Engineering Stable Discrete-Time Quantum Dynamics via a Canonical QR Decomposition

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Abstract—We analyze the asymptotic behavior of discrete-time, Markovian quantum systems with respect to a subspace of interest. Global asymptotic stability of subspaces is relevant to quantum information processing, in particular for initializing the system in pure states or subspace codes. We provide a linear-algebraic characterization of the dynamical properties leading to invariance and attractivity of a given quantum subspace. We then construct a design algorithm for discrete-time feedback control that allows to stabilize a target subspace, proving that if the control problem is feasible, then the algorithm returns an effective control choice. In order to prove this result, a canonical QR matrix decomposition is derived, and also used to establish the control scheme potential for the simulation of open-system dynamics.

Index Terms—Quantum control, QR decomposition, invariance principle, quantum information.

I. INTRODUCTION

SINCE the pioneering intuitions of R. P. Feynmann [14], Quantum Information (QI) has been the focus of an impressive research effort. Its potential has been clearly demonstrated, not only as a new paradigm for fundamental physics, but also as the key ingredient for a new generation of information technologies. Today the goal is to design and produce quantum chips, quantum memories, and quantum secure communication protocols [12], [25], [4]. The main difficulties in building effective QI processing devices are mainly related to scalability issues and to the disruptive action of the environment on the quantum correlations that embody the key advantage of QI. Many of these issues do not appear to be fundamental, and their solution is becoming mainly an engineering problem.

Most of the proposed approaches to realize quantum information technology require the ability to perform sequences of a limited number of fundamental operations. Two typical key tasks are concerned with the preparation of states of maximal information [11], [25], [34] and engineering of protected realization of quantum information [35], [22], [18], [17], i.e. the realization of information encodings that preserve the fragile quantum states from the action of noise. This paper will focus on these issues, providing a design strategy for engineering stable quantum subspaces.

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In [28] and [30] the seminal linear-algebraic approach of [22] to study noise-free subspaces has been extended to the general setting of noiseless subsystems [18] (which usually entails an operator-algebraic approach) and developed in two different directions, both concerned with the *robustness* of the encoded quantum information. The first ([28]) studies the cases in which the encoded information does not degenerate in the presence of initialization errors, the other ([30]) aims to ensure that the chosen encoding is an invariant, asymptotically stable set for the dynamics in presence of the noise. The latter tightly connects the encoding task to a set of familiar *stabilization* control problems.

Feedback state stabilization, and in particular pure-state stabilization problems, have been tackled in the quantum domain under a variety of modeling and control assumptions, with a rapidly growing body of work dealing with the Lyapunov approach, see e.g. [15], [2], [38], [32], [13], [24], [30], [31], [36], [37], [8] and references therein. Here we embrace the approach of [30], [31], extending these results to Markovian discrete-time evolutions.

A good review of the role of discrete-time models for quantum dynamics and control problems can be found in [3], to which we refer for a discussion of the relevant literature which is beyond the scope of this paper. In fact, we will assume from the very beginning discrete-time quantum dynamics described by sequences of trace-preserving quantum operations in Kraus representation [19], [25]. This assumption implies the Markovian character of the evolution [20], which, along with a forward composition law, ensures a semigroup structure. We introduce the class of dynamics of interest and the relevant notation in Section II. A basic analysis of kinematic controllability for Kraus map has been provided in [39].

After introducing the key concepts relative to *quantum subspaces* and dynamical stability in Section III, Section IV is devoted to the analysis of the dynamics. The results provide us with necessary and sufficient conditions on the dynamical model that ensure global stability of a certain quantum subspace. We employ a Lyapunov approach, exploiting the linearity of the dynamics, as well as the convex character of the state manifold. Lyapunov analysis of quantum discrete-time semigroups has been also considered, with emphasis on ergodicity properties, in [6].

The control scheme we next consider modifies the underlying dynamics of the system by indirectly measuring it, and applying unitary control actions, conditioned on the outcome of the measurement. If *we average over the possible outcomes*, we obtain a new semigroup evolution where the choice of the

control can be used to achieve the desired stabilization. We make use of the generalized measurement formalism, which is briefly recalled in Appendix A. This control scheme can be seen as an instance of discrete-time *Markovian reservoir engineering*: the use of "noisy" dynamics to obtain a desired dynamical behavior has long been investigated in a variety of contexts, see *e.g.* [26], [7], [10], [33].

The synthesis results of Section VI include a simple characterization of the controlled dynamics that can be enacted, and an algorithm that builds unitary control actions stabilizing the desired subspace. If such controls cannot be found, it is proven that no choice of controls can achieve the control task for the same measurement. The main tools we employ come from the stability theory of dynamical systems, namely LaSalle's Invariance principle [21], and linear algebra, namely the QR matrix decomposition [16]. We shall construct a "special form" of the QR decomposition: In particular, we prove that the upper triangular factor R can be rendered a *canonical form* with respect to the left action of the unitary matrix group. This result and the related discussion is presented in Section V.

II. DISCRETE–TIME QUANTUM DYNAMICAL SEMIGROUPS

Let $\mathcal{I}$ denote the physical quantum system of interest. Consider the associated separable Hilbert space $\mathcal{H}_I$ over the complex field $\mathbb{C}$. In what follows, we consider finite-dimensional quantum systems, i.e. $\dim(\mathcal{H}_I) < \infty$. In Dirac's notation, vectors are represented by a *ket* $|\psi\rangle \in \mathcal{H}_I$, and linear functionals by a *bra*, $\langle\psi| \in \mathcal{H}_I^\dagger$ (the adjoint of $\mathcal{H}_I$), respectively. The inner product of $|\psi\rangle, |\varphi\rangle$ is then represented as $\langle\psi|\varphi\rangle$.

Let $\mathfrak{B}(\mathcal{H}_I)$ represent the set of linear bounded operators on $\mathcal{H}_I$, $\mathfrak{H}(\mathcal{H}_I)$ denoting the real subspace of hermitian operators, with $\mathbb{I}$ and $\mathbb{O}$ being the identity and the zero operator, respectively. Our (possibly uncertain) knowledge of the state of the quantum system is condensed in a density operator, or *state* ρ, with $\rho \geq 0$ and $\mathrm{Tr}\rho = 1$. Density operators form a convex set $\mathfrak{D}(\mathcal{H}_I) \subset \mathfrak{H}(\mathcal{H}_I)$, with one-dimensional projectors corresponding to extreme points (pure states, $\rho_{|\psi\rangle} = |\psi\rangle\langle\psi|$). Given an $X \in \mathfrak{H}(\mathcal{H}_I)$, we indicate with $\ker(X)$ its kernel (0-eigenspace) and with $\mathrm{supp}(X) := \mathcal{H}_I \ominus \ker(X)$ its range, or *support*. If a quantum system $\mathcal{Q}$ is obtained by composition of two *subsystems* $\mathcal{Q}_1, \mathcal{Q}_2$, the corresponding mathematical description is carried out in the tensor product space, $\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$ [27], observables and density operators being associated with Hermitian and positive-semidefinite, normalized operators on $\mathcal{H}_{12}$, respectively.

In the presence of coupling between subsystems, quantum measurements (see Appendix A), or interaction with surrounding environment, the dynamics of a quantum system cannot in general be described by Schrödinger's dynamics: The evolution is no longer unitary and reversible, and the formalism of open quantum systems is required [9], [5], [1], [25]. An effective tool to describe these dynamical systems, of fundamental interest for QI, is given by quantum operations [25], [19]. The most general, linear and physically admissible evolutions which take into account interacting quantum systems and measurements, are described by Completely Positive

(CP) maps, that via the Kraus-Stinespring theorem [19] admit a representation of the form

$$\mathcal{T}[\rho] = \sum_k M_k \rho M_k^\dagger \tag{1}$$

(also known as operator-sum representation of $\mathcal{T}$), where ρ is a density operator and $\{M_k\}$ a family of operators such that the completeness relation

$$\sum_k M_k^\dagger M_k = I \tag{2}$$

is satisfied. Under this assumption the map is then Trace-Preserving and Completely-Positive (TPCP), and hence maps density operators to density operators. We refer the reader to e.g. [1], [25], [5], [9] for a detailed discussions of the properties of quantum operations and the physical meaning of the complete-positivity property.

