial_t^2 x, \frac{\partial x}{\partial \vec{u}} \rangle dt + \left[\langle \partial_t x, \frac{\partial x}{\partial \vec{u}} \rangle \right]_0^1.$$

The last term of this expression is called the boundary term and we define nice histories for this variational problem by fixing the starting and ending point of trajectories to annihilate this boundary term:

$$H = \{x \in \underline{\Gamma}(M, C),\ x(0) = x_0,\ x(1) = x_1\}$$

for x_0 and x_1 some given points in $\mathbb{R}^3$. In this case, one has

$$T = \{x \in H,\ d_x S = 0\} \cong \underline{\mathrm{Sol}}(\mathcal{A}/\mathcal{I}_{dS})$$

where $\mathcal{I}_{dS}$ is the $\mathcal{D}$-ideal in $\mathcal{A}$ generated by $-mx_2$, i.e., by Newton's differential equation for the motion of a free particle in $\mathbb{R}^3$. The critical space is thus given by

$$T = \{x \in H,\ \partial_t x \text{ is constant on } [0,1]\},$$

i.e., the free particle is moving on the line from x_0 to x_1 with constant speed.

Definition 12.2.5. A *general gauge theory* is a local variational problem with nice histories.

12.3 Regularity conditions and higher Noether identities

We now describe regularity properties of gauge theories, basing our exposition on the article [Pau11a]. We will moreover use the language of homotopical and derived geometry in the sense of Toen-Vezzosi [TV08] to get geometric insights on the spaces in play in this section (see Chapter 9 for an overview). We denote $\mathcal{A} \mapsto Q\mathcal{A}$ a cofibrant replacement functor in a given model category. Recall that all differential graded algebras are fibrant for their standard model structure.

In all this section, we set $\pi : C \to M$, $H \subset \underline{\Gamma}(M, C)$, $\mathcal{A} = \mathrm{Jet}(\mathcal{O}_C)$ and $S \in h(\mathcal{A})$ a gauge theory. The kernel of its insertion map

$$i_{dS} : \Theta^\ell_{\mathcal{A}} \to \mathcal{A}$$

is called the space $\mathcal{N}_S$ of Noether identities. Its right version

$$\mathcal{N}^r_S = \mathrm{Ber}_M \otimes \mathcal{N}_S \subset \Theta_{\mathcal{A}}$$

is called the space of Noether gauge symmetries.

Definition 12.3.1. The *derived critical space* of a gauge theory is the differential graded $\mathcal{A}$-space

$$\begin{array}{rccc} P := \mathrm{Spec}(\mathcal{A}_P) : & dg - \mathcal{A} - \mathrm{ALG} & \to & \mathrm{SSETS} \\ & \mathcal{R} & \mapsto & s\mathrm{Hom}_{dg-Alg_{\mathcal{D}}}(\mathcal{A}_P, \mathcal{R}). \end{array}$$

whose coordinate differential graded algebra is

$$\mathcal{A}_P := \mathrm{Sym}_{dg}([\Theta^\ell_{\mathcal{A}}[1] \xrightarrow{i_{dS}} \mathcal{A}]).$$

A *non-trivial Noether identity* is a class in $H^1(\mathcal{A}_P, i_{dS})$.

We refer to Beilinson and Drinfeld's book [BD04] for the following proposition.

Proposition 12.3.2. *The local Lie bracket of vector fields extends naturally to an odd local (so-called Schouten) Poisson bracket on the dg-$\mathcal{A}$-algebra $\mathcal{A}_P$ of coordinates on the derived critical space.*

The following corollary explains why we called $\mathcal{N}_S^r$ the space of Noether gauge symmetries.

Corollary 12.3.3. *The natural map*

$$\mathcal{N}_S^r \boxtimes \mathcal{N}_S^r \to \Delta_* \Theta_{\mathcal{A}}$$

induced by the local bracket on local vector fields always factors through $\Delta_*\mathcal{N}_S^r$ and the natural map

$$\mathcal{N}_S^r \boxtimes \mathcal{A}^r/\mathcal{I}_S^r \to \Delta_*\mathcal{A}^r/\mathcal{I}_S^r$$

is a local Lie $\mathcal{A}$-algebroid action.

The *trivial Noether identities* are those in the image of the natural map

$$\wedge^2 \Theta_{\mathcal{A}}^{\ell} \to \Theta_{\mathcal{A}},$$

and these usually don't give a finitely generated $\mathcal{A}[\mathcal{D}]$-module because of the simple fact that $\mathcal{D} \otimes_{\mathcal{O}} \mathcal{D}$ is not $\mathcal{D}$-coherent. This is a very good reason to consider only non-trivial Noether identities, because these can usually (i.e., in all the applications we have in mind) be given by a finitely generated $\mathcal{A}[\mathcal{D}]$-module.

Definition 12.3.4. The *proper derived critical space* of a gauge theory is the differential graded space

$$\begin{array}{rccc} \mathbb{R}\mathrm{Spec}(\mathcal{A}/\mathcal{I}_S): & dg-\mathcal{A}-\mathrm{ALG} & \to & \mathrm{SSETS} \\ & \mathcal{R} & \mapsto & s\mathrm{Hom}_{dg-Alg_{\mathcal{D}}}(\mathcal{B},\mathcal{R}). \end{array}$$

where $\mathcal{B} \xrightarrow{\sim} \mathcal{A}/\mathcal{I}_{dS}$ is a cofibrant resolution of $\mathcal{A}/\mathcal{I}_S$ as a dg-$\mathcal{A}$-algebra in degree 0.

From the point of view of derived geometry, differential forms on the cofibrant resolution $\mathcal{B}$ give a definition of the cotangent complex of the $\mathcal{D}$-space morphism

$$i : \underline{\mathrm{Spec}}_{\mathcal{D}}(\mathcal{A}/\mathcal{I}_S) \to \underline{\mathrm{Spec}}_{\mathcal{D}}(\mathcal{A})$$

of inclusion of critical points of the action functional in the $\mathcal{D}$-space of general trajectories. This notion of cotangent complex gives a well behaved way to study infinitesimal deformations of the above inclusion map i (see Illusie [Ill71]), even if it is not a smooth morphism (i.e., even if the critical space is singular).

We will see how to define a finer, so-called Koszul-Tate resolution, that will have better finiteness properties, by using generating spaces of Noether identities. These can be defined by adapting Tate's construction [Tat57] to the local context. We are inspired here by Stasheff's paper [Sta97].

Definition 12.3.5. A *generating space of Noether identities* is a tuple $(\mathfrak{g}_S, \mathcal{A}_n, i_n)$ composed of

1. a negatively graded projective $\mathcal{A}[\mathcal{D}]$-module $\mathfrak{g}_S$,
2. a negatively indexed family $\mathcal{A}_n$ of dg-$\mathcal{A}$-algebras with $\mathcal{A}_0 = \mathcal{A}$, and
3. for each $n \leq -1$, an $\mathcal{A}[\mathcal{D}]$-linear morphism $i_n : \mathfrak{g}_S^{n+1} \to Z^n\mathcal{A}_n$ to the n-cycles of $\mathcal{A}_n$,

such that if one extends $\mathfrak{g}_S$ by setting $\mathfrak{g}_S^1 = \Theta_{\mathcal{A}}^{\ell}$ and if one sets

$$i_0 = i_{dS} : \Theta_{\mathcal{A}}^{\ell} \to \mathcal{A},$$

1. one has for all $n \leq 0$ an equality
$$\mathcal{A}_{n-1} = \mathrm{Sym}_{\mathcal{A}_n}([\mathfrak{g}_S^{n+1}[-n+1] \otimes_{\mathcal{A}} \mathcal{A}_n \overset{i_n}{\to} \underset{0}{\mathcal{A}_n}]),$$
2. the natural projection map
$$\mathcal{A}_{KT} := \varinjlim \mathcal{A}_n \to \mathcal{A}/\mathcal{I}_S$$
is a cofibrant resolution, called the *Koszul-Tate algebra*, whose differential is denoted d_{KT}.

We are now able to define the right regularity properties for a given gauge theory. These finiteness properties are imposed to make the generating space of Noether identities dualizable as an $\mathcal{A}[\mathcal{D}]$-module (resp. as a graded $\mathcal{A}[\mathcal{D}]$-module). Without any regularity hypothesis, the constructions given by homotopical Poisson reduction of gauge theories, the so-called derived covariant phase space, don't give $\mathcal{A}$-algebras, but only $\mathbb{R}$-algebras, that are too poorly behaved and infinite dimensional to be of any (even theoretical) use. We thus don't go through the process of their definition, that is left to the interested reader.

We now recall the language used by physicists (see for example [HT92]) to describe the situation. This can be useful to relate our constructions to the one described in physics books.

Definition 12.3.6. A gauge theory is called *regular* if there exists a generating space of Noether identities $\mathfrak{g}_S$ whose components are finitely generated and projective. It is called *strongly regular* if this regular generating space is a bounded graded module. Suppose given a regular gauge theory. Consider the inner dual graded space (well-defined because of the regularity hypothesis)

$$\mathfrak{g}_S^{\circ} := \mathcal{H}om_{\mathcal{A}[\mathcal{D}]}(\mathfrak{g}_S, \mathcal{A}[\mathcal{D}])^{\ell}.$$

1. The generators of $\Theta_{\mathcal{A}}^{\ell}$ are called *antifields* of the theory.
2. The generators of $\mathfrak{g}_S$ of higher degree are called *antifields of ghosts*, or (non-trivial) *higher Noether identities* of the theory.

3. The generators of the graded $\mathcal{A}^r[\mathcal{D}^{op}]$-module $\mathfrak{g}_S^r$ are called (non-trivial) *higher gauge symmetries* of the theory.

4. The generators of the graded $\mathcal{A}[\mathcal{D}]$-module $\mathfrak{g}_S^\circ$ are called *ghosts* of the theory.

Remark that the natural map $\mathfrak{g}_S^{0,r} \to \mathcal{N}_S^r \subset \Theta_{\mathcal{A}}$ identifies order zero gauge symmetries with (densities of) local vector fields that induce tangent vectors to the $\mathcal{D}$-space $\underline{\mathrm{Spec}}_{\mathcal{D}}(\mathcal{A}/\mathcal{I}_S)$ of solutions to the Euler-Lagrange equation. This explains the denomination of higher gauge symmetries for $\mathfrak{g}_S^r$.

We now define an important invariant of gauge theories, called the Batalin-Vilkovisky bigraded algebra. This will be used in next section on the derived covariant phase space.

Definition 12.3.7. Let $\mathfrak{g}_S$ be a regular generating space of the Noether gauge symmetries. The bigraded $\mathcal{A}[\mathcal{D}]$-module

$$\mathcal{V}_{BV} := \begin{bmatrix} \mathfrak{g}_S[2] & \oplus & \Theta_{\mathcal{A}}^\ell[1] & \oplus & 0 \\ & & & & \oplus \\ & & & & {}^t\mathfrak{g}_S^\circ[-1] \end{bmatrix},$$

where ${}^t\mathfrak{g}_S^\circ$ is the vertical chain graded space associated to $\mathfrak{g}_S^\circ$, is called the module of *auxiliary fields*. The completed bigraded symmetric algebra

$$\hat{\mathcal{A}}_{BV} := \widehat{\mathrm{Sym}}_{\mathcal{A}\text{-bigraded}}(\mathcal{V}_{BV})$$

is called the *completed Batalin-Vilkovisky algebra* of the given gauge theory. The corresponding symmetric algebra

$$\mathcal{A}_{BV} := \mathrm{Sym}_{\mathcal{A}\text{-bigraded}}(\mathcal{V}_{BV})$$

is called the *Batalin-Vilkovisky algebra*.

In practical situations, physicists usually think of ghosts and antifields as sections of an ordinary graded bundle on spacetime itself (and not only on jet space). This idea can be formalized by the following.

Definition 12.3.8. Let $\mathfrak{g}$ be a regular generating space of Noether symmetries for $S \in h(\mathcal{A})$. Suppose that all the $\mathcal{A}[\mathcal{D}]$-modules $\mathfrak{g}^i$ and $\Theta_{\mathcal{A}}^\ell$ are locally free on M. A *Batalin-Vilkovisky bundle* is a bigraded vector bundle

$$E_{BV} \to C$$

with an isomorphism of $\mathcal{A}[\mathcal{D}]$-modules

$$\mathcal{A}[\mathcal{D}] \otimes_{\mathcal{O}_C} \mathcal{E}_{BV}^* \to \mathcal{V}_{BV},$$

where $\mathcal{E}_{BV}$ are the sections of $E_{BV} \to C$. The sections of the graded bundle $E_{BV} \to M$ are called the *fields-antifields variables* of the theory.

Recall that neither $C \to M$, nor $E_{BV} \to M$ are vector bundles in general. To illustrate the above general construction by a simple example, suppose that the action $S \in h(\mathcal{A})$ has no non-trivial Noether identities, meaning that for all $k \geq 1$, one has $H^k(\mathcal{A}_P) = 0$. In this case, one gets

$$\mathcal{V}_{BV} = \Theta_{\mathcal{A}}^{\ell}[1]$$

and the relative cotangent bundle $E_{BV} := T^*_{C/M} \to C$ gives a BV bundle because

$$\Theta_{\mathcal{A}}^{\ell} \cong \mathcal{A}[\mathcal{D}] \otimes_{\mathcal{O}_C} \Theta_{C/M}.$$

The situation simplifies further if $C \to M$ is a vector bundle because then, the vertical bundle $VC \subset TC \to M$, given by the kernel of $TC \to \pi^*TM$, is isomorphic to $C \to M$. Since one has $T^*_{C/M} \cong (VC)^*$, one gets a natural isomorphism

$$E_{BV} \cong C \oplus C^*$$

of bundles over M. This linear situation is usually used as a starting point for the definition of a BV theory (see for example Costello's book [Cos11]). Starting from a non-linear bundle $C \to M$, one can linearize the situation by working with the bundle

$$C_{x_0}^{linear} := x_0^* T_{C/M} \to M$$

with $x_0 : M \to C$ a given solution of the equations of motion (sometimes called the vacuum).

Proposition 12.3.9. *Let $E_{BV} \to C$ be a BV bundle. There is a natural isomorphism of bigraded algebras*

$$\mathrm{Jet}(\mathcal{O}_{E_{BV}}) \xrightarrow{\sim} \mathcal{A}_{BV} = \mathrm{Sym}_{bigrad}(\mathcal{V}_{BV}).$$

Proof. Since $E_{BV} \to C$ is a graded vector bundle concentrated in non-zero degrees, one has

$$\mathcal{O}_{E_{BV}} = \mathrm{Sym}_{\mathcal{O}_C}(\mathcal{E}^*_{BV}).$$

The natural map

$$\mathcal{E}^*_{BV} \to \mathcal{V}_{BV}$$

induces a morphism

$$\mathcal{O}_{E_{BV}} = \mathrm{Sym}_{\mathcal{O}_C}(\mathcal{E}^*_{BV}) \to \mathcal{A}_{BV}.$$

Since $\mathcal{A}_{BV}$ is a $\mathcal{D}$-algebra, one gets a natural morphism

$$\mathrm{Jet}(\mathcal{O}_{E_{BV}}) \to \mathcal{A}_{BV} = \mathrm{Sym}_{bigrad}(\mathcal{V}_{BV}).$$

Conversely, the natural map $\mathcal{E}^*_{BV} \to \mathcal{O}_{E_{BV}}$ extends to an $\mathcal{A}[\mathcal{D}]$-linear map

$$\mathcal{V}_{BV} \to \mathrm{Jet}(\mathcal{O}_{E_{BV}}),$$

that gives a morphism

$$\mathcal{A}_{BV} \to \mathrm{Jet}(\mathcal{O}_{E_{BV}}).$$

The two constructed maps are inverse of each other. □

The main interest of the datum of a BV bundle is that it allows to work with non-local functionals of the fields and antifields variables. This is important for the effective renormalization of gauge theories, that involves non-local functionals.

Definition 12.3.10. Let $E_{BV} \to C$ be a BV bundle. Denote $\mathbb{A}^1(A) := A$ the graded affine space. The space of *non-local functionals of the fields-antifields* is defined by

$$\mathcal{O}_{BV} := \underline{\mathrm{Hom}}(\underline{\Gamma}(M, E_{BV}), \mathbb{A}^1)$$

of (non-local) functionals on the space of sections of E_{BV}. The image of the natural map

$$h(\mathcal{A}_{BV}) \cong h(\mathrm{Jet}(\mathcal{O}_{E_{BV}})) \longrightarrow \mathcal{O}_{BV}$$

is called the space of *local functionals of the fields-antifields* and denoted $\mathcal{O}_{BV}^{loc} \subset \mathcal{O}_{BV}$. The image of the period map

$$h^*(\mathrm{DR}(\mathcal{A}_{BV})) \times H_{*,c}(M) \to \mathcal{O}_{BV}$$

is called the space of *quasi-local functionals* and denoted $\mathcal{O}_{BV}^{qloc} \subset \mathcal{O}_{BV}$.

12.4 The derived covariant phase space

In all this section, we set $\pi : C \to M$, $H \subset \underline{\Gamma}(M, C)$, $\mathcal{A} = \mathrm{Jet}(\mathcal{O}_C)$ and $S \in h(\mathcal{A})$ a gauge theory. Suppose given a strongly regular generating space of Noether symmetries $\mathfrak{g}_S$ for S, in the sense of definitions 12.3.5 and 12.3.6.

The idea of the BV formalism is to define a (local and odd) Poisson dg-$\mathcal{A}$-algebra $(\mathcal{A}_{BV}, D, \{.,.\})$ whose spectrum $\mathbb{R}\underline{\mathrm{Spec}}_{\mathcal{D}}(\mathcal{A}_{BV}, D)$ can be though as a kind of homotopical space of leaves

$$\mathbb{R}\mathrm{Spec}(\mathcal{A}/\mathcal{I}_S)\underset{\mathbb{R}}{/}\mathcal{N}_S^r$$

of the foliation induced by the action (described in corollary 12.3.3) of Noether gauge symmetries $\mathcal{N}_S^r$ on the derived critical space $\mathbb{R}\underline{\mathrm{Spec}}_{\mathcal{D}}(\mathcal{A}/\mathcal{I}_S)$. It is naturally equipped with a homotopical Poisson structure, which gives a nice starting point for quantization. From this point of view, the above space is a wide generalization of the notion extensively used by DeWitt in his covariant approach to quantum field theory [DeW03] called the covariant phase space. This explains the title of this section.

We will first define the BV Poisson dg-algebra by using only a generating space for Noether identities, and explain in more details in the next section how this relates to the above intuitive statement.

Proposition 12.4.1. *The local Lie bracket and local duality pairings*

$$[.,.] : \Theta_{\mathcal{A}} \boxtimes \Theta_{\mathcal{A}} \to \Delta_*\Theta_{\mathcal{A}} \qquad \text{and} \qquad \langle .,. \rangle : (\mathfrak{g}_S^n)^r \boxtimes (\mathfrak{g}_S^{n\circ})^r \to \Delta_*\mathcal{A}^r,\ n \geq 0,$$

induce an odd local Poisson bracket

$$\{.,.\} : \hat{\mathcal{A}}_{BV}^r \boxtimes \hat{\mathcal{A}}_{BV}^r \to \Delta_*\hat{\mathcal{A}}_{BV}^r$$

called the BV-antibracket on the completed BV algebra

$$\hat{\mathcal{A}}_{BV} = \widehat{\mathrm{Sym}}_{bigrad}\left(\left[\begin{array}{ccccc} \mathfrak{g}_S[2] & \oplus & \Theta^\ell_{\mathcal{A}}[1] & \oplus & \mathcal{A} \\ & & & & \oplus \\ & & & & {}^t\mathfrak{g}_S^\circ[-1] \end{array}\right]\right)$$

and on the BV algebra $\mathcal{A}_{BV}$.

Definition 12.4.2. Let $\mathfrak{g}_S$ be a regular generating space of Noether identities. A *formal solution to the classical master equation* is an $S_{cm} \in h(\hat{\mathcal{A}}_{BV})$ such that

1. the degree $(0,0)$ component of S_{cm} is S,
2. the component of S_{cm} that is linear in the ghost variables, denoted S_{KT}, induces the Koszul-Tate differential $d_{KT} = \{S_{KT}, .\}$ on antifields of degrees $(k, 0)$, and
3. the *classical master equation*
$$\{S_{cm}, S_{cm}\} = 0,$$
(meaning $D^2 = 0$ for $D = \{S_{cm}, .\}$) is fulfilled in $h(\hat{\mathcal{A}}_{BV})$.

A *solution to the classical master equation* is a formal solution that comes from an element in $h(\mathcal{A}_{BV})$.

Definition 12.4.3. Let $S \in h(\mathcal{A})$ be a gauge theory, and suppose given a solution $S_{cm} \in h(\widehat{\mathcal{A}}_{BV})$ of the classical master equation. The *anomalies of the gauge theory* are defined as the degree 1 cohomology classes of the complex

$$(h(\widehat{\mathcal{A}}_{BV}), \{S_{CM}, -\}).$$

The main theorem of homological perturbation theory, given in a physical language in Henneaux-Teitelboim [HT92], Chapter 17 (DeWitt indices), can be formulated in our language by the following.

Theorem 12.4.4. *Let $\mathfrak{g}_S$ be a regular generating space of Noether symmetries. There exists a formal solution to the corresponding classical master equation, constructed through an inductive method. If $\mathfrak{g}_S$ is further strongly regular and the inductive method ends after finitely many steps, then there exists a solution to the classical master equation.*

Proof. One can attack this theorem conceptually using the general setting of homotopy transfer for curved local L_∞-algebroids (see Schaetz's paper [Sch09b] for a finite dimensional analog). We only need to prove the theorem when $\mathfrak{g}$ has all $\mathfrak{g}^i$ given by free $\mathcal{A}[\mathcal{D}]$-modules of finite rank since this is true locally on M. We start by extending S to a generator of the Koszul-Tate differential $d_{KT} : \mathcal{A}_{KT} \to \mathcal{A}_{KT}$. Remark that the BV bracket with S on $\mathcal{A}_{BV}$ already identifies with the insertion map

$$\{S, .\} = i_{dS} : \Theta^\ell_{\mathcal{A}} \to \mathcal{A}.$$

We want to define $S_{KT} := \sum_{k\geq 0} S_k$ with $S_0 = S$ such that

$$\{S_{KT}, .\} = d_{KT} : \mathcal{A}_{KT} \to \mathcal{A}_{KT}.$$

Let $C^*_{\alpha_i}$ be generators of the free $\mathcal{A}[\mathcal{D}]$-modules $\mathfrak{g}^i$ and C^{α_i} be the dual generators of the free $\mathcal{A}[\mathcal{D}]$-modules $(\mathfrak{g}^i)^\circ$. We suppose further that all these generators correspond to closed elements for the de Rham differential. Let $n_{\alpha_i} := d_{KT}(C^*_{\alpha_i})$ in $\mathcal{A}_{KT}$. Then setting $S_k = \sum_{\alpha_k} n_{\alpha_k} C^{\alpha_k}$, one gets

$$\begin{array}{rcl}\{S_i, C^*_{\alpha_i}\} & = & \{n_{\alpha_i} C^{\alpha_i}, C^*_{\alpha_i}\} \\ & = & n_{\alpha_i} \\ & = & d_{KT}(C^*_{\alpha_i})\end{array}$$

so that $\{S_{KT}, .\}$ identifies with d_{KT} on $\mathcal{A}_{KT}$. Now let m_{α_j} denote the coordinates of n_{α_i} in the basis $C^*_{\alpha_i}$, so that

$$n_{\alpha_i} = \sum_j m_{\alpha_j} C^*_{\alpha_j}.$$

One gets in these coordinates

$$S_i = \sum_{\alpha_i, \alpha_j} C^*_{\alpha_j} m_{\alpha_j} C^{\alpha_i}.$$

The next terms in $S = \sum_{k\geq 0} S_k$ are determined by the recursive equation

$$2d_{KT}(S_k) + D_{k-1} = 0$$

where D_{k-1} is the component of Koszul-Tate degree (i.e., degree in the variables $C^*_{\alpha_i}$) $k-1$ in $\{R_{k-1}, R_{k-1}\}$, with

$$R_{k-1} = \sum_{j\leq k-1} S_j.$$

These equations have a solution because D_{k-1} is d_{KT}-closed, because of Jacobi's identity for the odd bracket $\{., .\}$ and since d_{KT} is exact on the Koszul-Tate components (because it gives, by definition, a resolution of the critical ideal), these are also exact. If we suppose that the generating space $\mathfrak{g}_S$ is strongly regular (i.e., bounded) and the inductive process ends after finitely many steps, one can choose the solution S in $h(\mathcal{A}_{BV})$. □

12.5 General gauge fixing procedure

We refer to Section 7.4, and particularly example 7.4.4 for some background on canonical transformations in classical hamiltonian mechanics. We are giving here a generalization of this notion to the $\mathcal{D}$-geometrical setting. Remark that in some particular cases, it may also be interesting to use derived Lagrangian subspaces (see Definition 9.7.10) in derived critical spaces of action functionals on spaces of stacky fields, that have a more intrinsic flavour. We however want to present the physicists' approach, that is also appealing because it is entirely based on the detailed study of the action functional's properties, and because it is at the heart of the functional integral quantization method.

Let $S \in h(\mathcal{A})$ be a strongly regular gauge theory and $S_{cm} \in h(\mathcal{A}_{BV})$ be a classical master action. Recall that the algebra $\mathcal{A}_{BV}$ may be defined as the bigraded symmetric algebra

$$\mathcal{A}_{BV} := \mathrm{Sym}_{\mathcal{A}-bigraded}(\mathcal{V}_{BV})$$

where

$$\mathcal{V}_{BV} := \begin{bmatrix} \mathfrak{g}_S[2] & \oplus & \Theta^\ell_{\mathcal{A}}[1] & \oplus & 0 \\ & & & & \oplus \\ & & & & {}^t\mathfrak{g}_S^\circ[-1] \end{bmatrix}$$

and $\mathfrak{g}_S$ is the graded $\mathcal{A}[\mathcal{D}]$-module of noether identities. We denote

$$X_{BV} := \underline{\mathrm{Spec}}_{\mathcal{D}}(\mathcal{A}_{BV})$$

the associated $\mathcal{D}$-space.

Definition 12.5.1. The *minimal space of extended fields* is the $\mathcal{D}$-space

$$X^{min}_{fields} := \underline{\mathrm{Spec}}_{\mathcal{D}}(\mathcal{A}^{min}_{fields})$$

where

$$\mathcal{A}^{min}_{fields} := \mathrm{Sym}_{\mathcal{A}}(\mathfrak{g}^\circ[-1])$$

is the *algebra of minimal fields.*

Proposition 12.5.2. *There is a graded symplectic isomorphism*

$$X_{BV} \cong T^*[-1]X^{min}_{fields}.$$

Proof. This follows from the graded biduality isomorphism

$$(\mathfrak{g}_S^\circ[-1])^\circ[1] \cong \mathfrak{g}_S[2],$$

due to the finiteness hypothesis imposed on $\mathfrak{g}_S$ and the symplectic isomorphism

$$T^*[-1]\underline{\mathrm{Spec}}_{\mathcal{D}}(\mathcal{A}) \cong \underline{\mathrm{Spec}}_{\mathcal{D}}(\mathrm{Sym}_g(\Theta^\ell_{\mathcal{A}}[1] \oplus \mathcal{A})).$$

□

Definition 12.5.3. A *graded gauge fixing* is the datum of a triple $(\mathcal{A}^{nm}_{BV}, i, L)$ composed of

1. a dg-$\mathcal{A}$-Poisson algebra $\mathcal{A}^{nm}_{BV}$ called the *non-minimal Batalin-Vilkovisky algebra*,
2. a quasi-isomorphism

$$i : \mathcal{A}^{nm}_{BV} \xrightarrow{\sim} \mathcal{A}_{BV}$$

of Poisson dg-$\mathcal{D}$-algebras, and

3. a graded Lagrangian $\mathcal{D}$-subspace
$$\psi : L \hookrightarrow \underline{\mathrm{Spec}}_{\mathcal{D}}(\mathcal{A}_{BV}^{nm})$$
called the *gauge fixing condition*.

An *admissible gauge fixing* is a gauge fixing such that the associated *gauge fixed action* $S_\psi \in h(\mathcal{O}_L)$ defined by
$$S_\psi := S_{cm} \circ i^{-1} \circ \psi : L \to \underline{\mathbb{R}},$$
as a degree zero graded map, has an invertible second order derivative.

The original (so-called minimal) algebra $\mathcal{A}_{BV}$ is usually not big enough to have a gauge fixing, and it is necessary to extend the original bundle $\mathcal{V}_{BV}$ by additional non-minimal generators, that form cohomologically trivial pairs, to get an admissible gauge fixing condition. We will describe here the original recipe of Batalin and Vilkovisky [BV81] to get a gauge fixing Lagrangian, that will be given by a hamiltonian perturbation of the zero section of an odd cotangent bundle.

Definition 12.5.4. The *(non-minimal) bundle of extended fields* is the bundle
$$\mathcal{V}_{fields} := \mathfrak{g}_S^\circ[-1] \oplus \mathfrak{a}_S[1] \oplus \mathfrak{l}_S,$$
where $\mathfrak{g}_S$ is called the space of *ghosts*, $\mathfrak{a}_S$ is the space of *antighosts*, $\mathfrak{l}_S$ the space of *lagrange multipliers*, and one has
$$\mathfrak{a}_S \cong \mathfrak{l}_S \cong \mathfrak{g}_S.$$
The *standard non-minimal Batalin-Vilkovisky space* is the space
$$X_{BV}^{nm} := T^*[-1]X_{fields},$$
with algebra of functions $\mathcal{A}_{BV}^{nm}$, where
$$X_{fields} := \underline{\mathrm{Spec}}_{\mathcal{D}}(\mathrm{Sym}_{dg}(\mathcal{V}_{fields}))$$
is the *space of non-minimal fields* with algebra of functions $\mathcal{A}_{fields}$.

Proposition 12.5.5. *There is a natural structure of dg-$\mathcal{D}$-symplectic space on X_{BV}^{nm} such that the natural injection*
$$i : X_{BV} \to X_{BV}^{nm}$$
is an equivalence.

Proof. The graded symplectic structure is given by the fact that X_{BV}^{nm} is a shifted cotangent space. One extends the classical master function $S_{cm} \in h(\mathcal{A}_{BV})$ by adding a trivial pairing
$$\sum_i l_i.a_i^*$$
where l_i are generators of the space $\mathfrak{l}_S$ of Lagrange multipliers and a_i^* are duals to the generators a_i of the space $\mathfrak{a}_S$ of antighosts. This addition does not change the cohomology because it makes the pair (a_i, l_i) cohomologically trivial. This means that the projection graded morphism
$$i : \mathcal{A}_{BV}^{nm} \to \mathcal{A}_{BV}$$
is then a dg-Poisson quasi-isomorphism. □

Remark that the solution spaces of the $\mathcal{D}$-spaces X_{BV}^{nm} and X_{fields} may be seen as the smooth spaces $\underline{\Gamma}(M, E_{BV})$ and $\underline{\Gamma}(M, E_{fields})$ of bundles $E_{BV}^{nm} \to C \to M$ and $E_{fields} \to C \to M$. We use the same notation for these underlying smooth spaces.

By definition, the classical master action $S_{cm} \in h(\mathcal{A}_{BV})$, being of total degree zero, induces a (partially defined) real valued functional

$$S_{cm} : X_{BV}^{nm} \to \underline{\mathbb{R}}.$$

If $\mathbb{A}$ is the graded smooth affine space and $\mathbb{A}^{\widehat{0}}$ is the subspace with zero real coordinate, the value of S_{cm} on $\mathbb{A}^{\widehat{0}}$ is a function

$$S_{cm}(\mathbb{A}^{\widehat{0}}) : X_{BV}^{nm}(\mathbb{A}^{\widehat{0}}) \to \underline{\mathbb{R}}(\mathbb{A}^{\widehat{0}}) = \mathbb{R},$$

that gives the value of the above functional on a non-trivial family of fields

$$(\varphi, \varphi^{\dagger}) : M \times \mathbb{A}^{\widehat{0}} \to E_{BV}^{nm},$$

that is concretely given by a true family of sections of the underlying bundle E_{BV}^{nm}. This gives a clear relation of our formalism with DeWitt's global approach [DeW03].

There is a natural projection

$$\pi : X_{BV}^{nm} := T^*[-1]X_{fields} \to X_{fields}$$

and a zero section 0 for π. The functional

$$S_{cm}(-, 0) : X_{fields} \to \mathbb{R}$$

has a highly degenerate derivative, and the aim of the gauge fixing procedure is to move the zero section $0 : X_{fields} \to X_{BV}$ along a Hamiltonian vector field in the antifield direction to get a better behaved function, without changing the differential graded algebra of extended fields, i.e., without changing classical local observables.

Definition 12.5.6. A *gauge fixing fermion* is a function $\psi \in \mathcal{A}_{fields}$ whose differential induces a *vertical canonical transformation*

$$f_{\psi} : T^*[-1]X_{fields} \to T^*[-1]X_{fields}$$

given by the formula

$$f_{\psi} = \mathrm{id} + \pi \circ d\psi$$

where the addition is given relatively to the projection π (i.e., in the odd cotangent fiber). This transformation is local, i.e., given by a local Poisson $\mathcal{D}$-algebra morphism

$$f_{\psi}^* : (\mathcal{A}_{BV}^{nm}, \{-,-\}) \to (\mathcal{A}_{BV}^{nm}, \{-,-\}).$$

The *gauge fixed action* associated to a gauge fixing ψ is the function

$$S_{\psi} := (S_{cm}) \circ h(f_{\psi}) \in h(\mathcal{A}_{BV}^{nm}) : X_{BV}^{nm} \to \mathbb{R}.$$

The fact that the gauge fixing can be made an $\mathbb{R}$-valued transformation is due to the presence of fields of negative degrees, called the antighosts. Without them, the composition of S_ψ with the zero section

$$0 : X_{fields} \to T^*[-1]X_{fields}$$

would give a function $S_\psi(-,0) : X_{fields} \to \mathbb{R}$ with only non-trivial component the degree zero component, defined on the space $\Gamma(M, C)$ of fields we started from, and the ghost components would be zero.

We now explain that fixing the gauge does not change the classical local observables.

Theorem 12.5.7. *The vertical canonical transformation f_ψ induces an isomorphism of differential graded $\mathcal{D}$-algebras*

$$f_\psi : (\mathcal{A}_{BV}^{nm}, \{S_{cm}, -\}, \{-,-\}) \xrightarrow{\sim} (\mathcal{A}_{BV}^{nm}, \{S_\psi, -\}, \{-,-\}).$$

Proof. This follows from the fact that f_ψ is a canonical transformation and the definition of S_ψ. □

We may define informally an admissible gauge fixing fermion as a gauge fixing fermion such that, when the antifield variable $\varphi^\dagger$ is fixed to zero, the second order derivative of the function

$$S_\psi(-,0) : X_{fields} \to \mathbb{R}$$

is non-zero at its critical points (and thus allows to initiate a perturbative construction of the functional integral, through a renormalization procedure). We will give a more abstract definition, that takes into account the higher gauge symmetries, and thus assures us to avoid any possible pathology.

Remark that one may also see $S_\psi(-,0)$ as a functional on X_{BV} by compositing it with π. We will use the same notation for this composition, as an element of $h(\mathcal{A}_{BV})$. Remark now that there is a natural isomorphism

$$(\mathcal{A}_{BV}, \{S_\psi(-,0), -\}) \xrightarrow{\sim} \mathrm{Sym}_{dg-\mathcal{A}_{fields}}\left(\Theta^\ell_{\mathcal{A}_{fields}}[1] \xrightarrow{i_{dS_\psi(-,0)}} \mathcal{A}_{fields}\right)$$

between the Batalin-Vilkovisky algebra with its differential given by bracketing with $S_\psi(-,0) \in h(\mathcal{A}_{BV})$, and the derived critical space of the functional $S_\psi(-,0) \in h(\mathcal{A}_{fields})$.

We will use the above isomorphism to give a natural admissibility condition on the gauge fixing.

Definition 12.5.8. An *admissible gauge fixing fermion* is a gauge fixing fermion $\psi \in \mathcal{A}_{fields}$ such that, if 0 denotes the zero section of the natural projection

$$\pi : T^*[-1]X_{fields} \to X_{fields},$$

the associated gauge fixed differential graded algebra

$$(\mathcal{A}_{BV}, \{S_\psi(-,0), -\})$$

has no non-zero cohomology.

Remark that the above definition is compatible with the fact that $(\mathcal{A}_{BV}, \{S_\psi, -\})$ is quasi-isomorphic to the usual Batalin-Vilkovisky algebra: the differential is not the same.

Recall that the $\mathcal{D}$-space X_{fields} may be seen as corresponding to the smooth space (that we will denote by the same name)

$$X_{fields} := \underline{\Gamma}(M, E_{fields})$$

of sections of the bundle $E_{fields} \to C \to M$ of field variables. The function

$$S_\psi(-, 0) : X_{fields} \to \mathbb{R}$$

may and will then be used to define a perturbative functional integral because it does not have any gauge symmetries left.

Part II

Classical trajectories and fields

Chapter 13

Variational problems of experimental classical physics

We now define the basic Lagrangian variational problems of experimental classical physics and explain a bit their physical interpretation.

13.1 Newtonian mechanics

In Newtonian mechanics, trajectories are given by smooth maps

$$x : [0,1] \to \mathbb{R}^3,$$

which are the same as section of the fiber bundle

$$\boxed{\pi : C = \mathbb{R}^3 \times [0,1] \to [0,1] = M.}$$

The trajectories represent one material point moving in $\mathbb{R}^3$. The space of histories is given by fixing a pairs of starting and ending points for trajectories $\{x_0, x_1\}$, i.e.,

$$H = \{x \in \Gamma(M, C),\ x(0) = x_0, x(1) = x_1\}.$$

If $\langle , \rangle$ is the standard metric on $\mathbb{R}^3$ and $V : \mathbb{R}^3 \to \mathbb{R}$ is a given "potential" function, one defines the action functional by

$$S(x) = \int_M \frac{1}{2} m \|\partial_t x\|^2 - V(x(t))dt.$$

The tangent space to H is given by smooth functions $\vec{h} : \mathbb{R} \to \mathbb{R}^3$ that fulfill $\vec{h}(0) = \vec{h}(1) = 0$. Defining

$$d_x S(\vec{h}) := \lim_{\epsilon \to 0} \frac{S(x + \epsilon h) - S(x)}{\epsilon},$$

one gets

$$d_x S(\vec{h}) = \int_M \langle m\partial_t x, \partial_t \vec{h}\rangle - \langle d_x V(x), \vec{h}\rangle dt$$

and by integrating by parts using that $h(0) = h(1) = 0$, finally,

$$d_x S(\vec{h}) = \int_M \langle m\partial_t^2 x - d_x V(x), \vec{h}\rangle dt$$

The space of physical trajectories is thus the space of maps $x : [0,1] \to \mathbb{R}^3$ such that

$$m\partial_t^2 x = -V'(x).$$

This is the standard law of Newtonian mechanics. For example, if $V = 0$, the physical trajectories are those with constant speed, which corresponds to free galilean inertial bodies. Their trajectories are also the geodesics in $\mathbb{R}^3$, i.e., lines.

Remark that in this simple case, there are no gauge symmetries. We work here in the algebraic setting: the smooth setting follows by the geometrization procedure. If

$$\mathcal{A} = \mathrm{Jet}(\mathcal{O}_C) = \mathbb{R}[t, x_0, x_1, \dots],$$

is the jet $\mathcal{D}$-algebra of the given bundle, the action functional is given by the class

$$S = \frac{1}{2}m\|x_1\|^2 - V(x(t))dt \in h(\mathcal{A})$$

of the usual Lagrangian density modulo the subspace $\{D_t F dt\}_{F\in\mathcal{A}}$ of total derivatives. The critical $\mathcal{D}$-algebra is given by $\mathcal{A}/(mx_2 + V'(x_0))$. The tangent algebroid to the jet space is the rank one module

$$\Theta_{\mathcal{A}} \cong \mathcal{A}^r[\mathcal{D}^{op}]\,\langle\partial_x\rangle\,.$$

The insertion map

$$i_{dS} : \Theta_{\mathcal{A}}^{\ell} \cong \mathcal{A}[\mathcal{D}]\,\langle\partial_x\rangle \to \mathcal{A}$$

sends the generator $\langle\partial_x\rangle$ to the Euler-Lagrange equation $mx_2 + V'(x_0)$, that is the generator of the Euler-Lagrange $\mathcal{D}$-ideal $\mathcal{I}_S \subset \mathcal{A}$. The derived critical algebra

$$\mathrm{Sym}_{dg}([\Theta_{\mathcal{A}}^{\ell}[1] \xrightarrow{i_{dS}} \mathcal{A}])$$

has no cohomology at all, and is thus quasi-isomorphic to the usual critical algebra $\mathcal{A}/\mathcal{I}_S$. So in this case, there are no gauge symmetries and the BV action functional S_{CM} is simply given by $S_{CM} = S$. The (Schouten-Nijenhuis) bracket with S indeed induces the Koszul-Tate differential S_{KT} on $\mathcal{A}_P$ because

$$i_{dS}(\vec{v}) = \{S, \vec{v}\}$$

and the classical master equation $\{S, S\}$ is fulfilled by definition.

This setting can be easily generalized to a configuration space $C = [0,1] \times X$ where X is a manifold equipped with a Euclidean metric

$$g : TX \times_X TX \to \mathbb{R}_X.$$

If $V : X \to \mathbb{R}$ is a smooth "potential" function, the action functional on $x : M \to C$ is given by

$$S(x) = \int_M \frac{1}{2} m.x^*g(Dx, Dx) - x^*V dt$$

where $Dx : TM \to x^*TX$ is the differential and $x^*g : x^*TX \times x^*TX \to \mathbb{R}_M$ is the induced metric. This generalization has to be done if one works with constrained trajectories, for example with a pendulum.

Remark that the free action functional above may be formally replaced by

$$S(x) = \int_M \frac{1}{2} \sqrt{m} \|\partial_t x\| dt,$$

and this gives the same drawings in C but there is then a natural (gauge) symmetry of S by reparametrizations, i.e., diffeomorphisms of M. This means that the trajectories are not necessarily of constant speed. However, they correspond to the smallest path in C between two given points for the given metric on C, also called the geodesics.

13.2 Relativistic mechanics

Special relativity is based on the following simple Lagrangian variational problem, that one can take as axiom of this theory. The space of configurations for trajectories is not ordinary space $\mathbb{R}^3$, but spacetime $\mathbb{R}^{3,1} := \mathbb{R}^4$, equipped with the Minkowski quadratic form

$$g(t, x) = -c^2t^2 + \|x\|^2.$$

The space of parameters for trajectories is $M = [0, 1]$, and its variable is called the proper time of the given particle and denoted τ to make it different of the usual time of Newtonian mechanics t, which is one of the coordinates in spacetime. The fiber bundle in play is thus given by

$$\boxed{\pi : C = \mathbb{R}^{3,1} \times [0, 1] \to [0, 1] = M.}$$

The action functional is exactly the same as the action of Newtonian mechanics, except that the quadratic form used to write is Minkowski's metric g. It is given on a section $x : M \to C$ by

$$S(x) = \int_M \frac{1}{2} mg(\partial_\tau x, \partial_\tau x) - V(x(\tau))$$

for $V : \mathbb{R}^{3,1} \to \mathbb{R}$ a given potential function.

The equations of motion are given as before by

$$m\partial_\tau^2 x = -V'(x),$$

but they are now valid only if x is in the light cone (domain where the Minkowski quadratic form is positive)

$$g(t, x) \geq 0.$$

One can think of the free action functional

$$S(x) = \int_M \frac{1}{2} m g(\partial_t x, \partial_t x)$$

as measuring the (square of) Minkowski length of the given trajectory, which is called proper time by physicists. Minimizing this action functional then means making your proper time the smallest possible to go from one point to another in spacetime. This means that you are going on a line. The free trajectories (without potential) are here of constant speed.

In physics, one usually replaces the above (free) action functional by

$$S(x) = \int_M \frac{1}{2} m \sqrt{g(\partial_\tau x, \partial_\tau x)}.$$

It has a (gauge) symmetry by reparametrization of $[0,1]$, i.e., by diffeomorphisms of $[0,1]$ that fix 0 and 1. This means that the free trajectories are following geodesics (smallest proper time, i.e., length of the path in Minkowski's metric), but they can have varying speed. It is in some sense problematic to allow varying speed because the true theory of accelerated particles is not special relativity but general relativity: special relativity only treats of inertial frames (with no acceleration). However, for a general relativistic particle, which can be accelerated, this is the correct version of the action functional.

One has nothing more to know to understand special relativity from a mathematical viewpoint. It is important to have a take on it because it is the basis of quantum field theory.

13.3 Electromagnetism

A simple physical example of a Lagrangian variational problem whose parameter space for trajectories is not a segment, as this was the case for Newtonian and special relativistic mechanics, is given by Maxwell's theory of electromagnetism. We refer to Landau and Lifchitz's book [LL66] for more details and to Derdzinski's book [Der92] for the covariant formulation.

13.3.1 Flat space formulation

Let M be an oriented 4-dimensional manifold. A Minkowski metric on M is a symmetric non-degenerate bilinear form

$$g : TM \times_M TM \to \mathbb{R}_M$$

that is locally of signature $(3,1)$., i.e., isomorphic to the Minkowski metric in the fiber at every point $x \in M$. We can see g as a bundle isomorphism $g : TM \xrightarrow{\sim} T^*M$, that can be extended to a bundle isomorphism

$$g : \wedge^* TM \to \wedge^* T^* M.$$

There is a natural operation of contraction of multi-vectors by the volume form

$$i_\bullet \omega_{\text{vol}} : \wedge^* TM \xrightarrow{\sim} \wedge^* T^* M$$

and the Hodge $*$-operator is defined by

$$* := i_\bullet \text{vol} \circ g^{-1} : \wedge^* T^* M \to \wedge^* T^* M.$$

If $d : \Omega^i \to \Omega^{i+1}$ is the de Rham differential, we denote $d^* : \Omega^3 \to \Omega^2$ the adjoint of d for the given metric g given by $g(d^*\omega, \nu) = g(\omega, d\nu)$. The bundle $\pi : C \to M$ underlying classical electromagnetism in a flat space is the bundle

$$\boxed{\pi : C = T^* M \to M}$$

of differential one forms on M. Let $J \in \Omega^3_M$ be a fixed 3-form (called the charge-current density or the source) that fulfills the compatibility condition

$$dJ = 0.$$

The Lagrangian density is the section L_J of

$$\text{Jet}^1 T^* M \otimes \wedge^4 T^* M$$

given in jet coordinates (A, A_1) by

$$L_J(A, A_1) = \frac{1}{2} A_1 \wedge *A_1 + A \wedge J.$$

It gives the action functional $S : \Gamma(M, C) \to \mathbb{R}$ given by

$$S_J(A) = \int_M L_J(A, dA).$$

Its variable A is called the electromagnetic potential and the derivative $F = dA$ is called the electromagnetic field. If one decomposes the spacetime manifold M in a product $T \times N$ of time and space, one can define the electric field $E_A \in \Omega^1(N)$ and magnetic field $B_A \in \Omega^2(N)$ by

$$F = B_A + dt \wedge E_A.$$

One then gets the Lagrangian density

$$L(A, dA) = \frac{1}{2} \left(\frac{|E_A|^2}{c^2} - |B_A|^2 \right)$$

if the source J is zero. Minimizing the corresponding action functional means that variations of the electric field induce variation of the magnetic field and vice versa, which is the induction law of electromagnetism.

Proposition 13.3.1.1. *The equations of motion for the electromagnetic field Lagrangian $S_J(A)$ are given by*

$$d * dA = J.$$

Proof. As explained in Chapter 12, one can compute the equations of motion from the Lagrangian density in terms of differential forms by

$$\frac{\partial L}{\partial A} - d\left(\frac{\partial L}{\partial A_1}\right) = 0.$$

Recall that the Lagrangian density is

$$L_J(A, A_1) = \frac{1}{2} A_1 \wedge *A_1 + A \wedge J.$$

On has $\frac{\partial L}{\partial A} = J$ and $\frac{\partial L}{\partial A_1} = *A_1$ because of the following equalities, that are true for $\epsilon^2 = 0$:

$$\begin{array}{rcl}(A_1 + \epsilon A_1') \wedge *(A_1 + \epsilon A_1') - A_1 \wedge *A_1 & = & \epsilon(A_1' \wedge *A_1 + A_1 \wedge *A_1'), \\ & = & \epsilon(A_1' \wedge *A_1 - *A_1' \wedge A_1), \\ & = & 2\epsilon(A_1' \wedge *A_1).\end{array}$$

This gives

$$d * dA = J.$$

□

The equation of motion is equivalent to

$$\Box A = J$$

where $\Box := d * d$ is the d'Alembertian (or wave) operator. On Minkowski's space, this identifies with $d^*d + dd^*$, which gives in local coordinates

$$\Box = -\frac{1}{c^2}\partial_t^2 + \partial_x^2.$$

In classical physics, $F = dA$ is the object of interest and the solution to $\Box A = J$ is not unique because for any function α, one also has $\Box(A + d\alpha) = 0$ since $d^2 = 0$. One can however fix a section of the projection $A \mapsto F := dA$ (called a Gauge fixing) by setting for example $\sum_\mu A_\mu = 0$. This implies the unicity of the solution A to the equation of motion $\Box A = J$.

In quantum physics, the electromagnetic potential itself can be thought of as the wave function of the photon light particle.

If one wants to study the motion of a charged particle in a given fixed electromagnetic potential $A \in \Omega^1_X$ on a given Lorentzian manifold (X, g), one uses the action functional on relativistic trajectories $x : [0, 1] \to X$ defined by

$$S(x) = \int_M \frac{1}{2} mc\sqrt{g(\partial_\tau x, \partial_\tau x)}$$

and add to it a so called Coulomb potential term of the form

$$\int_M \frac{e}{c} x^* A$$

where the constant e, called the charge of the particle, can be of positive or negative sign.

13.3.2 Generally covariant formulation

Let M be an oriented manifold of dimension n equipped with a non-degenerate metric $g : TM \times_M TM \to \mathbb{R}_M$ (the corresponding volume form is denoted ω_g), and a connection ∇ on the tangent bundle given on sections by a map

$$\nabla : \mathcal{T}_M \to \mathcal{T}_M \otimes \Omega^1_M$$

that fulfills Leibniz's rule. The bundle $\pi : C \to M$ underlying generalized electromagnetism is one of the following linear bundles:

$$\mathbb{R}_M,\ \wedge^n T^*M,\ T^*M,$$

(the case of interest for electromagnetism is $C = T^*M$ but other cases can be useful in particle physics). We will now write down the Lagrangian density for a generalized electromagnetism action functional

$$S : \Gamma(M, C) \to \mathbb{R}.$$

Since the bundle C is linear, we will denote it F. The metric and connection on TM induces a metric $g : F \times_M F \to \mathbb{R}_M$ and a connection

$$\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_M.$$

Definition 13.3.2.1. Fix an element $J \in \mathcal{F}$ called the *source*. The generalized *electromagnetism Lagrangian* (F, g, ∇) with source J sends a section $A \in \mathcal{F}$ to the differential form of maximal degree

$$L_J(A, \nabla A) := g(\nabla A, \nabla A)\omega_g + g(J, A)\omega_g$$

where g is here extended naturally to $\mathcal{F} \otimes \Omega^1_M$.

The usual electromagnetic Lagrangian (in not necessarily flat space) is obtained by using for F the cotangent bundle T^*M.

Definition 13.3.2.2. The *d'Alembertian operator* of the pair (g, ∇) is the operator $\Box : \mathcal{F} \to \mathcal{F}$ defined by contracting the second order derivative with the metric

$$\Box := \mathrm{Tr}\left((\mathrm{id}_{\mathcal{F}\otimes\Omega^1} \otimes g^{-1}) \circ \nabla_1 \circ \nabla\right).$$

Let us explain this in details. The inverse of the metric gives an isomorphism $g^{-1} : \Omega^1_M \xrightarrow{\sim} \mathcal{T}_M$ and thus an isomorphism

$$\mathrm{id} \otimes g^{-1} : \Omega^1 \otimes \Omega^1_M \to \Omega^1 \otimes \mathcal{T}_M \cong \mathrm{End}(\mathcal{T}_M).$$

If we denote

$$\nabla_1 : \mathcal{F} \otimes \Omega^1 \to Fc \otimes \Omega^1 \otimes \Omega^1$$

the extension $\nabla_1 := \nabla \otimes \mathrm{id} + \mathrm{id} \otimes \nabla$ of ∇ to differential 2-forms, one gets by composition

$$\mathcal{F} \xrightarrow{\nabla} \mathcal{F} \otimes \Omega^1 \xrightarrow{\nabla_1} \mathcal{F} \otimes \Omega^1 \otimes \Omega^1 \xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\mathfrak{g}^{-1}} \mathcal{F} \otimes \mathrm{End}(\mathcal{T})$$

a map $\mathcal{F} \to \mathcal{F} \otimes \mathrm{End}(\mathcal{T})$ of which we can compute the trace to get the d'Alembertian.

If we suppose that $F = T^*M$, $\nabla g = 0$ and ∇ is torsion free, meaning that

$$\nabla_x y - \nabla_y x = [x, y],$$

we get as equations of motion for the above Lagrangian the equation

$$\Box A = J$$

that is the wave equation on the given metric manifold with connection.

