# Chain of Separable Binary Goppa Codes and their Minimal Distance

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***Abstract*—It is shown that subclasses of separable binary Goppa codes , $\Gamma(L,G)$ - codes, with $L=\{\alpha \in GF(2^{2l}) : G(\alpha) \neq 0\}$ and special Goppa polynomials $G(x)$ can be presented as a chain of embedded codes. The true minimal distance has been obtained for all codes of the chain.**

***Index Terms*—Goppa codes, quasi-cyclic Goppa codes, minimal distance of separable binary codes.**

## I. Introduction

Any $q$-ary Goppa code $\Gamma(L,G)$ -code can be defined by two objects: Goppa polynomial $G(x)$ where $G(x)$ is a polynomial of a degree $t$ over $GF(q^m)$ and location set $L$ where $L=\{\alpha \in GF(q^m) : G(\alpha) \neq 0\}$.

*Definition 1:* A $q$-ary vector $a=(a_1...a_n)$ of a length $n$ where $n$ is a cardinality of the set $L=(\alpha_1,...,\alpha_n), \alpha_i \in GF(q^m)$ is a codeword of the $\Gamma(L,G)$ Goppa code if and only if the following equation is satisfied:

$$\sum_{i=1}^{n} a_i \frac{1}{x-\alpha_i} \equiv 0 \; mod G(x).$$

The parity check matrix of the $\Gamma(L,G)$ -code can be presented in the following form:

$$H=\begin{bmatrix} \frac{1}{G(\alpha_1)} & \cdots & \frac{1}{G(\alpha_n)} \\ \frac{\alpha_1}{G(\alpha_1)} & \cdots & \frac{\alpha_n}{G(\alpha_n)} \\ \cdot & \cdot & \cdot \\ \frac{\alpha_1^{t-1}}{G(\alpha_1)} & \cdots & \frac{\alpha_n^{t-1}}{G(\alpha_n)} \end{bmatrix}.$$

It is known that the $\Gamma(L,G)$ -code has following parameters [1]:

the length of the code is equal to the cordinality $n$ of the location set $L$, $n \leq q^m$

the dimension is $k \geq n-tm$ and

the true minimal distance is $d \geq t+1$.

The $\Gamma(L,G)$-code is a binary Goppa code if $q=2$.

The $\Gamma(G,L)$- code is a separable Goppa code if all roots of its Goppa polynomial $G$ are different.

The following estimation of the true minimal distance for binary separable (L,G)-codes is valid [1]:

$$d \geq 2t+1.$$

Binary separable Goppa codes have been studied by many authors. The binary separable codes with the location set over $GF(2^{sl})$ where $s=2,3,...$ are of the greatest interest.

M. Loeloeian and J. Conan were the first who considered the code of this class. In 1984 they presented [2] the best known (55,16,19)- Goppa code with the Goppa polynomial:

$$G(x)=(x-\alpha^9)\dot{(}x-\alpha^{12})\dot{(}x-\alpha^{30})\dot{(}x-\alpha^{34})\dot{(}x-\alpha^{42})\dot{(}x-\alpha^{43})\dot{(}x-\alpha^{50})\dot{(}x-\alpha^{54})$$

where $\alpha$ is a primitive element of $GF(2^6)$.

In 1986 we considered [3] this code as a code from a subclass of Goppa codes with the Goppa polynomial:
$G(x)=x^{t+1}+V^tx^t+Vx+1$
where $V \in GF(2^{2l})$ , $t=2^l$ , $L \subset GF(2^{2l})$ and $n=2^{2l}-t-1$.
We have proved [3], [4] that the dimension of these codes is $k \geq n-2l(t-\frac{3}{2})$.

In 1987 M. Loeloeian and J. Conan [5] considered a subclass of Goppa codes with the Goppa polynomial:
$G(x)=x^t+x$
where $t=2^l$ ,$L \subset GF(2^{2l})$ and $n=2^{2l}-t$.
They also gave the estimation for the dimension of these codes: $k \geq n-2l(t-\frac{3}{2})-1$.

The same estimation for the dimension of these codes was obtained by A.M.Roseiro, J.I.Hall, J.E.Adney and M.Siegel in [6] by using the kernel of an associated trace map. In this paper, the subclass of codes with polynomial $G(x)=x^t+x$ was called as quadratic trace Goppa codes.

In 1995 we described [7] the subclass of Goppa codes with the polynomial $G(x)=x^{t-1}+1$ and we have proved that the minimal distance of these codes is equal to their design distance.

In 2001 P. Veron [8] investigated the structure of trace Goppa codes and proved that the true dimension for these codes is equal to the estimation obtained previously:
$k \geq n-2l(t-\frac{3}{2})-1$

In 2005 P. Veron [9] proved that the estimation of the code dimension for Goppa codes with $G(x)=x^{t+1}+V^tx^t+Vx+1$ and $G(x)=x^{t-1}+1$ is the true dimension for the codes from this subclasses [10].

In [11] and also in [12] (G. Bommer and F. Blanchet) and in [13] (P. Veron) it was proved that all the mentioned above codes are a quasi-cyclic binary Goppa codes.

In 2007 G.Maatouk, A.Shokrollahi and M.Cheraghchi [14] tried to prove that the class of codes which was described in [7] achieved the GV bound.

In this paper, we present all codes that were mentioned above as a chain of embedded codes. We obtain the true minimal distance for these codes. The rest of the paper is organized as follows.

Section 2 describes the chain of Goppa codes subclasses.

Section 3 gives several Lemmas for the true minimal distance for subfield subcodes:
$G_5(x)=Cx^{t+1}+A^tx^t+Ax$,

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$G_6(x) = Rx^{t+1} + V^t x^t + Vx + 1$
where $R \in GF(2^l), V \in GF(2^{2l})$ and $G_7(x) = x^{t+1} + 1$.

In Section 4 similar Lemmas are presented for quadratic trace Goppa codes:
$G_2(x) = A^t x^t + Ax$ and
$G_3(x) = A^t x^t + Ax + C$,
where $A \in GF(2^{2l})$ and $C \in GF(2^l)$).

In Section 5 we obtain the true minimal distance for codes that are not subfield or trace codes: these new codes with a Goppa polynomial
$G_4(x) = A^t x^t + A^{t-1} x^{t-1} + 1$
have not been described before.

In Conclussion a table with parameters of codes from the code chain and table of binary quasi-cyclic codes from this chain are presented.

## II. Code chain

Let us show how to obtain one code from another and to create a family of embedded codes, a so-called chain of codes.