One can then consider the discrete-time dynamical semi-group, acting on $\mathfrak{D}(\mathcal{H}_I)$, induced by iteration of a given TPCP map. The resulting discrete-time quantum system is described by

$$\rho(t + 1) = \mathcal{T}[\rho(t)] = \sum_k M_k \rho(t) M_k^\dagger. \tag{3}$$

Given the initial conditions $\rho(0)$ for the system, we can then write

$$\rho(t) = \mathcal{T}^t[\rho(0)] \quad t = 1, 2, \ldots$$

where $\mathcal{T}^t[\cdot]$ indicates t applications of the TPCP map $\mathcal{T}[\cdot]$. Hence, the evolution obeys a forward composition law and, in the spirit of [1], is called a Discrete-time Quantum Dynamical Semigroup (DQDS). Notice that while the dynamic map is linear, the "state space" $\mathfrak{D}(\mathcal{H}_I)$ is a convex, compact subset of the cone of the positive elements in $\mathfrak{H}(\mathcal{H}_I)$.

While a TPCP maps can indeed represent general dynamics, assuming dynamics of the form (3), with M_k's that do not depend on the past states, is equivalent to assume Markovian dynamics (see [20] for a discussion of Markovian properties for quantum evolutions). From a probabilistic viewpoint, if density operators play the role of probability distributions, TPCP maps are the analogue of transition operators for classical Markov chains.

III. QUANTUM SUBSPACES, INVARIANCE AND ATTRACTIVITY

In this section we recall some definitions of quantum subspaces invariance and attractivity. We follow the subsystem approach of [30], [31], focusing on the case of subspaces. This is motivated by the fact that the general subsystem case is derived in the continuous-time case as a specialization with some additional constraints, and that for many applications of interest for the present work, namely *pure-state preparation* and *engineering of protected quantum information*, the subspace case is enough, as it is suggested by the results in [30].

Definition 1 (Quantum subspace): A quantum subspace $\mathcal{S}$ of a system $\mathcal{I}$ with associated Hilbert space $\mathcal{H}_I$ is a quantum system whose Hilbert space is a subspace $\mathcal{H}_S$ of $\mathcal{H}_I$,

$$\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R, \tag{4}$$

for some remainder space $\mathcal{H}_R$. The set of linear operators on $\mathcal{S}$, $\mathcal{B}(\mathcal{H}_S)$, is isomorphic to the algebra on $\mathcal{H}_I$ with elements of the form $X_I = X_S \oplus \mathbb{O}_R$.

Let $n = \dim(\mathcal{H}_I)$, $m = \dim(\mathcal{H}_S)$, and $r = \dim(\mathcal{H}_R)$, and let $\{|\phi\rangle_j^S\}_{j=1}^m$, $\{|\phi\rangle_k^R\}_{k=1}^r$ denote orthonormal bases for $\mathcal{H}_S$ and $\mathcal{H}_R$, respectively. Decomposition (4) is then naturally associated with the following basis for $\mathcal{H}_I$:

$$\{|\varphi_l\rangle\} = \{|\phi\rangle_j^S\}_{j=1}^m \cup \{|\phi\rangle_k^R\}_{k=1}^r.$$

This basis induces a block structure for matrices representing operators acting on $\mathcal{H}_I$:

$$X = \left[\begin{array}{c|c} X_S & X_P \\ \hline X_Q & X_R \end{array} \right].$$

In the rest of the paper the subscripts S, P, Q and R will follow this convention. Let moreover Π_S and Π_R be the projection operators over the subspaces $\mathcal{H}_S$ and $\mathcal{H}_R$, respectively. The following definitions are independent of the choices of $\{|\phi\rangle_j^S\}_{j=1}^m$, $\{|\phi\rangle_k^R\}_{k=1}^r$.

Definition 2 (State initialization): The system $\mathcal{I}$ with state $\rho \in \mathcal{D}(\mathcal{H}_I)$ is initialized in $\mathcal{S}$ with state $\rho_S \in \mathcal{D}(\mathcal{H}_S)$ if ρ is of the form

$$\rho = \left[\begin{array}{c|c} \rho_S & 0 \\ \hline 0 & 0 \end{array} \right]. \qquad (5)$$

We will denote with $\mathcal{J}_S(\mathcal{H}_I)$ the set of states of the form (5) for some $\rho_S \in \mathcal{D}(\mathcal{H}_S)$.

Definition 3 (Invariance): Let $\mathcal{I}$ evolve under iterations of a TPCP map. The subsystem $\mathcal{S}$ supported on the subspace $\mathcal{H}_S$ of $\mathcal{H}_I$ is invariant if the evolution of any initialized $\rho \in \mathcal{J}_S(\mathcal{H}_I)$ obeys

$$\rho(t) = \left[\begin{array}{c|c} \mathcal{T}_S^t[\rho_S] & 0 \\ \hline 0 & 0 \end{array} \right] \in \mathcal{J}_S(\mathcal{H}_I)$$

$\forall t \geq 0$, and with $\mathcal{T}_S$ being a TPCP map on $\mathcal{H}_S$.

Definition 4 (Attractivity): Let $\mathcal{I}$ evolve under iterations of a TPCP map $\mathcal{T}$. The subsystem $\mathcal{S}$ supported on the subspace $\mathcal{H}_S$ of $\mathcal{H}_I$ is attractive if $\forall \rho \in \mathcal{D}(\mathcal{H}_I)$ we have:

$$\lim_{t \to \infty} \left\| \mathcal{T}^t(\rho) - \Pi_S \mathcal{T}^t[\rho] \Pi_S \right\| = 0.$$

Definition 5 (Global asymptotic stability): Let $\mathcal{I}$ evolve under iterations of a TPCP map $\mathcal{T}$. The subsystem $\mathcal{S}$ supported on the subspace $\mathcal{H}_S$ of $\mathcal{H}_I$ is *Globally Asymptotically Stable* (GAS) if it is invariant and attractive.

IV. ANALYSIS RESULTS

This section is devoted to the derivation of necessary and sufficient conditions on the form of the TPCP map $\mathcal{T}$ for *a given* quantum subspace $\mathcal{S}$ to be GAS. We start by focusing on the invariance property.

A. Invariance of $\mathcal{J}_S(\mathcal{H}_I)$

The following proposition gives a sufficient and necessary condition on $\mathcal{T}$ such that $\mathcal{J}_S(\mathcal{H}_I)$ is invariant.

Proposition 1: Let the TPCP transformation $\mathcal{T}$ be described by the Kraus map (1). Let the matrices M_k be expressed in their block form

$$M_k = \begin{bmatrix} M_{k,S} & M_{k,P} \\ M_{k,Q} & M_{k,R} \end{bmatrix}$$

according to the state space decomposition (4). Then the set $\mathcal{J}_S(\mathcal{H}_I)$ is invariant if and only if

$$M_{k,Q} = 0 \quad \forall k. \qquad (6)$$

Proof: Verifying Definition 3 is equivalent to verifying that there exists a TPCP map $\mathcal{T}_S$ such that

$$\mathcal{T}\left[\begin{bmatrix} \rho_S & 0 \\ 0 & 0 \end{bmatrix} \right] = \begin{bmatrix} \mathcal{T}_S(\rho_S) & 0 \\ 0 & 0 \end{bmatrix} \qquad (7)$$

for all ρ_S in $\mathcal{D}(\mathcal{H}_S)$. By exploiting the block form of the M_k matrices in (1) given by the decomposition (4), we have

$$\begin{aligned} \mathcal{T}\left[\begin{bmatrix} \rho_S & 0 \\ 0 & 0 \end{bmatrix} \right] &= \sum_k M_k \begin{bmatrix} \rho_S & 0 \\ 0 & 0 \end{bmatrix} M_k^\dagger \\ &= \sum_k \begin{bmatrix} M_{k,S} & M_{k,P} \\ M_{k,Q} & M_{k,R} \end{bmatrix} \begin{bmatrix} \rho_S & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} M_{k,S}^\dagger & M_{k,Q}^\dagger \\ M_{k,P}^\dagger & M_{k,R}^\dagger \end{bmatrix} \\ &= \sum_k \begin{bmatrix} M_{k,S}\rho_S M_{k,S}^\dagger & M_{k,S}\rho_S M_{k,Q}^\dagger \\ M_{k,Q}\rho_S M_{k,S}^\dagger & M_{k,Q}\rho_S M_{k,Q}^\dagger \end{bmatrix} \end{aligned}$$

$$(8)$$

Sufficiency of (6) to have invariance of $\mathcal{J}_S(\mathcal{H}_I)$ is trivial. Necessity is given by the fact that the lower right blocks $M_{k,Q}\rho_S M_{k,Q}^\dagger$ are positive semi-definite for all k's, and therefore, for (7) to hold, it has to be $M_{k,Q}\rho_S M_{k,Q}^\dagger = 0 \; \forall k$. For $M_{k,Q}\rho_S M_{k,Q}^\dagger$ to be zero for any state $\rho_S \in \mathcal{D}(\mathcal{H}_S)$, it has then to be $M_{k,Q} = 0$. Equation (7) then implies that the completely-positive transformation

$$\mathcal{T}_S[\rho_S] = \sum_k M_{k,S}\rho_S M_{k,S}^\dagger$$

is also trace preserving. ∎

B. Global asymptotic stability of $\mathcal{J}_S(\mathcal{H}_I)$

The main tool we are going to use in deriving a characterization of TPCP maps that render a certain $\mathcal{H}_S$ GAS, is LaSalle's invariance principle, which we recall here in its discrete time form [21].

