Adaptive Neural Backstepping Control of Nonlinear Fractional-Order Systems with Input Quantization

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Abstract This article addresses the tracking control problem of uncertain fractional-order nonlinear systems in the presence of input quantization and external disturbance by combining with radial basis function(RBF) neural networks(NNs), fractional-order disturbance observer(FODO) and backstepping method. The unknown nonlinearities of fractional-order systems is approximated by RBF NNs. The design of hysteretic quantizer achieves quantification of input signal and avoids chattering. The FODO is utilized to evaluate the external disturbance exist in fractional-order systems. According to fractioanl-order Lyapunov stability analysis, the bounds of all the signals in the closed-loop system is proved. The effectiveness of the proposed method is confirmed by the simulation results.

Keywords Adaptive backstepping · Fractional-order system · Radial basis function neural networks · Input quantization · Disturbance observer

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1 Introduction

Fractional-order calculus was proposed more than three hundred years [1], and it still attracts the research interest of a great many researchers thanks to its unique properties and great potentials in many applications [2–7]. In the field of engineering, the models can be established more concisely and precisely by using fractional-order calculus. In recent years, an increasing number of researchers focus on the study of fractional-order systems, especially in stability analysis and controller design [8–13].

As we know, adaptive backstepping has been widely used in the control of integer-order nonlinear systems [14–17]. In recent years, some researchers have extend backstepping method to solve the control problem of fractional-order nonlinear systems [18, 19]. In [18], an adaptive fuzzy backstepping controller was design to deal with a kind of uncertain fractional-order nonlinear system. In [19], adaptive backstepping method was used to solve the control problem of a triangular fractional order systems with non-commensurate orders.

Until now, the quantized control is widely used in linear and nonlinear systems since its theoretical and practical significance in modern engineering. In order to reduce the chattering, a kind of hysteretic quantizer is introduced in [20]. In [21], the author studied a fractional-order nonlinear systems with input quantization, and proposed an output feedback tracking controller. An adaptive backstepping controller with hysteretic quantizer was proposed in [22]. In our paper, the hysteresis quantizer will be further studied in fractional-order systems .

In actual application, systems are often affected by external interference and uncertain parameters, disturbance observer can used to estimate the disturbance and attenuate their effects. There are many integer-order disturbance observer techniques has been reported. Two different design methods of nonlinear integer-order disturbance observer was given to handle disturbance in [23] and [24]. Recently, the studies of fractional-order disturbance observer have received much consideration [25–27]. A class of uncertain fractional-order chaotic systems with unknown disturbance and input saturation is studied in [25]. In [27], a FODO based sliding mode control was studied, the control problem of with matched and mismatched disturbances was solved. In our paper, the FODO is designed to estimate both of disturbance and uncertain parameters for nonlinear fractional-order system.

Inspired by the above discussions, we study the backstepping control problem for fractional-order nonlinear systems with disturbance and input quantization. The main contributions of this paper can be highlighted as the following. Firstly, a hysteresis quantizer is designed to compensate for disturbance and uncertain parameters, which can simplify the design process. Second, the input quantization is considered in the design of fractional order controllers. Third, the system to be concerned with is very common in practice.

The organization of the remaining paper is as follows. The problem formulation and preliminaries are given in Section 2. The FODO and backstepping controller design for the system are presented in Section 3. The simulation results can be found in Section 4. And, we conclude this article in Section 5.

2 Problem Formulation and Preliminaries

2.1 Fractional Calculus

There are several different forms for the definition of fractional derivative, among them, the most used in engineering applications is the Caputo definition. The Caputo definition form we use is as follows.

$$ {}_{0}^{c}\mathcal{D}_{t}^{\alpha} f = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau \tag{1} $$

where $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ denotes the Euler's Gamma function, and $\Gamma(z+1) = z\Gamma(z)$. For the differential order α, we only consider it on the set $(0, 1)$. For simplicity of description, ${}_{0}^{c}\mathcal{D}_{t}^{\alpha}$ is abbreviated as $\mathcal{D}^\alpha$. The Laplace transform of (1) is

$$ \begin{aligned} \mathcal{L}(\mathcal{D}^\alpha f = &\int_{0}^{\infty} e^{-st}\mathcal{D}^\alpha f dt \\ =&s^\alpha F(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0) \end{aligned} \tag{2} $$

where $F(s)$ is the Laplace transform of f.

Definition 1 [28] The Mittag-Leffler function is defined by

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)} \tag{3} $$

where $\alpha > 0$ and $\beta \in R, z \in R$. Its Laplace transform can be given by

$$ (\mathcal{L}t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha))(s) = \frac{s^{\alpha-\beta}}{s^\alpha - \lambda} \tag{4} $$

where $Re\ s > 0, \lambda \in C$ and $|\lambda s^{-\alpha}| < 1$.