*Definition 2:* Let matrix $H_1^*$ be a parity check matrix of the binary Goppa code with a location set $L_1^* = \{\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_n}\}$ of different nonzero elements from $GF(2^{2l})$ such that $\alpha_{i_j}^{t-1} \neq 1$ for all $j = 1, .., n$ and Goppa polynomial $G_1(x) = x^{t-1} + 1, t = 2^l$ :

$$H_1^* = \begin{bmatrix} \frac{1}{\alpha_{i_1}^{t-1}+1} & \cdots & \frac{1}{\alpha_{i_n}^{t-1}+1} \\ \frac{\alpha_{i_1}}{\alpha_{i_1}^{t-1}+1} & \cdots & \frac{\alpha_{i_n}}{\alpha_{i_n}^{t-1}+1} \\ . & . & . \\ \frac{\alpha_{i_1}^{t-1}}{\alpha_{i_1}^{t-1}+1} & \cdots & \frac{\alpha_{i_n}^{t-1}}{\alpha_{i_n}^{t-1}+1} \end{bmatrix}.$$

*Lemma 1:* A row $\begin{bmatrix} \frac{1}{\alpha_{i_1}(\alpha_{i_1}^{t-1}+1)} & \cdots & \frac{1}{\alpha_{i_n}(\alpha_{i_n}^{t-1}+1)} \end{bmatrix}$ can be represented as a linear combination of corresponding rows from the matrix $H_1^*$.

*Proof:*

For any $\alpha \in L_1^*$

$$\frac{\alpha^{2^{l-1}-1}}{\alpha^{t-1}+1} = \frac{\alpha^{2^{l-1}-1}}{\alpha^{(t-1)}(\alpha^{-(t-1)}+1)} = \frac{1}{\alpha^{2^{l-1}}(\alpha^{-(t-1)}+1)} = \frac{1}{\alpha^{2^{l-1}}(\alpha^{t-1}+1)^{2^l}} = \left(\frac{1}{\alpha(\alpha^{t-1}+1)^2}\right)^{2^{l-1}}.$$

Therefore the row $\begin{bmatrix} \frac{1}{\alpha_{i_1}(\alpha_{i_1}^{t-1}+1)^2} & \cdots & \frac{1}{\alpha_{i_n}(\alpha_{i_n}^{t-1}+1)^2} \end{bmatrix}$ can be obtained from the row $\begin{bmatrix} \frac{\alpha_{i_1}^{2^{l-1}-1}}{\alpha_{i_1}^{t-1}+1} & \cdots & \frac{\alpha_{i_n}^{2^{l-1}-1}}{\alpha_{i_n}^{t-1}+1} \end{bmatrix}$ of the matrix $H_1^*$ .

For any $\alpha \in L_1^*$

$$\frac{1}{\alpha(\alpha^{t-1}+1)^2} + \left(\frac{\alpha^{2^{l-1}-1}}{(\alpha^{t-1}+1)}\right)^2 = \frac{1}{\alpha(\alpha^{t-1}+1)^2} + \frac{\alpha^{2^l-2}}{(\alpha^{t-1}+1)^2} = \frac{1}{\alpha(\alpha^{t-1}+1)^2} + \frac{\alpha^{2^l-1}}{\alpha(\alpha^{t-1}+1)^2} = \frac{1}{\alpha(\alpha^{t-1}+1)}.$$

Therefore the row $\begin{bmatrix} \frac{1}{\alpha_{i_1}(\alpha_{i_1}^{t-1}+1)} & \cdots & \frac{1}{\alpha_{i_n}(\alpha_{i_n}^{t-1}+1)} \end{bmatrix}$ can be obtained from the row $\begin{bmatrix} \frac{\alpha_{i_1}^{2^{l-1}-1}}{\alpha_{i_1}^{t-1}+1} & \cdots & \frac{\alpha_{i_n}^{2^{l-1}-1}}{\alpha_{i_n}^{t-1}+1} \end{bmatrix}$ of the matrix $H_1^*$. ■

*Corollary 1:* By using the result of *Lemma 1* we can rewrite the matrix $H_1^*$ in the following form:

$$H_1^* = \begin{bmatrix} \frac{1}{\alpha_{i_1}\left(\alpha_{i_1}^{t-1}+1\right)} & \cdots & \frac{1}{\alpha_{i_n}\left(\alpha_{i_n}^{t-1}+1\right)} \\ \frac{1}{\alpha_{i_1}^{t-1}+1} & \cdots & \frac{1}{\alpha_{i_n}^{t-1}+1} \\ \frac{\alpha_{i_1}}{\alpha_{i_1}^{t-1}+1} & \cdots & \frac{\alpha_{i_n}}{\alpha_{i_n}^{t-1}+1} \\ . & . & . \\ \frac{\alpha_{i_1}^{t-1}}{\alpha_{i_1}^{t-1}+1} & \cdots & \frac{\alpha_{i_n}^{t-1}}{\alpha_{i_n}^{t-1}+1} \end{bmatrix}.$$

Now let us obtain the parity check matrix $H_1$ for a code $\Gamma(L_1, G_1)$ with $G_1(x) = x^{t-1} + 1,\ t = 2^l$ and $L_1 = \{\alpha_1, \alpha_2, ..., \alpha_{n_1-1}, 0\},\ n_1 = 2^{2l} - 2^l + 1$ :

$$H_1 = \begin{bmatrix} \frac{1}{\alpha_1^{t-1}+1} & \cdots & \frac{1}{\alpha_{n_1-1}^{t-1}+1} & 1 \\ \frac{\alpha_1}{\alpha_1^{t-1}+1} & \cdots & \frac{\alpha_{n_1-1}}{\alpha_{n_1-1}^{t-1}+1} & 0 \\ . & . & . & . \\ \frac{\alpha_1^{t-1}}{\alpha_1^{t-1}+1} & \cdots & \frac{\alpha_{n_1-1}^{t-1}}{\alpha_{n_1-1}^{t-1}+1} & 0 \end{bmatrix}.$$

By using the power $2^l$ of the first row from $H_1$ we obtain:

$$\begin{bmatrix} \frac{\alpha_1^{t-1}}{\alpha_1^{t-1}+1} & \cdots & \frac{\alpha_{n_1-1}^{t-1}}{\alpha_{n_1-1}^{t-1}+1} & 1 \end{bmatrix}.$$

The sum of this row and the first row from the matrix $H_1$ gives us

$$\begin{bmatrix} \frac{\alpha_1^{t-1}}{\alpha_1^{t-1}+1} & \cdots & \frac{\alpha_{n_1-1}^{t-1}}{\alpha_{n_1-1}^{t-1}+1} & 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{\alpha_1^{t-1}+1} & \cdots & \frac{1}{\alpha_{n_1-1}^{t-1}+1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & ... & 1 & 0 \end{bmatrix}.$$

Therefore the parity check matrix $H_1$ can be rewritten:

$$H_1 = \begin{bmatrix} \frac{1}{\alpha_1^{t-1}+1} & \cdots & \frac{1}{\alpha_{n_1-1}^{t-1}+1} & 1 \\ \frac{\alpha_1}{\alpha_1^{t-1}+1} & \cdots & \frac{\alpha_{n_1-1}}{\alpha_{n_1-1}^{t-1}+1} & 0 \\ . & . & . & . \\ \frac{\alpha_1^{t-1}}{\alpha_1^{t-1}+1} & \cdots & \frac{\alpha_{n_1-1}^{t-1}}{\alpha_{n_1-1}^{t-1}+1} & 0 \\ 1 & ... & 1 & 0 \end{bmatrix}.$$