Theorem 1 (La Salle's theorem for discrete-time systems): Consider a discrete-time system

$$x(t+1) = \mathcal{T}[x(t)]$$

Suppose V is a $\mathcal{C}^1$ function of $x \in \mathbb{R}^n$, bounded below and satisfying

$$\Delta V(x) = V(\mathcal{T}[x]) - V(x) \leq 0, \quad \forall x \qquad (9)$$

i.e. $V(x)$ is non-increasing along forward trajectories of the plant dynamics. Then any bounded trajectory converges to the largest invariant subset W contained in the locus $E = \{x | \Delta V(x) = 0\}$.

Being any TPCP map a map from the compact set of density operators to itself, any trajectory is bounded. Let's then consider the function

$$V(\rho) = \text{Tr}(\Pi_R \rho) \geq 0. \qquad (10)$$

The function $V(\rho)$ is $\mathcal{C}^1$ and bounded from below, and it is a natural candidate for a Lyapunov function for the system.

In fact, it represents the probability of the event Π_R (see Appendix A), that is, the probability that the system is found in $\mathcal{H}_R$ after the measurement.

Lemma 1: Let $\mathcal{T}$ be the generator of a DQDS, and assume a given quantum subsystems $\mathcal{S}$ to be invariant. Then $V(\rho) = \mathrm{Tr}(\Pi_R \rho)$ ($\mathcal{H}_R$ being the remainder space) satisfies the hypothesis (9) of Theorem 1.

Proof: The variation of $V(\rho)$ along forward trajectories of the system (3) is

$$
\begin{aligned}
\Delta V(\rho) &= \mathrm{Tr}\left(\Pi_R \mathcal{T}[\rho]\right) - \mathrm{Tr}(\Pi_R \rho) \\
&= \mathrm{Tr}\left[\Pi_R \left(\sum_k M_k \rho M_k^\dagger - \rho\right)\right]
\end{aligned}
\tag{11}
$$

Notice that $\mathrm{Tr}(\sum_k M_k \rho M_k^\dagger - \rho) = 0$, and that $V(\rho) = 0$ for all ρ's that have support in $\mathcal{H}_S$. Let us express $\sum_k M_k \rho M_k^\dagger - \rho$ in its block form, using the fact that $M_Q = 0$ by assuming invariance of $\mathcal{J}_S(\mathcal{H}_I)$. We get

$$
\sum_k M_k \rho M_k^\dagger - \rho =
$$
$$
= \sum_k \begin{bmatrix} M_{k,S} & M_{k,P} \\ 0 & M_{k,R} \end{bmatrix} \begin{bmatrix} \rho_S & \rho_P \\ \rho_P^\dagger & \rho_R \end{bmatrix} \begin{bmatrix} M_{k,S}^\dagger & 0 \\ M_{k,P}^\dagger & M_{k,R}^\dagger \end{bmatrix} - \rho
$$
$$
= \sum_k \begin{bmatrix} M_{k,S}\rho_S M_{k,S}^\dagger + M_{k,P}\rho_P^\dagger M_{k,S}^\dagger + M_{k,S}\rho_P M_{k,P}^\dagger + M_{k,P}\rho_R M_{k,P}^\dagger \\ M_{k,R}\rho_P M_{k,S}^\dagger + M_{k,R}\rho_R M_{k,P}^\dagger \end{bmatrix}
$$
$$
\left.\begin{matrix} M_{k,S}\rho_P M_{k,R}^\dagger + M_{k,P}\rho_R M_{k,R}^\dagger \\ M_{k,R}\rho_R M_{k,R}^\dagger \end{matrix}\right] - \begin{bmatrix} \rho_S & \rho_P \\ \rho_P^\dagger & \rho_R \end{bmatrix}
\tag{12}
$$

Therefore

$$
\begin{aligned}
\Delta V(\rho) &= \mathrm{Tr}\left[\Pi_R \left(\sum_k M_k \rho M_k^\dagger - \rho\right)\right] \\
&= \mathrm{Tr}\left[\sum_k M_{k,R}\rho_R M_{k,R}^\dagger - \rho_R\right],
\end{aligned}
\tag{13}
$$

so that in order to get $\Delta V \leq 0$ the map $\mathcal{T}_R[\rho_R] := \sum_k M_{k,R}\rho_R M_{k,R}^\dagger$ has to be trace non-increasing.

Note that this condition is automatically verified, once $\mathcal{T}$ is a TPCP map. Indeed, consider the application of $\mathcal{T}$ on a state $\bar{\rho}$ which has support on $\mathcal{H}_R$. According to the block from in (12) we have that the total trace of $\mathcal{T}[\bar{\rho}]$ is

$$
\mathrm{Tr}\left(\mathcal{T}[\bar{\rho}]\right) = \mathrm{Tr}\left(\sum_k M_{k,P}\bar{\rho}_R M_{k,P}^\dagger\right) + \mathrm{Tr}\left(\sum_k M_{k,R}\bar{\rho}_R M_{k,R}^\dagger\right).
$$

Therefore, as both the terms are positive, being $\bar{\rho}_R \geq 0$, and as $\mathcal{T}$ is TP, we have for any $\bar{\rho}_R \in \mathcal{D}(\mathcal{H}_R)$

$$
\mathrm{Tr}\left(\sum_k M_{k,R}\bar{\rho}_R M_{k,R}^\dagger\right) \leq \mathrm{Tr}\left(\mathcal{T}[\bar{\rho}]\right) = \mathrm{Tr}\left(\bar{\rho}_R\right)
$$

and thus $\mathcal{T}_R$ is trace non-increasing. ∎

This leaves us with determining when $\mathcal{J}_S(\mathcal{H}_I)$ contains the largest invariant set in E. We shall derive conditions that ensure that no other invariant set W exists in $E = \{\rho | \Delta V(\rho) = 0\}$ such that $\mathcal{J}_S(\mathcal{H}_I) \subset W$. We start by giving some preliminary results.

Lemma 2: Let $\mathcal{T}$ be a TPCP transformation described by the Kraus map (1). Consider an orthogonal subspace decomposition $\mathcal{H}_S \oplus \mathcal{H}_R$. Then the set $\mathcal{J}_R(\mathcal{H}_I)$ contains an invariant subset if and only if it contains an invariant state.

Proof: The "if" part is trivial. On the other hand, $\mathcal{J}_R(\mathcal{H}_I)$ is convex and compact, hence if it contains an invariant subset W it also contains the closure of its convex hull, call it $\bar{W}$. The map $\mathcal{T}$ is linear and continuous, so the convex hull of an invariant subset is invariant, and so is its closure. Hence, by Brouwer's fixed point theorem [40] it admits a fixed point $\bar{\rho} \in \bar{W} \subseteq \mathcal{J}_R(\mathcal{H}_I)$. ∎

Lemma 3: Let W be an invariant subset of $\mathcal{D}(\mathcal{H}_I)$ for the TPCP transformation $\mathcal{T}$, and let

$$
\mathcal{H}_W = \mathrm{supp}(W) = \bigcup_{\rho \in W} \mathrm{supp}(\rho).
$$

Then $\mathcal{J}_W(\mathcal{H}_I)$ is invariant.