Lemma 1 *[28] For the above Mittag-Leffer function, if $\beta > 0$, then*

$$ \int_{0}^{z} E_{\alpha,\beta}(\lambda t^\alpha) t^{\beta-1} dt = z^\beta E_{\alpha,\beta+1}(\lambda z^\alpha). \tag{5} $$

Lemma 2 *[3] Assuming that the origin is an equilibrium point of a nonautonomous fractional-order nonlinear system*

$$ \mathcal{D}^\alpha x = f(t, x) \tag{6} $$

where $f : \mathcal{I} \times \Omega \to \mathcal{R}$ is Lipschitz continuous. If there exist some class-K functions g_1, g_2, g_3 and a Lyapunov function $V(t, h)$ such that

$$\begin{aligned} & g_1(\|x\|) \leq V(t,x) \leq g_2(\|x\|) \\ & \mathcal{D}^{\alpha} V(t,x) \leq -g_3(\|x\|) \end{aligned} \tag{7}$$

then (6) *is asymptotic stability.*

Lemma 3 *[7] There is a smooth function $z \in R^n$, you can get*

$$\frac{1}{2}\mathcal{D}^{\alpha}\left(z^T z\right) \leq z^T \mathcal{D}^{\alpha} z \quad (\forall t \in \mathcal{I}) \tag{8}$$

where $B \in \mathcal{R}^{n \int n}$ is positive definite.

2.2 System Formulation

This paper considered the fractional-order nonlinear system described in the following form .

$$\begin{cases} \mathcal{D}^{\alpha} x_1 = x_2 + f_1(x_1) \\ \mathcal{D}^{\alpha} x_2 = x_3 + f_2(x_1, x_2) \\ \vdots \\ \mathcal{D}^{\alpha} x_{n-1} = x_n + f_{n-1}(x_1, x_2, \ldots, x_{n-1}) \\ \mathcal{D}^{\alpha} x_n = f_n(x_1, x_2, \ldots, x_n) + d(t) + q(u(t)) \\ y = x_1 \end{cases} . \tag{9}$$

where α is the system commensurate order, $x = [x_1, x_2, \ldots, x_n]^T \in R^n$ is the system state vector, and we only consider the case of $0 < \alpha < 1$. $f_i(x_i)(i = 1, 2, \ldots, n)$ are unknown smooth nonlinear functions. $u(t)$ is a designed input, and $q(u(t))$ is the actual quantitative input. $y \in R$ is the output, and $d(t) \in R$ is the external disturbance.

Without loss of generality, we can rewrite the system formulation as

$$\begin{cases} \mathcal{D}^{\alpha} x_1 = x_2 + f_1(\check{x}_1) \\ \mathcal{D}^{\alpha} x_2 = x_3 + f_2(\check{x}_2) \\ \vdots \\ \mathcal{D}^{\alpha} x_{n-1} = x_n + f_{n-1}(\check{x}_{n-1}) \\ \mathcal{D}^{\alpha} x_n = q(u) + d_n \\ y = x_1 \end{cases} . \tag{10}$$

where $\check{x}_i = [x_1, \ldots, x_i](i = 1, 2, \ldots, n)$, and $d_n = f_n(\check{x}_n) + d$. Then, FODO will be designed to estimate both of external interference d and unknown parameter $f_n(\check{x}_n)$.

Assumption 1 *For system* (10), *d_n and its fractional order derivatives are bounded with $|d_n| < \check{d}_n$ and $|D^{\alpha} d_n| \leq \eta$, where $\check{d}_n > 0$ and $\eta > 0$ are unknown positive constants.*

2.3 Quantizer

We use a hysteretic type of quantizer to quantitatively control the input of fractional order systems, which has the following form as in [22] and [29].

$$q(u)=\begin{cases} u_i\,\mathrm{sgn}(u), & \frac{u_i}{1+\delta}<|u|\le u_i, \dot{u}<0,\ \text{or} \\ & u_i<|u|\le \frac{u_i}{1-\delta}, \dot{u}>0 \\ u_i(1+\delta)\,\mathrm{sgn}(u), & u_i<|u|\le \frac{u_i}{1-\delta}, \dot{u}<0,\ \text{or} \\ & \frac{u_i}{1-\delta}<|u|\le \frac{u_i(1+\delta)}{1-\delta}, \dot{u}>0 \\ 0, & 0\le |u|<\frac{u_{\min}}{1+\delta},\quad \dot{u}<0,\ \text{or} \\ & \frac{u_{\min}}{1+\delta}\le |u|\le u_{\min}, \dot{u}>0 \\ q\left(u\left(t^-\right)\right), & \dot{u}=0 \end{cases} \tag{11}$$

where $u_i=\eta^{1-i}u_{min}(i=1,2,\dots),\delta=\frac{1-\eta}{1+\eta}$ with $u_{min}>0, 0<\eta<1$. $q(u)$ is in the set $U=\{0,\pm u_i,\pm u_i(1+\delta)\}$. A picture of the map of $q(u)$ for $u>0$ is shown in Fig.1

