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# Robust Stability of Time-varying Polytopic Systems by the Attractive Ellipsoid Method

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**Abstract**—This paper concerns the robust stabilization of continuous-time polytopic systems subject to unknown but bounded perturbations. To tackle this problem, the attractive ellipsoid method (AEM) is employed. The AEM aims to determine an asymptotically attractive (invariant) ellipsoid such that the state trajectories of the system converge to a small neighborhood of the origin despite the presence of non-vanishing perturbations. An alternative form of the elimination lemma is used to derive new LMI conditions, where the state-space matrices are decoupled from the stabilizing Lyapunov matrix. Then a robust state-feedback control law is obtained by semi-definite convex optimization, which is numerically tractable. Further, the gain-scheduled state-feedback control problem is considered within the AEM framework. Numerical examples are given to illustrate the proposed AEM and its improvements over previous works. Precisely, it is demonstrated that the minimal size ellipsoids obtained by the proposed AEM are smaller compared to previous works, and thus the proposed control design is less conservative.

## I. INTRODUCTION

Robust stabilization and control of affine linear parameter varying systems is an active area of research where stability,  $\mathcal{H}_2$ ,  $\mathcal{H}_\infty$ , gain-scheduling and multi-objective control are the most important problems of study. One of the main goals is to obtain less conservative linear matrix inequality (LMI) conditions [1], [2], [3]. This can be achieved by decoupling the state-space matrices and Lyapunov functions to establish extended LMI conditions [4], [5], [6], [7]. The influence of uncertainty (parametric type, unmodeled dynamics, external perturbations) on the performance of dynamical systems has been also extensively studied using different techniques such as  $\mathcal{H}_\infty$  [8], robust maximum principle [9], sliding-mode control [10], and active disturbance rejection control [11].

Among these techniques, the invariant ellipsoid method [12], [13] for linear systems and the more recently developed technique, the *attractive ellipsoid method* (AEM) for nonlinear systems [14], [15], employ the concept of the *asymptotically attractive (invariant) ellipsoid*. The AEM, which is based on Lyapunov arguments, is a robust control design technique that minimizes the effect of non-vanishing perturbations in nonlinear systems. In the presence of non-vanishing disturbances, it is well known that it is not possible to keep the state of the system at the origin. The goal of the AEM is to find a control law and an asymptotically attractive ellipsoid such that state trajectories of the system converge to

a small (in a given sense) neighborhood of the origin. These ellipsoidal regions characterise the effect of the exogenous disturbances on systems trajectories of the dynamical system.

In this paper, we extend the AEM to the robust constrained stabilization problem of linear continuous-time polytopic systems subject to unknown but bounded perturbations. We use an alternative form of the elimination lemma and derive new parameterized LMI conditions for robust stability. The obtained LMI conditions extend our previous work on the robust stabilization of continuous-time and discrete-time systems by the AEM [16], [17]. These conditions provide the minimum size of the corresponding attractive ellipsoid, solve the stabilization problem, and ensure convergence of system state trajectories to a minimal ellipsoidal set.

The employed form of the elimination Lemma allows to decouple the state-space matrices from the Lyapunov matrix. Thus the optimization variables associated with the controller are independent of the symmetric matrix that defines the Lyapunov function to test stability. This feature is of particular importance since it can be used to develop Lyapunov functions to prove the stability of uncertain systems where the uncertainty is a bounded and convex polytope or ellipsoid. Contrary, the AEM literature is dominated by complex bi-linear matrix inequality (BMI) or LMI conditions where matrices are not decoupled and the associated control gain matrices depend on the Lyapunov matrix (see e.g., [18], [19], [20]). In the proposed AEM, synthesis of the robust control law is reduced to a semi-definite optimization problem (SDP), which can be readily solved using interior-point algorithms [21], [22]. The search space of solutions for the corresponding SDP problem is restricted by a non-negative parameter which is determined by a Armijo-like step-size reduction rule. Two numerical examples are presented to illustrate the feasibility of the proposed approach.

## II. PRELIMINARIES

*Notation.* For matrices and vectors  $(\cdot)^T$  indicates transpose and  $A^\dagger := A + A^T$  denotes the Hermitian operator on  $A$ . For matrix elements  $\star$  denotes the transposed symmetric element. For symmetric matrices,  $X \prec 0$  indicates that  $X$  is negative definite and  $X \preceq 0$  indicates that  $X$  is negative semi-definite.  $\mathbb{S}^n$  denotes the space of square and symmetric real matrices of dimension  $n$ . For square matrices  $\text{trace}(\cdot)$  denotes the trace of  $(\cdot)$ .

