

Thin film limits for Ginzburg–Landau with strong applied magnetic fields

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Abstract. In this work, we study thin-film limits of the full three-dimensional Ginzburg–Landau model for a superconductor in an applied magnetic field oriented obliquely to the film surface. We obtain Γ –convergence results in several regimes, determined by the asymptotic ratio between the magnitude of the parallel applied magnetic field and the thickness of the film. Depending on the regime, we show that there may be a decrease in the density of Cooper pairs. We also show that in the case of variable thickness of the film, its geometry will affect the effective applied magnetic field, thus influencing the position of vortices.

Keywords: Ginzburg–Landau; thin-films; superconductivity.

Mathematics Subject Classification 2000: 35J50, 49J40

1. Introduction

In this paper we consider superconducting thin films subjected to an external magnetic field, using the Ginzburg–Landau model. We assume the superconductor occupies a domain $\Omega_\varepsilon \subset \mathbb{R}^3$ of variable but small thickness, which projects to a smooth planar domain $\omega \subset \mathbb{R}^2$,

$$x = (x', x_3) \in \Omega_\varepsilon \iff x' \subset \omega, \varepsilon f(x') < x_3 < \varepsilon g(x'),$$

for given smooth functions $f, g : \omega \rightarrow \mathbb{R}$ with $\inf_\omega (g - f) > 0$. Here, and throughout, we denote the projection of $x \in \mathbb{R}^3$ to the plane by $x' = (x_1, x_2) \in \mathbb{R}^2$. The state of the superconductor is described by a complex-valued order parameter, $\mathbf{u} : \Omega_\varepsilon \rightarrow \mathbb{C}$ defined inside the sample, and the magnetic vector potential $\mathbf{A} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, which determines the magnetic field $\mathbf{h} = \nabla \times \mathbf{A}$. We assume that the superconductor is placed in a constant magnitude, externally applied magnetic field $\mathbf{h}_\varepsilon^{\text{ex}}$, which may be oriented *obliquely* with respect to the plane of ω . With these choices, the Ginzburg–Landau energy functional is given by

$$\mathbf{I}_{\kappa,\varepsilon}(\mathbf{u}, \mathbf{A}) := \frac{1}{\varepsilon} \left(\frac{1}{2} \int_{\Omega_\varepsilon} \left(|\nabla_A \mathbf{u}|^2 + \frac{\kappa^2}{2} (1 - |\mathbf{u}|^2)^2 \right) dx + \frac{1}{2} \int_{\mathbb{R}^3} |\mathbf{h} - \mathbf{h}_\varepsilon^{\text{ex}}|^2 dx \right),$$

We note that the factor $1/\varepsilon$ which multiplies the energy is not traditionally present, but is useful here since the energy of minimizers will be order-one with this normalization.

Motivated by recent work on the Lawrence–Doniach model ([ABS08], [ABS]) we are particularly interested in the behavior of the thin film superconductor in applied fields which are *parallel* (or nearly parallel) to the plane of ω . In order to see the effect of strong parallel fields, we allow the parallel component of the applied field $\mathbf{h}_\varepsilon^{\text{ex}} (\in \mathbb{R}^2)$ to depend on the thickness parameter ε ,

$$\mathbf{h}_\varepsilon^{\text{ex}} = (\rho_\varepsilon h^{\text{ex}'}, h_3^{\text{ex}}), \tag{1.1}$$

We will identify different Γ –limits, in the sense of De Giorgi (see [DG75,GF75,DM93,Bra02]), depending on the magnitude of ρ_ε . The limiting behavior of minimizers of $\mathbf{I}_{\kappa,\varepsilon}$ with applied fields of fixed magnitude

($\rho_\varepsilon = 1$) was studied by Chapman, Du & Gunzburger [CDG96]. By means of an asymptotic expansion using the Euler–Lagrange equations and estimates on the minimum energy they show that the vertical averages of the order parameters \mathbf{u}_ε and potentials \mathbf{A}_ε converge (weakly in H^1) to a solution of a simplified two-dimensional Ginzburg–Landau model, in which the limiting vector potential produces the vertical component \mathbf{h}_3^{ex} of the applied field. Our results (below) reproduce this outcome as part of a more general Γ –convergence setting, in the appropriate (“subcritical”) regime. The critical case, $\rho_\varepsilon = O(\varepsilon^{-1})$, and supercritical cases produce very different and interesting results, which we will describe below.

In preparing this manuscript we have learned of very recent work by Contreras & Sternberg [CS] on Γ –limits for thin film superconductors, but with a very different point of view. They consider thin shells based on fixed closed manifolds in \mathbb{R}^3 , with magnetic fields independent of ε .

To identify the correct scales in the problem, we introduce the following rescaled coordinates:

$$\begin{aligned} x = (x', x_3) &= (x_1, x_2, x_3) = \left(\mathbf{x}_1, \mathbf{x}_2, \frac{x_3}{\varepsilon} \right) \\ A(x) &= (\mathbf{A}_1, \mathbf{A}_2, \varepsilon \mathbf{A}_3) \left(\mathbf{x}_1, \mathbf{x}_2, \frac{x_3}{\varepsilon} \right), \\ u(x) &= \mathbf{u}(\mathbf{x}). \end{aligned}$$

In the new coordinates, the magnetic field $h = \nabla \times A$ transforms in a straightforward way,

$$\mathbf{h} = \nabla \times \mathbf{A} = \left(\frac{1}{\varepsilon}(\partial_2 A_3 - \partial_3 A_2), \frac{1}{\varepsilon}(\partial_3 A_1 - \partial_1 A_3), (\partial_1 A_2 - \partial_2 A_1) \right) = \left(\frac{1}{\varepsilon} h', h_3 \right),$$

and similarly for $\mathbf{h}^{\text{ex}} = (\frac{1}{\varepsilon} h^{\text{ex}'}, h_3^{\text{ex}})$. Note also that the divergence free condition $\nabla \cdot h = 0$ is preserved under this rescaling.

Denote the rescaled domain

$$\Omega := \Omega_1 = \{(x', x_3) \in \mathbb{R}^3 : f(x') < x_3 < g(x'), \quad x' \in \omega\}.$$

Then, the Ginzburg–Landau energy becomes:

$$\begin{aligned} I_{\kappa, \varepsilon}(u, A) &:= \frac{1}{2} \int_{\Omega} \left(|\nabla' - iA'|u|^2 + \left| \frac{1}{\varepsilon}(\partial_3 - iA_3)u \right|^2 + \frac{\kappa^2}{2}(1 - |u|^2)^2 \right) dx \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^3} \left(|h_3 - h_3^{\text{ex}}|^2 + \frac{1}{\varepsilon^2} |h' - \varepsilon \rho_\varepsilon h^{\text{ex}'}|^2 \right) dx. \quad (1.2) \end{aligned}$$

In keeping with our notation above, $\nabla' = (\partial_1, \partial_2)$.

We must also define function spaces for our configurations (u, A) . This is complicated both by the fact that A is defined in the whole space \mathbb{R}^3 and the gauge invariance of the energy. The natural space for the order parameter is $u \in H^1(\Omega; \mathbb{C})$. To define a space for the vector potential A we must essentially fix an appropriate gauge, which also captures the behavior of the field at infinity. First, we fix a representative for the constant effective external field, $(\varepsilon \rho_\varepsilon h^{\text{ex}'}, h_3^{\text{ex}})$,

$$A_\varepsilon^{\text{ex}} = \frac{1}{2}(\varepsilon \rho_\varepsilon h^{\text{ex}'}, h_3^{\text{ex}}) \times (x_1, x_2, x_3) = \frac{1}{2}(\varepsilon \rho_\varepsilon h_2^{\text{ex}} x_3 - h_3^{\text{ex}} x_2, h_3^{\text{ex}} x_1 - \varepsilon \rho_\varepsilon h_1^{\text{ex}} x_3, \varepsilon \rho_\varepsilon (h_1^{\text{ex}} x_2 - h_2^{\text{ex}} x_1)). \quad (1.3)$$

Then, we assume $A - A_\varepsilon^{\text{ex}} \in \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3)$, defined as the completion of the space of smooth, compactly supported, divergence free vector fields $C_0^\infty(\mathbb{R}^3; \mathbb{R}^3)$, in the Dirichlet norm, $\|F\|_{\check{H}_{\text{div}}^1} = [\int_{\mathbb{R}^n} |DF|^2 dx]^{1/2}$. (See Giorgi & Phillips [GP99].)

With the energy of the form (1.2), we may now identify the different limiting regimes as $\varepsilon \rightarrow 0$. We identify the *subcritical regime* with $\varepsilon \rho_\varepsilon \rightarrow 0$, the *critical regime* corresponds to $\varepsilon \rho_\varepsilon \rightarrow L \neq 0$, and $\varepsilon \rho_\varepsilon \rightarrow \infty$

in the *supercritical regime*. We prove a Γ -convergence result for each regime: Assume $\varepsilon_n \rightarrow 0^+$ is any sequence, and (u_n, A_n) with $u \in H^1(\Omega_1; \mathbb{C})$ and $A_n - A_{\varepsilon_n}^{\text{ex}} \in \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3)$ is a sequence with bounded energy $\sup_n I_{\kappa, \varepsilon_n}(u_n, A_n) < \infty$.

The critical regime

By adjusting the constant values of $h^{\text{ex}'}$, we may simplify our condition to $\varepsilon \rho_\varepsilon \rightarrow 1$, and neglect the ε dependence of A^{ex} . This is the most interesting case, as it leads to two new phenomena in the limiting energy.

First, we obtain a compactness result: there exists $v \in H^1(\omega; \mathbb{C})$ and $b \in L^2(\omega; \mathbb{C})$ so that

$$\begin{aligned} u_n \rightharpoonup u &= v(x') \exp\left(i \int_0^{x_3} A_3^{\text{ex}}(t) dt\right) \quad \text{in } H^1(\Omega_1; \mathbb{C}) \\ A_n - A^{\text{ex}} &\rightharpoonup 0 \quad \text{in } \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3) \\ \frac{1}{\varepsilon d(x')} \int_{f(x')}^{g(x')} (\partial_3 - iA_{n3})u_n dx_3 &\rightharpoonup b(x') \quad \text{in } H^1(\omega; \mathbb{C}). \end{aligned}$$

Here $d(x') := g(x') - f(x')$, the rescaled thickness of the film. We observe that the limit $u(x)$ is gauge-equivalent to a function $v(x')$ defined in the 2D domain ω .

The functionals $I_{\kappa, \varepsilon}$ Γ -converge to the two-dimensional Ginzburg–Landau functional,

$$I_{\kappa, 0}(v, b) = \frac{1}{2} \int_{\omega} d(x') \left(|\nabla' - iB^{\text{ex}'} v|^2 + |b|^2 + \frac{d^2(x')}{12} |h^{\text{ex}'}|^2 |v|^2 + \frac{\kappa^2}{2} (1 - |v|^2)^2 \right) dx',$$

with fixed magnetic vector potential

$$B^{\text{ex}'} := \frac{h_3^{\text{ex}}}{2} (-x_2, x_1) - \left(\frac{f+g}{2} \right) (-h_2^{\text{ex}}, h_1^{\text{ex}}). \quad (1.4)$$

The quantity b measures the deviation of the gauge-invariant derivative of u_n in the vertical direction, and plays the role of the ‘‘Cossarat vectors’’ in limits of elastic membranes (see [BFM03],[FFL07],[GaSM].)

We note two features of the limiting energy. First, we may recomplete the square in the potential term,

$$\frac{d^2(x')}{12} |h^{\text{ex}'}|^2 |v|^2 + \frac{\kappa^2}{2} (1 - |v|^2)^2 = \frac{\kappa^2}{2} \left(\left[1 - \frac{d^2(x') |h^{\text{ex}'}|^2}{12\kappa^2} \right] - |v|^2 \right)^2 + \left[\left(1 - \frac{d^2(x') |h^{\text{ex}'}|^2}{12\kappa^2} \right)^2 - 1 \right]. \quad (1.5)$$

Thus, the presence of a strong (order $\rho_\varepsilon \sim \varepsilon^{-1}$) parallel applied field reduces the density of superconducting electrons in the sample, even in the absence of a perpendicular applied field component. Assume for simplicity that the sample has uniform thickness, $d(x') = 1$. Then, a simple application of the maximum principle shows that any solution of the Euler–Lagrange equations corresponding to the energy $I_{\kappa, 0}$ must satisfy

$$|v| \leq \arg \min_{\rho > 0} \frac{1}{12} |h^{\text{ex}'}|^2 \rho^2 + \frac{\kappa^2}{2} (1 - \rho^2)^2 = \arg \min_{\rho > 0} \left[\rho^2 - \left(1 - \frac{|h^{\text{ex}'}|^2}{12\kappa^2} \right) \right]^2 = \begin{cases} 0 & \text{if } |h^{\text{ex}'}|^2 \geq 12\kappa^2 \\ \sqrt{1 - \frac{|h^{\text{ex}'}|^2}{12\kappa^2}} & \text{if } |h^{\text{ex}'}|^2 < 12\kappa^2 \end{cases}$$

In particular, we conclude that the normal state $v \equiv 0$ is the only solution to the Euler–Lagrange equations for $I_{\kappa, 0}$ with $h^{\text{ex}'} \geq \sqrt{12}\kappa$, that is $\mathbf{h}^{\text{ex}'} \gtrsim \frac{\sqrt{12}\kappa}{\varepsilon}$ in the original coordinates.

