Rank and crank analogs for some colored partitions

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Abstract

We establish some rank and crank analogs for partitions into distinct colors and give combinatorial interpretations for colored partitions such as partitions defined by Toh, Zhang and Wang congruences modulo 5, 7.

Keywords: Partition congruences; rank analogs; Jacobi's triple product identity; Winquist's product identity

1 Introduction and Motivation

Let $p(n)$ be the number of unrestricted partitions of n, where n is nonnegative integer. In 1921, Ramanujan [20] discovered the following congruences

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5 \\ p(7n+5) &\equiv 0 \pmod 7. \end{aligned}$$

There exist many proofs in mathematical literature, for example [6, 7, 19].

In 1944, F. J. Dyson [10] defined the rank of a partition as the largest part minus the number of parts. Let $N(m,n)$ denote the number of partitions of n with rank m and let

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$N(m,t,n)$ denote the number of partitions of n with rank congruent to m modulo t. In 1953, A. O. L. Atkin and H. P. F. Swinnerton-Dyer [3] proved

$$N(0,5,5n+4) = N(1,5,5n+4) = \cdots = N(4,5,5n+4) = \frac{p(5n+4)}{5}$$

and

$$N(0,7,7n+5) = N(1,7,7n+5) = \cdots = N(6,7,7n+5) = \frac{p(7n+5)}{7}.$$

Following from the fact that the operation of conjugation reverses the sign of the rank, the trivial consequences are

$$N(m,n) = N(-m,n) \quad \text{and} \quad N(m,t,n) = N(t-m,t,n).$$

Hammond and Lewis [14] defined birank and explained that the residue of the birank mod 5 divided 2-colored partitions of n into 5 equal classes provided $n \equiv 2, 3 \,\text{or}\, 4 \pmod 5$. F. G. Garvan [12] found two other analogs the Dyson-birank and the 5-core-birank.

In 2010, Chan [8] introduced the cubic partition $a(n)$ as the number of 2-color partitions of n with colors red and blue subjecting to the restriction that the color blue appears only in even parts, and obtained the following congruence

$$a(3n+2) \equiv 0 \pmod 3.$$

Another proof has been given by B. Kim [16]. He defined a crank analog $M'(m,N,n)$ for $a(n)$ and proved that

$$M'(0,3,3n+2) \equiv M'(1,3,3n+2) \equiv M'(2,3,3n+2) \pmod 3,$$

for all nonnegative integers n, where $M'(m,N,n)$ is the number of partition of n with crank congruent to m modulo N. Later, B. Kim [17] gave two partition statistics which explained the partition congruences about cubic partition pairs $b(n)$. Here, $b(n)$ is the number of 4-color partitions of n with colors red, yellow, orange, and blue subjecting to the restriction that the colors orange and blue appear only in even parts.

About further research of arithmetic properties of cubic partitions, overcubic partitions and other colored partitions, some interesting results can be found in [18, 21, 23, 24]. The first author [25] of the present paper generalized Hammond-Lewis birank and gave combinatorial interpretations for some colored partitions.

The paper is organized as follows. In Section 2, we introduce necessary notation and some preliminary results. In Section 3, we aim to provide two partition statistics for two colored partition congruences modulo 5. We establish six rank or crank analogs for six colored partition with modulus 7 and give combinatorial interpretations in Section 4.

2 Preliminary results

For the two indeterminates q and z with $|q| < 1$, the q-shifted factorial of infinite order is defined by

$$(z;q)_\infty = \prod_{n=0}^{\infty}(1-zq^n)$$

where the multi-parameter expression for the former will be abbreviated as

$$[\alpha, \beta, \cdots, \gamma; q]_\infty = (\alpha; q)_\infty (\beta; q)_\infty \cdots (\gamma; q)_\infty.$$

The main purpose of this paper is to define rank and crank analogs for partition into colors and prove colored partition congruences applying the method of [11], which uses roots of unity. Jacobi triple product identity, the modified Jacobi triple product identity and Winquist product identity are given as follows:

- Jacobi triple product identity [1, 4, 5, 13, 15]:

$$\sum_{n=-\infty}^{+\infty} (-1)^n q^{\binom{n}{2}} x^n = [q, x, q/x; q]_\infty. \tag{1}$$

- Modified Jacobi triple product identity [14]:

$$[q, zq, q/z; q]_\infty = \sum_{n \geqslant 0} (-1)^n (z^n + z^{n-1} + \cdots + z^{-n}) q^{\binom{n+1}{2}}. \tag{2}$$

- Winquist product identity [9, 22]:

$$\begin{aligned} &(q; q)_\infty^4 [x, q/x; q]_\infty^2 \left[x^2, q/x^2; q\right]_\infty \\ &= \sum_{i,j=-\infty}^{+\infty} (-1)^{i+j} q^{3\binom{i}{2}+3\binom{j}{2}+j} (1 - 3i + 3j)\{x^{3i+3j} - x^{4-3i-3j}\}. \end{aligned} \tag{3}$$

By replacing q by q^2 in (3), splitting into two bilateral sums on right hand side of the resulting equation, and replacing $j \to j-1$ in the first double sum, and $i \to i+1$ in the second double sum, the resulting formula can be transformed as

- Modified Winquist product identity

$$\begin{aligned} &(q^2; q^2)_\infty^4 \left[x, q^2/x; q^2\right]_\infty^2 \left[x^2, q^2/x^2; q^2\right]_\infty \\ &= \sum_{i,j=-\infty}^{+\infty} (-1)^{i+j} q^{6\binom{i}{2}+6\binom{j}{2}+3i-j+1} (2 + 3i - 3j)\{(x/q)^{3i+3j-3} - (x/q)^{1-3i-3j}\}. \end{aligned} \tag{4}$$

Dividing both sides by $1+x$ in (3) and applying L'Hôpital's rule for the limit $x \to -1$, we have

$$(q; q)_\infty^2 (q^2; q^2)_\infty^4 = \sum_{i,j=-\infty}^{+\infty} q^{3\binom{i}{2}+3\binom{j}{2}+j} \frac{(1 - 3i + 3j)(2 - 3i - 3j)}{4}. \tag{5}$$

Divide both sides by $1 - q^2/x^2$ in (4) and utilize L'Hôpital's rule for the limit $x \to q$ to obtain

$$(q; q)_\infty^4 (q^2; q^2)_\infty^2 = \sum_{i,j=-\infty}^{+\infty} (-1)^{i+j} q^{6\binom{i}{2}+6\binom{j}{2}+3i-j+1} (2 + 3i - 3j)(3i + 3j - 2). \tag{6}$$

After Andrews and Garvan [2], for a partition λ, we define $\#(\lambda)$ is the number of parts in λ and $\sigma(\lambda)$ is the sum of the parts of λ with the convention $\#(\lambda) = \sigma(\lambda) = 0$ for the empty partition λ. Let $\mathcal{P}$ be the set of all ordinary partitions, $\mathcal{DO}$ be the set of all partitions into distinct odd parts.