*Problem formulation.* Consider a continuous-time linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) + D\omega(t), \quad x_0 \text{ given}, \quad (1)$$

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where the pair  $(A, B)$  is controllable,  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $D \in \mathbb{R}^{n \times p}$ ,  $x(t) \in \mathbb{R}^n$  is the system state,  $u(t) \in \mathbb{R}^m$  is the control input, and  $\omega \in \mathbb{R}^p$  is an unknown but bounded (at each time instant) perturbation expressed as,

$$\omega(t)^\top W \omega(t) \leq 1, \quad \forall t \geq 0, \quad (2)$$

where the matrix  $W = W^\top \succ 0$  is given. No other constraints are imposed on the perturbation  $\omega(t)$ , however it is not considered to be random.

The main objective is to design a robust state-feedback controller of the form  $u = Kx$ , where  $K \in \mathbb{R}^{m \times n}$  is a gain matrix, for the system (1) to compensate the influence of external perturbations (2) on the system state such that closed-loop system trajectories

$$\dot{x}(t) = (A + BK)x(t) + D\omega(t), \quad (3)$$

converge asymptotically to a minimal size ellipsoid, which includes the origin. This minimum size ellipsoid guarantees that the state trajectories of the system will remain within a neighborhood of the origin despite the presence of non-vanishing perturbations (2).

The following definition characterizes this minimal region.

*Definition 1 (Ellipsoidal set):* An ellipsoid  $\mathcal{E}(P, \bar{x}) \subset \mathbb{R}^n$  with center  $\bar{x}$  and shape matrix  $P$  is a set of the form,

$$\mathcal{E}(P, \bar{x}) := \{x \in \mathbb{R}^n : (x - \bar{x})^\top P^{-1}(x - \bar{x}) \leq 1\}, \quad (4)$$

where  $P \in \mathbb{S}^n$  is a positive definite matrix.

*Definition 2 (Robustly controlled invariant set):* The set  $\Omega \subseteq \mathcal{X}$ , where  $\mathcal{X}$  is the set of admissible states, is *robustly controlled invariant* for the system (1) if for all  $x(t) \in \Omega$ , there exists a control value  $u(t)$  such that, for all  $\omega(t)$  in (2), with  $W \in \mathbb{S}^n \succ 0$ ,

$$\dot{x}(t) = Ax(t) + Bu(t) + \omega(t) \in \Omega, \quad \forall t \geq 0.$$

If the control value is constrained as  $u(t) \in \mathcal{U}$ , where  $\mathcal{U}$  is the set of admissible controls, such a control action is called *admissible*. If  $\bar{x} = 0$  then the ellipsoid can be written as  $\mathcal{E}(P) := \{x \in \mathbb{R}^n : x^\top P^{-1}x \leq 1\}$ ,  $P \in \mathbb{S}^n \succ 0$ , and we assume it is a *robustly controlled invariant set* of (1).

Also, consider the following distance metric from a point  $x$  to a set  $\mathcal{E}$ ,  $\|x\|_{\mathcal{E}} := \inf_{y \in \mathcal{E}} \|x - y\|$ ,  $\forall x \in \mathbb{R}^n$ .

*Definition 3 (Asymptotically Attractive Ellipsoid):* The set  $\mathcal{E}(P)$  is an asymptotically attractive ellipsoid for the system (1) if  $\|x(t, x_0)\|_{\mathcal{E}(P)} \rightarrow 0$ , as  $t \rightarrow \infty$ , for any  $x_0 \in \mathbb{R}^n$ , where  $x(t, x_0)$  is a trajectory of the system for a given admissible control.

For any initial condition  $x_0$ , convergence of state trajectories in (1) to a minimal size ellipsoid is guaranteed by the asymptotic attractivity of the set  $\mathcal{E}(P)$ .

### III. MAIN RESULTS

#### A. Time-Invariant Continuous-time Linear Systems

Consider the continuous-time linear time-invariant system (1) and the quadratic Lyapunov function  $V(x) = x^\top P^{-1}x$ ,

$P \succ 0$ . The derivative of  $V$  along the trajectories of (3) is

$$\begin{aligned} \dot{V}(x, \omega) &= \dot{x}^\top P^{-1}x + x^\top P^{-1}\dot{x} \\ &= x^\top [(A + BK)^\top P^{-1} + P^{-1}(A + BK)]x \\ &\quad + \omega^\top D^\top P^{-1}x + x^\top P^{-1}D\omega \end{aligned} \quad (5)$$

We aim to prove that  $\dot{V}(x, \omega) < 0$  for all  $(x, \omega) \neq 0$  along with  $\omega(t)^\top W \omega(t) \leq 1$ , for all  $t \geq 0$ . Let  $z^\top = [x^\top \ \omega^\top]$ , then  $\dot{V}(x) = z^\top \Omega_1 z < 0$ , where

$$\Omega_1 := \begin{bmatrix} (A + BK)^\top P^{-1} + P^{-1}(A + BK) & P^{-1}D \\ D^\top P^{-1} & 0 \end{bmatrix}.$$