Let us define a parity check matrix for a subcode $\Gamma(L_1^*, G_1)$ of the code $\Gamma(L_1, G_1)$ ,
$L_1^* = L_1 \backslash \{0\} = \{\alpha_1, \alpha_2, ..., \alpha_{n_1-1}\}$, $n_1^* = n_1 - 1$.

$$\begin{bmatrix} \frac{1}{\alpha_1^{t-1}+1} & \cdots & \frac{1}{\alpha_{n_1-1}^{t-1}+1} \\ \frac{\alpha_1}{\alpha_1^{t-1}+1} & \cdots & \frac{\alpha_{n_1-1}}{\alpha_{n_1-1}^{t-1}+1} \\ . & . & . \\ \frac{\alpha_1^{t-1}}{\alpha_1^{t-1}+1} & \cdots & \frac{\alpha_{n_1-1}^{t-1}}{\alpha_{n_1-1}^{t-1}+1} \\ 1 & ... & 1 \end{bmatrix} = \begin{bmatrix} H_1^* \\ 1...1 \end{bmatrix}.$$

*Lemma 2:* $\Gamma(L_1^*, G_1) \equiv \Gamma(L_2, G_2)$ where $G_1(x) = x^{t-1} + 1$ and $G_2(x) = A^t x^t + Ax,\ t = 2^l,\ A \in GF(2^{2l})$.

*Proof:*

Obviously, $\Gamma(L_1, G_2) \equiv \Gamma(L_2, G_2^*)$ where $G_2^*(x) = x^t + x = x * G_1(x)$ and the Goppa polynomial $G_2(x) = A^t x^t + Ax$ can be obtained from $G_2^*(x)$ by using the $Ax$ substitution for a variable $x$, $A \in GF(2^{2l})$.

A parity check matrix $H_2$ for the code $\Gamma(L_2, G_2^*)$ with $G_2^*(x) = x^t + x,\ t = 2^l$ and $L_2 = \{\alpha_1, \alpha_2, ..., \alpha_{n_2}\},\ n_2 = 2^{2l} - 2^l$ is :

$$H_2 = \begin{bmatrix} \frac{1}{\alpha_1^t+\alpha_1} & \cdots & \frac{1}{\alpha_{n_2}^t+\alpha_{n_2}} \\ \frac{\alpha_1}{\alpha_1^t+\alpha_1} & \cdots & \frac{\alpha_{n_2}}{\alpha_{n_2}^t+\alpha_{n_2}} \\ \cdot & \cdot & \cdot \\ \frac{\alpha_1^{t-1}}{\alpha_1^t+\alpha_1} & \cdots & \frac{\alpha_{n_2}^{t-1}}{\alpha_{n_2}^t+\alpha_{n_2}} \end{bmatrix}.$$

It is easy to see that this matrix can be rewritten in the following form:

$$H_2 = \begin{bmatrix} \frac{1}{\alpha_1(\alpha_1^{t-1}+1)} & \cdots & \frac{1}{\alpha_{n_2}(\alpha_{n_2}^{t-1}+1)} \\ \frac{1}{\alpha_1^{t-1}+1} & \cdots & \frac{1}{\alpha_{n_2}^{t-1}+1} \\ \cdot & \cdot & \cdot \\ \frac{\alpha_1^{t-2}}{\alpha_1^{t-1}+1} & \cdots & \frac{\alpha_{n_2}^{t-2}}{\alpha_{n_2}^{t-1}+1} \end{bmatrix}$$

From *Corollary 1* the matrix $H_2$ is equal to the matrix $H_1^*$ , therefore $\Gamma(L_1^*, G_1) \equiv \Gamma(L_2, G_2^*) \equiv \Gamma(L_2, G_2)$. ■
(This statement has been proved by P.Veron in [13], [9] by using another approach.)

*Lemma 3:* $\Gamma(L_2, G_2) \equiv \Gamma(L_3, G_3)$ where $G_3(x) = A^t x^t + Ax + C$, $C \in GF(2^l)$ and $A \in GF(2^{2l})$.

*Proof:*

Using the $x + \beta$ substitution for a variable $x$ where $\beta \in GF(2^{2l})$ and $\beta \neq A^{-(t-1)}$ we obtain:

$G_2(x+\beta) = A^t(x+\beta)^t + A(x+\beta) = A^t x^t + Ax + (A^t\beta^t + A\beta)$

where $(A^t\beta^t + A\beta)^{2^l} = (A\beta + A^t\beta^t)$ follows from the conditions of *Lemma 3*. Therefore $C = A^t\beta^t + A\beta$ , $C \in GF(2^l)$ and $G_2(x + \beta) = A^t x^t + Ax + C = G_3(x)$ . ■

*Lemma 4:* All codewords of the code $\Gamma(L_4, G_4)$ with $G_4(x) = A^t x^t + A^{t-1}x^{t-1} + 1$ have the zero value on the position correspondig to the element $0$ from $L_4 = \{\alpha_1, \alpha_2, ..., \alpha_{n_4-1}, 0\}$.

*Proof:*

Obviously, by substituting $x$ to $A^{-1}x$ in $G_4(x)$ we obtain the same $\Gamma(L_4, G_4)$ code with a more simple Goppa polynomial: $G_4(x) = x^t + x^{t-1} + 1$.

Let us consider the parity check matrix of this code:

$$H_4 = \begin{bmatrix} \frac{1}{\alpha_1^t+\alpha_1^{t-1}+1} & \cdots & \frac{1}{\alpha_{n_4-1}^t+\alpha_{n_4-1}^{t-1}+1} & 1 \\ \frac{\alpha_1}{\alpha_1^t+\alpha_1^{t-1}+1} & \cdots & \frac{\alpha_{n_4-1}}{\alpha_{n_4-1}^t+\alpha_{n_4-1}^{t-1}+1} & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\alpha_1^{t-1}}{\alpha_1^t+\alpha_1^{t-1}+1} & \cdots & \frac{\alpha_1^{t-1}}{\alpha_{n_4-1}^t+\alpha_{n_4-1}^{t-1}+1} & 0 \end{bmatrix}.$$

By using the $t-$th degree of the first row of this parity check matrix we can obtain the following parity check row for $\Gamma(L_4, G_4)$ :

$$r = \begin{bmatrix} \frac{\alpha_1^{t-1}}{\alpha_1^t+\alpha_1^{t-1}+1} & \cdots & \frac{\alpha_1^{t-1}}{\alpha_{n_4-1}^t+\alpha_{n_4-1}^{t-1}+1} & 1 \end{bmatrix}.$$

For this row and for the last row of the parity check matrix $H_4$ and parity check row $r$ for any codeword $a = (a_1...a_{n_4})$ of the code $\Gamma(L_4, G_4)$ the following expressions are valid:

$\sum_{i=1}^{n_4-1} a_i \frac{\alpha_i^{t-1}}{\alpha_i^t+\alpha_i^{t-1}+1} = 0$ and $\sum_{i=1}^{n_4-1} a_i \frac{\alpha_i^{t-1}}{\alpha_i^t+\alpha_i^{t-1}+1} = a_{n_4}$.