13.3.3 Gauge theoretic formulation

Let M be an oriented manifold of dimension n equipped with a non-degenerate metric $g : TM \times_M TM \to \mathbb{R}_M$ (the corresponding volume form is denoted ω_g) and a connection $\nabla : \mathcal{T}_M \to \mathcal{T}_M \otimes \Omega^1_M$ (that is the Levi-Civita connection for example). Let $G := \mathrm{SU}(1) \subset \mathbb{C}^*$ be the group of complex numbers of norm 1 and consider a principal G-bundle P. Recall that the Lie algebra $\mathfrak{g}$ of G is identified with $2i\pi\mathbb{R}$ by the exponential map. It is equipped with a natural S^1 -equivariant inner product $\langle ., . \rangle$. The variable of the gauge theoretic formulation of electromagnetism is a principal G-connection A on P, i.e., an equivariant differential form

$$A \in \Omega^1(P, \mathfrak{g})^G$$

such that the following condition is fulfilled:

- (non-degeneracy) the natural map $\mathfrak{g} \to TP \to \mathfrak{g}$ induced by the action of G on P and ω is an isomorphism.

This means that the bundle underlying this theory is the bundle of non-degenerate equivariant differential forms

$$\boxed{\pi : C = [(T^*P \times \mathfrak{g})/G]_{nd} \to M.}$$

The curvature of A is defined as the $\mathfrak{g}$-valued 2 form

$$F := dA + [A \wedge A] \in \Omega^2(P, \mathfrak{g})^G.$$

Remark that the choice of a section $s : M \to P$ gives an identification

$$s^* : \Omega^1(P, \mathfrak{g})^G \to \Omega^1(M, \mathfrak{g}) = \Omega^1(M)$$

of the space of G-equivariant principal connections on P with differential forms on M by pull-back. This relates this gauge theoretic formulation with the covariant formulation of Section 13.3.2.

The Yang-Mills action for electromagnetism is given by

$$S(A) := \int_M \langle F \wedge *F \rangle + \langle A \wedge *J \rangle,$$

where $* : \Omega^*(M, \mathfrak{g}) \to \Omega^*(M, \mathfrak{g})$ is the operator obtained by combining the usual star operator on M with the star operator on $\mathfrak{g}$ induced by the given metric.

The corresponding equation is

$$d_A * F = *J,$$

where d_A is the combination of the gauge derivative and the covariant derivative.

13.4 General relativity

This section is mainly inspired by the excellent old book of Landau and Lifchitz [LL66].

13.4.1 The Einstein-Hilbert-Palatini action

We here give a version of general relativity that is due (in another language) to Palatini. It is the one used by people studying quantization of gravity, and it is better adapted to the introduction of fermionic ("matter") fields in general relativity. It is called the first order formulation of relativity because the equations of motion are of first order, contrary to the classical Einstein-Hilbert approach that gives an equation of order two.

Let M be an oriented and compact 4-dimensional manifold. A Minkowski metric on M is a symmetric non-degenerate bilinear forms

$$g : TM \times_M TM \to \mathbb{R}_M$$

that is locally of signature $(3, 1)$, i.e., isomorphic to the Minkowski metric. We will denote

$$\mathrm{Sym}^2_{mink}(T^*M) \to M$$

the bundle whose sections are metrics of this kind. It is a sub-bundle of the linear bundle $\mathrm{Sym}^2(T^*M) \to M$ of all bilinear forms. Recall that a connection on a linear bundle $F \to M$ is given by a covariant derivation

$$\nabla : \mathcal{F} \to \Omega^1_M \otimes \mathcal{F}$$

on the space $\mathcal{F} = \Gamma(M, F)$ of section of F, i.e., an $\mathbb{R}$-linear map that fulfills

$$\nabla(gf) = dg \otimes f + g\nabla(f).$$

We will consider the space $\mathrm{Conn}(F)$ of (parametrized families of) connections on M whose points with values in a smooth algebra B are

$$\mathrm{Conn}(F)(B) := \left\{ (x, \nabla) \;\middle|\; \begin{array}{l} x : \mathrm{Spec}(B) \to M \\ \nabla : \mathcal{F} \otimes_{\mathcal{C}^\infty(M)} B \to \Omega^1_M \otimes (\mathcal{F} \otimes_{\mathcal{C}^\infty(M)} B) \end{array} \right\}.$$

There is a natural projection
$$\pi : \mathrm{Conn}(F) \to M$$
that sends the pair (x, ∇) to x. There is also a natural action of sections A of $\Omega^1_M \otimes_{\mathcal{O}} \mathrm{End}(\mathcal{F})$ on the space $\mathrm{Conn}(F)$ of connections on F given by

$$\nabla \mapsto \nabla + A$$

that makes it a principal homogeneous space. In the case $M = \mathbb{R}^4$ and $F = TM$, this means that every connection can be written as

$$\nabla = d + A$$

where d is the de Rham differential extended to the trivial bundle $TM = \mathbb{R}^4 \times \mathbb{R}^4$ and A is a 1-form on M with values in $\mathrm{End}(\mathbb{R}^4)$.

The bundle $\pi : C \to M$ that is at the heart of the Lagrangian formulation of general relativity is the bundle

$$\boxed{\pi : C = \mathrm{Conn}_{tf}(TM) \times_M \mathrm{Sym}^2_{mink}(T^*M) \to M}$$

whose sections are pairs (∇, g) with

- $\nabla : \mathcal{T}_M \to \Omega^1_M \otimes \mathcal{T}_M$ a connection on the tangent bundle (often called an affine connection) that is supposed to be torsion free, i.e., such that

$$\nabla_x y - \nabla_y x = [x, y],$$

  and

- $g : TM \times_M TM \to \mathbb{R}_M$ a metric of Minkowski's type.

The Riemann curvature tensor of the connection is given by the composition

$$R_\nabla := \nabla_1 \circ \nabla : \mathcal{T} \to \mathcal{T} \otimes \Omega^2.$$

It measures the local difference between ∇ and the de Rham trivial connection on a trivial bundle. Contrary to the original connection, this is a $\mathcal{C}^\infty(M)$-linear operator that can be seen as a tensor
$$R_\nabla \in \Omega^1 \otimes \mathcal{T} \otimes \Omega^2 \subset \mathrm{End}(\mathcal{T}) \otimes \mathrm{Sym}^2 \Omega^1_M,$$
where we pair the $\mathcal{T}$ component with one of the Ω^1 components of Ω^2 to find $\mathrm{End}(\mathcal{T})$ and the two remaining Ω^1 are paired in the symmetric power. The Ricci curvature is simply the trace
$$\mathrm{Ric}(\nabla) := -\mathrm{Tr}(R_\nabla) \in \mathrm{Sym}^2 \Omega^1_M.$$
One can also interpret it as a morphism

$$\mathrm{Ric} : \mathcal{T} \to \Omega^1_M.$$

The metric g, being non-degenerate, is an isomorphism $g : \mathcal{T} \to \Omega^1$ and its inverse g^{-1} is a morphism $g^{-1} : \Omega^1 \to \mathcal{T}$. One can then define the scalar curvature of the pair (∇, g) as the function on M defined by

$$R(\nabla, g) := \mathrm{Tr}(g^{-1} \circ \mathrm{Ric}).$$

The given metric on M gives a fundamental class $d\mu_g$ and one defines the Einstein-Hilbert action functional $S : \Gamma(M, C) \to \mathbb{R}$ by

$$S(\nabla, g) := \int_M R(\nabla, g) d\mu_g.$$

One sometimes adds a constant Λ to get an action

$$S(\nabla, g) := \int_M [R(\nabla, g) - \Lambda] d\mu_g$$

that takes into account the “expansion of universe” experiment (red shift in old/far-away starlight).

The space of histories H is usually fixed to be a subspace of $\Gamma(M, C)$ with fixed value and normal derivative for the metric and ∇ along a starting and ending space-like hyper-surface. The Cauchy problem is well posed in small time once that starting datum is fixed, by the work of Choquet-Bruhat [CB72]. The problem of finding solutions in long time with special asymptotic properties is presently an active domain of research in mathematics.

Proposition 13.4.1.1. *The space $T \subset H$ of physical trajectories for the Einstein-Hilbert action functional, defined by*

$$T := \{(\nabla, g) \in H,\ d_{(\nabla, g)} S = 0\}$$

is given by the space of solutions of the following partial differential equations

∇ is the Levi-Civita connection for g:

$$\nabla g = 0,$$

and

Einstein's equations:

$$\mathrm{Ric}(\nabla) - \tfrac{1}{2} R(\nabla, g) g = 0.$$

Proof. One may adapt the proof given in Landau and Lifchitz [LL66]. □

13.4.2 Moving frames and the Cartan formalism

The main advantage of the Cartan approach is that it allows to combine easily general relativity with fermionic matter particles, and that it is also the base of the present work on quantum gravity and on the covariant formulation of supersymmetric gravity theory. We use Section 6 for background material on connections and in particular, Section 6.4 for background on Cartan geometry.

We just recall the definition of a Cartan connection for the reader's convenience.

Definition 13.4.2.1. Let M be a manifold, $H \subset G$ be two groups. A *Cartan connection* on M is the data of

1. a principal G-bundle Q on M,
2. a principal G-connection A on Q,
3. a section $s : M \to E$ of the associated bundle $E = Q \times_G G/H$ with fibers G/H,

such that the pullback

$$e = s^*A \circ ds : TM \to VE,$$

called the moving frame (vielbein), for $A : TE \to VE$ the associated connection, is a linear isomorphism of bundles.

The role of the section s here is to "break the G/H symmetry". It is equivalent to the choice of a principal H-sub-bundle $P \subset Q$. This section s may also be encoded in a stack morphism

$$s : M \to [G\backslash(G/H)]$$

where (G/H) is the quotient manifold and the left quotient by G is the stacky quotient. The associated principal G-bundle P is given by the composition

$$P_s : M \xrightarrow{s} [G\backslash(G/H)] \to [G\backslash\{*\}] =: BG.$$

In general relativity, we are mostly interested in the case of a four dimensional space-time M, with the group H being the Lorentz group $O(3,1)$, G being the Poincaré group $\mathbb{R}^4 \rtimes O(3,1)$. There is a canonical GL_4 principal bundle on the manifold M given by the bundle

$$P_{TM} := \mathrm{Isom}(TM, \mathbb{R}^4_M)$$

of frames in the tangent bundle TM. From a given metric g on TM, one can define a principal $O(3,1)$ bundle by considering

$$P := PO_{(TM,g)} := \mathrm{Isom}_{isom}((TM, g), (\mathbb{R}^4_M, g_{lorentz}))$$

of isometries between (TM, g) and the trivial bundle with its standard Lorentzian metric

$$g(t, x) = -c^2t^2 + \|x\|^2.$$

One then defines the underlying principal G-bundle of the Cartan connection by

$$Q = P \times_H G := PO_{(TM,g)} \times_{O(3,1)} (\mathbb{R}^4 \rtimes O(3,1)),$$

and the G/H bundle is given by

$$E = Q \times_G (G/H) = P \times_H (G/H) := PO_{(TM,g)} \underset{O(3,1)}{\times} \mathbb{R}^4.$$

The bundle E is a vector bundle equipped with a natural metric $g : E \times_M E \to E$ and the induced connection $A : TE \to VE$ being G-equivariant is also linear, so that it corresponds to an ordinary Koszul connection on E.

In fact, since E is linear, the natural projection $VE \to E$ is an isomorphism. This gives an isomorphism of TM with E, which is actually the identity in this particular case because E was constructed from (TM, g).

We thus find back from the cartan connection $(Q, A, s : M \to E)$ on E the data of a Koszul connection ∇ and a metric on TM, which are the fields of the Einstein-Hilbert-Palatini formulation of gravity. The curvature of the G-equivariant Ehresmann connection A is directly related to the Riemann curvature tensor through this relation.

The bundle underlying the Cartan-Palatini action is the categorical bundle

$$\boxed{\pi : C = \mathrm{Conn}_{\mathrm{Cartan}}(M, O(3,1), \mathbb{R}^4 \rtimes O(3,1)) \to M}$$

whose sections are tuples Cartan connections $(Q, A, s : M \to E)$ composed of a principal G-bundle, a principal G-connection $A \in \Omega^1(Q, TQ)^G$ on Q and a section $s : M \to E = Q \times_G G/H$ fulfilling Cartan's condition: $e = s^*A \circ ds : TM \to VE$ is an isomorphism. The morphisms between these objects are given by isomorphisms of Cartan connections.

The above construction of a Cartan connection from the pair (g, ∇) of a connection metric and a connection on TM gives an equivalence of bundles

$$\begin{array}{ccc} \mathrm{Conn}(TM) \times_M \mathrm{Sym}^2(T^*M) & \xrightarrow{\sim} & \mathrm{Conn}_{\mathrm{Cartan}}(M, O(3,1), \mathbb{R}^4 \rtimes O(3,1)) \\ & \searrow \quad M \quad \swarrow & \end{array}$$

Remark that the G-connection A on the principal G-bundle Q is equivalent to an equivariant $\mathfrak{g}$-valued differential form

$$A : TQ \to \mathfrak{g},$$

and its restriction to $P \subset Q$ gives an H-equivariant differential form

$$A : TP \to \mathfrak{g}.$$

This is the original notion of Cartan connection form. One can decompose $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$ in an H-equivariant way (the Cartan geometry is called reductive). In fact, $\mathfrak{h}$ is the special orthogonal Lie algebra and $\mathfrak{g}/\mathfrak{h} = \mathbb{R}^4$. The Cartan connection form thus can be decomposed in

$$A = \omega + e$$

for $\omega \in \Omega^1(P, \mathfrak{h})$ and $e \in \Omega^1(P, \mathfrak{g}/\mathfrak{h})$. In this particular case, e can also be seen as

$$e \in \Omega^1(M, \underline{\mathfrak{g}/\mathfrak{h}})$$

for $\underline{\mathfrak{g}/\mathfrak{h}}$ the H-bundle associated to $\mathfrak{g}/\mathfrak{h}$. This form is called the vielbein by cartan. It gives an isomorphism

$$e : TM \to \underline{\mathfrak{g}/\mathfrak{h}}.$$

In terms of these coordinates, the Palatini action

$$S_{pal} : \Gamma(M, C) \to \mathbb{R}$$

is given by

$$S_{pal}(\omega, e) = \int_M *(e \wedge e \wedge R)$$

where

- $R \in \Omega^2(P, \mathfrak{h})$ is the curvature of $\omega : TP \to \mathfrak{h}$,
- we use the isomorphism $\mathfrak{h} = \mathfrak{so}(3,1) \cong \wedge^2\mathbb{R}^{3,1} = \wedge^2\mathfrak{g}/\mathfrak{h}$ to identify R with an element of $\Omega^2(P, \wedge^2\underline{\mathfrak{g}/\mathfrak{h}})$,
- the wedge product acts on both space-time indices and internal Lorentz indices and
- $* : \Omega^4_M(\wedge^4\underline{\mathfrak{g}/\mathfrak{h}}) \to \Omega^4_M$ is the Hodge $*$-operator.

The equations of motions for this action (obtained by deriving with respect to ω and e) are

$$\begin{cases} e \wedge R &= 0 \\ d_\omega e &= 0. \end{cases}$$

The first equation is Einstein's equation.

Remark that, up to a simple change of variable, this Cartan action for M of dimension 3 and a Riemannian metric is equivalent to the Chern-Simons action functional to be studied in Section 15.4. Indeed, these two theories have the same equations of motion. The Chern-Simons theory is a topological field theory (the equations do not depend on the metric and are diffeomorphism invariants). Its quantum partition function gives important topological invariants like for example the Jones polynomial for varieties of dimension 3.

As explained before, one of the main motivations for the Cartan formalism is to treat the case of fermionic variables in general relativity. So let $PSpin$ be a principal $\mathrm{Spin}(3,1)$-bundle on M, $H = \mathrm{Spin}(3,1)$, $V \cong \mathbb{R}^4$ be its orthogonal representation and $G = V \rtimes \mathrm{Spin}(3,1)$. Remark that the representation of $\mathrm{Spin}(3,1)$ on V is not faithful. We consider the fiber bundle

$$Q = PSpin \underset{\mathrm{Spin}(3,1)}{\times} G$$

over M and a principal G-connection A on Q. The zero section s of the vector bundle

$$V = Q \times_G (G/H) = PSpin \times_{\mathrm{Spin}(3,1)} V$$

gives the principal H-sub-bundle $PSpin \subset Q$. The pull-back of the induced connection gives an isomorphism

$$TM \to V$$

of the tangent bundle with V which is naturally equipped with a metric (induced by the Minkowski metric on $V \cong \mathbb{R}^4$) and a connection (induced by the given connection A). We thus get a metric g and a connection ∇ on TM. Moreover, if S is the standard spin representation of $\mathrm{Spin}(3,1)$, the G-connection form A induced a connection on the linear spin bundle

$$\underline{S} := PSpin \underset{\mathrm{Spin}(3,1)}{\times} S.$$

Remark that, in the construction above, one could use any principal G-bundle Q. This means that one could replace the bundle

$$\pi : C = \mathrm{Conn}_{\mathrm{Cartan}}(Q, P, s) \to M$$

by the categorical bundle

$$\boxed{\pi : C = \mathrm{B\textsc{un}Conn}_{G,\mathrm{Cartan}}(M) \to M}$$

whose sections are triples (Q, A, s) of a principal G-bundle, Q, a principal G-connection A on Q and a section $s : M \to E = Q \times_G G/H$ of the associated bundle fulfilling Cartan's condition. One may interpret the datum (s, Q, A) of a Cartan connection as a point of the stack (see Section 9.3 for a precise definition of this object)

$$(s, Q, A) \in \underline{\mathrm{Hom}}(M, [G\backslash(G/H)]_{conn})$$

fulfilling the additional non-triviality condition.

This shows that even in classical general relativity, the tools of homotopical geometry are mandatory to properly understand the differential geometric properties of the space of gravitational fields.

13.4.3 A black hole solution with simple applications

The most important solution of Einstein's equations for experimental matters was found by Schwarzschild in 1916. With some symmetry condition on the background manifold M of general relativity, one can explicitely compute particular solutions of the general relativity equations. We refer to Landau-Lifchitz [LL66], 12.100, Besse [Bes08], Chapter III Section F and Sachs-Wu [SW77], example 1.4.2 for a detailed study of the Schwarzschild solution, that we call a black hole solution because it is used to formalize black holes. This solution is also the one used to explain the bending of light by a massive star: if you wait for the moon to make a solar eclipse, and look at a star that is far away behind the sun, the light of the star will be bended by the sun's gravitational field. The bending angle can be computed using Schwarzschild's solution. One can also use this model to explain Mercury's perihelion precession.

We here give a description of the Schwarzschild solution by first describing the background manifold, and then giving the metric. This is not very satisfactory from the physicist viewpoint because the geometry of spacetime should be dynamically obtained from Einstein's equations and some additional physical hypothesis. The physical hypothesis here is that

1. spacetime is static, i.e., there is a non-zero time-like Killing vector field X on M (i.e., a vector field such that $L_X g = 0$),

2. spacetime is spherically symmetric, i.e., there is an action of SO(3) on M whose orbits are either points or space-like hyper-surfaces.

We now construct the most simple static and spherically symmetric spacetime. Let S^2 be the unit two sphere, equipped with its metric h induced by the embedding $S^2 \subset \mathbb{R}^3$ and its fundamental form ω. Let $\mu \in]0, +\infty[$ be given. Consider the space

$$M = (]0, \mu[\cup]\mu, +\infty[) \times S^2 \times \mathbb{R}$$

with coordinates (r, θ, t) (θ being the vector of $\mathbb{R}^3$ representing a point in S^2). One must -not- think of t as being a time coordinate in all of M. Remark that $(1 - \frac{2\mu}{r})$ is a smooth function from M to $]-\infty, 0[\cup]0, 1[$. Denote by $p : M \to S^2$ the natural projection. The Schwarzschild metric on M is defined from the metric h on S^2 by

$$g := \left(1 - \frac{2\mu}{r}\right)^{-1} d^2r + r^2 p^* h - \left(1 - \frac{2\mu}{r}\right) d^2t.$$

The Schwarzschild spacetime is the subspace N of M defined by

$$N =]\mu, +\infty[\times S^2 \times \mathbb{R},$$

with coordinates (r, θ, t). The complement of N in M is identified with

$$B =]0, \mu[\times S^2 \times \mathbb{R}.$$

If $r_0 >> 8\pi\mu$, one can interpret the open sub-manifold of N defined by $r > r_0$ as an excellent history of the exterior of a star of radius r_0 and active mass $8\pi\mu$. The interior of the star is not modeled by any sub-manifold of (M, g).

This model is used to explain Mercury's perihelion precession. The space around the sun is modeled by a Schwarzschild spacetime (N, g). Consider the earth turning around the sun, given by a geodesic

$$x : \mathbb{R} \to N$$

lying in a totally geodesic sub-manifold $]\mu, +\infty[\times S^1 \times \mathbb{R}$ where S^1 is the S^2 component of $\mathbb{R}$. If one supposes that the radius r is constant, meaning that the orbit is circular, we get (see [Bes08], III.G) Keppler's law: the orbit of the planet around a star of mass M takes proper time

$$T = \frac{4\pi^2}{M} R$$

where R is the distance between the planet and the center of the star. Using almost circular geodesics $x : \mathbb{R} \to N$, one gets (see [Bes08], III.H) that the period of a Mercury planet (of mass m) turning around the sun (of mass M) is different of the Newtonian one by a factor

$$\left(1 - \frac{6m}{M}\right)^{-1/2}.$$

This is the most famous experimental prediction of general relativity theory.

The study of null geodesics (light rays) in N is used to explain the bending of light coming from a far away star behind the sun to the earth (see [Bes08], III.J).

The subspace B of M is called the Schwarzschild black hole of active mass $8\pi\mu$. Remark that the vector field $(\partial/\partial r)_{|B}$ is time-like, so that an observer close to B in N will tend to B, and he will experiment an infinite curvature of spacetime when arriving to the boundary of B. We advise the reader not to try this.

13.4.4 The big bang solution

We refer to Landau-Lifchitz [LL66], Chapter 14, for a very nice account of this theory. This solution, called the Friedmann-Lemaitre-Robertson-Walker solution, or the standard model of cosmology, is a solution that can be obtained by assuming that spacetime M is

1. homogeneous, i.e., of the form $M = N \times I$ with I an interval in $\mathbb{R}$ and (N, h) a simply connected Riemannian space,

2. and isotropic, meaning that (N, h) is of constant curvature.

We refer to Wolf's book [Wol84] for a complete study of spaces of constant scalar curvature. If $p : M \to N$ is the projection, the lorentzian metric on M is then supposed to be of the form

$$g = -c^2dt^2 + a(t)^2 p^* h.$$

The main examples of Riemannian spaces (N, h) like the one above are S^3 (positive curvature, defined by the equation $\|x\|^2 = a^2$ in Euclidean $\mathbb{R}^4$), $\mathbb{R}^3$ (zero curvature) and $\mathbb{H}^3$ (negative curvature, defined as the positive cone in $\mathbb{R}^{3,1}$). The above metric gives a theoretical explanation of the expansion of universe, which is experimentally explained by remarking that the spectrum of light of far away stars is shifted to the red.

One can also give examples that have a Klein geometric description as homogeneous spaces under a group. This allows to add geometrically a cosmological constant in the Lagrangian, by using the Cartan formalism (see [Wis06]).

- The de Sitter space, universal covering of $\mathrm{SO}(4, 1)/\mathrm{SO}(3, 1)$, is with positive cosmological constant,

- The Minkowski space $\mathbb{R}^{3,1} \rtimes \mathrm{SO}(3, 1)/\mathrm{SO}(3, 1)$ has zero cosmological constant.

- The anti de Sitter space, universal covering of $\mathrm{SO}(3, 2)/\mathrm{SO}(3, 1)$ has negative cosmological constant.

Recent fine measurements of the red shift show that the expansion of universe is accelerating. The standard model of cosmology with cosmological constant is able to explain these two experimental facts by using a FLRW spacetime with positive cosmological constant.

Chapter 14

Variational problems of experimental quantum physics

The following variational problems are not interesting if they are not used to make a quantum theory of the corresponding fields. They can't be used in the setting of classical physics. However, their extremal trajectories are often called classical fields. This is because in the very special case of electromagnetism, the classical trajectories correspond to the motion of an electromagnetic wave in spacetime.

14.1 The Klein-Gordon Lagrangian

14.1.1 The classical particle and the Klein-Gordon operator

One can see the Klein-Gordon operator as the canonical quantization of the homological symplectic reduction of the classical particle phase space. This relation is explained in Polchinski's book [Pol05], page 129.

Let X be an ordinary manifold equipped with a metric g and $M = [0,1]$. The bundle underlying the classical particle is the bundle

$$\boxed{\pi : C = X \times M \to M}$$

of spaces. Its sections correspond to maps $x : [0,1] \to X$. The free particle action functional is given by

$$S(x) = \int_{[0,1]} \frac{1}{2}\sqrt{x^*g(Dx, Dx)}.$$

The space of initial conditions (so called phase space of the corresponding Hamiltonian system) for the corresponding Euler-Lagrange equation is the cotangent bundle T^*X, with algebra of polynomial functions

$$\mathcal{O}_{T^*X} = \mathrm{Sym}^*_{\mathcal{O}_X}(\Theta_X)$$

given by the symmetric algebra on the $\mathcal{O}_X$-module $\Theta_X = \Gamma(X, TX)$ of vector fields on X.

There is a canonical two form $\omega \in \Omega^2_{T^*X}$ that gives a symplectic structure

$$\omega : \Theta_{T^*X} \otimes \Theta_{T^*X} \to \mathcal{O}_{T^*X}.$$

The Weyl algebra $\mathcal{W}_{T^*X}$ is given by the universal property

$$\mathrm{Hom}_{\mathrm{ALG}(\mathcal{O}_{T^*X})}(\mathcal{W}_{T^*X}, B) \cong \{j \in \mathrm{Hom}_{\mathrm{MOD}(\mathcal{O}_{T^*X})}(\Theta_{T^*X}, B) | j(v).j(w) - j(w).j(v) = \omega(v,w).1_B\}.$$

It is defined as a quotient of the tensor algebra $T_{\mathcal{O}_{T^*X}}(\Theta_{T^*X})$. This gives a natural filtration and the corresponding graded algebra is

$$\mathrm{gr}^F \mathcal{W}_{T^*X} \cong \mathrm{Sym}^*_{\mathcal{O}_{T^*X}}(\Theta_{T^*X}) = \mathcal{O}_{T^*(T^*X)}.$$

Remark that the algebra $\mathcal{D}_{T^*X}$ of differential operators on T^*X also has a filtration whose graded algebra is

$$\mathrm{gr}^F \mathcal{D}_{T^*X} = \mathcal{O}_{T^*(T^*X)}.$$

The algebra of differential operators $\mathcal{D}_X$ is in some sense a generalization of the Weyl algebra to non-symplectic manifolds, since one always has that its graded algebra

$$\mathrm{gr}^F \mathcal{D}_X \cong \mathrm{Sym}^*_{\mathcal{O}_X}(\Theta_X) = \mathcal{O}_{T^*X}$$

is isomorphic to the algebra of functions on T^*X. The action of $\mathcal{D}_X$ on $L^2(X)$ is thought by physicists as the canonical quantization of the algebra $\mathcal{O}_{T^*X}$ of functions on phase space.

Because of the symmetries of the Lagrangian of the free particle by reparametrization, the complete explanation of the relation between the classical particle and the Klein-Gordon operator can only be done in the setting of the BRST formalism for fixing the gauge symmetry. This is explained in Polchinski's book [Pol05], page 129.

14.1.2 The Klein-Gordon Lagrangian

The Klein-Gordon Lagrangian is a slight generalization of the Lagrangian of electromagnetism, that includes a mass for the field in play. We here use the book of Derdzinski [Der92]. Let (M, g) be a metric manifold, equipped with a connection $\nabla : \Omega^1 \to \Omega^1 \otimes \Omega^1$. The bundle $\pi : C \to M$ underlying the Klein-Gordon Lagrangian is one of the following linear bundles

1. $\mathbb{R}_M$: the corresponding particle is called of spin 0 (because the action of the orthogonal group $O(g)$ on it is trivial) and of parity 1. Its classical state is thus represented simply by a real valued function on M.

2. $\wedge^n T^*M$: the corresponding particle is also of spin 0 but of parity -1.

3. T^*M: the corresponding particle is of spin 1 (the standard representation of $O(g)$). The electromagnetic particles, called Photons, are of this kind.

The metric and connection on TM can be extended to C and to $C \otimes T^*M$. We will now define the Klein-Gordon action functional

$$S : \Gamma(M, C) \to \mathbb{R}$$

by its Lagrangian density.

Definition 14.1.3. The *Klein-Gordon Lagrangian* of mass m is given on sections $\varphi \in \Gamma(M, C)$ by

$$L(\varphi) = -\frac{1}{2}\left[g(\nabla\varphi, \nabla\varphi) + \frac{m^2c^2}{\hbar^2}g(\varphi, \varphi)\right]\omega_g.$$

The equation of motion for this Lagrangian is given by

$$\left(\Box - \frac{m^2c^2}{\hbar^2}\right)\varphi = 0,$$

where the d'Alembertian $\Box$ was introduced in the Section 13.3.2 on the covariant formulation of electromagnetism.

14.2 The Dirac Lagrangian

We refer to Section 4.4 for details on spinors and the Clifford algebra.

14.2.1 Fermionic particles and the Dirac operator

The Dirac operator can be seen as the canonical quantization of the fermionic particle. This relation is explained in Polchinski's book [Pol98] on superstring theory.

Let X be an ordinary manifold equipped with a metric g and $M = \mathbb{R}^{0|1}$. The bundle underlying the fermionic particle is the bundle

$$\boxed{\pi : C = X \times M \to M}$$

of superspaces. Its sections correspond to maps $x : \mathbb{R}^{0|1} \to X$.

The fermionic particle action functional is given by

$$S(x) = \int_{\mathbb{R}^{0|1}} \frac{1}{2} x^* g(Dx, Dx).$$

Remark that the superspace $\underline{\Gamma}(M, C)$ of such maps is identified with the odd tangent bundle $T[1]X$ of X, whose super-functions are given by the super-algebra

$$\mathcal{O}_{T[1]X} = \wedge^* \Omega^1_X$$

of differential forms. This space $T[1]X$ plays the role of a phase (i.e. initial condition) space for the fermionic particle and superfunctions on it are classical observables for this system. If one fixes a metric g on X (for example if X is ordinary spacetime), one can think (as explained in Section 4.4 about Clifford algebras) of $\Gamma(X, \mathrm{Cliff}(TX, g))$ as a canonical quantization of the algebra $\mathcal{O}_{T[1]X}$, and the spinor bundle S on which it acts as the state space for the canonical quantization of the fermionic particle. This means that fermions in the setting of field theory are obtained by canonically quantizing the fermionic particle.

14.2.2 Generally covariant formulation

Let (M, g, ∇) be an oriented pseudo-metric manifold with a connection and let $S \subset \mathrm{Cliff}(TM_{\mathbb{C}}, g)$ be an irreducible sub-representation of the action of the complexified Clifford algebra bundle on itself over (M, g). We suppose that S is of minimal dimension (equal to the dimension of a column of a fiber). It is then called a Dirac spinor bundle. The connection ∇ is supposed to extend to S (this is the case for the Leci-Civita connection). Denote $\mathcal{S}$ the space of sections of S. The Clifford multiplication induces a natural map

$$c : \Omega^1_{M,\mathbb{C}} \otimes \mathcal{S} \to \mathcal{S}.$$

The Dirac operator is defined by

$$\not{D} = c \circ \nabla : \mathcal{S} \to \mathcal{S}.$$

Suppose that there exists a real spinor bundle S whose extension of scalars to $\mathbb{C}$ gives the one used above. The Clifford multiplication gives a real Dirac operator

$$\not{D} : \mathcal{S} \to \mathcal{S}.$$

There are natural pairings

$$\tilde{\Gamma} : S \times_M S \to TM$$

or equivalently

$$\tilde{\Gamma} : S \times_M (S \times_M T^*M) \to \mathbb{R}_M,$$

and

$$\epsilon : S \times_M S \to \mathbb{R}_M.$$

If $m \in \mathbb{R}$ is a fixed number, we define the Dirac Lagrangian acting on sections of the super-bundle

$$\boxed{\pi : C = \Pi S \to M}$$

by

$$L(\psi) = \left(\frac{1}{2}\psi \not{D}\psi - \frac{1}{2}\psi m \psi\right) d^n x,$$

where the kinetic term is given by

$$\psi \not{D} \psi := \tilde{\Gamma}(\psi, \nabla\psi)$$

and the mass term is given by

$$\psi m \psi := m\epsilon(\psi, \psi).$$

Let us describe the above construction in the case of $M = \mathbb{R}^{3,1}$ and g is Minkowski's metric, following Dedzinski's book [Der92], Section 1.3. Let (V, q) be the corresponding quadratic real vector space. In this case, we have

$$\mathrm{Cliff}(V, q) = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathrm{M}_{4,\mathbb{C}} \text{ and } \mathrm{Cliff}^0(V, q) = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathrm{M}_{2,\mathbb{C}},$$

the spinor group is $\mathrm{Spin}(3,1) = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{SL}_{2,\mathbb{C}}$ and its real spinor representation is $S = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{C}^2$. An element of S is called a Weyl spinor and an element of $S_{\mathbb{C}}$ is called a Dirac spinor. One has a canonical symplectic form ω on S given by

$$\omega(x,y) = x_1y_2 - x_2y_1.$$

Now consider the vector space

$$V' := \{f \in \mathrm{End}_{\mathbb{R}}(S),\ f(\lambda x) = \bar{\lambda} f(x),\ \omega(-,f-) \text{ is hermitian}\}.$$

Remark that V' is a real vector space of dimension 4 with a lorentz form q', such that

$$fg + gf = q'(f,g).\mathrm{id}_S.$$

There is an identification between (V,q) and (V',q') and an identification

$$S \otimes \bar{S} \cong \mathrm{Res}_{\mathbb{C}/\mathbb{R}} V_{\mathbb{C}}.$$

One defines $\epsilon : S \otimes S \to \mathbb{R}$ by

$$\epsilon = \mathrm{Im}(\omega).$$

The Dirac Lagrangian can then be written

$$L(\psi) = \frac{1}{2}\mathrm{Im}\,\omega(\psi, (\not{D} + m)\psi).$$

14.2.3 The Dirac Lagrangian in Cartan formalism

One can also formulate this in a more gauge theoretical viewpoint by using Cartan's formalism. Let (V,q) be a quadratic space, $H = \mathrm{Spin}(V,q)$ and $G = V \rtimes \mathrm{Spin}(V,q)$. We choose a section of the exact sequence

$$0 \to V \to G \to H \to 1,$$

i.e., an embedding $H \subset G$. Let Q be a principal G-bundle equipped with a principal G-connection and $s : M \to P \times_G G/H$ be a reduction of the structure group of P to H, i.e., a sub-principal H-bundle P of Q. The hypothesis for doing Cartan geometry is that the pull-back of the connection A along $P \subset Q$ gives an isomorphism

$$e : TM \to \underline{V}$$

between the tangent bundle and the fake tangent bundle

$$\underline{V} := P \times_{\mathrm{SO}(V,q)} V.$$

If S is a spinorial representation of $\mathrm{Spin}(V,q)$, the pairings

$$\tilde{\Gamma} : S \otimes S \to V$$

and

$$\epsilon : S \otimes S \to \mathbb{R},$$

being $\mathrm{SO}(V,q)$-equivariant, induce natural pairings on the associated bundles $\underline{S}$ and $\underline{V}$. The connection A induces a connection on both of them.

In this setting, the underlying bundle of the Dirac Lagrangian is the odd super-bundle

$$\boxed{\pi : C := \Pi \underline{S} \to M}$$

and the Dirac Lagrangian is defined as before by

$$L(\psi) = \left(\frac{1}{2}\psi \not{D}_A \psi - \frac{1}{2}\psi m \psi\right) d^n x.$$

We may also allow the gravitational field (P, Q, A) to vary. This corresponds (in the *flat* case) to adding the space of fields

$$\text{BUNConn}_{G,\text{Cartan}}(M) \rightarrowtail \underline{\text{Hom}}(M_{DR}, [G\backslash(G/H)])$$

for the Cartan formalism for general relativity.

14.3 Classical Yang-Mills theory

We already presented the gauge theoretical version of electromagnetism in Section 13.3.3.

Let M be a manifold and G be a connected lie group. Suppose given a principal G-bundle P on M. The bundle underlying Yang-Mills theory is the bundle

$$\boxed{\pi : C = \text{Conn}_G(P) \to M}$$

of principal G-connections on P. Recall that such a connection is given by a (non-degenerate) G-equivariant $\mathfrak{g}$-valued 1-form

$$A \in \Omega^1(P, \mathfrak{g})^G$$

and that its curvature is the G-equivariant $\mathfrak{g}$-valued 2-form

$$F_A := dA + [A \wedge A] \in \Omega^2(P, \mathfrak{g})^G.$$

We suppose given a bi-invariant inner product $\langle ., . \rangle$ on $\mathfrak{g}$. The pure Yang-Mills action is given by the formula

$$S(A) := -\int_M \frac{1}{4}\langle F_A \wedge *F_A \rangle.$$

The principal G-connection A induces a covariant derivation

$$d_A : \Omega^*(P, \mathfrak{g}) \to \Omega^*(P, \mathfrak{g}).$$

The equations of motion are given by

$$d_A * F_A = 0.$$

The choice of a section $s : M \to P$, called a gauge fixing, gives an isomorphism

$$s^* : \Omega^1(P, \mathfrak{g})^G \supset \text{Conn}_G(P) \to \Omega^1(M, \mathfrak{g})$$

between G-principal connections on P and Lie algebra valued differential forms on M. However, such a choice is not always possible and we will see later that the obstruction to this choice plays an important role in quantization of Yang-Mills gauge theories. If we suppose given a gauge fixing, the bundle associated to Yang-Mills theory is the bundle

$$\boxed{\pi : C = T^*M \otimes \mathfrak{g} \to M}$$

whose space of sections is $\Omega^1(M, \mathfrak{g})$ and the action functional is

$$S(A) = \int_M \langle dA \wedge *dA \rangle.$$

One can show (by extending [Cos11], Chapter 5, 7.2 to the case of a base manifold M of dimension n) that the corresponding BV bundle is given by the $\mathcal{O}_M$-module map

$$\begin{array}{ccc} \mathcal{W}_{BV} & := & \Omega^0(M)\otimes\mathfrak{g}[1]\oplus\Omega^1(M)\otimes\mathfrak{g}\oplus\Omega^{n-1}(M)\otimes\mathfrak{g}[-1]\oplus\Omega^n(M)\otimes\mathfrak{g}[-2] \\ \uparrow & & \uparrow \\ \mathcal{C} & := & \Omega^1(M)\otimes\mathfrak{g} \end{array}$$

Remark that the BV bundle is defined as the $\mathcal{O}_C$-module

$$\mathcal{V}_{BV} := \left[\Omega^0(M)\otimes\mathfrak{g}[1]\oplus\Omega^{n-1}(M)\otimes\mathfrak{g}[-1]\oplus\Omega^n(M)\otimes\mathfrak{g}[-2]\right]\otimes_{\mathcal{O}_M}\mathcal{O}_C.$$

The BV algebra is the algebra on $\mathcal{A} = \mathrm{Jet}(\mathcal{O}_C)$ given by

$$\mathcal{A}_{BV} = \mathrm{Jet}(\mathrm{Sym}_{\mathcal{O}_C}(\mathcal{V}_{BV})).$$

One then has identifications of the various terms of the BV algebra with the corresponding local constructions

$$\begin{array}{ccccc} \Theta^\ell_{\mathcal{A}} & \cong & \Omega^{n-1}(M,\mathfrak{g})\otimes_{\mathcal{O}_M}\mathcal{A}[\mathcal{D}], & & \\ \mathfrak{g}^0 & \cong & \Omega^0(M,\mathfrak{g})\otimes_{\mathcal{O}_M}\mathcal{A}[\mathcal{D}] & \cong & \mathfrak{g}\otimes\mathcal{A}[\mathcal{D}], \\ (\mathfrak{g}^0)^\circ & \cong & \Omega^n(M,\mathfrak{g})\otimes_{\mathcal{O}_M}\mathcal{A}[\mathcal{D}]. & & \end{array}$$

The BV Lagrangian is then given (see [HT92],) by

$$L(A, A^*, C_*, C^*) = -\frac{1}{4}\langle F_A \wedge *F_A\rangle + \langle A^* \wedge dC_*\rangle.$$

Remark that in this case, the BV formalism can be formulated purely in terms of the graded linear bundle $\mathcal{W}_{BV}$ on M. One usually does this in perturbative quantum field theory (see for example Costello's book [Cos11]).

There is no reason to choose a particular principal bundle so that the bundle underlying Yang-Mills theory could be the moduli space

$$\pi : C = \mathrm{B{\scriptsize UN}Conn}_G(M) \to M$$

of pairs (P, A) composed of a principal bundle P on M and a principal connection. It is simply the stack of maps (see Sections 9.2 and 9.4 for a precise definition of this object)

$$\mathrm{B{\scriptsize UN}Conn}_G(M) := \underline{\mathrm{Hom}}(M, BG_{conn}).$$

This shows that the natural setting of Yang-Mills gauge theory is homotopical geometry.

14.4 Classical matter and interaction particles

One can easily combine the Klein-Gordon, Dirac and Yang-Mills action functionals. We will do this in the Cartan formalism that is practically easier to use with fermions. The generally relativistic version can be found by supposing that the Cartan connection is the Levi-Civita connection. We refer to Derdzinski's book [Der92] for another presentation that avoids the use of Cartan connections.

Let (V, q) be an even dimensional quadratic space and M be a manifold of the same dimension.

Suppose given a Lie group K with Lie algebra $\mathfrak{k}$, and a bi-invariant scalar product $\langle ., . \rangle$. Let R be a principal K-bundle on M. We keep the notations of 14.2.3 for the Dirac Lagrangian in Cartan formalism, so that Q is a principal $G = V \rtimes \mathrm{Spin}(V, q)$-bundle and P is a principal $H = \mathrm{Spin}(V, q)$ bundle.

Suppose given a representation S of $K \times \mathrm{Spin}(V, q)$ such that the two components commute. We associate to S the bundle

$$\underline{S} := (R \times P) \times_{K \times \mathrm{Spin}(V,q)} S.$$

If B is a Cartan connection on (P, Q) (that is supposed to be fixed here) and A is a principal K-connection on R, one gets a covariant derivation

$$d_{(A,B)} : \mathcal{S} \to \mathcal{S} \otimes \Omega^1_M$$

on the space $\mathcal{S}$ of sections of $\underline{S}$, that can be combined with the Clifford multiplication map

$$\mathcal{S} \otimes \Omega^1_M \to \mathcal{S}$$

to yield the covariant Dirac operator

$$\not{D}_{(A,B)} : \mathcal{S} \to \mathcal{S}.$$

In the physicists language, the Cartan connection (which is in practice the combination of the metric and the Leci-Civita connection on spacetime) is called the fixed background, the principal K-connection A on R is called the gauge field and a section ψ of $\Pi\underline{S} \to M$ is called an interacting spinor field.

The fiber bundle of Yang-Mills theory in a fixed gravitational background and with interaction is the bundle

$$\boxed{\pi : C = \mathrm{Conn}_K(R) \times_M \Pi\underline{S} \to M}$$

whose sections are pairs (A, ψ) of a principal K-connection on the gauge bundle R and of a section ψ of the super-bundle $\Pi\underline{S}$ of fermionic interacting particles.

The Yang-Mills theory with fermionic matter Lagrangian is then given by simply adding the pure Yang-Mills and the Dirac Lagrangian in the above setting

$$L(A, \psi) := -\frac{1}{2}\langle F_A \wedge *F_A \rangle + \psi \not{D}_A \psi + \psi m \psi.$$

Remark that the Dirac operator now depends on the gauge field A.

For example, in the case of electromagnetism, we can work with the representation given by complex spinors $S_{\mathbb{C}}$ equipped with the action of SU_1 by multiplication and the standard action of $\mathrm{Spin}(V, q)$. This gives the Quantum Electrodynamics Lagrangian.

Remark that it is not really natural to fix once and for all the principal bundles used to formalize the Lagrangian. A Cartan connection may be described as a stack morphism (see Section 9.3 for more details)

$$(s, Q, A) : M \to [G\backslash(G/H)]_{conn}.$$

The Yang-Mills field is given by a morphism

$$(R, A) : M \to BK_{conn}.$$

If we give a representation S of $G \times K$, we get a bundle

$$[S/(G \times K)] \to B(G \times K)$$

such that a morphism

$$M \to [S/(G \times K)]_{conn}$$

is the same as a principal $(G \times K)$ bundle $Q \times R$ with connection, equipped with a section ψ of the associated bundle $\mathcal{S}$. The space of classical histories (we neglect boundary conditions here) of the Yang-Mills theory with gravity and matter is thus simply given by homotopy fiber

$$\begin{array}{ccc} T_{YM} & \longrightarrow & \underline{\mathrm{Hom}}(M, [S/(G \times K)]_{conn}) \\ \downarrow & & \downarrow \\ \{*\} & \xrightarrow{TM} & \underline{\mathrm{Hom}}(M, B\mathrm{SL}(V)) \end{array}$$

where TM denotes the classifying map for the tangent bundle.

14.5 The standard model

We refer to de Faria and de Melo [dFdM10], Chapter 9, for the description of the standard model representation and Lagrangian. A refined description of the (modified) standard model Lagrangian is also given in Connes-Marcolli [CM08], Chapter 9. For a coordinate free presentation, one can also use Derdzinski [Der92].

The Yang-Mills theory underlying the standard model of elementary particle has gauge group $K = U(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$. For $i = 1, 2, 3$, denote $\det_i$ the determinant representations of these three groups on $\mathbb{C}$, V_i their standard representation on $\mathbb{C}^i$, $S = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{C}^2$ the spinorial representation of $\mathrm{Spin}(3, 1)$ and V its standard representation. Let $G = K \times (V \rtimes \mathrm{Spin}(3, 1))$ be the full gauge group of gravity paired with matter. We denote $\bar{S}$ and $\bar{V}_i$ the complex conjugated representations.

The representation of G underlying the standard model is given by defining

1. The left-quark doublet representation

$$Q_L = V_1 \otimes V_2 \otimes V_3 \otimes S,$$

2. The Up-quark singlet representation

$$U_R = V_1 \otimes \det_2 \otimes \bar{V}_3 \otimes S,$$

3. The Down-quark singlet representation

$$D_R = V_1 \otimes \det_2 \otimes \bar{V}_3 \otimes S,$$

4. The left lepton doublet representation

$$L_L = V_1 \otimes V_2 \otimes \det_3 \otimes S,$$

5. The right-electron singlet representation

$$E_R = V_1 \otimes \det_2 \otimes \det_3 \otimes S,$$

6. The Higgs representation

$$H = V_1 \otimes V_2 \otimes \det_3 \otimes S.$$

and admitting the experimental fact that there are three generations of quarks and leptons, the full standard model representation for matter particles in interaction is

$$V_{sm} := (Q_L \oplus U_R \oplus D_R \oplus L_L \oplus E_R)^{\oplus 3} \oplus H.$$

The bundle underlying the standard model is thus the Yang-Mills bundle

$$\boxed{\pi : C = \mathrm{Conn}_K(R) \times_M \Pi \underline{V}_{sm} \to M}$$

where $\underline{V}_{sm}$ is the bundle associated to a given principal bundle $R \times Q$ under

$$G = (\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)) \times (V \rtimes \mathrm{Spin}(3,1))$$

and to the representation V_{sm} by the associated bundle construction

$$\underline{V}_{sm} := (R \times Q) \times_G V_{sm}.$$

The standard model Lagrangian (before its Batalin-Vilkovisky homotopical Poisson reduction) is given by a combination of a classical Yang-Mills Lagrangian with matter and a Higgs/Yukawa component, that gives masses to some interacting and matter particles. It is thus essentially of the form

$$L(A, \psi, \varphi) = -\frac{1}{2}\langle F_A \wedge *F_A \rangle + \psi \not{D}_A \psi + V(A, \psi, \varphi),$$

where $V(A, \psi, \varphi) = L_H(\varphi) + L_{Hg}(A, \varphi) + L_{Hf}(\psi, \varphi)$ is the Higgs potential, with

- L_H the Higgs self-coupling potential,
- L_{Hg} the coupling of Higgs with gauge fields, and
- L_{Hf} the (Majorana) coupling of Higgs with fermions.

These additional Higgs couplings are made to give certain fermions and gauge fields a mass, by the spontaneous symmetry breaking mechanism. For more details on this mechanism, we refer to Derdzinski [Der92], Chapter 11, Connes-Marcolli [CM08], Chapter 9 and de Faria-de Melo [dFdM10], Chapter 9.

Remark that the CERN experiment at LHC has shown in july 2012 the existence of the Higgs boson φ. This results gives a very robust experimental meaning to the standard model, that had been lacking for roughly thirty years.

Chapter 15

Variational problems of theoretical physics

We will now present, sometimes in a more sketchy fashion, some important models of theoretical/mathematical physics. We chose to only give here the superspace formulations of superfields, because of its evident mathematical elegance. We refer to the two IAS volumes [DEF$^+$99] for more examples.

15.1 Kaluza-Klein's theory

The aim of Kaluza-Klein's theory is to combine general relativity on spacetime M and electromagnetism in only one "generally relativistic" theory on a $4+1$ dimensional space, locally given by a product $M \times S^1$.

Let M be a 4-dimensional manifold equipped with a principal U(1)-bundle $p : P \to M$. Fix a $U(1)$-invariant metric g on $U(1)$, which is equivalent to a quadratic form $g_U : \mathfrak{u}(1) \times \mathfrak{u}(1) \to \mathbb{R}$ on its Lie algebra. Fix also a U(1)-invariant metric

$$g_P : TP \times TP \to \mathbb{R}_P$$

on P. The lengh of a fiber of P (isomorphic to $U(1)$) for this metric is denoted λ_P. The metric on P induces a Levi-Civita connexion on TP. This connection, being also an Ehresman connection, induces a splitting of the canonical exact sequence

$$0 \to VP \longrightarrow TP \xrightarrow{dp} p^*TM \to 0,$$

and thus a decomposition $TP = HP \oplus VP$ where $HP \cong p^*TM$ and $VP = \mathrm{Ker}(dp)$ is the space of vertical tangent vectors. Recall that a principal bundle being parallelizable, there is a canonical trivialization of the vertical tangent bundle

$$VP \cong \mathfrak{u}(1)_P := \mathfrak{u}(1) \times P.$$

We suppose that

1. the restriction of the metric g_P on P to VP induces the given fixed metric on $U(1)$.

2. the restriction of the metric g_P to $HP \cong p^*TM$ is induced by pull-back of a lorentzian metric g on M.

The Kaluza-Klein theory has underlying fiber bundle the bundle

$$\boxed{\pi : C = \mathrm{Sym}^2_{KK}(T^*P) \to M}$$

of such metrics on P. The Kaluza-Klein action functional

$$S : \Gamma(M, C) \to \mathbb{R}$$

is given by

$$S(g_P) := \int_P R_{scal}(g_P)\mathrm{vol}_{g_P},$$

where R_{scal} is the scalar curvature.

The scalar curvature then decomposes in coordinates in

$$R_{scal}(g_P) = p^*(R_{scal}(g) - \frac{\Lambda^2}{2}|F|^2)$$

and if on integrates on the fibers, one gets the action

$$S(g_P) := \Lambda \int_M (R(g) - \frac{1}{\Lambda^2}|F|^2)\mathrm{vol}(g).$$

If g is fixed and the action varies with respect to the primitive A of F, one gets Maxwell's theory. If one fixes F and vary the action with respect to g, one gets Einstein's equation for the Einstein-Hilbert action with an electromagnetic term.

15.2 Bosonic gauge theory

We refer to Freed's lectures on supersymmetry [Fre99]. Let M be an oriented manifold, G be a Lie group with Lie algebra $\mathfrak{g}$, and suppose given a G-invariant product $\langle ., . \rangle$ on $\mathfrak{g}$. Let (X, g) be a metric manifold on which G acts by isometries. Let $V : X \to \mathbb{R}$ be a G-invariant potential function. Suppose given a principal G-bundle P. The bundle underlying bosonic gauge theory is the bundle

$$\boxed{\pi : C = \mathrm{Conn}_G(P) \times (P \times_G X) \to M}$$

whose sections are pairs (A, φ) of a principal G-connection on P and of a section φ of the associated bundle $P \times_G X$.

The action functional $S = \Gamma(M, C) \to \mathbb{R}$ is given by

$$S(A, \varphi) = \int_M -\frac{1}{2}|F_A|^2 + \frac{1}{2}|d_A\varphi|^2 - \varphi^*V$$

where d_A is the covariant derivative associated to A.

15.3 Poisson sigma model

We describe the Poisson sigma model because it inspired Kontsevich's construction in [Kon03] of a deformation quantization formula for general Poisson manifolds (see the article by Cattaneo and Felder [CF01], and the survey article [CKTB05], Part III for more details on the Poisson sigma model underlying Kontsevich's ideas on deformation quantization).