Adding and subtracting  $\alpha x^\top P^{-1}x$  and  $\alpha \omega^\top W \omega$  in (5), where  $\alpha > 0$ , yields

$$\begin{aligned} \dot{V} &= z^\top \Omega_2 z - \alpha x^\top P^{-1}x + \alpha \omega^\top W \omega \\ &\leq z^\top \Omega_2 z + \alpha(1 - V), \end{aligned}$$

where<sup>1</sup>

$$\Omega_2 := \begin{bmatrix} (A + BK)^\top P^{-1} + P^{-1}(A + BK) + \alpha P^{-1} & P^{-1}D \\ \star & -\alpha W \end{bmatrix}.$$

If  $\Omega_2 \prec 0$ , this implies that  $z^\top \Omega_2 z < 0$  for all  $z \neq 0$  and the corresponding Lyapunov function  $V(x)$  satisfies the inequality  $z^\top \Omega_2 z + \alpha(1 - V) < \alpha(1 - V)$ , while  $\dot{V}$  is upper bounded by  $\dot{V} < -\alpha(V - 1)$ , and  $V > 1$  guarantees that  $\mathcal{E}(P)$  is an attractive ellipsoid of the closed-loop system (3). Moreover, if  $x(0) \in \mathcal{E}(P)$  then  $V(x_t) = x_t^\top P^{-1}x_t \leq e^{-\alpha t}V(x_0) + 1 - e^{-\alpha t} \rightarrow 1$ , as  $t \rightarrow \infty$ .

The following lemma [23], which is based on the elimination lemma, will allow us to derive the main result.

*Lemma 1:* Let us define a symmetric matrix  $\Phi$  and matrices  $\mathcal{N}$ ,  $\mathcal{M}$  with appropriate dimensions. Then the following conditions are equivalent:

- 1)  $\Phi \prec 0$  and  $\Phi + \mathcal{N}\mathcal{M}^\top + \mathcal{M}\mathcal{N}^\top \prec 0$ .
- 2) The LMI problem

$$\begin{bmatrix} \Phi & \mathcal{M} + \mathcal{N}F \\ \mathcal{M}^\top + F^\top \mathcal{N}^\top & -F - F^\top \end{bmatrix} \prec 0,$$

is feasible with respect to  $F$ .

*Proof:* Proof is omitted; see [23]. ■

The following LMI condition establishes that the ellipsoid  $\mathcal{E}(P)$  is an attractive ellipsoid of the closed-loop system (3) with gain matrix  $K = LF^{-1}$ , where  $L$  and  $F$  are design matrices of appropriate dimension.

*Theorem 1:* If there exists a positive definite matrix  $P \in \mathbb{S}^n$ , matrices  $F \in \mathbb{R}^{n \times n}$  (non-singular) and  $L \in \mathbb{R}^{m \times n}$ , and a constant scalar  $\alpha > 0$ , such that:

$$\begin{bmatrix} -\alpha P & D & P + AF + BL + \alpha F \\ \star & -\alpha W & 0 \\ \star & \star & -F - F^\top \end{bmatrix} \prec 0, \quad (6)$$

then the ellipsoid  $\mathcal{E}(P)$  is an attractive ellipsoid of system (3) with feedback gain matrix  $K = LF^{-1}$ .

<sup>1</sup>Notation:  $\star$  denotes the transposed symmetric element.

*Proof:* Using the congruence transformation  $T = \text{diag}(P, I)$  pre-and post-multiplying both sides of  $\Omega_2$  by  $T$ , yields

$$\Omega_3 = \begin{bmatrix} P(A+BK)^\top + (A+BK)P + \alpha P & D \\ \star & -\alpha W \end{bmatrix}.$$

To apply Lemma 1,  $\Omega_3$  is decomposed as

$$\begin{aligned} \Omega_3 &= \begin{bmatrix} -\alpha P & D \\ D^\top & -\alpha W \end{bmatrix} + \begin{bmatrix} A+BK + \alpha I \\ 0 \end{bmatrix} \begin{bmatrix} P & 0 \end{bmatrix} \\ &+ \begin{bmatrix} P \\ 0 \end{bmatrix} \begin{bmatrix} A^\top + K^\top B^\top + \alpha I & 0 \end{bmatrix} \prec 0. \end{aligned}$$