For a given partition λ, the crank $c(\lambda)$ of a partition is defined as

$$c(\lambda) := \begin{cases} \ell(\lambda), & \text{if } r = 0; \\ \omega(\lambda) - r, & \text{if } r \geqslant 1, \end{cases}$$

where r is the number of 1's in λ, $\omega(\lambda)$ is the number of parts in λ that are strictly larger than r and $\ell(\lambda)$ is the largest part in λ. By extending the set of partitions $\mathcal{P}$ to a new set $\mathcal{P}^*$ by adding two additional copies of the partition 1, say 1^* and 1^{**}, B. Kim [16, 17] obtains

$$\frac{(q;q)_\infty}{[zq, z^{-1}q; q]_\infty} = \sum_{\lambda \in \mathcal{P}^*} wt(\lambda) z^{c^*(\lambda)} q^{\sigma^*(\lambda)},$$

where $wt(\lambda)$, $c^*(\lambda)$, and $\sigma^*(\lambda)$ are defined as follows. Denote the weight $wt(\lambda)$ for $\lambda \in \mathcal{P}^*$ by

$$wt(\lambda) := \begin{cases} 1, & \text{if } \lambda \in \mathcal{P}, \lambda = 1^*, \text{or } \lambda = 1^{**}; \\ -1, & \text{if } \lambda = 1, \end{cases}$$

and denote the extended crank $c^*(\lambda)$ by

$$c^*(\lambda) := \begin{cases} c(\lambda), & \text{if } \lambda \in \mathcal{P}; \\ 0, & \text{if } \lambda = 1; \\ 1, & \text{if } \lambda = 1^*; \\ -1, & \text{if } \lambda = 1^{**}. \end{cases}$$

Finally, denote the extended sum parts function $\sigma^*(\lambda)$ in the following way:

$$\sigma^*(\lambda) := \begin{cases} \sigma(\lambda), & \text{if } \lambda \in \mathcal{P}; \\ 1, & \text{otherwise.} \end{cases}$$

3 Rank analogs for colored partitions congruences modulo 5

In this section, we establish two statistics that divide the relevant partitions into equinumerous classes and present the combinatorial interpretation for colored partition congruences modulo 5.

We denote

$$C_{1^2 3^2} = \{(\lambda_1, \lambda_2, 3\lambda_3, 3\lambda_4) \mid \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathcal{P}\}.$$

For $\lambda \in C_{1^2 3^2}$, we define the sum of parts $s_{1^2 3^2}(\lambda)$, and rank analog $r_{1^2 3^2}(\lambda)$ by

$$\begin{aligned} s_{1^2 3^2}(\lambda) &= \sigma(\lambda_1) + \sigma(\lambda_2) + 3\sigma(\lambda_3) + 3\sigma(\lambda_4) \\ r_{1^2 3^2}(\lambda) &= \#(\lambda_1) - \#(\lambda_2) + \#(\lambda_3) - \#(\lambda_4). \end{aligned}$$

The number of 4-colored partitions of n if $s_{1^23^2}(\lambda) = n$ having $r_{1^23^2}(\lambda) = m$ will be written as $N_{C_{1^23^2}}(m,n)$, and $N_{C_{1^23^2}}(m,t,n)$ is the number of such 4-colored partitions of n having rank analog $r_{1^23^2}(\lambda) \equiv m \pmod{t}$. Now, summing over all 4-colored partitions $\lambda \in C_{1^23^2}$ gives

$$N_{C_{1^23^2}}(m,n) = \sum_{\substack{\lambda \in C_{1^23^2}, s_{1^23^2}(\lambda)=n, \\ r_{1^23^2}(\lambda)=m}} 1.$$

Since

$$r_{1^23^2}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = -r_{1^23^2}(\lambda_2, \lambda_1, \lambda_4, \lambda_3),$$

hence

$$N_{C_{1^23^2}}(m,n) = N_{C_{1^23^2}}(-m,n) \quad \text{and} \quad N_{C_{1^23^2}}(m,t,n) = N_{C_{1^23^2}}(t-m,t,n).$$

Then we have

$$\sum_{m\in\mathbb{Z}} \sum_{n=0}^{\infty} N_{C_{1^23^2}}(m,n) z^m q^n = \frac{1}{(zq;q)_\infty (z^{-1}q;q)_\infty (zq^3;q^3)_\infty (z^{-1}q^3;q^3)_\infty}. \tag{7}$$

By putting $z = 1$ in the identity (7), we find

$$\sum_{m=-\infty}^{\infty} N_{C_{1^23^2}}(m,n) = c(n),$$

where $c(n)$ is defined by $\sum_{n=0}^{\infty} c(n)q^n = \frac{1}{(q;q)_\infty^2 (q^3;q^3)_\infty^2}$.

Theorem 1. *For $n \geqslant 0$,*

$$N_{C_{1^23^2}}(0,5,5n+2) = N_{C_{1^23^2}}(1,5,5n+2) = N_{C_{1^23^2}}(2,5,5n+2) = \frac{c(5n+2)}{5}.$$

It can also prove the identity in Zhang and Wang [23]: $c(5n+2) \equiv 0 \pmod{5}$.