It is possible only in case when $a_{n_4} = 0$ for all codewords of the code $\Gamma(L_4, G_4)$. ■

*Corollary 2:* The code $\Gamma(L_4, G_4)$ is equal to the code $\Gamma(L_4^*, G_4)$ with $n_4^* = n_4 - 1$, $k_4^* = k_4$ and $L_4^* = \{\alpha_1, \alpha_2, ..., \alpha_{n_4-1}\}$.

The parity check matrix $H_4^*$of the code $\Gamma(L_4^*, G_4)$ is:

$$H_4^* = \begin{bmatrix} \frac{1}{\alpha_1^t+\alpha_1^{t-1}+1} & \cdots & \frac{1}{\alpha_{n_4-1}^t+\alpha_{n_4-1}^{t-1}+1} \\ \frac{\alpha_1}{\alpha_1^t+\alpha_1^{t-1}+1} & \cdots & \frac{\alpha_{n_4-1}}{\alpha_{n_4-1}^t+\alpha_{n_4-1}^{t-1}+1} \\ \cdot & \cdot & \cdot \\ \frac{\alpha_1^{t-1}}{\alpha_1^t+\alpha_1^{t-1}+1} & \cdots & \frac{\alpha_1^{t-1}}{\alpha_{n_4-1}^t+\alpha_{n_4-1}^{t-1}+1} \end{bmatrix}.$$

*Lemma 5:* A row $\begin{bmatrix} \frac{1}{\alpha_{i_1}(\alpha_1^t+\alpha_1^{t-1}+1)} & \cdots & \frac{1}{\alpha_{i_n}(\alpha_1^t+\alpha_1^{t-1}+1)} \end{bmatrix}$ can be represented as a linear combination of the corresponding rows from the matrix $H_4^*$.

*Proof:*

For any $\alpha \in L_4^*$

$\frac{\alpha^{2^{l-1}-1}}{(\alpha^t+\alpha^{t-1}+1)} = \frac{\alpha^{2^{l-1}-1}}{\alpha^{(t-1)}(\alpha+\alpha^{-(t-1)}+1)} = \frac{1}{\alpha^{2^{l-1}}(\alpha+\alpha^{-(t-1)}+1)} = \frac{1}{\alpha^{2^{l-1}}(\alpha^t+\alpha^{t-1}+1)^{2^l}} = \left(\frac{1}{\alpha(\alpha^t+\alpha^{t-1}+1)^2}\right)^{2^{l-1}}$.

Therefore a row $\begin{bmatrix} \frac{1}{\alpha_1(\alpha_1^t+\alpha_1^{t-1}+1)^2} & \cdots & \frac{1}{\alpha_{n_4}(\alpha_{n_4-1}^t+\alpha_{n_4-1}^{t-1}+1)^2} \end{bmatrix}$ can be obtained from the row $\begin{bmatrix} \frac{\alpha_1^{2^{l-1}-1}}{\alpha_1^t+\alpha_1^{t-1}+1} & \cdots & \frac{\alpha_{n_4}^{2^{l-1}-1}}{\alpha_{n_4-1}^t+\alpha_{n_4-1}^{t-1}+1} \end{bmatrix}$ of the matrix $H_4^*$ .

For any $\alpha \in L_4^*$

$\frac{1}{\alpha(\alpha^t+\alpha^{t-1}+1)^2} + \left(\frac{\alpha^{2^{l-1}-1}}{(\alpha^t+\alpha^{t-1}+1)}\right)^2 = \frac{1}{\alpha(\alpha^t+\alpha^{t-1}+1)^2} + \frac{\alpha^{2^l-2}}{(\alpha^t+\alpha^{t-1}+1)^2} = \frac{1}{\alpha(\alpha^t+\alpha^{t-1}+1)^2} + \frac{\alpha^{2^l-1}}{\alpha(\alpha^t+\alpha^{t-1}+1)^2} = \frac{1}{\alpha(\alpha^t+\alpha^{t-1}+1)} + \frac{\alpha^{2^l}}{\alpha(\alpha^t+\alpha^{t-1}+1)^2}$ .

For any $\alpha \in L_4^*$

$\frac{\alpha^{2^l}}{\alpha(\alpha^t+\alpha^{t-1}+1)^2} = \frac{\alpha^{2^l-1}}{(\alpha^t+\alpha^{t-1}+1)^2} = \left(\frac{\alpha^{2^{l-1}}}{(\alpha^t+\alpha^{t-1}+1)}\right)^2 + \left(\frac{1}{(\alpha^t+\alpha^{t-1}+1)}\right)^2 + \frac{1}{(\alpha^t+\alpha^{t-1}+1)}$ .

Therefore a row $\begin{bmatrix} \frac{1}{\alpha_1(\alpha_1^t+\alpha_1^{t-1}+1)} & \cdots & \frac{1}{\alpha_{n_4}(\alpha_{n_4-1}^t+\alpha_{n_4-1}^{t-1}+1)} \end{bmatrix}$ can be obtained from the rows

$\begin{bmatrix} \frac{\alpha_1^{2^{l-1}-1}}{\alpha_1^t+\alpha_1^{t-1}+1} & \cdots & \frac{\alpha_{n_4}^{2^{l-1}-1}}{\alpha_{n_4-1}^t+\alpha_{n_4-1}^{t-1}+1} \end{bmatrix}$,

$\begin{bmatrix} \frac{\alpha_1^{2^{l-1}}}{\alpha_1^t+\alpha_1^{t-1}+1} & \cdots & \frac{\alpha_{n_4}^{2^{l-1}}}{\alpha_{n_4-1}^t+\alpha_{n_4-1}^{t-1}+1} \end{bmatrix}$

and

$\begin{bmatrix} \frac{1}{\alpha_1^t+\alpha_1^{t-1}+1} & \cdots & \frac{1}{\alpha_{n_4-1}^t+\alpha_{n_4-1}^{t-1}+1} \end{bmatrix}$

of the matrix $H_4^*$. ■

*Corollary 3:* By using the result of *Lemma 5* we can rewrite matrix $H_4^*$ in the following form:

$$H_4^* = \begin{bmatrix} \frac{1}{\alpha_1(\alpha_1^t+\alpha_1^{t-1}+1)} & \cdots & \frac{1}{\alpha_{n_4}(\alpha_{n_4-1}^t+\alpha_{n_4-1}^{t-1}+1)} \\ \frac{1}{\alpha_1^t+\alpha_1^{t-1}+1} & \cdots & \frac{1}{\alpha_{n_4-1}^t+\alpha_{n_4-1}^{t-1}+1} \\ \frac{\alpha_1}{\alpha_1^t+\alpha_1^{t-1}+1} & \cdots & \frac{\alpha_{n_4-1}}{\alpha_{n_4-1}^t+\alpha_{n_4-1}^{t-1}+1} \\ \cdot & \cdot & \cdot \\ \frac{\alpha_1^{t-1}}{\alpha_1^t+\alpha_1^{t-1}+1} & \cdots & \frac{\alpha_1^{t-1}}{\alpha_{n_4-1}^t+\alpha_{n_4-1}^{t-1}+1} \end{bmatrix}.$$