The second curious consequence in the critical case is the effect of the potential $B^{\text{ex}'}$. For films which are appropriately bent (so that $\nabla'(f+g) \neq 0$), the deflection of the film’s vertical center essentially converts the horizontal component of the applied field to the vertical, creating a spatially dependent effective field. Thus, even in the absence of a perpendicular applied field component ($h_3^{\text{ex}} = 0$) we may observe (vertical) vortices

in the thin film limit. For very special domain shapes and applied field strengths, we may even observe vortex concentration on curves in the limit $\kappa \rightarrow \infty$, as has been studied by Alama, Bronsard, & Millot [ABM]. We present some illustrative examples in section 2. The proof of the compactness and Γ -convergence results will be presented in section 3.

We note that a similar phenomenon, whereby inhomogeneities in a thin domain lead to a curious dependence on the direction of an applied field, has been observed by Shieh [Shi08] in the context of thin three-dimensional domains which shrink as $\varepsilon \rightarrow 0$ to closed space curves. Shieh also considers Γ -limits with applied fields on the order of ε^{-1} . The limiting functional is supported on a closed loop, and it contains a new potential term determined by all three components of the applied field and the geometry of the underlying curve.

The subcritical regime

The subcritical regime, $\varepsilon\rho_\varepsilon \rightarrow 0$, subdivides in two cases. When $\rho_\varepsilon \rightarrow \rho < \infty$, we obtain Γ -convergence results along the lines of the model derived in [CDG96]. In this case, the magnetic field converges (weakly) to $(0, 0, h_3^{\text{ex}})$, and through a ‘‘Cosserat vector’’ $c = (c_1, c_2)$, we recover the deviation of the parallel magnetic field, $h \approx (\varepsilon c_1, \varepsilon c_2, h_3^{\text{ex}})$. We note that these vectors depend on all three spatial variables, they retain some of the effect of the actual thickness of the film on the deviation of the magnetic field from the vertical, inside and nearby the sample. The resulting Γ -limit is the two dimensional Ginzburg–Landau functional

$$I_{\kappa,-}^\rho(u, b, c) = \frac{1}{2} \int_\omega d(x') \left(|(\nabla' - iA_\perp^{\text{ex}'})u|^2 + |b|^2 + \frac{\kappa^2}{2}(1 - |u|^2)^2 \right) dx' + \frac{1}{2} \int_{\mathbb{R}^3} |(c_1, c_2) - \rho h^{\text{ex}'}|^2 dx,$$

with fixed magnetic potential $A_\perp^{\text{ex}} = h_3^{\text{ex}}(-\frac{x_2}{2}, \frac{x_1}{2}, 0)$.

In the case when $\rho_\varepsilon \rightarrow \infty$, the magnetic field also converges (weakly) to $(0, 0, h_3^{\text{ex}})$, but its parallel deviation is of higher order: $h \approx (\varepsilon\rho_\varepsilon c_1, \varepsilon\rho_\varepsilon c_2, h_3^{\text{ex}})$, but it doesn’t contribute to the energy. In this case, the functionals $I_{\kappa,\varepsilon}$ Γ -converge to the Ginzburg–Landau functional

$$I_{\kappa,-}^\infty(u, b) = \frac{1}{2} \int_\omega d(x') \left(|(\nabla' - iA_\perp^{\text{ex}'})u|^2 + |b|^2 + \frac{\kappa^2}{2}(1 - |u|^2)^2 \right) dx'.$$

Notice that when the external magnetic field is only applied parallel to the limiting plane ($h_3^{\text{ex}} = 0$) we recover the simple functional of Bethuel, Brezis, & Hélein [BBH94], but with natural (Neumann) boundary conditions. A precise statement of the compactness and convergence results is in section 4.

The case $\rho_\varepsilon \rightarrow \rho < \infty$ leads to an interesting auxiliary question about divergence-free vector fields: given the first two components $v' = (v_1, v_2) \in L^2(\mathbb{R}^3; \mathbb{R}^2)$ of a vector field on \mathbb{R}^3 , can it be completed as a divergence-free vector field $v \in L^2(\mathbb{R}^3; \mathbb{R}^3)$? It turns out that the answer is no, and we provide an example of a smooth compactly supported v' which may not be completed to a divergence-free L^2 vector field. Fortunately, to construct our upper bounds in the subcritical regime we do not require such a strong result: it suffices that v' be obtained as a weak limit of divergence-free L^2 vector fields, while allowing some unboundedness in the third component. In section 4.3 we show that any $v' = (v_1, v_2) \in L^2(\mathbb{R}^3; \mathbb{R}^2)$ may be obtained in this way.

The supercritical regime

In the supercritical regime, $\varepsilon\rho_\varepsilon \rightarrow \infty$, the Γ -limit is trivial:

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} I_{\kappa,\varepsilon}(u_\varepsilon, A_\varepsilon) = \begin{cases} \frac{\kappa^2}{4} |\Omega| & \text{if } u \equiv 0 \text{ and } h = h^{\text{ex}'} \\ \infty & \text{otherwise.} \end{cases}$$

This is consistent with the critical case, as taking $\varepsilon\rho_\varepsilon \rightarrow L \gg 1$ is equivalent to multiplying $h^{\text{ex}'}$ by a factor L in the previous paragraph. As described above, when the parallel component of the field is too strong (compared with ε^{-1}) only the normal state is admissible. A complete analysis of this case will be done in section 5.

2. Minimizers of the limit energies

Before providing the details of the Γ -convergence results, we discuss some interesting, and in some cases, surprising, consequences for global minimizers of the thin-film limits of Ginzburg–Landau. The two-dimensional Ginzburg–Landau model has been extensively studied, and so we present some relevant examples and indicate where the pertinent results may be found in the literature.

First we observe that in this section the domains and functions are two-dimensional, and so we use the usual notation $\nabla = (\partial_1, \partial_2)$, $x = (x_1, x_2)$. The only exception is the applied magnetic field h^{ex} which is three-dimensional, but the energies yield effective magnetic fields that are vertical, although they may depend on the parallel part of h^{ex} .

Energy minimizers will (in the Γ -limit) minimize a two-dimensional functional of the type

$$G_{\kappa,0}(v) = \int_{\omega} d(x) \left\{ \frac{1}{2} |(\nabla - i\lambda A_0)v|^2 + \frac{\kappa^2}{4} (|v|^2 - \gamma_\kappa^2)^2 \right\} dx. \quad (2.1)$$

In the subcritical case, we may take $\lambda = h_3^{\text{ex}}$ and $A_0 = \frac{1}{2}(-x_2, x_1)$. For the critical case there are three free parameters, so to reduce their number we fix the direction of the vector field h^{ex} as follows,

$$h^{\text{ex}} = (h_1^{\text{ex}}, h_2^{\text{ex}}, h_3^{\text{ex}}) = \lambda(\alpha_1, \alpha_2, \alpha_3),$$

for a constant vector $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. In the critical case we thus write

$$A_0 = \lambda^{-1}B = (\alpha_2, -\alpha_1) \left[\frac{f+g}{2} \right] + \frac{\alpha_3}{2}(-x_2, x_1). \quad (2.2)$$

The constant $\gamma_\kappa = 1$ in the subcritical cases, and is given by

$$\gamma_\kappa^2 = 1 - \frac{d^2(x)|h^{\text{ex}'|^2}}{12\kappa^2}$$

in the critical case. We will assume that the magnitude of $|h^{\text{ex}'|^2} \ll \kappa^2$ in the following discussion, and so we may effectively think of $\gamma_\kappa = 1$ in all cases.

We specialize to the case of applied fields on the order of the lower critical field, the value at which vortices first appear in the minimizing configurations. As is well-known (see [SS07],) this occurs at fields of order $\ln \kappa$. In this section, we briefly indicate the characteristics of minimizers with vortices in the London limit for general cases and for some interesting examples. We do not provide proofs, but refer the reader to previous work which applies with few modifications.

Assume first that ω is simply connected; multiply connected domains require different treatment. It is possible to choose a gauge for $G_{\kappa,0}$ for which

$$\operatorname{div}(d(x)A_0) = 0 \quad \text{in } \omega, \quad A_0 \cdot \nu|_{\partial\omega} = 0.$$

This is proven in [DD02]: one replaces A_0 by $A_0 - \nabla'\eta$, and obtains a Neumann problem for η . The resulting magnetic field is unchanged, as well as the other gauge-invariant quantities of minimizers. By this gauge

choice, it is possible to find ξ_0 with $\nabla^\perp \xi_0 = d(x)A_0$, where $\nabla^\perp = (-\partial_2, \partial_1)$, and $\xi_0|_{\partial\omega} = 0$. Indeed, ξ_0 will solve the Dirichlet problem,

$$-\operatorname{div} \left(\frac{1}{d(x)} \nabla \xi_0 \right) = -\nabla^\perp \cdot A_0, \quad \xi_0 \in H_0^1(\omega). \quad (2.3)$$

It is this auxilliary function ξ_0 which will determine the location of the first vortices. By writing $d(x)A_0$ as a curl, one may expand the square in the energy and integrate by parts to derive an asymptotic expansion of the minimum energy in terms of the vortices, using the vortex ball construction of [SS07] and the compactness of the weak Jacobian proved by Jerrard & Soner [JS02]. In particular, the value of the lower critical field and the location of the first vortices will be determined by the maximum value of $|\xi_0/d|$ and the point(s) where it is attained.

In the subcritical case, $A_0 = A_\perp^{\text{ex}'}$ corresponds to the constant vertical field h_3^{ex} , and $\gamma_\kappa = 1$. Numerical simulations of this model have been undertaken in [CDG96,LD97], and in the case of simply-connected domains ω , a study of global minimizers with vortices has been undertaken by Ding & Du [DD02,DD06], in the limit $\kappa \rightarrow \infty$. In this setting, $\nabla^\perp \cdot A_0 \equiv 1$, so by the maximum principle, $\xi_0 < 0$ in ω . Assuming $d(x)$ is real-analytic, ξ_0/d attains its global minimum at a finite number of points interior to ω . In this case, the lower critical field, the value of the vertical component of the applied field at which vortices first appear,

$$H_{c1} = \frac{1}{2 \max_\omega |\xi_0(x)/d(x)|} \ln \kappa + O(1).$$

Moreover, for applied fields sufficiently close to H_{c1} , $h_3^{\text{ex}} = H_{c1} + K \ln \ln \kappa$, a finite number of vortices (the number uniformly bounded in κ) will concentrate as $\kappa \rightarrow \infty$ near the set of minimizers of ξ_0/d . (See [AAB05,AB05,DD06,SS07], for example.)

The hypothesis that ω be simply-connected is implicit in the arguments of [DD02,DD06], which no longer hold for multiply-connected domains. As was observed in [AAB05], in a multiply-connected domain the holes act as ‘‘giant vortices’’ at *bounded* applied field strength h_3^{ex} . To analyze the creation of vortices in the interior of ω the effect of the holes must be taken into account, modifying the choice of auxilliary function which determines the critical field and the vortex locations. This analysis was done for a circular annulus (in the context of Bose-Einstein condensates) in [AAB05], and extended to more general multiply-connected domains and the full Ginzburg–Landau functionals (with or without inhomogeneities) in [AB05,AB06]. In these papers it has been observed that vortices may concentrate on *curves* in multiply-connected ω as $\kappa \rightarrow \infty$. The asymptotic distribution of vortices along the limiting curve is studied in [ABM].