Proof. Suppose ζ is primitive $5th$ root of unity. By setting $z = \zeta$ in (7), we write

$$\sum_{m\in\mathbb{Z}} \sum_{n=0}^{\infty} N_{C_{1^23^2}}(m,n) \zeta^m q^n = \frac{1}{[\zeta q, \zeta^{-1}q; q]_\infty [\zeta q^3, \zeta^{-1}q^3; q^3]_\infty}$$
$$= \frac{1}{(q^5;q^5)_\infty (q^{15};q^{15})_\infty} \times \left[q, \zeta^2 q, \zeta^{-2} q; q\right]_\infty \left[q^3, \zeta^2 q^3, \zeta^{-2} q^3; q^3\right]_\infty.$$

Using modified Jacobi triple product identity (2), the last two infinite products have the following series representation

$$\sum_{i,j=0}^{\infty} (-1)^{i+j} q^{\binom{i+1}{2} + 3\binom{j+1}{2}} \{\zeta^{2i} + \zeta^{2i-2} + \cdots + \zeta^{-2i}\}\{\zeta^{2j} + \zeta^{2j-2} + \cdots + \zeta^{-2j}\}.$$

Observe the congruence relation

$$\binom{i+1}{2}+3\binom{j+1}{2}+3\equiv 8\Big\{\binom{i+1}{2}+3\binom{j+1}{2}+3\Big\}\equiv (2i+1)^2+3(2j+1)^2\equiv 0 \pmod 5,$$

which can be reached only if $i\equiv 2 \pmod 5$ and $j\equiv 2 \pmod 5$ since the corresponding residues modulo 5 read respectively as

$$(2i+1)^2\equiv 0,1,4 \pmod 5, \quad \text{and} \quad 3(2j+1)^2\equiv 0,2,3 \pmod 5.$$

When $i\equiv 2 \pmod 5$ and $j\equiv 2 \pmod 5$, we have $\{\zeta^{2i}+\zeta^{2i-2}+\cdots+\zeta^{-2i}\}\{\zeta^{2j}+\zeta^{2j-2}+\cdots+\zeta^{-2j}\}=0$. We see that in the q-expansion on the right side of the last equation the coefficient of q^n is zero when $n\equiv 2 \pmod 5$. The proof of Theorem 1 has been finished. □

Let

$$C_{2^23^2}=\{(2\lambda_1,2\lambda_2,3\lambda_3,3\lambda_4)\mid \lambda_1,\lambda_2,\lambda_3,\lambda_4\in\mathcal{P}\}.$$

For $\lambda\in C_{2^23^2}$, we define the sum of parts $s_{2^23^2}(\lambda)$, and rank analog $r_{2^23^2}(\lambda)$ by

$$\begin{aligned} s_{2^23^2}(\lambda) &= 2\sigma(\lambda_1)+2\sigma(\lambda_2)+3\sigma(\lambda_3)+3\sigma(\lambda_4)\\ r_{2^23^2}(\lambda) &= \#(\lambda_1)-\#(\lambda_2)+\#(\lambda_3)-\#(\lambda_4). \end{aligned}$$

Define $N_{C_{2^23^2}}(m,n)$ as the number of 4-colored partitions of n if $s_{2^23^2}(\lambda)=n$ having $r_{2^23^2}(\lambda)=m$, and $N_{C_{2^23^2}}(m,t,n)$ as the number of such 4-colored partitions of n having rank analog $r_{2^23^2}(\lambda)\equiv m \pmod t$. Now, summing over all 4-colored partitions $\lambda\in C_{2^23^2}$ gives

$$N_{C_{2^23^2}}(m,n)=\sum_{\substack{\lambda\in C_{2^23^2}, s_{2^23^2}(\lambda)=n,\\ r_{2^23^2}(\lambda)=m}} 1.$$

Since

$$r_{2^23^2}(\lambda_1,\lambda_2,\lambda_3,\lambda_4)=-r_{2^23^2}(\lambda_2,\lambda_1,\lambda_4,\lambda_3),$$

hence

$$N_{C_{2^23^2}}(m,n)=N_{C_{2^23^2}}(-m,n) \quad \text{and} \quad N_{C_{2^23^2}}(m,t,n)=N_{C_{2^23^2}}(t-m,t,n).$$

Then we have

$$\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty} N_{C_{2^23^2}}(m,n)z^mq^n=\frac{1}{(zq^2;q^2)_\infty(z^{-1}q^2;q^2)_\infty(zq^3;q^3)_\infty(z^{-1}q^3;q^3)_\infty}. \tag{8}$$

By putting $z=1$ in the identity (8), we find

$$\sum_{m=-\infty}^{\infty} N_{C_{2^23^2}}(m,n)=\rho(n),$$

where $\rho(n)$ is defined by $\sum_{n=0}^{\infty}\rho(n)q^n=\frac{1}{(q^2;q^2)_\infty^2(q^3;q^3)_\infty^2}$.

Theorem 2. *For $n \geqslant 0$,*

$$N_{C_{2^23^2}}(0,5,5n+k) = N_{C_{2^23^2}}(1,5,5n+k) = N_{C_{2^23^2}}(2,5,5n+k) = \frac{\rho(5n+k)}{5};\ k=1,4.$$

It can also prove the identity in Zhang and Wang [23]: $\rho(5n+1) \equiv 0 \pmod 5$ and $\rho(5n+4) \equiv 0 \pmod 5$.

Proof. The proof of Theorem 2 is similar to Theorem 1. Replacing z by ζ in (8), we get

$$\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty} N_{C_{2^23^2}}(m,n)\zeta^m q^n = \frac{1}{[\zeta q^2, \zeta^{-1}q^2; q^2]_\infty\,[\zeta q^3, \zeta^{-1}q^3; q^3]_\infty}$$
$$= \frac{1}{(q^{10};q^{10})_\infty (q^{15};q^{15})_\infty} \times \left[q^2, \zeta^2 q^2, \zeta^{-2}q^2; q^2\right]_\infty \left[q^3, \zeta^2 q^3, \zeta^{-2}q^3; q^3\right]_\infty.$$

Applying modified Jacobi triple product identity (2), we transform the last two infinite products as follows

$$\sum_{i,j=0}^{\infty} (-1)^{i+j} q^{2\binom{i+1}{2}+3\binom{j+1}{2}} \{\zeta^{2i}+\zeta^{2i-2}+\cdots+\zeta^{-2i}\}\{\zeta^{2j}+\zeta^{2j-2}+\cdots+\zeta^{-2j}\}.$$

It is not hard to check that the residues of q-exponent in the formal power series just displayed $2\binom{i+1}{2}+3\binom{j+1}{2}$ modulo 5 are given by the following table:

$j\backslash i$	0	1	2	3	4
0	0	2	1	2	0
1	3	0	4	0	3
2	4	1	0	1	4
3	3	0	4	0	3
4	0	2	1	2	0

When $i \equiv 2 \pmod 5$ or $j \equiv 2 \pmod 5$, we have $\{\zeta^{2i}+\zeta^{2i-2}+\cdots+\zeta^{-2i}\}\{\zeta^{2j}+\zeta^{2j-2}+\cdots+\zeta^{-2j}\} = 0$. We observe that in the q-expansion on the right side of the last equation the coefficient of q^n is zero when $n \equiv 1 \pmod 5$ and $n \equiv 4 \pmod 5$. The proof of Theorem 2 has been completed. □

4 Rank and crank analogs for colored partitions congruences modulo 7

In this section, we define statistics that divide the relevant partitions into equinumerous classes and provide the combinatorial interpretation according to [17] for colored partitions congruences modulo 7 given in Toh [21], Zhang and Wang [23].