Proof: The proof follows the one for the continuous-time case in [31, Lemma 8]. Let $\hat{W}$ be the convex hull of W. By linearity of dynamics it is easy to show that $\hat{W}$ is invariant too. Furthermore, from the definition of $\hat{W}$, there exists a $\hat{\rho} \in \hat{W}$ such that $\mathrm{supp}(\hat{\rho}) = \mathrm{supp}(\hat{W}) = \mathcal{H}_W$. Consider the decomposition $\mathcal{H}_I = \mathcal{H}_W \oplus \mathcal{H}_W^\perp$, and the corresponding matrix partitioning

$$
X = \begin{bmatrix} X_W & X_L \\ X_M & X_N \end{bmatrix}.
$$

With respect to this partition, $\hat{\rho}_W$ is full rank while $\hat{\rho}_{L,M,N}$ are zero blocks. The state $\hat{\rho}$ is then mapped by $\mathcal{T}$ according to (8) and therefore, as $\hat{\rho}_W$ is full rank, it has to be $M_{k,Q} = 0$ for all k's. Comparing it with the conditions given in proposition 1, we then infer that $\mathcal{J}_W(\mathcal{H}_I)$ is invariant. ∎

Proposition 2: Consider $\rho \in \mathcal{J}_R(\mathcal{H}_I)$ and evolving under the TPCP transformation $\mathcal{T}$ described by the Kraus map (1). Let the matrices M_k be expressed in the block form

$$
M_k = \begin{bmatrix} M_{k,S} & M_{k,P} \\ 0 & M_{k,R} \end{bmatrix}
$$

according to the state space decomposition $\mathcal{H}_S \oplus \mathcal{H}_R$, with $\mathcal{J}_S(\mathcal{H}_I)$ invariant. Then $\rho \in E = \{\rho \in \mathcal{D}(\mathcal{H}_I) | \Delta V(\rho) = 0\}$, where $V(\rho)$ is defined by (10), if and only if its ρ_R block satisfies

$$
\mathrm{supp}(\rho_R) \subseteq \bigcap_k \ker\left(M_{k,P}\right).
$$

Proof: By direct computation, see (12), we have

$$
\sum_k M_k \rho M_k^\dagger = \sum_k \begin{bmatrix} M_{k,P}\rho_R M_{k,P}^\dagger & M_{k,P}\rho_R M_{k,R}^\dagger \\ M_{k,R}\rho_R M_{k,P}^\dagger & M_{k,R}\rho_R M_{k,R}^\dagger \end{bmatrix}
\tag{14}
$$

as ρ has support on $\mathcal{H}_R$ alone. Note that as $V(\rho) = 1$, $\Delta V(\rho) = 0$ is equivalent to $\mathcal{T}[\rho]$ having support on $\mathcal{H}_R$. Given the form of the upper-left block of (14), this is true if and only if $\mathrm{supp}(\rho_R) \subseteq \bigcap_k \ker\left(M_{k,P}\right)$. ∎

Proposition 2 allows then to state the following key characterization of global, asymptotical stability of $\mathcal{J}_S(\mathcal{H}_I)$.

Theorem 2: Let the TPCP transformation $\mathcal{T}$ be described by the Kraus map (1). Consider an orthogonal subset decomposition $\mathcal{H}_S \oplus \mathcal{H}_R$, with $\mathcal{J}_S(\mathcal{H}_I)$ invariant. Let the matrices M_k be expressed in their block form

$$M_k = \begin{bmatrix} M_{k,S} & M_{k,P} \\ 0 & M_{k,R} \end{bmatrix}$$

according to the same state space decomposition. Then the set $\mathcal{J}_S(\mathcal{H}_I)$ is GAS if and only if there are no invariant states with support on

$$\bigcap_k \ker\left(M_{k,P} \right).$$

Proof: Necessity is immediate: if there was an invariant state with support on $\bigcap_k \ker\left(M_{k,P} \right)$, It would have non trivial support on $\mathcal{H}_R$, and therefore $\mathcal{H}_S$ could not be attractive. In order to prove the other implication, consider LaSalle's theorem. By hypotesis, $\mathcal{J}_S(\mathcal{H}_I)$ is invariant and is contained in E (see Proposition 2), therefore it is contained in the largest invariant set W in the zero-difference locus E.

Let us suppose that $\mathcal{J}_S(\mathcal{H}_I) \subset W$, but $\mathcal{J}_S(\mathcal{H}_I) \neq W$. That is, there exists a set $W \subseteq E$ which is invariant and strictly contains $\mathcal{J}_S(\mathcal{H}_I)$. Therefore its support has to be

$$\mathcal{H}_W = \mathcal{H}_S \oplus \mathcal{H}_{R'}$$

with $\mathcal{H}_{R'}$ subspace of $\mathcal{H}_R$, and by Lemma 3 $\mathcal{J}_W(\mathcal{H}_I)$ must be invariant too. Consider then a state $\hat{\rho}$ which belongs to $\mathcal{J}_W(\mathcal{H}_I)$, with non trivial support on $\mathcal{H}_{R'}$, and define

$$\tilde{\rho} = \frac{\Pi_{R'} \hat{\rho} \Pi_{R'}}{\mathrm{Tr}(\Pi_{R'} \hat{\rho})} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{\rho}_R \end{bmatrix}$$

which has support on $\mathcal{H}_{R'}$ only. By construction, $\tilde{\rho}$ is in $\mathcal{J}_W(\mathcal{H}_I)$, and therefore its trajectory is contained in $\mathcal{J}_W(\mathcal{H}_I)$. It is also in E, that is $\Delta V(\tilde{\rho}) = 0$. As we have $V(\tilde{\rho}) = 1$, then its evolution must be also remain in $\mathcal{H}_{R'} \subseteq \mathcal{H}_R$ at any time. Therefore an invariant set with support on $\mathcal{H}_R$ exists. By reversing the implication, this means that if does not exist an invariant set with support on $\mathcal{H}_R$, then $\mathcal{J}_S(\mathcal{H}_I)$ is the largest invariant set in E. Furthermore, Proposition 2 indicates that if there is an invariant set in E with support on $\mathcal{H}_R$, its support must actually be contained in

$$\bigcap_k \ker\left(M_{k,P} \right).$$

Therefore, if no such subset exists, we have attractivity of $\mathcal{J}_S(\mathcal{H}_I)$ by LaSalle's theorem. By Lemma 2, the existence of an invariant set is equivalent to the existence of an invariant state with support on $\bigcap_k \ker\left(M_{k,P} \right)$. ∎

Given the usual decomposition $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$, let us further decompose $\mathcal{H}_R$ in

$$\mathcal{H}_{R'} = \mathcal{H}_R \ominus \bigcap_k \ker\left(M_{k,P} \right) \quad \text{and} \quad \mathcal{H}_{R''} = \bigcap_k \ker\left(M_{k,P} \right)$$

and consider the operation elements M_k in a basis induced by the decomposition $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_{R'} \oplus \mathcal{H}_{R''}$:

$$M_k = \left[\begin{array}{c|c|c} M_{k,S} & M_{k,P'} & 0 \\ \hline 0 & M_{k,R1} & M_{k,R2} \\ \hline 0 & M_{k,R3} & M_{k,R4} \end{array} \right].$$

Density operators ρ which have support on the bottom right block clearly belong to $\mathcal{J}_{R''}(\mathcal{H}_I)$. Sufficient, although not necessary, condition to be sure that no invariant sets have support on that subspace is that

$$\bigcap_k \ker\left(M_{k,R2} \right) = \{0\} \quad \text{and} \quad M_{k,R3} = 0 \,\forall k.$$

This way, the states that have support on $\bigcap_k \ker\left(M_{k,P} \right)$ will be mapped into states which has non-trivial support on $\left[\bigcap_k \ker\left(M_{k,P} \right) \right]^\perp$, and therefore no invariant set will exist in $\bigcap_k \ker\left(M_{k,P} \right)$. This intuition will be further developed in the Section VI, where a control design tool capable of achieving attractivity of a given subspace is obtained.

V. A CANONICAL MATRIX FORM BASED ON THE QR DECOMPOSITION

A. On the uniqueness of QR decomposition

Definition 6 (QR decomposition [16]): A QR decomposition of a complex-valued square matrix A is a decomposition of A as

$$A = QR,$$

where Q is an orthogonal matrix (meaning that $Q^\dagger Q = I$) and R is an upper triangular matrix.

We investigate here the uniqueness of the QR decomposition of a complex-valued matrix A, both in the case in which A is non-singular and in the case in which it is singular. While the real-matrix case is well known (see e.g. [16]), a little extra care is needed in the complex case. Let's first analyze the case of A non-singular. We have the following result.

Lemma 4: Let

$$A = Q_1 R_1, \quad A = Q_2 R_2$$

be two QR decompositions of the same *non-singular* square matrix A. Then R_1 and R_2 only differ for the phase of their rows, that is

$$R_1 = \Phi R_2, \qquad Q_1 = Q_2 \Phi^{-1}$$

where $\Phi = \mathrm{diag}\left(e^{j\phi_1}, \ldots, e^{j\phi_n} \right)$.

Proof: Using the fact that A and its factors are non singular, we have

$$Q_2^\dagger Q_1 = R_2 R_1^{-1} = \Phi$$

where Φ is upper triangular because it is the product of two upper triangular matrices. The matrix Φ must also be orthonormal, because it is the product of two orthonormal matrices. Therefore, starting from the first column of Φ, we have that $|\Phi_{11}| = 1$. where Φ_{ij} is the element of Φ in position (i, j).