In the critical regime more interesting phenomena may be observed. As mentioned above, $\gamma_\kappa \sim 1$, and so the reduction of $|v|$ by the modification of the potential (1.5) is negligible for applied fields $h^{\text{ex}} = O(\ln \kappa)$. However, the effective vector potential (see (2.2)) yields some new, unexpected results for the London limit. Indeed, the equation for ξ_0 now reads as:

$$-\operatorname{div} \left(\frac{1}{d(x)} \nabla \xi_0 \right) = -\nabla^\perp \cdot A_0 = \alpha \cdot \left(\frac{\partial}{\partial x_1} \left[\frac{f+g}{2} \right], \frac{\partial}{\partial x_2} \left[\frac{f+g}{2} \right], -1 \right) \quad \xi_0 \in H_0^1(\omega). \quad (2.4)$$

Note that the effective magnetic field $\nabla^\perp \cdot A_0$ coincides with the projection of the field direction α onto the familiar area-weighted normal vector to the centroid surface $x_3 = \frac{1}{2}(f(x) + g(x))$. In particular, notice that if the film’s centroid is not planar the lower critical field and location of vortices will differ from the subcritical case, due to the presence of the parallel field components $h^{\text{ex}'}$.

Since the right-hand side of (2.4) may not be sign definite, we cannot conclude from the Maximum principle that ξ_0 is sign definite, leading to the possibility that the maximum of $|\xi_0/d|$ could occur at a

positive or negative value of ξ_0 . Denote by

$$\Lambda := \left\{ p \in \omega : \left| \frac{\xi_0(p)}{d(p)} \right| = \max_{\omega} \left| \frac{\xi_0}{d} \right| \right\}.$$

In case the maxima of $|\xi_0/d|$ occur at finitely many points in ω , an analysis similar to that of [SS07,DD06] applies, and we may prove:

Theorem 2.1. *Assume Λ consists of finitely many points, and there exist constants $C, M > 0$ for which*

$$\left| \frac{\xi_0(x)}{d(x)} \right| \leq \max_{\omega} \left| \frac{\xi_0}{d} \right| - C[\text{dist}(x, \Lambda)]^M, \quad (2.5)$$

for x in some neighborhood of Λ . Let

$$h^{\text{ex}} = \alpha \lambda(\kappa) = \alpha \left[\frac{1}{\max_{\omega} |\xi_0/d|} \ln \kappa + K \ln \ln \kappa \right],$$

with fixed constant K . For any sequence $\kappa_n \rightarrow \infty$, let v_n be the minimizer of the energy $I_{\kappa_n, 0}$, with A_0 as in (2.2). Then:

- (1) there exists $K_* \in \mathbb{R}$ so that if $K < K_*$, v_n has no vortices for all large n .
- (2) for any $K \geq K_*$, v_n has finitely many vortices, and the sum of the absolute values of their degrees is uniformly bounded in terms of K .
- (3) the vortices concentrate at points in Λ , in the sense that their distance to Λ is bounded by $(\ln \kappa)^{-\alpha}$ for a constant $\alpha > 0$.
- (4) if $p \in \Lambda$ and $\xi_0(p) < 0$, the vortices concentrating at p have positive degrees. If $\xi_0(p) > 0$, the degrees are negative.

The proof of this result follows that of [SS03], except it is necessary to treat points of Λ in two groups, those with positive and negative values of ξ_0 . We note that hypothesis (2.5) holds when $d(x), f(x), g(x)$ are real-analytic.

We note that in this context, it is possible (and natural) that the maximum of $|\xi_0/d|$ is attained at both positive and negative values of ξ_0 , in which case minimizers would exhibit *both* vortices and antivortices. This will be the case if we choose $\omega = D_1(0)$, the unit disk, with $f(x) = \frac{1}{2}|x|^2$, $g(x) = f(x) + 1$ (and thus $d(x) = 1$). Then, taking a horizontal field, $\alpha = (1, 0, 0)$, we may solve the equation for ξ_0 exactly, $\xi_0(x) = \frac{1}{8}x_1(1 - |x|^2)$. The maximum absolute value is attained at $x = (\pm \frac{1}{\sqrt{3}}, 0)$, giving positive degree vortices concentrating at $(-\frac{1}{\sqrt{3}}, 0)$ and negative degree (anti-)vortices at $(\frac{1}{\sqrt{3}}, 0)$. The infinitesimal curvature of the film thus engenders *vertical* vortex lines in response to a purely *horizontal* applied field!

Furthermore, it is also possible to find settings in which the maximum of $|\xi_0/d|$ is attained on a curve inside ω , either a closed curve or a collection of compactly contained arcs. For instance, if we again consider the case of a disk $\omega = D_1(0)$, but now choose a different thickness profile $f(x) = \frac{|x|x_2}{2} + \frac{x_1^2}{2} \ln\left(\frac{|x|+x_2}{|x_1|}\right)$, $g(x) = f(x) + 1$ (so again $d(x) = 1$), with applied field generated by $\alpha = (1, 0, 0)$, we may again solve for ξ_0 explicitly, obtaining:

$$\xi_0(x) = \frac{1}{8}r(1 - r^2), \quad r = |x|.$$

The maximum value is obtained on the circle $r = |x| = 1/\sqrt{3}$. In this setting, we may apply the following Γ -convergence theorem of Alama, Bronsard, & Millot [ABM]: define

$$J_{\kappa}(v) := G_{\kappa, 0}(v) - \frac{1}{2} \int_{\omega} d(x) |\lambda A_0|^2.$$

Then the following theorem follows from [ABM]:

Theorem 2.2. *Assume Λ is a C^2 Jordan curve or embedded arc in ω , (2.5) holds, and*

$$h^{\text{ex}} = \alpha \lambda(\kappa) = \alpha \left[\frac{1}{\max_{\omega} |\xi_0/d|} \ln \kappa + \beta(\kappa) \right],$$

with $1 \ll \beta(\kappa) \ll \ln \kappa$. Let $\kappa_n \rightarrow \infty$. Then:

(1) *for any v_n with $\sup_n \frac{J_{\kappa_n}(v_n)}{\beta(\kappa_n)} < \infty$, there is a subsequence and a nonnegative Radon measure $\mu \in H^{-1}(\omega)$ supported on Λ so that*

$$\frac{1}{\beta(\kappa_n)} \text{curl}(iv_n, \nabla' v_n) \rightarrow \mu \quad \text{strongly in } (C_0^{0,1}(\omega))^*.$$

(2) *The family $\frac{1}{\beta^2(\kappa)} J_{\kappa}$ of functionals Γ -converges to*

$$J_{\infty}(\mu) = \frac{1}{2} \iint_{\omega \times \omega} G(x, y) d\mu(x) d\mu(y),$$

where G is the Dirichlet Green's function of the domain ω .

As a consequence, energy minimizers in this setting will have $O(\beta(\kappa))$ vortices concentrating near the curve Λ , and their normalized weak Jacobians $J(v_n) = \text{curl}(iv_n, \nabla' v_n)$ converge to a nonnegative Radon measure μ supported on Λ . The limiting measure minimizes J_{∞} , the electrostatic potential corresponding to the charge distribution μ . Thus, the normalized limiting Jacobian (which measure the distribution of the vortices for $\kappa \rightarrow \infty$) solves a classical equilibrium measures problem from potential theory (see [Ran95, ST97].) In the above example, Λ is a circle in the disk $\omega = D_1(0)$, and the measure μ is normalized arclength. Thus, the vortices will be asymptotically uniformly distributed on the circle.

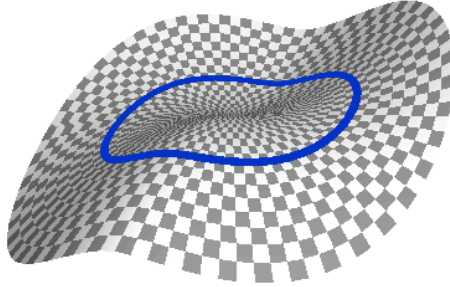


Fig. 1. The centroid given by $x_3 = f(x') = \frac{|x'|x_2}{2} + \frac{x_1^2}{2} \ln\left(\frac{|x'|+x_2}{|x_1|}\right)$, with external field directions $\alpha = (\alpha_1, 0, 0)$. Near the lower critical field, the vortices concentrate near the circle shown.

3. Critical Case

We begin proving the Γ -convergence results, starting with the critical case, $\varepsilon \rho_\varepsilon \rightarrow L \in (0, \infty)$. For simplicity we assume $\rho_\varepsilon = \varepsilon^{-1}$; for other limits L we incorporate the value of L in h^{ex} . Following [GP99], we define the Hilbert space $\check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3)$ as the completion of the space $C_0^\infty(\mathbb{R}^3; \mathbb{R}^3)$ of smooth, compactly supported divergence-free vector fields in the Dirichlet norm, $\|F\|_{\check{H}_{\text{div}}^1} = [\int_{\mathbb{R}^n} |DF|^2 dx]^{1/2}$. It follows that

$F \in \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3)$ is divergence-free in the sense of distributions. We may not have $F \in L^2(\mathbb{R}^3)$ (and so $\check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3) \neq H_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3)$), but by the Sobolev embedding $F \in L^6(\mathbb{R}^3; \mathbb{R}^3)$. We will require the following useful result on $\check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3)$ from [GP99].

Lemma 3.1.

- (1) Let $g \in L^2(\mathbb{R}^3; \mathbb{R}^3)$ such that $\text{div } g = 0$ in $\mathcal{D}'(\mathbb{R}^3)$. Then there is a unique $F \in \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3)$ such that $\nabla \times F = g$ and $\text{div } F = 0$.
- (2) For any $F \in \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3)$, there exists a constant C with

$$\|F\|_{L^6} \leq C \|F\|_{\check{H}_{\text{div}}^1} \leq C \|\nabla \times F\|_{L^2}.$$

Here and throughout the paper, we denote by $\nabla^\perp = (-\partial_2, \partial_1, 0)$ and hence for a vector field F ,

$$\nabla^\perp \cdot F = \partial_1 F_2 - \partial_2 F_1,$$

a shorthand for the third component of the curl of F .

With our simplifying assumption $\varepsilon \rho_\varepsilon = 1$, we consider vector potentials of the form $A = B + A^{\text{ex}}$, with $B \in \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3)$ and fixed (ε -independent) external potential

$$A^{\text{ex}} = \frac{1}{2} h^{\text{ex}} \times (x_1, x_2, x_3) = \frac{1}{2} (h_2^{\text{ex}} x_3 - h_3^{\text{ex}} x_2, h_3^{\text{ex}} x_1 - h_1^{\text{ex}} x_3, h_1^{\text{ex}} x_2 - h_2^{\text{ex}} x_1).$$

We now state the complete Γ -convergence result in three parts: the compactness of sequences of bounded energy; the lower semicontinuity of the limit; and the existence of sequences ε_n , (u_n, A_n) for which the energies converge. The appropriate limiting space is:

$$\mathcal{V}_0 := H^1(\omega; \mathbb{C}) \times L^2(\omega; \mathbb{C}), \quad (3.1)$$

Theorem 3.2 (Compactness). Let $\varepsilon_n \rightarrow 0^+$ as $n \rightarrow +\infty$ and let $u_n \in H^1(\Omega; \mathbb{C})$ and $A_n - A^{\text{ex}} \in \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3)$, $n \in \mathbb{N}$, be such that

$$\sup_{n \in \mathbb{N}} I_{\varepsilon_n}(u_n, A_n) < +\infty.$$

Then there exist a subsequence $\{\varepsilon_n\}$ (not relabeled) and $(v, b) \in \mathcal{V}_0$ such that

$$\begin{aligned} u_n &\rightharpoonup u = v e^{i A_3^{\text{ex}} x_3} \text{ in } H^1(\Omega; \mathbb{C}), \\ A_n - A^{\text{ex}} &\rightharpoonup 0 \text{ in } \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3), \\ v_n &:= e^{-i \int_0^{x_3} (A_n)_3(x', t) dt} u_n \rightharpoonup v \text{ in } H^1(\Omega; \mathbb{C}), \\ b_n &:= \frac{1}{\varepsilon_n d(x')} \int_{f(x')}^{g(x')} \partial_3 v_n dx_3 \rightharpoonup b \text{ in } L^2(\omega; \mathbb{C}), \end{aligned}$$

where $d(x') := g(x') - f(x')$.