If we denote

$$C_{1^32^{-2}} = \{(2\lambda_1, 2\lambda_2, \lambda_3, \lambda_4, \lambda_5) \mid \lambda_1, \lambda_2 \in \mathcal{P}, \lambda_3, \lambda_4, \lambda_5 \in \mathcal{P}^*\},$$

then we can call them as partitions into 5-colors.

For the set of the colored partitions, we define the sum of parts $s_{1^32^{-2}}(\lambda)$, a weight $wt_{1^32^{-2}}(\lambda)$ and a crank analog $c_{1^32^{-2}}(\lambda)$ by

$$\begin{aligned} s_{1^32^{-2}}(\lambda) &= 2\sigma(\lambda_1)+2\sigma(\lambda_2)+\sigma^*(\lambda_3)+\sigma^*(\lambda_4)+\sigma^*(\lambda_5) \\ wt_{1^32^{-2}}(\lambda) &= (-1)^{\#(\lambda_1)+\#(\lambda_2)}wt(\lambda_3)wt(\lambda_4)wt(\lambda_5) \\ c_{1^32^{-2}}(\lambda) &= c^*(\lambda_3)+2c^*(\lambda_4)+3c^*(\lambda_5), \end{aligned}$$

where the definitions of $\mathcal{P}^*$, $\sigma^*(\lambda)$, $wt(\lambda)$ and $c^*(\lambda)$ are presented in section 2. Let $M_{C_{1^32^{-2}}}(m,n)$ denote the number of 5-colored partitions of n if $s_{1^32^{-2}}(\lambda)=n$ (counted according to the weight $wt_{1^32^{-2}}(\lambda)$) with analog of crank $c_{1^32^{-2}}(\lambda)=m$, and $M_{C_{1^32^{-2}}}(m,t,n)$ denote the number of 5-colored partitions of n with analog of crank $c_{1^32^{-2}}(\lambda)$ congruent to $m \pmod t$, so that

$$M_{C_{1^32^{-2}}}(m,n)=\sum_{\substack{\lambda\in C_{1^32^{-2}},s_{1^32^{-2}}(\lambda)=n,\\ c_{1^32^{-2}}(\lambda)=m}} wt_{1^32^{-2}}(\lambda).$$

Then we have

$$\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty} M_{C_{1^32^{-2}}}(m,n)z^mq^n=\frac{(q^2;q^2)_\infty^2(q;q)_\infty^3}{[zq,z^{-1}q,z^2q,z^{-2}q,z^3q,z^{-3}q;q]_\infty}. \tag{9}$$

By putting $z=1$ in the identity (9), we find

$$\sum_{m=-\infty}^{\infty} M_{C_{1^32^{-2}}}(m,n)=Q_{(po,\overline{p})}(n),$$

where $Q_{(po,\overline{p})}(n)$ is defined by $\sum_{n=0}^{\infty}Q_{(po,\overline{p})}(n)q^n := \frac{(-q;q)_\infty}{(q;q^2)_\infty(q;q)_\infty}=\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^3}$.

Suppose ϖ is primitive seventh root of unity. By letting $z=\varpi$ in (9), we have

$$\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty} M_{C_{1^32^{-2}}}(m,n)\varpi^mq^n=\frac{(q^2;q^2)_\infty^2(q;q)_\infty^3}{[\varpi q,\varpi^{-1}q,\varpi^2q,\varpi^{-2}q,\varpi^3q,\varpi^{-3}q;q]_\infty}=\frac{(q^2;q^2)_\infty^2(q;q)_\infty^4}{(q^7;q^7)_\infty}.$$

Utilizing product identity (6), we compute the numerator of the right hand side of last identity as follows

$$\sum_{i,j=-\infty}^{+\infty}(-1)^{i+j}q^{6\binom{i}{2}+6\binom{j}{2}+3i-j+1}(2+3i-3j)(3i+3j-2). \tag{10}$$

We illustrate that the residues of q-exponent in the formal power series just displayed $6\binom{i}{2}+6\binom{j}{2}+3i-j+1$ modulo 7 are given by the following table:

$j\backslash i$	0	1	2	3	4	5	6
0	1	4	6	0	0	6	4
1	0	3	5	6	6	5	3
2	5	1	3	4	4	3	1
3	2	5	0	1	1	0	5
4	5	1	3	4	4	3	1
5	0	3	5	6	6	5	3
6	1	4	6	0	0	6	4

The power of q is congruent to 2 modulo 7 only when $i \equiv_7 0$ and $j \equiv_7 3$. Since the coefficient of q^n on the right side of the last identity is a multiple of 7 when $n \equiv 2 \pmod 7$, and $1+\varpi+\varpi^2+\varpi^3+\varpi^4+\varpi^5+\varpi^6$ is a minimal polynomial in $\mathbf{Z}[\varpi]$, we must have the result following as

Theorem 3. *For $n \geqslant 0$ and $0 \leqslant i < j \leqslant 6$, we have*

$$M_{C_{1^3 2^{-2}}}(i,7,7n+2) \equiv M_{C_{1^3 2^{-2}}}(j,7,7n+2) \pmod 7.$$

It can also prove the identity in Toh [21]: $Q_{(po,\overline{p})}(7n+2) \equiv 0 \pmod 7$.