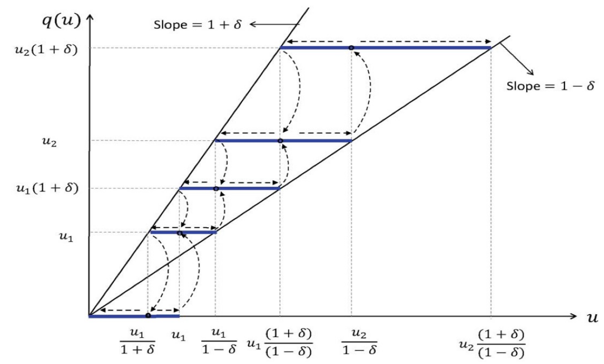

Fig. 1 Map of hysteretic quantizer q(u) for $u>0$.

Remark 1 In Comparison to logarithmic quantizer, hysteretic type of quantizer gets more quantization grades, thereby avoiding chattering. η is reviewed as the measure of the density of quantitative, which determines coarseness of quantizer. Some detailed description can be found in [30].

In order to make the controller design more suitable, the hysteresis type of quantizer will be rewritten as follows [30]

$$q(u)=h(u)u+g \tag{12}$$

where $h(u)$ and g satisfy

$$1-\delta\le h(u)\le 1+\delta, |g|\le u_{min} \tag{13}$$

2.4 RBF NNs

We used the following RBF NNs to appropriate the unknown nonlinear terms.

$$f(\varsigma) = \Phi^T \xi(\varsigma) \tag{14}$$

where $\Phi = [\phi_1, \phi_2, \ldots, \phi_l]^T \in R^l$ is the weight vector, in which $l > 1$ is node number of neural networks . $\varsigma \in \Xi \subset R^q$ is neural networks input vector, and q is input dimension. $\xi(\varsigma) = [\xi_1(\varsigma), \xi_2(\varsigma), \ldots, \xi_l(\varsigma)]^T$ means the radial basis function, which is generally selected as the Gaussian function as follows

$$\xi_i(\varsigma) = \exp\left[-\frac{(\varsigma - \iota_i)^T (\varsigma - \iota_i)}{\kappa^2}\right], i = 1, 2, \ldots, l \tag{15}$$

where $\iota_i = [\iota_{i1}, \iota_{i2}, \ldots, \iota_{iq}]^T$ represents the center of the receptive field and κ denotes the width of the basis function $\xi_i(\varsigma)$. By choosing a large enough number of l, RBF NNs can estimation $f(\varsigma)$ to arbitrary accuracy in the compact set $\Omega_\varsigma \in R^l$ with arbitrary accuracy $\epsilon > 0$ [31]:

$$f(\varsigma) = \Phi^{*T} \xi(\varsigma) + \epsilon(\varsigma), \forall \varsigma \in \Xi \subset R^q \tag{16}$$

Φ^* is ideal constant parameter vector, and the definition is

$$\Phi^* := \arg \min_{\Phi \in R^l} \left\{ \sup_{\varsigma \in \Xi} \left| f(\varsigma) - \Phi^T \xi(\varsigma) \right| \right\} \tag{17}$$

and ϵ denotes the approximation error and satisfies $|\epsilon(\varsigma)| \leq \varepsilon (\varepsilon > 0)$.

3 Main Results

3.1 FODO

In this subsection, we will propose a FODO to evaluate d_n in system (10) as in [23] and [27]. first, we introduce an auxiliary variable

$$\theta = d_n - \sigma x_n \tag{18}$$

where $\sigma > 0$ is a constant to be designed. Then, the Caputo derivative of θ is

$$\begin{aligned} D^\alpha \theta =& D^\alpha d_n - \sigma \mathcal{D}^\alpha x_n \\ =& -\sigma (d_n + q(u)) + \mathcal{D}^\alpha d_n \\ =& -\sigma \theta - \sigma (\sigma x_n + q(u)) + \mathcal{D}^\alpha d_n \end{aligned} \tag{19}$$