Theorem 3.3 (Γ -limit). Let $(v, b) \in L^1(\omega; \mathbb{C}) \times L^1(\omega; \mathbb{C})$. Then

$$\Gamma - \lim_{\varepsilon \rightarrow 0^+} I_\varepsilon(v, b) = \begin{cases} \frac{1}{2} \int_\omega d(x') \left(|(\nabla' - i B^{\text{ex}'}) v|^2 + |b|^2 + \frac{d^2(x')}{12} |h^{\text{ex}'}|^2 |v|^2 + \frac{\kappa^2}{2} (1 - |v|^2)^2 \right) dx & \text{if } (v, b) \in \mathcal{V}_0, \\ \infty & \text{otherwise,} \end{cases}$$

where $B^{\text{ex}} := A_\perp^{\text{ex}} + \frac{f+g}{2} (-h_2^{\text{ex}}, h_1^{\text{ex}}, 0)$ and $A_\perp^{\text{ex}} := \frac{h_3^{\text{ex}}}{2} (-x_2, x_1, 0)$, so $\nabla \times A_\perp^{\text{ex}} = (0, 0, h_3^{\text{ex}})$.

3.1. Compactness

Proof of Theorem 3.2.

Let $K := \sup_{n \in \mathbb{N}} I_{\varepsilon_n}(u_n, A_n) < \infty$. Then

$$\int_{\mathbb{R}^3} \left(\frac{1}{\varepsilon_n^2} |h'_n - h^{\text{ex}}|^2 + |h_3 - h_3^{\text{ex}}|^2 \right) dx \leq K. \quad (3.2)$$

In particular, Lemma 3.1 implies that

$$\sup_{n \in \mathbb{N}} \|\nabla(A_n - A^{\text{ex}})\|_{L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 3})} < \infty.$$

Thus, we deduce that $\{A_n - A^{\text{ex}}\}$ is bounded in \check{H}_{div}^1 and in $L^6(\mathbb{R}^3; \mathbb{R}^3)$, thus there exists a subsequence (not relabeled) such that

$$A_n - A \rightharpoonup 0 \quad \text{in } \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3).$$

By weak convergence we have $\text{div } A = 0$, and by the uniform bound (3.2) we may conclude that $\|h'_n - h^{\text{ex}}\|_{L^2} \rightarrow 0$, and thus

$$\nabla \times A = (h_1^{\text{ex}}, h_2^{\text{ex}}, \nabla^\perp \cdot A).$$

Since

$$\text{div } \nabla \times A = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^3),$$

so we conclude that

$$\partial_3[\nabla^\perp \cdot A] = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^3).$$

Also, $\nabla^\perp A - h_3^{\text{ex}} \in L^2(\mathbb{R}^3)$, which implies that $\nabla^\perp \cdot A \equiv h_3^{\text{ex}}$.

This means that $\nabla \times A = h^{\text{ex}}$, hence by the uniqueness in Lemma 3.1, we deduce that $A = A^{\text{ex}}$.

Moreover, we know that $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $L^4(\Omega; \mathbb{C})$, and because ∇A_n is bounded in $L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 3})$,

$$\begin{aligned} u_n &\text{ is bounded in } L^2(\Omega; \mathbb{C}), \\ \nabla u_n &\text{ is bounded in } L^2(\Omega; \mathbb{C}^3), \end{aligned}$$

so there exists a further subsequence (not relabeled) such that

$$u_n \rightharpoonup u \quad \text{in } H^1(\Omega; \mathbb{C}).$$

Also, if we define $v_n(x) = u_n(x) e^{-i \int_0^{x_3} (A_n)_3(x', t) dt}$, then we have

$$\begin{aligned} |v_n| &= |u_n|, \text{ which is bounded in } L^2(\Omega; \mathbb{C}), \\ \nabla v_n &\text{ is bounded in } L^2(\Omega; \mathbb{C}^3), \\ \partial_3 v_n &\rightarrow 0 \quad \text{in } L^2(\mathbb{R}^3; \mathbb{C}), \end{aligned}$$

so we deduce that there is a further subsequence (not relabeled) such that

$$v_n \rightharpoonup v \quad \text{in } H^1(\Omega; \mathbb{C})$$

with $\partial_3 v = 0$. We then have that

$$u = v e^{i \int_0^{x_3} A_3(x', t) dt} = v e^{i A_3^{\text{ex}} x_3}. \quad (3.3)$$

Let $b_n(x') := \frac{1}{\varepsilon_n d(x')} \int_{f(x')}^{g(x')} \partial_3 v_n(x', t) dt \in L^2(\omega; \mathbb{C})$. Then, we know that b_n is bounded in $L^2(\omega; \mathbb{C})$, hence there exists a subsequence (not relabeled) and a function $b \in L^2(\omega; \mathbb{C})$ such that

$$b_n \rightharpoonup b \quad \text{in } L^2(\omega; \mathbb{C}).$$

■

3.2. The Γ – lim inf inequality

Proposition 3.4 (Γ – lim inf inequality).

Let $(v, b) \in \mathcal{V}_0$ and consider sequences $\{\varepsilon_n\} \subset \mathbb{R}$, $\{u_n\} \subset H^1(\Omega; \mathbb{C})$, and $\{A_n - A^{\text{ex}}\} \subset \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3)$ satisfying

$$\begin{aligned} \varepsilon_n &\rightarrow 0^+, \\ u_n &\rightharpoonup u := v e^{iA_3^{\text{ex}} x_3} \text{ in } H^1(\Omega; \mathbb{C}), \\ v_n &:= u_n e^{-i \int_0^{x_3} (A_n)_3(x', t) dt} \rightharpoonup v \text{ in } H^1(\Omega; \mathbb{C}), \\ b_n &:= \frac{1}{\varepsilon_n d(x')} \int_{f(x')}^{g(x')} \partial_3 v_n dx_3 \rightharpoonup b \text{ in } L^2(\omega; \mathbb{C}), \\ A_n - A^{\text{ex}} &\rightharpoonup 0 \text{ in } \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3). \end{aligned}$$

Then

$$\liminf_{n \rightarrow \infty} I_{\varepsilon_n}(u_n, A_n) \geq \frac{1}{2} \int_{\omega} d(x') \left(|(\nabla' - iB^{\text{ex}})v|^2 + |b|^2 + \frac{d^2(x')}{12} |h^{\text{ex}}|^2 |v|^2 + \frac{\kappa^2}{2} (1 - |v|^2)^2 \right) dx',$$

where B^{ex} is as on Theorem 3.3.

Proof.

Since $v_n \rightarrow v$ in $H^1(\Omega; \mathbb{C})$ and $|u_n| = |v_n|$, we know that $v_n \rightarrow v$ in $L^2(\Omega; \mathbb{C}) \cap L^6(\Omega; \mathbb{C})$, hence

$$\lim_{n \rightarrow \infty} \int_{\Omega} \frac{\kappa^2}{4} (1 - |u_n|^2)^2 dx = \lim_{n \rightarrow \infty} \int_{\Omega} \frac{\kappa^2}{4} (1 - |v_n|^2)^2 dx = \int_{\Omega} \frac{\kappa^2}{4} (1 - |v|^2)^2 dx = \int_{\omega} d(x') \frac{\kappa^2}{4} (1 - |v|^2)^2 dx'.$$

Then because $b_n \rightharpoonup b$ in $L^2(\omega; \mathbb{C})$ and

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{2} \int_{\Omega} \left| \frac{1}{\varepsilon_n} (\partial_3 - i(A_n)_3) u_n \right|^2 dx &\geq \liminf_{n \rightarrow \infty} \frac{1}{2} \int_{\omega} \left| \int_f^g \frac{1}{\varepsilon_n} (\partial_3 - i(A_n)_3) u_n dx_3 \right|^2 dx' \\ &= \liminf_{n \rightarrow \infty} \frac{1}{2} \int_{\omega} d(x') |b_n|^2 dx' \geq \frac{1}{2} \int_{\omega} d(x') |b|^2 dx', \end{aligned}$$

using Fubini's theorem, Hölder's inequality, and Fatou's lemma.

Moreover,

$$\nabla \times A_n - h^{\text{ex}} \rightharpoonup 0 \text{ in } L^2(\mathbb{R}^3; \mathbb{R}^3),$$

so we write

$$|(\nabla' - iA'_n)u_n|^2 = |\nabla' u_n|^2 + |A'_n u_n|^2 + 2 \Re(A'_n \bar{u}_n \cdot \nabla' u_n).$$

Using the fact that $\nabla' u_n \rightharpoonup \nabla' u$ in $L^2(\Omega; \mathbb{C})$,

$$\liminf_{n \rightarrow \infty} \int_{\Omega} |\nabla' u_n|^2 dx \geq \int_{\Omega} |\nabla' u|^2 dx,$$

and since $A_n \rightarrow A^{\text{ex}}$ in H_{loc}^1 and $u_n \rightarrow u$ in $L^4(\Omega; \mathbb{C})$,

$$\lim_{n \rightarrow \infty} \int_{\Omega} |A'_n u_n|^2 dx = \int_{\Omega} |A^{\text{ex}} u|^2 dx.$$

Also $A'_n \bar{u}_n \rightarrow A^{\text{ex}'\bar{u}}$ in $L^2(\Omega; \mathbb{C})$, so

$$\lim_{n \rightarrow \infty} \int_{\Omega} 2 \Re(A'_n \bar{u}_n \cdot \nabla' u_n) dx = \int_{\Omega} 2 \Re(A^{\text{ex}'\bar{u}} \cdot \nabla' u) dx.$$

This yields

$$\liminf_{n \rightarrow \infty} \int_{\Omega} |(\nabla' - iA'_n)u_n|^2 dx \geq \int_{\Omega} |(\nabla' - iA^{\text{ex}'})u|^2 dx.$$

To complete the proof, we write the last term in a different form:

$$\begin{aligned} |(\nabla' - i(A^{\text{ex}'})u|^2 &= |(\nabla' - i(A_{\parallel}^{\text{ex}'} + A_{\perp}^{\text{ex}'}))u|^2 \\ &= |(\nabla' - i(A_{\parallel}^{\text{ex}'} + A_{\perp}^{\text{ex}'} + \frac{1}{2}(h_2^{\text{ex}}, -h_1^{\text{ex}})x_3)v|^2 \\ &= |(\nabla' - i(A_{\perp}^{\text{ex}'} + (h_2^{\text{ex}}, -h_1^{\text{ex}})x_3))v|^2 \\ &= |(\nabla' - iA_{\perp}^{\text{ex}'})v|^2 + \frac{1}{2}|h^{\text{ex}'|^2|v|^2x_3^2 + 2\Im((\nabla' - iA_{\perp}^{\text{ex}'})v \cdot (h_2^{\text{ex}}, -h_1^{\text{ex}})\bar{v})x_3, \end{aligned} \quad (3.4)$$

where we recall that $A_{\parallel}^{\text{ex}} = \frac{1}{2}(h_2^{\text{ex}}x_3, -h_1^{\text{ex}}x_3, h_1^{\text{ex}}x_2 - h_2^{\text{ex}}x_1)$, so $\nabla \times A_{\parallel}^{\text{ex}} = (h_1^{\text{ex}}, h_2^{\text{ex}}, 0)$, and we recall that $A_{\perp}^{\text{ex}} = \frac{h_3^{\text{ex}}}{2}(-x_2, x_1, 0)$. Since we know that none of the terms in (3.4) depends on x_3 , and

$$\int_f^g x_3 dx_3 = \frac{g^2 - f^2}{2} = \frac{d(f+g)}{2} \quad \text{and} \quad \int_f^g x_3^2 dx_3 = \frac{g^3 - f^3}{3} = \frac{d(f^2 + fg + g^2)}{3},$$

we deduce that

$$\begin{aligned} \int_{\Omega} |(\nabla' - iA_{\perp}^{\text{ex}'})u|^2 dx &= \int_{\omega} \int_{-\frac{1}{2}}^{\frac{1}{2}} |(\nabla' - iA')u|^2 dx_3 dx' \\ &= \int_{\omega} d(x') \left(|(\nabla' - iA_{\perp}^{\text{ex}'})v|^2 + 2\Im((\nabla' - iA_{\perp}^{\text{ex}'})v \cdot (h_2^{\text{ex}}, -h_1^{\text{ex}})\bar{v}) \left(\frac{f+g}{2} \right) \right. \\ &\quad \left. + |h^{\text{ex}'|^2|v|^2 \left(\frac{f^2 + fg + g^2}{3} \right) \right) dx' \\ &= \int_{\omega} d(x') \left(|(\nabla' - iB^{\text{ex}'})v|^2 + |h^{\text{ex}'|^2|v|^2 \left(\frac{f^2 + fg + g^2}{3} - \frac{(f+g)^2}{4} \right) \right) dx' \\ &= \int_{\omega} d(x') \left(|(\nabla' - iB^{\text{ex}'})v|^2 + \frac{d^2(x')}{12}|h^{\text{ex}'|^2|v|^2 \right) dx', \end{aligned} \quad (3.5)$$

where $B^{\text{ex}'} = A_{\perp}^{\text{ex}'} + \frac{f+g}{2}(-h_2^{\text{ex}}, h_1^{\text{ex}})$ and on the third equality we completed the square. This completes the proof. ■

3.3. The Γ – lim sup inequality

Proposition 3.5 (Γ – lim sup inequality).