Next we define

$$C_{1^4 4^4 2^{-7}} = \{(\lambda_1,\lambda_2,\lambda_3,\lambda_4,2\lambda_5,2\lambda_6,2\lambda_7,2\lambda_8,2\lambda_9)\ |\lambda_1,\lambda_2,\lambda_3,\lambda_4 \in \mathcal{DO},\\ \lambda_5,\lambda_6 \in \mathcal{P}, \lambda_7,\lambda_8,\lambda_9 \in \mathcal{P}^*\}.$$

For $\lambda = (\lambda_1,\lambda_2,\lambda_3,\lambda_4,2\lambda_5,2\lambda_6,2\lambda_7,2\lambda_8,2\lambda_9)$, we denote the sum of parts $s_{1^4 4^4 2^{-7}}(\lambda)$, a weight $wt_{1^4 4^4 2^{-7}}(\lambda)$ and a crank analog $c_{1^4 4^4 2^{-7}}(\lambda)$ by

$$\begin{aligned} s_{1^4 4^4 2^{-7}}(\lambda) =& \sigma(\lambda_1)+\sigma(\lambda_2)+\sigma(\lambda_3)+\sigma(\lambda_4)\\ &+2\sigma(\lambda_5)+2\sigma(\lambda_6)+2\sigma^*(\lambda_7)+2\sigma^*(\lambda_8)+2\sigma^*(\lambda_9)\\ wt_{1^4 4^4 2^{-7}}(\lambda) =& (-1)^{\#(\lambda_5)+\#(\lambda_6)} wt(\lambda_7)wt(\lambda_8)wt(\lambda_9)\\ c_{1^4 4^4 2^{-7}}(\lambda) =& c^*(\lambda_7)+2c^*(\lambda_8)+3c^*(\lambda_9). \end{aligned}$$

Finally define $M_{C_{1^4 4^4 2^{-7}}}(m,n)$ as the number of 9-colored partitions of n if $s_{1^4 4^4 2^{-7}}(\lambda) = n$ with crank analog $c_{1^4 4^4 2^{-7}}(\lambda) = m$ counted according to the weight $wt_{1^4 4^4 2^{-7}}(\lambda)$ as follows:,

$$M_{C_{1^4 4^4 2^{-7}}}(m,n) = \sum_{\substack{\lambda \in C_{1^4 4^4 2^{-7}}, s_{1^4 4^4 2^{-7}}(\lambda)=n,\\ c_{1^4 4^4 2^{-7}}(\lambda)=m}} wt_{1^4 4^4 2^{-7}}(\lambda).$$

Let $M_{C_{1^4 4^4 2^{-7}}}(m,t,n)$ denote the number of 9-colored partitions of n with crank analog $c_{1^4 4^4 2^{-7}}(\lambda)$ congruent to $m \pmod t$.

Then we have

$$\sum_{m\in\mathbb{Z}} \sum_{n=0}^{\infty} M_{C_{1^4 4^4 2^{-7}}}(m,n) z^m q^n = \frac{(-q;q^2)_\infty^4 (q^2;q^2)_\infty^5}{[zq^2, z^{-1}q^2, z^2q^2, z^{-2}q^2, z^3q^2, z^{-3}q^2; q^2]_\infty}. \quad (11)$$

By replacing z by 1 in the identity (11), we discover

$$\sum_{m=-\infty}^{\infty} M_{C_{1^44^42^{-7}}}(m,n) = \gamma(n),$$

where $\gamma(n)$ is defined by $\sum_{n=0}^{\infty}\gamma(n)q^n = \frac{(-q;q^2)_\infty^4}{(q^2;q^2)_\infty}$. (see [23]).

Theorem 4. *For $n \geqslant 0$,*

$$M_{C_{1^44^42^{-7}}}(0,7,7n+2) \equiv M_{C_{1^44^42^{-7}}}(1,7,7n+2) \equiv \cdots \equiv M_{C_{1^44^42^{-7}}}(6,7,7n+2) \pmod 7.$$

It can also prove the identity $\gamma(7n+2) \equiv 0 \pmod 7$.

Proof. Put $z = \varpi$ in (11) and apply product identity (6) substituting $q \to -q$ to obtain

$$\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty} M_{C_{1^44^42^{-7}}}(m,n)\varpi^m q^n = \frac{(-q;q^2)_\infty^4(q^2;q^2)_\infty^5}{[\varpi q^2, \varpi^{-1}q^2, \varpi^2 q^2, \varpi^{-2}q^2, \varpi^3 q^2, \varpi^{-3}q^2; q^2]_\infty}$$

$$= \frac{(-q;q^2)_\infty^4(q^2;q^2)_\infty^6}{(q^{14};q^{14})_\infty} = \frac{-1}{(q^{14};q^{14})_\infty}\sum_{i,j=-\infty}^{+\infty} q^{6\binom{i}{2}+6\binom{j}{2}+3i-j+1}(2+3i-3j)(3i+3j-2).$$

We discover that the double sum of the last identity is similar as (10). Then we can use the same congruence relations. Since the coefficient of q^n on the right side of the last identity is a multiple of 7 when $n \equiv 2 \pmod 7$, and $1+\varpi+\varpi^2+\varpi^3+\varpi^4+\varpi^5+\varpi^6$ is a minimal polynomial in $\mathbf{Z}[\varpi]$, we deduce the theorem. □

If we denote

$$C_{1^22} = \{(\lambda_1, \lambda_2, 2\lambda_3, 2\lambda_4, 2\lambda_5) \mid \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in \mathcal{P}\}.$$

It can be said as partitions into 5-colors. For $\lambda = (\lambda_1, \lambda_2, 2\lambda_3, 2\lambda_4, 2\lambda_5) \in C_{1^22}$, we define the sum of parts $s_{1^22}(\lambda)$, a weight $w_{1^22}(\lambda)$ and a rank analog $r_{1^22}(\lambda)$ by

$$\begin{aligned} s_{1^22}(\lambda) &= \sigma(\lambda_1)+\sigma(\lambda_2)+2\sigma(\lambda_3)+2\sigma(\lambda_4)+2\sigma(\lambda_5) \\ w_{1^22}(\lambda) &= (-1)^{\#(\lambda_5)} \\ r_{1^22}(\lambda) &= \#(\lambda_1)-\#(\lambda_2)+3\#(\lambda_3)-3\#(\lambda_4). \end{aligned}$$

Let $N_{C_{1^22}}(m,n)$ denote the number of 5-colored partitions of n if $s_{1^22}(\lambda) = n$ (counted according to the weight $w_{1^22}(\lambda)$) with rank analog $r_{1^22}(\lambda) = m$, and $N_{C_{1^22}}(m,t,n)$ denote the number of 5-colored partitions of n with rank analog $r_{1^22}(\lambda)$ congruent to $m \pmod t$, hence

$$N_{C_{1^22}}(m,n) = \sum_{\substack{\lambda\in C_{1^22}, s(\lambda)=n,\\ r_{1^22}(\lambda)=m}} w_{1^22}(\lambda).$$

By considering the transformation that interchanges λ_1 and λ_2, λ_3 and λ_4, we get

$$N_{C_{1^22}}(m,n) = N_{C_{1^22}}(-m,n), \quad N_{C_{1^22}}(m,t,n) = N_{C_{1^22}}(t-m,t,n).$$

Then we have

$$\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty} N_{C_{1^2 2}}(m,n)z^m q^n = \frac{(q^2;q^2)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty(z^3q^2;q^2)_\infty(z^{-3}q^2;q^2)_\infty}. \tag{12}$$

By putting $z=1$ in the identity (12), we check

$$\sum_{m=-\infty}^{\infty} N_{C_{1^2 2}}(m,n) = \alpha(n),$$

where $\alpha(n)$ is defined by $\sum_{n=0}^{\infty}\alpha(n)q^n = \frac{1}{(q;q)_\infty^2(q^2;q^2)_\infty}$. (see [23]).