The fractional order disturbance observer is suggested as

$$\begin{cases} D^\alpha \hat{\theta} = -\sigma \hat{\theta} - \sigma (\sigma x_n + q(u)) \\ \hat{d_n} = \hat{\theta} + \sigma x_n \end{cases} \tag{20}$$

Define the disturbance estimation error as

$$\tilde{d}_n = d_n - \hat{d}_n \tag{21}$$

In order to verify the feasibility of the above FODO, we choose the Lyapunov function cadidate to perform stability analysis of the interference estimation error.

$$V_d = \frac{1}{2}\tilde{d}_n^2 \tag{22}$$

Based on lemma 3, the Caputo derivative of V_d is

$$D^{\alpha}V_d \leq \tilde{d}_n D^{\alpha}\tilde{d}_n \tag{23}$$

According to (20), (10) and assumption 1, we have

$$\begin{aligned} D^{\alpha}V_d &\leq \tilde{d}_n\left(\mathcal{D}^{\alpha}d_n - \mathcal{D}^{\alpha}\hat{d}_n\right) \\ &= \tilde{d}_n\mathcal{D}^{\alpha}d_n - \sigma\tilde{d}_n^2 \\ &\leq -\sigma\tilde{d}_n^2 + \frac{1}{2}\tilde{d}_n^2 + \frac{1}{2}\eta^2 \\ &= -\left(\sigma - \frac{1}{2}\right)\tilde{d}_n^2 + \frac{1}{2}\eta^2 \\ &= -B_0V_d + B_1 \end{aligned} \tag{24}$$

Where $B_0 = 2\sigma - 1$ and $B_1 = \frac{1}{2}\eta^2$. To ensure the estimated error $\tilde{d}_n$ is bounded, the gian σ should be chosen to make $2\sigma - 1 > 0$. Considering the following design, we choose $\sigma > 1$. So, the estimation yielded by the disturbance observer approaches to the disturbance d_n globally exponentially.

3.2 Controller Design

Define the error variables

$$\begin{cases} e_1 = x_1 - y_d \\ e_i = x_i - \tau_{i-1} \end{cases} \tag{25}$$

Where τ_i are virtual controllers.

Then, apply the RBF NNs to approximate nonlinear function. We get the approximate of unknown smooth function $f_i(x)(i = 1, 2, \ldots, n-1)$ as follows:

$$\hat{f}\left(\check{x}_i\right) = \hat{\Phi}_i^T\xi_i\left(\check{x}_i\right) \tag{26}$$

Define $\tilde{\Phi}_i = \Phi_i^* - \hat{\Phi}_i$, where $\hat{\Phi}_i^*$ is the optimal parameter vector. As far as we know, the derivative of parameter $\hat{\Phi}_i^*$ is zero, we can obtain

$$\mathcal{D}^{\alpha}\tilde{\Phi}_i = \mathcal{D}^{\alpha}\Phi_i^* - \mathcal{D}^{\alpha}\hat{\Phi}_i = -\mathcal{D}^{\alpha}\hat{\Phi}_i \tag{27}$$

The RBF NNs error is defined by

$$\epsilon_i\left(\check{x}_i\right)=f_i\left(\check{x}_i\right)-\hat{f}_i\left(\check{x}_i, \Phi_i^*\right) \tag{28}$$

where $\epsilon_i\left(\check{x}_i\right) \leq \varepsilon_i\ (\varepsilon_i>0)$. Then, we can get

$$\begin{aligned}
& f_i\left(\check{x}_i\right)-\hat{f}_i\left(\check{x}_i, \Phi_i\right) \\
= & f_i\left(\check{x}_i\right)-\hat{f}_i\left(\check{x}_i, \Phi_i^*\right)+\hat{f}_i\left(\check{x}_i, \Phi_i^*\right)-\hat{f}_i\left(\check{x}_i, \Phi_i\right) \\
= & \Phi_i^{*T} \xi_i\left(\check{x}_i\right)+\epsilon_i\left(\check{x}_i\right)-\hat{\Phi}_i^T \xi_i\left(\check{x}_i\right) \\
= & \tilde{\Phi}_i^T \xi_i\left(\check{x}_i\right)+\epsilon_i\left(\check{x}_i\right)
\end{aligned} \tag{29}$$

The design process of fractional-order backstepping controller will be introduced as the following steps.