Let $(v, b) \in \mathcal{V}_0$ and let $\{\varepsilon_n\} \in \mathbb{R}$ be a sequence such that $\varepsilon_n \rightarrow 0^+$. Then, there exist sequences $\{u_n\} \subset H^1(\Omega; \mathbb{C})$ and $\{A_n - A^{\text{ex}}\} \subset \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3)$ such that

$$\begin{aligned} \varepsilon_n &\rightarrow 0^+, \\ u_n &\rightharpoonup u := ve^{iA_3^{\text{ex}}x_3} \text{ in } H^1(\Omega; \mathbb{C}), \\ v_n &:= u_n e^{-i \int_0^{x_3} (A_n)_3(x', t) dt} \rightharpoonup v \text{ in } H^1(\Omega; \mathbb{C}), \\ b_n &:= \frac{1}{\varepsilon_n d(x')} \int_{f(x')}^{g(x')} \partial_3 v_n dx_3 \rightharpoonup b \text{ in } L^2(\omega; \mathbb{C}), \\ A_n - A^{\text{ex}} &\rightharpoonup 0 \text{ in } \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3). \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} I_{\varepsilon_n}(u_n, A_n) = \frac{1}{2} \int_{\omega} d(x') \left(|(\nabla' - iB^{\text{ex}'})v|^2 + |b|^2 + \frac{d^2(x')}{12} |h^{\text{ex}'|^2} |v|^2 + \frac{\kappa^2}{2} (1 - |v|^2)^2 \right) dx',$$

where B^{ex} is as on Theorem 3.3.

Proof.

Define

$$A_n(x) := A^{\text{ex}}(x),$$

and

$$u_n(x) = e^{iA_3^{\text{ex}}(x')x_3} (v(x') + \varepsilon_n b(x')x_3).$$

We prove first that the convergences in the proposition hold. Note that

$$|u_n - u| = \left| \varepsilon_n e^{iA_3^{\text{ex}}(x')x_3} b(x')x_3 \right| = \varepsilon_n |b(x')x_3|$$

so that

$$\|u_n - u\|_{L^2(\Omega; \mathbb{C})}^2 \leq \varepsilon_n \|b\|_{L^2(\omega; \mathbb{C})}^2 \rightarrow 0.$$

Since $\{\nabla u_n\}$ is bounded in $L^2(\Omega; \mathbb{C}^3)$, we know that

$$u_n \rightharpoonup u \text{ in } H^1(\Omega; \mathbb{C}).$$

The other convergences are trivial, since $b_n \equiv b$ and $A_n \equiv A^{\text{ex}}$. Moreover,

$$I_{\varepsilon_n}(u_n, A_n) = \frac{1}{2} \int_{\Omega} \left(|(\nabla' - iA_n)u_n|^2 + |b|^2 + \frac{\kappa^2}{2} (1 - |u_n|^2)^2 \right) dx.$$

We know that

$$(1 - |u_n|^2)^2 = (1 - |v|^2)^2 + O_{L^1}(\varepsilon_n)$$

so we have

$$\lim_{n \rightarrow \infty} \frac{1}{2} \int_{\Omega} (1 - |u_n|^2)^2 dx = \frac{1}{2} \int_{\omega} d(x') (1 - |v|^2)^2 dx'.$$

As for the remaining term, we follow an analogous reasoning as in (3.5) to deduce

$$\int_{\Omega} |(\nabla' - iA_n')u_n|^2 dx = \int_{\omega} d(x') \left(|(\nabla' - iB^{\text{ex}'})v|^2 + \frac{d^2(x')}{12} |h^{\text{ex}'|^2} |v|^2 \right) dx' + O(\varepsilon_n).$$

We conclude that

$$\lim_{n \rightarrow \infty} I_{\varepsilon_n}(u_n, A_n) = \frac{1}{2} \int_{\omega} d(x') \left(|(\nabla' - iB^{\text{ex}'})v|^2 + |b|^2 + \frac{d^2(x')}{12} |h^{\text{ex}'}|^2 |v|^2 + \frac{\kappa^2}{2} (1 - |v|^2)^2 \right) dx'.$$

This completes the proof. ■

4. Subcritical Case

This case, when $\varepsilon \rho_\varepsilon \rightarrow 0$, is itself split into two subcases, when $\rho_\varepsilon \rightarrow \rho \in [0, \infty)$ and $\rho_\varepsilon \rightarrow \infty$. We recall the definition of $A_\varepsilon^{\text{ex}}$ from (1.3); note that in this regime $A_\varepsilon^{\text{ex}} \rightarrow A_\perp^{\text{ex}}$ (in H_{loc}^1) with limiting potential

$$A_\perp^{\text{ex}} := \frac{1}{2} h_3^{\text{ex}} \vec{e}_3 \times (x_1, x_2, x_3) = h_{\text{ex}}^3 \left(-\frac{x_2}{2}, \frac{x_1}{2}, 0 \right).$$

To capture the Cosserat vectors in the limit we must have some control on the order of ε at which the vector fields are converging or diverging. We thus define the space

$$\mathcal{V}_- := H^1(\omega; \mathbb{C}) \times L^2(\omega; \mathbb{C}) \times L^2(\mathbb{R}^3; \mathbb{R}^2). \quad (4.1)$$

We consider sequences $\varepsilon_n \rightarrow 0$, and write $\rho_n = \rho_{\varepsilon_n}$ and $A_n^{\text{ex}} = A_{\varepsilon_n}^{\text{ex}}$ throughout.

Theorem 4.1 (Compactness). *Let $\varepsilon_n \rightarrow 0^+$ as $n \rightarrow +\infty$ and let $u_n \in H^1(\Omega; \mathbb{C})$ and $A_n - A_n^{\text{ex}} \in \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3)$, $n \in \mathbb{N}$, be such that*

$$\sup_{n \in \mathbb{N}} I_{\varepsilon_n}(u_n, A_n) < +\infty.$$

(i) *If $\rho_n \rightarrow \rho$, define*

$$c'_n := \frac{1}{\varepsilon_n} (\nabla \times A_n)'. \quad (4.2)$$

Then there exist a subsequence $\{\varepsilon_n\}$ (not relabeled) and $(u, b, c' - \rho h^{\text{ex}'}) \in \mathcal{V}_-$ such that

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } H^1(\Omega; \mathbb{C}), \\ A_n - A_n^{\text{ex}} &\rightharpoonup 0 \text{ in } \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3), \\ b_n := \frac{1}{\varepsilon_n d(x')} \int_{f(x')}^{g(x')} \partial_3 v_n dx_3 &\rightharpoonup b \text{ in } L^2(\omega; \mathbb{C}), \\ c'_n - \rho_n h^{\text{ex}'} &\rightharpoonup c' - \rho h^{\text{ex}'} \text{ in } L^2(\mathbb{R}^3; \mathbb{R}^2). \end{aligned}$$

(ii) *If $\rho_\varepsilon \rightarrow \infty$, define*

$$c'_n := \frac{1}{\varepsilon_n \rho_n} (\nabla \times A_n)'. \quad (4.3)$$

Then there exist a subsequence $\{\varepsilon_n\}$ (not relabeled) and $(u, b) \in \mathcal{V}_0$ such that

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } H^1(\Omega; \mathbb{C}), \\ A_n - A_n^{\text{ex}} &\rightharpoonup 0 \text{ in } \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3), \\ b_n := \frac{1}{\varepsilon_n d(x')} \int_{f(x')}^{g(x')} \partial_3 v_n dx_3 &\rightharpoonup b \text{ in } L^2(\omega; \mathbb{C}), \\ c'_n - h^{\text{ex}'} &\rightharpoonup 0 \text{ in } L^2(\mathbb{R}^3; \mathbb{R}^2). \end{aligned}$$

Note that in either case the magnetic fields $h_n = \nabla \times A_n \rightharpoonup (0, 0, h_3^{\text{ex}})$ in L^2_{loc} .

Theorem 4.2 (Γ -limit).

(i) Let $\rho_\varepsilon \rightarrow \rho$, and let $(u, b, c') \in L^1(\Omega; \mathbb{C}) \times L^1(\Omega; \mathbb{C}) \times L^1(\mathbb{R}^3; \mathbb{R}^2)$. Then

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0^+} I_\varepsilon(u, b, c') = \begin{cases} \frac{1}{2} \int_\omega d(x') \left(|(\nabla' - iA_\perp^{\text{ex}'})u|^2 + |b|^2 + \frac{\kappa^2}{2} (1 - |u|^2)^2 \right) dx' \\ \quad + \frac{1}{2} \int_{\mathbb{R}^3} |c' - \rho h^{\text{ex}'}|^2 dx & \text{if } (u, b, c' - \rho h^{\text{ex}'}) \in \mathcal{V}_-, \\ \infty & \text{otherwise.} \end{cases}$$

(ii) Let $\rho_\varepsilon \rightarrow \infty$, and let $(u, b) \in L^1(\Omega; \mathbb{C}) \times L^1(\Omega; \mathbb{C})$. Then

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0^+} I_\varepsilon(u, b) = \begin{cases} \frac{1}{2} \int_\omega d(x') \left(|(\nabla' - iA_\perp^{\text{ex}'})u|^2 + |b|^2 + \frac{\kappa^2}{2} (1 - |u|^2)^2 \right) dx' & \text{if } (u, b) \in \mathcal{V}_0, \\ \infty & \text{otherwise.} \end{cases}$$

Corollary 4.3.

Theorem 4.2, in all three cases, implies that the Ginzburg–Landau model in 3D

$$\min_{\substack{u \in H^1(\Omega; \mathbb{C}) \\ A \in H^1_{\text{div}}(\mathbb{R}^3; \mathbb{R}^3)}} I_\varepsilon(u, A)$$

converges, in the thin-film limit, to the model

$$\min_{u \in H^1(\omega; \mathbb{C})} \frac{1}{2} \int_\omega d(x') \left(|(\nabla' - iA_\perp^{\text{ex}'})u|^2 + \frac{\kappa^2}{2} (1 - |u|^2)^2 \right) dx',$$

where we let $b \equiv 0$ in ω and $c' \equiv h^{\text{ex}'}$ in \mathbb{R}^3 .

4.1. Compactness

Proof of Theorem 4.1.

Let $K := \sup_{n \in \mathbb{N}} I_{\varepsilon_n}(u_n, A_n) < \infty$. Then

$$\int_{\mathbb{R}^3} \frac{1}{\varepsilon_n^2} |h'_n - \varepsilon_n \rho_n h^{\text{ex}'}|^2 + |h_3 - h_3^{\text{ex}}|^2 dx \leq K, \quad (4.4)$$

This implies that $\nabla \times (A_n - A_n^{\text{ex}})$ is bounded in L^2 , and by Lemma 3.1 we conclude that $(A_n - A_n^{\text{ex}})$ is bounded in $\check{H}^1_{\text{div}}(\mathbb{R}^3; \mathbb{R}^3)$, and therefore there exists a subsequence (not relabeled) such that

$$B_n := (A_n - A_n^{\text{ex}}) \rightharpoonup B \quad \text{in } \check{H}^1_{\text{div}}(\mathbb{R}^3; \mathbb{R}^3).$$

Let $A := B + A_\perp^{\text{ex}}$. Then, by weak convergence, $\text{div } A = \text{div } B = 0$, and by the estimate (4.4) we conclude that $\nabla \times B = (0, 0, \nabla^\perp \cdot B)$. This implies that $\partial_3[\nabla^\perp \cdot B] = 0$ in $\mathcal{D}'(\mathbb{R}^3)$. Also, from Fatou's Lemma in (4.4), we deduce that $\nabla^\perp \cdot B \in L^2(\mathbb{R}^3)$, thus $\nabla^\perp \cdot B \equiv 0$, that is $\nabla^\perp \cdot A \equiv h_3^{\text{ex}}$. Because $\nabla \times (A - A_\perp^{\text{ex}}) = 0$, the uniqueness in Lemma 3.1 implies that $A \equiv A_\perp^{\text{ex}}$.

This means that in the thin film limit, the magnetic field is vertical. The Cosserat vector for the magnetic field should give the direction which the magnetic field takes to get vertical in the limit.