Suppose ϖ is primitive $7th$ root of unity. Substituting $z=\varpi$ into (12), we have

$$\begin{aligned}\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty} N_{C_{1^2 2}}(m,n)\varpi^m q^n &= \frac{(q^2;q^2)_\infty}{[\varpi q, q/\varpi;q]_\infty\,[\varpi^3q^2, q^2/\varpi^3;q^2]_\infty}\\
&= \frac{[q,\varpi^2q,q/\varpi^2;q]_\infty\,[q^2,\varpi^3q,q/\varpi^3;q^2]_\infty}{(q^7;q^7)_\infty}\\
&= \frac{\sum_{i\geqslant 0}(-1)^i(\varpi^{2i}+\varpi^{2i-2}+\cdots+\varpi^{-2i})q^{\binom{i+1}{2}}\sum_{j=-\infty}^{\infty}(-1)^j\varpi^{3j}q^{2\binom{j}{2}+j}}{(q^7;q^7)_\infty}.\end{aligned}$$

The last line depends only on modified Jacobi identity (2) and classical Jacobi identity (1).

If and only if $i\equiv_7 3$, we have $\varpi^{2i}+\varpi^{2i-2}+\cdots+\varpi^{-2i}=0$. Obviously

$$\binom{i+1}{2}\equiv_7\begin{cases}0, & i\equiv_7 0,6;\\ 1, & i\equiv_7 1,5;\\ 3, & i\equiv_7 2,4;\\ 6, & i\equiv_7 3;\end{cases}\quad\text{and}\quad 2\binom{j}{2}+j\equiv_7\begin{cases}0, & j\equiv_7 0;\\ 1, & j\equiv_7 1,6;\\ 4, & j\equiv_7 2,5;\\ 2, & m\equiv_7 3,4.\end{cases} \tag{13}$$

The power of q is congruent to 6 modulo 7 only when $\binom{i+1}{2}\equiv_7 6$ and $2\binom{j}{2}+j\equiv_7 0$ in which case $i\equiv_7 3$ and $j\equiv_7 0$ and the coefficient of q^{7n+6} in the last identity is zero. Since $1+\varpi+\varpi^2+\varpi^3+\varpi^4+\varpi^5+\varpi^6$ is a minimal polynomial in $\mathbf{Z}[\varpi]$, our main result is as follows.

Theorem 5. *For $n\geqslant 0$,*

$$\begin{aligned}N_{C_{1^2 2}}(0,7,7n+6) =& N_{C_{1^2 2}}(1,7,7n+6) = N_{C_{1^2 6}}(2,7,7n+6) = \cdots = N_{C_{1^2 6}}(6,7,7n+6)\\ =& \frac{\alpha(7n+6)}{7}.\end{aligned}$$

It can also prove the identity $\alpha(7n+6)\equiv 0 \pmod 7$.

Denote

$$C_{1^5 4^5 2^{-7}} = \{(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,2\lambda_6,2\lambda_7,2\lambda_8)\mid \lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5\in\mathcal{DO},\lambda_6,\lambda_7,\lambda_8\in\mathcal{P}^*\}.$$

We call the elements of $C_{1^54^52^{-7}}$ 8-colored partitions. For $\lambda \in C_{1^54^52^{-7}}$, we define the sum of parts $s_{1^54^52^{-7}}(\lambda)$, a weight $wt_{1^54^52^{-7}}(\lambda)$ and a crank analog $c_{1^54^52^{-7}}(\lambda)$ by

$$\begin{aligned} s_{1^54^52^{-7}}(\lambda) &= \sigma(\lambda_1)+\sigma(\lambda_2)+\sigma(\lambda_3)+\sigma(\lambda_4)+\sigma(\lambda_5)+2\sigma^*(\lambda_6)+2\sigma^*(\lambda_7)+2\sigma^*(\lambda_8) \\ wt_{1^54^52^{-7}}(\lambda) &= wt(\lambda_6)wt(\lambda_7)wt(\lambda_8) \\ c_{1^54^52^{-7}}(\lambda) &= c^*(\lambda_6)+2c^*(\lambda_7)+3c^*(\lambda_8). \end{aligned}$$

Let $M_{C_{1^54^52^{-7}}}(m,n)$ denote the number of 8-colored partitions of n if $s_{1^54^52^{-7}}(\lambda)=n$ (counted according to the weight $wt_{1^54^52^{-7}}(\lambda)$) with crank analog $c_{1^54^52^{-7}}(\lambda)=m$, and $M_{C_{1^54^52^{-7}}}(m,t,n)$ denote the number of 8-colored partitions of n with crank analog $c_{1^54^52^{-7}}(\lambda)\equiv t \pmod m$, so that

$$M_{C_{1^54^52^{-7}}}(m,n)=\sum_{\substack{\lambda\in C_{1^54^52^{-7}}, s_{1^54^52^{-7}}(\lambda)=n,\\ c_{1^54^52^{-7}}(\lambda)=m}} wt_{1^54^52^{-7}}(\lambda).$$

Then the generating function is

$$\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty} M_{C_{1^54^52^{-7}}}(m,n)z^mq^n=\frac{(-q;q^2)_\infty^5(q^2;q^2)_\infty^3}{[zq^2,z^{-1}q^2,z^2q^2,z^{-2}q^2,z^3q^2,z^{-3}q^2;q^2]_\infty}. \quad (14)$$

By putting $z=1$ in the identity (14), we discover

$$\sum_{m=-\infty}^{\infty} M_{C_{1^54^52^{-7}}}(m,n)=\nu(n),$$

where $\nu(n)$ is defined by $\sum_{n=0}^{\infty}\nu(n)q^n=\frac{(-q;q^2)_\infty^5}{(q^2;q^2)_\infty^3}$.