Step 1: The Caputo derivative of e_1 is

$$\begin{aligned}
D^\alpha e_1 & =D^\alpha x_1-D^\alpha y_d \\
& =x_2+f_1\left(\check{x}_1\right)-D^\alpha y_d \\
& =x_2+f_1\left(x_1\right)-\hat{f}_1\left(x_1\right)+\hat{f}_1\left(x_1\right)-D^\alpha y_d \\
& =e_2+\tau_1+\tilde{\Phi}_1^T \xi_1\left(\check{x}_1\right)+\epsilon_1\left(\check{x}_1\right)+\hat{\Phi}_1^T \xi_1\left(\check{x}_1\right)-D^\alpha y_d
\end{aligned} \tag{30}$$

The Layapunov function is chosen as

$$V_1=\frac{1}{2} e_1^2+\frac{1}{2 \gamma_1} \tilde{\Phi}_1^T \tilde{\Phi}_1 \tag{31}$$

Let the virtual control law τ_1 be

$$\tau_1=-\left(c_1+\frac{1}{2}\right) e_1-\hat{\Phi}_1^T \xi_1(\check{x}_1)+D^\alpha y_d \tag{32}$$

and the fractional-order adaptation law be

$$\mathcal{D}^\alpha \hat{\Phi}_1=\gamma_1 e_1 \xi_1(\check{x}_1)-\rho_1 \hat{\Phi}_1 \tag{33}$$

where $c_1>0$, $\gamma_1>0$ and $\rho_1>0$ are designed parameters.

By applying Lemma 3, (32) , (33), and Young's inequality, the Caputo derivative of V_1 is such that

$$\begin{aligned}
\mathcal{D}^\alpha V_1 \leq & e_1 \mathcal{D}^\alpha e_1+\frac{1}{\gamma_1} \tilde{\Phi}_1^T \mathcal{D}^\alpha \tilde{\Phi}_1 \\
= & e_1\Big(e_2+\tau_1+\tilde{\Phi}_1^T \xi_1\left(\check{x}_1\right)+\epsilon_1\left(\check{x}_1\right) \\
& +\hat{\Phi}_1^T \xi_1\left(\check{x}_1\right)-D^\alpha y_d\Big)-\frac{1}{\gamma_1} \tilde{\Phi}_1 \mathcal{D}^\alpha \hat{\Phi}_1 \\
\leq & -c_1 e_1^2-\frac{1}{2} e_1^2+e_1 e_2+\frac{\rho_1}{\gamma_1} \tilde{\Phi}_1^T\left(\Phi_1^*-\tilde{\Phi}_1\right)+e_1 \varepsilon_1 \\
\leq & -c_1 e_1^2+e_1 e_2+\frac{1}{2} \varepsilon_1^2+\frac{\rho_1}{2 \gamma_1} \Phi_1^{*T} \Phi_1^*-\frac{\rho_1}{2 \gamma_1} \tilde{\Phi}_1^T \tilde{\Phi}_1
\end{aligned} \tag{34}$$

Step $i(2 \leq i \leq n-1)$: The Caputo derivative of e_i is

$$\begin{aligned} D^{\alpha} e_i =& D^{\alpha} x_i - D^{\alpha} \tau_{i-1} \\ =& x_{i+1} + f_i\left(\check{x}_i\right) - D^2 \tau_{i-1} \\ =& x_{i+1} + f_i\left(x_i\right) - \hat{f}_i\left(x_i\right) + \hat{f}_i\left(x_i\right) - D^{\alpha} \tau_{i-1} \\ =& x_{i+1} + \tau_i + \tilde{\Phi}_i^T \xi_i\left(\check{x}_i\right) + \epsilon_1\left(\check{x}_1\right) \\ &+ \hat{\Phi}_1^T \xi_1\left(\check{x}_1\right) - D^{\alpha} \tau_{i-1} \end{aligned} \tag{35}$$

The Layapunov function is chosen as

$$V_i = V_{i-1} + \frac{1}{2} e_i^2 + \frac{1}{2\gamma_i} \tilde{\Phi}_i^T \tilde{\Phi}_i \tag{36}$$

Let the virtual control law τ_i be

$$\tau_i = -\left(c_i + \frac{1}{2}\right) e_i - \hat{\Phi}_i^T \xi_i(\check{x}_i) - e_{i-1} + D^{\alpha} \tau_{i-1} \tag{37}$$

and the fractional-order adaptation law be

$$D^{\alpha} \Phi_i = \gamma_i e_i \xi_i(\check{x}_i) - \rho_i \hat{\Phi}_i \tag{38}$$

where $\gamma_i > 0, \rho_i > 0, c_i > 0$ are designed parameters.