Let (X, θ) be a Poisson manifold, i.e., a manifold equipped with a bivector $\theta \in \Gamma(X, \wedge^2 TX)$. Let M be a smooth surface. The bundle underlying this variational problem is the bundle

$$\boxed{\pi : C \to M}$$

whose sections are pairs (x, η) composed of a map $x : M \to X$ and of a differential form $\eta \in \Omega^1(M, x^*T^*X)$ with values in the pull-back by x of the cotangent bundle on X.

The action functional is given by

$$S(x, \eta) = \int_M \langle \eta \wedge Dx \rangle + \frac{1}{2} \langle \eta \wedge x^*\theta(\eta) \rangle$$

where

- $Dx : TM \to x^*TX$ is the differential of x,
- θ is viewed as a morphism $\theta : T^*X \to TX$ so that $x^*\theta$ is a map $x^*\theta : x^*T^*X \to x^*TX$, and
- $\langle , \rangle : T^*X \times TX \to R_X$ is the standard duality pairing.

15.4 Chern-Simons field theory

Let M be a three dimensional manifold and G be a lie group. Suppose given an invariant pairing $\langle , \rangle$ on the Lie algebra $\mathfrak{g}$ of G and a principal G-bundle P. The bundle underlying this variational problem is the bundle

$$\boxed{\pi : C = \mathrm{Conn}_G(P) \to M}$$

of principal G-connections on P. Such a connection is given by a G-invariant differential form

$$A \in \Omega^1(P, \mathfrak{g})^G$$

so that one has an identification

$$\mathrm{Conn}_G(P) = (T^*P \times \mathfrak{g})/G.$$

The Chern-Simons action functional was introduced by Witten in [Wit89] to study topological invariants of three manifold from a quantum field theoretical viewpoint. One says that it is a topological action functional because it does not depend on a metric and it is diffeomorphism invariant. It is given by

$$S(A) = \int_M \mathrm{Tr}(A \wedge dA + \frac{1}{3} A \wedge A \wedge A).$$

One can show that this action is classically equivalent (meaning has same equations of motions) to the Cartan formalism version of Euclidean general relativity in dimension 3. This shows that general relativity in dimension 3 is a purely topological theory.

15.5 Higher Chern-Simons field theory

We refer to [Sch11] for a complete modern treatment of this and other higher topological field theories. We will use here the formalism of non-abelian and differential cohomology presented in Sections 9.3 and 9.4.

We first start with a simple example. Let $G = U(n)$ and $A = U(1)$ be smooth unitary groups, and consider the determinant morphism

$$\det : G = U(n) \to U(1) = A.$$

The induced morphism

$$B\det : BG \to BA$$

corresponds, for each smooth manifold M, to a morphism

$$c_1 : \underline{\mathrm{Hom}}(M, BG) \longrightarrow \underline{\mathrm{Hom}}(M, BA),$$

that may be thought as a smooth geometric refinement of the first Chern class, that is with values in

$$\pi_0(\underline{\mathrm{Mor}}(X, BA)) \cong H^1(X, U(1)) \cong H^2(X, \mathbb{Z}),$$

where the last isomorphism is given by the boundary of the exponential exact sequence. This smooth map c_1 has a differential refinement

$$\hat{c}_1 : BU(n)_{conn} \to BU(1)_{conn}$$

that applies to the classifying spaces of principal bundles with invariant connections.

If M is a one-dimensional compact manifold, the holonomy integral

$$\int_M : \underline{\mathrm{Mor}}(M, BU(1)_{conn}) \longrightarrow U(1)$$

induces an action functional S_{c_1} by the diagram

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(M, BU(n)_{conn}) & \xrightarrow{\hat{c}_1} & \underline{\mathrm{Hom}}(M, BU(1)) \\ & \searrow^{e^{iS_{c_1}}} & \downarrow \int_M \\ & & U(1) \end{array}$$

This thus gives a morphism

$$e^{iS_{c_1}} : \mathrm{B{\scriptstyle UN}}_{conn}(M, U(n)) \to U(1),$$

from the moduli space of $U(n)$ bundles with connections on M, that may be used for a functional integral quantization. Remark that only the exponential of the action functional is canonical in this setting.

The basic idea of higher Chern-Simons field theory is to take advantage of the abstract setting of non-abelian and differential cohomology to generalize the above example by using arbitrary higher groups and higher differential cohomology classes. Let G be a smooth ∞-group with delooping BG, and let $A = U(1)$ (it may actually be an arbitrarily deloopable ∞-group). Let

$$c : BG \to B^n A$$

be a non-abelian cohomology class in $H^n(BG, A)$, and suppose given a differential refinement

$$\hat{c} : BG_{conn} \to B^n A_{conn}.$$

If M is an n-dimensional compact manifold, the higher holonomy integral

$$\int_M : \underline{\mathrm{Hom}}(M, B^n A_{conn}) \to U(1)$$

induces an action function S_c by the diagram

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(M, BG_{conn}) & \xrightarrow{\hat{c}_1} & \underline{\mathrm{Hom}}(M, BU(1)) \; . \\ & \searrow^{e^{iS_c}} & \downarrow \int_M \\ & & U(1) \end{array}$$

Definition 15.5.1. The above defined functional

$$e^{iS_c} : \mathrm{B{\scriptstyle UN}}_{conn}(M, G) \to U(1)$$

on the moduli space of ∞-G-bundles with connections is called the *higher Chern-Simons functional* associated to the refined characteristic class

$$\hat{c} : BG_{conn} \to B^n A_{conn}.$$

15.6 Supersymmetric particle

We refer to Freed's lectures on supersymmetry [Fre99] for the two classical presentations (in component and using superspaces) of the supersymmetric particle.

Let (X, g) be the Minkowski space. The super bundle underlying the supersymmetric particle is the bundle

$$\boxed{\pi : C = X \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}}$$

whose sections are morphisms

$$\varphi : \mathbb{R}^{1|1} \to X.$$

We introduce on $\mathbb{R}^{1|1}$ the three vector fields

$$\partial_t,\ D = \partial_\theta - \theta\partial_t,\ \text{and } \tau_Q = \partial_\theta + \theta\partial_t.$$

The action functional for the supersymmetric particle is the integral

$$S(\varphi) = \int_{\mathbb{R}^{1|1}} -\frac{1}{2}\langle D\varphi, \partial_t\varphi\rangle$$

with respect to the standard super-measure $|dt|d\theta$. The action is invariant with respect to ∂_t, D and τ_Q, so that it is called supersymmetric.

15.7 Superfields

We refer to the IAS volume [DF99], Chapter "Supersolutions", for the description of the following example.

Let M be ordinary Minkowski space, C a positive cone of time-like vectors in V, S a real representation of $\mathrm{Spin}(V)$, and

$$\Gamma : S^* \otimes S^* \to V$$

a pairing which is positive definite in the sense that $\Gamma(s^*, s^*) \in \bar{C}$ for $s^* \in S^*$ with $\Gamma(s^*, s^*) = 0$ only for $s^* = 0$. Associated to this, there is a morphism

$$\tilde{\Gamma} : S \otimes S \to V.$$

Definition 15.7.1. The *super translation group* is the supergroup with underlying space $V^s = V \times \Pi S^*$ and multiplication given by

$$(v_1, s_1^*)(v_2, s_2^*) = (v_1 + v_2 - \Gamma(s_1^*, s_2^*), s_1^* + s_2^*).$$

Its points with value in a super-commutative ring R is the even part of

$$(V + \Pi S^*) \otimes R.$$

The *super Minkowski space* is the underlying manifold of the super translation group. The *super Poincaré group* is the semi-direct product

$$P^s := V^s \rtimes \mathrm{Spin}(V).$$

The super Minkowski space is usually called the "N superspace" if S is the sum of N irreducible real spin representations of $\mathrm{Spin}(V)$.

The bundle underlying superfield gauge theory is the bundle

$$\boxed{\pi : C = \mathrm{Conn}_K(R) \times X \to M^s}$$

with X an ordinary manifold, K a super-group and R a super-K-principal bundle over M^s.

We only give here two examples of Lagrangian action functional on such a super-Minkowski space $M^{4|4}$ that is based on ordinary Minkowski space $\mathbb{R}^{3,1}$ with 4 supersymmetries (be careful, this notation could be misleading: it has nothing to do with super-affine space $\mathbb{A}^{4|4}$; the first 4 is for spacetime dimension 4 and the second 4 is for 4 supersymmetries, i.e., one has 4 copies of the standard real spin representation S of $\mathrm{Spin}(3,1)$).

First consider the case of Chiral superfield φ, that is a section of

$$\boxed{\pi : C = \mathbb{C} \times M^{4|4} \to M^{4|4},}$$

i.e., a supermap $\varphi : M^{4|4} \to \mathbb{C}$. We use the standard volume form on $S = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{C}^2$ and on M^4 to get a supertranslation invariant volume form on $M^{4|4}$. The action functional is given by

$$S(\varphi) = \int_{M^{4|4}} \frac{1}{4}\bar{\varphi}\varphi.$$

Now let X be a Kähler manifold, i.e.,

1. a holomorphic manifold X,

2. equipped with a symplectic form $\omega \in \Omega^2_X$,

such that

$$g(v, w) = \omega(v, Jw)$$

is a Riemannian metric.

We suppose given locally on X a Kähler potential, i.e., a real valued function K on X such that the Kähler form is

$$\omega = i\partial\bar{\partial}K.$$

The Lagrangian action functional applies to sections of the bundle

$$\boxed{\pi : C = X \times M^{4|4} \to M^{4|4},}$$

and is given by

$$S(\varphi) = \int_{M^{4|4}} \frac{1}{2}K(\bar{\varphi}, \varphi).$$

15.8 Bosonic strings

We carefully inform the reader that the action given here is not supposed to be quantized through a functional integral. There is at the time being no proper definition of an hypothetic Lagrangian field theory (called M-theory) whose quantum field theoretic perturbative expansions would give the perturbative series defined *ad-hoc* by string theorists to define quantum string propagation amplitudes.

We refer to Polchinski's book [Pol05] for a complete physical introduction to bosonic string theory. Let M be a two dimensional manifold and let

$$\mathrm{Sym}^2_{min}(T^*M) \to M$$

be the bundle of minkowski metrics on M Let (P, g_P) be a given space with a fixed metric. For example, this could be a Kaluza-Klein $U(1)$-fiber bundle (P, g_P) over ordinary spacetime. The fiber bundle underlying bosonic string theory on a fixed string topology M is the bundle

$$\boxed{\pi : C = P \times \mathrm{Sym}^2_{min}(T^*M) \to M}$$

whose sections are pairs (x, g) composed of a map $x : M \to P$ and a lorentzian metric on M.

The Polyakov action functional $S : \Gamma(M, C) \to \mathbb{R}$ for bosonic strings is given by the formula

$$S(x, g) = \int_M \mathrm{Tr}(g^{-1} \circ x^* g_P) d\mu_g,$$

where

- $g^{-1} : T^*M \to TM$ is the inverse of the metric map g,
- $x^* g_P : TM \to T^*M$ is the inverse image of $g_P : TP \to T^*P$,
- and the trace applies to $g^{-1} \circ x^* g_P \in \mathrm{End}(TM)$.

If one fixes x and varies g, the extremal points for this action are given by $g = x^* g_P$, but in a quantization process, it is important to allow "quantum" fluctuations of the metric g around these extremal points.

One can also add a potential function $V : P \to \mathbb{R}$ to the Lagrangian to get more general bosonic string action functionals.

Remark that the Euclidean version of this theory, with the bundle $\mathrm{Sym}^2_+(T^*M) \to M$ is directly related to the moduli space of curves of a given genus. Indeed, it is the moduli space of curves equipped with a quadratic differential on them. If the curve is non-compact, the moduli spaces in play will be the moduli spaces

$$T^*\mathcal{M}_{g,n} \to \mathcal{M}_{g,n}$$

of quadratic differentials on curves of genus g with n marked points.

15.9 Superstrings

The same remark as in Section 15.8 applies to this section. There are two ways to define classical superstrings. We refer to the introductory book [McM09] for a basic introduction to superstrings and to [Pol98] and [GSW88] for more specialized presentations.

In the Ramond-Neveu-Schwarz formalism, superstrings are a special case of superfields on a Lorentzian curve with N supersymmetries $M^{2|N}$. For $N = 1$, for example, one considers the super-space extension $M^{2|1}$ of the surface M that appear in the description of the Bosonic string in Section 15.8 with two fermionic variables that correspond to the real two dimensional real spinorial representation of $\mathrm{Spin}(1,1)$. The strings are then given by superfields with values in say a complex target manifold X with given local Kähler potential K as in Section 15.7

$$x : M^{2|N} \to X.$$

So the fields of the theory are superfields on the supersymmetric extension of a given surface, meaning that the underlying bundle is

$$\boxed{\pi : C = X \times M^{2|N} \to M^{2|N}.}$$

In the Green-Schwarz formalism, superstrings are given by maps

$$x : M^2 \to X_s^N$$

from a surface to an N-super-Poincaré extension X_s^N of a given Lorentzian manifold X. In this case, the fields are classical fields on a surface but with values in a super-space, meaning that the underlying bundle is

$$\boxed{\pi : C = X_s^N \times M^2 \to M^2.}$$

In the two cases, the space of fields $\Gamma(M, C)$ is a super-space that can be quantized by a functional integral approach.

15.10 Supergravity

We refer to Lott's paper [Lot90] and to Egeileh's thesis [Ege07] for the superspace formulation of supergravity.

The fields of supergravity are sections of the supergravity bundle, fulfilling some torsion constraints. We only define here the underlying bundle, and warn the reader that constraints (that we won't describe here), play an essential role in the theory.

We need the definition of a Cartan super-connection on $M^{4|4}$ for the pair $(H, G) = (\mathrm{Spin}(3,1), P^s)$ composed of ordinary spinor group and super-Poincaré group. It is given by

1. a principal G-bundle Q on $M^{4|4}$,
2. a principal H-sub-bundle $P \subset Q$ of Q,
3. a section $s : M \to Q \times_G G/H$ of the associated bundle,
4. a principal G-connection $A \in \Omega^1(Q, \mathfrak{g})$ on Q,

such that the pull-back s^*A induces an isomorphism

$$e_A : TM^{4|4} \cong Q \times_G V^s$$

called the Cartan coframe, are also sometimes the supervielbein. One has to suppose moreover that the torsion of the given Cartan connection fulfill some first order constraint, for which we refer the reader to Lott's paper cited above.

The bundle underlying super-gravity theory is the bundle

$$\boxed{\pi : C = \mathrm{Conn}_{\mathrm{Cartan},tc}(Q, P, s) \to M^{4|4} = M}$$

whose sections are Cartan connections with torsion constraints.

The super-gravity action functional is then essentially the same as the Cartan version of Einstein-Hilbert-Palatini action given in Section 13.4.2.

Part III

Quantum trajectories and fields

Chapter 16

Quantum mechanics

For historical reasons, we present the first quantum theories as they were thought by Heisenberg, Schrödinger, von Neumann and Dirac, among others. These quantization methods are essentially restricted to systems without interactions, and the most mathematically developed theory based on them is the deformation quantization program that says that for any Poisson manifold (M, π), the Poisson bracket on $\mathcal{O}_M$ can be extended to an associative multiplication operation $*$ on $\mathcal{O}_M[[\hbar]]$ such that one has

$$f * g = fg + \hbar\{f, g\} + O(\hbar^2).$$

This general result is due to Kontsevich [Kon03], and was found using functorial integral methods (applied to a convenient Poisson sigma model). The author's viewpoint is that the covariant methods, using functional integrals, have better functorial properties than the Hamiltonian methods, and generalize better to systems with (complicated) gauge symmetries, that are important in mathematical applications and in theoretical physics.

Remark that these deformation quantization methods can be extended to perturbative covariant quantum field theory (following Fredenhagen and his collaborators). We refer to Chapter 21 for an introduction to this causal approach to perturbative quantum field theory. The deformation quantization methods also generalize to Chiral/factorization algebras (see Beilinson-Drinfeld's book [BD04], Costello's book [Cos11] and Costello-Gwilliam's book [CG10]). We refer to Chapters 20 and 23 for an introduction to these generalized deformation quantization problems.

A hike through the Hamiltonian methods is however useful to better understand the physicists intuitions with quantum fields. The lazy reader can pass directly to Chapter 17. We will use systematically spectral theory, referring to Section 3.4 for some basic background.

16.1 Principles in von Neumann's approach

We recall the von Neumann presentation of Heisenberg/Schrödinger approach to quantum mechanics, that one can find in the still very modern reference [vN96].

Definition 16.1.1. A *quantum mechanical system* is a tuple

$$(\mathcal{H}, \mathcal{A}, I, \mathrm{Hist}, H : \mathcal{H} \to \mathcal{H})$$

made of the following data:

1. an interval I of $\mathbb{R}$ that represents the physical *time parameter*,
2. a Hilbert space $\mathcal{H}$ that plays the role of the *phase space* T^*X of classical Hamiltonian systems, and whose norm 1 vectors are the physical states of the system.
3. A C*-algebra $\mathcal{A} \subset \mathcal{L}(H)$ of (not necessarily continuous) operators on $\mathcal{H}$ called the *observables* of the system.
4. two sets of applications $\mathrm{Hist}_{\mathcal{H}} \subset \mathrm{Hom}(I, \mathcal{H})$ and $\mathrm{Hist}_{\mathcal{A}} \subset \mathrm{Hom}(I, \mathcal{A})$ respectively called the space of *possible evolutions* for the states and observables of the system (analogous to the space of histories of Hamiltonian mechanics).
5. A self-adjoint operator $H : \mathcal{H} \to \mathcal{H}$ called the *Hamiltonian* of the system.

If A is an observable, the elements of its spectrum $\lambda \in \mathrm{Sp}(A) \subset \mathbb{R}$ are the possible *measures* for the value of A, i.e., the real numbers that one can obtain by measuring A with a machine. The spectrum $\mathrm{Sp}(A)$ is thus the *spectrum of possible measures* for the value of A.

From a quantum mechanical system, one defines the space $\mathcal{T} \subset \mathrm{Hist}_{\mathcal{H}}$ of possible trajectories for the states of the system as the solutions of the *Schrödinger equation*

$$\varphi_{t_0} = \varphi \text{ and } \frac{\hbar}{2i\pi}\frac{\partial \varphi_t}{\partial t} = -H\varphi_t.$$

This translates more generally on the evolution of observables $A_. \in \mathrm{Hist}_{\mathcal{A}}$ of the system by

$$i\hbar\frac{\partial A}{\partial t} = [H, A],$$

which corresponds to the naïve quantization of the equation

$$\frac{\partial a}{\partial t} = \{H, a\}$$

of Hamiltonian mechanics obtained by replacing the Poisson bracket by commutators (see Section 16.2).

One can consider that quantum mechanics is based on the following physical principles, that give the link with experiment:

Principle 3. 1. If $A \in \mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is an observable (self-adjoint operator) of the physical system and $[b, c] \subset \mathbb{R}$ is an interval, the probability that one finds a number in the interval $[b, c]$ by measuring with a device the value of $A \in \mathcal{B}(\mathcal{H})$ on the system in the state $\varphi \in \mathcal{H}$ is

$$\mathbb{P}(A, [b, c]) := \sqrt{\langle \varphi, \mathbb{1}_{[b,c]}(A)\varphi \rangle}.$$

2. Two observables can be measured simultaneously if and only if they commute.

3. A measure of the observable A on a state φ changes the state φ in the state $A\varphi$.

4. A projector P is an observable that gives a true or false property for a state of the system.

If R and S are two observables that don't commute, on associates to them the operator P of projection on the kernel M of the commutator $[R, S]$. The projector then corresponds to the property that the two observables are measurable simultaneously. If one supposes that $M = \{0\}$, this means that R and S cannot be measured simultaneously. One can restrict to operators as the P and Q of quantum mechanics that fulfill the relation

$$[P, Q] = \frac{\hbar}{2i\pi}\mathrm{Id}.$$

One can then evaluate the dispersion of the measures of P and Q on a state φ by setting for $R = P, Q$,

$$d(R, \varphi) = \|R\varphi - \langle R\varphi, \varphi\rangle\varphi\|.$$

This shows the difference between the state $R\varphi$ obtained after the measure R and the state $\langle R\varphi, \varphi\rangle\varphi$ essentially equivalent to φ (the physical states being unitary). One then gets the Heisenberg uncertainty principle:

$$d(P, \varphi).d(Q, \varphi) \geq \frac{\hbar}{4\pi}$$

that means that the less the value of P on φ changes the state φ, the more the measure of the value of Q on φ change the state φ of the given system. This can also be translated by saying that the more your measure P with precision, the less you can measure Q with precision.

16.2 Canonical quantization of Heisenberg/Schrödinger

There is a strong analogy between quantum mechanical systems and classical Hamiltonian systems, given by the following array:

	CLASSICAL	QUANTUM
System	$(P, \pi, I, \mathrm{Hist}, H)$	$(\mathcal{H}, \mathcal{A}, I, \mathrm{Hist}, H)$
States	$x \in P$	$x \in \mathcal{H},\ \|x\| = 1$
Observables	$a \in C^\infty(P)$	$a \in \mathcal{A}$
Bracket	$\{a, b\}$	$[a, b]$
Hamiltonian	$H \colon X \to \mathbb{R}$	$H : \mathcal{H} \to \mathcal{H}$ self-adjoint $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$
Time evolution	solution of the Hamiltonian vector field $\xi_H = \{H, .\}$	$\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$ $e^{itH}\pi(a)e^{-itH} = \pi(\sigma_t(a))$

If a classical Hamiltonian system $(P = T^*X, \pi, I, \mathrm{Hist}, H)$ is given, one can define a quantum mechanical system $(\mathcal{H}, \mathcal{A}, I, \mathrm{Hist}, H)$ by the following method:

1. one sets $\mathcal{H} = L^2(X)$,
2. to every function $q \in \mathcal{C}^\infty(X)$ one associates the operator/observable $Q = m_q$ of multiplication by q in $\mathcal{H}$. The coordinate functions on $X = \mathbb{R}^n$ are denoted q_i, and the corresponding operators, Q_i.
3. one associates to a vector field $v \in \theta_X$ the operator $P_v(-) = \frac{\hbar}{2i\pi} d_v(-)$ of Lie derivative along the vector field v given by

$$d_v(f) = Df \circ v : X \to \mathbb{R}.$$

   The derivations along coordinates are denoted P_i.
4. More generally, one would like to have a Lie algebra morphism

$$\varphi : (\mathcal{A}, \{.,.\}) \to (\mathrm{End}(\mathcal{H}), [.,.]),$$

   where $\mathcal{A} \subset \mathcal{C}^\infty(T^*X)$ is the algebra of interesting observables, that is supposed in particular to contain

   - the classical Hamiltonian H,
   - the coordinate functions q_i, and
   - the derivatives with respect to the q_i, that one can also see as the applications $\dot{q}_i(x, \omega) = \omega(\dot{q}_{i_x})$.

   Such a Lie algebra morphism, called a quantization, would allow to quantize all the useful algebras on $P = T^*X$ and give back the above explicitely given constructions as particular cases, to compute the bracket between observable quantities by setting

$$[A, B] = \frac{\hbar}{i}\{a, b\}.$$

   This problem is clarified and solved by the theory called deformation quantization (see [Kon03] for the main result and Chapter 22 for an overview of the proof).
5. In particular, one gets the quantum Hamiltonian by replacing, when possible, the variables q_i and p_i on T^*X by the corresponding operators Q_i and P_i.

The probability amplitude that a state $f \in \mathcal{H}$ of the system is situated in a Borelian B of the space X is given by the value

$$\mathbb{A}_B(f) := \int_B |f|^2 d\mu.$$

One can thus also think of the quantum states in $\mathcal{H}$ as wave functions for some given particles.

16.3 Algebraic canonical quantization

We now turn on to the algebraic approach to canonical quantization, that was invented in the fermionic case by Dirac [Dir82], and for which we also refer to Section 4.4 on Clifford algebras. This section can be seen as an algebraic explanation of the fact that the so-called "first canonical quantization" of classical Newtonian particles give the free field equations of field theory, that we will use later.

The quantization of the algebra of polynomial functions on the cotangent bundle $P = T^*X$ of a Riemannian manifold (X, g), that is the phase space for a particle $x : [0, 1] \to X$ moving in X, can be given by the algebra of differential operators $\mathcal{D}_X$. This algebra acts on $L^2(X)$ and has a filtration F whose graded algebra is the algebra

$$\mathrm{gr}^F\mathcal{D}_X = \mathcal{O}_P$$

of functions on phase space. Moreover, the bracket on $\mathcal{D}_X$ induces the Poisson bracket on $\mathcal{O}_P$. The quantum Hamiltonian can thus be seen as a differential operator $H \in \mathcal{D}_X$ acting on the state space $L^2(X)$.

In the case of a fermionic particle $x : \mathbb{R}^{0|1} \to X$, the phase space is the supermanifold $P = T^*[1]X$ with coordinate algebra $\mathcal{O}_P = \wedge^*\Theta_X$. The Clifford algebra $\mathrm{Cliff}(\Theta_X, g)$ gives a nice canonical quantization of $\mathcal{O}_P$, since it also has a filtration whose graded algebra is the algebra

$$\mathrm{gr}^F\mathrm{Cliff}(TX, g) = \wedge^*\Theta_X = \mathcal{O}_P$$

of functions on phase space. Moreover, the bracket on $\mathrm{Cliff}(\Theta_X, g)$ induces the Schouten-Nijenhuis super Poisson bracket on $\mathcal{O}_P$. The quantum Hamiltonian is then an element $H \in \mathrm{Cliff}(\Theta_X, g)$, and it acts naturally on the complex spinor representation S of the Clifford algebra, that is the state space of the quantum fermionic particle.

16.4 Weyl quantization and pseudo-differential calculus

Let X be a manifold equipped with a metric $g : TX \times_X TX \to \mathbb{R}_X$. One can think of (X, g) as a solution of Einstein's equations with g of Minkowski's signature. To perform the Weyl quantization, one needs a Fourier transform and a linear structure on spacetime. The easiest way to replace a curved spacetime by a flat one is to work in the tangent space of spacetime at a given point, which is the best linear approximation to spacetime. This procedure may be implemented locally by using the exponential map associated to the given metric. This remark is very important, since it shows that from a mathematical viewpoint,

it does not really makes sense to combine gravity and quantum mechanics: one of them lives on spacetime, and the other lives in the fiber of the tangent bundle of spacetime at a given point.

We denote (x,q) the local coordinates on TX and (x,p) the local coordinates on T^*X. We denote

$$\langle .,.\rangle : T^*X \times_X TX \to \mathbb{R}_X$$

the natural pairing given by

$$\langle (x,p),(x,q)\rangle = (x,p(q)).$$

One has a fiber-wise Fourier transform

$$\mathcal{F} : \mathcal{S}(T^*X) \to \mathcal{S}(TX)$$

on the Schwartz spaces (functions with compact support on the base X and rapid decay along the fibers equipped with their linear Lebesgue measure) given by $f \in \mathcal{S}(T^*X)$ by

$$\mathcal{F}(f)(x,q) = \int_{T_x^*X} e^{i\langle p,q\rangle} f(x,p)dp.$$

If one gives a symbol $H(x,p,q) \in \mathcal{S}(T^*X \times_X TX)$ (for quantization, this will typically be the Hamiltonian, that depends only on p et q and not on x) and a function $f(x,q)$ of $\mathcal{S}(TX)$, one can associate to it the product $H.(f \circ \pi) \in \mathcal{S}(T^*X \times_X TX)$ where $\pi : T^*X \times_X TX \to TX$ is the natural projection.

Definition 16.4.1. We denote $L^2_{\infty,c}(TX)$ the space of functions on TX that are smooth with compact support on X and square integrable in the fibers of $TX \to X$. The operator $\hat{H}$ on $L^2_{\infty,c}(TX)$ associated to H is then given by the formula

$$(\hat{H}.f)(x,q) := \int_{T_xX}\int_{T_x^*X} e^{i\langle p,q-q_0\rangle} H(x,p,q).f(x,q_0)d\mu_g(q_0)d\mu_{g^*}(p).$$

This map

$$\begin{array}{ccc} \mathcal{S}(T^*X \times_X TX) & \to & \mathcal{L}(L^2_{\infty,c}(TX)) \\ H & \mapsto & \hat{H} \end{array}$$

is called *Weyl quantization* or infinitesimal pseudo-differential calculus.

The (non-canonical) relation with ordinary pseudo-differential calculus is the following: if $U \subset \mathbb{R}^n$ is an open and $h : T^*U \cong U \times (\mathbb{R}^n)^* \to \mathbb{R}$ is a symbol $h(x,p)$ on U, one associates to it the function

$$H : T^*U \times_U TU \cong U \times (\mathbb{R}^n)^* \times \mathbb{R}^n \to \mathbb{R}$$

by replacing x by q in $h(x,p)$, i.e., by the formula

$$H(x,p,q) := h(p,q).$$

This is obtained geometrically by identifying U with a subspace of the fiber of its tangent bundle $\mathbb{R}^n$, by the implicit function theorem (this space is a plane in $\mathbb{R}^n \times \mathbb{R}^n$). One gets an operator $\hat{H}$ on $L^2_{\infty,c}(TU)$, that is constant on U and thus gives an operator on the fibers, that are $L^2(\mathbb{R}^n)$.

In practice, one often supposes $X = \mathbb{R}^n$, which gives $TX = \mathbb{R}^n \times \mathbb{R}^n$ and one can then consider the functions on TX that depend only on q in the coordinates (x, q) on TX. The choice of this decomposition is clearly not canonical, but Weyl quantization is often described as the map

$$\begin{array}{ccc} \mathcal{S}(T^*X) & \to & \mathcal{L}(L^2(X)) \\ H & \mapsto & \hat{H}. \end{array}$$

With a convenient normalization of the Fourier transform, the Weyl quantization gives back the Heisenberg quantization described in Section 16.2. Indeed, a linear coordinate is a function $q_n : TX \to \mathbb{R}$ (coordinate in the fiber of TX) that one can compose with the projection $\pi_T : T^*X \times_X TX \to TX$, which gives by quantization an operator

$$Q_n := \widehat{q_n \circ \pi_T}$$

in $\mathcal{C}^\infty(X, \mathcal{L}(L^2_{\infty,c}(TX)))$ that is simply the multiplication by the coordinate q_n in the fibers.

Given a vector field $\vec{v}_n \in \Theta_X = \Gamma(X, TX)$ (derivation in the x_n coordinate direction on X, and not on TX, this time: it is at this point that the identification between coordinates on X and on TX, $x_n = q_n$ used by physicists, is non-canonical), one can see this as the application $p_{n,h} : T^*X \to \mathbb{C}$, given by $p_{n,h}(x,p) = \frac{h}{i}\langle p, \vec{v}_n\rangle$, and composable with the projection $\pi_{T^*} : T^*X \times_X TX \to T^*X$, that gives by quantization an operator

$$P_n := \widehat{p_{n,h} \circ \pi_{T^*}}$$

in $\mathcal{C}^\infty(X, \mathcal{L}(\mathcal{S}(TX)))$ that is nothing else than $P_n = \frac{1}{2i\pi}\frac{\partial}{\partial q_n}$.

16.5 Quantization of the harmonic oscillator

The quantum harmonic oscillator is at the base of the free quantum field theory. Its resolution by the ladder method (creation and annihilation operators) is due to Dirac [Dir82].

The classical harmonic oscillator on $X = \mathbb{R}$ is given by the Hamiltonian system whose trajectories are maps from $\mathbb{R}$ into the Poisson manifold $P = T^*X \cong \mathbb{R} \times \mathbb{R}$ (with coordinates (q, p) called position and impulsion) and whose Hamiltonian is the function $H : P \to \mathbb{R}$ given by

$$H(q,p) = \frac{1}{2m}(p^2 + m^2\omega^2 q^2).$$

Recall that the classical solution of the Hamilton equations are then given by

$$\left\{ \begin{array}{ccc} \frac{\partial q}{\partial t} & = & \frac{\partial H}{\partial p} = \frac{p}{m} \\ \frac{\partial p}{\partial t} & = & -\frac{\partial H}{\partial q} = m\omega^2 q, \end{array} \right.$$

which gives the Newton equations of motion

$$\frac{\partial^2 q}{\partial t^2} = \omega^2 q.$$

Canonical quantization of this system is obtained by working with the Hilbert space $\mathcal{H} = L^2(X)$ and replacing the position and impulsion by the operators $Q = m_q$ of multiplication by the function $q : X \to \mathbb{R}$ and $P = -i\hbar\partial_q$ of derivation with respect to q.

One then compute the spectrum of the Hamiltonian (whose elements are possible energy levels for the system)

$$H(Q,P) = \frac{1}{2m}(P^2 + m^2\omega^2 Q^2),$$

which also means that one solves the eigenvalue equation

$$H\psi = E\psi.$$

To find the solutions of this eigenvalue equation, one uses, following Dirac's ladder method, two auxiliary operators

$$\begin{aligned} a &= \sqrt{\frac{m\omega}{2\hbar}}\left(Q + \frac{i}{m\omega}P\right) \\ a^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}\left(Q - \frac{i}{m\omega}P\right) \end{aligned}$$

This gives the relations

$$\begin{aligned} x &= \sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right) \\ p &= i\sqrt{\frac{\hbar m\omega}{2}}\left(a^\dagger - a\right) \end{aligned}$$

Theorem 16.5.1. *One can solve the eigenvalue equation for the operator H by using the operators a and $a^\dagger$. More precisely,*

1. *one has $H = \hbar\omega\left(a^\dagger a + 1/2\right)$.*
2. *The spectrum of $a^\dagger a$ is the set $\mathbb{N}$ of natural numbers and the spectrum of H is $\hbar\omega(\mathbb{N} + 1/2)$.*
3. *If ψ_0 is nonzero and such that $a\psi_0 = 0$, we denote $\psi_n = (a^\dagger)^n\psi_0$ and $\psi_0 = |\emptyset\rangle$ is called the empty state. One then has*
$$H\psi_n = \hbar\omega(n + 1/2)\psi_n.$$
4. *One has the commutation relations*
$$\begin{aligned} \left[a, a^\dagger\right] &= 1 \\ \left[a^\dagger, H\right] &= \hbar\omega a^\dagger \\ [a, H] &= -\hbar\omega a \end{aligned}$$

Remark that the empty state ψ_0 has as energy (eigenvalue) $\frac{\hbar\omega}{2}$ and is given by the function (solution of the differential equation in q of first order $a\psi_0 = 0$)

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}.$$

16.6 Canonical quantization of the free scalar field

We refer to the Folland's book [Fol08] for a very complete treatment of this issue.

The free scalar field of mass m is a function on Minkowski space $X = \mathbb{R}^{3,1}$, that is solution of the Klein-Gordon equation

$$\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\varphi = \left(\frac{1}{c^2}\partial_t^2 - \partial_x^2 + \frac{m^2c^2}{\hbar^2}\right)\varphi = 0.$$

The partial Fourier transform on the variable x of $\varphi(t,x)$, denoted $(\mathcal{F}_x\varphi)(t,\xi)$ fulfills, at each point $\xi \in \mathbb{R}^3$, the differential equation of the classical harmonic oscillator

$$\left(\frac{1}{c^2}\partial_t^2 + \xi^2 + \frac{m^2c^2}{\hbar^2}\right)(\mathcal{F}_x\varphi)(t,\xi) = 0$$

whose classical Hamiltonian is the function $h(q,p) = p^2 + \omega^2 q^2$ with, for ξ fixed, $\omega^2 = \xi^2 + \frac{m^2c^4}{\hbar^2}$, $q = (\mathcal{F}_x\varphi)(t,\xi)$ and $p = \frac{\partial q}{\partial t}$.

The quantification of the free scalar field can be essentially done by replacing the classical harmonic oscillator by a quantum harmonic oscillator with Hamiltonian $H_\xi(Q,P) = P^2 + \omega^2 Q^2$ acting on $L^2(\mathbb{R}^3)$. We remark that this Hamiltonian depends on $\xi \in \mathbb{R}^3$ that is the Fourier dual variable to the position variable x in space.

There exists a nice way to formalize the above construction more canonically. One works in relativistic units, i.e., $c = \hbar = 1$. Let φ be a tempered distribution on $\mathbb{R}^4$ that solves the Klein-Gordon equation

$$(\Box + m^2)\varphi = 0.$$

Its Lorentz-covariant Fourier transform is a tempered distribution that fulfills $(-p^2 + m^2)\hat{\varphi} = 0$ so that it is supported on the two fold hyperboloid

$$X_m = \{p \in \mathbb{R}^{3,1} | p^2 = m^2\}$$

called the mass hyperboloid. One can decompose X_m in two components X_m^+ and X_m^- corresponding to the sign of the first coordinate p_0 (Fourier dual to time). One puts on X_m^+ the normalized measure

$$d\lambda = \frac{d^3\mathbf{p}}{(2\pi)^3\omega_{\mathbf{p}}}$$

with $\omega_{\mathbf{p}} = p_0 = \sqrt{|\mathbf{p}|^2 + m^2}$. One then considers the Hilbert space

$$\mathcal{H} = L^2(X_m^+, \lambda).$$

Definition 16.6.1. The *total (resp. bosonic, resp. fermionic) Fock space* on $\mathcal{H}$, denoted $\mathcal{F}(\mathcal{H})$ (resp. $\mathcal{F}_s(\mathcal{H})$, $\mathcal{F}_a(\mathcal{H})$) is the completed tensor algebra $\mathcal{F}^0(\mathcal{H})$ (resp. symmetric algebra $\mathcal{F}_s^0(\mathcal{H}) := \mathrm{Sym}_{\mathbb{C}}(\mathcal{H})$, resp. exterior algebra $\mathcal{F}_a^0(\mathcal{H}) := \wedge_{\mathbb{C}}\mathcal{H}$). The subspaces given by the algebraic direct sums $\mathcal{F}^0$, $\mathcal{F}_s^0$ et $\mathcal{F}_a^0$ are called *finite particle subspaces.*

One defines the number operator N on $\mathcal{F}^0(\mathcal{H})$ by

$$N = kI \text{ sur } \otimes^k \mathcal{H}.$$

For $v \in \mathcal{H}$, one defines the operators B and $B^\dagger$ on $\mathcal{F}^0(\mathcal{H})$ by

$$\begin{aligned} B(v)(u_1 \otimes \cdots \otimes u_k) &= \langle v|u_1\rangle.u_2 \otimes u_k, \\ B(v)^\dagger(u_1 \otimes \cdots \otimes u_k) &= v \otimes u_1 \otimes \cdots \otimes u_k, \end{aligned}$$

and one shows that these operators are adjoint for $v \in \mathcal{H}$ fixed. The operator $B(v)^\dagger$ does not preserve the symmetric subspace $\mathcal{F}^0_s(\mathcal{H})$ but $B(v)$ preserves it. One also defines $A(v) = B(v)\sqrt{N}$ and denotes $A^\dagger(v)$ the adjoint of the operator $A(v)$ on the bosonic Fock space $\mathcal{F}^0_s(\mathcal{H})$. One can also see it as the operator

$$A(v)^\dagger = P_s B(v)^\dagger \sqrt{N+1} P_s$$

with $P_s : \mathcal{F}^0(\mathcal{H}) \to \mathcal{F}^0(\mathcal{H})$ the projection on the bosonic subspace $\mathcal{F}^0_s(\mathcal{H})$.

One has the following commutation relations of operators on the bosonic Foch space: $[A(v), A(w)^\dagger] = \langle v|w\rangle I$ and $[A(v), A(w)] = [A(v)^\dagger, A(w)^\dagger] = 0$.

One has a natural map $R : \mathcal{S}(\mathbb{R}^4) \to \mathcal{H}$ defined by

$$Rf = \hat{f}_{|X_m^+}$$

with $\hat{f}(p) = \int e^{ip_\mu x^\mu} f(x) d^4x$ the Lorentz-covariant Fourier transform.

Definition 16.6.2. The *neutral free quantum scalar field of mass m* is the distribution on $\mathbb{R}^4$ valued in operators on the bosonic Foch finite particle space $\mathcal{F}^0_s(\mathcal{H})$ given by

$$\Phi(f) = \frac{1}{\sqrt{2}} \left[A(Rf) + A(Rf)^\dagger\right].$$

Remark that Φ is a distribution solution of the Klein-Gordon equation, i.e., $\Phi((\Box + m^2)f) = 0$ for every $f \in \mathcal{S}(\mathbb{R}^4)$ because $(-p^2 + m^2)\hat{f} = 0$ on X_m^+.

For charged scalar field (i.e. with complex values), it is necessary to use the two Hilbert spaces $\mathcal{H}_+ = L^2(X_m^+)$ and $\mathcal{H}_- = L^2(X_m^-)$ and the anti-unitary operator $C : \mathcal{H}_\pm \to \mathcal{H}_\mp$ given by $Cu(p) = u(-p)^*$. One defines the creation and annihilation operators for $v \in \mathcal{H}_-$ on $\mathcal{H}_+$ by

$$B(v) = A(Cv) \text{ et } B(v)^\dagger = A(Cv)^\dagger.$$

Now here comes the trick. One considers as before the Fock space $\mathcal{F}^0_s(\mathcal{H}_+)$ and see it as defined over $\mathbb{R}$ by considering $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathcal{F}^0_s(\mathcal{H}_+)$. The operators $B(v)$ and $B(v)^\dagger$ are $\mathbb{R}$-linear because the operator $C : \mathcal{H}_\pm \to \mathcal{H}_\mp$ is, and that's why one takes the scalar restriction.

One then extends the scalars from $\mathbb{R}$ to $\mathbb{C}$ by applying $.\otimes_\mathbb{R}\mathbb{C}$ to get a space that we will denote $\mathcal{F}^0_s$, and that can be decomposed as a tensor product of two Foch spaces isomorphic to $\mathcal{F}^0_s(\mathcal{H}_+)$, and exchanged by an automorphism induced by complex conjugation. One then defines, for $f \in \mathcal{S}(\mathbb{R}^4)$, two maps $R_\pm : \mathcal{S}(\mathbb{R}^4) \to \mathcal{H}_\pm$ by

$$R_\pm f = \hat{f}_{|X_m^\pm}.$$

Definition 16.6.3. The *charged free scalar field of mass m* is the data of the two distributions on $\mathbb{R}^4$ with values in operators on the Foch space $\mathcal{F}_s^0$ defined by

$$\begin{array}{rcl}\Phi(f) & = & \frac{1}{\sqrt{2}}\left[A(R_+f) + B(R_-f)^\dagger\right], \\ \Phi(f)^\dagger & = & \frac{1}{\sqrt{2}}\left[A(R_+f)^\dagger + B(R_-f)\right].\end{array}$$

To quantize non-free fields, one uses perturbative methods by expressing the Hamiltonian $H = F + \lambda V$ as a sum of a free (quadratic) part, that can be quantized freely as before and a potential λV, with λ a formal variable. One then uses the free state space (Fock space) of the previous section and the action of some operators constructed from the free field using the potential. This method is purely perturbative (in that the parameter λ is purely formal). More on this can be found in Folland's book [Fol08].

We prefer to use covariant and functional integral methods, because

- they are closer to geometry and allow a uniform treatment of fermionic and bosonic variables, starting from classical Lagrangian field theory, and
- they allow a very general treatment of gauge symmetries.

Chapter 17

Mathematical difficulties of perturbative functional integrals

In this chapter, we authorize ourselves to use the physicists' notations

and some of their problematic computations,

describing where are the mathematical problems, so that one has a good idea of what needs to be done to solve them.

The author's viewpoint is that functional integrals (and particularly the nonperturbative renormalization group and Dyson-Schwinger equations, to be described in Chapter 19) give the description of interacting quantum fields that is presently the easiest entry door for mathematicians into the modern applications of quantum field theory methods in mathematics (and in particular, into Witten's work).

The road to such a theory is tricky but paved by physicists, at least in each and all of the examples of physical relevance. It thus gives an interesting area of research for mathematicians. It seems impossible to explain quantum gauge fields to a mathematician without using this clumsy road. We will see later in the renormalization Chapters 18, 19, 20 and 21 that one can get at the end to proper mathematical definitions in some interesting examples.

17.1 Perturbative expansions in finite dimension

The main idea of perturbative methods in quantum field is to use explicit computations of gaussian integrals in finite dimension as definitions of the analogous gaussian integrals in field theory. We thus review perturbative computations of gaussian integrals in finite dimension.

17.1.1 Gaussian integrals with source

To understand the notations used by physicists, it is good to understand their normalizations. One can however use other normalization that the ones commonly used in physics to escape the infiniteness problem of normalization of gaussian integrals in infinite dimension. This is explained in details in Cartier and DeWitt-Morette's book [CDM06] and expanded to the case of fields in J. LaChapelle's thesis [LaC06].

We essentially follow here the excellent presentation of Chapter 1.2.2 of Zinn-Justin's book [ZJ05]. Let S be a quadratic form on $\mathbb{R}^d$ and $\Delta = S^{-1}$ its inverse (that we will later call the propagator or the kernel), that induces a quadratic form on $(\mathbb{R}^d)^\vee$. For $b \in (\mathbb{R}^d)^\vee$ a linear form, one denotes

$$Z(S,b) = \int_{\mathbb{R}^d} \exp\left(-\frac{1}{2}S(x) + b.x\right) dx.$$

The change of variable $x = y + \Delta.b$ gives that this integral is

$$Z(S,b) = \exp\left(\frac{1}{2}\Delta(b)\right) \int_{\mathbb{R}^d} \exp\left(-\frac{1}{2}S(y)\right) = \exp\left(\frac{1}{2}\Delta(b)\right) Z(S,0).$$

The value of $Z(S,0)$ is obtained by diagonalizing from the dimension 1 value and gives

$$Z(S,0) = \frac{(2\pi)^{d/2}}{\sqrt{\det(S)}}.$$

One remarks that the limit as d goes to infinity of $Z(S,b)$ makes no sense but the limit of the quotient $\frac{Z(S,b)}{Z(S,0)}$ can exist, because one has the formula

$$\frac{Z(S,b)}{Z(S,0)} = \exp\left(\frac{1}{2}\Delta(b)\right).$$

If F is a function on $\mathbb{R}^n$, we denote

$$Z_F(S,b) := \int_{\mathbb{R}^d} F(x) d\mu_{S,b}(x)$$

where

$$d\mu_{S,b}(x) := \frac{\exp\left(-\frac{1}{2}S(x) + b.x\right)}{Z(S,0)} d\mu(x)$$

is the Laplace transform of the gaussian measure and the gaussian mean value of F is given by

$$\langle F \rangle := Z_F(S,b)|_{b=0} = \frac{\int_{\mathbb{R}^n} F(x) e^{-\frac{1}{2}S(x)} dx}{Z(S,0)}.$$

Remark in particular that

$$\langle \exp(b.x) \rangle = \frac{Z(S,b)}{Z(S,0)}.$$

If we denote $b.x = \sum b_i.x_i$, we can find the mean values of monomials by developing $\exp(b.x)$ in power series:

$$\langle x_{k_1} \dots x_{k_r} \rangle = \left[\frac{\partial}{\partial b_{k_1}} \dots \frac{\partial}{\partial b_{k_r}} \frac{Z(S,b)}{Z(S,0)}\right]\Bigg|_{b=0} = \left[\frac{\partial}{\partial b_{k_1}} \dots \frac{\partial}{\partial b_{k_r}} \exp(\frac{1}{2}\Delta(b))\right]\Bigg|_{b=0}.$$

Each partial derivative $\frac{\partial}{\partial b_{k_i}}$ can either take a linear factor b_i out, or differentiate one that is already present. The result is zero if there are the same number of these two operations, so that

$$\langle x_{k_1} \dots x_{k_r} \rangle = \sum_{\text{index pairings}} w_{i_{j_1}, i_{j_2}} \dots w_{i_{j_{l-1}}, i_{j_l}}$$

with $w_{i,j}$ the coefficients of the matrix w. This formula is called Wick's formula.

More generally, if F is a power series in the x_i's, one finds the identity

$$\langle F \rangle = \left[F\left(\frac{\partial}{\partial b}\right) \exp\left(\frac{1}{2}\Delta(b)\right)\right]\Big|_{b=0}.$$

This is this mean value function that physicists generalize to infinite dimensional situations. The auxiliary linear form b is often denoted J and called the external source.

There is even another way to look at this mean value, that allows to replace the linear pairing $b.x$ by an arbitrary function of x. This type of viewpoint may be useful when one works on a non-linear space X, instead of $\mathbb{R}^d$. Consider the partition function as a (partially defined) function

$$Z : \mathcal{O}(\mathbb{R}^d) \to \mathbb{R}$$

on functions on $\mathbb{R}^d$ given for $S \in \mathcal{O}(\mathbb{R}^d)$ by the formula

$$Z(S) = \int_{\mathbb{R}^d} \exp\left(S(x)\right) dx.$$

This formula has a meaning even if the space parametrizing the variable x is nonlinear. The relation with the usual partition function Z_{lin} for $x \in \mathbb{R}^d$ is given by

$$Z_{lin}(b) = Z(S + b).$$

One may then see the mean value of $f \in \mathcal{O}(\mathbb{R}^d)$ with respect to the quadratic form S as the derivative

$$\langle F \rangle_S = \frac{\delta Z}{\delta F}(S) := \frac{d}{d\epsilon}\Big|_{\epsilon=0} Z(S + \epsilon F).$$

This means that the observable F may be seen as a tangent vector to the space of functions at the point given by S. This viewpoint is at the basis of Costello and Gwilliam's constructions [CG10].

17.1.2 The defining equation of the Fourier-Laplace transform

Another take at the computation of the gaussian integrals is given by the use of their defining equation. The analog in quantum field theory of this equation is called the Dyson-Schwinger equation and is very practical for the definition of Fermionic or graded functional integrals (whose variables are, for example, sections of an odd fiber bundle of spinors on spacetime or sections of a graded bundle obtained by applying the BV formalism to an ordinary action functional).

Let F be function on $\mathbb{R}^n$ with rapid decay at infinity. One then has, for all $i = 1, \ldots, n$, the integration by part fundamental identity

$$\int_{\mathbb{R}^n} \frac{\partial}{\partial x_i} F(x) dx = 0.$$

Applying this to the definition of $Z(S, b)$, we get

$$Z_i(S, b) := \int_{\mathbb{R}^d} \frac{\partial}{\partial x_i} \exp\left(-\frac{1}{2}S(x) + b.x\right) dx = 0,$$

and

$$Z_i(S, b) = \int_{\mathbb{R}^d} \frac{\partial}{\partial x_i} \left(-\frac{1}{2}S(x) + b.x\right) \exp\left(-\frac{1}{2}S(x) + b.x\right) dx.$$

The basic property of the Laplace transform is that

$$\int_{\mathbb{R}^d} F(x)G(x) \exp(b.x) dx = F\left[\frac{\partial}{\partial b}\right] \int_{\mathbb{R}^d} G(x) \exp(b.x) dx.$$

Applying this to the above computation, we find

$$Z_i(S, b) = \frac{\partial}{\partial x_i} \left(-\frac{1}{2}S(x) + b.x\right) \left[\frac{\partial}{\partial b}\right] Z(S, b),$$

where the bracket means replacing a function (symbol) of x by the corresponding differential operator in the universal derivation $\frac{\partial}{\partial b}$. Finally, the defining equations for the gaussian with source $Z(q, b)$ are given, for $i = 1, \ldots, d$, by

$$\left(-\frac{1}{2}\frac{\partial S}{\partial x_i} + b_i\right) \left[\frac{\partial}{\partial b}\right] Z(S, b) = 0.$$

Now if F is a function on $\mathbb{R}^n$, by applying the fundamental identity of integration by part and the basic property of the Laplace transform, we find the defining equations for the gaussian mean value with source

$$Z_F(S, b) := \int_{\mathbb{R}^d} F(x) \frac{\exp\left(-\frac{1}{2}S(x) + b.x\right)}{Z(S, 0)} d\mu(x).$$

These are given, for $i = 1, \ldots, d$, and if one already has found $Z(S, b)$, by

$$\frac{\partial F}{\partial x_i} \left[\frac{\partial}{\partial b}\right] Z(S, b) + \left(-\frac{1}{2}\frac{\partial q}{\partial x_i} + b_i)\right) \left[\frac{\partial}{\partial b}\right] Z_F(S, b) = 0.$$

17.1.3 Perturbative gaussians

We refer to the introductory article [Phi01]. The aim of Feynman diagrams is to encode the combinatorics of Wick's formula for the value of mean values

$$\langle F \rangle = \left[F\left(\frac{\partial}{\partial b}\right) \exp\left(\frac{1}{2}\Delta(b)\right)\right]\Big|_{b=0}$$

of functions for a gaussian measure.

To compute these mean values in the case where F is a monomial, one develops the exponential $\exp\left(\frac{1}{2}\Delta(b)\right)$ in formal power series:

$$\exp\left(\frac{1}{2}\Delta(b)\right) = \sum_n \frac{\Delta(b)^n}{2^n n!} = \sum_n \frac{1}{2^n n!}\left(\sum_{i,j=1}^d \Delta_{i,j} b_i b_j\right)^n .$$

The terms of this series are homogeneous in the b_i of degree $2n$. If one differentiates k times such a monomial, and evaluate it at 0, on gets 0, except if $k = 2n$. It is thus enough to compute the value of $2n$ differentiations on a monomial of the form $(\sum_{i,j=1}^d \Delta_{i,j} b_i b_j)^n$.

The differentiation most frequently used in this computation is

$$\frac{\partial}{\partial b_k}\left(\frac{1}{2}\sum_{i,j=1}^d \Delta_{i,j} b_i b_j\right) = \sum_{i=1}^d \Delta_{i,j} b_i,$$

where one uses the symmetry of the matrix of quadratic form Δ, inverse of the quadratic form S. We denote $\partial_i = \frac{\partial}{\partial b_i}$.