Case (i): If $\rho_n \rightarrow \rho \in [0, \infty)$, then we know that

$$\int_{\mathbb{R}^3} |c'_n - \rho_n h^{\text{ex}'}|^2 dx = \int_{\mathbb{R}^3} \left| \frac{1}{\varepsilon_n} h'_n - \rho_n h^{\text{ex}'} \right|^2 dx \leq K,$$

which implies that we can find a further subsequence (not relabeled) such that

$$c'_n - \rho_n h^{ex'} \rightharpoonup c' - \rho h^{ex'} \quad \text{in } L^2(\mathbb{R}^3; \mathbb{R}^2).$$

case (ii): If $\rho_n \rightarrow \infty$, then we know that

$$\rho_n^2 \int_{\mathbb{R}^3} |c_n - h^{ex'}|^2 dx = \rho_n^2 \int_{\mathbb{R}^3} \left| \frac{1}{\varepsilon_n \rho_n} h'_n - h^{ex'} \right|^2 dx \leq K,$$

which implies that we can find a further subsequence (not relabeled) such that

$$c_n - h^{ex'} \rightharpoonup 0 \quad \text{in } L^2(\mathbb{R}^3; \mathbb{R}^2).$$

For both cases, we know that $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $L^4(\Omega; \mathbb{C})$, and because ∇A_n is bounded in $L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 3})$,

$$\begin{aligned} u_n, \text{ which is bounded in } L^2(\Omega; \mathbb{C}), \\ \nabla u_n \text{ is bounded in } L^2(\Omega; \mathbb{C}^3), \end{aligned}$$

so there exists a further subsequence (not relabeled) such that

$$u_n \rightharpoonup u \quad \text{in } H^1(\Omega; \mathbb{C}).$$

Also, if we define $v_n(x) = u_n(x) e^{-i \int_0^{x_3} (A_n)_3(x', t) dt}$, then we have

$$\begin{aligned} |v_n| &= |u_n|, \text{ which is bounded in } L^2(\Omega; \mathbb{C}), \\ \nabla v_n &\text{ is bounded in } L^2(\Omega; \mathbb{C}^3), \\ \partial_3 v_n &\rightarrow 0 \quad \text{in } L^2(\mathbb{R}^3; \mathbb{C}), \end{aligned}$$

so we deduce that there is a further subsequence (not relabeled) such that

$$v_n \rightharpoonup v \quad \text{in } H^1(\Omega; \mathbb{C})$$

with $\partial_3 v = 0$. We then have that

$$u = v e^{i \int_0^{x_3} A_{\perp 3}^{ex}(x', t) dt} = v. \tag{4.5}$$

Let $b_n := \frac{1}{\varepsilon_n d(x')} \int_{f(x')}^{g(x')} \partial_3 v_n dx_3 \in L^2(\omega; \mathbb{C})$. Then, we know that b_n is bounded in $L^2(\omega; \mathbb{C})$, hence there exists a subsequence (not relabeled) and a function $b \in L^2(\omega; \mathbb{C})$ such that

$$b_n \rightharpoonup b \quad \text{in } L^2(\omega; \mathbb{C}).$$

■

4.2. The Γ – lim inf inequality

Proposition 4.4 (Γ – lim inf inequality).

(i) Let $(u, b, c' - \rho h^{ex'}) \in \mathcal{V}_-$ and consider sequences $\{\varepsilon_n\} \subset \mathbb{R}$, $u_n \in H^1(\Omega; \mathbb{C})$ and $A_n - A_n^{ex} \in \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3)$, $n \in \mathbb{N}$, satisfying

$$\begin{aligned} \varepsilon_n &\rightarrow 0^+, \quad \rho_n \rightarrow \rho, \\ u_n &\rightharpoonup u \text{ in } H^1(\Omega; \mathbb{C}), \\ b_n &:= \frac{1}{\varepsilon_n d(x')} \int_{f(x')}^{g(x')} \partial_3 \left(u_n e^{-i \int_0^{x_3} (A_n)_3(x', t) dt} \right) dx_3 \rightharpoonup b \text{ in } L^2(\omega; \mathbb{C}), \\ A_n - A_n^{ex} &\rightharpoonup 0 \text{ in } \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3), \\ c'_n - \rho_n h^{ex'} &\rightharpoonup c' - \rho h^{ex'} \text{ in } L^2(\mathbb{R}^3; \mathbb{R}^2), \end{aligned}$$

with c'_n as in (4.2). Then

$$\liminf_{n \rightarrow \infty} I_{\varepsilon_n}(u_n, A_n) \geq \frac{1}{2} \int_{\omega} d(x') \left(|(\nabla' - iA_{\perp}^{ex'})u|^2 + |b|^2 + \frac{\kappa^2}{2} (1 - |u|^2)^2 \right) dx' + \frac{1}{2} \int_{\mathbb{R}^3} |c' - \rho h^{ex'}|^2 dx.$$

(ii) Let $(u, b) \in \mathcal{V}_0$ and consider sequences $\{\varepsilon_n\} \subset \mathbb{R}$, $u_n \in H^1(\Omega; \mathbb{C})$ and $A_n - A_n^{ex} \in \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3)$, $n \in \mathbb{N}$, satisfying

$$\begin{aligned} \varepsilon_n &\rightarrow 0^+, \quad \rho_n \rightarrow \infty, \\ u_n &\rightharpoonup u \text{ in } H^1(\Omega; \mathbb{C}), \\ b_n &:= \frac{1}{\varepsilon_n d(x')} \int_{f(x')}^{g(x')} \partial_3 \left(u_n e^{-i \int_0^{x_3} (A_n)_3(x', t) dt} \right) dx_3 \rightharpoonup b \text{ in } L^2(\omega; \mathbb{C}), \\ A_n - A_n^{ex} &\rightharpoonup 0 \text{ in } \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3), \\ c'_n - h^{ex'} &\rightharpoonup 0 \text{ in } L^2(\mathbb{R}^3; \mathbb{R}^2), \end{aligned}$$

with c'_n as in (4.3). Then

$$\liminf_{n \rightarrow \infty} I_{\varepsilon_n}(u_n, A_n) \geq \frac{1}{2} \int_{\omega} d(x') \left(|(\nabla' - iA_{\perp}^{ex'})u|^2 + |b|^2 + \frac{\kappa^2}{2} (1 - |u|^2)^2 \right) dx.$$

Proof.

Since $u_n \rightarrow u$ in $H^1(\Omega; \mathbb{C})$, we know that $u_n \rightarrow u$ in $L^2(\Omega; \mathbb{C}) \cap L^6(\Omega; \mathbb{C})$, hence

$$\lim_{n \rightarrow \infty} \int_{\Omega} \frac{\kappa^2}{4} (1 - |u_n|^2)^2 dx = \int_{\Omega} \frac{\kappa^2}{4} (1 - |u|^2)^2 dx = \int_{\omega} d(x') \frac{\kappa^2}{4} (1 - |v|^2)^2 dx'.$$

Then because $b_n(x) \rightharpoonup b$ in $L^2(\Omega; \mathbb{C})$ and

$$\liminf_{n \rightarrow \infty} \frac{1}{2} \int_{\Omega} \left| \frac{1}{\varepsilon} (\partial_3 - i(A_n)_3) u_n \right|^2 dx \geq \liminf_{n \rightarrow \infty} \frac{1}{2} \int_{\omega} d(x') |b_n|^2 dx \geq \frac{1}{2} \int_{\omega} d(x') |b|^2 dx',$$

using Fubini's theorem, Hölder's inequality, and Fatou's lemma.

For the covariant term, we write

$$|(\nabla' - iA'_n)u_n|^2 = |\nabla' u_n|^2 + |A'_n u_n|^2 + 2 \Re(A'_n \bar{u}_n \cdot \nabla' u_n).$$

Using the fact that $\nabla' u_n \rightharpoonup \nabla' u$ in $L^2(\Omega; \mathbb{C})$,

$$\liminf_{n \rightarrow \infty} \int_{\Omega} |\nabla' u_n|^2 dx \geq \int_{\Omega} |\nabla' u|^2 dx,$$

and since $A_n \rightarrow A_\perp^{\text{ex}}$ in H_{loc}^1 , and $u_n \rightarrow u$ in $L^4(\Omega; \mathbb{C})$,

$$\lim_{n \rightarrow \infty} \int_{\Omega} |A'_n u_n|^2 dx = \int_{\Omega} |A_\perp^{\text{ex}} u|^2 dx.$$

Also $A'_n \bar{u}_n \rightarrow A_\perp^{\text{ex}} \bar{u}$ in $L^2(\Omega; \mathbb{C})$, so

$$\lim_{n \rightarrow \infty} \int_{\Omega} 2 \Re(A'_n \bar{u}_n \cdot \nabla' u_n) dx = \int_{\Omega} 2 \Re(A_\perp^{\text{ex}} \bar{u} \cdot \nabla' u) dx.$$

This yields

$$\liminf_{n \rightarrow \infty} \int_{\Omega} |(\nabla' - iA'_n)u_n|^2 dx \geq \int_{\Omega} |(\nabla' - iA_\perp^{\text{ex}})u|^2 dx' = \int_{\omega} d(x') |(\nabla' - iA_\perp^{\text{ex}})u|^2 dx'.$$

Finally, in case (i) we apply Fatou's Lemma to the last term,

$$\liminf_{n \rightarrow \infty} \frac{1}{2} \int_{\mathbb{R}^3} \left| \frac{1}{\varepsilon_n} (\nabla \times A_n)' - \rho_n h^{\text{ex}'} \right|^2 dx \geq \frac{1}{2} \int_{\mathbb{R}^3} |c' - \rho h^{\text{ex}'}|^2 dx.$$

This completes the proof. ■

4.3. The Γ – lim sup inequality

As mentioned in the Introduction, the Cosserat vectors in the case $\rho_n \rightarrow \rho$ are the rescaled limit of the x' -component of the internal magnetic field. More specifically, by the compactness result, Theorem 4.1, in case $\rho_n \rightarrow \rho \geq 0$,

$$w'_n := \frac{1}{\varepsilon_n} (\nabla \times A_n)' - \rho_n h^{\text{ex}'} \rightharpoonup c' - \rho h^{\text{ex}'} =: w',$$

and $w' \in L^2(\mathbb{R}^3; \mathbb{R}^2)$. In order to construct upper bound sequences we need to recover sequences $w_n \in L^2_{div}(\mathbb{R}^3; \mathbb{R}^3)$ whose first two components converge to w' .

As a first attempt, we may ask whether a given w' may be extended to $w = (w', w_3)$, a divergence-free $L^2(\mathbb{R}^3; \mathbb{R}^3)$ vector field. It turns out that this is not possible, even for smooth compactly supported $w' \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^2)$. Consider the following example: let $\varphi(x) \in C_c^\infty(\mathbb{R}^3)$ with

$$\begin{cases} \varphi(x) = 1, & \text{for } \max_{j \in \{1,2,3\}} |x_j| \leq 1, \\ \varphi(x) = 0, & \text{for } \max_{j \in \{1,2,3\}} |x_j| \geq 2, \\ \varphi(x) \geq 0, & \text{in } \mathbb{R}^3, \end{cases}$$

and $w'(x) = (x_1, x_2)\varphi(x)$. Assume that we can find $w_3(x)$ so that $w = (w_1, w_2, w_3) \in L^2(\mathbb{R}^3; \mathbb{R}^3)$ with divergence zero. In that case, we calculate $\partial_{x_3} w_3 = -2\varphi + (x_1, x_2) \cdot \nabla' \varphi$. For $(x_1, x_2) = (0, 0)$ we conclude $\partial_{x_3} w_3 = -2\varphi \leq 0$ for all $x_3 \in \mathbb{R}$, and $\partial_{x_3} w_3 = -2$ for $x_3 \in [-1, 1]$. In particular, $w_3(0, 0, x_3)$ has distinct limits as $x_3 \rightarrow \pm\infty$, and thus $w \notin L^2(\mathbb{R}^3; \mathbb{R}^3)$.

Fortunately, we do not require $w' \in L^2(\mathbb{R}^3; \mathbb{R}^2)$ to be the restriction of a divergence-free L^2 vector field, and we may indeed recover any $w' \in L^2(\mathbb{R}^3; \mathbb{R}^2)$ as a limit of divergence-free vector fields in the sense described in Theorem 4.1.