Now consider a code $\Gamma(L_3, G_3)$ with $G_3(x) = A^t x^t + Ax + C$, $t = 2^l$ and $L_1 = \{\alpha_1, \alpha_2, ..., \alpha_{n_3-1}, 0\}$, $n_3 = 2^{2l} - 2^l$. Obviously, by substituting $x$ to $A^{-1}Cx$ in $G_3(x)$ we obtain the same $\Gamma(L_3, G_3)$ code with a more simple Goppa polynomial $G_3(x) = x^t + x + 1$. The parity check matrix for this code is:

$$H_3 = \begin{bmatrix} \frac{1}{\alpha_1^t+\alpha_1+1} & \cdots & \frac{1}{\alpha_{n_3-1}^t+\alpha_{n_3-1}+1} & 1 \\ \frac{\alpha_1}{\alpha_1^t+\alpha_1+1} & \cdots & \frac{\alpha_{n_3-1}}{\alpha_{n_3-1}^t+\alpha_{n_3-1}+1} & 0 \\ \cdots & \cdots & \cdots & 0 \\ \frac{\alpha_1^{t-1}}{\alpha_1^t+\alpha_1+1} & \cdots & \frac{\alpha_{n_3-1}^{t-1}}{\alpha_{n_3-1}^t+\alpha_{n_3-1}+1} & 0 \end{bmatrix}.$$

It is easy to show that if we have a row $r_1$ in the parity check matrix $H_3$ :

$$r_1 = \left[ \begin{array}{ccccccc} \frac{\alpha_1}{\alpha_1^t+\alpha_1+1} & \cdots & \frac{\alpha_i}{\alpha_i^t+\alpha_i+1} & \cdots & \frac{\alpha_{n_3-1}}{\alpha_{n_3}^t+\alpha_{n_3}+1} & & 0 \end{array} \right],$$

then the row

$$r_t = \left[ \begin{array}{ccccccc} \frac{\alpha_1^t}{\alpha_1^t+\alpha_1+1} & \cdots & \frac{\alpha_i^t}{\alpha_i^t+\alpha_i+1} & \cdots & \frac{\alpha_{n_3-1}^t}{\alpha_{n_3-1}^t+\alpha_{n_3-1}+1} & & 0 \end{array} \right]$$

is in the parity check matrix of this code, too.

Therefore we obtain the following row:

$$r^* = \left[ \begin{array}{ccccccc} 1 & \ldots & 1 & \ldots & 1 \end{array} \right]$$

from these two rows and the first row of matrix $H_3$ :

$$r_0 = \left[ \begin{array}{ccccccc} \frac{1}{\alpha_1^t+\alpha_1+1} & \cdots & \frac{1}{\alpha_i^t+\alpha_i+1} & \cdots & \frac{1}{\alpha_{n_3-1}^t+\alpha_{n_3-1}+1} & & 1 \end{array} \right].$$

By using the substitution $\alpha_i \longrightarrow \alpha_i^{-1}$ the matrix $H_3$ will become:

$$H_3 = \left[ \begin{array}{ccc} \frac{\alpha_1^{t-1}}{\alpha_1^t+\alpha_1^{t-1}+1} & \cdots & \frac{\alpha_{n_3-1}^{t-1}}{\alpha_{n_3-1}^t+\alpha_{n_3-1}^{t-1}+1} \quad 1 \\ \frac{\alpha_1^{i-2}}{\alpha_1^t+\alpha_1^{t-1}+1} & \cdots & \frac{\alpha_{n_3-1}^{t-2}}{\alpha_{n_3-1}^t+\alpha_{n_3-1}^{t-1}+1} \quad 0 \\ \cdots & \cdots & \cdots \quad 0 \\ \frac{\alpha_1}{\alpha_1^t+\alpha_1^{t-1}+1} & & \frac{\alpha_{n_3-1}}{\alpha_{n_3-1}^t+\alpha_{n_3-1}^{t-1}+1} \\ \frac{1}{\alpha_1^t+\alpha_1^{t-1}+1} & \cdots & \frac{1}{\alpha_{n_3-1}^t+\alpha_{n_3-1}^{t-1}+1} \quad 0 \end{array} \right]$$

and the row $r_t$ will be rewritten as:

$$r_t = \left[ \begin{array}{ccccccc} \frac{\alpha_1^t}{\alpha_1^t+\alpha_1^{t-1}+1} & \cdots & \frac{\alpha_i^t}{\alpha_i^t+\alpha_i^{t-1}+1} & \cdots & \frac{\alpha_{n_3-1}^t}{\alpha_{n_3-1}^t+\alpha_{n_3-1}^{t-1}+1} & & 0 \end{array} \right].$$

The first row of the new matrix $H_3$ will be the sum of its last row, the row $r_t$ and row $r^*$.

For the i-th element of the first row we will obtain:

$$\frac{\alpha_i^{t-1}}{\alpha_i^t+\alpha_i^{t-1}+1} = \frac{\alpha_i^t}{\alpha_i^t+\alpha_i^{t-1}+1} + \frac{1}{\alpha_i^t+\alpha_i^{t-1}+1} + 1$$

and for the element in the last column of the first row:

$$1 = 0 + 0 + 1.$$

Consequently, it is possible to rewrite the matrix $H_3$:

$$H_3 = \left[ \begin{array}{ccc} 1 & \ldots & 1 \quad 1 \\ \frac{\alpha_1^{i-1}}{\alpha_1^t+\alpha_1^{t-1}+1} & \cdots & \frac{\alpha_{n_3-1}^{t-1}}{\alpha_{n_3-1}^t+\alpha_{n_3-1}^{t-1}+1} \quad 0 \\ \frac{\alpha_1^{i-2}}{\alpha_1^t+\alpha_1^{t-1}+1} & \cdots & \frac{\alpha_{n_3-1}^{t-2}}{\alpha_{n_3-1}^t+\alpha_{n_3-1}^{t-1}+1} \quad 0 \\ \cdots & \cdots & \cdots \quad \cdots \\ \frac{\alpha_1}{\alpha_1^t+\alpha_1^{t-1}+1} & \cdots & \frac{\alpha_{n_3-1}}{\alpha_{n_3-1}^t+\alpha_{n_3-1}^{t-1}+1} \quad 0 \\ \frac{1}{\alpha_1^t+\alpha_1^{t-1}+1} & \cdots & \frac{1}{\alpha_{n_3-1}^t+\alpha_{n_3-1}^{t-1}+1} \quad 0 \end{array} \right] = \left[ \begin{array}{cc} 1..1 & 1 \\ H_4^* & 0 \end{array} \right].$$