The proof is concluded by using the following matrix assignments in Lemma 1:

$$\begin{aligned} \Phi &\leftarrow \begin{bmatrix} -\alpha P & D \\ D^\top & -\alpha W \end{bmatrix}, & \mathcal{N} &\leftarrow \begin{bmatrix} A+BK + \alpha I \\ 0 \end{bmatrix} \\ \mathcal{M} &\leftarrow \begin{bmatrix} P & 0 \end{bmatrix}^\top, & F &\in \mathbb{R}^{n \times n}, \end{aligned}$$

which results, after some algebra, in the LMI condition (6). The control is obtained by applying the change of variables  $L := KF$  and  $u(t) = LF^{-1}x(t)$ , for  $F$  non-singular. ■

In the LMI condition (6), observe that  $P \succ 0$  is guaranteed by the  $(1, 1)$ -block,  $-\alpha P \prec 0$ . Also, the  $(3, 3)$ -block,  $-F - F^\top$ , implies that  $F$  is nonsingular.

An important feature of condition (6), in contrast to the existing literature in AEM, is that the state-space matrices and Lyapunov matrix are separated, and the feedback gain  $K$  does not depend on the Lyapunov matrix  $P$ . This feature is of particular importance since it can be used to develop Lyapunov functions to prove the stability of time-variant uncertain systems with polytopic uncertainty. The slack matrix  $F$  can be seen as an additional degree of freedom.

*Remark 1 (Line search subproblem):* Due to the presence of the decision variable  $\alpha > 0$ , condition (6) is not an LMI. However, for fixed  $\alpha$ , this condition actually becomes an LMI. To find a suitable  $\alpha$ , a line-search algorithm can be used such that it keeps increasing the value of  $\alpha$  until the problem becomes feasible, or stops when  $\alpha$  reaches a certain threshold value. The Armijo rule is essentially a successive reduction rule, suitable for this line-search subproblem. The idea here is to find the maximum  $\alpha$  that minimizes the  $\text{trace}(P)$  subject to feasibility of (6) and  $\Omega_2 \succ 0$  (see e.g., [16], [17]).

*Remark 2 (Handling control and state constraints):* Linear polyhedral state and control constraints can be readily handled using ellipsoidal sets dealing with control and state constraints directly, where the LMI condition of Theorem 1 and additional LMI conditions are combined to a coupled system of LMIs, and can be formulated as a Semi-definite Programming (SDP) problem. Suppose that the magnitude of the control signal  $u(t) = Kx(t)$  is constrained inside an ellipsoid  $\mathcal{E}(\Omega_u) \subseteq \mathcal{U}$  and the state constraints are satisfied if  $\mathcal{E}(\Omega_x) \subseteq \mathcal{X}$ . We can then impose that  $\Omega_x, \Omega_u \succeq P$ . In this case, the attractive ellipsoid corresponding to  $P$  is nested inside the bigger ellipsoids  $\Omega_x$  and  $\Omega_u$  (see e.g. [21], [16]).

An optimal attractive ellipsoid with minimal size can be found using the trace criterion due to linearity of the trace

function. The following SDP (with fixed  $\alpha$ ) problem provides LMI-based conditions for optimal robust stabilization,

$$\begin{aligned} \min_{P, L, F, \Omega_u, \Omega_x} & \text{trace}(P) \\ \text{subject to:} & \quad (6), \text{LMI}_x, \text{LMI}_u, \end{aligned} \quad (7)$$

where  $\text{LMI}_x$  and  $\text{LMI}_u$  (omitted here due to space limitations, see e.g., [16]) are LMI conditions satisfying  $\mathcal{E}(\Omega_x) \subseteq \mathcal{X}$  and  $\mathcal{E}(\Omega_u) \subseteq \mathcal{U}$ , respectively. Then the linear feedback gain  $K = LF^{-1}$  minimizes the size of the attractive ellipsoid of the closed-loop system (3).

## B. Time-varying Continuous-time Polytopic Systems

The aim of this section is to derive a finite-dimensional set of LMI conditions for the design of static feedback controllers of linear parameter varying polytopic systems. Consider the class of continuous-time linear parameter varying (LPV) systems of the form

$$\dot{x}(t) = A(\lambda)x(t) + B(\lambda)u(t) + D(\lambda)\omega(t), \quad t \geq 0, \quad (8)$$

where matrices  $A(\lambda)$ ,  $B(\lambda)$  and  $D(\lambda)$  depend affinely on the unknown but *measurable* time-invariant vector of parameters  $\lambda$ . The vector  $\lambda$  takes values in the unit simplex  $\Lambda_N$ ,  $\lambda \in \Lambda_N \subset \mathbb{R}^N$ ,  $N \in \mathbb{N}$ ,  $N \geq 2$ , where  $N$  is the number of vertices and  $\Lambda_N$  may be expressed as,

$$\Lambda_N = \left\{ \lambda \in \mathbb{R}^N : \sum_{i=1}^N \lambda_i = 1, \lambda_i \geq 0 \right\}. \quad (9)$$