We proceed by induction on the column index j. Assume that all the columns Φ_k with $k < j$ satisfy $\Phi_{lk} = 0$ for any $l \neq k$. In order for Φ_j to satisfy $\Phi_k^\dagger \Phi_j = 0 \,\forall k < j$, it must be

$$\Phi_{1j} = \cdots = \Phi_{j-1,j} = 0.$$

Moreover, as Φ is upper triangular, we must also have $\Phi_{j+1,j} = \cdots = \Phi_{n,j} = 0$. Therefore, by othonormality of Φ, it has to be $|\Phi_{jj}| = 1$. ∎

In the case in which A is singular, on the other hand, the QR decomposition is not just unique up to a phase of the rows of R.

Example 1. Consider the following matrix:

$$M = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Since it is already upper triangular, a valid QR decomposition is given by $Q = I, R = M$. On the other hand, we can consider

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \ R = \begin{bmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix},$$

which clearly also give $QR = M$, with $Q^\dagger Q = I$.

However, introducing some conditions on the QR decomposition, it is possible to obtain a *canonical form* for the QR decomposition in a sense that will be explained later in this section. A useful lemma in this sense is the following.

Lemma 5: Consider a QR decomposition of a square matrix A of dimension n, and an index $\bar{j}$ in $[1, n]$, such that

$$r_{ij} = 0 \quad \forall j \le \bar{j}, \forall i > \rho_j \tag{15}$$

where ρ_j is the rank of the first j columns of A. Let a_i and q_i, be the i-th column of A and Q respectively. Then

$$<a_1, \ldots, a_j> = <q_1, \ldots, q_{\rho_j}> \quad \forall j = 1, \ldots, \bar{j}.$$

Proof: Consider the expression for the j-th column of A

$$a_j = Q r_j.$$

By the hypothesis, the last $n - \rho_j$ elements of r_j are zeros, hence it results

$$a_j \in <q_1, \ldots, q_{\rho_j}> \quad \forall j = 1, \ldots, \bar{j}$$

and therefore

$$<a_1, \ldots, a_j> \subseteq <q_1, \ldots, q_{\rho_j}> \quad \forall j = 1, \ldots, \bar{j}.$$

As the rank of the first j columns is ρ_j, which is also the dimension of $<q_1, \ldots, q_{\rho_j}>$, equality of the two subspaces holds. ∎

We next show as a particular choice of the QR decomposition, suggested by lemma 5, gives a canonical form on $\mathbb{C}^{n \times n}$ with respect to left-multiplication for elements of the unitary matrix group $\mathcal{U}(n)$. We construct the QR decomposition through the Gram-Schmidt orthonormalization process, fixing the degrees of freedom of the upper-triangular factor R and verifying that the resulting decomposition satifies the hypotesis of lemma 5 for $\bar{j} = n$.

B. Construction of the QR decomposition by orthonormalization

Theorem 3: Given any (complex) square matrix A of dimension n, it is possible to derive a QR decomposition $A = QR$ such that hypoteses of Lemma 5 are satisfied, and

such that the first nonzero element of each row of R is real and positive.

Proof: We explicitly construct the QR decomposition of A column by column. We denote by A, Q, R the matrices, with a_i, q_i, r_i their i-th columns and with $a_{i,j}, q_{i,j}, r_{i,j}$ their elements, respectively. Let us start from the first non zero column of $A \in \mathbb{C}^{n \times n}$, a_{i_0}, and define

$$q_1 = \frac{a_{i_0}}{\|a_{i_0}\|}, \qquad r_{1,i_0} = \|a_{i_0}\|, \qquad r_{2,i_0} = \ldots = r_{n,i_0} = 0. \tag{16}$$

Also fix $r_j = 0$ for all $j < i_0$.

The next columns of Q, R are constructed by an iterative procedure. Define ρ_{i-1} as the rank of the first $i - 1$ columns of A. We can assume (by induction) to have the first ρ_{i-1} columns of Q and the first $i - 1$ columns of R constructed in such a way that $r_{k,j} = 0$ for $k > \rho_j$ and $j \le i - 1$.

Consider the next column of A, a_i. Assume that a_i is linearly dependent with the previous columns of A, that is $\rho_i = \rho_{i-1}$. Since lemma 5 applies, a_i can be written as

$$a_i = \sum_{j=1}^{i-1} \alpha_j a_j = \sum_{j=1}^{i-1} \alpha_j \sum_{\ell=1}^{\rho_j} r_{\ell,j} q_\ell$$

and therefore, being a_i a linear combination of the columns $\{q_1, \ldots, q_{\rho_{i-1}}\}$, the elements of r_i are defined as

$$r_{\ell,i} = q_\ell^\dagger a_i, \quad \text{for } \ell = 1, \ldots, \rho_i.$$

On the other hand, if the column a_i is linearly independent from the previous columns of A, then the rank $\rho_i = \rho_{i-1} + 1$. As before, the first ρ_{i-1} coefficients of r_i must be defined as

$$r_{\ell,i} = q_\ell^\dagger a_i, \quad \text{for } \ell = 1, \ldots, \rho_i - 1,$$

and let set $r_{\ell,i} = 0$ for $\ell = \rho_i + 1, \ldots, n$. Let us also introduce

$$\tilde{a}_i := a_i - \sum_{\ell=1}^{\rho_i} r_{\ell,i} q_\ell \ne 0 \tag{17}$$

and define

$$q_{\rho_i} = \frac{\tilde{a}_i}{\|\tilde{a}_i\|} \quad r_{\rho_i,i} = \|\tilde{a}_i\|, \tag{18}$$

It is immediate to verify that the obtained q_{ρ_i} is orthonormal to the columns $q_1, \ldots, q_{\rho_i-1}$, and that $a_i = Q r_{\rho_i}$.

After iterating until the last column of R is defined, we are left to choose the remaining columns of Q so that the set $\{q_1, \ldots, q_n\}$ is an orthonormal basis for $\mathbb{C}^{n \times n}$. By construction, $A = QR$. ∎

C. R is a canonical form

Let $\mathcal{G}$ be a group acting on $\mathbb{C}^{n \times n}$. Let $A, B \in \mathbb{C}^{n \times n}$. If there exists a $g \in \mathcal{G}$ such that $g(A) = B$, we say that A and B are $\mathcal{G}$-equivalent, and we write $A \sim_{\mathcal{G}} B$.

Definition 7: A canonical form with respect to $\mathcal{G}$ is a function $\mathcal{F} : \mathbb{C}^{n \times n} \to \mathbb{C}^{n \times n}$ such that for every $A, B \in \mathbb{C}^{n \times n}$:

  i. $\mathcal{F}(A) \sim_{\mathcal{G}} A$;
  ii. $\mathcal{F}(A) = \mathcal{F}(B)$ if and only if $A \sim_{\mathcal{G}} B$.

Let us consider the unitary matrix group $\mathcal{U}(n) \subset \mathbb{C}^{n \times n}$ and consider its action on $\mathbb{C}^{n \times n}$ through left-multiplication, that is, for any $U \in \mathcal{U}(n)$, $M \in \mathbb{C}^{n \times n}$:

$$U(M) = UM.$$

We are now ready to prove the following.

Theorem 4: Define $\mathcal{F}(A) = R$, with R the upper-triangular matrices obtained by the procedure described in the proof of theorem 3. Then $\mathcal{F}$ is a canonical form with respect to $\mathcal{U}(n)$ (and its action on $\mathbb{C}^{n \times n}$ by left multiplication).

Proof: By construction $A = QR$, with unitary Q, so $\mathcal{F}(A) \sim_{\mathcal{U}(n)} A$. If $A, B \in \mathcal{C}^{n \times n}$ are such that $\mathcal{F}(A) = \mathcal{F}(B) = R$, thus $A = QR$ and $B = VR$ for some $Q, V \in \mathcal{U}(n)$, and hence $A = QV^{-1}B$.