By applying Lemma 3, (37) , (38), and Young's inequality, the Caputo derivative of V_i can be described as

$$\begin{aligned} D^{\alpha} V_i \leq& \mathcal{D}^{\alpha} V_{i-1} + e_i \mathcal{D}^{\alpha} e_i + \frac{1}{\gamma_i} \tilde{\Phi}_i^T \mathcal{D}^{\alpha} \tilde{\Phi}_i \\ =& \mathcal{D}^{\alpha} V_{i-1} + e_i \Big(e_{i+1} + \tau_i + \tilde{\Phi}_i^T \xi_i\left(\check{x}_i\right) + \epsilon_i\left(\check{x}_i\right) \\ &+ \hat{\Phi}_1^T \xi_1\left(\check{x}_1\right) - D^{\alpha} \tau_{i-1} \Big) - \frac{1}{\gamma_i} \tilde{\Phi}_i^T \mathcal{D}^{\alpha} \hat{\Phi}_i \\ \leq& \mathcal{D}^{\alpha} V_{i-1} - c_i e_i^2 - \frac{1}{2} e_i^2 + e_i e_{i+1} \\ &+ \frac{\rho_k}{\gamma_k} \tilde{\Phi}_i \left(\Phi_i^* - \tilde{\Phi}_i^T \right) + e_i \varepsilon_i \\ \leq& -\sum_{k=1}^{i} c_k e_k^2 + \sum_{k=1}^{i} \frac{1}{2} \varepsilon_k^2 + e_i e_{i+1} \\ &+ \sum_{k=1}^{i} \frac{\rho_k}{2\gamma_k} {\Phi_k^*}^T \Phi_k^* - \sum_{k=1}^{i} \frac{\rho_k}{2\gamma_k} \tilde{\Phi}_k^T \tilde{\Phi}_k \end{aligned} \tag{39}$$

Step n: The Caputo derivative of e_n is

$$\begin{aligned} \mathcal{D}^{\alpha} e_n =& \mathcal{D}^{\alpha} \check{x}_n - \mathcal{D}^{\alpha} \tau_{n-1} \\ =& q(u) + d_n - D^{\alpha} \tau_{n-1} \\ =& h(u) u + g + d_n - D^{\alpha} \tau_{n-1} \end{aligned} \tag{40}$$

The Layapunov function is chosen as

$$\begin{aligned} V_n =& V_{n-1} + \frac{1}{2}e_n^2 + \frac{1}{2}\tilde{d}_n^2 \\ =& \sum_{k=1}^{n} \frac{1}{2}e_k^2 - \sum_{k=1}^{n-1} \frac{\rho_k}{2\gamma_k}\tilde{\Phi}_k^T\tilde{\Phi}_k + \frac{1}{2}\tilde{d}_n^2 \end{aligned} \tag{41}$$

The disturbance observer is designed as

$$\begin{cases} D^\alpha\hat{\theta} = -\sigma\hat{\theta} - \sigma\left(\sigma x_n + q(u)\right) \\ \hat{d}_n = \hat{\theta} + \sigma x_n \end{cases} \tag{42}$$

where $c_n > 0$ is designed parameter.

Let the input control law u be

$$u = \frac{1}{1-\delta}\Big(-\left(c_n+1\right)e_n - e_{n-1} - \hat{d}_n + D^\alpha\tau_{n-1}\Big) \tag{43}$$

where $\sigma > 0$ is designed parameter.

According to (13), we have

$$h(u)u \leq -\left(c_n+1\right)e_n - e_{n-1} - \hat{d}_n + D^\alpha\tau_{n-1} \tag{44}$$

By applying Lemma 3, (42) , (43), and Young's inequality, the Caputo derivative V_n can be described as

$$\begin{aligned} \mathcal{D}^\alpha V_n \leq& \mathcal{D}^\alpha V_{n-1} + e_n\mathcal{D}^\alpha e_n + \tilde{d}_n D^\alpha\tilde{d}_n \\ =& \mathcal{D}^\alpha V_{n-1} + e_n\left(h(u)u + g + d_n - D^\alpha\tau_{n-1}\right) \\ &+ \tilde{d}_n D^\alpha d_n - \tilde{d}_n D^\alpha\hat{d}_n \\ \leq& \mathcal{D}^\alpha V_{n-1} - c_n e_n^2 - e_n^2 - e_{n-1}e_n \\ &+ e_n\tilde{d}_n + e_n g(u) + \tilde{d}_n\eta - \sigma\tilde{d}_n^2 \\ \leq& -\sum_{k=1}^{n} c_k e_k^2 - \sum_{k=1}^{n-1}\frac{\rho_k}{2\gamma_k}\tilde{\Phi}_k^T\tilde{\Phi}_k - \left(\sigma-1\right)\tilde{d}_n^2 \\ &+ \sum_{k=1}^{n-1}\frac{\rho_k}{2\gamma_k}{\Phi_k^*}^T\Phi_k^* + \sum_{k=1}^{n-1}\frac{1}{2}\varepsilon_k^2 + \frac{1}{2}\eta^2 + \frac{1}{2}u_{min}^2 \end{aligned} \tag{45}$$

We can rewrite (45) as

$$\mathcal{D}^\alpha V_n \leq -cV_n + k \tag{46}$$

in which $c = \min\left\{2c_i, \rho_i, 2\left(\sigma-1\right), i = 1, 2, \cdots, n\right\}$ and $k = \sum_{k=1}^{n-1}\frac{\rho_k}{2\gamma_k}{\Phi_k^*}^T\Phi_k^* + \sum_{k=1}^{n-1}\frac{1}{2}\varepsilon_k^2 + \frac{1}{2}\eta^2 + \frac{1}{2}u_{min}^2$ are two positive constants.