The affine assumption implies that matrices  $A(\lambda)$ ,  $B(\lambda)$  and  $D(\lambda)$  are matrix polytopes and can be written as

$$(A, B, D) = \sum_{i=1}^N \lambda_i (A_i, B_i, D_i). \quad (10)$$

It should be noted that  $\lambda_i$  might be time-varying. Using state-feedback control, the closed loop system reads:

$$\dot{x}(t) = (A(\lambda) + B(\lambda)K)x(t) + D(\lambda)\omega(t). \quad (11)$$

Assume that  $\mathcal{E}(P(\lambda))$  given by (4) with  $\bar{x} = 0$  and,

$$P(\theta)^{-1} := \left( \sum_{i=1}^N \lambda_i P_i \right)^{-1}, \quad (12)$$

is a robustly controlled invariant set of (11).

The following theorem summarises the main result.

*Theorem 2:* The ellipsoid  $\mathcal{E}(P)$  is the attractive ellipsoid of the closed-loop system (11) with feedback gain matrix  $K = LF^{-1}$  if and only if there exist positive definite matrices  $P_i \in \mathbb{S}^n$ ,  $i = 1, \dots, N$ , matrices  $F \in \mathbb{R}^{n \times n}$  and  $L \in \mathbb{R}^{m \times n}$ , and constant scalar  $\alpha > 0$ , such that the following LMI conditions are satisfied

$$\begin{bmatrix} -\alpha P_i & D_i & P + A_i F + B_i L + \alpha F \\ \star & -\alpha W & 0 \\ \star & \star & -F - F^\top \end{bmatrix} \prec 0, \quad (13)$$

for all vertices,  $i = 1, \dots, N$ .

*Proof:* The necessity of this condition is directly obtained from Theorem 1 and (6). Indeed, if we satisfy

$$\begin{bmatrix} -\alpha P & D(\lambda) & P + A(\lambda)F + B(\lambda)L + \alpha F \\ \star & -\alpha W & 0 \\ \star & \star & -F - F^\top \end{bmatrix} \prec 0,$$

for all  $\lambda \in \Lambda_N$ , then this implies that the LMI must hold on the vertices. Now suppose that (13) hold for all  $A_i, B_i$ , and  $D_i, i = 1, \dots, N$ . Multiplying each inequality matrix by  $\lambda_i$  and summing for all  $i = 1, \dots, N$ , by convexity of the sets, we conclude that (13) must hold for all  $\lambda \in \Lambda_N$ . ■

Minimization of trace  $P(\lambda)$  can be equivalently expressed as the minimization of a parameter  $\eta > 0$ , subject to,

$$\text{trace}(P_i) \leq \eta, \quad \forall i = 1, \dots, N \quad (14)$$

The following SDP problem (for given  $\alpha > 0$ ) summarises,

$$\begin{aligned} \min_{P_i, L, F} \quad & \eta \\ \text{subject to:} \quad & (13), (14). \end{aligned} \quad (15)$$

### C. Gain-scheduled Dynamic Feedback Control

This section studies the problem of designing gain-scheduled dynamic feedback controllers in the special case where  $B_i = B$ , for all  $i = 1, 2, \dots, N$ . Assume gain-scheduled controllers of the form

$$u(x(t)) = K(\lambda)x(t), \quad K(\lambda) = \sum_{i=1}^N \lambda_i K_i, \quad (16)$$

where  $K_i, i = 1, \dots, N$ , are gain matrices, such that the closed-loop system

$$\dot{x}(t) = [A(\lambda) + BK(\lambda)]x(t) + D(\lambda)\omega(t) \quad (17)$$

converges asymptotically to a minimal size ellipsoid, which includes the origin for all  $\lambda \in \lambda_N \subseteq \mathbb{R}^N$  in (9).

*Theorem 3:* The ellipsoid  $\mathcal{E}(P)$  is the attractive ellipsoid of the closed-loop system (11) with gain-scheduled controllers (16) and feedback gains  $K_i = L_i F^{-1}, i = 1, \dots, N$ ,

$$u(t) = \left( \sum_{i=1}^N \lambda_i L_i \right) F^{-1} x(t), \quad t \geq 0, \quad (18)$$

if and only if there exist positive definite matrices  $P_i \in \mathbb{S}^n$ , matrices  $L_i \in \mathbb{R}^{m \times n}, i = 1, \dots, N, F \in \mathbb{R}^{n \times n}$  and constant scalar  $\alpha > 0$ , such that:

$$\begin{bmatrix} -\alpha P_i & D_i & P_i + A_i F + B L_i + \alpha F \\ \star & -\alpha W & 0 \\ \star & \star & -F - F^\top \end{bmatrix} \prec 0, \quad (19)$$

for all vertices,  $i = 1, \dots, N$ .