*Definition 3:* Let us define a subcode $\Gamma(L_3^*, G_3)$ of the code $\Gamma(L_3, G_3)$ as shortened by a position corresponding to the element 0 from $L_3 = \{\alpha_1, \alpha_2, ..., \alpha_{n_3-1}, 0\}$. Hence $\Gamma(L_3^*, G_3) \subset \Gamma(L_3, G_3)$ and $n_3^* = n_3 - 1$ , $k_3^* = k_3 - 1$ and $L_3^* = L_3 \backslash \{0\}$ .

*Lemma 6:* $\Gamma(L_3^*, G_3) \subset \Gamma(L_4, G_4)$ where $G_4(x) = A^t x^t + A^{t-1} x^{t-1} + 1$, $A \in GF(2^{2l})$.

*Proof:*

It follows directly from the above presentation of the matrix $H_3$. ■

*Corollary 4:* $\left[ \begin{array}{cccc} & H_1^* & & \\ 1 & 1 & \ldots & 1 \end{array} \right] = \left[ \begin{array}{cc} H_4^* & 0 \\ 1 \ldots 1 & 1 \end{array} \right]$.

*Proof:*

It follows directly from Lemma 3 where we have proved the equivalence of two codes: $\Gamma(L_2, G_2)$ and $\Gamma(L_3, G_3)$ . ■

*Lemma 7:* $\Gamma(L_4, G_4) \equiv \Gamma(L_5, G_5)$ where $G_5(x) = Cx^{t+1} + A^t x^t + Ax$, $C \in GF(2^l)$, and $A \in GF(2^{2l})$ .

*Proof:*

It is easy to show that $G_5(x) = x * G_4(x)$ and $L_5 = L_4 \backslash \{0\}$.

Obviously, by substiting $x$ to $A^{-1}Cx$ in $G_5(x)$ we obtain the same $\Gamma(L_5, G_5^*)$ code with a more simple $G_5^*(x) = x^{t+1} + x^t + x$.

The parity-check matrix for this code is:

$$H_5 = \left[ \begin{array}{ccc} \frac{1}{\alpha_1^{t+1}+\alpha_1^t+\alpha_1} & \cdots & \frac{1}{\alpha_{n_5}^{t+1}+\alpha_{n_5}^t+\alpha_{n_5}} \\ \frac{\alpha_1}{\alpha_1^{t+1}+\alpha_1^t+\alpha_1} & \cdots & \frac{\alpha_{n_5}}{\alpha_{n_5}^{t+1}+\alpha_{n_5}^t+\alpha_{n_5}} \\ \cdot & \cdot & \cdot \\ \frac{\alpha_1^t}{\alpha_1^{t+1}+\alpha_1^t+\alpha_1} & \cdots & \frac{\alpha_{n_5}^t}{\alpha_{n_4-1}^{t+1}+\alpha_{n_4-1}^t+\alpha_{n_5}} \end{array} \right] =$$

$$\left[ \begin{array}{ccc} \frac{1}{\alpha_1(\alpha_1^t+\alpha^{t-1}+1)} & \cdots & \frac{1}{\alpha_{n_5}(\alpha_{n_5}^t+\alpha_{n_5}^{t-1}+1)} \\ \frac{1}{\alpha_1^t+\alpha_1^{t-1}+\alpha_1} & \cdots & \frac{}{\alpha_{n_5}^t+\alpha_{n_5}^{t-1}+\alpha_{n_5}} \\ \cdot & \cdot & \cdot \\ \frac{\alpha_1^{t-1}}{\alpha_1^t+\alpha_1^{t-1}+\alpha_1} & \cdots & \frac{\alpha_{n_5}^{t-1}}{\alpha_{n_4-1}^t+\alpha_{n_4-1}^{t-1}+\alpha_{n_5}} \end{array} \right].$$

Therefore $H_5 = H_4^*$ according to *Corollary 2* . ■

*Lemma 8:* $\Gamma(L_5, G_5) \equiv \Gamma(L_6, G_6)$ where $G_6(x) = Rx^{t+1} + V^t x^t + Vx + 1$, $R \in GF(2^l)$, and $V \in GF(2^{2l})$ .

*Proof:*

Using the $x + \beta$ substitution for a variable $x$ we obtain:

$$G_5(x+\beta) = Cx^{t+1} + (A^t + C\beta)x^t + (A + C\beta^t)x + (C\beta^{t+1} + A^t\beta^t + A\beta)$$

where $\beta : \beta \in GF(2^{2l})$ and $\beta \neq \frac{A^t}{C}$. Notice that $(A + C\beta^t) = (A^t + C\beta)^t$ and

$$(C\beta^{t+1} + A^t\beta^t + A\beta)^t = C\beta^{t+1} + A\beta + A^t\beta^t.$$

Therefore $(C\beta^{t+1} + A^t\beta^t + A\beta) \in GF(2^l)$.

$$G_6(x) = \frac{1}{C\beta^{t+1}+A^t\beta^t+A\beta} G_5(x+\beta) = \frac{C}{(C\beta^{t+1}+A^t\beta^t+A\beta)} x^{t+1} + \frac{(A^t+C\beta)}{(C\beta^{t+1}+A^t\beta^t+A\beta)} x^t + \frac{(A+C\beta^t)}{(C\beta^{t+1}+A^t\beta^t+A\beta)} x + 1,$$

$$G_6(x) = Rx^{t+1} + V^t x^t + Vx + 1$$

where $R = \frac{C}{(C\beta^{t+1}+A^t\beta^t+A\beta)}$ and $V = \frac{(A+C\beta^t)}{(C\beta^{t+1}+A^t\beta^t+A\beta)}$.

This means that $\Gamma(L_5, G_5) \equiv \Gamma(L_6, G_6)$. ■

*Lemma 9:* $\Gamma(L_6, G_6) \equiv \Gamma(L_7, G_7)$ where $G_7(x) = Bx^{t+1} + 1$, $B = \alpha^{2^l-1}$, and $\alpha$ is a primitive element of $GF(2^{2l})$.

*Proof:*

It can be proved in the same way as the previous Lemma by using the $x+\beta$ substitution for a variable $x$ where $\beta = \frac{V^t}{R}$ . ■

In Figure 1 we present the structure of the code chain. It is possible to define the similar code chain for the codes described in paper [15].