Let's compute on examples:

1. The term of order $n = 1$ gives

$$\langle x_1 x_2 \rangle = \partial_2 \partial_1 \left(\frac{1}{2}\sum_{i,j=1}^d \Delta_{i,j} b_i b_j\right) = \partial_2\left(\sum_j \Delta_{1,j} b_j\right) = \Delta_{1,2}$$

by using the symmetry of the matrix Δ, and

$$\langle x_1 x_1 \rangle = \partial_1 \partial_1 \left(\frac{1}{2}\sum_{i,j=1}^d \Delta_{i,j} b_i b_j\right) = \Delta_{1,1}.$$

2. The term of order $n = 2$ gives

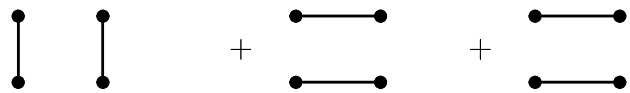
$$\begin{array}{rcl}
\langle x_1 x_2 x_3 x_4 \rangle & = & \partial_4 \partial_3 \partial_2 \partial_1 \left(\frac{1}{2^2 2!}\sum_{i,j=1}^d \Delta_{i,j} b_i b_j\right)^2 \\
& = & \partial_4 \partial_3 \partial_2 \left(\frac{1}{2}(\sum \Delta_{i,j} b_i b_j)(\sum \Delta_{1,j} b_j)\right) \\
& = & \partial_4 \partial_3 \left[(\sum \Delta_{2,j} b_j)(\sum \Delta_{1,j} b_j) + \frac{1}{2}(\sum \Delta_{i,j} b_i b_j)\Delta_{1,2}\right] \\
& = & \partial_4 \left[\Delta_{2,3}(\sum \Delta_{1,j} b_j) + \Delta_{1,3}(\sum \Delta_{2,j} b_j) + \Delta_{1,2}(\sum \Delta_{3,j} b_j))\right] \\
& = & \Delta_{2,3}\Delta_{1,4} + \Delta_{2,4}\Delta_{1,3} + \Delta_{3,4}\Delta_{1,2}.
\end{array}$$

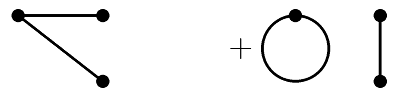
Similarly, one finds

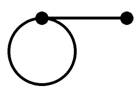
$$\begin{array}{rcccl}
\langle x_1 x_1 x_3 x_4 \rangle & = & \partial_4 \partial_3 \partial_1 \partial_1 \left(\frac{1}{2^2 2!}\sum_{i,j=1}^d \Delta_{i,j} b_i b_j\right)^2 & = & 2\Delta_{1,4}\Delta_{1,3} + \Delta_{3,4}\Delta_{1,1}, \\
\langle x_1 x_1 x_1 x_4 \rangle & = & \partial_4 \partial_1 \partial_1 \partial_1 \left(\frac{1}{2^2 2!}\sum_{i,j=1}^d \Delta_{i,j} b_i b_j\right)^2 & = & 3\Delta_{1,4}\Delta_{1,1}, \\
\langle x_1 x_1 x_4 x_4 \rangle & = & \partial_4 \partial_4 \partial_1 \partial_1 \left(\frac{1}{2^2 2!}\sum_{i,j=1}^d \Delta_{i,j} b_i b_j\right)^2 & = & 2\Delta_{1,4}\Delta_{1,4} + \Delta_{4,4}\Delta_{1,1}, \\
\langle x_1 x_1 x_1 x_1 \rangle & = & \partial_1 \partial_1 \partial_1 \partial_1 \left(\frac{1}{2^2 2!}\sum_{i,j=1}^d \Delta_{i,j} b_i b_j\right)^2 & = & 3\Delta_{1,1}\Delta_{1,1}.
\end{array}$$

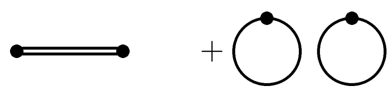
The combinatorics of these computations being quite complex, it is practical to represent each of the products that appear there by a graph, whose vertices correspond to indices of the coordinates x_i that appear in the functions whose mean values are being computed and each $\Delta_{i,j}$ becomes an edge between the i and the j vertex.

One thus gets the diagrams:

$\langle x_1x_2x_3x_4\rangle =$ 

$\langle x_1x_1x_3x_4\rangle = 2.$ 

$\langle x_1x_1x_1x_4\rangle = 3.$ 

$\langle x_1x_1x_4x_4\rangle = 2.$ 

$\langle x_1x_1x_1x_1\rangle = 3.$ 

17.1.4 Finite dimensional Feynman rules

We keep the notations of the preceding sections and give a polynomial V in the coordinates $x_1, \ldots, x_d$. The integrals whose generalizations are interesting for physics are of the type

$$Z(S,V) := \int_{\mathbb{R}^d} \exp\left(-\frac{1}{2}S(x) + \hbar V(x)\right) dx,$$

that one can rewrite in the form

$$Z(S,V) = \int_{\mathbb{R}^d} \exp\left(-\frac{1}{2}S(x)\right) \left[\sum_n \frac{1}{n!}(\hbar V(x))^n\right] dx.$$

One can evaluate these integrals as before by using the formal formula

$$Z(S,V) = Z(S,0) \exp\left(\hbar V\left(\frac{\partial}{\partial b}\right)\right) \exp\left(\frac{1}{2}\Delta(b)\right)\Big|_{b=0},$$

whose power series expansion can be computed by using the preceding computations by the n-points function. The combinatorics of these computations is encoded in diagrams whose vertices correspond to the coefficients $a_{i_1,\ldots,i_d}$ of the monomials of the polynomial $V = \sum_{i\in\mathbb{N}^d} a_i x^i$ and whose edges correspond to the coefficients $\Delta_{i,j}$ of the quadratic forms that appears in the Wick formula for these monomials.

For example, one can consider a polynomial of the form $V(x) = \sum_{i,j,k} v_{ijk}x_ix_jx_k$ (situation similar to the famous "φ^3" toy model) and compute

$$\begin{array}{rcl} Z(S,V) & = & \int_{\mathbb{R}^d} \exp\left(-\frac{1}{2}S(x) + \hbar \sum_{i,j,k} v_{ijk}x_ix_jx_k\right) dx \\ & = & Z(S,0)\exp\left(\hbar \sum_{i,j,k} v_{ijk}\partial_i\partial_j\partial_k\right)\exp\left(\frac{1}{2}S(b)\right)\Big|_{b=0}. \end{array}$$

Let's compute the terms of degree 2 in $\hbar$. This terms have 6 derivatives and their sum is

$$\sum_{i,j,k}\sum_{i',j',k'} v_{ijk}v_{i'j'k'}\partial_i\partial_j\partial_k\partial_{i'}\partial_{j'}\partial_{k'} \exp\left(\frac{1}{2}\Delta(b)\right)\Big|_{b=0}.$$

By Wick's theorem, one can rewrite this sum as

$$\sum_{i,j,k}\sum_{i',j',k'} \Delta_{i_1,i_2}\Delta_{i_3,i_4}\Delta_{i_5,i_6}v_{ijk}v_{i'j'k'}$$

where the internal sum is taken on all pairings $\{(i_1,i_2),(i_3,i_4),(i_5,i_6)\}$ of $\{i,j,k,i',j',k'\}$.

One associates to these pairings the graphs whose trivalent vertices correspond to the factors v_{ijk} and whose edges correspond to the $\Delta_{i,j}$. In this case, one gets exactly to distinct graphs called respectively the dumbbell and the theta:

One then has to number the edges at each vertices to encode the corresponding monomial.

17.2 Feynman's interpretation of functional integrals

We recall here the main idea of Feynman's sum over histories approach to quantum physics. Let M be a spacetime with its Lorentz metric g, let $\pi : C = M \times F \to M$ be a trivial linear bundle with fiber F, and $S : \Gamma(M,C) \to \mathbb{R}$ be a local action functional. Let $\iota_i : \Sigma_i \hookrightarrow M$, $i = 0,1$ be two space-like hyper-surfaces (that one can think as the space at times t_0 and t_1). If we suppose that the action functional is of order 1 (first order derivatives only), one can specify a space of histories by choosing two sections $\varphi_i \in \Gamma(\Sigma_i, \iota_i^*C)$, $i = 0,1$ and saying that

$$H = H_{(\Sigma_0,\varphi_0),(\Sigma_1,\varphi_1)} := \{\varphi \in \Gamma(M,C),\ \varphi_{|\Sigma_0} = \varphi_0,\ \varphi_{|\Sigma_1} = \varphi_1\}.$$

We now suppose that $S(\varphi) = S_{free}(\varphi) + S_{int}(\varphi)$ is a sum of a free/quadratic part in the field, whose equations of motion give a linear partial differential equation

$$D\varphi = 0$$

and of an interaction part that depend only on the values of the field and not of its derivatives.

One can think of the value $\varphi_0 : \Sigma_0 \to \iota_0^* C$ as specifying the value of the classical free field $\tilde{\varphi}_0$ on the spacetime region before the time corresponding to Σ_0, which is the starting time for the collision experiment. The value of $\tilde{\varphi}_0$ is measured experimentally in the preparation of the machine and its data is essentially equivalent to the datum of the pair (S_{free}, φ_0) composed of the free Lagrangian and the initial value for the experiment. One gets $\tilde{\varphi}_0$ from (S_{free}, φ_0) by solving the backward in time Cauchy problem for the hyperbolic linear partial differential equation of free motion

$$D\tilde{\varphi}_0 = 0, \ \varphi_{|\Sigma_0} = \varphi_0$$

with initial condition φ_0.

Similarly, the value $\varphi_1 : \Sigma_1 \to \iota_1^* C$ specifies the value of the classical free field $\tilde{\varphi}_1$ after the final time Σ_1 of the collision experiment, by solving the forward in time Cauchy problem for the free equation of motion. It is also measured after the experiment by the apparatus.

To sum up, what you put in the machine is a free field and what you get after the experiment is also a free field.

The quantum process essentially takes place between Σ_0 and Σ_1, and is composed of collisions and various particle creations and annihilations. The information that one can gather about the probabilistic property of this quantum process with respect to a given functional $A : H \to \mathbb{R}$, called an observable, are all contained in Feynman's sum over history, also called functional integral:

$$\langle \varphi_1 | A | \varphi_0 \rangle = \frac{\int_H A(\varphi) e^{\frac{i}{\hbar} S(\varphi)} [d\varphi]}{\int_H e^{iS(\varphi)} [d\varphi]}.$$

One can understand this formal notation by taking a bounded box $B \subset M$ whose time-like boundaries are contained in the spacetime domain between Σ_0 and Σ_1, and that represent a room (or an apparatus' box) in which one does the experiment. We now suppose that $M = \mathbb{R}^{3,1}$, B is a 0-centered square and $\Lambda = \mathbb{Z}^{3,1} \subset \mathbb{R}^{3,1}$ is the standard lattice. An observable is simple enough if it can be restricted to the space

$$\mathrm{Hom}(\Lambda \cap B, F) = F^{\Lambda \cap B} \subset \Gamma(M, C)$$

of functions on the finite set of points in $\Lambda \cap B$ with values in the fiber F of the bundle $\pi : C \to M$. For simple enough observables, one can make sense of the above formula by using ordinary integration theory on the finite dimensional linear space $F^{\Lambda \cap B}$ with its Lebesgue measure.

The main problem of renormalization theory is that if one makes the size L of the box B tend to infinity and the step ℓ of the lattice Λ tend to 0, one does not get a well defined limit. The whole job is to modify this limiting process to get well defined values for the above formal expression.

17.3 Functional derivatives

In this course, we used well defined functional derivatives on the space of histories H of a field theory $(\pi : C \to M, S : \Gamma(M, C) \to \mathbb{R})$ along a given vector field $\vec{v} : H \to TH$. This kind of functional derivatives are necessary to understand computations with fermionic variables.

We will now explain the analysts' approach to functional derivatives, even if it has some mathematical drawbacks. This will be necessary to understand properly the mathematical problems that appear in the manipulation of functional integrals for scalar fields (i.e., real valued functions).

Definition 17.3.1. Let $\mathcal{H}$ and $\mathcal{G}$ be two complete locally convex topological vector spaces, whose topology are given by a families of seminorms. If $F : \mathcal{H} \to \mathcal{G}$ is a functional, $f \in \mathcal{H}$ and $h \in \mathcal{H}$, one defines its *gâteaux derivative*, if it exists, by

$$\frac{\delta F}{\delta \vec{h}}(f) := \lim_{\epsilon \to 0} \frac{F(f + \epsilon.h) - F(f)}{\epsilon}.$$

One can apply this definition for example to functionals $F : \mathcal{H} \to \mathbb{R}$ on a topological space of fields $\mathcal{H} = \Gamma(M, C)$ of a bosonic field theory where $\pi : C = F \times M \to M$ is a trivial linear bundle on spacetime with fiber F, since it is enough to understand the main difficulties. In particular, one can work with $\mathcal{H} = \mathcal{C}_c^\infty(\mathbb{R}^n)$ or $\mathcal{H} = \mathcal{S}(\mathbb{R}^n)$ (equipped with their ordinary topologies).

One can also work with spaces of distributions. For example, the Gâteaux derivative of a functional $F : \mathcal{S}'(\mathbb{R}^n) \to \mathbb{R}$ along Dirac's δ_x distribution (that one often finds in physics books), if it exists, defines a map

$$\frac{\delta}{\delta \vec{\delta}_x} : \mathcal{S}'(\mathbb{R}^n) \to \mathbb{R},$$

that associates for example to a locally integrable function φ a derivative

$$\frac{\delta F}{\delta \vec{\delta}_x}(\varphi) \in \mathbb{R}$$

of F at φ along the Dirac δ distribution, denoted $\frac{\delta F(\varphi)}{\delta \varphi(x)}$ by physicists. For example, if $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ is fixed, the corresponding functional $[\varphi_0](\varphi) := \int_{\mathbb{R}^n} \varphi_0(x)\varphi(x)dx$ on $\mathcal{S}(\mathbb{R}^n)$ can be extended to a functional of distributions $[\varphi_0] : \mathcal{S}'(\mathbb{R}^n) \to \mathbb{R}^n$ by setting

$$[\varphi_0](D) := D(\varphi_0).$$

One then has a well defined functional derivative for the functional $[\varphi_0]$ given by

$$\frac{\delta[\varphi_0]}{\delta \varphi(x)}(\varphi) := \frac{\delta[\varphi_0]}{\delta \vec{\delta}_x} = \varphi_0(x).$$

One can also consider the evaluation functional $F = ev_x : \mathcal{S}(\mathbb{R}^n) \to \mathbb{R}$ given by $f \mapsto f(x)$. One then has

$$\frac{\delta F}{\delta \vec{h}}(f) := \lim_{\epsilon \to 0} \frac{F(f + \epsilon.h) - F(f)}{\epsilon} = \lim_{\epsilon \to 0} \frac{f(x) + \epsilon.g(x) - f(x)}{\epsilon} = g(x)$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$, so that the functional derivative of F at f exists and is

$$\frac{\delta F}{\delta \vec{h}}(f) = F(\vec{h}).$$

One here remarks that the functional derivative has nothing in common with the derivative of distributions. One also remarks that the function $F = ev_x$ does not extend to tempered distributions and that one can't compute its functional derivative along δ_x, because $F(\delta_x) = \mathrm{ev}_x(\delta_x)$ isn't well defined. This is one of the explanation of the infinities that appear in the physicists' computations of functional integrals, that impose the use of renormalization methods.

For the rest of this section, we will suppose that all our functionals extend to distributions and use the following definition of functional derivatives:

$$\frac{\delta F(f)}{\delta f(x)} := \frac{\delta F(f)}{\delta(\delta_x)} = \lim_{\epsilon \to 0} \frac{F(f + \epsilon \delta_x) - F(f)}{\epsilon}.$$

The most useful functional derivatives in field theory are given by the following proposition (that one can find in Folland's book [Fol08], page 269).

Proposition 17.3.2. *One has the following computations:*

Functional	*Formula*	*Functional derivative*
Integral	$F(f) = \int f(y)h(y)dy$	$\frac{\delta F(f)}{\delta f(x)} = h(x)$
Exponential integral	$\exp\left[\int f(y)h(y)dy\right]$	$\frac{\delta^J F(f)}{\delta f(x_1)\ldots\delta f(x_J)} = h(x_1)\ldots h(x_J)F(f)$
Quadratic integral	$F(f) = \int\int f(x)K(x,y)g(y)dxdy$	$\frac{\delta F(f)}{\delta f(u)} = 2\int K(u,z)f(z)dz$
Quadratic exponential	$F(f) = \exp\left(\int\int f(x)K(x,y)g(y)dxdy\right)$	$\frac{\delta F(f)}{\delta f(u)} = \left[2\int K(u,z)f(z)dz\right]F(f)$

Proof. We will only give the proof for the two first integrals. The functional $F(f) = \int f(y)h(y)dy$ can be extended from $\mathcal{S}(\mathbb{R})$ to distributions f on $\mathcal{S}(\mathbb{R})$ by $F(f) = f(h)$: this is simply the evaluation of distributions at the given function h. It is a linear functional in the distribution f. One can thus compute $F(f + \epsilon\delta_x) = F(f) + \epsilon F(\delta_x)$ and get

$$\frac{\delta F(f)}{\delta(\delta_x)} = \delta_x(h) = h(x).$$

Similarly, for $F(f) = \exp\left[\int f(y)h(y)dy\right]$, one can extend it to distributions by $F(f) = \exp(f(h))$, getting

$$\begin{aligned}\frac{\delta F(f)}{\delta(\delta_x)} &= \lim_{\epsilon\to 0}\frac{\exp(f(h)+\epsilon\delta_x(h))-\exp(f(h))}{\epsilon}\\ &= \lim_{\epsilon\to 0}\exp(f(h)).\frac{\exp(\epsilon\delta_x(h))-1}{\epsilon}\\ &= h(x)\exp(f(h)).\end{aligned}$$

The other equalities are obtained by similar methods. □

17.4 Perturbative definition of functional integrals

We refer to Connes-Marcolli's book [CM08] for a smooth mathematical presentation of the perturbative expansions. One can also use Cartier and DeWitt-Morette's book [CDM06]. The main idea about perturbative expansions is to define the functional integral by their expansions in terms of Feynman diagrams, using the analogy with finite dimensional gaussian integrals. In this section, as in Chapter 17, we authorize ourselves to use physicists notations

and some of their problematic computations,

describing where are the mathematical problems, so that one has a good idea of what needs to be done to solve them.

One thus considers the action functional $S(\varphi)$ of a given system $\pi : C \to M$ without gauge symmetries as a perturbation of a free action functional $S_{free}(\varphi) = \int_M \langle\varphi, D\varphi\rangle$ that is quadratic in the fields with the linear equation $D\varphi = 0$ as Euler-Lagrange equation, by an interaction term $S_{int}(\varphi) = \int_M V(\varphi)$ with V a potential function. One writes

$$S(\varphi) = S_{free}(\varphi) + \lambda S_{int}(\varphi).$$

If Δ is the Schwartz kernel of D, also called the free propagator, one then defines

$$Z_{free}(J) = \int_H e^{\frac{i}{\hbar}(S_{free}(\varphi)+\int J(\varphi))}[d\varphi] := e^{-\frac{i}{\hbar}J\Delta J}.$$

If the kernel Δ is a smooth function, this free partition function fulfills the free Dyson-Schwinger equation

$$\left(\frac{\delta S_{free}}{\delta\varphi(x)}\left[-\frac{i}{\hbar}\frac{\delta}{\delta J}\right] + J(x)\right).Z_{free}(J) = 0$$

that is perfectly well defined in this case because there are no higher order functional derivatives in play. If Δ is only a distribution, one can not apply the functional derivative to $Z_{free}(J)$ because it involves evaluating a distribution at the δ function. The solution of this problem is given by the regularization procedure for propagators.

One then formally writes

$$Z(J) = \int_H e^{\frac{i}{\hbar}(S(\varphi)+\int J(\varphi))}[d\varphi] = e^{\frac{i}{\hbar}S_{int}}\left[-\frac{i}{\hbar}\frac{\delta}{\delta J}\right].Z_{free}(J)$$

and permute ordinary and functional integrals to finally define

$$\begin{aligned} Z(J) &:= \sum_{k=0}^{\infty} \frac{i^k}{k!\hbar^k} \underbrace{\int \dots \int}_{k \text{ terms}} \mathcal{L}_{int}^k \left[\frac{\delta}{\delta J}\right] e^{-\frac{i}{\hbar} J\Delta J} dx_1 \dots dx_k \\ &= \sum_{k=0}^{\infty} \frac{i^k}{k!} \int J(x_1)\dots J(x_k) G_k(x_1,\dots,x_k) dx_1 \dots dx_k. \end{aligned}$$

The terms that appear in this formal power series in $\hbar$ are integrals of some expressions that are ill defined because they involve the evaluation of distributions at δ functions. One can however check that this expression formally fulfills the Dyson-Schwinger equation

$$\left(\frac{\delta S}{\delta\varphi(x)}\left[-\frac{i}{\hbar}\frac{\delta}{\delta J(x)}\right] + J(x)\right).Z(J) = 0.$$

One then has to regularize, i.e., replace these distributions by usual families of functions that converge to them, compute everything with these regularized solutions and then pass to the distributional limit. The green functions G_k that appear in the perturbative expansion can be described (or actually defined) by analogy with the usual gaussian case as sums

$$G_k(x_1,\dots,x_k) = \sum_{\Gamma} \int \frac{V(\Gamma)(p_1,\dots,p_k)}{\sigma(\Gamma)} e^{i(x_1p_1+\dots+x_kp_k)} \prod_i \frac{dp_i}{(2\pi)^d},$$

indexed by Feynman graphs that are not necessarily connected. The problem is that the rational fractions $V(\Gamma)(p_1,\dots,p_k)$ are usually not integrable, so that one has to replace them by their renormalized values, that are obtained by adding to them some counter-terms (obtained by adding local counter terms to the Lagrangian), to make them convergent. This is the theory of renormalization.

Remark that it is also interesting to consider the connected partition function defined by the equality

$$Z(J) =: e^{\frac{i}{\hbar}W(J)}$$

since its perturbative expansion is described by connected Feynman graphs, and so combinatorially simpler. One actually uses the more simple 1PI partition function, also called the effective action $S_{eff}(\varphi)$. It is formally defined as a kind of Legendre transform by

$$S_{eff}(\varphi) = [J_E(\varphi) - W(J_E)]_{|J_E=J(\varphi)}$$

where J_E is chosen so that

$$\frac{\delta W}{\delta J}(J_E) = \varphi.$$

One can properly define it by taking the unrenormalized values

$$U(\Gamma)(\varphi) = \frac{1}{k!}\int_{\sum p_j=0} \hat{\varphi}(p_1)\dots\hat{\varphi}(p_k) U(\Gamma(p_1,\dots,p_k)) \prod_i \frac{dp_i}{(2\pi)^d},$$

and defining

$$\Gamma(\varphi) = S(\varphi) - \sum_{\Gamma\in 1\mathrm{PI}} \frac{U(\Gamma)(\varphi)}{\sigma(\Gamma)}.$$

In this perturbative setting, where one defines functional integrals by their Feynman graph expansions, one has the equality

$$\int F(\varphi)e^{\frac{i}{\hbar}S(\varphi)}[d\varphi] = \int_{\text{tree level}} F(\varphi)e^{\frac{i}{\hbar}\Gamma(\varphi)}[d\varphi]$$

where the tree-level functional integral means that one only uses diagrams that are trees. This shows that the knowledge of the effective action is enough to determine all quantum observables.

17.5 Schwinger's quantum variational principle

Another way of giving a mathematical meaning to the expectation value $\langle\varphi_1|A|\varphi_0\rangle$ is to consider the dual linear bundle $C^\dagger$ of C and to add to the action functional a source term

$$S_{source}(\varphi) = S(\varphi) + \int_M \langle J, \varphi\rangle(x)d^4x.$$

The aim is to make sense of the partition function

$$Z_A(J) = \langle\varphi_1|A|\varphi_0\rangle_J = \frac{\int_H A(\varphi)e^{\frac{i}{\hbar}S_{source}(\varphi)}[d\varphi]}{\int_H e^{iS(\varphi)}[d\varphi]},$$

whose value at $J = 0$ gives the sum over histories. The main advantage of this new formal expression is that it replaces the problem of defining an integral on the space of histories by the problem of defining the functional differential equation on the dual space whose solution is $Z_A(J)$. This equation is called the Dyson-Schwinger equation and may be used to define implicitely the partition function, since this is usually how physicist using functional integral proceed (see for example Zinn-Justin's book [ZJ93], Chapter 7 or Rivers' book [Riv90]).

The idea behind the Dyson-Schwinger approach can be explained quickly in the case of a scalar field theory $\pi : C = \mathbb{R} \times M \to M$ on spacetime. The space $\mathcal{O}_H$ of functions on H is equipped with a (partially defined) functional derivation $\frac{\delta}{\delta\varphi(x)}$ defined by

$$\frac{\delta F}{\delta\varphi(x)} = \lim_{t\to 0} \frac{F(\varphi + t\delta_x) - F(\varphi)}{t}.$$

The hypothetic functional integral $\int_H [d\varphi] : \mathcal{O}_H \to \mathbb{R}$ is supposed to fulfill the following formal equalities, that are inspired by ordinary properties of finite dimensional integrals and Fourier transforms:

$$\int_H [d\varphi]\frac{\delta}{\delta\varphi(x)}F(\varphi) = 0$$

and

$$\int_H [d\varphi]F(\varphi)G(\varphi)e^{\frac{i}{\hbar}J(\varphi)} = F\left[-\frac{i}{\hbar}\frac{\delta}{\delta J(x)}\right]\int_H G(\varphi)e^{\frac{i}{\hbar}J(\varphi)}[d\varphi].$$

Here, the expression $F\left[-\frac{i}{\hbar}\frac{\delta}{\delta J(x)}\right]$ is a functional differential operator obtained from $F(\varphi)$ by replacing φ by $-\frac{i}{\hbar}\frac{\delta}{\delta J(x)}$. Applying this to the (normalized) functional integral, one gets

$$\int_H \frac{\delta}{\delta\varphi(x)} e^{-\frac{i}{\hbar}[S(\varphi)+J(\varphi)]}[d\varphi] = 0,$$

and finally

$$\left(\frac{\delta S}{\delta\varphi(x)}\left[-\frac{i}{\hbar}\frac{\delta}{\delta J(x)}\right] + J(x)\right).Z(J) = 0$$

and more generally

$$\frac{\delta A}{\delta\varphi(x)}\left[-\frac{i}{\hbar}\frac{\delta}{\delta J(x)}\right].Z(J) + \left(\frac{\delta S}{\delta\varphi(x)}\left[-\frac{i}{\hbar}\frac{\delta}{\delta J(x)}\right] + J(x)\right).Z_A(J) = 0.$$

17.6 Theories with gauge freedom: Zinn-Justin's equation

We refer to Henneaux and Teitelboim [HT92], chapters XVII and XVIII. The formalism of this section was first used by Zinn-Justin to study the renormalizability of Yang-Mills theory, and independently by Batalin-Vilkovisky. It is called the Zinn-Justin equation, or the quantum BV formalism in the physics literature.

Let $\pi : C \to M$ be a bundle and $S \in h(\mathcal{A})$ be a local action functional with $\mathcal{A} = \mathrm{Jet}(\mathcal{O}_C)$. In Section 12.4, we explained the methods of homological symplectic reduction in local field theory. Roughly speaking, the result of this procedure is a dg-$\mathcal{D}$-algebra $(\mathcal{A}^\bullet, D)$ equipped with a local bracket

$$\{.,.\} : \mathcal{A}^\bullet \boxtimes \mathcal{A}^\bullet \to \Delta_*\mathcal{A}^\bullet$$

and a particular element $S_{cm} \in \mathcal{A}^\bullet$ called the classical master action, or also the BRST generator, such that

$$D = \{S_{cm}, .\}.$$

This construction works in the most general context of a local field theory on a superspace. The space of classical observables is then given by the algebra

$$H^0(\mathcal{A}^\bullet, D).$$

Roughly speaking, the canonical version of the BV quantization (operator formalism) is given by a quantization of the graded algebra $\mathcal{A}^\bullet$, denoted $\hat{\mathcal{A}}^\bullet$, that gives also a quantization of the BRST generator S_{cm} and thus a quantization $\hat{D}$ of the BRST differential. The algebra of quantum observables is then given by

$$H^0(\hat{\mathcal{A}}^\bullet, \hat{D}).$$

The functional integral quantization is a bit more involved. We denote by φ_i the fields and ghost variables and by φ_i^* the antifield variables. In physicists' notations, the duality between fields and antifields is given by

$$(\varphi^i, \varphi_i^*) = \delta_j^i.$$

This means that the antibracket is extended to the algebra $\mathcal{A}^\bullet$ by

$$(A, B) = \frac{\delta^R A}{\delta\varphi^i}\frac{\delta^L B}{\delta\varphi_i^*} - \frac{\delta^R A}{\delta\varphi_i^*}\frac{\delta^L B}{\delta\varphi^i}.$$

Here, the left and right functional derivatives are distributions defined by

$$\frac{d}{dt}F(\varphi + t\psi_\varphi) = \int_M \frac{\delta^L F}{\delta\varphi^a(x)}\psi_\varphi^a(x)dx = \int_M \psi_\varphi^a(x)\frac{\delta^R F}{\delta\varphi^a(x)}dx$$

for ψ_φ tangent vectors to the graded field variable φ.

One can not naively use the functional integral (Dyson-Schwinger solution)

$$\int [d\varphi] \exp\frac{i}{\hbar}S_{cm}(\varphi, \varphi^* = 0)$$

because this would give infinite values because of the gauge freedom. The idea is to replace this by an expression

$$\int [d\varphi] \exp\frac{i}{\hbar}S_{qm}\left(\varphi, \psi^* + \frac{\delta\psi}{\delta\varphi}\right)$$

for some given functional $\psi(\varphi)$ of the fields variables called the gauge fixing (the antifields are seen here as background fields and don't enter into the dymanics, i.e., into the integration variables). This gives a canonical transformation of the antibracket that does not change the corresponding differential graded algebra, but such that the corresponding equations of motion are well-posed, so that one can perform the usual perturbative quantization, as in the scalar field case.

The classical master action has to be expanded to a quantum master action $S_{qm} = S_{cm} + \sum_{n\geq 1}\hbar^n S_n$ that fulfills the quantum master equation

$$i\hbar\Delta S_{qm} - \frac{1}{2}(S_{qm}, S_{qm}) = 0,$$

where $\Delta = \frac{\delta^R}{\delta\varphi^i}\frac{\delta^R}{\delta\varphi_i^*}$ is the super Laplacian on $\mathcal{A}^\bullet$ (that is not well defined and needs to be regularized because it involves double functional derivatives). This equation guaranties that the above functional integral does not depend on the gauge fixing functional ψ. It is equivalent to the equation

$$\Delta \exp\frac{i}{\hbar}S_{qm} = 0.$$

If $A_0(\varphi^i) \in \mathcal{A}$ is a classical local gauge invariant function, a classical BRST extension is given by an element $A(\varphi, \varphi^*)$ of $\mathcal{A}^\bullet$ solution of $(A, S_{cm}) = 0$. A quantum BRST extension of A is given by an element $\alpha \in \mathcal{A}^\bullet[[\hbar]]$ of the form

$$\alpha = A + \sum_{n \geq 1} \hbar \alpha_i,$$

such that the quantum master observable equation

$$\Delta \alpha \exp \frac{i}{\hbar} S_{qm} = 0$$

is fulfilled, the quantum master action S_{qm} being already fixed. This quantum master observable equation is again equivalent to

$$i\hbar \Delta \alpha - (\alpha, S_{qm}) = 0.$$

The function $A_\psi(\varphi) = A(\varphi, \frac{\delta \psi}{\delta \varphi})$ is called the gauge fixed version of A. The expectation value of A_0 is finally given by

$$\langle A_0 \rangle = \int \alpha \left(\varphi, \frac{\delta \psi}{\delta \varphi} \right) e^{\frac{i}{\hbar} S_{qm}\left(\varphi, \frac{\delta \psi}{\delta \varphi}\right)} [d\varphi].$$

The main problem with this definition is that the BV Laplacian does not make sense because it involves second order functional derivatives. One of course also gets into trouble when writing down the Dyson-Schwinger equation for this functional integral since it involves higher order functional derivatives.

Chapter 18

Connes-Kreimer-van Suijlekom view of renormalization

We follow here the Hopf algebra / Riemann-Hilbert description of renormalization, due to Connes and Kreimer (see [CK00] and [CK01]). This method is described in [CM08] and in [dM98] for theories without gauge symmetries.

We give here an overview of van Suijlekom's articles on this formalism (see [van08a] and [van08b]) for theories with gauge symmetries.

These constructions give a nice mathematical formulation of the BPHZ renormalization procedure in the dimensional regularization scheme, that is the main method for the renormalization of the standard model.

18.1 Diffeographism groups for gauge theories

We thank W. van Suijlekom for allowing us to include in this section some diagrams from his paper [van08a].

Suppose we are working with the classical BV action of a given gauge theory, given by an element S_{cm} in $h(\mathcal{A}^\bullet)$ for $\mathcal{A}^\bullet$ the BV dg-algebra of fields and antifields. As a graded $\mathcal{A}[\mathcal{D}]$-algebra, this algebra is generated by an even family

$$\Phi = \{\varphi^1, \dots, \varphi^n, \varphi_1^*, \dots, \varphi_n^*\}$$

of generators that will be called the fields. One can actually define a linear bundle $E^{tot} = E \oplus E^*$ over spacetime M with E and E^* both graded such that $\mathrm{Jet}(\mathcal{O}_{E^{tot}}) = \mathcal{A}^\bullet$ as a graded $\mathcal{A}[\mathcal{D}]$-algebra.

We suppose that S_{cm} is the cohomology class of a polynomial Lagrangian density $\mathcal{L}$ (which is the case for Yang-Mills with matter).

We denote R the set of monomials in $\mathcal{L}$, and decompose it into the free massless terms R_E, called the set of edges, and the interaction and mass terms R_V, called the set of vertices. Each edge in R_E will be represented by a different type of line, called the propagator, that corresponds to the free massless motion of the corresponding particle. Each vertex in R_V will be denoted by a vertex whose entering edges are half lines of the same type as the line in R_E that corresponds to the given type of field.

The datum of the Lagrangian density is thus equivalent to the datum of the pair $R = (R_E, R_V)$ of sets of colored edges and vertices and of a map

$$\iota : R_E \coprod R_V \to \mathcal{A}^\bullet.$$

Indeed, one can find back the Lagrangian density by setting

$$\mathcal{L}_{free-massless} = \sum_{e \in R_E} \iota(e), \qquad \mathcal{L}_{int-mass}(\varphi) = \sum_{v \in R_V} \iota(v),$$

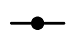
and finally

$$\mathcal{L}(\varphi) = \mathcal{L}_{free-massless} + \mathcal{L}_{int-mass}.$$

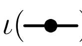
Example 18.1.1. The Lagrangian density of quantum electrodynamics is a $U(1)$-gauge invariant Yang-Mills Lagrangian (see Section 14.3) of the form

$$L(A, \psi) = -\frac{1}{2}\langle F_A \wedge *F_A\rangle + \bar{\psi}\not{D}_A\psi + \bar{\psi}m\psi$$

where A is the photon field given by an $i\mathbb{R} = \mathrm{Lie}(U(1))$-valued 1-form $A \in \Omega^1_M$ and a section $\psi : M \to S$ of the (here supposed to be) trivial spinor bundle $M \times \mathbb{C}^2$ on flat spacetime that represents the electron field. Once a basis for the space of connections is fixed, the A-dependent Dirac operator decomposes as a sum of a Dirac propagator $i\bar{\psi}\not{D}\psi$ and an interaction term $-e\bar{\psi}\gamma \circ A\psi$. The space of vertices and edges for this Lagrangian are thus given by

$R_V = \{$  ,  $\}$; $R_E = \{$, $\}$.

The corresponding monomials in $\mathcal{A}^\bullet$ are

$\iota($$) = -e\bar{\psi}\gamma \circ A\psi$, $\iota($$) = -m\bar{\psi}\psi$,

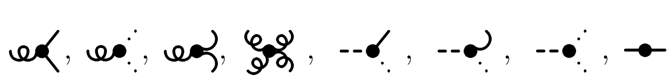
$\iota($$) = i\bar{\psi}\gamma \circ d\psi$, $\iota($$) = -dA * dA$.

with e and m the electric charge and mass of the electron (coupling constants), respectively.

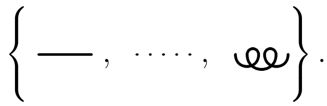
Example 18.1.2. Similarly, the Lagrangian density of quantum chromodynamics SU(3)-gauge invariant Yang-Mills Lagrangian (see Section 14.3) of the form

$$L(A, \psi) = -\frac{1}{2}\langle F_A \wedge *F_A\rangle + \bar{\psi}\not{D}_A\psi + \bar{\psi}m\psi$$

where A is the quarks field, described by wiggly lines and ψ is the gluon field, described by straight lines. In addition, one has the ghost fields ω and $\bar{\omega}$, indicated by dotted lines, as well as BRST antifields K_ψ, K_A, K_ω and $K_{\bar{\omega}}$. Between these fields, there are four interactions, three BRST-source terms, and a mass term for the quark. This leads to the following sets of vertices and edges,

$R_V = \Big\{$  $\Big\}$

with the dashed lines representing the BRST-source terms, and

$R_E =$ 

Note that the dashed edges do not appear in R_E, i.e. the source terms do not propagate and in the following will not appear as internal edges of a Feynman graph.

Definition 18.1.3. A *Feynman graph* Γ is a pair composed of a set Γ_V of vertices each of which is an element in R_V and Γ_E of edges in R_E, and maps

$$\partial_0, \partial_1 : \Gamma_V \to \Gamma_E \cup \{1, 2, \ldots, N\},$$

that are compatible with the type of vertex and edge as parametrized by R_V and R_E, respectively. One excludes the case that ∂_0 and ∂_1 are both in $\{1, \ldots, N\}$. The set $\{1, \ldots, N\}$ labels the external lines, so that $\sum_j \mathrm{card}(\partial_j^{-1}(v)) = 1$ for all $v \in \{1, \ldots, N\}$. The set of *external lines* is $\Gamma_E^{ext} = \partial_0^{-1}(\{1, \ldots, N\}) \cup \partial_1^{-1}(\{1, \ldots, N\})$ and its complement Γ_E^{int} is the set of *internal lines*.

One sees the external lines Γ_E^{ext} as labeled $e_1, \ldots, e_N$ where $e_k = \partial_0^{-1}(k) \cup \partial_1^{-1}(k)$.

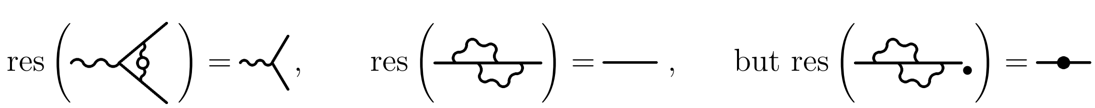
Definition 18.1.4. An *automorphism* of a Feynman graph Γ is given by as a pair of isomorphisms $g_V : \Gamma_V \to \Gamma_V$ and $g_E : \Gamma_E \to \Gamma_E$ that is the identity on Γ_E^{ext} that are compatible with the boundary maps, meaning that for all $e \in \Gamma_E$,

$$\cup_j g_V(\partial_j(e)) = \cup_j \partial_j(g_E(e)).$$

Moreover, we require g_V and g_E to respect the type of vertex/edge in the set R. The automorphism group $\mathrm{Aut}(\Gamma)$ of Γ consists of all such automorphisms; its order is called the *symmetry factor* of Γ and denoted $\mathrm{Sym}(\Gamma)$.

Definition 18.1.5. A Feynman graph is called a *one-particle irreducible graph* (1PI) if it is not a tree and can not be disconnected by removal of a single edge. The *residue* $\mathrm{res}(\Gamma)$ of a Feynman graph Γ is obtained by collapsing all its internal edges and vertices to a point.

If a Feynman graph Γ has two external lines, both corresponding to the same field, we would like to distinguish between propagators and mass terms. In more mathematical terms, since we have vertices of valence two, we would like to indicate whether a graph with two external lines corresponds to such a vertex, or to an edge. A graph Γ with two external lines is dressed with bullet when it corresponds to a vertex, i.e. we write $\Gamma_\bullet$. The above correspondence between Feynman graphs and vertices/edges is given by the residue. For example, we have:



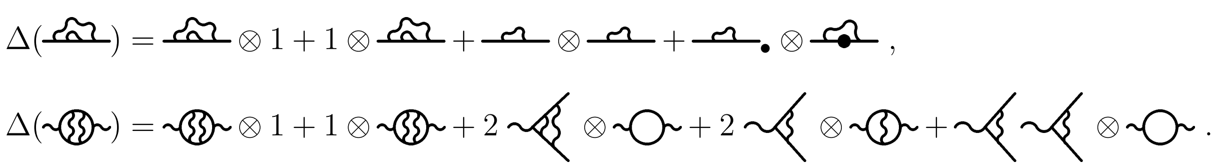
Definition 18.1.6. The *diffeographism group scheme* G is the affine group scheme whose algebra of functions A_G is the free commutative algebra generated by all 1PI graphs, with counit $\epsilon(\Gamma) = 0$ unless $\Gamma = \emptyset$, and $\epsilon(\emptyset) = 1$, coproduct

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subsetneq \Gamma} \gamma \otimes \Gamma/\gamma,$$

and antipode given recursively by

$$S(\Gamma) = -\Gamma - \sum_{\gamma \subsetneq \Gamma} S(\gamma)\Gamma/\gamma.$$

In the above definition, the graph Γ/γ is obtained by contracting in Γ the connected components of the subgraph γ to the corresponding vertex/edge. If the connected component γ' of γ has two external lines, there are possibly two contributions corresponding to the valence two vertex and the edge; the sum involves the two terms $\gamma'_{\bullet} \otimes \Gamma/(\gamma' \to \bullet)$ and $\gamma' \otimes \Gamma/\gamma'$. To illustrate this with the QED example, one has



The Hopf algebra A_G is graded (meaning has a grading that respects the product and coproduct) by the loop number $L(\Gamma) := b_1(\Gamma)$ of a graph. This gives a decomposition

$$A_G = \oplus_{n \in \mathbb{N}} A_G^n.$$

It has also a multi-grading indexed by the group $\mathbb{Z}^{(R_V)}$. For $R = R_E \coprod R_V$, one defined $m_{\Gamma,r}$ as the number of vertices/internal edges of type $r \in R$ appearing in Γ, and $n_{\gamma,r}$ the number of connected component of γ with residue r. For each $v \in R_V$, one defines the degree d_v by setting

$$d_v(\Gamma) = m_{\Gamma,v} - n_{\Gamma,v}.$$

The multi-degree indexed by R_V, given by

$$d : A_G \to \mathbb{Z}^{(R_V)}$$

is compatible with the Hopf algebra structure. This gives a decomposition

$$A_G = \oplus_{\alpha \in \mathbb{Z}^{(R_V)}} A_G^{\alpha}.$$

For $\alpha \in \mathbb{Z}^{(R_V)}$, we denote $p_\alpha : A_G \to A_G^\alpha$ the corresponding projection.

18.2 The Riemann-Hilbert correspondence

Let $X = \mathbb{P}^1(\mathbb{C}) - \{x_0, \ldots, x_n\}$. A flat bundle on X is a locally free $\mathcal{O}_X$-module $\mathcal{F}$ with a connection

$$\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_X$$

whose curvature is zero, i.e., such that $\nabla_1 \circ \nabla = 0$. Equivalently, this is a $\mathcal{D}_X$-module that is also a locally free $\mathcal{O}_X$-module.

Given a flat bundle $(\mathcal{F}, \nabla)$ its space of solutions (also called horizontal sections) is the locally constant bundle (local systems) on X (locally isomorphic to $\mathbb{C}^n_X$ given by

$$\mathcal{F}^\nabla = \{f \in \mathcal{F},\ \nabla f = 0\}.$$

The functor

$$\begin{array}{rccc} \mathrm{Sol}: & \textsc{Flat Bundles} & \to & \textsc{Local systems} \\ & (\mathcal{F}, \nabla) & \to & \mathcal{F}^\nabla \end{array}$$

is fully faithful. Remark that a local system is equivalent to a representation of the fundamental group $\pi_1(X)$. One of Hilbert's problem is to show that it is essentially surjective, meaning that to every representation V of $\pi_1(X)$, one can associate a differential equation whose corresponding local system is given by the local system

$$\underline{V} := V \underset{\pi_1(X)}{\times} \tilde{X}$$

where $\tilde{X} \to X$ is the universal covering of X. Results in this direction are called Riemann-Hilbert correspondences.

Remark that if $(\mathcal{F}, \nabla)$ is a local system, and if $\gamma : x \to y$ is a path in X (whose homotopy class is an element in the fundamental groupoid $\Pi(X)$), the analytic continuation of solutions of $(\mathcal{F}, \nabla)$ along γ gives a linear map, called the monodromy transformation

$$M_\gamma : \mathcal{F}^\nabla_x \to \mathcal{F}^\nabla_y$$

between the two finite dimensional $\mathbb{C}$-vector spaces give by the fibers of the solution space at the two given point. If

$$\mathrm{Isom}(\mathcal{F}^\nabla) \to X \times X$$

is the algebraic groupoid whose fiber over (x, y) is the space of linear isomorphisms $\mathrm{Isom}(\mathcal{F}^\nabla_x, \mathcal{F}^\nabla_y)$, there is a natural groupoid morphism

$$M : \Pi(X) \to \mathrm{Isom}(\mathcal{F}^\nabla)$$

given by the monodromy representation.

A theorem due to Cartier and independently to Malgrange is that the Zariski-Closure of the image of the monodromy representation M in the relatively algebraic groupoid $\mathrm{Isom}(\mathcal{F}^\nabla)$ is equal to the differential Galois groupoid. To explain this result, one has to define the differential Galois groupoid of $(\mathcal{F}, \nabla)$. It is the groupoid whose fiber over a point $(x, y) \in X \times X$ is the space

$$\mathrm{Isom}(\omega_x, \omega_y)$$

between the fiber functors (i.e., symmetric monoidal functors between rigid symmetric monoidal categories, defined in Section 1.3; see Deligne [Del90] for more details on the general theory of tannakian categories and their fundamental groups)

$$\omega_x, \omega_y : \langle(\mathcal{F}, \nabla)\rangle \to (\mathbb{C} - \text{Vector spaces}, \otimes)$$

given by

$$\omega_{x,y}(\mathcal{G}, \nabla) = \mathcal{G}^{\nabla}_{x,y}.$$

where $\langle(\mathcal{F}, \nabla)\rangle$ denotes the sub-monoidal category of the category of all flat bundles generated by the given flat bundle.

One can interpret the diffeographism groups of the Connes-Kreimer theory as some differential Galois groups associated to the defining equations of time-ordered exponentials (iterated integrals). This relation between differential Galois theory and the Connes-Kreimer theory is due to Connes and Marcolli and explained in detail in [CM08].

We will now explain a particular version of the Riemann-Hilbert correspondence that is directly related to the renormalization procedure in the dimensional regularization scheme.

Let $\gamma : C \to G$ be a loop with values in an arbitrary complex Lie group G, defined on a smooth simple curve $C \subset \mathbb{P}^1(\mathbb{C})$. Let $C_\pm$ be the two complements of C in $\mathbb{P}^1(\mathbb{C})$, with $\infty \in C_-$. A Birkhoff decomposition of γ is a factorization of the form

$$\gamma(z) = \gamma_-(z)^{-1}\gamma(z),\ (z \in C),$$

where $\gamma_\pm$ are (boundary values of) two holomorphic maps on $C_\pm$, respectively with values in G.

Definition 18.2.1. The value of $\gamma_+(z)$ at $z = 0$ is called the *renormalized value* of the given loop and the terme $\gamma_-(z)$ is called the *counter-term* associated to the given loop.

Theorem 18.2.2. *Let G be a group scheme whose hopf algebra is graded commutative (this means that G is pro-unipotent). Then any loop $\gamma : C \to G$ admits a Birkhoff decomposition.*

18.3 Connes-Kreimer's view of renormalization

We refer to [CM08] for a complete overview of the Connes-Kreimer theory and for a description of its relations with Riemann-Hilbert correspondence and motivic Galois theory.

We consider here the effective action from Section 17.4. It is given as a formal Legendre transform of the connected partition function that is itself given by the logarithm of the full partition function. One expresses this effective action in terms of the unrenormalized values of the Feynman graphs of the given theory.

If S is the original (gauge fixed) action functional, one writes

$$\Gamma(\varphi) = S(\varphi) - \sum_{\Gamma \in 1\text{PI}} \frac{U(\Gamma)(\varphi)}{\text{Sym}(\Gamma)}$$

where

$$U(\Gamma)(\varphi) = \frac{1}{N!}\int_{\sum p_j=0} \hat{\varphi(p_1)}\dots\hat{\varphi}(p_N)U(\Gamma(p_1,\dots,p_N))\prod_j \frac{dp_j}{(2\pi)^d}.$$

where $U(\Gamma(p_1,\dots,p_N))$ actually denotes a multilinear distribution on the fields, called the bare value of the graph, and is specified by the Feynman rules of the theory. More precisely, one has

$$U(\Gamma(p_1,\dots,p_N)) = \int I_\Gamma(k_1,\dots,k_L,p_1,\dots,p_N)d^dk_1\dots d^dk_L$$

where $L = b_1(\Gamma)$ is the loop number of the graph, $k_i \in \mathbb{R}^d$ are the momentum variables assigned to the internal edges. The rational fraction $I_\Gamma(k_1,\dots,k_L,p_1,\dots,p_N)$ is obtained by applying the Feynman rules. Roughly speaking, this is done by assigning

- a fundamental solution (or its Fourier transform) of the linear differential operator that corresponds to each internal line (actually, one regularizes this solution to get a smooth function that approximate it),
- a momentum conservation rule to each vertex,
- a power of the coupling constant to each more than 3-valent vertex,
- a factor m^2 to each mass 2-point vertex,

The rational fraction $I_\Gamma(k_1,\dots,k_L)$ is given by the the product of all these terms, except some distributional terms. It is supposed to be (when regularized) a smooth function. The problem is that if one sends the regularization parameters to 0, one gets divergencies. It is thus necessary to modify these bare values $U(\Gamma)$ to get meaningful limits at the end. This is the aim of the BPHZ renormalization procedure.

Now if G is the diffeographism group of a given field theory, the bare values $U(\Gamma)(z)$ associated to the Feynman graphs give algebra morphisms from A_G to $\mathbb{C}$ and thus point of G with values in $\mathbb{C}$. If C is a curve around $0 \in \mathbb{P}^1(\mathbb{C})$, one can define a loop in G by $\gamma(z)(\Gamma) := U(\Gamma)(z)$.

Applying the Birkhoff decomposition theorem 18.2.2 to the above loop in the diffeographism group, one gets finite values from the dimensionally regularized bare Feynman amplitudes. Connes and Kreimer showed in [CK00] that these are exactly the values computed by physicists by the BPHZ renormalization procedure.

18.4 Zinn-Justin's equation and Slavnov-Taylor identities

Recall that we defined the BV Laplacian, anti-bracket and quantum BV formalism in Section 17.6.

The Zinn-Justin equation, also called the quantum master equation, is the equation that must be fulfilled by the deformed action functional S_{qm} associated to the classical master action S_{cm}, and the deformed observable A_{qm} associated to a classical observable $A_0 \in H^0(\mathcal{A}^\bullet)$, such that the corresponding functional integral with gauge fixing functional $\psi(\varphi)$

$$\langle A_0 \rangle = \int \alpha\left(\varphi, \frac{\delta\psi}{\delta\varphi}\right) e^{\frac{i}{\hbar} S_{qm}\left(\varphi, \frac{\delta\psi}{\delta\varphi}\right)} [d\varphi],$$

makes sense and does not depend of the choice of the extensions S_{qm} and A_{qm} and of the gauge fixing functional. This is the computation of the effective action associated to this functional integral that is achieved purely in terms of Feynman graphs in this section. The quantum master equation is usually expressed in terms of relations between the renormalized Green functions called the Slavnov-Taylor identities. The aim of this section is to explain, following van Suijlekom's work, how these identities are encoded in the Connes-Kreimer setting.

Let $\mathcal{A}_R := \mathbb{C}[[\lambda_{v_1}, \dots, \lambda_{v_k}]] \otimes \mathcal{A}^\bullet$, $k = |R_V|$, be the algebra obtained from the BV algebra by extension of scalars to the coupling constants algebra. We now suppose that all the interaction (and mass) terms of the Lagrangian density $\mathcal{L}$ of the given theory are affected with a corresponding coefficient that is now a formal variable among the λ_{v_i}. Let $A_G^{Green} \subset A_G$ be the Hopf sub-algebra generated by elements $p_\alpha(Y_v)$, $v \in R_V$ and $p_\alpha(G^e)$, $e \in R_E$, for $\alpha \in \mathbb{Z}^{R_V}$,

$$G^e = 1 - \sum_{\mathrm{res}(\Gamma)=e} \frac{\Gamma}{\mathrm{Sym}(\Gamma)}, G^v = 1 + \sum_{\mathrm{res}(\Gamma)=v} \frac{\Gamma}{\mathrm{Sym}(\Gamma)} \text{ and } Y_v := \frac{G^v}{\prod_\varphi} (G^\varphi)^{N_\varphi(v)/2}$$

and

$$G^{\varphi^i} = G^e,\ G^{\varphi_i^*} = (G^e)^{-1}.$$

We denote

$$G \twoheadrightarrow G_{Green} = \mathrm{Spec}(A_G^{Green})$$

the corresponding quotient of the diffeomorphism group.