*Proof:* Proof omitted since it is a direct application of Theorems 1 and 2. ■

Finally, we obtain the following SDP problem (given  $\alpha$ ),

$$\begin{aligned} \min_{P_i, L_i, F} \quad & \eta \\ \text{subject to:} \quad & (14), (19). \end{aligned} \quad (20)$$

The feedback gain-scheduled controllers  $L_i := K_i F$  minimize the size of the attractive ellipsoid of the closed-loop system (17).

## IV. NUMERICAL EXAMPLES

This section presents two numerical examples to demonstrate the efficiency of the proposed AEM approach when compared to the AEM in [15]. The first example involves a continuous-time linear time-invariant system with unknown but bounded perturbations. The second concerns the robust stabilization of an LPV system by gain-scheduled control.

### A. Example 1

Consider the system (1) with the following matrices

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 2 & 1 \\ -1 & 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

$$W = \begin{bmatrix} 2.2 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.0004 \end{bmatrix} \times 10^5, \quad D = R = I.$$

Let the exogenous disturbance input  $\omega(t)$  be

$$\omega(t) = 0.2 + 0.5 \sin(50t) + 0.4 \sin(100t) (0.001, 0.01, 0.1)^\top.$$

The system with  $u(t) = 0$  is unstable.

Suppose that the magnitude of the control signal  $u(t) = Kx(t)$  is constrained as,

$$\|u\|_R^2 := x^\top K^\top R^{-1} K x < \mu^2, \quad \forall x : x^\top \Omega_u^{-1} x \leq 1, \quad (21)$$

where  $\mu^2 = 100$  and  $R = 1$ . The linear feedback gain  $K$ , the ellipsoidal matrix  $P$  and its size resulting from the solution of the SDP problem (7) derived from Theorem 1 are:

$$K = [-24.36 \quad -36.24 \quad -10.69],$$

$$P = \begin{bmatrix} 0.015 & -0.0017 & -0.0217 \\ -0.0017 & 0.0006 & 0.0010 \\ -0.0217 & 0.0010 & 0.0400 \end{bmatrix},$$

with  $\text{tr}(P) = 0.0556$ . To assess the performance of the proposed AEM, we compare its results with the results obtained by the AEM in [15]. For the AEM in [15], the feedback gain  $K$ , the ellipsoidal matrix  $P$  and its size are:

$$K = [-29.83 \quad -37.94 \quad -9.28],$$

$$P = \begin{bmatrix} 0.1285 & -0.0402 & -0.1099 \\ -0.0402 & 0.0258 & 0.0226 \\ -0.1099 & 0.0226 & 0.1525 \end{bmatrix},$$

with  $\text{tr}(P) = 0.3068$ . To compare the two AEM approaches, consider the initial condition  $x_0 = [-1 \quad 1 \quad -2]^\top$ . Figs 1 and 2 compare the proposed AEM with the AEM in [15], denoted as AEM\*. As can be seen, the proposed approach indicates smooth and fast convergence of the system states to the origin (cf. Fig 1(a) with Fig. 1(b)) compared to the AEM in [15]. Also it needs less control effort to stabilise the system (cf. Fig. 1(c) with Fig. 1(d)).

Fig 2 illustrates the projection onto the subspaces  $(x_1, x_2)$  and  $(x_2, x_3)$  and minimal ellipsoids. As can be seen, the obtained minimal size ellipsoid with Theorem 1 and SDP (7) is substantially smaller than the one in [15] (cf. Fig 2(a) with Fig. 2(b) and Fig 2(c) with Fig. 2(d)). This demonstrates that the proposed AEM approach is less conservative (compared