On the other hand, if $A = UB$, $U \in \mathcal{U}(n)$, we have to prove that the upper-triangular matrix in the canonical QR decompositions $A = QR^{(A)}$ and $B = VR^{(B)}$ is the same. If the first non zero column of B is b_{i_0}, then the first column nonzero column of A is, being U unitary, $a_{i_0} = Ub_{i_0}$. One then finds from (16)

$$v_1 = \frac{U^\dagger a_{i_0}}{\|U^\dagger a_{i_0}\|} = U^\dagger q_1 \quad r_{1,i_0}^{(B)} = \|U^\dagger a_{i_0}\| = r_{1,i_0}^{(A)}. \quad (19)$$

Hence the first i_0 columns of $R^{(A)}$ and $R^{(B)}$ are identical. We then proceed by induction. Assume that $r_j^{(A)} = r_j^{(B)}$, $q_j = Uv_j$ for $j = 1, ..., i-1$. If the column a_i is linearly dependent from the previous $i-1$ so it must be b_i. The elements of $r_i^{(A)}$ are defined as

$$r_{k,i}^{(A)} = q_k^\dagger a_i = q_k^\dagger UU^\dagger a_i = v_k^\dagger b_i = r_{k,i}^{(B)}, \quad \text{for } k = 1, \ldots, \rho_i - 1.$$

On the other hand, if the column a_i is linearly independent from the previous columns of A, then the rank $\rho_i = \rho_{i-1} + 1$. As before, the first $\rho_i - 1$ coefficients of r_i are defined as

$$r_{k,i}^{(A)} = q_k^\dagger a_i = q_k^\dagger UU^\dagger a_i = v_k^\dagger b_i = r_{k,i}^{(B)}, \quad \text{for } k = 1, \ldots, \rho_i - 1,$$

and $r_{k,i}^{(A)} = r_{k,i}^{(B)} = 0$ for $k = \rho_i + 1, \ldots, n$. Let us consider as before

$$\tilde{a}_i := a_i - \sum_{k=1}^{\rho_i - 1} r_{ki}^{(A)} q_k \neq 0.$$

By using the equivalent definition and the inductive hypothesis it follows that $\tilde{b}_i = U^\dagger \tilde{a}_i$ and

$$v_{\rho_i} = \frac{U^\dagger \tilde{a}_i}{\|U^\dagger \tilde{a}_i\|} = U^\dagger q_{\rho_i} \quad r_{\rho_i,i}^{(B)} = \|U^\dagger \tilde{a}_i\| = r_{\rho_i,i}^{(A)}.$$

Hence $r_i^{(A)} = r_i^{(B)}$, and by induction $R^{(A)} = R^{(B)}$. ∎

VI. ENGINEERING ATTRACTIVE SUBSPACES VIA CLOSED-LOOP CONTROL

A. The controlled dynamics

In this section we deal with the problem of stabilization of a given quantum subspace by discrete-time measurements and unitary control. The control scheme we employ follows the ideas of [23], [29].

Suppose that a generalized measurement operation can be performed on the system at times $t = 1, 2, \ldots$, resulting

in an open system, discrete-time dynamics described by a given Kraus map, with associated Kraus operators $\{M_k\}$. This can be realized, for example, when the system is coupled to an auxiliary measurement apparatus, it is manipulated coherently, and then a projective measurement is performed on the auxiliary system (see appendix A). Suppose moreover that we are allowed to unitarily control the state of the system, i.e. $\rho_{controlled} = U\rho U^\dagger$, $U \in \mathcal{U}(\mathcal{H}_I)$. We shall assume that the control is fast with respect to the measurement time scale, or the measurement and the control acts in distinct time slots.

We can then use the generalized measurement outcome k to condition the control choice, that is, a certain coherent transformation U_k is applied after the k-th output is recorded. The measurement-control loop is then iterated: If we average over the measurement results at each step, this yields a different TPCP map, which dependens on the design of the set of unitary controls $\{U_k\}$ and describes the evolution of the state *immediately after* each application of the controls:

$$\rho(t+1) = \sum_k U_k M_k \rho(t) M_k^\dagger U_k^\dagger.$$

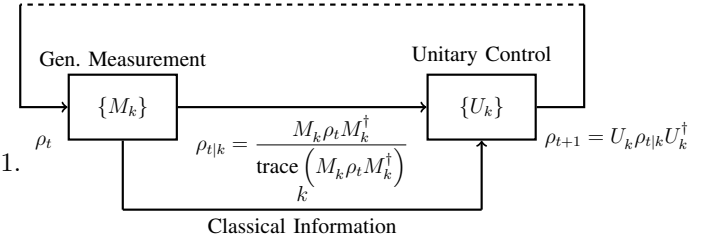Figure 1 depicts the feedback control loop (before the averaging).

Fig. 1. A measurement-dependent unitary control scheme.

In this section we will tackle the problem of characterizing the set of open loop dynamics that can be engineered through this feedback setup *by designing the set $\{U_k\}$ with fixed measurement operator $\{M_k\}$*. Moreover, on the basis of the analysis results of section IV, we will derive an algorithm that allows to design the set of unitary controls such that global asymptotical stability of a *given* subspace is achieved. Notice that the desired result is going to be achieved for the averaged time-evolution that describes the system immediately after the control step. It is easy to show that if a certain subspace is GAS for the averaged dynamics, it must be so also for the conditional ones.

B. Simulating generalized measurements

A first straightforward application of the canonical form we derived in the previous section is the following. Assume we are able to perform a generalized measurement, with associated operators $\{M_k\}_{k=1}^m$, and we would like to actually implement a different measurement with associated operators $\{N_k\}_{k=1}^m$, by using the unitary control loop as above. Notice that the control scheme we considered allows to modify only the conditioned states, not the probability of the outcomes, since $\text{trace}(M_k^\dagger M_k \rho) = \text{trace}(M_k^\dagger U_k^\dagger U_k M_k \rho)$. The following holds:

Proposition 3: A measurement with associated operators $\{N_k\}_{k=1}^m$ can be simulated by a certain choice of unitary controls from a measurement $\{M_k\}_{k=1}^m$, if and only if there exist a reordering $j(k)$ of the first m integers such that:

$$\mathcal{F}(N_k) = \mathcal{F}(M_{j(k)}),$$

where $\mathcal{F}$ returns the canonical R factor of the argument, as described in the section V.

Proof: Let us first assume that for a given reordering $j(k)$ it holds $\mathcal{F}(N_k) = \mathcal{F}(M_{j(k)}) = R_k$. Therefore the canonical QR decomposition of N_k and $M_{j(k)}$ gives

$$N_k = U_k R_k, \qquad M_{j(k)} = V_{j(k)} R_k.$$

Let then $U_k V_{j(k)}^\dagger$ be the unitary control associated with the measurement outcome k. We have

$$
\begin{aligned}
\mathcal{T}_{\text{closed loop}}[\rho] &= \sum_k U_k V_{j(k)}^\dagger M_{j(k)} \rho M_{j(k)}^\dagger V_{j(k)} U_k^\dagger \\
&= \sum_k U_k R_k \rho R_k^\dagger U_k^\dagger = \sum_k N_k \rho N_k^\dagger
\end{aligned}
$$

and therefore it simulates the measurement associated with the operators $\{N_k\}_{k=1}^m$.

On the other hand, suppose that there exists a set of unitary controls $\{Q_k\}_{k=1}^m$ and there is a reordering $j(k)$ of the first m integers such that

$$Q_{j(k)} M_{j(k)} = N_k.$$

According to Theorem 4, $\mathcal{F}$ is a canonical form with respect to $\mathcal{U}(n)$ and its action on $\mathbb{C}^{n \times n}$ by left multiplication, and therefore if $Q_{j(k)} M_{j(k)} = N_k$ then $\mathcal{F}(M_{j(k)}) = \mathcal{F}(N_k)$. ∎

C. Global asymptotic stabilization of a quantum subspace

Suppose that the operators $\{M_k\}$ are given, corresponding to a measurement that is performed on the quantum system, with corresponding outcomes $\{k\}$. We are then looking for a set of unitary transformations $\{U_k\}$ such that, once they are applied to the system, the resulting transformation

$$\mathcal{T}[\rho] = \sum_k U_k M_k \rho M_k^\dagger U_k^\dagger$$

makes a given subsystem $\mathcal{S}$ GAS. Let us introduce a preliminary result.

Lemma 6: Let R be the upper triangular factor of a *canonical QR decomposition* in the form

$$R = \begin{bmatrix} R_S & R_P \\ 0 & R_R \end{bmatrix}$$

(according to the block structure induced by (4)) and suppose $R_P = 0$. Consider the matrix N obtained my left multiplying R by a unitary matrix V:

$$N = VR = \begin{bmatrix} V_S & V_P \\ V_Q & V_R \end{bmatrix} \begin{bmatrix} R_S & 0 \\ 0 & R_R \end{bmatrix} = \begin{bmatrix} N_S & N_P \\ N_Q & N_R \end{bmatrix}.$$

Then $N_Q = 0$ implies $N_P = 0$.

Proof: Let first consider the case in which R_S has full rank. Let $r \times m$ be the dimension of V_Q, and $m \times m$ be the dimension of R_S. Since it must be $N_Q = V_Q R_S = 0$ and R_S is full rank, we have $V_Q = 0$.