Theorem 18.4.1. *The algebra $\mathcal{A}_R$ is a comodule BV-algebra for the Hopf algebra A_G^{green}.*

The BV-ideal $\mathcal{I}_{BV}$ in $\mathcal{A}_R$ is generated by the antibracket (S, S) (that corresponds to the classical BV equation $(S, S) = 0$. There is a natural Hopf ideal in A_G^{Green} such that the corresponding subgroup of diffeographisms

$$G_{ST} \subset G_{Green}$$

acts on $\mathcal{A}_R/\mathcal{I}_{BV}$. The corresponding group is given by the sub-group of G that fixes the ideal $\mathcal{I}$ by the above coaction. This takes into account the Slavnov Taylor identities corresponding to the quantum master equation (here obtained by using only the classical master equation). To conclude geometrically, one has a diagram of diffeographism group actions on spaces

$$\begin{array}{ccccc} G_{ST} & \rightarrowtail & G_{Green} & \twoheadleftarrow & G \\ \curvearrowright & & \curvearrowright & & \\ \mathrm{Spec}(\mathcal{A}_R/\mathcal{I}_{BV}) & \rightarrowtail & \mathrm{Spec}(\mathcal{A}_R) & & \end{array}$$

and G_{ST} is defined as the subgroup of G_{Green} that fixes the subspace of $\mathrm{Spec}(\mathcal{A}_R)$ that corresponds to the classical BV equation.

Chapter 19

Nonperturbative quantum field theory

19.1 The geometric meaning of perturbative computations

In this section, we describe the geometric objects underlying the computations of physicists in perturbative quantum field theory. We will give more details on the various perturbative renormalization procedures in Chapters 20 and 21.

In this section, we denote X a smooth space and $\pi : \mathbb{R}_X := \mathbb{R} \times X \to X$ the trivial bundle. The jet space of $\mathbb{R}_X$ is the space

$$\mathrm{Jet}(\mathbb{R}_X/X) := \varprojlim \mathrm{Jet}^n(\mathbb{R}_X/X).$$

The algebra of functions

$$\mathcal{O}(\mathrm{Jet}(\mathbb{R}_X/X)) := \varinjlim \mathcal{O}(\mathrm{Jet}^n(\mathbb{R}_X/X))$$

is thus naturally a filtered smooth algebra. The degree zero term of the filtration is

$$\mathcal{O}(\mathrm{Jet}^0(\mathbb{R}_X/X)) = \mathcal{O}(\mathbb{R}_X) = \mathcal{O}(X \times \mathbb{R}).$$

It is the smooth $\mathcal{O}(X)$-algebra associated to the bundle $\mathbb{R}_X$.

We suppose now given a dual affine connection

$$\nabla : \Omega^1_X \to \Omega^1_X \otimes \Omega^1_X,$$

where $\Omega^1_X \otimes \Omega^1_X$ denotes the module of sections so that we get a second order derivative

$$\nabla^2 : \mathcal{O}_X \to \mathrm{Sym}^2_{\mathcal{O}_X}(\Omega^1_X)$$

and more generally higher order derivatives ∇^n, that give an isomorphism

$$\mathrm{Jet}(\mathbb{R}_X/X) := \varprojlim_n \mathrm{Jet}^n(\mathbb{R}_X/X) \xrightarrow[\sim]{\nabla} \varprojlim_n \oplus_{k=0}^n \mathrm{Hom}_{\mathbb{R}_X}(\mathrm{Sym}^k(TX), \mathbb{R}_X)$$

of bundles over X.

Remark 19.1.1. If $X = \underline{\Gamma}(M, C)$ is the infinite dimensional space of fields of a given classical field theory, the bundle $\mathrm{Sym}^k(TX)$ has fiber at $\varphi \in X$ the space

$$\mathrm{Sym}^k(TX)_\varphi \cong \Gamma(M^k, \varphi^* TC^k)_{S_k}.$$

Now we make the stronger assumption that the tangent bundle TX is trivialized by an isomorphism

$$TX \overset{i}{\underset{\sim}{\longrightarrow}} X \times V$$

with V a given vector space. This gives in particular an affine connection as above (the trivial one). This hypothesis is true if X is an affine space, i.e., a principal homogeneous space under a vector space V (if X itself is a vector space, one gets a canonical identification $X \cong V$). We then have an isomorphism

$$\mathrm{Jet}(\mathbb{R}_X/X) \xrightarrow{\sim} \varprojlim X \times \oplus_{k=0}^n \mathrm{Hom}_{\mathbb{R}}(\mathrm{Sym}^k(V), \mathbb{R}).$$

Now we are able to explain the geometric meaning of perturbative computations. in quantum field theory. One works with the jet bundle $\mathrm{Jet}(\mathbb{R}_X)$ of functions on the space $X = \underline{\Gamma}(M, C)$ of fields around a given so-called background field φ_0, solution of the Euler-Lagrange equation. If the bundle C is linear, one usually choses simply $\varphi_0 = 0$. This situation is exactly the one postulated by Costello in his book [Cos11] and Epstein-Glaser in their original paper [EG73].

Now to explain fully the computations of perturbative renormalization, one also has to work on the bigger space Y of theories, given by $Y := \mathcal{O}(X) = \mathrm{Hom}(X, \mathbb{R})$, or of local theories, $Y = \mathcal{O}^{loc}(X)$. One then makes perturbative computations for functionals $F \in \mathcal{O}(Y)$ (i.e., functionals $F(S)$ of the action functionals S) around the gaussian theory S_{free}, meaning we are working with $\mathrm{Jet}(\mathbb{R}_Y)$. This situation is exactly the one postulated by Borcherds [Bor10] (he works on the space $\mathcal{O}^{loc}(X)$ of local Lagrangian densities, also sometimes called composite fields in the physics litterature).

We now see clearly the interest of working in the setting of functorial analysis: it is much more easy to deal with complicated spaces of functionals on functionals on functions, like $\mathcal{O}(\mathcal{O}^{loc}(\underline{\Gamma}(M, C)))$, that are hard to consider using functional analysis and topological vector spaces.

We refer to Chapter 21 for a detailled application of these methods to the Epstein-Glaser causal perturbative quantum field theory approach, and to Chapter 20 for Costello's description of the perturbative Wilson effective field theory for gauge theories.

We will first present in the next section the nonperturbative definition of a quantum field theory, whose perturbative expansion in our setting gives back Costello's effective quantum BV formalism.

19.2 Nonperturbative quantum field theory

We will use the Wilson effective field theory approach to quantum field theory. We refer to Delamotte's survey [Del07] for a physical introduction to this formalism, that was worked out by Polchinski and Wetterich. This approach is called non-perturbative because it allows to choose truncations that are different of the usual one, around the gaussian and around a background field. We won't go into these technical details, and further refer to Gies' survey [Gie06] for complementary informations (see also [Ros12] and [Paw07]).

Let $\pi : C \to M$ be a bundle and $X = \underline{\Gamma}(M, C)$ be its space of sections. Suppose given an action functional $S : X \to \mathbb{R}$. The Wetterich version of the renormalization group equation is a vector field on the space $Y = \mathcal{O}_X$ of theories, that one wants to solve starting from the initial condition $S \in Y$. The definition of this vector field depends on an auxiliary function R_k of the scale parameter of the theory (a family of real numbers k), and looks like:

$$\partial_k S_k \quad = \quad \frac{1}{2}\,\tilde{\partial}_k \mathrm{Tr}\,\mathrm{Log}\left(S_k^{(2)} + R_k\right),$$

where $\tilde{\partial}_k := \frac{\partial R_k}{\partial k}\frac{\partial}{\partial R_k}$. The application of this operator means that the above equation is actually a vector field living on the space $Y \times Z$ of pairs (S, R) composed of an effective action S and a regularization operator R, that must be of the same nature as the second order field derivative of S_k. One wants to solve it using the initial condition

$$S_0(\varphi) = S(\varphi).$$

Recall that if $S : X \to \mathbb{R}$ is a functional, seen as living in $\Gamma(X, \mathbb{R}_X)$, its jet is a section

$$j_\infty S : X \to \mathrm{Jet}(\mathbb{R}_X).$$

If we want to see the second order derivative as a well defined expression, we need an affine connection on the space of fields, that may be induced by a (partially defined) metric on the space of fields by the Levi-Civita method. We thus suppose given a metric

$$g_X : TX \times_X TX \to \mathbb{R}_X$$

and a compatible affine connection

$$\nabla : \Theta_X \to \Theta_X \otimes \Omega^1_X,$$

that is supposed to be *flat*. In this flat affine situation, we have an "identification"

$$\mathrm{Jet}(\mathbb{R}_X) \cong \prod_{n \geq 0} \mathrm{Hom}_{\mathbb{R}_X}(TX^{\otimes n}, \mathbb{R}_X)^{S_n}$$

of fiber bundles over X given by the symmetric powers of ∇, and we can define the second order derivative of S as a section

$$D^2 S \in \Gamma(X, \mathrm{Hom}_{\mathbb{R}_X}(TX^{\otimes 2}, \mathbb{R}_X)) \cong \Gamma\left(X, \mathrm{Hom}_{\mathbb{R}_X - sym}(TX, \mathrm{Hom}_{\mathbb{R}_X}(TX, \mathbb{R}_X))\right).$$

Using the metric $g_X : TX \to \mathrm{Hom}_{\mathbb{R}_X}(TX, \mathbb{R}_X)$, or more exactly its (partially defined) inverse

$$g_X^{-1} : \mathrm{Hom}_{\mathbb{R}_X}(TX, \mathbb{R}_X) \to TX,$$

we can send D^2S to a section

$$g_X^{-1} \circ D^2S \in \Gamma\left(X, \mathrm{Hom}_{\mathbb{R}_X}(\mathrm{End}(TX))\right).$$

We suppose also given another section

$$R \in \Gamma(X, \mathrm{Hom}_{\mathbb{R}_X}(TX^{\otimes 2}, \mathbb{R}_X))$$

that is the regularization parameter. This allows us to define

$$g_X^{-1} \circ (D^2S + R) \in \Gamma(X, \mathrm{Hom}_{\mathbb{R}_X}(\mathrm{End}(TX))).$$

There is a partially defined logarithm $\mathbb{R}_X$-linear morphism of bundles

$$\mathrm{Log} : \mathrm{End}(TX) \to \mathrm{End}(TX)$$

given by the usual power series

$$\mathrm{Log}(M) := \sum_{n\geq 0} (-1)^n \frac{(M-I)^{n+1}}{n+1}.$$

There is also a trace map

$$\mathrm{Tr} : \mathrm{End}(TX) \to \mathbb{R}_X,$$

given on the fibers by seeing an operator

$$T_{\varphi_0}X \to T_{\varphi_0}X$$

as a continuous (i.e., often, L^2) operator for the given metric g_{X,φ_0}. Applying this trace to the above section, we get

$$\mathrm{Tr}\left(\mathrm{Log}(g_X^{-1} \circ (D^2S + R))\right) \in \Gamma(X \times Z, \mathbb{R}_{X\times Z}).$$

One can then write down without coordinates the nonperturbative renormalization group equation

$$\partial_k S_k = \frac{1}{2}\tilde{\partial}_k \mathrm{Tr}\,\mathrm{Log}(g_X^{-1} \circ (D^2S_k + R_k)),$$

where $\tilde{\partial}_k := \frac{\partial R_k}{\partial k}\frac{\partial}{\partial R_k}$. This equation is essentially equivalent to

$$\partial_k S_k = \frac{1}{2}\mathrm{Tr}\left(\partial_k R_k (D^2S_k + R_k)^{-1}\right).$$

Remark that the expression $\partial_k R_k(D^2S_k + R_k)^{-1}$ in the above trace may be interpreted as a regularized version of the ill-defined nonperturbative propagator $(D^2S)^{-1}$, that correspond to the solutions of the equations of motion. This interpretation is important to understand the relation of the above equation with its perturbation around the gaussian.

19.3 The Dyson-Schwinger quantum equations of motion

One can base on the nonperturbative renormalization group equation

$$\partial_k S_k = \frac{1}{2}\mathrm{Tr}\left(\partial_k R_k (D^2 S_k + R_k)^{-1}\right)$$

to give a regularized version of the Dyson-Schwinger equation, by taking functional derivatives of it.

Definition 19.3.1. The *regularized Dyson-Schwinger equations* is the family of equations

$$\frac{\delta}{\delta \vec{v}_1 \ldots \delta \vec{v}_n}\partial_k S_k = \frac{1}{2}\frac{\delta}{\delta \vec{v}_1 \ldots \delta \vec{v}_n}\mathrm{Tr}\left(\partial_k R_k (D^2 S_k + R_k)^{-1}\right)$$

derived from the nonperturbative renormalization group equation by functional derivation along a family $\{\vec{v}_i\}_{i\in\mathbb{N}}$ of vector fields on the space X of fields.

Usually, physicists use the ill-defined vector fields given by derivation along δ functions. This is well defined only when one works with local functionals.

19.4 Nonperturbative quantum gauge theory

We now use Section 12.5, where we described the Batalin-Vilkovisky gauge fixing procedure, to give a presentation of non-perturbative quantum gauge theories (that is explain in perturbative physical terms in [HT92], without a precise regularization procedure; see also [Bar00] for a perturbative formalism that takes care of anomalies). We refer to Gies' survey [Gie06] for a physical introduction to nonperturbative quantum gauge theory.

In the following, if X is a graded space, we denote $\mathcal{O}(X) := \mathrm{Hom}_{\mathrm{PAR}}(X, \mathbb{A})$ where $\mathbb{A}$ is the *full* graded affine space whose points in a smooth graded algebra A are given by $\mathbb{A}(A) = A$.

We recall the notations of Section 12.5. Let $\mathcal{A}$ be the jet algebra of a bundle $C \to M$ and $S \in h(\mathcal{A})$ be a local action functional with gauge symmetries. Suppose given a Batalin-Vilkovisky algebra $\mathcal{A}_{BV}$, a solution S_{cm} to the classical master equation and a graded gauge fixing

Definition 19.4.1. A *nonperturbative quantization* of the gauge theory S is the datum of

1. a graded gauge fixing in the sense of Definition 12.5.3, i.e., a symplectic quasi-isomorphism

$$i : \mathcal{A}_{BV}^{nm} \xrightarrow{\sim} \mathcal{A}_{BV}$$

and a graded Lagrangian subspace

$$\psi : L \hookrightarrow \underline{\mathrm{Spec}}_{\mathcal{D}}(\mathcal{A}_{BV}^{nm}),$$

that gives a gauge fixed action $S_\psi \in h(\mathcal{O}_L)$.

2. a family Γ_k of functionals $\Gamma_k : X^{nm}_{BV} \to \mathbb{R}$, where X^{nm}_{BV} is the graded space of solutions of the graded $\mathcal{D}$-algebra $\mathcal{A}^{nm}_{BV}$.

3. a family R_k of regularization variables
$$R_k : L \to \mathrm{Hom}_{\mathbb{R}_L}(TL^{\otimes 2}, \mathbb{R}_L)^{S_2},$$
such that $\Gamma_k(-,0)$ gives a solution to the nonperturbative renormalization group equation
$$\partial_k \Gamma_k(-,0) = \frac{1}{2}\mathrm{Tr}\left(\partial_k R_k (D^2\Gamma_k(-,0) + R_k)^{-1}\right),$$

4. with initial condition $\Gamma_0 = S_\psi$,

5. and further fulfils the *Zinn-Justin equation*
$$\{\Gamma_k, \Gamma_k\} = \Delta(R_k),$$
where Δ is a functional of the regularization variable whose limit at infinity is 0.

We now describe the concrete implementation of the above construction using the original Batalin-Vilkovisky gauge fixing procedure. Let
$$\mathcal{A}_{fields} = \mathrm{Sym}_{\mathcal{A}}(\mathfrak{g}^\circ[-1] \oplus \mathfrak{a}_S[1] \oplus \mathfrak{l}_S)$$
be the algebra of fields variables (with coordinates the ghosts, antighosts and their lagrange multipliers). For simplicity, we will denote
$$X_{BV} = \underline{\mathrm{Spec}}_{\mathcal{D}}(\mathcal{A}_{BV}) := T^*[-1]X_{fields}$$
the non-minimal Batalin-Vilkovisky graded symplectic space. We will denote E_{BV} and E_{fields} the bundles over C whose jet algebras give the algebras $\mathcal{A}_{BV}$ and $\mathcal{A}_{fields}$, and $S_{cm} \in h(\mathcal{A}_{BV})$ the (extended) classical master solution, so that the associated dg-$\mathcal{D}$-structure is the usual one on X_{BV}. We thus have a graded gauge fixing condition.

We will also denote $X_{fields} := \underline{\Gamma}(M, E_{fields})$ and $X_{BV} := \underline{\Gamma}(M, E_{BV})$ the associated spaces of sections. A morphism between these spaces is called local if it comes from a morphism of the underlying graded $\mathcal{D}$-spaces. One may identify the de Rham cohomologies of the jet algebras with algebras of functions on those spaces with values in $\mathbb{A}$, meaning that there are natural injections
$$h(\mathcal{A}_{fields}) \to \mathcal{O}(X_{fields}) \text{ and } h(\mathcal{A}_{BV}) \to \mathcal{O}(X_{BV}).$$

There is a natural local projection morphism
$$\pi : X_{BV} \to X_{fields},$$
and we fix a local gauge fixing $\psi : X_{fields} \to \mathbb{A}$. This gives an associated local canonical transformation
$$f_\psi : X_{BV} \to X_{BV}$$

and Lagrangian subspace

$$L = X_{fields} \xrightarrow{f_\psi \circ 0} X_{BV} \ .$$

The gauge fixed action is the local functional

$$S_\psi := S_{cm} \circ f_\psi \circ 0 : X_{fields} \to \mathbb{R}.$$

If we solve perturbatively the renormalization group equation around a background field φ_0, finding a family of formal functionals: $\Gamma_k : \widehat{X_{BV}}_{\varphi_0} \to \mathbb{R}[[\hbar]]$, we may define observables as solutions to the associated regularized Dyson-Schwinger equations.

Chapter 20

Perturbative renormalization à la Wilson

We shortly present here the results of Costello's book [Cos11], that gives a neat combination of the Batalin-Vilkoviski formalism with Wilson's effective field theory (see also [CG10] and [Gwi05] for applications to factorization algebras).

Remark that Wilson's renormalization group method can be also applied in non-perturbative settings, where $\hbar$ is not a formal variable but a real number, as was explained in Chapter 19. This nonperturbative formalization allows to use other types of truncation schemes, that sometimes have better analytic properties than the one we present here.

20.1 Energetic effective field theory

One usually considers that a (euclidean) effective quantum field theory is given by an effective action $S_{eff}(\Lambda, \varphi)$ that depends on a cut-off parameter Λ. Observables are then functionals

$$O : \mathcal{C}^\infty(M)_{\leq\Lambda} \to \mathbb{R}[[\hbar]]$$

on the space of fields (here smooth functions) whose expression in the basis of eigenvalues for the Laplacian contain only eigenvalues smaller that the cut-off energy Λ. One clearly has an inclusion of observables at energy Λ' into observables at energy Λ for $\Lambda' \leq \Lambda$. The working hypothesis for Wilson's method is that the mean value of an observable O at energy Λ is given by the functional integral of O on the space $\mathcal{C}^\infty_{\leq\Lambda}$ (which is an ordinary integral since this space is finite dimensional)

$$\langle O\rangle := \int_{\mathcal{C}^\infty_{\leq\Lambda}} O(\varphi)e^{S_{eff}(\Lambda,\varphi)}[d\varphi]$$

where $S_{eff}(\Lambda, \varphi)$ is a functional called the effective action. The equality between mean values for $\Lambda' \leq \Lambda$ induces an equality

$$S_{eff}(\Lambda, \varphi_L) = \hbar \log \int_{\varphi_H \in \mathcal{C}^\infty(M)_{[\Lambda',\Lambda]}} \exp\left(\frac{1}{\hbar}S_{eff}(\Lambda, \varphi_L + \varphi_H)\right).$$

This equation is called the renormalization group equation. The relation of the effective action with the local action S (ordinary datum defining a classical field theory) is given by an ill-defined infinite dimensional functional integral

$$S_{eff}(\Lambda,\varphi_L)=\hbar\log\left(\int_{\varphi_H\in\mathcal{C}^\infty_{[\Lambda,\infty]}}\exp\left(\frac{1}{\hbar}S(\varphi_L+\varphi_H)\right)\right).$$

A drawback of this energetic approach is that it is global on the manifold M because the eigenvalues of the Laplacian give global information. We will see in Section 20.4 how to use parametrices to give a localized version of this formalism.

20.2 Effective propagators

We refer to Section 3.7 for an introduction to heat kernel regularizations. The starting point is to remark that, if $\Delta+m^2$ is the linear operator for the equations of motion of the free Euclidean scalar field, its inverse (free propagator) can be computed by using the heat kernel K_τ, that is the fundamental solution of the heat equation

$$\partial_\tau u(\tau,x)=\Delta u(\tau,x),$$

with Δ the Laplacian (we here denote τ the time variable, since one think of it as the proper time of a Euclidean particle). This amounts to describe $e^{-\tau(\Delta+m^2)}$ as the convolution by a function $K_\tau(x,y)$ (called the heat kernel) to obtain the propagator

$$P(x,y)=\frac{1}{\Delta+m^2}=\int_0^\infty e^{-\tau(\Delta+m^2)}:=\int_0^\infty e^{-\tau m^2}K_\tau(x,y)d\tau.$$

The heat kernel can be interpreted as the mean value of a (Wiener) measure on the space of path on M, meaning that one can give a precise meaning to the equality

$$K_\tau(x,y)=\int_{f:[0,1]\to M,f(0)=x,f(1)=y}\exp\left(-\int_0^\tau\|df\|^2\right).$$

The propagator can then be interpreted as the probability for a particle to start at x and end at y along a given random path. This also permits to see Feynman graphs as random graphical paths followed by families of particles with prescribed interactions. It is also defined on a compact Riemannian manifold with the operator Δ_g+m^2, where g is the metric.

Theorem 20.2.1. *The heat kernel has a small τ asymptotic expansion, in normal coordinates near a point (identify M with a small ball B), of the form*

$$K_\tau\equiv\tau^{-\dim(M)/2}e^{-\|x-y\|^2}\sum_{i\geq 0}t^i\Phi_i$$

with $\Phi_i\in\Gamma(B,M)\otimes\Gamma(B,M)$. More precisely, if K_τ^N is the finite sum appearing in the asymptotic expansion, one has

$$\|K_\tau-K_\tau^N\|_{C^l}=O(t^{N-\dim(M/2)-l}).$$

One can moreover define a weak notion of heat kernel for non-compact varieties, whose use do not pose existential problems.

The effective propagator of the theory is then given by a cut-off of the heat kernel integral

$$P_{\epsilon,L}(x,y) := \int_{\epsilon}^{L} e^{-\tau m^2} K_\tau(x,y) d\tau.$$

20.3 The renormalization group equation

Let $E \to M$ be a graded fiber bundle and $\mathcal{E} = \Gamma(M, E)$ be its space of sections (nuclear Fréchet space). We denote $\mathcal{E}^{\otimes n}$ the space

$$\mathcal{E}^{\otimes n} := \Gamma(M^n, E^{\boxtimes n}).$$

Definition 20.3.1. A *functional* is an element of the space $\mathcal{O}$ of formal power series on $\mathcal{E}$, given by

$$\mathcal{O} = \prod_{n \geq 0} \mathrm{Hom}(\mathcal{E}^{\otimes n}, \mathbb{R})^{S_n},$$

where morphisms are supposed to be multilinear continuous maps. We denote $\mathcal{O}^+[[\hbar]] \subset \mathcal{O}_{\mathcal{E}}[[\hbar]]$ the space of functionals I that are at least cubic modulo $\hbar$. A *local functional* is an element of the space $\mathcal{O}_{\mathcal{E}}^{loc}$ of functionals that can be written

$$\Phi = \sum_n \Phi_n$$

with $\Phi_n : \mathcal{E}^{\otimes n} \to \mathbb{R}$ of the form

$$\Phi_n(e_1, \ldots, e_n) = \sum_{j=1}^{k} \int_M (D_{1,j} e_1) \ldots (D_{n,j} e_n) d\mu$$

with the $D_{i,j} : \mathcal{E} \to \mathcal{C}_M^\infty$ some linear differential operators. One can identify these local functionals to global sections of the tensor product over $\mathcal{D}_M$ of the algebra of formal power series on jets by the maximal differential forms on M.

One has a natural algebra structure on $\mathcal{O}_{\mathcal{E}}$.

Let $I \in \mathcal{O}^+[[\hbar]]$ be a functional and denote

$$I = \sum_{i,k} \hbar^i I_{i,k}$$

with $I_{i,k}$ homogenous of degree k in $\mathcal{O}$. We denote $\mathrm{Sym}^n \mathcal{E}$ the space of S_n-invariants in $\mathcal{E}^{\boxtimes n}$.

Definition 20.3.2. Let $P \in \mathrm{Sym}^2 \mathcal{E}$ and $I \in \mathcal{O}^+[[\hbar]]$ be given. Let ∂_P be the contraction with P. The *P-renormalization group* evolution of I is given by the functional

$$W(P, I) = \hbar \log(\exp(\hbar \partial_P) \exp(I/\hbar)) \in \mathcal{O}^+[[\hbar]].$$

Example 20.3.3. If one starts with the standard local interaction functional $I \in \mathrm{Hom}(\mathcal{C}^\infty(M^3), \mathbb{R})$, given by

$$I = \int_M \varphi^3(x)dx = \int_{M^3} \varphi(x)\varphi(y)\varphi(z)\delta_{\Delta_{123}} dxdydz,$$

of the φ^3 theory (where $\Delta_{123} = M \subset M^3$ is the diagonal) and $P(x,y) \in \mathcal{C}^\infty(M \times M)$ is symmetric in x and y, the contraction of I by P is the distribution $\partial_P(I) \in \mathcal{C}^\infty(M)'$ defined as

$$\langle \partial_P(I), \varphi \rangle = \sigma \int_M P(x,x)\varphi(x)dx = \sigma \int_{M^3} P(x,y)\varphi(z)\delta_{\Delta_{123}} dxdydz,$$

where σ is the symmetrization operator.

20.4 Effective theory with interaction

Definition 20.4.1. Let D be a generalized Laplacian on a graded bundle $E \to M$. A *parametrix* for the operator D is a distributional section $P \in \mathrm{Hom}(\mathcal{E}^{\otimes 2}, \mathbb{R})$ of $E \boxtimes E \to M \times M$, which is symmetric, smooth away from the diagonal, and such that

$$(D \otimes 1)P - K_0 \in \Gamma_{\mathcal{C}^\infty}(M \times M, E \boxtimes E),$$

where K_0 is the time zero heat operator (δ_M for the Laplacian on flat space).

For example, if $P(0,L)$ is the heat kernel propagator, $P = P(0,L)$ is a parametrix. The difference $P - P'$ between two parametrices is a smooth function. Parametrices are partially ordered by their support.

Definition 20.4.2. A function $J \in \mathcal{O}(\mathcal{E})$ has *smooth first derivative* if the continuous linear map

$$\mathcal{E} \to \mathcal{O}(\mathcal{E}), \varphi \mapsto \frac{\partial J}{\partial \varphi}$$

extends to the space $\mathcal{E}'$ of distributional sections of $\mathcal{E}$.

Definition 20.4.3. A *quantum field theory* is a collection of functionals

$$I[P] \in \mathcal{O}^+(\mathcal{E})[[\hbar]]$$

indexed by parametrices, such that

1. (Renormalization group equation) If P, P' are parametrices, then

   $$W(P - P', I[P']) = I[P].$$

   This expression makes sense , because $P - P'$ is smooth in $\Gamma(M \times M, E \boxtimes E)$.

2. (locality) For any (i,k), the support $\mathrm{supp}(I_{i,k}[P]) \subset M^k$ can be made as close as we like to the diagonal by making the parametrix P small (for any small neighborhood U of the diagonal, there exists a parametrix such that the support is in U for all $P' \leq P$).

3. The functionals $I[P]$ all have smooth first derivative.

One denotes $\mathcal{T}^{(n)}$ the space of theories defined modulo $\hbar^{n+1}$.

If $I[L]$ is a family of effective interactions in the length formulation,

$$I[P] = W(P - P(0, L), I[L])$$

gives a quantum field theory in the parametrix sense.

Remark that the effective action functional is given by

$$S_{eff}[P] = P(\varphi \otimes \varphi) + I[P](\varphi).$$

20.5 Feynman graph expansion of the renormalization group equation

We now explain the relation of Costello's approach to the physicists' Feynman graph computations. This is not strictly necessary to prove the basic theorems, but may be helpful to understand the relation with other approaches.

Definition 20.5.1. A *stable graph* is a graph γ, with external edges, and for each vertex v, an element $g(v) \in \mathbb{N}$ called the *genus* of the vertex, with the property that any vertex of genus 0 is at least trivalent.

If $P \in \mathrm{Sym}^2 X$ (a propagator) and $I \in \mathcal{O}_X^+[[\hbar]]$ (an interaction), for every stable graph γ, one defines

$$w_\gamma(P, I) \in \mathcal{O}_X$$

by the following.

Let $T(\gamma)$ be the set of tails, $H(\gamma)$ be the set of half edges, $V(\gamma)$ the set of vertices and $E(\gamma)$ the set of internal edges. We denote $b_1(\gamma)$ the first Betti number of the graph. The tensor product of interactions $I_{i,k}$ at vertices of γ with valency k and genus i define an element of

$$\mathrm{Hom}(X^{\boxtimes H(\gamma)}, \mathbb{R}).$$

One can define an element of

$$X^{\boxtimes 2E(\gamma)} \boxtimes X^{\boxtimes T(\gamma)} \cong X^{\boxtimes H(\gamma)}$$

by associating to each internal edge a propagator and to each tail a field $\varphi \in X$. The contraction of these two elements give an element

$$w_\gamma(P, I)(\varphi) \in \mathbb{R}.$$

To say it differently, propagators give an element of $X^{\boxtimes 2E(\gamma)}$ that can be contracted to get

$$w_\gamma(P, I) \in \mathrm{Hom}(X^{\boxtimes T(\gamma)}, \mathbb{R}).$$

One can thus define

$$W(P, I) = \sum_\gamma \frac{1}{|\mathrm{Aut}(\gamma)|} \hbar^{b_1(\gamma)} w_\gamma(P, I) \in \mathcal{O}_X^+[[\hbar]].$$

Example 20.5.2. Take a scalar φ^3 theory with

$$I = I_{3,0} = \int_{M^3} \varphi_1(x)\varphi_2(y)\varphi_3(z)\delta_\Delta(x,y,z)d\mu_3 := \int_M \varphi_1(x)\varphi_2(x)\varphi_3(x)d\mu,$$

where $\Delta : M \subset M^3$ is the diagonal. The effective propagator $P(\epsilon, L)$ for the Laplacian is a distribution on M^2, that is given by a smooth function on M^2, leaving in $\mathcal{C}^\infty(M^2, \mathbb{R}^{\boxtimes 2}) = \mathcal{C}^\infty(M^2)$. One can also see it as an element of the tensor product $\mathcal{C}^\infty(M) \otimes \mathcal{C}^\infty(M)$, i.e., of $X^{\boxtimes 2}$. One associates to the vertex of valency 3 simply

$$I_{3,0} \in \mathrm{Hom}(X^{\boxtimes 3}, \mathbb{R}).$$

If one has a graph with only one propagator and two vertices of valency 3 and genus 0, one associate to it the contraction of the tensor product

$$I_{3,0} \otimes I_{3,0} \in \mathrm{Hom}(X^{\boxtimes 6}, \mathbb{R})$$

with propagator $P = P(\epsilon, L) \in X^{\boxtimes 2}$, that gives a morphism

$$w(P,\gamma) \in \mathrm{Hom}(X^{\boxtimes 4}, \mathbb{R}).$$

The explicit expression for $w(P,\gamma)$ is given by

$$\begin{array}{rcl} w(P,\gamma)(\varphi_1,\ldots,\varphi_6) & := & \sigma \int_{M^6} \varphi_1(x_1)\varphi_2(x_2)P(x_3,x_4)\varphi_5(x_5)\varphi_6(x_6)\delta_{\Delta_{123}}(x_i)\delta_{\Delta_{456}}(x_i)d\mu_6 \\ & := & \sigma \int_{M^2} \varphi_1(x_1)\varphi_2(x_1)P(x_1,x_2)\varphi_5(x_2)\varphi_6(x_2)d\mu_2, \end{array}$$

were $\Delta_{123} : M \subset M^6$ and $\Delta_{456} : M \subset M^6$ are the diagonal maps into the indexed subspaces and σ is the symmetrization operator. Remark that this new functional $w(P,\gamma)$ is not supported on the diagonal $\Delta_{123456} : M \subset M^6$, so that it is not anymore a local functional.

One can associate to $P \in \mathrm{Sym}^2 X$ a differential operator

$$\partial_P : \mathcal{O}_X \to \mathcal{O}_X$$

that gives a contraction in degree n. One then gets

$$W(P,I) = \hbar \log(\exp(\hbar\partial_P)\exp(I/\hbar)),$$

that is the usual equation for the definition of Feynman graphs.

20.6 The quantization theorem

Period variations are functions $f(t)$ on $]0,\infty[$ of the form

$$f(t) = \int_{\gamma_t} \omega_t$$

with ω a relative logarithmic differential form on a family of varieties $(X,D) \to U$ with normal crossing divisor defined on an open subset of the algebraic affine line $\mathbb{A}^1_{\mathbb{R}}$ that contains $]0,\infty[$. We denote $\mathcal{P} = \mathcal{P}(]0,\infty[)$ the set of period variations. These are smooth functions of t. We denote $\mathcal{P}_{\geq 0} \subset \mathcal{P}$ the sub-space of period variations that have a limit at 0.

Definition 20.6.1. A *renormalization scheme* is a choice of a complementary subspace (singular parts) $\mathcal{P}_{<0}$ to $\mathcal{P}_{\geq 0}$ in $\mathcal{P}$, i.e., of a direct sum decomposition

$$\mathcal{P} = \mathcal{P}_{\geq 0} \oplus \mathcal{P}_{<0}.$$

The *singular part* of a period variation for this scheme is its projection

$$\mathrm{sing} : \mathcal{P} \to \mathcal{P}_{<0}.$$

Theorem 20.6.2. *The space $\mathcal{T}^{(n+1)} \to \mathcal{T}^{(n)}$ is canonically equipped with a principal bundle structure under the group $\mathcal{O}_X^{loc}$ of local functionals. Moreover, $\mathcal{T}^{(0)}$ is canonically isomorphic to the space $\mathcal{O}^{+,loc}$ of local functionals that are at least cubic modulo $\hbar$.*

Proof. Remark first that the proof of the theorem uses a renormalization scheme, but that its statement doesn't. So let's fix a renormalization scheme. If $I[P]$ and $J[P]$ are two theories that coincide modulo $\hbar^{n+1}$, then

$$I_{0,*}[P] + \delta\hbar^{-(n+1)}(I[P] - J[P]) \in \mathcal{O}[\delta]/\delta^2$$

define a tangent vector to the space $\mathcal{T}^{(0)}$ of classical theories, that is canonically isomorphic to $\mathcal{O}^{+,loc}$. To prove the surjectivity, one needs to show that if

$$I[P] = \sum_{i,k} \hbar^i I_{i,k}[P] \in \mathcal{T}^{(n)}$$

is a theory defined modulo $\hbar^{n+1}$, there exist counter-terms

$$I^{CT}(\epsilon) = \sum_{i,k} \hbar^i I_{i,k}^{CT}(\epsilon) \in \mathcal{O}^{loc}[[\hbar]] \otimes \mathcal{P}_{<0}$$

such that

$$I[P] := \lim_{\epsilon \to 0} W(P - P(0,\epsilon), I - I^{CT}(\epsilon))$$

exists and defines a theory in $\mathcal{T}^{(n+1)}$. Recall that the renormalization group equation reads

$$W(P - P', I) = \hbar \log(\exp(\hbar\partial_{P-P'}) \exp(I/\hbar))$$

for P and P' parametrices. One has by construction

$$W(P - P', W(P' - P", I[P"])) = W(P - P", I[P"])$$

and decomposing in homogeneous components in $\hbar$ and $\mathcal{E}$, one also has

$$W(P - P', I)_{i,k} = W(P - P', I_{<(i,k)})_{i,k} + I_{i,k}.$$

Suppose that the limit of

$$W_{<(i,k)}\left(P - P(0,\epsilon), I - \sum_{(r,s)<(i,k)} \hbar^r I_{r,s}^{CT}(\epsilon)\right)$$

as ϵ tends to 0 exists. Using the above relations, one have to set

$$I^{CT}_{(i,k)}(\epsilon) := \mathrm{sing}_\epsilon\left(W_{(i,k)}(P - P(0,\epsilon), I - \sum_{(r,s)<(i,k)} \hbar^r I^{CT}_{(r,s)}(\epsilon))\right).$$

One has to check that this expression does not depend on P. This can be proven in the case $P = P(0, L)$ by showing that derivative in L of what is inside the singular part is non-singular: it is given by graphs with vertices containing small degree expressions that are non-singular by induction and one just add a propagator $P(\epsilon, L)$ between them that is also non-singular. Some more work is needed to prove the locality axiom, on the base of the asymptotic expansion of the heat kernel. □

20.7 The effective quantum BV formalism

Definition 20.7.1. A *free BV theory* is given by

1. A $\mathbb{Z}$-graded vector bundle E on M, whose graded space of global sections is denoted $\mathcal{E}$.
2. An odd antisymmetric pairing of graded vector bundles
$$\langle,\rangle_{loc} : E \otimes E \to \mathrm{Dens}(M)[-1]$$
that is supposed to be fiber-wise non-degenerate.
3. A differential operator $Q : \mathcal{E} \to \mathcal{E}[1]$ of square zero that is anti-self-adjoint for the given pairing such that $H^*(\mathcal{E}, Q)$ is finite dimensional.
4. a solution $I \in \mathcal{O}^+_{loc}$ of the classical master equation
$$Q(I) + \frac{1}{2}\{I, I\} = 0.$$

A *gauge fixation* on a BV theory is the datum of an operator

$$Q^{GF} : \mathcal{E} \to \mathcal{E}[-1]$$

such that

1. Q^{GF} is odd, of square zero and self-adjoint,
2. the commutator
$$D := [Q, Q^{GF}]$$
is a (generalized) Laplacian.

One thinks of Q^{GF} as a homological version of the ordinary gauge fixing, that gives a way to fix a Lagrangian subspace in $\mathcal{E}$ on which the quadratic form $\int \varphi D\varphi$ is non-degenerate.

Definition 20.7.2. A *parametrix* Φ for a BV theory is a symmetric distributional section of $(E \boxtimes E)[1]$ on $M \times M$ closed under the differential $Q \otimes 1 + 1 \otimes Q$, with proper support and such that $(D \otimes 1)\Phi - K_0$ is a smooth section of $E \boxtimes E$. The *propagator* of a parametrix is

$$P(\Phi) = (Q^{GF} \otimes 1)\Phi$$

and its *fake heat kernel* is

$$K(\Phi) = K_0 - (D \otimes 1)\Phi.$$

The difference between two parametrix is smooth because it is in the kernel of the elliptic operator D.

The effective BV Laplacian is defined as the insertion of $K(\Phi)$ by

$$\Delta_\Phi := -\partial_{K(\Phi)} : \mathcal{O} \to \mathcal{O}.$$

The ill-defined BV Laplacian would be

$$\Delta = \lim_{L \to 0} \Delta_{P(0,L)}.$$

One defines a bracket $\{,\}_\Phi$ on $\mathcal{O}$ by the formula

$$\{\alpha, \beta\}_\Phi = \Delta_\Phi(\alpha\beta) - (\Delta_\Phi \alpha)\beta - (-1)^{|\alpha|}\alpha \Delta_\Phi \beta$$

that measures the obstruction for Δ_Φ to be a graded-derivation.

Definition 20.7.3. A functional $I[\Phi] \in \mathcal{O}^+[[\hbar]]$ satisfies the *scale* Φ *quantum master equation* if

$$QI[\Phi] + \{I[\Phi], I[\Phi]\}_\Phi + \hbar \Delta_\Phi I[\Phi] = 0.$$

One can also show that

$$[\partial_{P(\Psi)-P(\Phi)}, Q] = \partial_{K_\Psi} - \partial_{K_\Phi} = \Delta_\Phi - \Delta_\Psi$$

so that $\partial_{P(\Psi)-P(\Phi)}$ defines a homotopy operator between the BV Laplacians at different scales. Remark that if $I[\Phi]$ satisfies the scale Φ quantum master equation, then

$$I[\Psi] = W(\Psi - \Phi, I[\Phi])$$

satisfies the scale Ψ quantum master equation.

Definition 20.7.4. A *BV theory* is the datum of a family $I[\Phi] \in \mathcal{O}[[\hbar]]$ such that

1. the collection defines a theory in the classical sense.

2. the collection fulfills the quantum master equation.

We denote $\mathcal{T}^n_{BV}$ the space of BV theories defined up to order n in $\hbar$.

By Theorem 20.6.2, one can always associate an effective theory to the gauge fixed local theory (that has no symmetries anymore). However, there exist obstructions for this to be constructed and to also fulfill the quantum master equation. More precisely, if one looks at the application

$$\mathcal{T}_{BV}^{n+1} \longrightarrow \mathcal{T}_{BV}^{n}$$

between theories defined up to order $n+1$ in $\hbar$ and theories defined up to order n, the obstruction to its lifting is of local cohomological nature, i.e., is given by a simplicial map

$$O_{n+1} : \mathcal{T}^n \longrightarrow \mathcal{O}_{loc}$$

for families of theories parametrized by $\Omega^*(\Delta^*)$.

Theorem 20.7.5. *The obstruction to lift a solution to quantum master up to order n to a solution to quantum master up to order $n+1$ is given by a cohomology class in the classical local BV algebra equipped with the differential $\{S_{BV}, .\}$.*

Proof. Let $I[P]$ be a theory defined up to order $\hbar^{n+1}$ that fulfills quantum BV and $\tilde{I}[P]$ be a theory defined up to order $\hbar^{n+2}$. Let

$$O_{n+1}[P] := \hbar^{-(n+1)}\left(Q(\tilde{I}) + \frac{1}{2}\{I, I\}_P + \hbar\Delta_P(\tilde{I})\right)$$

be the obstruction for I to fulfill the QME. Let $\tilde{J}$ be another lifting and $J := \hbar^{-(n+1)}(\tilde{I} - \tilde{J})$. □

20.8 Observables and the partition function

We refer to [CG10] for the following considerations.

Let X denote the space of fields and $\mathcal{T}$ be the space of theories. One may see the partition function as a function

$$Z : \mathcal{T} \to \mathbb{R}$$

on the space of functionals (i.e., theories), given by the formal formula

$$Z(T) := \int_X e^{S(\varphi)+T(\varphi)}[d\varphi],$$

that makes sense when M is discrete. The advantage of this presentation is that we don't suppose that T is given by a linear interaction of the form

$$T_J(\varphi) = \int_M J(x)\varphi(x)dx,$$

so that we don't have to suppose that the space of fields is linear.

Now remark that if f is an observable, we have the equality

$$\int_X f(\varphi)e^{S(\varphi)}[d\varphi] = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int e^{S(\varphi)+\epsilon f(\varphi)}[d\varphi] =: \frac{\delta Z}{\delta f}(0).$$

This implies that the mean value

$$\langle f\rangle_S := \frac{\int_X f(\varphi)e^{S(\varphi)}[d\varphi]}{Z(0)}$$

may be computed by the formula

$$\langle f\rangle_S := \frac{1}{Z(0)}\frac{\delta Z}{\delta f}(0).$$

The conclusion is that we may interpret observables f for a theory S as living in the tangent space

$$T_S\mathcal{T}$$

at S of the space of theories.

One may formulate the above formal computations properly by defining the mean value of an observable $f \in T_S\mathcal{T}$ (for a given decomposition $S = Q + I$ of S in a quadratic part Q and an interaction part I), by the formula

$$\langle f\rangle_I := \frac{d}{d\epsilon}\left(\log e^{\hbar\partial_P}e^{(I+\epsilon f)/\hbar}\right) = \frac{d}{d\epsilon}W(P, I+\epsilon f).$$

20.9 The local renormalization group action

We now consider theories on $\mathbb{R}^n$, that is not compact. One thus has to restrict to functionals with some decreasing conditions at infinity for the homogeneous components. Remark that Hollands and Wald [HW10] also use a renormalization group in curved spacetime, by acting directly on the metric itself.

Consider the operation

$$R_l : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

defined by

$$R_l(\varphi)(x) = l^{n/2-1}\varphi(lx).$$

If $I \in \mathcal{O}(\mathcal{S}(\mathbb{R}^n))$ is a functional on $\mathcal{S}(\mathbb{R}^n)$, one defines

$$R_l^*(I)(\varphi) := I(R_{l^{-1}}(\varphi)).$$

Definition 20.9.1. The *local renormalization group* is the action

$$\mathcal{RG}_l : \mathcal{T}^{(\infty)} \to \mathcal{T}^{(\infty)}$$

on the space of theories defined by

$$\mathcal{RG}_l(\{I_L\}) := \{R_l^*(I_{l^2L})\}.$$

One shows that the dependence on l is in $\mathbb{C}[\log l, l, l^{-1}]$.

Definition 20.9.2. A theory is called *valuable* if the terms of its effective action vary in $l^k(\log l)^r$ when $k \geq 0$ for $r \in \mathbb{Z}_{\geq 0}$. We denote $\mathcal{R}^{(\infty)}$ the space of valuable theories. It is called *renormalizable* if it is valuable and, term by term in $\hbar$, it has only a finite number of deformations, i.e., the tangent space $T_{\{I_L\}}\mathcal{R}^{(n)}$ is finite dimensional.

Costello uses the above notion to give in [Cos11], Theorem 8.1.1, a complete classification of renormalizable scalar field theories on $\mathbb{R}^n$. In particular, he gets the following theorem on $\mathbb{R}^4$.

Theorem 20.9.3. *Renormalizable scalar field theories on $\mathbb{R}^4$, invariant under $\mathrm{SO}(4) \rtimes \mathbb{R}^4$ and under $\varphi \mapsto -\varphi$, are in bijection with Lagrangians of the form*

$$\mathcal{L}(\varphi) = a\varphi D\varphi + b\varphi^4 + c\varphi^2$$

for $a, b, c \in \mathbb{R}[[\hbar]]$, where $a = -1/2$ modulo $\hbar$ and $b = 0$ modulo $\hbar$.

20.10 Application to pure Yang-Mills theory

In the Yang-Mills case of dimension 4, the graded space E is given by

$$\mathcal{E} = \Omega^0(M) \otimes \mathfrak{g}[1] \oplus \Omega^1(M) \otimes \mathfrak{g} \oplus \Omega^3(M) \otimes \mathfrak{g}[-1] \oplus \Omega^4(M) \otimes \mathfrak{g}[-2].$$

The elements of the four components of $\mathcal{E}$ are respectively called the ghosts, fields, antifields and antighosts and denoted X, A, $A^\vee$ and $X^\vee$.

The differential is given by

$$\Omega^0(M) \otimes \mathfrak{g} \xrightarrow{d} \Omega^1(M) \otimes \mathfrak{g} \xrightarrow{d*d} \Omega^3(M) \otimes \mathfrak{g} \xrightarrow{d} \Omega^4(M) \otimes \mathfrak{g}.$$

Recall that the Yang-Mills field is in $\Omega^1(M)\otimes\mathfrak{g}$ and the gauge symmetries are in $\Omega^0(M)\otimes\mathfrak{g}$.

The pairing on $\Omega^*(M) \otimes \mathfrak{g}$ is given by

$$\langle \omega_1 \otimes E_1, \omega_2 \otimes E_2 \rangle := \int_M \omega_1 \wedge \omega_2 (E_1, E_2)_{\mathfrak{g}}.$$

The Yang-Mills action functional is given by

$$S(A) = \frac{1}{2} \int_M \left[\langle F_A, F_A \rangle + \langle F_A, *F_A \rangle \right].$$

The first term is topological and the second is the usual Yang-Mills functional.

The solution of the classical master equation is obtained by adding to the Lagrangian the ghost terms

$$\frac{1}{2}\langle [X, X], X^\vee \rangle$$

and the terms that give the action of the ghosts on fields

$$\langle dX + [X, A], A^\vee \rangle.$$

One has to use a first order formulation of Yang-Mills theory to have a gauge fixing (such that $D = [G, G^{GF}]$ is a Laplacian, and not a differential operator of degree 4). This theory has two fields: one in $\Omega^1(M) \otimes \mathfrak{g}$ and the other auto-dual in $\Omega^2_+(M) \otimes \mathfrak{g}$.

The problem of proving the existence of an effective BV theory is then reduced to a computation of a cohomological obstruction. We refer to Costello's book for more details and for applications to renormalization of pure Yang-Mills in dimension 4.

Chapter 21

Causal perturbative quantum field theory

This section gives a short self contained and coordinate free presentation of causal perturbative quantum field theory in the spirit of the Bogoliubov-Epstein-Glaser renormalization method (see [EG73]), in the version developped on curved spacetime by Brunetti and Fredenhagen [BF09] (see also [Kel07] for more references and the relation with the Connes-Kreimer approach of Chapter 18), Brouder [Bro09], and Borcherds [Bor10] (see also [BBD$^+$12]).

Remark that these methods have been extended to gauge theories by Fredenhagen and Rejzner [FR11] (see also [Rej11]) and may also be applied to the euclidean setting, thanks to the work of Keller [Kel09].

We adapt the presentation of loc. cit. to our functorial setting, which allows us to clarify the mathematical status of the space of fields underlying these computations, and to precise the generality of the gauge fixing procedure.

21.1 Moyal type star products and Hopf algebras

Let M be a finite dimensional manifold and $\pi \in \wedge^2 \Theta_M$ be a Poisson bracket. Let $\mathbb{R}_M \to M$ be the trivial bundle and $\mathrm{Jet}(\mathbb{R}_M) \to M$ be the associated jet bundle. We suppose given a *flat* affine connection ∇ on M that is compatible with the Poisson bracket, meaning that π is horizontal for ∇. The symmetric powers of ∇ give an isomorphism

$$\mathrm{Jet}(\mathbb{R}_M) \xrightarrow{\sim} \prod_{n\geq 0} \mathrm{Hom}_{\mathbb{R}_M}(TM^{\otimes n}, \mathbb{R}_M)^{S_n}.$$

Recall that the Poisson bracket of two functions f and g is defined by

$$\{f, g\} := \langle df \wedge dg, \pi\rangle.$$

The jet bundle $\mathrm{Jet}(\mathbb{R}_M) \to \mathbb{R}_M$ is a vector bundle. Consider the exterior product

$$\mathrm{Jet}(\mathbb{R}_M) \wedge \mathrm{Jet}(\mathbb{R}_M)$$

over $\mathbb{R}_M$. One may generalize the above formula by computing

$$\langle D^n f \wedge D^n g, \pi^n \rangle.$$

This means that the exponential operator

$$P_\pi := \exp\left(\frac{i\hbar}{2}\pi\right) := \sum_{n\geq 0} \frac{1}{n!}\left(\frac{i\hbar}{2}\right)^n \langle D^n - \wedge D^n -, \pi^n \rangle,$$

where D^n is the projection of jets on their n-th component, and by convention,

$$\langle D^0 f \wedge D^0 g, \pi^0 \rangle := f.g,$$

gives a well defined operator

$$\Gamma(M, \mathrm{Jet}(\mathbb{R}_M) \wedge \mathrm{Jet}(\mathbb{R}_M)) \to \mathcal{C}^\infty(M)[[\hbar]].$$

Combining this operator with the jet map

$$j_\infty : \mathcal{C}^\infty(M) = \Gamma(M, \mathbb{R}_M) \to \Gamma(M, \mathrm{Jet}(\mathbb{R}_M)),$$

we get the Moyal star product

$$* : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)[[\hbar]].$$

Remark that the associativity of this product is true only because the connection is flat and the given Poisson structure π is horizontal with respect to the given affine connection ∇. The $*$-product associated to a general Poisson bracket may be obtained in a similar way, but one has to work with the corresponding Poisson sigma model in the Batalin-Vilkovisky formalism, as described in Section 22.