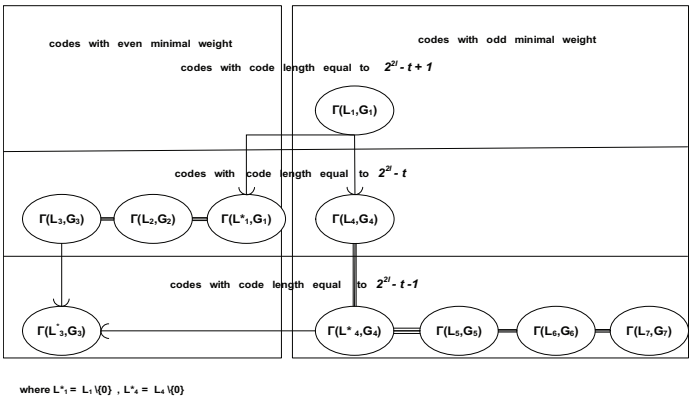


Fig. 1. Code chain

## III. Minimal distance of subfield subcodes

*Lemma 10:* The minimal distance of the last Goppa code in the chain $\Gamma(L_7, G_7)$ exactly equals to its design distance, i.e. $d = 2(t+1)+1$.

*Proof:*

It is easy to show that a polynomial $x^{2^l+1}-1$ can be presented as a product $\prod_{i=1}^{2^l+1}(x-\alpha^{i(2^l-1)})$ where $\alpha$ is a primitve element of $GF(2^{2l})$. Choose some element $A$ from $GF(2^{2l})$ such that $A^{2^l+1} \neq 1$ and let $B = A^{-1}$. Thus we can calculate two polynomials with all different roots $\{A\alpha^{i(2^l-1)}\}$ and $\{B\alpha^{i(2^l-1)}\}$ $i = 1, .., 2^l+1$.

$$x^{2^l+1} - A^{2^l+1} = \prod_{i=1}^{2^l+1}(x - A\alpha^{i(2^l-1)}),$$

$$x^{2^l+1} - B^{2^l+1} = \prod_{i=1}^{2^l+1}(x - B\alpha^{i(2^l-1)}).$$

The result of the multiplication of these two polynomials and $x$ :

$$x(x^{2^l+1} - A^{2^l+1})(x^{2^l+1} - B^{2^l+1}) = x^{2^{l+1}+3} - (A^{2^l+1} + B^{2^l+1})x^{2^l+2} - x.$$

A formal derivative of result of this multiplication:

$$x^{2^{l+1}+2} - 1.$$

Now consider a binary vector $a = (a_0 a_1 ... a_n)$ with nonzero elements on and only on positions $\beta_j$, $(j = 1, .., (2^{l+1}+3))$ from the following subset of $L$:

$$\{\{A\alpha^{i(2^l-1)}\}, i = 1, .., 2^l+1\} \cup \{\{B\alpha^{i(2^l-1)}\}, i = 1, .., 2^l+1\} \cup \{0\} .$$

From the definition of the Goppa code, this vector will be a codeword of $\Gamma(L_7, G_7)$ :

$$\sum_{j=1}^{2^{l+1}+3} a_i \frac{1}{x-\beta_j} \equiv \frac{x^{2^{l+1}+2}-1}{x^{2^{l+1}+3} - (A^{2^l+1}+B^{2^l+1})x^{2^l+2} - x} \equiv 0 mod x^{2^l+1} - 1$$

where $\beta_j \in \{\{A\alpha^{i(2^l-1)}\}, i = 1, .., 2^l+1\} \cup \{\{B\alpha^{i(2^l-1)}\}, i = 1, .., 2^l+1\} \cup \{0\}$.

Therefore the minimal distance of the Goppa code $\Gamma(L_7, G_7)$ is equal to the design distance $2^{l+1}+3$. ■

*Corollary 5:* The minimal distance of the equivalent Goppa codes $\Gamma(L_6, G_6)$ and $\Gamma(L_5, G_5)$ is exactly equal to its design distance, i.e. $d = 2(t+1)+1$.

## IV. Minimal distance of quadratic trace subcodes

*Lemma 11:* The minimal distance of the equivalent Goppa codes $\Gamma(L_2, G_2)$ and $\Gamma(L_3, G_3)$ is exactly equal to the minimal even weight of a codeword of the code $\Gamma(L_5, G_5) \equiv \Gamma(L_6, G_6) \equiv \Gamma(L_7, G_7)$ , i.e. $d \geq 2(t+1)+2$.

*Proof:*

It follows directly from the parity check matrixes $H_3, H_2$ and $H_5$. ■

It is necessary to note that P.Veron in [13] has proved that the Hamming weight of all codewords of these codes is even.

## V. Minimal distance of the new code

*Lemma 12:* The minimal distance of the $\Gamma(L_4, G_4)$ and $\Gamma(L_4^*, G_4)$ Goppa codes is exactly equal to its design distance, i.e. $d = 2(t+1)+1$ .

*Proof:* It follows directly from the equivalence of codes $\Gamma(L_4, G_4)$ and $\Gamma(L_5, G_5)$ (Lemma 7). ■

## VI. Conclusion

Parameters of the codes forming a chain are presented in Table 1.

In Table 2 we present the quasi-cyclic Goppa codes from our chain. It is easy to see that by substituting $x$ by $\beta x+\gamma$ we will obtain the same code if the Goppa polynomial is invariant to this substitution: $\alpha G(x) = G(\beta x+\gamma)$ where $\alpha, \beta, \gamma \in GF(2^{2l})$ .

In Table 2 we present such values of $\gamma$ and $\beta$ for the Goppa codes from our chain.

Therefore these quasi-cyclic codes have indexes $2^l-1$ and $2^l+1$.

## References

[1] V. D. Goppa, A new class of linear error correcting codes. Probl. Inform.Transm, Vol. 6, No. 3 , pp. 24-30,1970

[2] M.Loeloeian and J.Conan, A (55,16,19) binary Goppa code *IEEE Trans. on Information Theory*, vol. 30, p.773, 1984.

[3] S. V. Bezzateev, E. T. Mironchikov and N. A. Shekhunova, One subclass of binary Goppa codes, Proc. XI Simp. Po Probl. Izbit. v Inform. Syst. pp. 140-141, 1986.

[4] S.V.Bezzateev and N.A.Shekhunova , On the designed distance of the best known (55,16,19) Goppa code , Probl.Inform.Transm., vol. 23, No 4, p.352, 1987.