Theorem 6. *For $n\geqslant 0$,*

$$M_{C_{1^54^52^{-7}}}(0,7,7n+6)\equiv M_{C_{1^54^52^{-7}}}(1,7,7n+6)\equiv\cdots\equiv M_{C_{1^54^52^{-7}}}(6,7,7n+6) \pmod 7.$$

It can also prove the identity $\nu(7n+6)\equiv 0 \pmod 7$.

Proof. By replacing z by ϖ in (14), we write

$$\begin{aligned} \sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty} N_{C_{1^34^12^{-3}}}(m,n)\varpi^mq^n &= \frac{(-q;q^2)_\infty^5(q^2;q^2)_\infty^3}{[\varpi q^2,\varpi^{-1}q^2,\varpi^2q^2,\varpi^{-2}q^2,\varpi^3q^2,\varpi^{-3}q^2;q^2]_\infty} \\ &= \frac{(-q;q^2)_\infty^5(q^2;q^2)_\infty^4}{(q^{14};q^{14})_\infty}. \end{aligned}$$

Consider

$$(q;q)_\infty^3\left[q^2,q,q;q^2\right]_\infty=\sum_{i=0}^{\infty}\sum_{j=-\infty}^{\infty}(-1)^{i+j}(2i+1)q^{\binom{i+1}{2}+j^2},$$

which can be deduced by Jacobi identity (1) and (2).

Replacing q by $-q$ in the last identity, we have the following series representation

$$(-q;q^2)_\infty^5(q^2;q^2)_\infty^4=\sum_{i=0}^{\infty}\sum_{j=-\infty}^{\infty}(-1)^{\binom{i}{2}}(2i+1)q^{\binom{i+1}{2}+j^2}.$$

If and only if $i\equiv_7 3$, we have $2i+1\equiv_7 0$. We see that in the q-expansion on the right side of the last equation the coefficient of q^n is a multiple of 7 when $n\equiv 6 \pmod 7$ referring to (13). The proof of Theorem 6 has been finished. $\square$

Let

$$C_{1^52^3}=\{(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,2\lambda_6,2\lambda_7,2\lambda_8)\mid \lambda_1,\lambda_2,\lambda_3,\lambda_4\in\mathcal{P},\lambda_5,\lambda_6,\lambda_7,\lambda_8\in\mathcal{P}^*\}.$$

We can say them as partitions into 8-colors. For $\lambda\in C_{1^52^3}$, we denote the sum of parts $s_{1^52^3}(\lambda)$, a weight $wt_{1^52^3}(\lambda)$ and a crank analog $c_{1^52^3}(\lambda)$ by

$$\begin{aligned}
s_{1^52^3}(\lambda) &= \sigma(\lambda_1)+\sigma(\lambda_2)+\sigma(\lambda_3)+\sigma(\lambda_4)+\sigma^*(\lambda_5)+2\sigma^*(\lambda_6)+2\sigma^*(\lambda_7)+2\sigma^*(\lambda_8)\\
wt_{1^52^3}(\lambda) &= wt(\lambda_5)wt(\lambda_6)wt(\lambda_7)wt(\lambda_8)\\
c_{1^52^3}(\lambda) &= \#(\lambda_1)-\#(\lambda_2)+2\#(\lambda_3)-2\#(\lambda_4)+3c^*(\lambda_5)+c^*(\lambda_6)+2c^*(\lambda_7)+3c^*(\lambda_8).
\end{aligned}$$

The number of 8-colored partitions of n if $s_{1^52^3}(\lambda)=n$ with crank analog $c_{1^52^3}(\lambda)=m$ counted according to the weight $wt_{1^52^3}(\lambda)$ is denoted by $M_{C_{1^52^3}}(m,n)$, so that

$$M_{C_{1^52^3}}(m,n)=\sum_{\substack{\lambda\in C_{1^52^3},s_{1^52^3}(\lambda)=n,\\ c_{1^52^3}(\lambda)=m}} wt_{1^52^3}(\lambda).$$

The number of 8-colored partitions of n with crank analog $c_{1^52^3}(\lambda)$ congruent to $m \pmod t$ is denoted by $M_{C_{1^52^3}}(m,t,n)$. The following generating function for $M_{C_{1^52^3}}(m,n)$ is

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty}M_{C_{1^52^3}}(m,n)z^mq^n\\
&= \frac{(q;q)_\infty(q^2;q^2)_\infty^3}{[zq,z^{-1}q,z^2q,z^{-2}q,z^3q,z^{-3}q;q]_\infty\,[zq^2,z^{-1}q^2,z^2q^2,z^{-2}q^2,z^3q^2,z^{-3}q^2;q^2]_\infty}. \quad (15)
\end{aligned}$$

By setting $z=1$ in the identity (15), we find

$$\sum_{m=-\infty}^{\infty}M_{C_{1^52^3}}(m,n)=\mu(n),$$

where $\mu(n)$ is defined by $\sum_{n=0}^{\infty}\mu(n)q^n=\frac{1}{(q;q)_\infty^5(q^2;q^2)_\infty^3}$.

Theorem 7. *For $n\geqslant 0$,*

$$M_{C_{1^52^3}}(0,7,7n+6)\equiv M_{C_{1^52^3}}(1,7,7n+6)\equiv\cdots\equiv M_{C_{1^52^3}}(6,7,7n+6) \pmod 7.$$

It can also prove the identity $\mu(7n+6) \equiv 0 \pmod{7}$.

Proof. By letting $z = \varpi$ in (15), we get

$$\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty} M_{C_{1^52^3}}(m,n)\varpi^m q^n$$
$$= \frac{(q;q)_\infty (q^2;q^2)_\infty^3}{[\varpi q, \varpi^{-1}q, \varpi^2 q, \varpi^{-2}q, \varpi^3 q, \varpi^{-3}q; q]_\infty [\varpi q^2, \varpi^{-1}q^2, \varpi^2 q^2, \varpi^{-2}q^2, \varpi^3 q^2, \varpi^{-3}q^2; q^2]_\infty}$$
$$= \frac{(q;q)_\infty^2 (q^2;q^2)_\infty^4}{(q^7;q^7)_\infty (q^{14};q^{14})_\infty}.$$

Investigating product identity (5), splitting the bilateral sum with respect to i into two unilateral sums, the numerator infinite products on the last line of the last formula have the following series expression

$$(q;q)_\infty^2 (q^2;q^2)_\infty^4 = \sum_{i=0}^{\infty}\sum_{j=-\infty}^{+\infty} q^{3\binom{i+1}{2}+3\binom{j}{2}+j} \frac{(1+3i+3j)(2+3i-3j)}{2}. \tag{16}$$