Theorem 1 *Consider fractional order system 10 under Assumption 1, and the design of disturbance observer 42, the controller 43, the virtual controller 32, 37, and adaptive law 33 and 38, then all the signals in the closed-loop system remain semiglobally uniformly bounded, and the tracking error eventually converge to an arbitrary small region.*

Proof According to (45), there exists a nonnegative function m such that

$$\mathcal{D}V_n = -cV_n + k + m \tag{47}$$

Taking Laplace transform on (47), we obtain

$$V_n(s) = \frac{s^{\alpha-2}}{s^\alpha + a}V_n(0) + \frac{\mathcal{L}(m+k)}{s^\alpha + a} \tag{48}$$

where $V_n(0)$ is initial condition. Using definition 1, we can solve (48) as follows

$$\begin{aligned} V_n =& V_n(0)E_{\alpha,1}(-at^\alpha) \\ &+ \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-a(t-\tau)^\alpha)(m(\tau)+b)d\tau \end{aligned} \tag{49}$$

which yields that

$$\begin{aligned} ||V_n|| \leq & ||V_n(0)||E_{\alpha,1}(-at^\alpha) \\ &+ b\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-a(t-\tau)^\alpha)d\tau \end{aligned} \tag{50}$$

Using Lemma5, we can obtain

$$||V_n|| \leq ||V_n(0)||E_{\alpha,1}(-at^\alpha) + bt^\alpha E_{\alpha,\alpha+1}(-a\tau^\alpha) \tag{51}$$

According to [13], we can get that there is a constant $t_0 > 0$ for all $t \in (t_0, \infty)$

$$||V_n|| \leq \frac{2k}{c}. \tag{52}$$

Therefore, from (52) and V_n, we can conclude that all signals in system (10) keep bounded and tracking error e gradually approach arbitrary small range.

4 Simulation results

In this paper, a fractional-order backstepping controller design method with FODO and RBF NNs is proposed. In this section, we use a fractional-order nonlinear system to verigy the effectiveness of the design method.

$$\begin{cases} \mathcal{D}^\alpha x_1 = x_2 - 0.6x_1^2 \\ \mathcal{D}^\alpha x_2 = q(u) + \frac{x_2 - x_1^2}{1+x_1^2} \end{cases} \tag{53}$$

Let the fractional-order be $\alpha = 0.8$, and the initial condition be $x(0) = [0.3, 0.5]^T$. The design parameters are chosen as $\sigma = 30$, $\rho_1 = 1$, $c_1 = 20, c_2 = 30$, $\delta = 0.2$ and $u_{min} = 0.02$.

Assume there exist an external disturbances $d(t) = 0.5(1+sin(t))$ in system (53), so we can rewritten (53) as system (10). So we can get $d_n = \frac{x_2 - x_1^2}{1+x_1^2} + 0.5(1 + sin(t))$.

On the corresponding compact sets, select the certers and widths of RBF NNs on a regular lattice. In this simulation, a three-layer neural network with five nodes per layer is used, and the ceter of each of them spaced evenly in the interval $[-5, 5]$. Therefore, neural network $\Phi_1^T \xi_1$ contains 125 nodes and the widths of Gaussian functions equal to 2. The simulation results are shown in Figs.2-7.