The addition on the vector bundle TM induces a natural bundle coalgebra structure Δ on the power series algebra bundle $\mathrm{Jet}(\mathbb{R}_M)$, given fiberwisely by

$$\Delta(f)(v, w) := f(v + w).$$

Writing $\Delta(f) = \sum f_{(1)} \otimes f_{(2)}$ may be done by using the Taylor series expansion

$$\Delta(f)(v, w) = f(v + w) = \sum \frac{1}{n!}(D^n f)(v) \otimes w^n.$$

We may extend the Poisson pairing

$$\pi \in \Gamma(M, \mathrm{Hom}_{\mathbb{R}_M}(TM^{\otimes 2}, \mathbb{R}_M)$$

to a bicharacter (see Section 5.2)

$$\psi_\pi \in \Gamma(M, \mathrm{Hom}_{\mathbb{R}_M}(\mathrm{Jet}(\mathbb{R}_M) \otimes \mathrm{Jet}(\mathbb{R}_M), \mathbb{R}_M[[\hbar]])),$$

that we may use to twist the bundle comodule algebra $(\mathrm{Jet}(\mathbb{R}_M), \otimes)$ under the bialgebra

$$(\mathrm{Jet}(\mathbb{R}_M), \otimes, \Delta).$$

This twist is given by an associative product

$$\overset{\hbar}{*} : \Gamma(M, \mathrm{Jet}(\mathbb{R}_M)) \times \Gamma(M, \mathrm{Jet}(\mathbb{R}_M)) \to \Gamma(M, \mathrm{Jet}(\mathbb{R}_M))[[\hbar]]$$

of the form

$$f \overset{\hbar}{*} g := \sum \psi_\pi(f_{(1)}, g_{(1)}) f_{(2)} g_{(2)}.$$

One may check that the product of two jets $j_\infty f$ and $j_\infty g$ is also a jet, that we may denote $j_\infty(f * g)$. This thus, gives by restriction, a well defined $*$-product

$$\overset{\hbar}{*} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)[[\hbar]],$$

that identifies, when correctly normalized, with the Moyal star product.

Remark that all the above reasoning can be done in the setting where we replace M by a space of T of trajectories $T \subset \underline{\Gamma}(M, C)$ for a scalar field theory, and π is given by the Peierls-Dewitt bracket (see [Pei52], [DeW03], [DDM04]). The case of gauge theories can also be studied in a similar way in the setting of their homotopical Poisson reduction, described in Chapter 12. In these infinite dimensional cases, one has to compute carefully the insertions

$$\langle D^n f \wedge D^n g, \pi^n \rangle$$

because the Poisson bivector π is usually only a distributional bivector field. This leads to the regularization and renormalization problem.

21.2 Causal star products

We now use the formalism of Section 19.1 to give a geometric formulation of the Epstein-Glaser perturbative renormalization, inspired by Brunetti-Fredenhagen [BF09]. Our presentation using functors of points is however original, and adapted to more general situations.We denote $X = \underline{\Gamma}(M, C)$ the space of fields of a quantum field theory, $TX \to X$ its tangent space, and $\mathbb{R}_X := \mathbb{R} \times X \to X$ the trivial bundle over X, whose sections are smooth functions on X. We suppose given a flat affine connection ∇_X and a metric g_X on X that is flat for the given connection. We will actually suppose that the metric is given at each $\varphi_0 \in X$ by the integration pairing

$$\int_M : \Gamma(M, \varphi_0^* TC) \times \Gamma(M, \varphi_0^* TC) \to \mathbb{R}$$

associated to a given density on M.

Definition 21.2.1. A regularized propagator is a section $P \in \Gamma(X, \mathrm{Sym}^2(TX))$ that is horizontal for ∇_X.

Remark that the regularized propagators of physics usually do not depend on the variable $\varphi \in X$, and has constant value in the fiber of $\mathrm{Sym}^2(TX)$. This corresponds to the situation where X is a linear space, so that it carries a trivial affine connection, whose horizontal sections are simply the constant ones.

Using the trivialization

$$\mathrm{Jet}(\mathbb{R}_X/X) \cong \prod_{k\geq 0} \mathrm{Hom}_{\mathbb{R}_X}(\mathrm{Sym}^k(TX), \mathbb{R}_X)$$

induced by the affine connection on X, one can define a bundle algebra structure on $\mathrm{Jet}(\mathbb{R}_X)/X$, induced by the algebra structure on symmetric multilinear maps, that we will denote $\otimes$.

Every functional $f : X \to \mathbb{R}$ thus defines a family

$$f^{(n)} = D^n f : X \to \mathrm{Hom}_{\mathbb{R}_X}(\mathrm{Sym}^n(TX), \mathbb{R}_X)$$

of distributions parametrized by X, whose value at $\varphi_0 \in X$ is a symmetric distribution

$$D^n_{\varphi_0} f : \otimes^n (T_{\varphi_0} X) := \Gamma(M^n, \varphi_0^* TC^n) \to \mathbb{R}.$$

From now on, we restrict to functionals whose derivatives are Schwarz distributions.

Definition 21.2.2. The *support of a functional* $f : X \to \mathbb{R}$ is defined by

$$\mathrm{supp}(f) := \overline{\cup_{\varphi_0 \in X} \mathrm{supp}\left(f^{(1)}_{\varphi_0}\right)}.$$

Definition 21.2.3. The star product associated to the regularized propagator is the operation

$$\begin{array}{cccc} * : & \mathcal{O}(X)[[\hbar]] \times \mathcal{O}(X)[[\hbar]] & \to & \mathcal{O}(X)[[\hbar]] \\ & (f, g) & \mapsto & \langle j_\infty(f) \otimes j_\infty(g), \sum_{n\geq 0} \hbar^n P^{\otimes n} \rangle. \end{array}$$

We now want to define a coordinate free version of the Epstein-Glaser renormalization scheme. The basic idea of this construction is to take a family P_λ of regularized propagators that tends to a distributional propagator

$$P \in \Gamma(X, \mathrm{Sym}^2_{dist}(TX)),$$

and to try to make sense of the above star product, at the limit. This is a way to formulate the renormalization problem, that we will now explain with more details.

Definition 21.2.4. The bundle $\mathrm{Jet}_{dist}(\mathbb{R}_X/X)$ of *distributional jets* is defined by

$$\mathrm{Jet}_{dist}(\mathbb{R}_X/X) := \prod_{k\geq 0} \mathrm{Hom}_{\mathbb{R}_X}(\mathrm{Sym}^k(DTX^\vee), \mathbb{R}_X),$$

where $D \to X$ is the linear bundle over X given by

$$D := \underline{\Gamma}(M, \mathrm{Dens}_M) \times X \to X,$$

the duality is given at each point $\varphi_0 \in X$ by the finite dimensional bundle duality

$$T_{\varphi_0} X^\vee := \Gamma(M, \varphi_0^* TC^\vee)$$

over the base manifold M, and the product $DTX^\vee$ also denotes a fiberwise tensor product of linear bundles over M. We will also denote

$$\otimes^k_{dist}(TX) := \mathrm{Hom}_{\mathbb{R}_X}(\otimes^k DTX^\vee, \mathbb{R}_X)$$

and

$$\mathrm{Sym}^k_{dist}(TX) := \mathrm{Hom}_{\mathbb{R}_X}(\mathrm{Sym}^k(DTX^\vee), \mathbb{R}_X)$$

the bundles of *distributional tensors* on the space X. We also denote

$$\mathrm{Sym}^k_{dist}(\mathbb{R}_X) := \mathrm{Hom}_{\mathbb{R}_X}(\mathrm{Sym}^k(D), \mathbb{R}_X)$$

the bundle of *degree k distributional real numbers* on X, whose fiber at a point $\varphi_0 \in X$ are real distributions on M^k.

Lemma 21.2.5. *There are natural monomorphisms*

$$\otimes^k(TX) \hookrightarrow \otimes^k_{dist}(TX),\ \mathrm{Sym}^k(TX) \hookrightarrow \mathrm{Sym}^k_{dist}(TX).$$

of bundles over X.

Proof. At a given point φ_0, we have by definition

$$TX_{\varphi_0} = \Gamma(M, \varphi_0^* TC)$$

and

$$DTX^\vee_{\varphi_0} = \Gamma(M, \varphi_0^* TC^\vee \otimes \mathrm{Dens}_M).$$

The above monomorphisms are simply induced by the integration pairing

$$\begin{array}{cccc} \int_M : & TX_{\varphi_0} \times DTX^\vee_{\varphi_0} & \longrightarrow & \mathbb{R} \\ & (\varphi, \psi d\mu) & \longmapsto & \int_M \langle \psi, \varphi \rangle d\mu, \end{array}$$

by extending it to parametrized points. □

The definition of a coordinate free Epstein-Glaser renormalization scheme involves the use of distributional propagators

$$P \in \Gamma(X, \otimes^2_{dist}(TX)),$$

that we may think of as the limit of a family of regularized propagators

$$P_\lambda \in \Gamma(X, \otimes^2(TX)).$$

Let us first construct the most naive notion of 2-point time ordered product. The notion of Epstein-Glaser renormalization scheme will give a nice refinement of this construction. We also define the notion of $*$-product, that is a non-symmetric version of the time-ordered product.

Proposition 21.2.6. *Let $P \in \Gamma(X, \otimes^2_{dist}(TX))$ be a distributionally valued section. The naive $*$-product associated to P is the binary operation*

$$\begin{array}{cccc} *_2: & \mathcal{O}(X)[[\hbar]] \times \mathcal{O}(X)[[\hbar]] & \to & \mathcal{O}(X)[[\hbar]] \\ & (f,g) & \mapsto & \sum_{n\geq 0} \frac{i^n \hbar^n}{2^n n!} \Delta^*_{2n}\left(f^{(n)} \otimes P^{\otimes n} \otimes g^{(n)}\right)(1), \end{array}$$

If $P \in \Gamma(X, \mathrm{Sym}^2_{dist}(TX))$ is symmetric, the above product is called the naive time ordered product associated to P, and denoted τ_2. Its definition domain is given by the set of pairs (f,g) such that for all $n \geq 0$, the wave front set of the distribution

$$f^{(n)} \otimes P^{\otimes n} \otimes g^{(n)}$$

is orthogonal to the conormal bundle $T^*_{\Delta_{2n}} M^{4n}$ of the diagonal $\Delta_{2n} : M^{2n} \hookrightarrow M^{4n}$.

In the case of a globally hyperbolic field theory on a Lorentzian spacetime, the Hadamard parametrix P (see [BF09]) has the good property of having well defined powers, because its wave front set is in a time-positive cone. The definition domain of the associated $*$-product defined above is quite big and contains in particular multilocal (i.e., products of local) functionals. This $*$-product plays an important role in the Lorentzian causal Batalin-Vilkovisky formalism, to be discussed shortly in Section 21.3.

Definition 21.2.7. An *Epstein-Glaser renormalization scheme* is a tuple

$$\mathfrak{R} = (P, \leq, D, \{\mathrm{ext}_{2n}\}_{n\geq 0})$$

composed of

1. a section $P \in \Gamma(X, \mathrm{Sym}^2_{dist}(TX))$ called the *Epstein-Glaser parametrix*,
2. a partial ordering $\leq$ on M,
3. a definition domain $D \subset \mathcal{O}(X)^2$,
4. a family of distributional extension maps

$$\mathrm{ext}_{2n} : \Gamma(X, \mathbb{R}^{2n}_{X,dist}(\Delta^c_{thin}))) \longrightarrow \Gamma(X, \mathbb{R}^{2n}_{X,dist}),$$

   where $\Delta_{thin} \subset M^{2n}$ is the thin diagonal and $\mathbb{R}^{2n}_{X,dist}(\Delta^c_{thin})$ is the bundle on X whose fiber at a point $\varphi_0 \in X$ are real valued distributions defined outside of the thin diagonal $\Delta_{thin} : M \hookrightarrow M^{2n}$.

The above data is further supposed to fulfill the following compatibility conditions, under the support condition $\mathrm{supp}(f) \leq \mathrm{supp}(g)$ on a pair $(f,g) \in D$:

1. For all $n \geq 0$, the wave front set of the distribution

$$f^{(n)} \otimes P^{\otimes n} \otimes g^{(n)}$$

   is orthogonal to the conormal bundle $T^*_{\Delta_{2n} \backslash \Delta_{thin}} M^{4n}$ of the diagonal inclusion

$$\Delta_{2n\backslash\Delta_{thin}} : M^{2n} \backslash \Delta_{thin} = M^{2n} \backslash M \longrightarrow M^{4n}.$$

2. The pullback distribution

$$\Delta^*_{2n\setminus\Delta_{thin}}(f^{(n)} \otimes P^{\otimes n} \otimes g^{(n)})$$

is in the definition domain of the distributional extension map ext_{2n}.

Definition 21.2.8. The 2-points time ordered product associated to an Epstein-Glaser renormalization scheme $\mathfrak{R}$ is the (partially defined) operation

$$\begin{array}{cccc} \tau_2: & \mathcal{O}(X)[[\hbar]] \times \mathcal{O}(X)[[\hbar]] & \to & \mathcal{O}(X)[[\hbar]] \\ & (f,g) & \mapsto & \sum_{n\geq 0} \frac{i^n\hbar^n}{2^n n!}\mathrm{ext}_{2n}(\Delta^*_{2n\setminus\Delta_{thin}}\left(f^{(n)} \otimes P^{\otimes n} \otimes g^{(n)}\right)(1)), \end{array}$$

with definition domain $D[[\hbar]]$.

Example 21.2.9. The most simple renormalization scheme on an ordered spacetime is given by choosing the definition domain to be the set of pairs (f,g) such that, for all $n \geq 0$, the wave front set of the distribution

$$f^{(n)} \otimes P^{\otimes n} \otimes g^{(n)}$$

is orthogonal to the conormal bundle $T^*_{\Delta_{2n}}M^{4n}$ of the full diagonal. The distribution pullbacks are then canonically extended to the thin diagonal, giving back the naive time ordered product of Proposition 21.2.6. The problem is that this hypothesis is too strong in practice when P is a true distribution, to allow both f and g to be local functionals, which is the case we are mostly interested in. We will describe later some examples of more interesting renormalization schemes.

We start by the central geometric lemma of the Epstein-Glaser method, whose proof can be found in [Ber04] (see also [BBD+12]).

Lemma 21.2.10. *Let $(M,\leq)$ be a partially ordered set, and define the relation $\nleq$ by*

$$x \nleq y \Leftrightarrow \mathrm{not}(x \leq y).$$

Let N be a finite set and $I \subset N$ be a finite subset. Define $C_I \subset M^N$ by

$$C_I = \{x \in M^N,\ x_i \nleq x_j \text{ for all } i \in I \text{ and } j \in I^c\}.$$

Then there is a stratification

$$M^N \backslash \Delta_{thin} = \coprod_I C_I$$

of the complementary of the thin diagonal $\Delta_{thin} : M \to M^N$, where I runs over proper subsets of N.

We now give, in our formalism, the Epstein-Glaser construction of n-points time ordered products, that works uniformly in the Euclidean and Lorentzian setting. We are directly inspired here by the presentation of Keller [Kel09] and Brunetti-Fredenhagen [BF09].

Theorem 21.2.11. *Suppose given an Epstein-Glaser renormalization scheme*

$$\mathfrak{R} = (P, \leq, D, \{\mathrm{ext}_{2n}\}).$$

There is a natural family of maps

$$\tau_n : \mathcal{O}(X)[[\hbar]]^n \to \mathcal{O}(X)[[\hbar]],$$

called the renormalized n-points time ordered products, that are associative, commutative, and with definition domain given by the family (f_n) of functionals that are pairwise in D, i.e., such that $(f_i, f_j) \in D[[\hbar]]$ for all $i \neq j$.

Proof. This theorem is proved by induction on n. We already defined in Definition 21.2.8 the two points time ordered product τ_2. Suppose the maps τ_n to be well defined for $n \leq N-1$. Let now $n = N$ and let $(f_n) \in \mathcal{O}(X)[[\hbar]]^n$ be a family of functionals that are pairwise in D. Choose a partition of unity $\{\mathbb{1}_{C_I}\}$ for the stratification

$$M^n \backslash \Delta_{thin} = \coprod_I C_I$$

of Lemma 21.2.10. One may then define a section in $\Gamma(X, \mathbb{R}^{2n}_{X,dist}(\Delta^c_{thin}))$ from (f_n) by the formula

$$\tau_n(f_1, \ldots, f_n) := \sum_I \tau_2\left(\tau_I(\mathbb{1}_{C_I} f_I), \tau_{I^c}(\mathbb{1}_{C_{I^c}} f_{I^c})\right),$$

where $f_I = \otimes_{i \in I} f_i$. The above formula makes sense because the cardinals of I and I^c are both strictly smaller than n. □

We now define for the sake of completeness the S-matrix, that is often used by physicists.

Definition 21.2.12. The *time-ordered product* is the (partially defined) operation

$$\cdot_\tau : \mathcal{O}(X)[[\hbar]] \times \mathcal{O}(X)[[\hbar]] \to \mathcal{O}(X)[[\hbar]]$$

induced by the family of compatible operations τ_n. The *Bogolyubov S-matrix* of a theory $S = S_0 + I$ with interaction part I (equipped with an Epstein-Glaser renormalization scheme for the free theory S_0) is the time-ordered exponential

$$\mathbb{S}_I := \exp_{\cdot_\tau}(I).$$

Proposition 21.2.13. *Let (M, g) be a metric manifold that is either globally hyperbolic (i.e., $M = \mathbb{R} \times M_0$, with $\mathbb{R}$ timelike) or euclidean. There exists a natural Epstein-Glaser renormalization scheme $\mathfrak{R}(M, g)$ associated to scalar fields (i.e., functions) on M.*

Proof. In both cases, one uses parametrices for the associated laplacian Δ_g (that is elliptic in one case, and hyperbolic in the other). The partial ordering is given by equality in the euclidean case and by time ordering in the Lorentzian case (two points are time ordered if there is a Cauchy surface for the operator Δ_g, and a time-oriented path between them passing through this Cauchy surface, such that one is before it and the other after). The distribution extensions were described by Brunetti-Fredenhagen [BF09] and by Keller (see [Kel07] and [Kel09]). In the Euclidean case, a definition domain that is already worked out is given by pairs of local functionals. In the Lorentzian case, there is a bigger definition domain defined by the wave front set conditions we used, whose precise description can be found in [BF09]. □

The above theorem extends straightforwardly to vector valued fields, but since we have already described a similar result in Chapter 20, we don't repeat the argument here.

21.3 Causal renormalization for gauge theories

We give here a sketch, in our functorial setting for spaces of fields, of the theory developed by Fredenhagen and Rejzner in [FR11] (see also [Rej11] for a more detailed presentation). It gives a causal analog of the effective quantum BV formalism, presented in Section 20.7. We don't suppose here that the original bundle $\pi : C \to M$ is linear.

Let $\pi : E_{fields} \to M$ be a gauge fixed BV theory, as in Section 12.5, with BV bundle

$$E_{BV} = T^*[-1]E_{fields},$$

associated BV algebra $\mathcal{A}_{BV} := \mathrm{Jet}(\mathcal{O}_{E_{BV}})$. Choose a gauge fixed action $S_\psi \in h(\mathcal{A}_{BV})$ (of degree zero). We may see it as a functional

$$S_\psi : X_{BV} := \underline{\Gamma}(M, E_{BV}) \to \mathbb{R}$$

whose restriction to $X := X_{fields} := \underline{\Gamma}(M, E_{fields})$ through the zero section $0 : X \to T^*[-1]X$, also denoted

$$S_\psi : X \to \mathbb{R},$$

has a non-degenerate second order derivative.

The choice of an Epstein-Glaser renormalization scheme for a parametrix of $D^2 S_\psi$ (supposed to exist) gives by renormalization, a family of time ordered products

$$\tau_n : \mathcal{O}(X)[[\hbar]]^n \longrightarrow \mathcal{O}(X)[[\hbar]].$$

These time ordered products can be extended to the space X_{BV} by taking a family (f_n) of functionals, making the associated gauge fixed functionals $(f_n \circ \psi \circ 0)$, and computing the time ordered product on the space X. This gives a family of time ordered products

$$\tau_n : \mathcal{O}(X_{BV})[[\hbar]]^n \longrightarrow \mathcal{O}(X)[[\hbar]].$$

Another way to extend this is to consider the class of functionals in $\mathcal{O}(X_{BV})$ given by the space

$$\mathcal{O}_{poly}(X_{BV}) := \Gamma(X, \mathrm{Sym}_{\mathbb{R}_X}(T[1]X)),$$

the partially defined map

$$\mathcal{O}_{poly}(X_{BV}) \to \mathcal{O}(X_{BV})$$

being given by insertion of tensors.

One may then extend the time ordered product maps to

$$\tau_n : \mathcal{O}_{poly}(X_{BV})[[\hbar]]^n \longrightarrow \mathcal{O}_{poly}(X_{BV})[[\hbar]],$$

by simply using the usual product on the polynomial variables, and the time ordered product on coefficients, that are functions on X.

The (polynomial) local functionals $A \in h(\mathcal{A}_{BV})$ are in $\mathcal{O}_{X_{BV}}$, so that we can now define their time ordered product.

There is a natural Poisson bracket $\{-,-\}$ on $\mathcal{O}_{poly}(X_{BV})$ with definition domain containing local functionals. This Poisson bracket is actually given by the natural symplectic structure on

$$X_{BV} = T^*[1]X_{fields}.$$

One may deform this Poisson bracket by defining a time ordered odd Poisson bracket $\{-,-\}_\tau$ on $\mathcal{O}_{poly}(X_{BV})[[\hbar]]$. This extends naturally to functionals with good wave front set properties, and in particular to multilocal functionals, both in the Euclidean and Lorentzian setting.

Now we denote $\{-,-\}_*$ the operation with two entries on $\mathcal{O}_{poly}(X_{BV})[[\hbar]]$ given by:

- the usual Poisson bracket $\{-,-\}$, in the euclidean setting, and
- the $*$-product deformation of the usual Poisson bracket induced by a causal propagator (Hadamard parametrix)

  $$P \in \Gamma(X, \otimes^2_{dist}(TX))$$

  (whose wave front set is concentrated in the positive cone).

The quantum master equation in this situation is then given by

$$\{e^I_\tau, S_0\}_* = 0,$$

where $S = S_0 + I$ is the decomposition of the action functional in its quadratic and interacting part. One may also define, in this situation, a renormalized anomalous Batalin-Vilkovisky laplacian Δ_I that depends on the interaction term, so that the following quantum master equation

$$\frac{1}{2}\{S,S\}_\tau = \Delta_I(I)$$

is fulfilled. It is a renormalized equation, that looks like the equation used by physicists, but that is well defined (because Δ_I is renormalized) and that also works with an anomalous theory (the anomaly is included in the Δ_I term). For a related treatment of anomalous gauge theories, see [Bar00].

21.4 Deformation quantization and Hopf algebraic structures

We refer to Chapter 5 for the necessary background on Hopf algebras. This section is inspired by Brouder's Hopf theoretic approach in [Bro09] to quantum field theory computations (see also [BFFO04] and [BBD+12]), and by Borcherds' formalization [Bor10] of perturbative functional integrals using Hopf algebraic methods.

21.4.1 A finite dimensional toy model

We refer to [Bro09] for this section. We present here the twisting methods from Section 5.2 on a simple example, that will be our main inspiration for the next section. We will use Sweedler's notation for coproducts:

$$\Delta(a) = \sum a_{(1)} \otimes a_{(2)}.$$

Let V be a finite dimensional vector space and $C = \mathrm{Sym}(V^*)$ be the algebra of polynomial functions on V, equipped with its commutative coproduct given by

$$\Delta_C(v^*) = v^* \otimes 1 + 1 \otimes v^*$$

on generators $v^* \in V^*$, and extended by algebra morphism. We may also interpret this coproduct as the one induced by addition on V, by the formula

$$\Delta_C(v^*(v)) = v^*(v_1 + v_2),$$

where v, v_1 and v_2 are in V and v^* in V^*. The coproduct Δ_C may be naturally extended to $\mathrm{Sym}(C) = \mathrm{Sym}(\mathrm{Sym}(V^*))$ by setting

$$\Delta(a) = \Delta_C(a)$$

for $a \in C = \mathrm{Sym}(V^*)$, and by extending this by algebra morphism

$$\Delta : \mathrm{Sym}(C) \to \mathrm{Sym}(C) \otimes \mathrm{Sym}(C).$$

This means in particular that

$$\Delta(a.b) = \sum (a_{(1)}.b_{(1)}) \otimes (a_{(2)}.b_{(2)})$$

for all $a, b \in C$. If D is a comodule over C, one also gets by the same formula a comodule algebra $(\mathrm{Sym}(D \otimes C), \Delta)$ over the bialgebra $(\mathrm{Sym}(C), \Delta)$. This is the type of object that will be used to define the deformation quantization in perturbative quantum field theory, through Drinfeld's twisting method (described in Section 5.2).

There is another coalgebra structure on $\mathrm{Sym}(C)$ given on generators $a \in C$ by

$$\delta(a) = a \otimes 1 + 1 \otimes a.$$

This corresponds to the additive group structure on the dual linear space C^*, by

$$\delta(a(a^*)) = a(a_1^* + a_2^*).$$

This coalgebra structure will be useful to define the renormalization group. Remark that there is a natural coaction of (C, Δ_C) on $(\mathrm{Sym}(C), \delta)$, given by a map

$$\Delta_C : \mathrm{Sym}(C) \to C \otimes \mathrm{Sym}(C)$$

induced by comultiplication on C by extension to an algebra morphism. At the level of functions on C^*, this coaction corresponds to the action of translations in the affine space V on the algebra of functions C on V, given by

$$\begin{array}{ccc} (V,+) \times (C,+) & \to & (C,+) \\ (w, a(v)) & \mapsto & a(v+w). \end{array}$$

This action is clearly compatible with the addition of functions $a \in C$, so that one has a comodule structure over $(S(C), \delta)$.

If D is a comodule over C, given by a coaction map $\Delta_D : D \to C \otimes D$, we may also define a coaction

$$\Delta : \mathrm{Sym}(DC) \to \mathrm{Sym}(C) \otimes \mathrm{Sym}(DC)$$

of the coalgebra $(\mathrm{Sym}(C), \Delta)$ on the algebra $\mathrm{Sym}(DC)$ by extending the natural comodule structure

$$\Delta_{DC} : DC \to C \otimes DC$$

by algebra morphism.

If $\psi : V^* \otimes V^* \to \mathbb{R}$ is a bilinear form, it may be extended to a 2-cocycle on the bialgebras $C = \mathrm{Sym}(V^*)$ and $\mathrm{Sym}(C)$ by setting

$$\begin{array}{rcl} \psi(ab, c) & = & \sum \psi(a, c_{(1)})\psi(b, c_{(2)}), \\ \psi(a, bc) & = & \sum \psi(a_{(1)}, b)\psi(a_{(2)}, c). \end{array}$$

The associated twisted product (see Section 5.2) on the comodule algebra $\mathrm{Sym}(C)$ is defined by

$$a * b = \sum \psi(a_{(1)}, b_{(1)}) a_{(2)} b_{(2)},$$

where the Sweedler decomposition is written with respect to the $(\mathrm{Sym}(C), \delta)$-comodule structure map

$$\Delta : \mathrm{Sym}(C) \to C \otimes \mathrm{Sym}(C).$$

Morally, what we are doing here is a deformation quantization of the space C of functions on the affine space V, as a homogeneous space under the action of the group $(V, +)$.

Suppose now given a comodule D over C and define a coproduct δ on $\mathrm{Sym}(DC)$ by saying that DC is primitive. The coalgebra $(\mathrm{Sym}(DC), \delta)$ is a comodule under (C, Δ_C). In this abstract setting, the group G_{ren} of renormalizations is given by the group

$$G_{ren} := \mathrm{Aut}_{\mathrm{CoMod}(C, \Delta_C)}(\mathrm{Sym}(DC), \delta)$$

of automorphisms of the coalgebra $\mathrm{Sym}(DC)$, as a comodule over the coalgebra C.

21.4.2 Hopf algebraic methods in field theory

We now describe how to generalize the methods of the previous section to the infinite dimensional setting of field theory. We use here [Bor10] and [BBD+12]. In all this section, we denote by normal letters the bundles, and by calligraphic letters their sheaves of sections.

Let $E \to M$ be a vector bundle, with associated jet bundle $\mathrm{Jet}(E) \to M$, and space of sections $\underline{\Gamma}(M, E)$ denoted X. We denote $\mathcal{E}$ the sheaf on M of section of E, and $\mathrm{Jet}(\mathcal{E})$ the sheaf of sections of $\mathrm{Jet}(E)$. Let V be the jet bundle $\mathrm{Jet}(E)$ and denote

$$V^* := \mathrm{Hom}_{\mathbb{R}_M}(\mathrm{Jet}(E), \mathbb{R}_M),$$

the dual bundle. Consider the bundle algebra

$$C = \mathrm{Sym}_{\mathbb{R}_M}(V^*)$$

of polynomial Lagrangian functions on E, where the symmetric algebra is taken relatively to the symmetric monoidal category $(\mathrm{VECT}_M, \times_M)$ of (pro)-vector bundles on M with its fiber product. It is equipped with a coproduct Δ_C given on generators by

$$\Delta_C(v) = v \otimes 1 + 1 \otimes v,$$

that gives the algebraic group structure on V corresponding to its natural addition, given by the fact that we started with a vector bundle E, equipped with an addition map

$$+ : E \times_M E \to E.$$

We define a bundle coalgebra structure Δ on the bundle $\mathrm{Sym}(C) := \mathrm{Sym}_{\mathbb{R}_M}(C)$ by

$$\Delta(a) = \Delta_C(a)$$

for $a \in C$ and extending this to

$$\Delta : \mathrm{Sym}(C) \to \mathrm{Sym}(C) \times_M \mathrm{Sym}(C)$$

by algebra morphism. Now consider the trivial comodule D over C given by the bundle $D = \mathrm{Dens}_M$ of densities on M. We equip the algebra bundle

$$\mathrm{Sym}(DC) := \mathrm{Sym}_{\mathbb{R}_M}(\mathrm{Dens}_M \times_M C)$$

with the coproduct δ given on its generators by

$$\delta(a) = a \times_M 1 + 1 \times_M a.$$

The space

$$Y = \mathcal{O}^{loc}(X) := \Gamma(M, \mathrm{Sym}(DC))$$

is the space of polynomial local Lagrangian densities on the space $X = \underline{\Gamma}(M, E)$ of fields. Morally, the coproduct δ corresponds to the additive group structure on the dual bundle $(DC)^*$. There is a natural coaction

$$\Delta_C : \mathrm{Sym}(DC) \to C \otimes \mathrm{Sym}(DC),$$

that corresponds to the action of the additive group $(V, +)$ on the space $\mathrm{Sym}(DC)$ of formal products of functions on V.

Definition 21.4.3. The renormalization group is the sheaf of groups

$$\mathcal{G}_{ren} := \mathrm{Aut}_{\mathrm{CoMod}(\mathcal{C},\Delta_{\mathcal{C}})}(\mathrm{Sym}(\mathcal{DC}),\delta)$$

of automorphisms of the $\mathcal{C}^\infty_M$-sheaf of coalgebras $\mathrm{Sym}(\mathcal{DC}) := \Gamma(-,\mathrm{Sym}(DC))$ that commute with the coaction of the sheaf of coalgebras $\mathcal{C} = \Gamma(-,C) := \Gamma(-,\mathrm{Sym}_{\mathbb{R}_M}(\mathrm{Jet}(E)^*))$ of local Lagrangian functions.

We carefully inform the reader that renormalizations are *not* given by algebra automorphisms of $\mathrm{Sym}(DC)$, so that the renormalization group is a very big group of automorphisms of a sheaf of $\mathcal{C}^\infty_M$-module coalgebras (not of bialgebras).

Up to now, we have been working with objects living on the manifold M, that are "local" objects in the physicists' sense. We now describe the main non-local Hopf algebraic structures that are used in perturbative quantum field theory.

Remark first that the algebra

$$\mathrm{Sym}_{\mathbb{R}}(\Gamma_c(M,DC)) := \oplus_{n\geq 0}\Gamma_c(M^n,\boxtimes^n DC)_{S_n}$$

is equipped with a natural coproduct δ that makes generators primitive. A coproduct δ similar to this one may be defined more intuitively on the dual algebra of formal functionals

$$\widehat{\mathcal{O}}(Y) := \prod_{n\geq 0}\mathrm{Hom}_{\mathbb{R}}(Y^{\hat{\otimes} n},\mathbb{R})^{S_n} = \prod_{n\geq 0}\mathrm{Hom}_{\mathbb{R}}(\Gamma_c(M^n,(DC)^{\boxtimes n}),\mathbb{R})^{S_n}$$

on the space $Y = \mathcal{O}^{loc}(X) := \Gamma(M,DC)$ of Lagrangian densities, by simply writing

$$\delta(F(\varphi_1,\dots,\varphi_n)) := F(\varphi_1+\psi_1,\dots,\varphi_n+\psi_n).$$

Remark that this formula also extends naturally to the algebra $\widehat{\mathcal{T}}(Y)$ of noncommutative formal power series

$$\widehat{\mathcal{T}}(Y) := \prod_{n\geq 0}\mathrm{Hom}_{\mathbb{R}}(\Gamma(M^n,(DC)^{\boxtimes n}),\mathbb{R}).$$

The natural coaction map $\Delta_{DC} : DC \to DC \otimes C$ induces a coaction

$$\Delta : \mathrm{Sym}_{\mathbb{R}}(\Gamma_c(M,DC)) \to \mathrm{Sym}_{\mathbb{R}}(\Gamma_c(M,C)) \otimes \mathrm{Sym}_{\mathbb{R}}(\Gamma_c(M,DC)).$$

Definition 21.4.4. A propagator is a continuous bilinear map

$$G : \Gamma_c(M,DE^*) \otimes \Gamma_c(M,DE^*) \to \mathbb{R}.$$

We may extend a propagator G to a continuous bilinear map

$$G : \Gamma_c(M,\mathrm{Jet}(E)^*) \otimes \Gamma_c(M,\mathrm{Jet}(E)^*) \to \mathcal{D}'(M^2) := \mathrm{Hom}_{\mathbb{R}}(\Gamma_c(M^2,D^{\boxtimes 2}),\mathbb{R})$$

by

$$G(s^*_\alpha,t^*_\beta) := (\partial^\alpha,\partial^\beta).G(s^*-,t^*-).$$

We extend this bilinear pairing by the bicharacter condition for the Hopf algebra $(\Gamma_c(M,C),\Delta_C)$ to a Hopf pairing

$$\psi_G : \Gamma_c(M,C) \otimes \Gamma_c(M,C) \to \mathcal{D}'(M^2),$$

and further through the bicharacter condition for the Hopf algebra $(\mathrm{Sym}_{\mathbb{R}}(\Gamma_c(M,C)),\Delta)$ to a Hopf pairing

$$\psi_G : \mathrm{Sym}^n_{\mathbb{R}}(\Gamma_c(M,C)) \otimes \mathrm{Sym}^m_{\mathbb{R}}(\Gamma_c(M,C)) \to \mathcal{D}(M^n \times M^m).$$

Finally, if we multiply the propagator by $\hbar$, we get a bicharacter ψ_G on the Hopf algebra $\mathrm{Sym}_{\mathbb{R}}(\Gamma_c(M,C))$ with values in the ring

$$\mathcal{R}[[\hbar]] := \prod_{n\geq 0} \mathrm{Hom}_{\mathbb{R}}(\Gamma_c(M^n,D^n),\mathbb{R})[[\hbar]].$$

We may use this bicharacter to twist the comodule algebra $\mathrm{Sym}_{\mathbb{R}}(\Gamma_c(M,DC))$ under the bialgebra $(\mathrm{Sym}_{\mathbb{R}}(\Gamma_c(M,C)),\Delta)$, to get a partially defined $*$-product

$$f \overset{\hbar}{*} g := \sum \psi_G(f_{(1)},g_{(1)})f_{(2)}g_{(2)}.$$

This product is only well define when one has particular wave front set condition on f and g, because we used a distribution product to define it.

Definition 21.4.5. A *true $*$-product* associated to the propagator G is a continuous extension of the above star product to a fully defined associative product on $\mathrm{Sym}_{\mathbb{R}}(\Gamma(M,DC))[[\hbar]]$.

Proposition 21.4.6. *The renormalization group G_{ren} acts transitively on true $*$-products.*

The main theorem of Borcherds' paper [Bor10] is the following existence theorem. Borcherds actually formulates this result in terms of Feynman measures. This theorem gives an existence and unicity statement for the perturbative functional integral in Lorentzian quantum field theory. There is no reasonable doubt that it extends directly to the euclidean setting, by the use of the uniform treatment of causal methods presented in Section 21.2.

Theorem 21.4.7. *There exists a true $*$-product.*

Chapter 22

Topological deformation quantizations

In this chapter, we give a very short description of algebraic deformation quantization methods, that are based on the deformation theory described in Section 9.8.

To illustrate the relation between homotopy and deformation, we shortly describe the modern approach to deformation quantization of Poisson manifolds, due to Kontsevich [Kon03] and Tamarkin [Tam98] (see also [Kon99]). This approach is inspiring for the formulation of the quantization problem for factorization spaces, that will be discussed in Section 23.6. We refer to Section 9.8 for an introduction to the methods of deformation theory.

Suppose given a smooth Poisson manifold X, with Poisson algebra of functions $(A, \{-,-\})$. A deformation quantization of X is given by an associative product

$$* : A[[\hbar]] \times A[[\hbar]] \to A[[\hbar]]$$

that reduces to A modulo $\hbar$, and whose commutator gives back the Poisson bracket by the formula

$$\{a, b\} = \frac{[a, b]}{\hbar} \mod \hbar.$$

One also asks that $*$ is defined by polydifferential operators $b_n : A^{\otimes n} \to A$. One shows that equivalence classes of star products identify with solutions of the Mauer-Cartan equation of the dg-Lie algebra $C^*_{loc}(A, A)$ given by local Hochschild cochains on A. There is a natural isomorphism

$$H^*(C^*_{loc}(A, A)) \cong H^*(X, \wedge^* \Theta_X)$$

with the cohomology of multivector fields. Remark that this space has a natural dg-Lie algebra structure. Kontsevich's formality theorem says that the quasi-isomorphism

$$C^*_{loc}(A, A) \to H^*(X, \wedge^* \Theta_X)$$

may be made compatible with the dg-Lie structures on both sides. Since a Mauer-Cartan element on the right is the same as a Poisson bracket, one gets that every Poisson bracket corresponds to a $*$-product. Remark that one may interpret this formality theorem as an analog of the degeneration of the Hodge to de Rham spectral sequence at first page, that gives an algebraic formulation of Hodge theory.

Part of the above reasoning can be translated in a purely geometric one, as was done in [PTVV11]. This geometric translation allows one to treat also nicely the deformation quantization of Yang-Mills type gauge theories in any dimension (the deformation quantizations are only k-monoidal dg-categories, in this homotopical setting). The objects to be quantized are given by pairs (X, ω) composed of an Artin higher stack with a closed k-shifted symplectic 2-form ω (in the sense of Section 9.7) that induces an isomorphism

$$\Theta_\omega : \mathbb{T}_X \xrightarrow{\sim} \mathbb{L}_X[k]$$

between the tangent complex and the shifted cotangent complex of X.

A useful example of such a space is given by $X = BG$, equipped with the 2-shifted 2-form given by the natural G-invariant quadratic form

$$\omega : \mathfrak{g}[1] \wedge \mathfrak{g}[1] \cong \mathrm{Sym}^2(\mathfrak{g})[2] \to k[2].$$

The main result of loc. cit. implies that, if M is an oriented spacetime of dimension n, the above form induces a $2(1-n)$-shifted 2-form ω on the space

$$X = \mathrm{B\scriptsize{UN}}\mathrm{Conn}_G(M) := \underline{\mathrm{Hom}}(M_{dR}, BG)$$

of G-bundles on M with flat connection. This is the space of trajectories of pure Yang-Mills theory. For $n = 4$, the symmetric monoidal category of such a space with -6-shifted 2-form may be deformed to a 6-monoidal dg-category.

This categorical approach allows one to work also in the algebraic or holomorphic setting, and to better understand obstructions. The Hochshild complex $C^*(A, A)$ identifies with the tangent space to the space of deformations of (tempered) dg-categories up to Morita equivalences at the (tempered) derived category $D_t(A)$ of modules over A. All classes in Hochschild cohomology correspond to deformations of the dg-category $D_t(A)$, so that the deformation problem is unobstructed. Once Kontsevich formality theorem is proved, the problem is to show that a given deformation of the category of modules is actually also a category of modules on an actual associative algebra. In the case of a smooth Poisson manifold, this is true, but not anymore in the algebraic setting. The main idea of this geometric interpretation is that deforming categories of representations of algebras is easier than deforming algebras. This is also true when one works with deformations of Hopf algebras. Once this setting is settled, the problem is to prove that a given k-monoidal dg-category is the category of modules over an E_k-algebra.

In the case of a commutative algebra A of functions on a Poisson manifold, the corresponding commutative (E_∞-)coalgebra is simply the Chevalley-Eilenberg coalgebra of the trivial Lie bracket on A, i.e., the symmetric Hopf algebra $S(A)$, and a deformation of the dg-category of A-modules will be a category of modules over an E_2-algebra B.

To treat non-topological theories, one has to dive into the formalism of $\mathcal{D}$-geometry over the Ran space and factorization spaces, to be treated in Chapter 23. The type of objects that one obtain when doing the deformation quantization of a non-topological theory are Chiral categories, that generalize n-monoidal dg-categories.

We also mention the Cattaneo-Felder quantum field theoretic interpretation (see the original article [CF01] for the result and the survey article [CKTB05] for a refined presentation of the mathematical aspects) of Kontsevich's star product (Kontsevich actually found his formulas by computing functional integrals for this theory). We refer to Section 15.3 for a description of the classical Poisson sigma model. Let $D = \{z \in \mathbb{C}, |z| \leq 1\}$ be the Poincaré disc and (X, θ) be a Poisson manifold. Recall that the fields of the theory are given by pairs (x, η) composed of a morphism $x : D \to X$ and a differential form $\eta \in \Omega^1(D, x^*T^*X)$. We fix three points 0, 1 and ∞ on the boundary of D. One may interpret a function $f : X \to \mathbb{R}$ as a pair of observable for this theory, given by the formulae

$$(x, \eta) \mapsto f(x(0)) \text{ and } (x, \eta) \mapsto f(x(1)).$$

The Kontsevich formula for the *-product is formally given by the mean value (expressed by a functional integral)

$$(f * g)(y) := \frac{1}{Z(0)} \int_{x(\infty)=y} f(x(0))g(x(1))e^{\frac{i}{\hbar}S(x,\eta)}[dxd\eta].$$

The action functional having gauge symmetries, one needs to apply the Batalin-Vilkovisky formalism for gauge fixing. The detailed construction is provided in [CF01].

Chapter 23

Factorization spaces and quantization

In this chapter, M denotes a smooth variety in a differentially convenient site (see Definition 2.1.3). We will be mostly interested by real, complex analytic, or smooth manifolds.

Factorization spaces give a geometric way to formalize the notion of operator product expansion, that is used by physicists to formalize quantum field theory in a locally covariant way (see e.g., Hollands-Wald [HW10] for a their use in Lorentzian field theory on curved spacetime, and the Institute of Advanced Study semester notes [DEF$^+$99], Volume 2, for their use in string theory). This approach was grounded by Beilinson-Drinfeld [BD04] in dimension 1 (see [Roz09], [Gai99] and [FBZ04], Chapter 18, as additional references with concrete examples from conformal field theory). The generalization to a formal affine theory in higher dimension was developped by Francis-Gaitsgory [FG11]. The geometric approach through factorization spaces is presently in active development by Rozenblyum (see [Roz11], [Roz10]), and Gaitsgory-Rozenblyum [GR11]. Our presentation of factorization spaces also owes pretty much to discussions with N. Rozenblyum on his ongoing project [Roz12]. Remark however that the deformation quantization picture for factorization categories is original in our book.

All the chiral references work in the algebraic (or topological) setting, but we also need an analytic version of it. We will use the natural notion of derived stack in $\mathcal{D}$-geometry, obtained as a functor of point on homotopical $\mathcal{D}$-algebras, following the general methods of Chapter 9. This formalism applies equally well to the algebraic or analytic setting, and we need to work with analytic spaces to get the tools of algebraic analysis of Chapter 10, that relate the general formalism to usual computations of physicists, using distributions. One may also use manifolds with corners, basing on the formalism described in Example 2.2.8 and the general differential calculus formalism of Section 1.5. We will use the general geometry of non-linear partial differential equations, described in Section 11.5.

There is another approach to factorization quantization that is due to Costello and Gwilliam [CG10] and uses the notion of factorization algebra. The relation of what we explain in this chapter with this approach is described in Remark 23.4.9. We have chosen the factorization space approach because it is directly related to our functor of point viewpoint of the geometry of spaces of fields, but the reader is strongly advised to refer to their work for an alternative approach, that may be accessible to a larger audience. Remark that Costello and Gwilliam have a quite general existence theorem for quantizations of euclidean field theories in their setting, that still needs to be adapted to ours, combining techniques of algebraic microlocal analysis from Chapter 10, with methods close to those used for causal perturbative quantum field theory in Chapter 21. We will partly explain this relation in Remarks 23.4.9 and 23.4.10, in a langage that is directly adapted to the treatment of Lorentzian field theories, contrary to Costello and Gwilliam, that restrict their considerations to euclidean (aka elliptic) field theories.

23.1 $\mathcal{D}$-modules over the Ran space

We start our presentation of factorization methods by working with the perturbative theory of $\mathcal{D}$-spaces over the Ran space, that will play an important role in the perturbative theory. Recall that the Ran space is the derived stack defined by the ∞-colimit

$$\mathrm{Ran}(M) := \operatorname*{colim}_{I \twoheadrightarrow J} M^I$$

along diagonal embeddings. The category of $\mathcal{D}$-modules on the Ran space is the stable ∞-category

$$\mathcal{D}(\mathrm{Ran}(M)) := \textsc{Mod}(\mathrm{Ran}(M)_{DR})$$

of modules over $\mathrm{Ran}(M)_{DR}$. Remark that by definition, one has

$$\mathcal{D}(\mathrm{Ran}(M)) \cong \lim_{I \twoheadrightarrow J} \mathcal{D}(M^I)$$

in the ∞-category of stable ∞-categories, so that a (right) $\mathcal{D}$-module on the Ran space may be seen as the datum of a compatible family $\{\mathcal{M}^I\}$ of $\mathcal{D}$-modules on M^I with a homotopy transitive system of homotopy equivalences

$$\Delta^* \mathcal{M}^I \cong \mathcal{M}^J.$$

The operation

$$\mathrm{union} : \mathrm{Ran}(M) \times \mathrm{Ran}(M) \to \mathrm{Ran}(M)$$

of union of subsets and

$$j : (\mathrm{Ran}(M) \times \mathrm{Ran}(M))_{disj} \to \mathrm{Ran}(M) \times \mathrm{Ran}(M)$$

of inclusion of the disjoint pairs induce two monoidal structures

$$\otimes^* := \mathrm{union}_*(- \boxtimes -)$$

and

$$\otimes^{ch} := \mathrm{union}_* j_* j^* (- \boxtimes -),$$

respectively called the local and chiral monoidal structures. One also has the usual monoidal structure $\otimes^!$ on $\mathcal{D}(\mathrm{Ran}(M))$.

If $\mathcal{M}$ is a left $\mathcal{D}$-module on M, the associated de Rham functor

$$\mathrm{DR}_{\mathcal{M}} : (\mathcal{D}(\mathrm{Ran}(M)), \otimes^*) \to (\mathrm{DS}\textsc{h}(\mathrm{Ran}(M)), \otimes_{\mathbb{R}})$$

is defined by

$$\mathrm{DR}_{\mathcal{M}}(\{\mathcal{M}^I\})) := \{\mathrm{DR}_{\mathcal{M}^{\boxtimes I}}(\mathcal{M}^I)\} := \{\mathcal{M}^I \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_{M^I}} \mathcal{M}^{\boxtimes I}\}.$$

We will simply denote DR the de Rham functor associated to $\mathcal{O}$. It plays an important role in the euclidean case. Remark that the de Rham functor associated to the $\mathcal{D}$-module of hyperfunctions $\mathcal{B}_M$ (or hyperfunctions with support in a positive time cone C on a Lorentzian spacetime M, denoted $\mathcal{B}_C$) is more interesting in the case of hyperbolic spacetime.

The de Rham functor commutes to the main diagonal embedding, so that if $\mathcal{N}$ is a $\mathcal{D}$-module on M, then

$$\mathrm{DR}_{\mathcal{M}}(\Delta_*(\mathcal{N})) \cong \Delta_*(\mathrm{DR}_{\mathcal{M}}(\mathcal{N})).$$

This allows us to interpret the local functional calculus described in Section 11.2, directly on the Ran space.

23.2 Chiral Lie algebras and factorization coalgebras

We now give a short summary of the theory of chiral algebras and their Koszul duality, that is due to Beilinson-Drinfeld [BD04] in dimension 1 and to Francis-Gaitsgory [FG11] in higher dimension. This result allows us to relate formal affine factorization spaces (that are determined by their factorization coalgebra of "distributions") to Lie algebraic objects. It is also reminiscent of derived deformation theory, described in Section 9.8. The original property of this duality is that it is automatically an equivalence, because of the nilpotence property of the chiral tensor product on the Ran space. This may be compared with classical rational homotopy theory (see e.g., Quillen [Qui69]), where one needs to restrict the class of coalgebras in play to get an equivalence. This is avoided here because all factorization coalgebras already have the right nilpotence property.

Remark that there is also a family version of this theory, which relates a class of Lie algebroids over a base $\mathcal{D}$-space Z/M_{DR} to factorization coalgebras over $Z^{(Ran)}/\mathrm{Ran}(M)_{DR}$. We only describe here the original theory, over a trivial base.

Definition 23.2.1. A chiral (Lie) algebra is a Lie algebra in $(\mathcal{D}(\mathrm{Ran}(M)), \otimes^{ch})$ that is supported on the main diagonal M. A local Lie algebra (also called a Lie*-algebra) is a Lie algebra in $(\mathcal{D}(\mathrm{Ran}(M)), \otimes^*)$ that is supported on the main diagonal.

A Chiral Lie algebra is thus given by a model

$$\mathcal{A} : (\textsc{Lie}, \otimes) \longrightarrow (\mathcal{D}(\mathrm{Ran}(M)), \otimes^{ch})$$

of the symmetric linear theory of Lie algebras with values in $\mathcal{D}$-modules on the Ran space. We may think of this, in most cases, as the datum of a $\mathcal{D}$-module $\mathcal{A}$ on M together with bracket operations

$$[-, \cdots, -]_{i,ch} : \mathcal{A}^{\otimes^{ch} i} \to \mathcal{A},$$

that form a homotopy coherent L_∞-algebra. By adjunction, this may also be seen as operations

$$[-,\cdots,-]_{ch} : j_*j^*(\mathcal{A}^{\boxtimes i}) \to \Delta_*\mathcal{A}.$$

The natural adjunction map $\mathrm{id} \to j_*j^*$ gives a morphism

$$(\mathcal{D}(\mathrm{Ran}(M)), \otimes^{ch}) \longrightarrow (\mathcal{D}(\mathrm{Ran}(M)), \otimes^*),$$

that allows us to associate a local Lie algebra to a chiral Lie algebra, by composition.

Remark 23.2.2. If one wants to have, on curves, and equivalence between commutative chiral algebras and commutative $\mathcal{D}$-algebras, it is necessary to add a unit to the Lie algebras in play, by defining unital chiral algebras. We refer to Beilinson-Drinfeld's book [BD04], Gaitsgory's survey [Gai99], Chapter 18 of Frenkel and Ben-Zvi's book [FBZ04], and Rozenblyum's lecture notes [Roz09] for more details on unital chiral algebra structures.

Locally constant chiral algebras give nice geometric ways to encode algebraic structures, and their homotopy versions. Let us illustrate this general fact by simple examples.

Example 23.2.3. Let $M = \mathbb{R}$ be the real analytic affine line and A be an associative algebra. Let $\mathcal{A} := A \otimes \omega_{\mathbb{R}}$ be the associated right $\mathcal{D}$-module. The de Rham cohomology of $\mathcal{A}$ is the locally constant sheaf $\underline{A}$ on $\mathbb{R}$. There is a natural chiral bracket on $\mathcal{A}$ given by a morphism

$$\Delta^*j_*j^*(\mathcal{A} \boxtimes \mathcal{A}) \to \mathcal{A}$$

of $\mathcal{D}$-modules on $\mathbb{R}$. The left hand side is the $\mathcal{D}$-module $(\mathcal{A} \otimes^! \mathcal{A}) \oplus (\mathcal{A} \otimes^! \mathcal{A})$ with the two summand corresponding to two connected components of $\mathbb{R}^2\backslash\Delta$. The Chiral product is the one induced by the multiplication on one side and its opposite on the other. Its restriction to $\mathcal{A}\otimes^!\mathcal{A} = \Delta^*(\mathcal{A}\boxtimes\mathcal{A})$ gives the Lie bracket on $\mathcal{A}$. One may prove that locally constant chiral algebras on $\mathbb{R}$ correspond exactly to associative algebras up-to-homotopy, i.e., to E_1-algebras, through the above construction. More generally, locally constant chiral algebras on $\mathbb{R}^n$ are equivalent to E_n-algebras (see Figure 23.2 for a drawing of the space of degree n-operations in E_2, given by families of n discs in a disc), as was proved by Lurie [Lur09c], Chapter 5 (he works in the topological setting, but the local constancy hypothesis allows a direct translation of his results to the chiral setting).

Example 23.2.4. The chiral algebra $\mathcal{A}$, associated in Example 23.2.3 with an associative algebra A, being locally constant, it is in particular translation invariant. It thus descends to a chiral algebra $\mathcal{A}_{S^1}$ on $S^1 = \mathbb{R}/\mathbb{Z}$. One can show (see [Cos10] and [Lur09c] for the related construction in the situation of factorization algebras) that the chiral homology of $\mathcal{A}_{S^1}$ identifies with the Hochshild homology of the underlying associative algebra A.