As V is unitary, its column must be orthonormal. Being $V_Q = 0$, V_S must be itself an orthonormal block in order to have orthonormality of the first m columns of V. It then follows that $V_P = 0$, because any j-th column, $j > m$, must be orthonormal to all the first m columns. It then follows that

$$N_P = V_P R_R = 0.$$

Let us now consider the other case, in which R_S is singular. This implies that $\rho_m < m$ (ρ_m being the rank of the first m columns of R): Therefore, as R is a triangular factor of a canonical QR decomposition, the element $R_{\rho_m+1,m} = 0$.

Now, by construction of the canonical QR decomposition, if there were non-zero columns of index $j > m$, one of them would have a non-zero element on the row of index $\rho_m + 1$. By recalling that $R_P = 0$, we have that $R_{\rho_m+1,j} = 0$, $\forall j \in [m+1, m+r]$. Therefore, all the last r columns are zero-vectors, and in particular $R_R = 0$. It then follows that

$$N_P = V_P R_R = 0.$$

∎

This result will be instrumental in proving the main theorem of the section, which provides us with an iterative control design procedure that renders the desired subspace asymptotically stable whenever it is possible.

Theorem 5: Consider a subspace orthogonal decomposition $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$ and a given generalized measurement associated to Kraus operators $\{M_k\}$. If asymptotic stability of a subspace $\mathcal{S}$ can be achieved by any measurement-dependent unitary control $\{U_k\}$, it can be achieved by building U_k using the iterative algorithm below.

Control design algorithm

Let $\{|\phi\rangle_j^S\}_{j=1}^m$, $\{|\phi\rangle_k^R\}_{k=1}^r$ denote orthonormal bases for $\mathcal{H}_S$ and $\mathcal{H}_R$, and represent each M_k as a matrix with respect to the basis $\{|\phi\rangle_j^S\}_{j=1}^m \bigcup \{|\phi\rangle_k^R\}_{k=1}^r$. Compute a QR decomposition $M_k = Q_k R_k$ with canonical R_k for each k. Call $\mathcal{H}_R^{(0)} = \mathcal{H}_R$, $\mathcal{H}_S^{(0)} = \mathcal{H}_S$, $U_k^{(0)} = Q_k^\dagger$ and rename the matrix blocks $R_{S,k}^{(0)} = R_{S,k}$, $R_{P,k}^{(0)} = R_{P,k}$ and $R_{R,k}^{(0)} = R_{R,k}$.

If $R_{P,k} = 0$ $\forall k$, then the problem is not feasible and a unitary control law cannot be found. Otherwise define $V^{(0)} = I$, $Z^{(0)} = I$, and consider the following iterative procedure, starting from $i = 0$:

1) Define $\mathcal{H}_R^{(i+1)} = \bigcap_k \ker R_{P,k}^{(i)}$:
   If $\mathcal{H}_R^{(i+1)} = \{0\}$ then the iteration is successfully completed. Go to step 8).
   If $\mathcal{H}_R^{(i+1)} \subsetneq \mathcal{H}_R^{(i)}$, define $\mathcal{H}_S^{(i+1)} = \mathcal{H}_R^{(i)} \ominus \mathcal{H}_R^{(i+1)}$ and $Y^{(i+1)} = I$.
   If $\mathcal{H}_R^{(i+1)} = \mathcal{H}_R^{(i)}$ (i.e. $R_{P,k}^{(i)} = 0$ $\forall k$) then, if $\dim(\mathcal{H}_R^{(i)}) \geq \dim(\mathcal{H}_S^{(i)})$:

   a) Choose a subspace $\mathcal{H}_S^{(i+1)} \subseteq \mathcal{H}_R^{(i)}$ of the same dimension of $\mathcal{H}_S^{(i)}$. (Re)-define $\mathcal{H}_R^{(i+1)} = \mathcal{H}_R^{(i)} \ominus \mathcal{H}_S^{(i+1)}$.

b) Let $\mathcal{H}_T^{(i)} = \bigoplus_{j=0}^{i-1} \mathcal{H}_S^{(j)}$. Construct a unitary matrix Y with the following block form, according to a Hilbert space decomposition $\mathcal{H}_I = \mathcal{H}_T^{(i)} \oplus \mathcal{H}_S^{(i)} \oplus \mathcal{H}_S^{(i+1)} \oplus \mathcal{H}_R^{(i+1)}$:

$$Y^{(i+1)} = \left[\begin{array}{c|c|c|c} I & 0 & 0 & 0 \\ \hline 0 & 1/\sqrt{2}I & 1/\sqrt{2}I & 0 \\ \hline 0 & 1/\sqrt{2}I & -1/\sqrt{2}I & 0 \\ \hline 0 & 0 & 0 & I \end{array}\right]. \tag{20}$$

If instead $\dim(\mathcal{H}_R^{(i)}) < \dim(\mathcal{H}_S^{(i)})$:

a) Choose a subspace $\mathcal{H}_S^{(i+1)} \subseteq \mathcal{H}_S^{(i)}$ of the same dimension of $\mathcal{H}_R^{(i)}$.

b) Let $\mathcal{H}_T^{(i)} = \left(\bigoplus_{j=0}^{i-1} \mathcal{H}_S^{(j)}\right) \oplus \left(\mathcal{H}_S^{(i)} \ominus \mathcal{H}_S^{(i+1)}\right)$. Construct a unitary matrix Y with the following block form, according to a Hilbert space decomposition $\mathcal{H}_I = \mathcal{H}_T^{(i)} \oplus \mathcal{H}_S^{(i+1)} \oplus \oplus\mathcal{H}_R^{(i+1)}$:

$$Y^{(i+1)} = \left[\begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & 1/\sqrt{2}I & 1/\sqrt{2}I \\ \hline 0 & 1/\sqrt{2}I & -1/\sqrt{2}I \end{array}\right]. \tag{21}$$

c) Define $Z^{(i+1)} = Z^{(i)}Y^{(i+1)}$ and go to step 8).

2) Define $Z^{(i+1)} = Z^{(i)}Y^{(i+1)}$.

3) Rewrite $\tilde{R}_{R,k}^{(i)} = W^{(i+1)}R_{R,k}^{(i)}W^{(i+1)\dagger}$ in a basis according to the $\mathcal{H}_R^{(i)} = \mathcal{H}_S^{(i+1)} \oplus \mathcal{H}_R^{(i+1)}$ decomposition.

4) Compute the canonical QR decomposition of $\tilde{R}_{R,k}^{(i)} = Q_k^{(i+1)}R_k^{(i+1)}$. Compute the matrix blocks $R_{P,k}^{(i+1)}, R_{R,k}^{(i+1)}$ of $R_k^{(i+1)}$, again according to the decomposition $\mathcal{H}_R^{(i)} = \mathcal{H}_S^{(i+1)} \oplus \mathcal{H}_R^{(i+1)}$.

5) Define $U^{(i+1)} = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & W^{(i+1)\dagger}\left(Q_k^{(i+1)}\right)^{\dagger}W^{(i+1)} \end{array}\right]U^{(i)}.$

6) Define $V^{(i+1)} = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & W^{(i+1)} \end{array}\right]V^{(i)}.$

7) Increment the counter and go back to step 1).

8) Return the unitary controls $U_k = V^{(i)\dagger}Z^{(i)}V^{(i)}U_k^{(i)}$.

Proof: Let us first consider the case in which the algorithm stops before the iterations. This happens if for every k we have $R_{P,k} = 0$. Remember that each R_k has been put in canonical form, so Lemma 6 applies: This means that any control choice that ensures invariance of the desired subspace, that is $N_k = U_k R_k$ with $N_{Q,k} = 0$, makes also $\mathcal{J}_R(\mathcal{H}_I)$ invariant, since $N_{P,k} = 0$. Hence an invariant state with support on $\mathcal{H}_R$ always exists. This, via Theorem 2, precludes the existence of a control choice that renders $\mathcal{J}_S(\mathcal{H}_I)$ GAS.