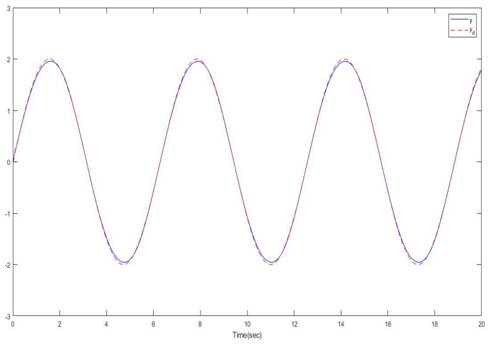

Fig. 2 The trajectory of $y(t), y_d(t)$

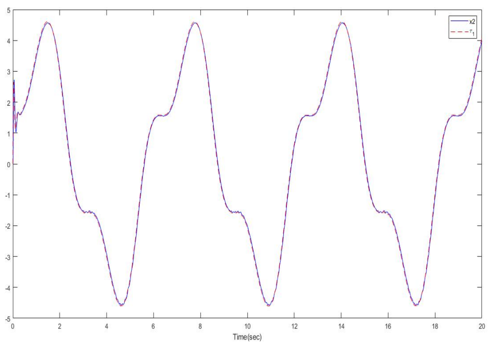

Fig. 3 The trajectory of $x_2(t), \tau_1(t)$

Figs.2 displays output signal y and desired reference signal y_d. Figs.3 shows the state $x2$ and the virtual control law τ_1. Fig.4 shows that the quantized input signal is bounded. From Fig.5 and Fig.6, we can see that the disturbance estimation error is bounded. Therefore, the FODO designed above can estimate the unknown disturbance and uncertain parameters well. The tracking error of e is shown in Fig.7. It is concluded that the simulation system 53 is bounded synchronization under the designed FODO based fractional-order

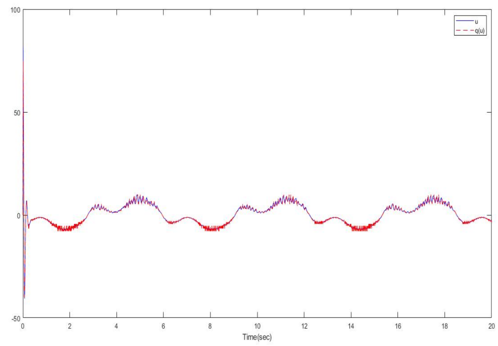

Fig. 4 The input $u(t)$ and hysteretic quantized input $q(u(t))$

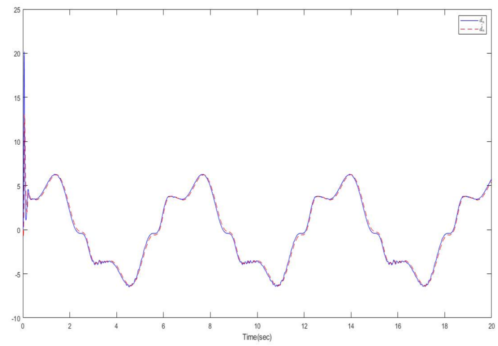

Fig. 5 The disturbance $d_n(t)$ and the approximation output of $\hat{d}_n(t)$

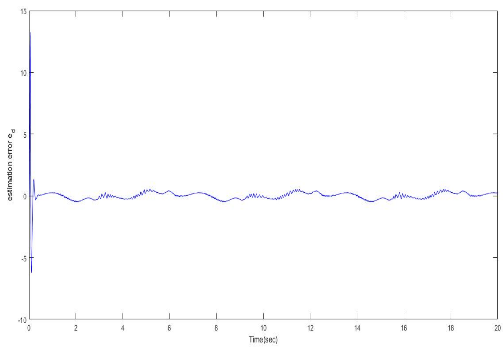

Fig. 6 The disturbance observation error $\tilde{d}_n(t)$

backstepping controller. Therefore, the proposed adaptive neural backstepping

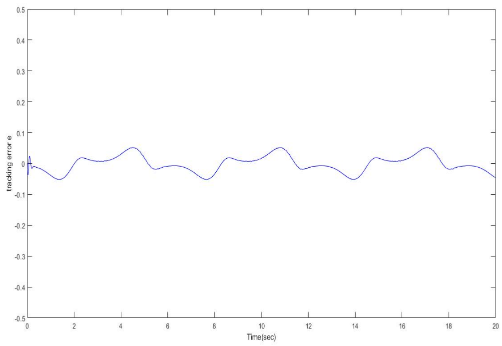

Fig. 7 The tracking error e

controller for fractional-order systems with disturbance and input quantization is effective.

5 conclusion

In this paper, the adaptive neural backstepping control scheme has been studied for fractional-order nonlinear systems with disturbance and quantized input. RBF NNs are used to approximate the unknown nonlinear smooth functions. A hysteretic type of quantizer was designed to evaluate the unknown disturbances and uncertain parameters. The RBF NNs, disturbance estimation and hysteresis quantization were used to construct backstepping control law. Under the designed controller, it has been proved that all the signals in the systems are bounded and convergent. Simulation results demonstrate the effectiveness of the proposed adaptive backstepping control scheme.

Conflict of interest

This work was supported by National Natural Science Foundation of China (Grant number 51405013).

The authors declare that they have no conflict of interest.

Data availability statement

The datasets generated during the current study are available from the corresponding author on reasonable request.

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