Example 23.2.5. A locally constant chiral algebra on the Poincaré half plane

$$\mathbb{H} := \{z \in \mathbb{C},\ \mathrm{Im}(z) > 0\}$$

is essentially the same as an E_2-algebra in complexes (recall that E_2-algebras are objects of a symmetric monoidal ∞-category equipped with two E_2-algebra structures that commute

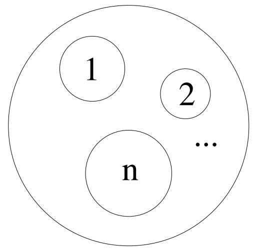


Figure 23.1: The space of n-operations of the little discs operad.

one with each other up-to-homotopy). Its homology may be equipped with a Gerstenhaber structure, given by an action of the graded operad $G = H^*(E_2)$. This is essentially a graded commutative odd Poisson algebra. Remark that Kontsevich's formality result over $\mathbb{R}$ from [Kon99] gives an explicit formula for the quasi-isomorphism of E_2 with $G = H^*(E_2)$, with coefficients in $\mathbb{R}$. A similar formula with coefficients in $\mathbb{C}$ is given by Drinfeld's associator [Dri87]. This formality result also gives an E_2 structure on the Hochshild cochain complex

$$\mathrm{CH}^*(A, A) := \mathbb{R}\mathrm{Hom}_{A\otimes A^{op}}(A, A)$$

of any associative algebra A, answering a conjecture of Deligne (whose validity is more generally known with coefficients in $\mathbb{Z}$, see [Lur09c], Section 6.1.4 for a proof conformal to our doctrinal approach to structures up-to-homotopy). One may intuitively understand the fact that this Hochshild complex has an E_2-algebra structure by thinking of it as a space of derived endomorphisms f of A, equipped with the two operations

$$(f, g) \mapsto f \circ g \text{ and } (f, g) \mapsto f *_A g.$$

These two operations get identified on the zero cohomology, that is simply the center $Z(A)$ of A, because of the Heckman-Hilton argument. However, they give two operations commuting up to homotopy on the full complex.

Example 23.2.6. If we work on the closure

$$\overline{\mathbb{H}} := \{z \in \mathbb{C},\ \mathrm{Im}(z) \geq 0\}$$

of the Poincaré half plane, that has boundary the real line $\mathbb{R}$, locally constant chiral algebras correspond to two sorted algebras: we have an E_2-algebra G coming from the $\mathbb{H}$ component and an E_1-algebra A coming from the $\mathbb{R}$ component, and they are linked by a morphism of E_2-algebras

$$f : G \to \mathrm{CH}^*(A, A).$$

The homology of these algebras form algebras over the Swiss-Cheese operad [Vor98] (see Figure 23.2 for a drawing of the space of degree n operations of the Swiss-Cheese operad). These are pairs (G, A) composed of a Gerstenhaber algebra (i.e., an algebra over the homology $H^*(E_2)$ of the differential graded E_2 operad) and an associative algebra (i.e.,

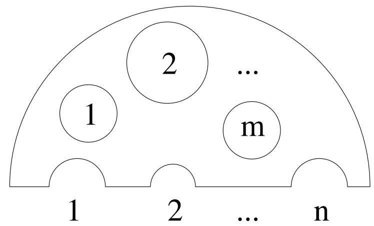


Figure 23.2: The space of n-operations of the swiss-cheese operad.

an algebra over the operad Ass $\cong H^*(E_1)$ of associative algebras) together with a map $f : G \to A$ between them fulfilling compatibility conditions with the given structures. This situation is directly related to Tamarkin's operadic proof [Tam98] of Kontsevich's formality theorem. It actually gives a conceptual understanding of the relation between the Poisson sigma model, described in [CF01] and Tamarkin's work.

The main theorem of [FG11] is the following result, that tells us that the natural derived Koszul duality adjunction between Lie and Com for the chiral monoidal structure is an equivalence. This theorem essentially follows from the nilpotence property of the chiral tensor product, that makes all commutative coalgebras formal, in some sense.

Theorem 23.2.7 (Chiral Koszul duality)**.** *The natural pair of adjoint ∞-functors*

$$\mathrm{CE} : \mathrm{Lie}^{ch}(\mathcal{D}(\mathrm{Ran}(M))) \leftrightarrows \mathrm{coAlg}^{ch}(\mathcal{D}(\mathrm{Ran}(M))) : \mathrm{Prim}[-1]$$

between Lie algebras and cocommutative coalgebras in $(\mathcal{D}(\mathrm{Ran}(M)), \otimes^{ch})$ *is an equivalence of ∞-categories.*

One may think of commutative factorization coalgebras as algebras of distributions on some distributional formal space.

Definition 23.2.8. Let $\mathcal{A}$ be a chiral commutative coalgebra. The distributional formal moduli space associated to $\mathcal{A}$ is the ∞-functor

$$\begin{array}{rccc} \underline{\mathrm{Spec}}_{distrib}(\mathcal{A}) : & \mathrm{coAlg}^{ch}(\mathcal{D}(\mathrm{Ran}(M))) & \to & \mathrm{Sh}_\infty(\mathrm{Ran}(M)) \\ & \mathcal{B} & \mapsto & \underline{\mathrm{Mor}}_{\mathrm{coAlg}}(\mathcal{B}, \mathcal{A}) \end{array}$$

with values in sheaves of ∞-groupoids on $\mathrm{Ran}(M)$.

One may think geometrically of the Chiral Lie algebra $\mathrm{Prim}[-1]A$ as the tangent complex to the formal moduli space $\underline{\mathrm{Spec}}_{distrib}(\mathcal{A})$.

Definition 23.2.9. A commutative coalgebra given by the Chevalley-Eilenberg image of a chiral Lie algebra is called a factorization coalgebra on M.

Thus a factorization coalgebra on M is a particular kind of commutative chiral coalgebra in $\mathcal{D}(\mathrm{Ran}(M))$. Remark that it almost never has support on the main diagonal, contrary to chiral Lie algebras.

Definition 23.2.10. A chiral algebra is called *commutative* if the associated local bracket is trivial. A factorization coalgebra has *commutative factorization structure* if the associated chiral Lie algebra is commutative.

We now explain how to associate a chiral Lie algebra to a commutative $\mathcal{D}$-algebra. With a right notion of (higher dimensional) unital chiral algebra in hand, the obtained functor would become an equivalence.

Proposition 23.2.11. *There is a functor between commutative derived $\mathcal{D}$-algebras and commutative chiral Lie algebras.*

Proof. Let $\mathcal{A}^\ell$ be a commutative derived $\mathcal{D}$-algebra on M and $\mathcal{A}$ be the corresponding right $\mathcal{D}$-algebra. By definition, $\mathcal{A}$ is a model

$$\mathcal{A} : (\textsc{Com}, \otimes) \longrightarrow (\textsc{DMod}(\mathcal{D}^{op}), \otimes^!)$$

of the commutative symmetric linear theory of commutative monoids into the monoidal ∞-category of right $\mathcal{D}$-modules.Using the natural morphisms

$$j_*j^*(\mathcal{A}^{\boxtimes n}) \longrightarrow \Delta_*(\mathcal{A}^{\otimes^! n}),$$

one can extend this to a chiral Lie algebra structure

$$\mathcal{A} : (\textsc{Lie}, \otimes) \longrightarrow (\mathcal{D}(\mathrm{Ran}(M)), \otimes^{ch}).$$

Using the fact that the sequence

$$\mathcal{A}^{\boxtimes n} \to j_*j^*(\mathcal{A}^{\boxtimes n}) \to \Delta_*(\mathcal{A}^{\otimes^! n}),$$

is a fibration sequence, we get that the obtained chiral algebra is commutative. □

Remark 23.2.12. Let us describe the $\mathcal{D}$-algebra of a free euclidean scalar theory, given by a linear differential operator

$$\mathcal{D} \xrightarrow{.D} \mathcal{D} \twoheadrightarrow \mathcal{M}_D,$$

the corresponding $\mathcal{D}$-algebra is simply the differential graded $\mathcal{D}$-algebra

$$\mathbb{R}\mathrm{Sym}(\mathcal{M}_D) := \mathrm{Sym}([\underset{1}{\mathcal{D}} \xrightarrow{.D} \underset{0}{\mathcal{D}}]),$$

and its solutions with values in a $\mathcal{D}$-algebra $\mathcal{A}$ identify with the derived solutions of $\mathcal{M}_D$ in the underlying $\mathcal{D}$-module. In this linear situation, one may take solutions of $\mathcal{M}_D$ with values in hyperfunctions, and if the operator is elliptic, one has an identification

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M}_D, \mathcal{O}_M) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M}_D, \mathcal{B}_M).$$

This isomorphism means that D induces an isomorphism

$$D : \mathcal{B}_M/\mathcal{A}_M \xrightarrow{\sim} \mathcal{B}_M/\mathcal{A}_M,$$

whose inverse H is called the parametrix of P. Microlocally, this means that microfunction solutions

$$\mathbb{R}\mathcal{H}om_{\mathcal{E}}(\mathcal{M}_D, \mathcal{C}_M) \cong 0$$

annihilate outside of the zero section of $T^*_M X$. If the operator is globally hyperbolic, the above statement is not true anymore, but one may reformulate it.

Remark 23.2.13. The easiest way to have a formalism that works both in the hyperbolic and elliptic situation is to see the parametrix H of a differential operator $D \in \mathcal{D}$ not as acting on a particular distributional space, but as a microdifferential operator (see Definition 10.5.7), given by a kernel on X^2 supported on the diagonal. This actually works for any operator, by seing it as a microdifferential operator

$$\mathcal{E} \xrightarrow{.D} \mathcal{E}.$$

One may then invert D outside of the characteristic variety $S = \mathrm{char}(D) \subset T^*X$, yielding a microdifferential operator $H = D^{-1} \in \mathcal{E}$, that can be called the parametrix. This formulation has the advantage of generalizing directly to systems, the microdifferential inverse being replaced by a local homotopy between the complex of $\mathcal{E}$-modules in play and zero. One may then get an analog of the isomorphism between analytic and hyperfunction solutions of the elliptic case by writing down the isomorphism

$$\mathbb{R}\mathcal{H}om_{\mathcal{E}}(\mathcal{M}_D, \mathcal{C}_M)_{|\mathrm{char}(\mathcal{M}_D)^c} \cong 0,$$

that comes from the fact that $\mathcal{M}_D$, as a microdifferential module, has support exactly equal to $\mathrm{char}(\mathcal{M}_D)$. In the physicist's language, the parametrix H is called the propagator, and it is usually represented by its kernel, leaving on X^2, with support on the diagonal. One can then make the microdifferential operator H act on the space

$$\mathcal{C}_{M^n \setminus S^n}$$

of microfunctions $f(x_1, \xi_1, \ldots, x_n, \xi_n)$ defined outside of the characteristic manifold. The formulas appearing in the $*$-products of quantum field theory are given by pullbacks to the diagonal $\{(x_i, \xi_i, y_i, \eta_i),\ x_i = y_i,\ \xi_i = \eta_i\}$ of expressions of the form

$$g(y_1, \eta_1, \ldots, y_n, \eta_n) H(x_1, \xi_1) \cdots H(x_n, \xi_n) f(x_1, \xi_1, \ldots, x_n, \xi_n),$$

where H is the microlocal parametrix operator. These pullbacks can be done only under some wave front set properties. The causal renormalization method, described in Chapter 21 gives a systematic way to deal with the problems that appear when the pullback is not well defined.

23.3 Operator product expansions

We now briefly describe (analytic) operator product expansions, by using an analytic version of Beilinson-Drinfeld's construction [BD04], Section 3.9. This gives a direct relation between chiral algebras and the physicists' operator product expansions. This definition looks natural because of recent results of Hollands and Kopper on φ^4 theory [HK11], in the setting developped by Hollands and Wald [HW10]. They obtained an analycity result for operator product expansions.

Let M be an analytic manifold and $(\mathcal{A}, [-,-]_{ch})$ be a Chiral algebra on M. We also denote $\mathcal{A} = \mathrm{CE}_{\otimes^{ch}}(\mathcal{A}, [-,-]_{ch})$ the associated chiral coalgebra on $\mathrm{Ran}(M)$. Its restriction to M is the original $\mathcal{A}$.

For each surjection $\pi : I \twoheadrightarrow J$, we denote $\Delta(\pi) : M^J \to M^I$ the associated diagonal map and $j(\pi) : U^J \to M^I$ the inclusion of the complement open subset. By definition, there is a natural homotopy equivalence

$$\Delta(\pi)^! \mathcal{A}^I \cong \mathcal{A}^J.$$

We denote $\hat{\Delta}(\pi)^! \mathcal{A}^I$ the restriction of $\mathcal{A}^I$ to the formal completion of M^I along the diagonal and

$$\tilde{\Delta}(\pi)^! \mathcal{A}^I := \hat{\Delta}(\pi)^! \mathcal{A}^I \otimes j_* \mathcal{O}_{U(\pi)}$$

its restriction to the pointed formal neighborhood of the diagonal.

By definition of a factorization coalgebra, the natural maps

$$j(\pi)^*(\mathcal{A}^I) \longrightarrow j(\pi)^*(\boxtimes_{i \in J} \mathcal{A}^{I_j})$$

are equivalences.

Definition 23.3.1. Let $\pi : I \to \{1\}$ the canonical projection. The I-th operator product expansion associated to $\mathcal{A}$ is the morphism

$$\circ_I : \mathcal{A}^{\ell \boxtimes I} \to j(\pi)_* j(\pi)^* \mathcal{A}^{\ell \boxtimes I} \xrightarrow{\sim} j(\pi)_* j(\pi)^* \mathcal{A}^{\ell I} \to \tilde{\Delta}^I_* \mathcal{A}^{\ell}.$$

23.4 Derived $\mathcal{D}$-geometry of factorization spaces

We now recall from Section 11.5, for the reader's convenience, the following basic constructions of differential calculus on the Ran space in derived geometry.

Definition 23.4.1. The Ran space is the derived stack defined by the ∞-colimit

$$\mathrm{Ran}(M) := \operatorname*{colim}_{I \twoheadrightarrow J} M^I$$

along diagonal embeddings.

We will denote $\mathrm{Ran}(M)_{DR}$ the associated de Rham space.

Let X be an Artin stack. We denote $C = M \times X$ and $\pi : C \to M$ the natural projection. One may define the jet space

$$\pi_\infty : \mathrm{Jet}_{DR}(C) \to M_{DR},$$

as the push-forward $p_* C \to M_{DR}$ for the natural map $p : M \to M_{DR}$. We will also denote $\mathrm{Jet}(C) := p^* p_* C$ the corresponding bundle over M with flat connection.

There is a natural inclusion

$$\underline{\Gamma}(M_{DR}, \mathrm{Jet}_{DR}(C)) \hookrightarrow \underline{\Gamma}(M, \mathrm{Jet}(C))$$

that identifies horizontal sections of the jet bundle, and an isomorphism

$$j_\infty : \underline{\Gamma}(M, C) \xrightarrow{\sim} \underline{\Gamma}(M_{DR}, \mathrm{Jet}_{DR}(C))$$

that sends a section to its infinite jet, that is horizontal for the canonical connection on $\mathrm{Jet}(C)$.

The above considerations may be extended by seing M and C as derived stacks. Derived mapping spaces are not always isomorphic to the usual mapping space, but we will suppose from now on that we work only with these derived mapping spaces.

One may give, following Rozenblyum [Roz11], another description of the derived stack $\underline{\Gamma}(M,C)$, as flat sections of the multijet bundle

$$\mathrm{Jet}^{Ran}_{DR}(C) \to \mathrm{Ran}(M)_{DR}$$

over the de Rham space of the Ran space. This multijet space has the property of being a so-called factorization space, that we now define.

By definition, the category of derived stacks over $\mathrm{Ran}(M)$ is the ∞-limit of the categories of derived stacks over M^I with respect to the diagonal maps associated to surjections. We will denote it $\mathrm{DS}\textsc{tacks}/\mathrm{Ran}(M)$.

Definition 23.4.2. A *factorization space* is a formal stack $Z^{(Ran)}$ over $\mathrm{Ran}(M)_{DR}$, i.e., a family of formal stacks

$$p_I : Z^{(I)} \to M^I$$

with a connection along M^I for each finite set I, that are compatible with pullbacks along diagonals and composition of surjections, up to coherent homotopy, together with compatible homotopy coherent factorization isomorphisms

$$Z^{(I)}_{|U_\alpha} \cong \left(\prod_{j\in J} Z^{I_j}\right)_{|U_\alpha},$$

for surjections $\alpha : I \twoheadrightarrow J$ and $U_\alpha \subset M^I$ is the subset

$$U_\alpha := \{x_i \neq x_{i'} \text{ if } \alpha(i) \neq \alpha(i')\}.$$

A factorization space $Z^{(Ran)}$ is called counital if in addition there are maps

$$M^{(I_1)} \times Z^{(I_2)} \to Z^{(I_1 \coprod I_2)}$$

that are compatible with factorization and restrictions to diagonals. We will denote $\mathrm{DS}\textsc{tacks}_{fact}$ the ∞-category of factorization spaces and $\mathrm{DS}\textsc{tacks}^{cu}_{fact}$ the ∞-category of counital factorization spaces.

There is a natural forgetful functor

$$\begin{array}{cccc} (\_)^{(1)}: & \mathrm{DS}\textsc{tacks}^{cu}_{fact}/\mathrm{Ran}(M)_{DR} & \to & \mathrm{DS}\textsc{tacks}/M_{DR} \\ & Z^{(Ran)} & \mapsto & Z^{(1)} \end{array}$$

from counital factorization spaces to $\mathcal{D}_M$-spaces (aka., spaces over M_{DR}).

Definition 23.4.3. The *multijet ∞-functor* is the right adjoint

$$(\_)^{(1)} : \mathrm{DS}\textsc{tacks}^{cu}_{fact}/\mathrm{Ran}(M)_{DR} \leftrightarrows \mathrm{DS}\textsc{tacks}/M_{DR} : \mathrm{Jet}^{Ran}_{DR}$$

to the forgetful functor. The *multijet space of a derived stack* F over M_{DR} is the counital factorization space

$$\mathrm{Jet}^{Ran}_{DR}(F) \to \mathrm{Ran}(M)_{DR}.$$

In particular, the *multijet space of a map* $\pi : C \to M$ is the multijet space of the derived stack

$$\mathrm{Jet}_{DR}(C) \to M_{DR}.$$

There is a natural isomorphism

$$\underline{\Gamma}(M, C) \xrightarrow{\sim} \underline{\Gamma}(\mathrm{Ran}(M)_{DR}, Jet^{Ran}_{DR}(C))$$

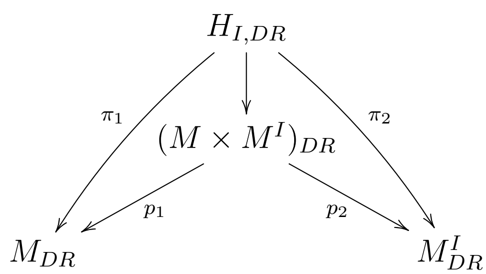
that identifies sections of π with horizontal sections of the multijet space, and a given PDE on these sections, given by a closed subspace Z of $\mathrm{Jet}_{DR}(C)$, corresponds to a closed subspace $Z^{(Ran)}$ of the corresponding multijet space, that is also a counital factorization space.

The following proposition can be found in [Roz10], Proposition 5.1.1.

Proposition 23.4.4. *The multijet functor is given by the formula*

$$\mathrm{Jet}^{Ran}_{DR}(F)^I = (\pi_{2,DR})_*(\pi_{1,DR})^*F \underset{M^I_{DR}}{\times} M^I,$$

where π_1 and π_2 are the morphisms making the following canonical diagram



commutative and $H_I \subset M \times M^I$ is the incidence subspace, defined by

$$H_I := \{(x, \{x_i\}), x = x_i\}.$$

Definition 23.4.5. Let $Z^{(Ran)} \to \mathrm{Ran}(M)_{DR}$ be a factorization derived $\mathcal{D}$-stack over $\mathrm{Ran}(M)$. The Chiral homology of Z is the derived stack

$$H(M, Z) := \underline{\Gamma}(\mathrm{Ran}(M)_{DR}, Z^{(Ran)}).$$

If Z is the derived $\mathcal{D}$-space $\mathbb{R}\underline{\mathrm{Sol}}_{\mathcal{D}}(\mathcal{I}_Z)$ of solutions of a PDE on $\underline{\Gamma}(M, C)$, and $Z^{(Ran)}$ is the corresponding multijet space, the Chiral homology of Z identifies with the corresponding derived space $\mathbb{R}\underline{\mathrm{Sol}}(\mathcal{I}_Z)$, as a usual derived stack.

By definition, there is a natural evaluation morphism

$$\mathrm{ev} : H(M, Z) \times M \to Z.$$

Definition 23.4.6. Let $\mathcal{M}$ be a module over Z. The *localization* of $\mathcal{M}$ is the module over $H(M,Z)$ given by de Rham cohomology along M of $\mathrm{ev}^*\mathcal{M}$.

Example 23.4.7. We present here an example due to Rozenblyum, that is related to the Witten genus construction. Let E be an elliptic curve and X be a derived Artin stack. Let $\pi : C = T^*X \times E \to E$ be the natural projection. Then $Z = \mathrm{Jet}_{DR}(T^*X)$ is equipped with a local Poisson structure, given by a bivector

$$\pi \in \Theta_Z \wedge^* \Theta_Z$$

on the Ran space. The localization of Θ_Z identifies with the tangent complex

$$\mathbb{T}_{H(E,Z)} \cong \mathbb{T}_{\underline{\mathrm{Hom}}(E,T^*X)}$$

and the symplectic structure on $\underline{\mathrm{Hom}}(E,T^*X)$ defined by Costello [Cos10] and Pantev-Toen-Vaquie-Vezzosi [PTVV11] comes from the above bivector by localization.

Example 23.4.8. Let M be a smooth proper manifold and G be a compact Lie group, with an invariant pairing on its Lie algebra $\mathfrak{g}$. Let $\pi : C = \Gamma_M(M \times BG, M \times \mathrm{Jet}^1 BG) \to M$ be the bundle whose sections are principal G-bundles equipped with a principal G-connection. Then $\mathrm{Jet}(C) \to M_{DR}$ is the $\mathcal{D}$-bundle whose horizontal sections are principal G-bundles with a connection, and one may define a PDE $Z \subset \mathrm{Jet}(C)$ by setting the curvature of the given connection to zero, in a derived sense, i.e., using the derived critical space of the curvature local functional. One then has

$$\mathbb{T}_{H(M,Z)} \cong \mathbb{T}_{\underline{\mathrm{Hom}}(M_{DR},BG)}.$$

There is a natural local symplectic structure on Z given by the given invariant pairing on the Lie algebra $\mathfrak{g}$ of G.

Remark 23.4.9. We want to present here a useful intuitive dictionary between Costello and Gwilliam's approach to the perturbative quantization [CG10] of factorization algebras, and the presentation we give here. The heart of this dictionnary is the fact that we work with sheaves on the Ran space, whereas Costello and Gwilliam work with cosheaves. The relation of their approach to ours is given by Lurie's non-abelian Poincaré duality theorem (see [Lur09c], Section 5.3 and [AFT12]), that sometimes allows to describe the space of compactly supported horizontal sections of a $\mathcal{D}$-space in a highly non-trivial way. Roughtly speaking, if $(Z \to M_{DR}, s)$ is a pair composed of a $\mathcal{D}$-space over M and a horizontal section $s : M_{DR} \to Z$ of it, fulfilling some additional *finiteness* and *connectivity* hypothesis, one may associate to it a space $\underline{\Gamma}_c(M_{DR}, Z)$ of compactly supported horizontal sections, and this space may be computed as the topological chiral homology

$$\underline{\Gamma}_c(M_{DR}, Z) \cong \int_M E^!$$

of a factorization M-cospace $E^!$ (essentially, a cosheaf on the Ran space fulfilling an additional factorization property). Lurie's theorem, that applies to a locally compact topological space, probably extends directly to the semianalytic and subanalytic topoi (maybe with some overconvergence hypothesis), that we used to formalize algebraic analysis (see Example 2.2.7 and Section 10.3). We mention this result because it plays an important role in relating locally constant factorization algebras with algebras over some classical operads, to give a geometrical explanation of the usefulness of operads in the perturbative quantization of topological quantum field theories (illustrated by Cattaneo-Felder's paper [CF01] and Tamarkin's proof [Tam98] of Kontsevich's formality theorem).

Remark 23.4.10. Let $\mathcal{L}$ be an elliptic local Lie algebra with an invariant pairing $\psi : \mathcal{L} \otimes \mathcal{L} \to \omega_{\mathcal{D}}$. The Chevalley-Eilenberg commutative coalgebra $\mathrm{CE}_{\otimes^*}(\mathcal{L})$ for the local monoidal structure may be seen as a chiral coalgebra through the natural transformation $\otimes^* \to \otimes^{ch}$. By [FG11], Proposition 6.1.2, the obtained factorization coalgebra is the Chevalley-Eilenberg algebra of the chiral envelope $U^{ch}(\mathcal{L})$ (this may actually be taken as a definition for the chiral envelope of a local Lie algebra). If $\mathcal{L}$ is dualizable for the $\otimes^*$ monoidal structure, with dual $\mathcal{L}^\circ$, the local invariant pairing ψ induces a Poisson bracket on the commutative $\mathcal{D}$-algebra $\mathcal{A} = \mathrm{CE}_{\otimes^*}(\mathcal{L}^\circ)$. By loc. cit., proof of proposition 6.3.6, we know that

$$\mathrm{DR}_{\mathcal{O}}(\mathrm{CE}_{\otimes^*}(\mathcal{L})) \cong \mathrm{CE}_{\otimes}(\mathrm{DR}_{\mathcal{O}}(\mathcal{L})).$$

Since $\mathcal{L}$ is elliptic on M, we get that

$$\mathrm{DR}_{\mathcal{O}_{\mathrm{Ran}}}(\mathcal{A}) \longrightarrow \mathrm{DR}_{\mathcal{B}_{\mathrm{Ran}}}(\mathcal{A})$$

is a homotopy equivalence. Now, one may take the compactly supported sections

$$\Gamma_c(\mathrm{DR}_{\mathcal{B}_{\mathrm{Ran}}}(\mathcal{A}))$$

to construct a cosheaf on the Ran space, that is directly related, in the case of a commutative Lie algebra $\mathcal{L}$, to the commutative factorization algebras considered by Costello and Gwilliam [CG10].

23.5 Generalized affine grassmanians and perturbative quantization

We shall explain here Rozenblyum's generalization [Roz12] of the affine grassmanian quantization of Beilinson and Drinfeld [BD04]. This construction gives, in some sense, an analog in the chiral world of Lurie's non-abelian Poincaré-Verdier duality (see [Lur09c], Section 5.3). It also plays an important role, as we will see, in the geometric quantization of perturbative local field theories.

To every reasonable (in the sense of deformation theory, see [GR11]) pointed formal factorization $\mathcal{D}$-space Z on M (pointed means here, equipped with an horizontal section $s_0 \in \Gamma(M_{DR}, Z)$, one may associate naturally another formal factorization $\mathcal{D}$-space Gr_Z on $\mathrm{Ran}(M)$ called the affine grassmanian $\mathcal{D}$-space of Z, with the following property: for $[S] \in \mathrm{Ran}(M)$ a point, we denote D_S the union of the formal discs around points $s \in S \subset M$ and D_S° the union of the corresponding punctured formal discs. The space Gr_Z has fiber at $[S]$ the fiber of the morphism

$$\underline{\Gamma}(D_S, Z) \longrightarrow \underline{\Gamma}(D_S^\circ, Z)$$

at the given fixed point s_0. The space Gr_Z is actually defined as the fiber of a morphism

$$Z^{(Ran)} \longrightarrow Z^{(0)}$$

at the given point s_0, where $Z^{(0)}$ is a punctured (aka local formal loop space) analogs of the multijet construction. There is a relation between the chiral homology of Gr_Z and that of Z, that is valid only under some strong connectivity hypothesis (Z must be n-connected, where $n = \dim(M)$).

One may also think of this operation as a kind of geometric analog of the one dimensional twisted chiral envelope of Beilinson-Drinfeld [BD04], Section 3.7.10, and a twisted version of the higher dimensional chiral envelope of Francis-Gaitsgory [FG11], Section 6.4.

We now translate Costello and Gwilliam's definition from [CG10] of a perturbative quantum field to the setting of $\mathcal{D}$-geometry.

Definition 23.5.1. A *perturbative classical field theory* is the datum of a local (homotopy) Lie algebra $\mathcal{L}$ with invariant bilinear form ψ. A perturbative classical field theory is called *coherent* (resp. *holonomic*) if the $\mathcal{D}$-module underlying the Lie algebra $\mathcal{L}$ is coherent (resp. holonomic).

One denotes $Z = B\mathcal{L}$ the formal $\mathcal{D}$-space with coalgebra of distributional coordinates the Chevalley-Eilenberg coalgebra $\mathrm{CE}_*(\mathcal{L})$. This coalgebra is equipped with a local Poisson bracket $\{-,-\}_\psi$ induced by the pairing ψ. If $\mathcal{L}$ is coherent, we may also define the algebra of coordinates on $B\mathcal{L}$ to be the Chevalley-Eilenberg algebra $\mathrm{CE}^*(\mathcal{L}^\circ)$.

Now if we apply the affine grassmanian construction to $Z = B\mathcal{L}$, we get a $\mathcal{D}$-space $\mathrm{Gr}_\mathcal{L}$, and the Poisson bracket $\{-,-\}_\mathcal{L}$ induces a line bundle $\mathcal{F}_\psi$ on $\mathrm{Gr}_\mathcal{L}$. The quantization of the formal Poisson $\mathcal{D}$-space $(B\mathcal{L}, \{-,-\}_\psi)$ is then given by the factorization algebra of global sections of this line bundle.

23.6 Categorical quantization of factorization spaces

We now turn on to the formal quantization question, to explain how to quantize $\mathcal{D}$-spaces that are not of the form $B\mathcal{L}$, for $\mathcal{L}$ a local Lie algebra equipped with an invariant pairing. The general approach that we give now may need some adaptations, in particular to have a well behaved notion of deformation theory (see [PTVV11], and [GR11] for discussions of related issues). The main issue at stake here is to have good finiteness properties for the categories in play.

The basic idea of deformation theory of Poisson algebras, that we discussed shortly in Chapter 22, is that it is much easier to quantize the category of modules over a commutative algebra with a given Poisson structure, than to quantize the algebra itself. This idea was already clear in Kontsevich's formality approach to deformation quantization [Kon03].

Similarly, quantum group deformations, discussed in Chapter 5, were actually defined by Drinfeld in [Dri87] as deformation quantizations of braided monoidal categories. More generally, the article [PTVV11], shortly discussed in Chapter 22, also consider deformations of (the monoidal dg-category of modules over) symplectic derived stacks of maps between particular geometric stacks as n-monoidal dg-categories.

It is also clear in the homological mirror symmetry program of Kontsevich [Kon95] that the main objects at stake in mirror symmetry are some kinds of stable ∞-categories (or their more concrete analogs, given by A_∞-categories).

The above various quantization problems for categories may be put in the setting of categorical factorization quantizations, using deformations of the stable ∞-category of modules (with additional finiteness conditions, like perfectness, say) over a given factorization algebra, as a factorization category. The advantage of the factorization approach is that, even if it generalizes naturally the above quantization problems for algebraically structured categories, it allows to give non-topological quantization of categories, with refined additional geometrical structures, encoded in factorization properties. This gives a neat combination of quantum field theory, in the general formulation we gave in this book (based on the geometry of non-linear partial differential equations), and deformation quantization of structured categories, that is at the basis of the above discussed (now standard) examples.

The precise relation with topological deformation quantizations of Chapter 22 is given by the fact that an n-monoidal differential graded category may be viewed as a locally constant factorization category over $\mathbb{R}^n$ (i.e., an E_n-algebra in the ∞-category of presentable stable ∞-categories enriched over dg-modules), and topological theories on a manifold (i.e., theories with a variable metric, that have a diffeomorphism gauge invariance) may be seen as giving such factorization categories over $\mathbb{R}^n$.

We first define the notion of factorization categories, by simply adapting Definition 23.4.2 of factorization spaces. We need a notion of tensor product of presentable ∞-categories (these have small colimits and are generated by a small set of objects; see [Lur09c], Section 1.4.5 and Section 6.3). Remark that the presentability property for module categories over factorization spaces may only be provable in very particular (like algebraic proper) cases, so that the following definition may need some modification to be adapted to smooth differential geometry.

Definition 23.6.1. A *factorization ∞-category* is a stack of stable presentable ∞-categories over $\mathrm{Ran}(M)_{DR}$, i.e., a family of stacks of ∞-categories

$$\mathcal{C}^{(I)} : (\mathrm{O}\textsc{pen}^{op}_{M^I}, \tau) \longrightarrow {}^{\infty}\mathrm{C}\textsc{at}^{cont}_{sp},$$

from the site of open subsets in M^I to the ∞-category of stable presentable ∞-categories (with continuous functors), with a categorical connection along M^I for each finite set I, i.e., an extension of $\mathcal{C}^{(I)}$ to

$$\mathcal{C}^{(I)}_{DR} \in \mathrm{S}\textsc{h}_{^\infty\mathrm{C}\textsc{at}_{sp}}(M^I_{DR}),$$

that are compatible with pullbacks along diagonals and composition of surjections, together with compatible homotopy coherent factorization equivalences

$$\mathcal{C}^{(I)}_{|U_\alpha} \cong \left(\boxtimes_{j\in J}\mathcal{C}^{(I_j)}\right)_{|U_\alpha},$$

for surjections $\alpha : I \twoheadrightarrow J$ and $U_\alpha \subset M^I$ is the subset

$$U_\alpha := \{x_i \neq x_{i'} \text{ if } \alpha(i) \neq \alpha(i')\}.$$

Example 23.6.2. The category $\mathcal{D}(\mathrm{Ran}(M))$ is the most simple example of a factorization category. It is the factorization category of modules over the (factorization algebra associated to the) chiral algebra ω_M.

Proposition 23.6.3. *Let $Z^{(Ran)}$ be a geometric factorization space over* $\mathrm{Ran}(M)$. *The category* $\mathrm{MOD}(Z^{(Ran)})$ *is a factorization ∞-category.*

Proof. This follows from the functoriality of module categories (see for example [BFN08]). □

Definition 23.6.4. A *categorical quantization* of a factorization category $\mathcal{C}$ is a factorization category $\mathcal{C}[[\hbar]]$ flat over $\mathbb{R}[[\hbar]]$, together with an equivalence

$$\mathcal{C} \cong \mathcal{C}' \mod \hbar.$$

If Z/M_{DR} is an Artin $\mathcal{D}$-space, a categorical quantization of Z is a categorical quantization of the factorization category $\mathrm{MOD}(Z^{(Ran)})$.

Example 23.6.5. To illustrate the above general definition, we give an example from Chapter 5. Let $\mathfrak{g}$ be a real Lie algebra and $U_\hbar(\mathfrak{g})$ be a quantum enveloping algebra of $U(\mathfrak{g})$. The category $\mathrm{MOD}(U_\hbar(\mathfrak{g}))$ is a braided monoidal (i.e., 2-monoidal) category that gives a formal deformation of the symmetric monoidal category $\mathrm{MOD}(U(\mathfrak{g}))$. Let $M = \mathbb{R}^2$ and consider the factorization categories

$$\mathcal{M}od(U_\hbar(\mathfrak{g})) \text{ and } \mathcal{M}od(U(\mathfrak{g}))$$

on $\mathrm{Ran}(M)_{DR}$ associated to these two categorical E_2-algebras. Then the first is a categorical quantization of the second.

The following definition may be adapted by choosing a particular class of modules, like perfect modules, say. One may then formulate a general quantization problem, which is to find mild finiteness hypothesis under which,

there exists a categorical quantization
of an Artin Poisson $\mathcal{D}$-space as a factorization category.

If we know the existence of a categorical quantization $\mathcal{C}$ of an Artin Poisson $\mathcal{D}$-space Z (this is known in special cases), we may then ask if it has a formality property in the sense that its quantization bundle over $\mathbb{R}[[\hbar]]$ has a natural splitting. This situation is quite analogous to the Hodge to de Rham spectral sequence degeneration, that is the conceptual heart of Hodge theory. Indeed, if X is a smooth complex projective variety, one always has a Hodge filtration on its de Rham cohomology, given, through the Rees construction, by a module over the affine line $\mathbb{A}^1$ that is equivariant under the action of GL_1. The Hodge to de Rham degeneration corresponds, in this case, to saying that there is a natural equivariant trivialization of this module (i.e., a splitting of the Hodge filtration).

Exercices sheets for the course

Appendix A

Categories and universal properties

Categories and universal properties

Definition 1.0.6. A *category* C is given by the following data:

1. a class $\mathcal{O}b(C)$ called the *objects* of C,
2. for each pair of objects X, Y, a set $\mathrm{Hom}(X, Y)$ called the set of *morphisms*,
3. for each object X a morphism $\mathrm{id}_X \in \mathrm{Hom}(X, X)$ called the *identity*,
4. for each triple of objects X, Y, Z, a composition law for morphisms

$$\circ : \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z).$$

One supposes moreover that this composition law is associative, i.e., $f \circ (g \circ h) = (f \circ g) \circ h$ and that the identity is a unit, i.e., $f \circ \mathrm{id} = f$ et $\mathrm{id} \circ f = f$.

Definition 1.0.7. A *universal property* [1] for an object Y of C is an explicit description (compatible to morphisms) of $\mathrm{Hom}(X, Y)$ (or $\mathrm{Hom}(Y, X)$) for every object X of C.

Example 1.0.8. Here are some well known examples.

1. Sets whose objects are sets and morphisms are maps.
2. Grp whose objects are groups and morphisms are group morphisms.
3. Grab whose objects are abelian groups and morphisms are group morphisms.
4. Rings whose objets are commutative unitary rings and whose morphisms are ring morphisms.
5. Top whose objets are topological spaces and morphisms are continuous maps.

Principle 4. (Grothendieck)

The main interest in mathematics are not the mathematical objects,
but their relations
(i.e., morphisms).

Exercice 1. (Universal properties) What is the universal property of

1. the empty set?
2. the one point set?

[1]Every object has exactly two universal properties, but we will usually only write the simplest one.

3. $\mathbb{Z}$ as a group?

4. $\mathbb{Z}$ as a commutative unitary ring?

5. $\mathbb{Q}$ as a commutative unitary ring?

6. the zero ring?

To be more precise about universal properties, we need the notion of "morphism of categories".

Definition 1.0.9. A (covariant) *functor* $F : C \to C'$ between two categories is given by the following data:

1. For each object X in C, an object $F(X)$ in C',

2. For each morphism $f : X \to Y$ in C, a morphism $F(f) : F(X) \to F(Y)$ in C'.

One supposes moreover that F is compatible with composition, i.e., $F(f \circ g) = F(f) \circ F(g)$, and with unit, i.e., $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

Definition 1.0.10. A *natural transformation* φ between two functors $F : C \to C'$ and $G : C \to C'$ is given by the following data:

1. For each object X in C, a morphism $\varphi_X : F(X) \to G(X)$,

such that if $f : X \to Y$ is a morphism in C, $G(f) \circ \varphi_X = \varphi_Y \circ F(f)$.

We can now improve definition 1.0.7 by the following.

Definition 1.0.11. A *universal property* for an object Y of C is an explicit description of the functor $\mathrm{Hom}(X, -) : C \to \textsc{Sets}$ (or $\mathrm{Hom}(-, X) : C \to \textsc{Sets}$).

The following triviality is at the heart of the understanding of what a universal property means.

Exercice 2. (Yoneda's lemma) Let $C^\vee$ be the category (whose morphisms form classes, not sets) whose objects are functors $F : C \to \textsc{Sets}$ and whose morphisms are natural transformations.

1. Show that there is a natural bijection

$$\mathrm{Hom}_C(X, Y) \to \mathrm{Hom}_{C^\vee}(\mathrm{Hom}(X, -), \mathrm{Hom}(Y, -)).$$

2. Deduce from this the an object X is determined by $\mathrm{Hom}(X, -)$ uniquely up to a unique isomorphism.

Exercice 3. (Free objects) Let C be a category whose objects are described by finite sets equipped with additional structures (for example, $\textsc{Sets}$, $\textsc{Grp}$, $\textsc{Grab}$, $\textsc{Rings}$ or $\textsc{Top}$). Let $\text{Forget} : C \to \textsc{Sets}$ be the forgetful functor. Let X be a set. A free object of C on X is an object $L(X)$ of C such that for all object Z on C, one has a natural bijection

$$\text{Hom}_C(L(X), Z) \cong \text{Hom}_{\text{Ens}}(X, \text{Forget}(Z)).$$

Let X be a given set. Describe explicitely

1. the free group on X,
2. the free abelian group on X,
3. the free $\mathbb{R}$-module on X,
4. the free unitary commutative ring on X,
5. the free commutative unitary $\mathbb{C}$-algebra on X,
6. the free associative unitary $\mathbb{C}$-algebra on X.
7. the free topological space on X.
8. the free smooth algebra on X.

Exercice 4. (Monads and rings) Let $\textsc{Rings}$ be the category of commutative unital rings. Let $\text{Endof}(\textsc{Sets})$ be the category of endofunctors $F : \textsc{Sets} \to \textsc{Sets}$. It is equipped with a monoidal structure

$$\begin{array}{rccc} \circ : & \text{Endof}(\textsc{Sets}) \times \text{Endof}(\textsc{Sets}) & \to & \text{Endof}(\textsc{Sets}) \\ & (F, G) & \mapsto & F \circ G. \end{array}$$

Let $F : \textsc{Rings} \to \text{Endof}(\textsc{Sets})$ be the functor defined by $X \mapsto \text{Forget}(A^{(X)})$.

1. Show that F is equipped with a multiplication $\mu : F \circ F \to F$ and a unit $1 : \text{Id}_{\textsc{Sets}} \to F$ that makes it a monoid in the monoidal category $(\text{Endof}(\textsc{Sets}), \circ)$.
2. A monad in $\textsc{Sets}$ is a monoid in the monoidal category $(\text{Endof}(\textsc{Sets}), \circ)$. Denote $\textsc{Monads}(\textsc{Sets})$ the category of monads in $\textsc{Sets}$. Show that the functor

   $$F : \textsc{Rings} \to \textsc{Monads}(\textsc{Sets})$$

   defined by the free module construction is fully faithful.

Exercice 5. (Products and sums) The product (resp. the sum) of two objects X and Y is an object $X \times Y$ (resp. $X \coprod Y$, sometimes denoted $X \oplus Y$) such that for all object Z, there is a bijection natural in Z

$$\text{Hom}(Z, X \times Y) \cong \text{Hom}(Z, X) \times \text{Hom}(Z, Y)$$
$$(\text{resp. } \text{Hom}(X \textstyle\coprod Y, Z) \cong \text{Hom}(X, Z) \times \text{Hom}(Y, Z)).$$

Explicitely describe

1. the sum and product of two sets,

2. the sum and product of two abelian groups, and then of two groups,

3. the sum and product of two unitary associative rings.

4. the sum and product of two unitary commutative rings.

Exercice 6. (Fibered products and amalgamated sums) The fibered product (resp. amalgamated sum) of two morphims $f : X \to S$ and $g : Y \to S$ (resp. $f : S \to X$ and $f : S \to Y$) is an object $X \times_S Y$ (resp. $X \coprod_S Y$, sometimes denoted $X \oplus_S Y$) such that for all object Z, there is a natural bijection

$$\begin{array}{rcl} \mathrm{Hom}(Z, X \times_S Y) & \cong & \{(h,k) \in \mathrm{Hom}(Z,X) \times \mathrm{Hom}(Z,Y) | f \circ h = g \circ k\} \\ \text{(resp. } \mathrm{Hom}(X \coprod_S Y, Z) & \cong & \{(h,k) \in \mathrm{Hom}(X,Z) \times \mathrm{Hom}(Y,Z) | h \circ f = k \circ g\}). \end{array}$$

1. Answer shortly the questions of the previous exercice with fibered products and amalgamated sums.

2. Let $a < b < c < d$ be three real numbers. Describe explicitely the sets

$$]a,c[\times_{]a,d[}]b,d[\qquad \text{and} \qquad]a,c[\coprod_{]b,c[}]b,d[.$$

3. Describe explicitely the abelian group

$$\mathbb{Z} \times_{\mathbb{Z}} \mathbb{Z}$$

   where $f, g : \mathbb{Z} \to \mathbb{Z}$ are given by $f : n \mapsto 2n$ and $g : n \mapsto 3n$.

4. Describe explicitely the fiber product of smooth spaces $L_1 \cap L_2 = L_1 \times_{\mathbb{R}^2} L_2$ for $L_1 = \{y = 0\}$ and $L_2 = \{x = 0\}$ the two axis in $\mathbb{R}^2$ (transversal case), and also for $L_1 = L_2 = \{y = 0\}$ (non transversal case).

Exercice 7. (Projective limits) Let $(I, \leq)$ be a partially ordered set. A projective system indexed by I is a family

$$A_\bullet = ((A_i)_{i \in I}, (f_{i,j})_{i \leq j})$$

of objects and for each $i \leq j$, morphims $f_{i,j} : A_j \to A_i$ such that $f_{i,i} = \mathrm{id}_{A_i}$ and $f_{i,k} = f_{i,j} \circ f_{j,k}$ (such a data is a functor $A_\bullet : I \to C$ to the given category C). A projective limit for $A_\bullet$ is an object $\varprojlim_I A_\bullet$ such that for all object Z, one has a natural bijection

$$\mathrm{Hom}(Z, \varprojlim_I A_\bullet) \cong \varprojlim_I \mathrm{Hom}(A_i, Z)$$

where $\varprojlim_I \mathrm{Hom}(A_i, Z) \subset \prod_i \mathrm{Hom}(A_i, Z)$ denotes the families of morphisms h_i such that $f_{i,j} \circ h_j = h_i$. One defines inductive limits $\varinjlim A_\bullet$ in a similar way by reversing source and target of the morphims.

1. Show that product and fibered products are particular cases of this construction.

2. Describe the ring $\varprojlim_n \mathbb{C}[X]/(X^n)$.

3. Describe the ring $\mathbb{Z}_p := \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$.

Exercice 8. (**Localization**) Let A be a unitary commutative ring, $S \subset A$ a multiplicative subset (stable by multiplication and containing 1_A). The localization $A[S^{-1}]$ of A with respect to S is defined by the universal property

$$\mathrm{Hom}_{\mathrm{RINGS}}(A[S^{-1}], B) = \{f \in \mathrm{Hom}_{\mathrm{RINGS}}(A, B) | \forall s \in S,\ f(s) \in B^\times\},$$

where $B^\times$ is the set of invertible elements in the ring B.

1. Describe $\mathbb{Z}[1/2] := \mathbb{Z}[\{2^{\mathbb{Z}}\}^{-1}]$.

2. Is the morphism $\mathbb{Z} \to \mathbb{Z}[1/2]$ finite (i.e. is $\mathbb{Z}[1/2]$ a finitely generated $\mathbb{Z}$-module)? Of finite type (i.e. can $\mathbb{Z}[1/2]$ be described as a quotient of a polynomial ring over $\mathbb{Z}$ with a finite number of variables)?

3. Construct a morphism $\mathbb{Z}[1/2] \to \mathbb{Z}_3$ where $\mathbb{Z}_3$ are the 3-adic integers defined in the previous exercice.

4. Does there exist a morphism $\mathbb{Z}[1/3] \to \mathbb{Z}_3$?

Appendix B

Differential calculus

Differential calculus

Exercice 9. (Smooth affine spaces) Let $\textsc{Open}_{\mathcal{C}^\infty}$ be the category of open subsets $U \subset \mathbb{R}^n$ for varying n and let $\textsc{Aff}_{\mathcal{C}^\infty}$ be the full subcategory given by the affine spaces $\mathbb{R}^n$. Let $\textsc{Alg}^{fin}_{\mathcal{C}^\infty}$ be the category of finitary smooth algebras (see [MR91] for a complete study of these objects), given by product preserving (we consider that the empty product is the final object) functors

$$A : (\textsc{Aff}_{\mathcal{C}^\infty}, \times) \to (\textsc{Sets}, \times).$$

Let $\textsc{Alg}_{\mathcal{C}^\infty}$ be the category of smooth algebras, given by functors

$$A : (\textsc{Open}_{\mathcal{C}^\infty}, - \overset{t}{\underset{-}{\times}} -) \to (\textsc{Sets}, - \underset{-}{\times} -)$$

that preserve transversal fiber products of open subsets. The underlying set $A(\mathbb{R})$ of a smooth algebra is also denoted A.

1. Show that the forgetful functor

$$\begin{array}{ccc} \textsc{Alg}^{fin}_{\mathcal{C}^\infty} & \to & \textsc{Sets} \\ A & \mapsto & A \end{array}$$

has an adjoint, called the free smooth algebra. Describe the free smooth algebra on the set $\{x_1, \dots, x_n\}$ of n variables. Same question for the set $\mathbb{N}$ of integers.

2. Show that the affine space

$$\mathbb{A}^n : \begin{array}{ccc} \textsc{Alg} & \to & \textsc{Sets} \\ A & \mapsto & A^n \end{array}$$

is representable for $\textsc{Alg}$ one of the two categories of smooth algebras.

3. Show that, for $f \in \mathcal{C}^\infty(\mathbb{R})$, the solution space

$$\underline{\mathrm{Sol}}(A) := \{x \in A,\ f(x) = 0\}$$

is representable by a finitary smooth algebra A.

4. Show that, for $U \subset \mathbb{R}^n$ open and $f \in \mathcal{C}^\infty(U)$, the solution space

$$\underline{\mathrm{Sol}}(A) := \{x \in A(U),\ f(x) = 0\}$$

is representable by a smooth algebra A.

5. Show that the forgetful functor $\textsc{Alg}_{\mathcal{C}^\infty} \to \textsc{Alg}_{\mathbb{R}}$ has an adjoint, called geometrization.

6. Let $V \to M$ be a smooth vector bundle and

$$\mathcal{O}^{alg}(V) := \Gamma_{\mathcal{C}^\infty}(M, \mathrm{Sym}^*(V^*)).$$

Compute its geometrization.

Exercice 10. (Structure of smooth algebras) We use the notations of exercice 9.

1. Show that if $U \subset \mathbb{R}^n$ is open, its finitary smooth algebra may be written

$$\mathcal{C}^\infty(U) \cong \mathcal{C}^\infty(\mathbb{R}^{n+1})/(x_{n+1}f - 1),$$

with $f \in \mathcal{C}^\infty(\mathbb{R}^{n+1})$ a given smooth function.

2. Show that for $I \subset A$ and $J \subset A$ two finitely generated ideals in a finitary smooth algebra A, one has

$$A/I \otimes_A A/J \cong A/(I, J).$$

3. Show that the functor $\textsc{Open}_{\mathcal{C}^\infty} \to \textsc{Alg}^{fin}_{\mathcal{C}^\infty}$ that sends U to $\mathcal{C}^\infty(U)/(f_1, \dots, f_p)$ for given functions f_i is local, i.e., sends unions along coverings to limits (hint: use partitions of unity; see [MR91], Lemma 2.7).

4. Show that the functor $\textsc{Open}_{\mathcal{C}^\infty} \to \textsc{Alg}^{fin}_{\mathcal{C}^\infty}$ sends transversal pullbacks to pushouts (hint: use the implicit function theorem to show that transversal pullbacks are locally described as zeroes of functions, and apply locality).

5. Let $A \in \textsc{Alg}^{fin}_{\mathcal{C}^\infty}$ and extend A to a functor $\tilde{A} : \textsc{Open}_{\mathcal{C}^\infty} \to \textsc{Sets}$ by

$$\tilde{A}(U) := \mathrm{Hom}_{\textsc{Alg}^{fin}_{\mathcal{C}^\infty}}(\mathcal{C}^\infty(U), A).$$

Show that $\tilde{A}$ commutes with transversal fiber products.

6. Show that the inclusion $\textsc{Aff}_{\mathcal{C}^\infty} \subset \textsc{Open}_{\mathcal{C}^\infty}$ induces an equivalence

$$\textsc{Alg}_{\mathcal{C}^\infty} \xrightarrow{\sim} \textsc{Alg}^{fin}_{\mathcal{C}^\infty}.$$

Exercice 11. (Tangent and jets) The tangent category $T\mathcal{C}$ of a category $\mathcal{C}$ with fiber products and final object is the category of abelian group objects in the arrow category $[I, \mathcal{C}]$. The first thickened category $\mathrm{Th}^1\mathcal{C}$ is the category of torsors under these, called first order thickenings. The k-thickened category $\mathrm{Th}^k\mathcal{C}$ is the category of morphism $D \to A$ that admit a decomposition

$$D = C_1 \to C_2 \to \dots C_n = A$$

in a sequence of first order thickenings. The cotangent functor Ω^1 is an adjoint to the source functor $T\mathcal{C} \to \mathcal{C}$. The k-th thickening functor Th^k is an adjoint to the source functor $\mathrm{Th}^k\mathcal{C} \to \mathcal{C}$, and the jet functor is $A \mapsto \mathrm{Jet}^k(A) := \mathrm{Th}^k(A \otimes A \to A)$.