Lemma 4.5. *Let $w' \in L^2(\mathbb{R}^3; \mathbb{R}^2)$. Then there is a sequence $\{B_\varepsilon\}_{\varepsilon>0} \subset \check{H}^1_{div}(\mathbb{R}^3; \mathbb{R}^3)$ such that $(\nabla \times B_\varepsilon)' \rightarrow w'$ in $L^2(\mathbb{R}^3; \mathbb{R}^2)$ and $\varepsilon \nabla \times B_\varepsilon \rightarrow 0$ in $L^2(\mathbb{R}^3; \mathbb{R}^3)$.*

Proof. We divide the proof in three steps.

Step 1. w' is the characteristic function of a compact set.

Assume that $w'(x) = (1, 1)\chi_K(x)$ where $K \subset \mathbb{R}^3$ is a compact set. Then, for all $\delta > 0$, define $w'_\delta := w' * \psi_\delta$ where $\psi_\delta(x) = \frac{1}{\delta^3}\psi(\frac{x}{\delta})$ and $\psi \in C_c^\infty(\mathbb{R}^3)$ is the standard mollifier. Consider $(w_\delta)_3(x) := -\int_0^{x_3} (\partial_1(w_\delta)_1(x', t) + \partial_2(w_\delta)_2(x', t)) dt \in C^\infty(\mathbb{R}^3; \mathbb{R}^2)$, and $\operatorname{div} w_\delta = 0$.

Consider the function $\chi_\eta \in C^\infty(\mathbb{R})$ such that $\chi_\eta(t) \equiv 1$ for $|t| \leq \eta$ and $\chi_\eta(t) \leq C \exp(-(t^2 - \eta^2)^2)$, $\chi_\eta \geq 0$, and $\|\chi'_\eta\|_\infty \leq C$. Now we define $W_{\eta, \delta}(x) := w_\delta(x)\chi_\eta(x_3) - \nabla\varphi_{\eta, \delta}(x)$, where $\varphi_{\eta, \delta} \in H^1(\mathbb{R}^3)$ is the solution of $\Delta\varphi_{\eta, \delta}(x) = (w_\delta)_3(x)\chi'_\eta(x_3)$.

Since $\operatorname{div} W_{\eta, \delta} = 0$, by Lemma 3.1, we find $B_{\eta, \delta} \in \check{H}_{\operatorname{div}}^1(\mathbb{R}^3; \mathbb{R}^3)$ such that $\nabla \times B_{\eta, \delta} = W_{\eta, \delta}$.

On the other hand, since $w_\delta \in L^\infty(\mathbb{R}^3; \mathbb{R}^3)$, we have that $\Delta\varphi_{\eta, \delta} \rightarrow 0$ as $\eta \rightarrow \infty$ in $L^2(\mathbb{R}^3)$, and

$$\int_{\mathbb{R}^3} |\nabla\varphi_{\eta, \delta}|^2 dx = - \int_{\mathbb{R}^3} (w_\delta)_3 \chi'_\eta(x_3) \varphi_{\eta, \delta} dx \leq \|(w_\delta)_3 \chi'_\eta\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \|\varphi_{\eta, \delta}\|_{L^6(\mathbb{R}^3)} \rightarrow 0 \text{ as } \eta \rightarrow \infty.$$

Thus $\varphi_{\eta, \delta} \rightarrow 0$ as $\eta \rightarrow \infty$ in $H^1(\mathbb{R}^3)$, which means that $(\nabla \times B_{\eta, \delta})' \rightarrow w'_\delta$ as $\eta \rightarrow \infty$ in $L^2(\mathbb{R}^3; \mathbb{R}^2)$. Then we can find $\eta_\delta \rightarrow \infty$ as $\delta \rightarrow 0^+$ such that

$$\|(\nabla \times B_{\eta_\delta, \delta})' - w'_\delta\|_{L^2(\mathbb{R}^3; \mathbb{R}^2)} \leq \delta.$$

Denote $B_\delta := B_{\eta_\delta, \delta}$ and $W_\delta := W_{\eta_\delta, \delta}$. Then,

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla \times B'_\delta - w'|^2 dx &\leq 2 \int_{\mathbb{R}^3} |\nabla \times B'_\delta - w'_\delta|^2 dx + 2 \int_{\mathbb{R}^3} |w'_\delta - w'|^2 dx \\ &\leq 2\delta^2 + 2\|w'_\delta - w'\|_{L^2(\mathbb{R}^3; \mathbb{R}^2)}^2 \rightarrow 0 \text{ as } \delta \rightarrow 0^+, \end{aligned}$$

so $(\nabla \times B_\delta)' \rightarrow w'$ in $L^2(\mathbb{R}^3; \mathbb{R}^2)$, which implies that $\varepsilon(\nabla \times B_{\delta_\varepsilon})' \rightarrow 0$ in $L^2(\mathbb{R}^3; \mathbb{R}^2)$ for all $\delta_\varepsilon \rightarrow 0^+$.

For the third component of the curl, we may choose $\delta_\varepsilon \rightarrow 0^+$ as $\varepsilon \rightarrow 0^+$ such that

$$\|(W_{\delta_\varepsilon})_3\|_{L^2(\mathbb{R}^3; \mathbb{R}^3)} \leq \frac{1}{\sqrt{\varepsilon}}.$$

This yields $\varepsilon W_{\delta_\varepsilon} \rightarrow 0$ in $L^2(\mathbb{R}^3; \mathbb{R}^3)$.

Step 2. w' is a simple function with compact support.

Since these functions are just a finite sum of characteristic functions of compact sets, the proof follows immediately from Step 1.

Step 3. General case.

Let $w' \in L^2(\mathbb{R}^3; \mathbb{R}^2)$. Then, we can find a sequence of simple functions with compact support $\{w'_n\}$ such that $w'_n \rightarrow w'$ in $L^2(\mathbb{R}^3; \mathbb{R}^2)$.

Then, following the construction in Step 1, we can find a sequence $B_{n, \delta} \in \check{H}_{\operatorname{div}}^1(\mathbb{R}^3; \mathbb{R}^3)$ satisfying

$$(\nabla \times B_{n, \delta})' \rightarrow w'_n \text{ as } \delta \rightarrow 0^+ \text{ in } L^2(\mathbb{R}^3; \mathbb{R}^2).$$

Hence we can find $\delta_n \rightarrow 0^+$ such that

$$\|(\nabla \times B_{n, \delta_n})' - w'_n\|_{L^2(\mathbb{R}^3; \mathbb{R}^2)} \leq \frac{1}{n},$$

thus $(\nabla \times B_{n, \delta_n})' \rightarrow w'$ in $L^2(\mathbb{R}^3; \mathbb{R}^2)$. We write $B_n := B_{n, \delta_n}$. Then $\varepsilon(\nabla \times B_{n_\varepsilon})' \rightarrow 0$ in $L^2(\mathbb{R}^3; \mathbb{R}^2)$ for all $n_\varepsilon \rightarrow \infty$.

As above, we choose $n_\varepsilon \rightarrow \infty$ as $\varepsilon \rightarrow 0^+$ such that

$$\|(W_{n_\varepsilon})_3\|_{L^2(\mathbb{R}^3; \mathbb{R}^3)} \leq \frac{1}{\sqrt{\varepsilon}},$$

and thus $\varepsilon W_{n_\varepsilon} \rightarrow 0$ in $L^2(\mathbb{R}^3; \mathbb{R}^3)$. Write $B_\varepsilon := B_{n_\varepsilon}$ and the proof is complete. ■

Proposition 4.6 (Γ – lim sup inequality).

(i) Let $(u, b, c' - \rho h^{\text{ex}'}) \in \mathcal{V}_-$ and let $\{\varepsilon_n\} \in \mathbb{R}$ be a sequence such that $\varepsilon_n \rightarrow 0^+$ and $\rho_n \rightarrow \rho$. Then, there exist sequences $\{u_n\} \subset H^1(\Omega; \mathbb{C})$ and $\{A_n - A_n^{\text{ex}}\} \subset \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3)$ such that

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } H^1(\Omega; \mathbb{C}), \\ b_n &:= \frac{1}{\varepsilon_n d(x')} \int_{f(x')}^{g(x')} \partial_3 \left(u_n e^{-i \int_0^{x_3} (A_n)_3(x', t) dt} \right) dx_3 \equiv b \text{ a.e. in } \omega, \\ A_n - A_n^{\text{ex}} &\rightarrow 0 \text{ in } \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3), \\ c'_n - \rho_n h^{\text{ex}'} &\rightarrow c' - \rho h^{\text{ex}'} \text{ in } L^2(\mathbb{R}^3; \mathbb{R}^2), \end{aligned}$$

with c'_n as in (4.2), and

$$\lim_{n \rightarrow \infty} I_{\varepsilon_n}(u_n, A_n) = \frac{1}{2} \int_{\omega} \left(|(\nabla' - iA_{\perp}^{\text{ex}'})u|^2 + |b|^2 + \frac{\kappa^2}{2} (1 - |u|^2)^2 \right) dx' + \frac{1}{2} \int_{\mathbb{R}^3} |c - \rho h^{\text{ex}'}|^2 dx.$$

(ii) Let $(u, b) \in \mathcal{V}_0$ and let $\{\varepsilon_n\} \in \mathbb{R}$ be a sequence such that $\varepsilon_n \rightarrow 0^+$ and $\rho_n \rightarrow \infty$. Then, there exist sequences $\{u_n\} \subset H^1(\Omega; \mathbb{C})$ and $\{A_n - A_n^{\text{ex}}\} \subset \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3)$ such that

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } H^1(\Omega; \mathbb{C}), \\ b_n &:= \frac{1}{\varepsilon_n d(x')} \int_{f(x')}^{g(x')} \partial_3 \left(u_n e^{-i \int_0^{x_3} (A_n)_3(x', t) dt} \right) dx_3 \equiv b \text{ a.e. in } \omega, \\ A_n - A_n^{\text{ex}} &\rightarrow 0 \text{ in } \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3), \\ c'_n &\equiv h^{\text{ex}'} \text{ a.e. in } \mathbb{R}^3, \end{aligned}$$

with c'_n as in (4.3), and

$$\lim_{n \rightarrow \infty} I_{\varepsilon_n}(u_n, A_n) = \frac{1}{2} \int_{\omega} \left(|(\nabla' - iA_{\perp}^{\text{ex}'})u|^2 + |b|^2 + \frac{\kappa^2}{2} (1 - |u|^2)^2 \right) dx'.$$

Proof.

We divide the proof in two parts.

(i) Applying Lemma 4.5 to $w' = c' - \rho h^{\text{ex}'}$, we find a sequence of potentials $B_n \in \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3)$. We define

$$A_n(x) := A_n^{\text{ex}}(x) + \varepsilon_n B_n(x),$$

so that

$$c'_n = \frac{1}{\varepsilon_n} (\nabla \times A_n)' = \rho_n h^{\text{ex}'} + (\nabla \times B_n)',$$

and $(\nabla \times B_n)' \rightarrow w' = c' - \rho h^{\text{ex}'}$. Then define

$$u_n(x) = e^{i\varepsilon_n \int_0^{x_3} (B_n)_3(x', t) dt} (u(x') + \varepsilon_n b(x') x_3).$$

Then, we prove first that the convergences in the proposition hold.

First, note that

$$\begin{aligned} |u_n - u| &= \left| u \left(e^{i\varepsilon_n \int_0^{x_3} (B_n)_3(x',t) dt} - 1 \right) + \varepsilon_n e^{i\varepsilon_n \int_0^{x_3} (B_n)_3(x',t) dt} b(x') x_3 \right| \\ &\leq |u| \left| e^{i\varepsilon_n \int_0^{x_3} (B_n)_3(x',t) dt} - 1 \right| + \varepsilon_n |b(x')| \end{aligned}$$

so that

$$\begin{aligned} |u_n - u| &\rightarrow 0 \text{ a.e. in } \Omega, \\ |u_n - u|^2 &\leq 4|u| + C\varepsilon_n \text{ which is integrable in } \Omega. \end{aligned}$$

Using Lebesgue's Dominated Convergence, we deduce that $u_n \rightarrow u$ in $L^2(\Omega; \mathbb{C})$. Since $\{\nabla u_n\}$ is bounded in $L^2(\Omega; \mathbb{C}^3)$, we know that $u_n \rightharpoonup u$ in $H^1(\Omega; \mathbb{C})$, and $b_n \equiv b$. Also, $\varepsilon_n B_n \rightarrow 0$ in $\check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3)$, so $A_n - A_n^{\text{ex}} \rightharpoonup 0$ in $\check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3)$.