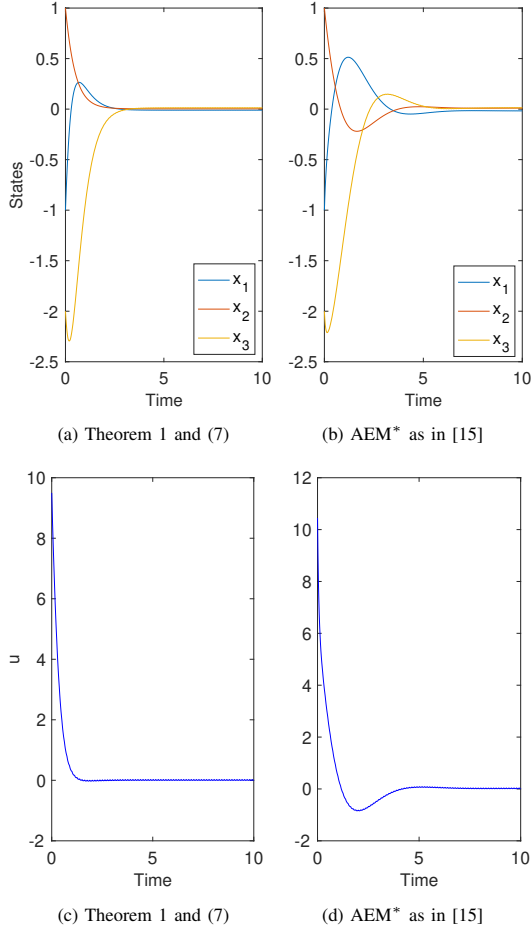


Fig. 1. Example 1: State and control trajectories.

to previous works) given that the feedback parameters of control design should be such that making the attractive ellipsoid of a smaller size. Note that every state in state space can be taken inside  $P$  (red ellipsoid). In this example,  $\Omega_u$  is equal to  $P$  but can be taken as  $\Omega_u \succ P$  and any state inside  $\Omega_u$  will satisfy the control constraint (21). State constraints can be also imposed as discussed in Remark 2.

### B. Example 2

Consider a continuous-time LPV system with the following affine parameter-dependent matrices

$$A(\theta) = \begin{bmatrix} -1.6 + 0.4\theta & 2 & \theta \\ -2 + \theta & -2\theta & \theta \\ -\theta & 1 & -2\theta \end{bmatrix},$$

$$B(\theta) = \begin{bmatrix} 0.9 \\ -0.81 \\ -0.8 \end{bmatrix}, \quad W = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.0005 \end{bmatrix} \times 10^5,$$

and  $D(\theta) = I$  where  $\theta$  is a time-invariant parameter with  $|\theta| \leq 1$ . Matrix  $A(\theta)$  is unstable for all  $\theta \leq 0$ . Let the exogenous disturbance input  $\omega(t)$  be

$$\omega(t) = 0.5 + 0.2 \sin(80t) + 0.2 \sin(100t) (0.001, 0.01, 0.1)^T.$$

To solve the stabilization problem, we consider a gain-scheduled feedback controller of the form (16) and express

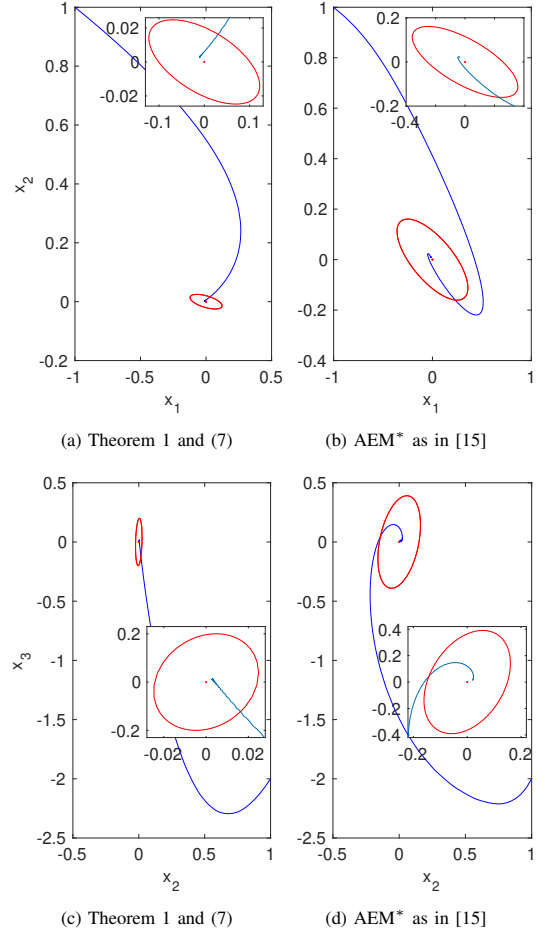


Fig. 2. Example 1: Projection onto the subspaces  $(x_1, x_2)$  and  $(x_2, x_3)$  and minimal ellipsoids.