We check that the residues of q-exponent in the formal power series displayed $3\binom{i+1}{2} + 3\binom{j}{2} + j$ modulo 7 are presented by the following table:

$j\backslash i$	0	1	2	3	4	5	6
0	0	3	2	4	2	3	0
1	1	4	3	5	3	4	1
2	5	1	0	2	0	1	5
3	5	1	0	2	0	1	5
4	1	4	3	5	3	4	1
5	0	3	2	4	2	3	0
6	2	5	4	6	4	5	2

If and only if $3\binom{i+1}{2} + 3\binom{j}{2} + j \equiv 6 \pmod{7}$, we have $i \equiv_7 3$ and $j \equiv_7 6$. Since the coefficient of q^n on the right side of the last identity is a multiple of 7 when $n \equiv 6 \pmod{7}$, and $1 + \varpi + \varpi^2 + \varpi^3 + \varpi^4 + \varpi^5 + \varpi^6$ is a minimal polynomial in $\mathbf{Z}[\varpi]$, we finish the proof of Theorem 7. □

Consider

$$C_{1^24^22^{-3}} = \{(\lambda_1, \lambda_2, 2\lambda_3, 2\lambda_4, 2\lambda_5, 2\lambda_6, 2\lambda_7) \mid \lambda_1, \lambda_2 \in \mathcal{DO}, \lambda_3, \lambda_4 \in \mathcal{P}, \lambda_5, \lambda_6, \lambda_7 \in \mathcal{P}^*\}.$$

We call them as partitions into 7-colors. For $\lambda \in C_{1^24^22^{-3}}$, we define the sum of parts $s_{1^24^22^{-3}}(\lambda)$, a weight $wt_{1^24^22^{-3}}(\lambda)$ and a crank analog $c_{1^24^22^{-3}}(\lambda)$ by

$$\begin{aligned} s_{1^24^22^{-3}}(\lambda) &= \sigma(\lambda_1) + \sigma(\lambda_2) + 2\sigma(\lambda_3) + 2\sigma(\lambda_4) + 2\sigma^*(\lambda_5) + 2\sigma^*(\lambda_6) + 2\sigma^*(\lambda_7) \\ wt_{1^24^22^{-3}}(\lambda) &= (-1)^{\#(\lambda_3)+\#(\lambda_4)} wt(\lambda_5) wt(\lambda_6) wt(\lambda_7) \\ c_{1^24^22^{-3}}(\lambda) &= c^*(\lambda_5) + 2c^*(\lambda_6) + 3c^*(\lambda_7). \end{aligned}$$

Let $M_{C_{1^2 4^2 2^{-3}}}(m,n)$ denote the number of 7-colored partitions of n if $s_{1^2 4^2 2^{-3}}(\lambda) = n$ (counted according to the weight $wt_{1^2 4^2 2^{-3}}(\lambda)$) with crank analog $c_{1^2 4^2 2^{-3}}(\lambda) = m$, so that

$$M_{C_{1^2 4^2 2^{-3}}}(m,n) = \sum_{\substack{\lambda \in C_{1^2 4^2 2^{-3}}, s_{1^2 4^2 2^{-3}}(\lambda)=n,\\ c_{1^2 4^2 2^{-3}}(\lambda)=m}} wt_{1^2 4^2 2^{-3}}(\lambda).$$

The number of 7-colored partitions of n with crank analog $c_{1^2 4^2 2^{-3}}(\lambda) \equiv m \pmod t$ is denoted by $M_{C_{1^2 4^2 2^{-3}}}(m,t,n)$. Then the two variable generating function for $M_{C_{1^2 4^2 2^{-3}}}(m,n)$ is

$$\sum_{m\in\mathbb{Z}} \sum_{n=0}^{\infty} M_{C_{1^2 4^2 2^{-3}}}(m,n) z^m q^n = \frac{(-q;q^2)_\infty^2 (q^2;q^2)_\infty^5}{[zq^2, z^{-1}q^2, z^2q^2, z^{-2}q^2, z^3q^2, z^{-3}q^2; q^2]_\infty}. \tag{17}$$

If we simply put $z = 1$ in the identity (17), and read off the coefficients of like powers of q, we find

$$\sum_{m=-\infty}^{\infty} M_{C_{1^2 4^2 2^{-3}}}(m,n) = \beta(n),$$

where $\beta(n)$ is defined by $\sum_{n=0}^{\infty} \beta(n) q^n = \frac{(-q;q^2)_\infty^2}{(q^2;q^2)_\infty}$.

Putting $z = \varpi$ in (17) gives

$$\sum_{m\in\mathbb{Z}} \sum_{n=0}^{\infty} M_{C_{1^2 4^2 2^{-3}}}(m,n) \varpi^m q^n = \frac{(-q;q^2)_\infty^2 (q^2;q^2)_\infty^5}{[\varpi q^2, \varpi^{-1}q^2, \varpi^2q^2, \varpi^{-2}q^2, \varpi^3q^2, \varpi^{-3}q^2; q^2]_\infty}$$
$$= \frac{(-q;q^2)_\infty^2 (q^2;q^2)_\infty^6}{(q^{14};q^{14})_\infty}.$$

Substituting $q \to -q$ into identity (16), the numerator infinite products have the following series expression

$$(-q;q^2)_\infty^2 (q^2;q^2)_\infty^6 = \sum_{i=0}^{\infty} \sum_{j=-\infty}^{+\infty} (-1)^{\binom{i+1}{2}+\binom{j+1}{2}} q^{3\binom{i+1}{2}+3\binom{j}{2}+j} \frac{(1+3i+3j)(2+3i-3j)}{2}.$$

It is easy to find that the power of q is congruent to 6 modulo 7 if and only if $i \equiv_7 3$ and $j \equiv_7 6$ considering the congruence relations in the proof of Theorem 7. Since the coefficient of q^n on the right side of the last identity is a multiple of 7 when $n \equiv_7 6$, and $1 + \varpi + \varpi^2 + \varpi^3 + \varpi^4 + \varpi^5 + \varpi^6$ is a minimal polynomial in $\mathbf{Z}[\varpi]$, our main result is as follows:

Theorem 8. *For $n \geqslant 0$ and $0 \leqslant i < j \leqslant 6$, we obtain*

$$M_{C_{1^2 4^2 2^{-3}}}(i,7,7n+6) \equiv M_{C_{1^2 4^2 2^{-3}}}(j,7,7n+6) \pmod 7.$$

It can also prove the identity $\beta(7n+6) \equiv 0 \pmod 7$.

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