1. Show that the tangent category to $\textsc{Alg}_{\mathbb{R}}$ is equivalent to the category of pairs (A, M) composed of an $\mathbb{R}$-algebra and a module over it.

2. Show that the cotangent functor (adjoint to the source functor on $T_A\textsc{Alg}_{\mathbb{R}}$)

$$\begin{array}{cccc} \Omega^1 : & \textsc{Alg}_{\mathbb{R}} & \to & T_A\textsc{Alg}_{\mathbb{R}} \\ & A & \mapsto & [A \oplus \Omega^1_A \to A] \end{array}$$

on the category $\textsc{Alg}_{\mathbb{R}}$ identifies to the functor of Kähler differentials

$$A \mapsto \Omega^{1,alg}_A := (A \otimes A)/I^2 \cong A \oplus I/I^2,$$

where $I = \mathrm{Ker}(A \otimes A \to A)$.

3. Compute the universal thickening $\mathrm{Th}^k(B \to A)$ of a given morphism of algebras (hint: use the graph of the corresponding morphism on spectra).

4. Compute the k-jet algebra of $A \in \textsc{Alg}_{\mathbb{R}}$, and describe it explicitely for $A = \mathbb{R}[x_1, \ldots, x_n]$.

5. Describe the cotangent functor on the category of smooth algebras.

6. Compute the smooth k-thickening of a morphism $A \to B$ of smooth algebras and the k-jet algebra of $A \in \textsc{Alg}_{\mathcal{C}^\infty}$.

7. If $C \to M$ is a bundle, let $\mathrm{Jet}^k(C/M) \to M$ be the bundle $C \times_M \mathrm{Jet}^k M$. Show that the bundle $\mathrm{Jet}(C/M) := \lim \mathrm{Jet}^k(C/M) \to M$ is equipped with an integrable connection, i.e., that it is defined on M_{DR}.

8. Show that $[C/M] \mapsto [\mathrm{Jet}(C)/M_{DR}]$ is a Jet functor, meaning that it is adjoint to the pullback functor $[X/M_{DR}] \mapsto [X \times_{M_{DR}} M/M]$.

Exercice 12. (Symmetric monoidal categories: examples)

1. Let $(\textsc{Vect}^s_{\mathbb{R}}, \otimes)$ be the monoidal category of $\mathbb{Z}/2$-graded vector spaces $V = V^0 \oplus V^1$ equipped with the commutativity isomorphism

$$\begin{array}{cccc} \mathrm{com}_{M,N} : & M \otimes N & \to & N \otimes M \\ & m \otimes n & \mapsto & (-1)^{\deg(m)\deg(n)} m \otimes n. \end{array}$$

Show that $(\textsc{Vect}^s_{\mathbb{R}}, \otimes)$ is closed, i.e., that it has internal homomorphism objects $\underline{\mathrm{Hom}}(M, N)$ (hint: describe the evaluation morphism $\underline{\mathrm{Hom}}(M, N) \otimes M \to N$).

2. Let $U \subset \mathbb{R}^n$ be an open subset and

$$\mathcal{D} = \mathcal{D}_U = \left\{ \sum_{|\alpha| \leq k} a_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha},\ a_\alpha \in \mathcal{C}^\infty(U) \right\} \subset \mathrm{End}_{\textsc{Vect}_{\mathbb{R}}}(\mathcal{C}^\infty(U))$$

be the algebra of differential operators on $\mathcal{O} = \mathcal{C}^\infty(U)$. If $\mathcal{M}$ and $\mathcal{N}$ are two $\mathcal{D}$-modules, we denote $\mathcal{M} \otimes \mathcal{N}$ the $\mathcal{O}$-module $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}$, equipped with the action of $\mathcal{D}$ induced by the action of derivations $\partial \in \Theta_U$ given by Leibniz's rule

$$\partial(m \otimes n) = \partial(m) \otimes n + m\partial(n).$$

Show that

$$(\textsc{Mod}(\mathcal{D}), \otimes)$$

is a symmetric monoidal category with internal homomorphisms.

Exercice 13. (Universal property of the de Rham algebra)

1. Let $U \subset \mathbb{R}^n$ be an open subset, $\mathcal{O} = \mathcal{C}^\infty(U)$,

$$\Omega^1 = \Omega^1(U) = \mathcal{C}^\infty(U)^{(dx^1,\dots,dx^n)}$$

   and $d : \mathcal{O} \to \Omega^1$ be the de Rham differential. Show that

$$\mathcal{D} \otimes_{\mathcal{O}} d : \mathcal{D} \to \mathcal{D} \otimes_{\mathcal{O}} \Omega^1$$

   is $\mathcal{D}^{op}$-linear (i.e., a linear differential operator).

2. Let $(\mathrm{Diff}_{\mathbb{R}}(\mathcal{O}), \otimes_{\mathcal{O}})$ be the category of $\mathcal{O}$-modules equipped with linear $\mathbb{R}$-differential operators, i.e.,

$$\mathrm{Hom}_{\mathrm{Diff}_{\mathbb{R}}(\mathcal{O})}(\mathcal{M}, \mathcal{N}) := \{D : \mathcal{M} \to \mathcal{N},\ \mathbb{R}\text{-linear, such that } \mathcal{D} \otimes_{\mathcal{D}} D \text{ is } \mathcal{D}^{op}\text{-linear}\}.$$

   Show that the functors $\mathcal{M} \mapsto \mathcal{M} \otimes_{\mathcal{O}} \mathcal{D}$ and $\mathcal{N} \mapsto h(\mathcal{N}) := \mathcal{N} \otimes_{\mathcal{D}} \mathcal{O}$ induce an equivalence between the category $\mathrm{Diff}_{\mathbb{R}}(\mathcal{O})$ and a full sub-category of the category $\textsc{Mod}(\mathcal{D}^{op})$ of $\mathcal{D}^{op}$-module.

3. Let $\Omega^* := \wedge^*_{\mathcal{O}} \Omega^1$ be the algebra of differential forms on U and $d : \Omega^* \to \Omega^*$ be the de Rham differential, given by extending $d : \mathcal{O} \to \Omega^1$ by the graded Leibniz rule

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg(\omega)} \omega \wedge d\eta.$$

   Show that there is a natural isomorphism of differential graded $(\mathrm{Diff}(\mathcal{O}), \otimes_{\mathcal{O}})$-algebras

$$\mathrm{Sym}_{dg-\mathrm{Diff}_{\mathbb{R}}(\mathcal{O})}([\underset{0}{\mathcal{O}} \xrightarrow{d} \underset{1}{\Omega^1(U)}]) \xrightarrow{\sim} (\Omega^*(U), d).$$

Exercice 14. (Affine super-spaces) Let $\textsc{Alg}^s_{\mathbb{R}} := \textsc{Alg}(\textsc{Vect}^s_{\mathbb{R}}, \otimes)$ be the category of algebras in the monoidal category of super-vector spaces (i.e., $\mathbb{Z}/2$-graded vector spaces). Define the affine super-space as the functor

$$\mathbb{A}^{n|m} : \textsc{Alg}_{s,\mathbb{R}} \to \textsc{Sets}$$

from super-algebras to sets sending $A = A^0 \oplus A^1$ to $(A^0)^n \times (A^1)^m$.

1. Show that $\mathbb{A}^{n|m}$ is representable in $\text{ALG}^s_{\mathbb{R}}$.

2. For $S = S^0 \oplus S^1 \to M$ a finite dimensional $\mathbb{Z}/2$-graded bundle on a smooth manifold M, we denote
$$\mathcal{O}^{alg}(S) := \Gamma_{\mathcal{C}^\infty}(M, \text{Sym}^*(S^0) \otimes \wedge^*(S^1)).$$
Compute $\mathbb{A}^{n|m}(\mathcal{O}^{alg}(S))$ and give the universal property of $\mathcal{O}^{alg}(S)$ in $\text{ALG}^s_{\mathbb{R}}$.

3. For A a super-algebra, its soul is the $\mathbb{R}$-algebra $|A| = A/(A^1)$ of A generated by the ideal generated by even elements.

   - Let $\text{ALG}^{sh}_{\mathbb{R}}$ be the category of super-algebras such that the morphism $p : A^0 \to |A|$ of A^0-algebras admits a section. These are called super-algebras with kind soul.
   - Let $\text{ALG}^{s,scg}_{\mathbb{R}}$ be the category of super-algebras A that are smoothly closed geometric, meaning that its soul $|A|$ is kind and smoothly closed geometric.

   Show that if A has kind soul, the section $s : |A| \to A^0$ is unique, and deduce that the forgetful functor Forget : $\text{ALG}^{s,scg}_{\mathbb{R}} \to \text{ALG}^{sh}_{\mathbb{R}}$ from smoothly closed geometric super-algebras to super-algebras with kind soul has an adjoint, called the geometric closure.

4. Show that if $S \to M$ is a $\mathbb{Z}/2$-graded vector bundle, the super-algebra $\mathcal{O}^{alg}(S)$ has kind soul but is not smoothly closed geometric in general. Compute its geometric closure.

5. Describe morphisms $f : \mathbb{A}^{n|0} \to \mathbb{A}^{p,q}$.

6. Describe morphisms $f : \mathbb{A}^{0|1} \to \mathbb{A}^{n|0}$.

7. Describe the super-space of (internal) homomorphisms $\underline{\text{Hom}}(\mathbb{A}^{0|1}, \mathbb{A}^{n|0})$ defined by
$$\underline{\text{Hom}}(\mathbb{A}^{0|1}, \mathbb{A}^{n|0})(A) := \text{Hom}(\mathbb{A}^{0|1} \times \text{Spec}(A), \mathbb{A}^{n|0}).$$

8. Compute all morphisms of algebraic super-spaces
$$f : \mathbb{A}^{n|m} \to \mathbb{A}^{r|s}.$$

Exercice 15. (Tangent bundle) Let ALG_{scg} be the category of geometric smoothly closed algebras, equipped with its Zariski topology. If $X : \text{ALG} \to \text{SETS}$ is a functor, its smooth tangent bundle is defined by
$$\begin{array}{cccc} TX : & \text{ALG}_{scg} & \to & \text{SETS} \\ & A & \mapsto & X(A[\epsilon]/(\epsilon^2)). \end{array}$$

1. Let M be a smooth manifold, seen as a functor $\text{Hom}(\mathcal{C}^\infty(M), -) : \text{ALG} \to \text{SETS}$. Show that the functor TM is representable.

2. Let M and N be two smooth manifolds. Describe the functor $T\underline{\mathrm{Hom}}(M, N) : \mathrm{ALG}_{scg} \to \mathrm{SETS}$.

3. If $x : M \to N$ is a smooth map, show that the fiber at x of $T\underline{\mathrm{Hom}}(M, N)$ is given by

$$T_x\underline{\mathrm{Hom}}(M, N) = \Gamma(M, x^*TN)$$

where $x^*TN := M \times_{x,N,p} TN$ is the pull-back of TN along x (hint: express this space of sections in terms of derivations $D : \mathcal{C}^\infty(N) \to \mathcal{C}^\infty(M)$).

4. Let $M = \underline{\mathrm{Spec}}(A)$ be a smooth affine algebraic variety over $\mathbb{R}$, seen as an algebraic super-variety. Show that $\underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, M)$ is representable.

5. The even (resp. odd) tangent bundle T^0X (resp. T^1X) of a super-space $X : \mathrm{ALG}^s \to \mathrm{SETS}$ is given by $X(A[\epsilon]/(\epsilon^2))$ where ϵ is an even (resp. odd) variable. Suppose given $x \in \mathrm{Hom}(\mathbb{R}^{0|1}, M)$. Describe the fiber at x of the even and odd tangent bundles to $\underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, M)$.

Appendix C

Groups and representations

Groups and representations

Exercice 16. (Algebraic and Lie groups) We work with spaces defined on the category $\mathrm{Alg}_{\mathbb{R}}$ or $\mathrm{Alg}_{scg,\mathbb{R}}$ of real algebras or real smoothly closed geometric algebras. An algebraic (resp Lie) group space is a functor $G : \mathrm{Alg}_{\mathbb{R}} \to \mathrm{Grp}$ (resp. $G : \mathrm{Alg}_{scg,\mathbb{R}} \to \mathrm{Grp}$).

1. Show that the group space $\mathrm{GL}_n = \{(M,N) \in M_n^2,\ MN = NM = \mathrm{id}\}$ is a representable algebraic and a representable Lie group space.

2. Show that the map $(M,N) \mapsto M$ induces an isomorphism

$$\mathrm{GL}_n \cong \{M \in M_n,\ \det(M) \text{ invertible}\}.$$

3. Use the above result to show that GL_n is a smooth manifold.

4. Applying the local inversion theorem to the exponential map $\exp : M_n \to \mathrm{GL}_n$, show that GL_n is a smooth manifold, i.e., a Lie group space covered by open subsets of $\mathbb{R}^{n^2}$.

Exercice 17. (Lie algebras) Recall that if $G : \mathrm{Alg}_{\mathbb{R}} \to \mathrm{Sets}$ is a group valued functor, one defines the space $\mathrm{Lie}(G)$ by

$$\mathrm{Lie}(G)(A) := \{g \in G(A[\epsilon]/(\epsilon^2)) \mid g = \mathrm{id} \mod \epsilon\}.$$

1. Show that the Lie algebra of GL_n is M_n.

2. Show that the zero section $0 : G \to TG$ induces a natural action of G on $\mathrm{Lie}(G)$, by conjugation.

3. Compute the derivative $\mathrm{ad} : \mathrm{Lie}(\mathrm{GL}_n) \to \mathrm{End}(\mathrm{Lie}(\mathrm{GL}_n))$ of the adjoint action $g \mapsto gmg^{-1}$ of GL_n on its Lie algebra.

4. Show that the operation $[.,.] : \mathrm{Lie}(\mathrm{GL}_n) \times \mathrm{Lie}(\mathrm{GL}_n) \to \mathrm{Lie}(\mathrm{GL}_n)$ given by

$$[h,k] = \mathrm{ad}(h)(k)$$

identifies with the commutator of matrices.

5. Let $g \in \mathrm{GL}_n$ be fixed. Show that the operation $h \mapsto ghg^{-1}$ of g on M_n is the derivative of the operation $h \mapsto ghg^{-1}$ on GL_n.

6. Compute the Lie algebra of SL_2.

7. Compute the Lie algebra of SL_n.

8. Let (V, b) be a bilinear space and

$$\mathrm{Sim}(V, b) := \{g \in \mathrm{End}(V) | b(gv, gw) = b(v, w), \forall v, w \in V\}.$$

Show that $\mathrm{Sim}(V, b)$ can be extended to a group space.

9. Compute the Lie algebra of $\mathrm{Sim}(V, b)$.

10. Deduce from the above computation the description of the Lie algebra of the standard orthogonal and symplectic groups, that correspond to the bilinear forms ${}^t vAw$ for $A = I$ on $\mathbb{R}^n$ and $A = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ on $\mathbb{R}^{2n}$ respectively.

Exercice 18. (Morphisms of algebraic groups)

1. Using the commutative diagram

$$\begin{array}{ccc} \mathrm{GL}_1(\mathbb{R}[x, x^{-1}]) & \xrightarrow{m^*} & \mathrm{GL}_1(\mathbb{R}[x_1, x_2, x_1^{-1}, x_2^{-1}]) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{GL}_1(\mathbb{R}[x, x^{-1}]) & \xrightarrow{m^*} & \mathrm{GL}_1(\mathbb{R}[x_1, x_2, x_1^{-1}, x_2^{-1}]) \end{array}$$

induced by the multiplication morphism $m : \mathrm{GL}_1 \times \mathrm{GL}_1 \to \mathrm{GL}_1$, show that the image of $x \in \mathbb{R}[x, x^{-1}]^\times$ by f is a sum $\sum_{n \in \mathbb{Z}} a_n x^n$ with $a_n^2 = a_n$ for all n and $a_n a_m = 0$ if $n \neq m$.

2. Deduce from the above all the morphisms of group spaces $\mathrm{GL}_1 \to \mathrm{GL}_1$.

3. Let $\mathbb{G}_a(A) = A$ equipped with addition. Show that if $f : \mathbb{G}_a \to \mathbb{G}_a$ is a morphism, the image of the universal point $x \in \mathbb{R}[x]$ by the multiplication map fulfills

$$\sum_n a_n (x_1 + x_2)^n = \sum_n a_n x_1^n + \sum_m a_m x_2^m.$$

4. Deduce from the above that $\mathrm{End}(\mathbb{G}_a) = \mathbb{R}$.

Exercice 19. (Structure and representations of SL_2) Let $T = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ be the maximal torus of SL_2.

1. Compute the roots of SL_2, i.e., the non-trivial characters of T (morphisms $T \to \mathrm{GL}_1$) that occur in the adjoint representation of SL_2 on $\mathrm{Lie}(\mathrm{SL}_2)$.

2. For each root $r : T \to \mathrm{GL}_1$, construct a morphism

$$r : \mathbb{G}_a \to \mathrm{SL}_2.$$

3. Show that SL_2 is generated by the images of r and T. Describe their relations.

4. Let $\rho : \mathrm{SL}_2 \to \mathrm{GL}(V)$ be an irreducible representation of SL_2. By using the commutation relation between the roots and the generator a of T, compute the possible weights of T on V.

5. Show that every irreducible representation of SL_2 is isomorphic to $\mathrm{Sym}^n(V)$ with V the standard representation.

Exercice 20. (Super-algebraic groups) We work with superspaces defined on the category $\mathrm{ALG}_{s,\mathbb{R}}$ of real super-algebras.

1. Show that one can define a super-group space structure on $\mathbb{R}^{1|1}$ by

$$(t_1, \theta_1).(t_2, \theta_2) = (t_1 + t_2 + \theta_1\theta_2, \theta_1 + \theta_2).$$

2. Recall that the free module of dimension $p|q$ on a super-algebra A is defined by

$$A^{p|q} := \mathbb{R}^{p|q} \otimes A \underset{\mathrm{VECT}_{\mathbb{R}}}{=} (\mathbb{R}^p \otimes A^0 \oplus \mathbb{R}^q \otimes A^1)^0 \oplus (\mathbb{R}^q \otimes A^0 \oplus \mathbb{R}^p \otimes A^1)^1.$$

Show that the super space $\mathrm{GL}_{p|q}(A) = \mathrm{Aut}_{\mathrm{MOD}(A)}(A^{p|q})$ is representable.

3. Show that the Berezinian gives a super-group morphism

$$\mathrm{GL}_{p|q} \to \mathrm{GL}_{1|0}.$$

Exercice 21. (Clifford algebras)

1. Describe the super-Clifford algebra of a supersymmetric bilinear space (V, g) where $V = V^0 \oplus V^1$ is a super vector space and $g : V \otimes V \to V$ is a non-degenerate supersymmetric bilinear form.

2. Show that one can define a bundle $\mathrm{Cliff}(V, g)$ over $\mathbb{R}$ indexed by $\hbar$, equipped with an action of $\mathbb{R}^*$, such that the fiber at $\hbar = 0$ is $\mathrm{Sym}_s(V[1])$ and the fiber at $\hbar = 1$ is $\mathrm{Cliff}(V, g)$.

3. Describe a filtration on $\mathrm{Cliff}(V, g)$ whose graded algebra is $\mathrm{Sym}_s(V[1])$.

4. Show that if $V = \mathbb{H}(W) = W \oplus W^*$ is the hyperbolic quadratic space, with the symmetric bilinear form given by ordinary duality on W, one has

$$\mathrm{Cliff}(V, g) \cong \mathrm{End}(W).$$

5. A spinor representation of the Clifford algebra is an irreducible representation of $\mathrm{Cliff}(V, g)$ acting on itself. Show that if $V = \mathbb{H}(W)$, the space W gives a spinor representation of $\mathrm{Cliff}(V)$.

6. Describe the Clifford algebra of $\mathbb{R}^{1,3}$, the corresponding spinor representation.

7. Describe the spinor group and its real spinor representation (Weyl spinors). Describe the corresponding complex representation (Dirac spinors).

Appendix D

Homotopical algebra

UPMC - Paris — Master Course

Year 2010-2011

Homotopical algebra

Recall that the simplicial category Δ is the category whose objects are the finite ordered sets $[n] = [0, \ldots, n-1]$ and whose morphisms are increasing maps, and the category of simplicial sets SSets is the category of contravariant functors $\Delta^{op} \to$ Sets.

Exercice 22. (Cylinders and homotopies) The category Top of Hausdorff compactly generated topological spaces has an internal homomorphism object defined by $\underline{\mathrm{Hom}}(Y, X)$ to be the set of continuous maps between Y and X equipped with the topology generated by the subsets $V(U, K) = \{f : Y \to X,\ f(K) \subset U\}$ indexed by compacts $K \subset Y$ and opens $U \subset X$. If X is in Top, denote $\mathrm{Cyl}(X) = X \times [0, 1]$ and $\mathrm{Cocyl}(X) := \underline{\mathrm{Hom}}([0, 1], X)$

1. Show that for X and Y locally compact, the internal $\underline{\mathrm{Hom}}$ of Top fulfills its ordinary adjunction property with products, given by a functorial isomorphism
$$\mathrm{Hom}(Z \times X, Y) \cong \mathrm{Hom}(Z, \underline{\mathrm{Hom}}(X, Y)).$$

2. Deduce that Cyl and Cocyl are adjoint, meaning that
$$\mathrm{Hom}(\mathrm{Cyl}(X), Y) \cong \mathrm{Hom}(X, \mathrm{Cocyl}(Y)).$$

3. Show that Cyl is indeed a cylinder in Top with its standard model structure, meaning that $X \coprod X \to \mathrm{Cyl}(X)$ is a cofibration (closed embedding) and $\mathrm{Cyl}(X) \to X$ is a weak equivalence.

4. Show that $\mathrm{Cocyl}(X)$ indeed is a cocylinder in Top, meaning that $\mathrm{Cocyl}(X) \to X \times X$ is a fibration (lifting property with respect to $Y \times \{0\} \to Y \times [0, 1]$) and $X \to \mathrm{Cocyl}(X)$ is a weak equivalence (see [DK01], theorem 6.15).

5. Deduce that right and left homotopies coincide in Top.

Exercice 23. (Mapping cylinder and mapping cone) We refer to [DK01], 6.6 for this exercice. Let $f : X \to Y$ be a morphism in Top. Define the mapping path space P_f of f as the fiber product

$$\begin{array}{ccc} P_f & \longrightarrow & \mathrm{Cocyl}(Y) \\ \downarrow & & \downarrow{\scriptstyle p_0} \\ X & \xrightarrow[f]{} & Y \end{array}$$

and the mapping path fibration $p : P_f \to Y$ by $p(x, \alpha) = \alpha(1)$.

1. Show that there exists a homotopy equivalence $h : X \to P_f$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & P_f \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle p} & \\ Y & & \end{array}$$

   commutes.

2. Show that the map $p : P_f \to Y$ is a fibration.

3. Show that if $f : X \to Y$ is a fibration, then h is a relative (or fiber) homotopy equivalence on Y.

4. We now work in TOP_* (see [DK01], 6.11 for this question). Let $E \to B$ be a fibration with fiber F. Let Z be the fiber of the mapping path fibration of $F \to E$. Show that Z is homotopy equivalent to the loop space $\Omega B := \underline{\text{Hom}}_{\text{TOP}_*}(S^1, B)$.

Exercice 24. (Simplicial sets)

1. Prove the same statements as those in exercice 22 for the cylinder $\text{Cyl}(X) := X \times \Delta^1$ and cocylinder $\text{Cocyl}(X) := \underline{\text{Hom}}(\Delta^1, X)$ of simplicial sets.

2. Show that the geometric realization and the singular simplex functors are adjoint.

3. Characterize fibrant simplicial spaces and show that if Y is a topological space, the singular simplex $S(Y)$ is fibrant.

Exercice 25. (Fibrations) Let X and Y be two topological spaces.

1. Show that a projection map $p : X \times Y \to X$ is a fibration.

2. Deduce that a fiber bundle $p : B \to X$ (locally isomorphic on X to a projection) is a fibration.

3. Show that if X is path connected, two fibers of a fibration $p : F \to X$ are homotopy equivalent.

Exercice 26. (Simplicial nerves of categories and groupoids) A groupoid is a category whose morphisms are all invertible. In this exercice, classes are called sets.

1. Show that a category C can be described by a tuple $(X_1, X_0, s, t, \epsilon, m)$ composed of two sets X_1 and X_0, equipped with source, target, unit, and composition maps:

$$X_1 \begin{array}{c} \xleftarrow{\epsilon} \\[-4pt] \xrightarrow{s} \\[-4pt] \xrightarrow[t]{} \end{array} X_0\,, \qquad\qquad X_1 \underset{s,X_0,t}{\times} X_1 \xrightarrow{m} X_1,$$

   and describe the conditions on these data to give a groupoid.

2. Show that the families of n composable arrows can be described by the set

$$X_n := \underbrace{X_1 \underset{s,X_0,t}{\times} \dots \underset{s,X_0,t}{\times} X_1}_{n \text{ times}}.$$

3. Let Δ be the simplicial category of finite totally ordered sets with order preserving maps between them. The morphisms

$$\begin{array}{lll} d^i : [n-1] \to [n] & 0 \leq i \leq n & \text{(cofaces)} \\ s^j : [n+1] \to [n] & 0 \leq j \leq n & \text{(codegeneracies)} \end{array}$$

given by

$$d^i([0, \dots, n-1]) = [0, \dots, i-1, i+1, \dots, n]$$

and

$$s^j([0, \dots, n+1]) = [0, \dots, j, j, \dots, n]$$

fulfill the so-called cosimplicial identities

$$\begin{cases} d^j d^i & = & d^i d^{j-1} & \text{if } i < j \\ s^j d^i & = & d^i s^{j-1} & \text{if } i < j \\ s^j d^j & = & 1 = s^j d^{j+1} & \\ s^j d^i & = & d^{i-1} s^j & \text{if } i > j+1 \\ s^j s^i & = & s^i s^{j+1} & \text{if } i \leq j \end{cases}$$

We admit that Δ is generated by cofaces and codegeneracies with these relations. Show that the map $[n] \mapsto X_n$ can be extended to a simplicial set, i.e., to a functor

$$X : \Delta^{op} \to \textsc{Sets}.$$

4. For m and n two objects, describe $\mathrm{Hom}(m, n)$ in terms of the simplicial maps.

5. Using the abstract Segal maps

$$X_n \to X_1 \underset{X_0}{\times} \dots \underset{X_0}{\times} X_1,$$

associated to $a_i : [1] \to [n], [0 \to 1] \mapsto [i \to i+1] \subset [n]$, describe the conditions for a simplicial set $X : \Delta^{op} \to \textsc{Sets}$ to come from a category by the above construction.

6. Describe the simplicial space BG associated to a group G, viewed as a groupoid with one object

$$G \underset{t}{\overset{\overset{\epsilon}{\longleftarrow}}{\underset{s}{\rightrightarrows}}} \{.\}, \qquad\qquad G \times G \xrightarrow{m} G.$$

Exercice 27. (Homotopical spaces of leaves for foliations) Let M be a smooth manifold. Suppose given a foliation $\mathcal{F} \subset \Theta_M$ on M, i.e., a Lie sub-algebroid of the vector fields on M (i.e., a sub-$\mathcal{O}_M$-module stable by the Lie bracket).

1. Denote $\Omega(\mathcal{F}) := \wedge^* \mathcal{F}^*$. Show that $\Omega(\Theta_M) = \Omega^*_M$.

2. Let $d : \mathcal{F}^* \to \wedge^2 \mathcal{F}^*$ be the dual of the Lie bracket operation $[.,.] : \mathcal{F} \wedge \mathcal{F} \to \mathcal{F}$. Show that d is a linear differential operator (i.e., a morphism in the category $\mathrm{Diff}(\mathcal{O}_M)$, meaning that $d \otimes_{\mathcal{O}_M} \mathcal{D}_M$ is $\mathcal{D}_M^{op}$-linear).

3. Identify d when $\mathcal{F} = \Theta_M$.

4. Show that $\Omega(\mathcal{F})$ is equipped with a differential induced by d (Hint: try this with $\Omega(\Theta_M)$ and use the projection $\Omega(\Theta_M) \to \Omega(\mathcal{F})$). Give another description of this differential using the functor $\mathrm{Sym}_{dg-\mathrm{Diff}(\mathcal{O}_M)}$.

5. Show that $(\mathcal{F}, [.,.]) \mapsto (\Omega(\mathcal{F}), d)$ is a fully faithful embedding of the category of foliations on M into the category of differential graded $\mathrm{Diff}(\mathcal{O}_M)$-algebras.

6. Denote by $|\Delta_\bullet| : \Delta \to \textsc{Sets}$ the functor defined by
$$|\Delta_n| := \{t_i \in [0,1]^{n+1},\ \sum_i t_i \leq 1\}$$
with maps $\theta_* : |\Delta^n| \to |\Delta^m|$ for $\theta : [n] \to [m]$ given by $\theta_*(t_0, \ldots, t_m) = (s_0, \ldots, s_n)$ where
$$s_i = \begin{cases} 0 & \text{if} \quad \theta^{-1}(i) = \emptyset \\ \sum_{j \in \theta^{-1}(i)} t_j & \text{if} \quad \theta^{-1}(i) \neq \emptyset. \end{cases}$$
Show that the so-called ∞-groupoid of $\mathcal{F}$, defined by
$$\Pi_\infty(\mathcal{F}) := \mathrm{Hom}_{dg-\textsc{Alg}}(\Omega(\mathcal{F}), \Omega(\Theta_{\Delta_\bullet})) \cong \mathrm{Hom}_{dg-\textsc{Alg}}(\Omega(\mathcal{F}), \Omega^*_{\Delta_\bullet})$$
defines a simplicial set.

7. Show that $\Pi_\infty(\mathcal{F})_n$ identities with the space of Lie algebroid morphisms $\Theta_{\Delta_n} \to \mathcal{F}$.

8. Deduce that $\Pi_\infty(\Theta_M) = \mathrm{Hom}(\Delta_\bullet, M)$.

9. Show that setting
$$\Pi_\infty(\mathcal{F})(U) := \mathrm{Hom}_{dg-\textsc{Alg}}\left(\Omega(\mathcal{F}), \Omega(\Theta_{\Delta_\bullet}) \otimes_{\mathcal{C}^\infty(\Delta_\bullet)} \mathcal{C}^\infty(\Delta_\bullet \times U)\right)$$
gives a natural extension of $\Pi_\infty(\mathcal{F})$ to a simplicial smooth space
$$\Pi_\infty(\mathcal{F}) : \Delta^{op} \to \textsc{Sh}(\textsc{Open}_{\mathcal{C}^\infty}, \tau).$$

10. Show that $\Pi_\infty(\mathcal{F})_1$ projects onto the space $\Pi_1(\mathcal{F})$ of homotopy classes of paths $x : [0,1] \to M$ tangent to $\mathcal{F}$.

11. Suppose that $\Pi_1(\mathcal{F})$ is equipped with a groupoid structure
$$\Pi_1(\mathcal{F}) \underset{t}{\overset{\overset{\epsilon}{\longleftarrow}}{\underset{s}{\rightrightarrows}}} M\ , \qquad\qquad \Pi_1(\mathcal{F}) \underset{s,M,t}{\times} \Pi_1(\mathcal{F}) \xrightarrow{m} \Pi_1(\mathcal{F}),$$
given by composition of homotopy classes of paths. Show that this structure is induced by the simplicial structure on $\Pi_\infty(\mathcal{F})$.

12. Let $|\Pi_1(\mathcal{F})|$ be the smooth variety given by the coarse quotient of M by the leaf-wise equivalence relation (given by $x \sim y$ if there is a path $\gamma \in \Pi_1(\mathcal{F})$ such that $\gamma(0) = x$ and $\gamma(1) = x$). Recall that its functions are defined as functions on M invariant by the action of $\Pi_1(\mathcal{F})$. Compute $|\Pi_1(\mathcal{F})|$ for $M = \mathbb{R}^2/\mathbb{Z}^2$, $\mathcal{F}_1 = \langle \frac{\partial}{\partial x} \rangle$ and $\mathcal{F}_2 = \langle \frac{\partial}{\partial x} + \sqrt{2}\frac{\partial}{\partial y} \rangle$.

13. Let $\pi_0(\Pi_1(\mathcal{F})) : \textsc{Open}_{\mathcal{C}^\infty} \to \textsc{Sets}$ be the sheaf associated to the presheaf

$$U \mapsto \mathrm{Hom}_{\mathcal{C}^\infty}(U, M)/\sim$$

where $\sim$ is the leaf-wise equivalence relation on U-valued points. Show that the natural map

$$|\Pi_1(\mathcal{F})| \to \pi_0(\Pi_1(\mathcal{F}))$$

is an isomorphism for $\mathcal{F} = \mathcal{F}_1$ but not for $\mathcal{F} = \mathcal{F}_2$ (hint: in the case of $\mathcal{F}_2$, define non-trivial automorphisms of the given sheaf).

Appendix E

Functional calculus

Functional calculus

Exercice 28. (Calculus of variation in mechanics) Let $\pi : C = \mathbb{R} \times [0,1] \to [0,1] = M$ be the trivial bundle with base coordinate t and fiber coordinate x, whose sections are trajectories of a particle in a line (i.e., functions $\mathbb{R} \to \mathbb{R}$). We use polynomial functions $\mathcal{O}_M = \mathbb{R}[t]$ and $\mathcal{O}_C = \mathbb{R}[t,x]$. Let $L(t,x_i) \in \mathrm{Jet}(\mathcal{O}_C) = \mathbb{R}[t,x_i]$ be a Lagrangian density and $H \subset \Gamma(M,C)$ be the subspace of trajectories that start and end at two fixed points x_0 and x_1 with fixed derivates up to the order of derivative that appear in L. We denote

$$S(x) = \int_0^1 L(t, x(t), \partial_t x(t), \dots)dt$$

the corresponding action functional.

1. Compute the tangent space to $\Gamma(M,C)$ and show that the tangent space to H has fiber at $x \in H$ the space
$$T_x H = \{h \in \Gamma(M,C),\ h^{(k)}(0) = h^{(k)}(1) = 0 \text{ for all } k \leq \mathrm{order}(L)\}.$$

2. Let
$$D_1 := \frac{\partial}{\partial t} + \sum_{n=0}^{\infty} x_{n+1}\frac{\partial}{\partial x_n}$$
be the total derivative operator on the algebra $\mathcal{A} = \mathrm{Jet}(\mathcal{O}_C) = \mathbb{R}[t,x_i]$. Show that D_1 is indeed a derivation on $\mathcal{A}$.

3. Give an explicit description of the differential forms and vector fields of the jet $\mathcal{D}$-space $\mathrm{Jet}(C)$.

4. Show that for every $L \in \mathcal{A}$ and $x \in H$,
$$\int_0^1 (D_1 L)(t, \partial_t^i x(t))dt = \left[L(t, \partial_t^i x(t))\right]_0^1$$
and deduce from that the variational version of integration by parts
$$\int_0^1 (F.D_1 G)(t, \partial_t^i x(t))dt = -\int_0^1 (D_1 F.G)(t, \partial_t^i x(t)) + \left[(FG)(t, \partial_t^i x)\right]_0^1.$$

5. We denote $\underline{x} = (x, x_1, \dots)$ and $\underline{h} = (h, h_1, \dots)$ two universal coordinates on jet space. Let $L(t, \underline{x}) = \sum_i \sum_\alpha a_{i,\alpha} t^i \underline{x}^\alpha$ be a general Lagrangian and suppose that $|\alpha| > 0$ for every term in L. Show that
$$(\underline{x} + \epsilon \underline{h})^\alpha := \prod_i (x_i + \epsilon h_i)^{\alpha_i} = \prod_i (x_i^{\alpha_i} + \epsilon \alpha_i x_i^{\alpha_i - 1} h_i) = \underline{x}^\alpha + \epsilon \sum_j \alpha_j \underline{x}^{\alpha - \underline{j}} \underline{h}^{\underline{j}}$$
for $\epsilon^2 = 0$ and $\underline{j}$ the multi-index nontrivial only in degree j.

6. Deduce that
$$L(\underline{x}+\epsilon \underline{h}) = L(\underline{x}) + \epsilon \sum_j \frac{\partial L}{\partial x_j}.D_j h.$$

7. By using integration by part on the above expression for L, deduce that
$$S(x+\epsilon h) - S(x) = \epsilon B(x,h) + \epsilon \int_0^1 \left[\sum_j (-1)^j D_j (\frac{\partial L}{\partial x_j})(x, \partial_t^i x(t))\right] h dt,$$
where B is a function, called the boundary term, given by
$$B(x,h) = \sum_j \sum_{k=1}^{j} (-1)^{k+1} \left[(\frac{\partial L}{\partial x_j})(t, \partial_t^i x(t)) \partial_t^k h(t) \right]_0^1.$$

8. Describe the space of critical points of $S : \Gamma(M,C) \to \mathbb{R}$.

9. Describe locally (PDE) the space of trajectories $T = \{x \in H | \ d_x S = 0\}$.

Exercice 29. (Newtonian mechanics) We will use here only polynomial functions. Let $M = [0,1]$ with functions $\mathcal{O}_M = \mathbb{R}[t]$ and $C = \mathbb{R} \times [0,1]$ with functions $\mathcal{O}_C = \mathbb{R}[t,x]$. Let $\pi : C \times M$ be the bundle whose sections are polynomial functions $x : [0,1] \to \mathbb{R}$ and let $H \subset \underline{\Gamma}(M,C)$ be given by fixing the values $x(0) = x_0$ and $x(1) = x_1$ for some given $x_0, x_1 \in \mathbb{R}$. We denote $\mathcal{A} := \mathrm{Jet}(\mathcal{O}_C) = \mathbb{R}[t, x_0, x_1, \ldots]$ and $\mathcal{I}_{dS}$ the Euler-Lagrange $\mathcal{D}$-ideal in $\mathcal{A}$ associated to $S = [Ldt] \in h(\mathcal{A}) := \mathcal{A}.dt$

1. Describe the space of trajectories of the Lagrangian of standard Newtonian mechanics $L(t, x_0, x_1) = \frac{1}{2}x_1^2 - V(x_0)$ for $V : \mathbb{R} \to \mathbb{R}$ a given function by computing the differential of the corresponding action functional.

2. For $L(t, x_0, x_1) = \frac{1}{2}x_1^2$, give the generator of $\mathcal{I}_{dS}$ as an $\mathcal{A}[\mathcal{D}]$-module. Give generators of $\mathcal{I}_{dS}$ as an $\mathcal{A}$-module.

3. Describe the local critical space $\mathcal{A}/\mathcal{I}_{dS}$ and show that it is a finite dimensional algebraic variety (over M).

4. Described the differential graded algebra $(\mathcal{A}_P, d)$ of coordinates on the derived critical space of S.

5. Show that the $\mathcal{D}$-ideal $\mathcal{I}_{dS}$ is regular (i.e., its generator is not a zero divisor) and that $H^k(\mathcal{A}_P, d) = 0$, i.e., that there are no non-trivial Noether identities.

6. Describe the space of trajectories of the Lagrangian
$$L(t, x_0, x_1) = |x_1|$$
(defined on the subspace $H_{ns} \subset H$ of functions $x : [0,1] \to \mathbb{R}$ whose derivative never vanish).

7. Describe the Euler-Lagrange ideal and the corresponding local critical space $\mathcal{A}/\mathcal{I}_S$.

8. Describe the derived critical space $(\mathcal{A}_P, d)$ and show that $H^1(\mathcal{A}_P, d) \cong \mathcal{A}[\mathcal{D}]$.

9. Show that the non-trivial Noether identities can be chosen to be

$$\mathfrak{g} = \mathcal{A}[\mathcal{D}]$$

and describe the corresponding Koszul-Tate differential graded algebra.

Exercice 30. (Electromagnetism) Describe the critical space and derived critical space of local electromagnetism (expressed in terms of the electromagnetic potential $A \in \Omega^1_M$ as a differential form on spacetime M).

Appendix F

Distributions and partial differential equations

Distributions and partial differential equations

Exercice 31. (Distributions and smooth functionals) Let Alg_{geom} be the category of geometric $\mathcal{C}^\infty$-algebras $\mathbb{R}$-algebras and $\mathrm{Legos} = \mathrm{Alg}_{geom}^{op}$. Let $\Omega \subset \mathbb{R}^n$ be an open subset and $\underline{\mathbb{R}}(U) := \mathcal{C}^\infty(U)$. Let $\underline{\mathrm{Hom}}(\Omega, \mathbb{R})$ be the space of functions on Ω whose points with values in a legos U are given by the space of smooth functions on Ω with parameter in U:

$$\underline{\mathrm{Hom}}(\Omega, \mathbb{R})(U) := \mathrm{Hom}(\Omega \times U, \mathbb{R}).$$

For X and Y two spaces, a partially defined function between X and Y is a subspace $R_f \subset X \times Y$ that defines a function $f : X \to Y$ with a definition domain. Denote $\mathrm{Hom}_{\mathrm{PAR}}(X, Y) \subset \mathcal{P}(X \times Y)$ the set of partially defined functions. A functional is a partially defined function $f : \underline{\mathrm{Hom}}(\Omega, \mathbb{R}) \to \underline{\mathbb{R}}$, i.e, a family of (partially defined) maps of sets

$$f_U : \mathcal{C}^\infty(\Omega \times U) \to \mathcal{C}^\infty(U)$$

compatible with the smooth change of parameter space U.

1. Show that a functional $f : \underline{\mathrm{Hom}}(\Omega, \mathbb{R}) \to \underline{\mathbb{R}}$ is always given by a composition $f(\varphi) = f_. \circ \varphi$ where $f_. : \mathcal{C}^\infty(\Omega) \to \mathbb{R}$ is the value of f on the point $U = \{.\}$ and $\varphi : U \to \mathrm{Hom}(\Omega, \mathbb{R})$ is a point in $\mathcal{C}^\infty(\Omega \times U)$.

2. Show that spaces with partially defined functions between them form a category.

3. Show that if $f \in L^1_{loc}(\Omega)$ is a locally integrable function, the map

$$\varphi(x, u) \longmapsto \left[u \mapsto \int_{\mathbb{R}} \varphi(x, u) f(x) dx\right]$$

defines a functional $f \in \mathrm{Hom}_{\mathrm{PAR}}(\mathrm{Hom}(\Omega, \mathbb{R}), \underline{\mathbb{R}})$. Use Lebesgue's dominated convergence theorem to describe its domain of definition.

4. Recall that the topological space of smooth functions with compact support $\mathcal{C}_c^\infty(\Omega)$ is equipped with a topological vector space structure given by the family of seminorms

$$N_{K,k}(\varphi) := \sum_{|\alpha| \leq k} \left\| \frac{\partial_\alpha \varphi}{\partial x^\alpha} \right\|_{\infty, K}$$

indexed by integers and compact subsets $K \subset \Omega$. A distribution is a continuous linear function $f : \mathcal{C}_c^\infty(\Omega) \to \mathbb{R}$. Show that f induces an $\underline{\mathbb{R}}$-linear functional in $\mathrm{Hom}_{\mathrm{PAR}, \mathbb{R}}(\underline{\mathrm{Hom}}(\Omega, \mathbb{R}), \underline{\mathbb{R}})$ given by

$$\varphi \mapsto [u \mapsto f(\varphi(x, u))].$$

Describe its domain of definition and show that it is non-empty.

5. Define, by using a taylor expansion of φ with integral remainder, a domain of definition $D_{distrib} \subset \underline{\mathrm{Hom}}(\Omega, \mathbb{R})$ common to all the functionals associated to distributions.

6. Using the closed graph theorem of functional analysis, show that the natural map

$$\mathcal{C}^{-\infty}(\Omega) \to \mathrm{Hom}_{\underline{\mathbb{R}}-lin}(D_{distrib}, \underline{\mathbb{R}})$$

from ordinary distributions to $\mathbb{R}$-linear functionals on the space $D_{distrib}$ is an isomorphism of real vector spaces.

7. Give an example of an $\underline{\mathbb{R}}$-linear functional

$$f : \underline{\mathrm{Hom}}(\Omega, \mathbb{R}) \to \underline{\mathbb{R}}$$

whose domain of definition does not contain $D_{distrib}$.

Exercice 32. (Abstract and concrete Schwartz kernel theorem)

1. For X and Y two $\underline{\mathbb{R}}$-spaces, define

$$\underline{\mathrm{Hom}}_{\mathrm{PAR},\underline{\mathbb{R}}-lin}(X,Y)(U) := \mathrm{Hom}_{\mathrm{PAR},\underline{\mathbb{R}}-lin}(X \times U, Y)$$

the space of functions that are both partially defined and $\underline{\mathbb{R}}$-linear in the X-variable. Show that for X and Y two $\underline{\mathbb{R}}$-vector spaces, the space $X \otimes_{\underline{\mathbb{R}}} Y$ defined by the universal property

$$\mathrm{Hom}_{\mathrm{PAR}}(X \otimes_{\underline{\mathbb{R}}} Y, Z) \cong \mathrm{Hom}_{\mathrm{PAR}}(X, \underline{\mathrm{Hom}}_{\mathrm{PAR}}(Y, Z))$$

exists.

2. Show that
$$\underline{\mathrm{Hom}}(U, \mathbb{R}) \otimes_{\underline{\mathbb{R}}} \underline{\mathrm{Hom}}(V, \mathbb{R}) \cong \underline{\mathrm{Hom}}(U \times V, \mathbb{R}).$$

3. Deduce from the above results the following generalized Schwartz kernel theorem: Let
$$P \in \mathrm{Hom}_{\mathrm{PAR},\underline{\mathbb{R}}-lin}(\underline{\mathrm{Hom}}(\Omega, \mathbb{R}), \underline{\mathrm{Hom}}_{\mathrm{PAR},\underline{\mathbb{R}}-lin}(\underline{\mathrm{Hom}}(\Omega, \mathbb{R}), \mathbb{R}))$$
be an operator from functions to linear functionals. There exists a linear functional

$$K \in \mathrm{Hom}_{\mathrm{PAR},\mathbb{R}-lin}(\underline{\mathrm{Hom}}(\Omega, \mathbb{R}) \otimes_{\underline{\mathbb{R}}} \underline{\mathrm{Hom}}(\Omega, \mathbb{R}), \mathbb{R})$$

such that for f and φ in the domain of definitions of the above functionals,

$$\langle Pf, \varphi \rangle = \langle K, f \otimes \varphi \rangle.$$

4. Deduce from the above theorem the usual Schwartz kernel theorem: for every continuous operator $P : \mathcal{C}_c^\infty(\mathbb{R}^n) \to \mathcal{C}^{-\infty}(\mathbb{R}^m)$, there exists a distribution $K \in \mathcal{C}^{-\infty}(\mathbb{R}^n \times \mathbb{R}^m)$ such that
$$\langle Pf, \varphi \rangle = \langle K, f \otimes \varphi \rangle.$$

Exercice 33. (Polynomial distributional functionals) We refer to the beginning of Douady's thesis [Dou66] as an inspiration for this exercice. Let X and Y be two $\underline{\mathbb{R}}$-linear spaces. A partially defined morphism $f : X \to Y$ is called polynomial of degree n if there exists a symmetric n-multilinear operation $\tilde{f} : X \otimes_{\underline{\mathbb{R}}} \cdots \otimes_{\underline{\mathbb{R}}} X \to Y$ such that $f(x) = \tilde{f}(x, \dots, x)$. Denote $\underline{\mathrm{Hom}}_{\mathrm{PAR},poly}(X, Y) := \oplus_{n \geq 0} \underline{\mathrm{Hom}}_{\mathrm{PAR},n-poly}(X^n, Y)$ the space of polynomial functionals on X with values in Y.

1. For any function $h : X \to Y$, denote
$$\Delta_x(h)(y) = \frac{1}{2}\left(h(y+x) - h(y-x)\right).$$
Show if $f : X \to Y$ is a polynomial function of degree n associated to $\tilde{f} : X^{\otimes n} \to Y$, one has
$$\tilde{f}(x_1, \dots, x_n) = \frac{1}{n!}\Delta_{x_n} \dots \Delta_{x_1} f.$$

2. Let $E \to M$ be a vector bundle and $\mathcal{E}$ denote its $\mathbb{R}$-module of smooth sections with compact support, equipped with its natural Fréchet topology. Let $X = \underline{\Gamma}(M, E)$. The space of Fréchet polynomial functions is the space
$$\mathcal{O}(\mathcal{E}) := \oplus \mathrm{Hom}_{\mathbb{R}-lin,cont}(\mathcal{E}^{\otimes n}, \mathbb{R})^{S_n}.$$
Show that there is an injective map
$$\mathcal{O}(\mathcal{E}) \to \underline{\mathrm{Hom}}_{\mathrm{PAR},poly}(X, Y).$$

3. Describe by an estimation similar to the one done for ordinary distributions in exercice 31 the image of the above map (hint: what is the domain of definition of all polynomial functionals?).

4. Denote
$$\underline{\mathrm{Hom}}_{\mathrm{PAR},formal}(X, Y) := \prod_{n \geq 0} \underline{\mathrm{Hom}}_{\mathrm{PAR},n-poly}(X^n, Y)$$
the space of formal polynomial functionals, and
$$\widehat{\mathcal{O}}(\mathcal{E}) := \prod_{n \geq 0} \mathrm{Hom}_{\mathbb{R}-lin,cont}(\mathcal{E}^{\otimes n}, \mathbb{R})^{S_n}$$
the space of Fréchet formal polynomial functionals. Show that there is an injective map
$$\widehat{\mathcal{O}}(\mathcal{E}) \to \underline{\mathrm{Hom}}_{\mathrm{PAR},formal}(X, Y).$$

5. Show that the extension of the above results to the super, graded and differential graded setting are also true.

Exercice 34. (Nonlinear weak solutions: original nonsense)

As in the previous exercice, we denote $\Omega \subset \mathbb{R}^n$ an open subset and $X = \underline{\mathrm{Hom}}(\Omega, \mathbb{R})$ the corresponding space of functions. We denote $\mathcal{O}_X := \underline{\mathrm{Hom}}_{\mathrm{PAR}}(X, \mathbb{R})$ the $\mathbb{R}$-algebra space of partially defined non-linear functionals on X and its Gelfand spectrum is denoted

$$X^{**} := \underline{\mathrm{Spec}}(\mathcal{O}_X)(\underline{\mathbb{R}}) := \underline{\mathrm{Hom}}_{\mathrm{PAR,ALG}/\underline{\mathbb{R}}}(\mathcal{O}_X, \underline{\mathbb{R}}).$$

One can think of $\mathrm{Spec}(\mathcal{O}_X)(\underline{\mathbb{R}})$ as a kind of non-linear bidual X^{**} of X. A nonlinear weak solution of an operator (morphism of spaces) $P : X \to X$ is an element $F \in \underline{\mathrm{Spec}}(\mathcal{O}_X)(\underline{\mathbb{R}})$ such that

$$(P^{**}F)(f(\varphi)) := F(f(P(\varphi))) = 0$$

for every f in a given domain in $\underline{\mathrm{Hom}}_{\mathrm{PAR}}(X, \underline{\mathbb{R}})$.

1. Let $f \in \underline{\mathrm{Hom}}_{\underline{\mathbb{R}}-lin}(D_{distrib}, \underline{\mathbb{R}})$ be a distribution such that $f({}^tP(D)\varphi) = 0$ for all test function $\varphi : \Omega \to \mathbb{R}$, ${}^tP(D)$ being the adjoint operator for the integration pairing $(f, g) \mapsto \int fg$. Show that the polynomial extension of

$$F(g(\varphi)) := \langle f, g \rangle$$

defines a nonlinear weak solution $F \in X^{**}$ of P with definition domain containing the space $X \subset \underline{\mathrm{Hom}}(X, \mathbb{R})$ of distributions that are defined by smooth functions with compact support.

2. Give an example of a nonlinear weak solution of a nonlinear equation.

Exercice 35. (Fundamental solutions) Recall that if $f(x)$ is a polynomial function, there exists a differential operator $P_f(D)$ and a complex polynomial $P(s)$ such that the Bernstein-Sato equation

$$P_f(D)(f(x))^{s+1} = P(s)(f(x))^s$$

is fulfilled.

1. Show that given the Bernstein-Sato equation for $(f(x))^2$, the constant term of the analytic continuation of $(f(x))^{2s}$ allows to compute a distributional inverse of $f(x)$, i.e., a distribution g such that $f.g = 1$.

2. Using Fourier transformation, show that given a polynomial differential operator with constant coefficients $P(D)$ and a Bernstein-Sato equation for $P(x)$, one can find a distribution f such that $P(D).f = \delta_0$

3. Describe the Bernstein-Sato equation for $f(x) = x^2$.

4. Describe the Bernstein-Sato polynomial for $f(x_1, \ldots, x_n) = \sum_{i=1}^n \xi_i^2$ and for f any non-degenerate quadratic form on $\mathbb{Q}^n$ (hint: use $P_f(D) = f^{-1}(D)$).

5. Use the above result to describe fundamental solutions of the differential equation $P(D)f = \delta$.

6. Show that the distribution $\varphi \mapsto \langle \mathrm{vp}(x), \varphi \rangle := \lim_{\epsilon \to 0} \int_{|\epsilon|>0} \frac{\varphi(x)}{x} dx$ gives a distribution that is an inverse of x.

7. Use the same method to define a distributional inverse of x^2 and compare this result to the one obtained using the Bernstein-Sato polynomial.

8. Give a distributional inverse of any non-degenerate quadratic form on $\mathbb{R}^n$ by a limit process and use this result to give another construction of fundamental solutions to the differential equation $P(D)f = \delta$.

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