Table 1 Parameters of the code chain

| $\Gamma$(L, G)-code with parity check matrix | code length | number of information symbols | minimal distance |
|---|---|---|---|
| $\Gamma(L_1, G_1)$, where $G_1(x) = x^{t-1} + 1$ parity check matrix: $H_1$ | $n_1 = 2^{2l} - t + 1$ | $k_1 = 2^{2l} - t - 2l(t - \frac{3}{2})$ [9] | $d_1 = 2t - 1$ [7] |
| $\Gamma(L_1^*, G_1)$ , where $G_1(x) = x^{t-1} + 1$ , $L_1^* = L_1 \backslash \{0\}$ parity check matrix: $\begin{bmatrix} H_1^* \\ 1...1 \end{bmatrix}$ | $n_1^* = 2^{2l} - t$ | $k_1^* = k_1 - 1$ | $d_1^*$ is even and equals to the minimal even weight of a codeword from the code $\Gamma(L_1, G_1)$ |
| $\Gamma(L_2, G_2)$, where $G_2(x) = A^t x^t + Ax$ parity check matrix: $\begin{bmatrix} H_1^* \\ 1...1 \end{bmatrix}$ | $n_2 = 2^{2l} - t$ | $k_2 = k_1 - 1$ (Lemma 2) $k_2 = 2^{2l} - t - 2l(t - \frac{3}{2}) - 1$ [8] | $d_2 = d_1^*$ |
| $\Gamma(L_3, G_3)$, where $G_3(x) = A^t x^t + Ax + C$ parity check matrix: $\begin{bmatrix} H_4^* & 0 \\ 1...1 & 1 \end{bmatrix}$ | $n_3 = 2^{2l} - t$ | $k_3 = k_2$ (Lemma 3) | $d_3 = d_1^*$ |
| $\Gamma(L_3^*, G_3)$, where $G_3(x) = A^t x^t + Ax + C$ , $L_3^* = L_3 \backslash \{0\}$ parity check matrix: $\begin{bmatrix} H_4^* \\ 1...1 \end{bmatrix}$ | $n_3^* = 2^{2l} - t - 1$ | $k_3^* = k_3 - 1$ (Definition 2) | $d_3 = d_1^*$ |
| $\Gamma(L_4, G_4)$, where $G_4(x) = A^t x^t + A^{t-1} x^{t-1} + 1$ parity check matrix: $H_4^*$ | $n_4 = 2^{2l} - t$ | $k_4 = k_4^*$ (Corollary 2) | $d_4 = d_7$ |
| $\Gamma(L_4^*, G_4)$ , where $G_4(x) = A^t x^t + A^{t-1} x^{t-1} + 1$ , $L_4^* = L_4 \backslash \{0\}$ parity check matrix: $H_4^*$ | $n_4^* = 2^{2l} - t - 1$ | $k_4^* = k_3$ (Lemma 6) | $d_4^* = d_7$ |
| $\Gamma(L_5, G_5)$, where $G_5(x) = Cx^{t+1} + A^t x^t + Ax$ parity check matrix: $H_4^*$ | $n_5 = 2^{2l} - t - 1$ | $k_5 = k_4$ (Lemma 7) | $d_5 = d_7$ |
| $\Gamma(L_6, G_6)$, where $G_6(x) = Rx^{t+1} + V^t x^t + Vx + 1$ parity check matrix: $H_4^*$ | $n_6 = 2^{2l} - t - 1$ | $k_6 = k_4$ (Lemma 8) | $d_6 = d_7$ |
| $\Gamma(L_7, G_7)$, where $G_7(x) = x^{t+1} + 1$ parity check matrix: $H_4^*$ | $n_7 = 2^{2l} - t - 1$ | $k_7 = k_4$ (Lemma 9) $k_7 = 2^{2l} - t - 2l(t - \frac{3}{2}) - 1$ [9] | $d_7 = 2t + 3$ (Lemma 11) |

Table 2 Quasi cyclic Goppa codes

| code | G(x) | $\gamma$ | $\beta$ |
|---|---|---|---|
| $\Gamma(L_1, G_1)$ | $x^{t-1} + 1$ | 0 | any nonzero element from $GF(2^l)$ |
| $\Gamma(L_2, G_2)$ | $A^t x^t + Ax$ | $A^{-1} or\ 0$ | any nonzero element from $GF(2^l)$ |
| $\Gamma(L_3, G_3)$ | $A^t x^t + Ax + C$ | $\gamma \in GF(2^{2l}) : A^t \gamma^t + A\gamma = C(1 + \beta)$ | any nonzero element from $GF(2^l)$ |
| $\Gamma(L_5, G_5)$ | $Cx^{t+1} + A^t x^t + Ax$ | $\gamma \in GF(2^{2l}) : C\gamma^{t+1} + A^t \gamma^t + A\gamma = 0$ | $\beta = \frac{\gamma C}{A^t} + 1$ |
| $\Gamma(L_7, G_7)$ | $x^{t+1} + 1$ | 0 | $\left(\alpha^{2^l - 1}\right)^i , i = 1, ..., 2^l + 1$ |

$\alpha$ is a primitive element of $GF(2^{2l})$ and $A \in GF(2^{2l})$.

[5] M.Loeloeian and J.Conan , A transform approach in Goppa codes, *IEEE Trans. on Information Theory*, vol. 35, pp.105-115, 1987.
[6] A.M.Roseiro, J.I.Hall, J.E.Adney and M.Siegel, The trace operator and redundancy of Goppa codes, *IEEE Trans. on Information Theory*, vol. 38,No.3, pp. 1130-1133, 1992.
[7] S.Bezzateev and N. Shekhunova , Subclass of binary Goppa codes with minimal distance equal to the design distance, *IEEE Trans. on Information Theory*, vol. 41, pp. 554-555, 1995.
[8] P. Veron, True dimension of some binary quadratic trace Goppa codes, Designs, Codes and Cryptography, 24, pp. 81-97, 2001.
[9] P. Veron, Proof of conjectures on the true dimension of some binary Goppa codes, Designs, Codes and Cryptography, 36, pp.317-325, 2005.
[10] N.A.Shekhunova, S.V.Bezzateev and E.T.Mironchikov, A subclass of binary Goppa codes, Probl.Inform.Transm., vol.25, no.3, pp.98-102, 1989.
[11] S.V.Bezzateev and N.A.Shekhunova, Quasi-cyclic Goppa codes, IEEE International Symposium on Information Theory, Canada, p.499, 1995.
[12] G. Bommer and F. Blanchet, Binary quasicyclic Goppa codes, Designs, Codes and Cryptography, 20, pp.107-124,2000.
[13] P. Veron, Goppa codes and trace operator, *IEEE Trans. on Information Theory*, vol. 44,No.1, pp. 290-295, 1998.
[14] G. Maatouk, A.Shokrollahi and M.Cheraghchi,Good Ensembles of Goppa Codes, *Ecole Polytachnique Federale De Lausanne,ALGO Lab*, , 2007, www. algo.epfl.ch/contents/output/sempr/Ghid_MAATOUK.pdf
[15] S.V.Bezzateev and N.A.Shekhunova , A subclass of binary Goppa codes with improved estimation of the code dimension, Designs, Codes and Cryptography, 14, pp.23-38, 1998