By convergence of B'_n , we have that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{\varepsilon_n^2} \int_{\mathbb{R}^3} |h' - \varepsilon_n \rho_n h^{\text{ex}'}|^2 &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |c'_n - \rho_n h^{\text{ex}'}|^2 dx \\ &= \int_{\mathbb{R}^3} |c' - \rho h^{\text{ex}'}|^2 dx, \end{aligned}$$

and because $\varepsilon_n \nabla \times B_n \rightarrow 0$ in $L^2(\mathbb{R}^3; \mathbb{R}^3)$, we have that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |(\nabla \times A_n)_3 - h_3^{\text{ex}}|^2 dx = 0.$$

Moreover, we know that

$$|u_n(x)|^2 = |u(x') + \varepsilon_n b(x') x_3|^2 = |u|^2 + O_{L^1}(\varepsilon_n)$$

thus

$$(1 - |u_n|^2)^2 = (1 - |u|^2)^2 + O_{L^1}(\varepsilon_n) \rightarrow (1 - |u|^2)^2 \text{ in } L^1(\Omega).$$

By Lebesgue's Dominated Convergence Theorem, we obtain that

$$\lim_{n \rightarrow \infty} \frac{1}{2} \int_{\Omega} (1 - |u_n|^2)^2 dx = \frac{1}{2} \int_{\omega} (1 - |u|^2)^2 dx'.$$

As for the covariant term, we have

$$|(\nabla' - iA'_n)u_n|^2 = |(\nabla' - iA_{\perp}^{\text{ex}'})u|^2 + O_{L^1}(\varepsilon_n),$$

hence

$$\lim_{n \rightarrow \infty} \frac{1}{2} \int_{\Omega} |(\nabla' - iA'_n)u_n|^2 dx = \frac{1}{2} \int_{\omega} |(\nabla' - iA_{\perp}^{\text{ex}'})u|^2 dx'.$$

This completes the proof for part (i).

(ii) In this case, when $\rho_n \rightarrow \infty$, we define

$$A_n(x) := A_{\perp}^{\text{ex}}(x') + \varepsilon_n \rho_n A_{\parallel}^{\text{ex}}(x),$$

and

$$u_n(x) := e^{i\varepsilon_n \rho_n \int_0^{x_3} A_{\parallel}^{\text{ex}}(x', t) dt} (u(x') + \varepsilon_n b(x') x_3).$$

Then, we prove first that the convergences in the proposition hold. As in the previous case, we deduce that $u_n \rightharpoonup u$ in $H^1(\Omega; \mathbb{C})$, and $b_n \equiv b$. Also, $A_n - A_{\perp}^{\text{ex}} \rightarrow 0$ in $\check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3)$, and $c'_n \equiv h^{\text{ex}}$ in $L^2(\mathbb{R}^3; \mathbb{R}^2)$. Moreover,

$$I_{\varepsilon_n}(u_n, A_n) = \frac{1}{2} \int_{\Omega} \left(|(\nabla' - iA'_n)u_n|^2 + |b|^2 + \frac{\kappa^2}{2}(1 - |u_n|^2)^2 \right) dx.$$

Following the same reasoning as in the other case, we obtain

$$\lim_{n \rightarrow \infty} I_{\varepsilon_n}(u_n, A_n) = \frac{1}{2} \int_{\omega} \left(|(\nabla' - iA_{\perp}^{\text{ex}})u|^2 + |b|^2 + \frac{\kappa^2}{2}(1 - |u|^2)^2 \right) dx'.$$

This completes the proof. ■

5. Supercritical Case

Theorem 5.1 (Compactness). *Let $\varepsilon_n \rightarrow 0^+$ as $n \rightarrow +\infty$ and let $\{u_n, A_n - A^{\text{ex}}\}_{n \in \mathbb{N}} \subset H^1(\Omega; \mathbb{C}) \times \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3)$ be such that*

$$\sup_{n \in \mathbb{N}} I_{\varepsilon_n}(u_n, A_n) < +\infty.$$

Then there exist a subsequence $\{\varepsilon_n\}$ (not relabeled) such that

$$\begin{aligned} u_n &\rightarrow 0 \text{ in } L^2(\Omega; \mathbb{C}), \\ \frac{1}{\varepsilon_n \rho_n} A_n - A_{\parallel}^{\text{ex}} &\rightarrow 0 \text{ in } \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3), \end{aligned}$$

Theorem 5.2 (Γ -limit). *Let $(u, A) \in L^1(\Omega; \mathbb{C}) \times L^1(\mathbb{R}^3; \mathbb{R}^3)$. Then*

$$\Gamma - \lim_{\varepsilon \rightarrow 0^+} I_{\varepsilon}(u, A) = \begin{cases} \frac{\kappa^2}{4} |\Omega| & \text{if } u \equiv 0 \text{ and } A = A_{\parallel}^{\text{ex}} \\ \infty & \text{otherwise.} \end{cases}$$

5.1. Compactness

Lemma 5.3. *Let $\{f_n\}, \{g_n\} \subset L^2(\Omega; \mathbb{C}^2)$ be such that*

$$\begin{aligned} f_n &\rightharpoonup 0 \text{ in } L^2(\Omega; \mathbb{C}^2), \\ g_n &\rightharpoonup g \text{ in } L^2(\Omega; \mathbb{C}^2). \end{aligned}$$

Assume further that $f_n - g_n \rightarrow 0$ in $L^2(\Omega; \mathbb{C}^2)$.

Then

$$g = 0.$$

Proof of Theorem 5.1.

Let $K := \sup_{n \in \mathbb{N}} I_{\varepsilon_n}(u_n, A_n) < \infty$. Then define

$$\begin{aligned} w_n &:= \frac{1}{\varepsilon_n \rho_n} u_n, \\ B_n &:= \frac{1}{\varepsilon_n \rho_n} A_n, \\ \ell_n &:= \nabla \times B_n = \frac{1}{\varepsilon_n \rho_n} h_n, \\ e_n &:= \lambda_n^{-1}(\ell_n)_3 = (h_n)_3. \end{aligned}$$

Then

$$\int_{\mathbb{R}^3} \left(\rho_n^2 |\ell'_n - h^{\text{ex}'}|^2 + |(h_n)_3 - h_3^{\text{ex}}|^2 \right) dx \leq K,$$

This implies that $\nabla \times (B_n - A^{\text{ex}})$ is bounded in L^2 , so since $\text{div } B_n = 0$ we know that

$$\sup_{n \in \mathbb{N}} \|\nabla(B_n - A^{\text{ex}})\|_{L^2(\mathbb{R}^3; \mathbb{R}^{3 \times 3})} < \infty,$$

and $(\ell_n)_3 \rightarrow 0$ in $L^2(\mathbb{R}^3)$.

By Lemma 3.1, we deduce that $\{B_n - A^{\text{ex}}\}$ is bounded in \check{H}_{div}^1 , thus there exists a subsequence (not relabeled) such that

$$B_n - A^{\text{ex}} \rightharpoonup B - A^{\text{ex}} \quad \text{in } \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3).$$

and

$$\text{div } B = 0, \quad \nabla \times B = (h_1^{\text{ex}}, h_2^{\text{ex}}, 0).$$

Moreover consider $B - A_{\parallel}^{\text{ex}}$, which satisfies $\nabla \times (B - A_{\parallel}^{\text{ex}}) = 0$. By the uniqueness in Lemma 3.1, we deduce that

$$B \equiv A_{\parallel}^{\text{ex}}.$$

On the other hand, we know that $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $L^4(\Omega; \mathbb{C})$, that ∇B_n is bounded in $L^2(\Omega; \mathbb{R}^{3 \times 3})$, and

$$\int_{\Omega} \left(|\nabla' u_n - i A'_n u_n|^2 + \left| \frac{1}{\varepsilon_n} \partial_3 u_n - i (A_n)_3 u_n \right|^2 \right) dx \leq K,$$

so

$$(\varepsilon_n \rho_n)^2 \int_{\Omega} \left(\left| \nabla' \left(\frac{u_n}{\varepsilon_n \rho_n} \right) - i B'_n u_n \right|^2 + \left| \frac{1}{\varepsilon_n} \partial_3 \left(\frac{u_n}{\varepsilon_n \rho_n} \right) - i (B_n)_3 u_n \right|^2 \right) dx \leq K.$$

This yields that $\{w_n\}$ is bounded in $H^1(\Omega; \mathbb{C})$, so we may extract a further subsequence (not relabeled) such that

$$w_n \rightharpoonup w \quad \text{in } H^1(\Omega; \mathbb{C}).$$

On the other hand, we know that $\{u_n\}$ is bounded in $L^2(\Omega; \mathbb{C})$, so we can extract another subsequence (not relabeled) such that

$$u_n \rightharpoonup u \quad \text{in } L^2(\Omega; \mathbb{C}),$$

which implies that $w = 0$. So

$$w_n \rightharpoonup 0 \text{ in } H^1(\Omega; \mathbb{C}).$$

We now know that

$$\begin{aligned} \nabla' w_n &\rightharpoonup 0 \text{ in } L^2(\Omega; \mathbb{C}^2), \\ iB'_n u_n &\rightharpoonup iA_{\parallel}^{\text{ex}'} u \text{ in } L^2(\Omega; \mathbb{C}^2), \\ \nabla' w_n - iB'_n u_n &\rightarrow 0 \text{ in } L^2(\Omega; \mathbb{C}^2), \end{aligned}$$

so by Lemma 5.3, we deduce that

$$iB'_n u_n \rightarrow 0 \text{ in } L^2(\Omega; \mathbb{C}^2).$$

Since $iB'_n u_n \rightarrow iA_{\parallel}^{\text{ex}'} u$ pointwise, and $A_{\parallel}^{\text{ex}'} \neq 0$, we conclude that $u = 0$, so we know that

$$u \rightarrow 0 \text{ in } L^2(\Omega; \mathbb{C}).$$

■

5.2. The Γ – lim inf inequality

Proposition 5.4 (Γ – lim inf inequality).

Consider sequences $\{\varepsilon_n\} \subset \mathbb{R}$, $\{u_n\} \subset H^1(\Omega; \mathbb{C})$, and $\{A_n - A^{\text{ex}}\} \subset \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3)$ satisfying

$$\begin{aligned} \varepsilon_n &\rightarrow 0^+, \\ u_n &\rightarrow 0 \text{ in } L^2(\Omega; \mathbb{C}), \\ \frac{1}{\varepsilon_n \rho_n} A_n - A_{\parallel}^{\text{ex}} &\rightharpoonup 0 \text{ in } \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3). \end{aligned}$$

Then

$$\liminf_{n \rightarrow \infty} I_{\varepsilon_n}(u_n, A_n) \geq \frac{\kappa^2}{4} |\Omega|.$$

Proof.

Since $u_n \rightarrow 0$ in $L^2(\Omega; \mathbb{C})$, we have

$$\liminf_{n \rightarrow \infty} I_{\varepsilon_n}(u_n, A_n) \geq \frac{\kappa^2}{4} \liminf_{n \rightarrow \infty} \int_{\Omega} (1 - |u_n|^2)^2 dx = \frac{\kappa^2}{4} |\Omega|.$$

This completes the proof.

■

5.3. The Γ – lim sup inequality

Proposition 5.5 (Γ – lim sup inequality).

Let $\{\varepsilon_n\} \in \mathbb{R}$ be a sequence such that $\varepsilon_n \rightarrow 0^+$. Then, there exist sequences $\{u_n\} \subset H^1(\Omega; \mathbb{C})$ and $\{A_n - A^{\text{ex}}\} \subset \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3)$ such that

$$\begin{aligned} u_n &\rightarrow 0 \text{ in } L^2(\Omega; \mathbb{C}), \\ \frac{1}{\varepsilon_n \rho_n} A_n - A_{\parallel}^{\text{ex}} &\rightharpoonup 0 \text{ in } H^1(\mathbb{R}^3; \mathbb{R}^3), \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} I_{\varepsilon_n}(u_n, A_n) = \frac{\kappa^2}{4} |\Omega|.$$

Proof.

First, define

$$\begin{aligned} u_n(x) &:= 0, \\ A_n(x) &:= \varepsilon_n \rho_n A_{\parallel}^{\text{ex}} + A_{\perp}^{\text{ex}}. \end{aligned}$$

Then

$$B_n - A_{\parallel}^{\text{ex}} := \frac{1}{\varepsilon_n \rho_n} A_n - A_{\parallel}^{\text{ex}} = \frac{A_{\perp}^{\text{ex}}}{\varepsilon_n \rho_n} \rightharpoonup 0 \text{ in } \check{H}_{\text{div}}^1(\mathbb{R}^3; \mathbb{R}^3).$$

Moreover,

$$I_{\varepsilon_n}(u_n, A_n) = \frac{\kappa^2}{4} |\Omega|.$$

This completes the proof. ■

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