the LPV system into the polytopic form (8). The state matrix  $A(\theta)$  can be expressed as in (10) with  $N = 2$  vertices

$$A_1 = \begin{bmatrix} -2 & 2 & -1 \\ -3 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1.2 & 2 & 1 \\ -1 & -2 & 1 \\ -1 & 1 & -2 \end{bmatrix},$$

$$\lambda_1 = \frac{\theta_{\max} - \theta}{\theta_{\max} - \theta_{\min}}, \quad \lambda_2 = \frac{\theta - \theta_{\min}}{\theta_{\max} - \theta_{\min}},$$

with  $\theta_{\min} = -1$  and  $\theta_{\max} = 1$ . The convex coordinates  $\lambda_i$ ,  $i = 1, 2$ , satisfy  $0 \leq \lambda_i \leq 1$ , and  $\lambda_1 + \lambda_2 = 1$ . The linear feedback gains  $K_1, K_2$ , the ellipsoidal matrix  $P$  and its size resulting from the solution of the SDP problem (20) are

$$K_1 = [335.94 \quad -19.77 \quad 626.08],$$

$$K_2 = [321.65 \quad -13.91 \quad 602.15],$$

$$P = \begin{bmatrix} 0.1314 & -0.0162 & -0.0780 \\ -0.0162 & 0.0356 & 0.0162 \\ -0.0780 & 0.0162 & 0.0486 \end{bmatrix},$$

with  $\text{tr}(P) = 0.2156$ . To illustrate the approach, consider an unstable system with  $\lambda_1 = 0.98$ ,  $\lambda_2 = 0.02$  and initial condition  $x_0 = [2 \quad 4.5 \quad -0.5]^T$ . Fig. 3 shows the obtained results. As can be seen, the proposed method provides fast and smooth convergence to the origin. It should be noted that the AEM in [15], when applied to the system above

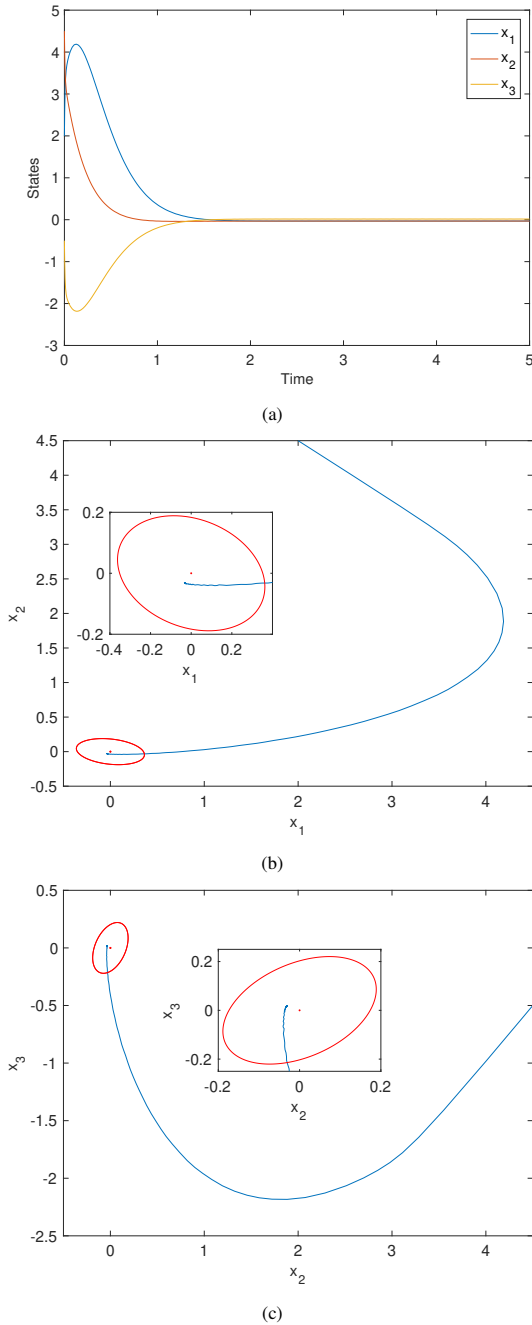


Fig. 3. Example 2: (a) state trajectories; (b) and (c) projection onto the subspaces  $(x_1, x_2)$  and  $(x_2, x_3)$ , respectively, and minimal ellipsoids.

with  $\lambda_1 = 0.98$  and  $\lambda_2 = 0.02$ , resulted in a state-feedback matrix  $K$  with gain values of the order of  $10^6$ .

## V. CONCLUSIONS

This paper presented new LMI conditions for the robust stabilization of continuous-time polytopic linear systems with unknown but bounded perturbations via the AEM. These LMI conditions can be used to derive state-feedback control laws via convex optimization. The proposed control laws ensure robust stabilization and convergence of state trajectories of the system to a minimal size ellipsoidal set

(by the trace criterion). Importantly, the derived stabilization conditions are less conservative compared to previous works (for example compared to [15]) and can simplify the complexity of other approaches presented in the literature (e.g. by employing the elimination Lemma 1). On going work extends the current results to the robust constrained stabilization of discrete-time linear systems [17].

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