If the algorithm does not stop, then at each step of the iteration the dimension of $\mathcal{H}_R^{(i)}$ is reduced by at at least 1,

hence the algorithm is completed in at most n steps. If the algorithm is successfully completed at a certain iteration j, we have built unitary controls $\{U_k^{(j)}\}$ and a unitary $V^{(j)}$ such that the controlled quantum operation element, under the change of basis $V^{(j)}$, is of the form:

$$\begin{aligned} \tilde{N}_k &= V^{(j)}U_k M_k V^{(j)\dagger} \\ &= Z^{(j)} \left[\begin{array}{c|c|c|c|c} R_{S,k}^{(0)} & \bar{R}_{P,k}^{(0)} & 0 & 0 & 0 \\ \hline 0 & R_{S,k}^{(1)} & \ddots & 0 & 0 \\ \hline 0 & 0 & \ddots & \bar{R}_{P,k}^{(j-1)} & 0 \\ \hline 0 & 0 & 0 & R_{S,k}^{(j)} & \bar{R}_{P,k}^{(j)} \\ \hline 0 & 0 & 0 & 0 & R_{R,k}^{(j)} \end{array}\right] \end{aligned} \tag{22}$$

where the block structure is consistent with the decomposition $\bigoplus_{i=0}^{j+1} \mathcal{H}_S^{(i)}$ (where to simplify the notation we set $\mathcal{H}_S^{(j+1)} = \mathcal{H}_R^{(j)}$). Let $\bar{R}_k$ be the block matrix above and consider its upper-triangular part. The rows have the form $\left[\bar{R}_{P,k}^{(i)} \quad 0 \quad \ldots \quad 0\right]$ because at each step of the iteration we choose a basis $W^{(i)}$ according to the decomposition $\mathcal{H}_S^{(i+1)} \oplus \mathcal{H}_R^{(i+1)}$, where $\mathcal{H}_R^{(i+1)} \subseteq \bigcap_k \ker R_{P,k}^{(i)}$, hence obtaining $R_{P,k}^{(i)}W^{(i)\dagger} = \left[\bar{R}_{P,k}^{(i)} \quad 0 \quad \ldots \quad 0\right]$. It is easy to verify that the subsequent unitary transformations have no effects on the blocks $\bar{R}_{P,k}^{(i)}$.

The upper-triangular form of each $\bar{R}_k$ and the form of $Z^{(j)}$ and $V^{(j)}$, both block-diagonal with respect to the orthogonal decomposition $\mathcal{H}_S \oplus \mathcal{H}_R$, ensure invariance of $\mathcal{H}_S$.

By construction, for all $i = 0, \ldots, j$, either $\bigcap_k \ker \bar{R}_{P,k}^{(i)} = \{0\}$ and $Y^{(i)} = I$, or $\bar{R}_{P,k}^{(i)} = 0$ for all k and $Y^{(i)}$ differs from the identity matrix and has the form (20) or (21).

Let us prove that no invariant state can have support on $\bigoplus_{i=1}^{j+1} \mathcal{H}_S^{(i)}$ by induction. First consider a state with support on $\mathcal{H}_S^{(j+1)} = \mathcal{H}_R^{(j)}$ alone:

$$\bar{\rho} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \bar{\rho}_R \end{array}\right].$$

If $\bigcap_k \ker \bar{R}_{P,k}^{(j)} = \{0\}$, then $\bar{\rho}$ is mapped by $\sum_k \bar{R}_k \cdot \bar{R}_k^{\dagger}$ into a state $\bar{\rho}'$ with non-trivial support on $\mathcal{H}_S^{(j)}$. Being in this case $Y^{(j)} = I$, $Z^{(j)}$ is block-diagonal with respect to the considered decomposition and we cannot thus get $Z^{(j)}\bar{\rho}'Z^{(j)\dagger} = \bar{\rho}$, for any $\bar{\rho}$ in $\mathfrak{D}(\mathcal{H}_R^{(j)})$.

On the other hand, if $\bar{R}_{P,k}^{(j)} = 0 \; \forall k$, then $Y^{(j)}$ contains off-diagonal full-rank blocks and maps the state

$$\bar{\rho}' = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \sum_k R_{R,k}^{(j)}\bar{\rho}_R R_{R,k}^{(j)\dagger} \end{array}\right]$$

into a state with non-trivial support on $\mathcal{H}_S^{(j)}$. The subsequent application of $Z^{(j-1)}$ will then map the state into a state with nontrivial support on $\bigoplus_{i=1}^{j} \mathcal{H}_S^{(i)}$, and therefore $\bar{\rho}$ cannot be invariant.

Let us now proceed with the inductive step, with m as the induction index. Assume that no invariant state can have support on $\bigoplus_{i=j+1-m}^{j+1} \mathcal{H}_S^{(i)}$ alone (induction hypothesis), and

consider the subspace $\bigoplus_{i=j-m}^{j+1} \mathcal{H}_S^{(i)}$. By the induction hypothesis if there were an invariant state with support on this subspace, it would be in the form

$$\bar{\rho} = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & \bar{\rho}_S & \bar{\rho}_P \\ \hline 0 & \bar{\rho}_P^{\dagger} & \bar{\rho}_R \end{array}\right]$$

with $\bar{\rho}_S \neq 0$ having support on $\mathcal{H}_S^{(j-m)}$. Let us rewrite

$$Z^{(j)} = Z^{(j-m-1)} Y^{(j-m)} Z_{(j-m+1)}$$

where $Z_{(j-m+1)} = Y^{(j-m+1)} \cdot \ldots \cdot Y^{(j)}$.

Again, we have two cases. If $\bigcap_k \ker \bar{R}_{P,k}^{(j-m-1)} = \{0\}$, then $Y^{(j-m)} = I$ and $\bar{\rho}$ is mapped by $\bar{R}_k$ into a state with non trivial support on $\mathcal{H}_S^{(j-m-1)}$. The subsequent application of $Z_{(j-m+1)}$ and of $Y^{(j-m)}$ does not affect this, and because of $Z^{(j-m-1)}$, the first complete iteration will map $\bar{\rho}$ into a state with non trivial support on $\bigoplus_{i=1}^{j-m-1} \mathcal{H}_S^{(i)}$. Therefore $\bar{\rho}$ cannot be invariant.

On the other hand, if $\bar{R}_{P,k}^{(j-m-1)} = 0 \ \forall k$, then $Y^{(j-m)}$ has the form (20) and the closed loop evolution of $\bar{\rho}$ is

$$\bar{\rho}' = \sum_k \left(Z^{(j-m-1)} Y^{(j-m)} \underbrace{Z_{(j-m+1)} \bar{R}_k \bar{\rho} \bar{R}_k^{\dagger} Z_{(j-m+1)}^{\dagger}}_{\tilde{\rho}_k} \right.$$
$$\left. Y^{(j-m)\dagger} Z^{(j-m-1)\dagger} \right).$$

If $\tilde{\rho}_k$ has support on $\bigoplus_{i=j+1-m}^{j+1} \mathcal{H}_S^{(i)}$ for all k, then $\bar{\rho}'$ will have the same support, and therefore $\bar{\rho}$ is not invariant. If instead $\tilde{\rho}_k$ has non trivial support on $\mathcal{H}_S^{(j-m)}$ for some k, then because of the subsequent application of $Z^{(j-m-1)} Y^{(j-m)}$, $\bar{\rho}'$ will have non trivial support on $\bigoplus_{i=1}^{j-m-1} \mathcal{H}_S^{(i)}$, and again $\bar{\rho}$ is not invariant.

When the induction process reaches $m = j-1$, then it states that no invariant states are supported on $\mathcal{H}_S^{(1)} \oplus \cdots \oplus \mathcal{H}_S^{(j)} \oplus \mathcal{H}_R^{(j)}$, and therefore according to Theorem 2 global asymptotic stability of the subspace S is achieved. ∎

The algorithm is clearly constructive. We then get the following:

Corollary 1: A certain subspace $\mathcal{H}_S$ can be made GAS if and only if the $R_{P,k}$ blocks of the canonical R-factors, computed with respect to the decomposition $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$, are not all zero.

VII. A TOY PROBLEM

We consider in this example a *two-qubit system*, defined on a Hilbert space $\mathcal{H}_I \simeq \mathbb{C}^2 \otimes \mathbb{C}^2$. Consider the task of stabilizing the maximally entangled state

$$\rho_d = \frac{1}{2} (|00\rangle + |11\rangle)(\langle 00| + \langle 11|) \tag{23}$$

which has the following representation in the *computational basis* $\mathcal{C} = \{|ab\rangle = |a\rangle \otimes |b\rangle \,|\, a,b = 0,1\}$:

$$\rho_d = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

In order to apply the proposed control design technique, let us consider a different basis $\mathcal{B}$ such that in the new representation $\rho_d^B = \mathrm{diag}([\,1\ 0\ 0\ 0\,])$. This can be achieved by considering the *Bell-basis*

$$\mathcal{B} = \left\{ \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \frac{|01\rangle - |10\rangle}{\sqrt{2}} \right\}.$$

Let B be the unitary matrix realizing the change of basis, i.e. $\rho_d^{\mathcal{B}} = B^{\dagger} \rho_d B$. Consider the space decomposition

$$\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$$

where $\mathcal{H}_S = \mathrm{span}\left\{ \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \right\}$ and $\mathcal{H}_R = \mathcal{H}_S^{\perp}$. We have then successfully casted the problem of stabilizing the maximally entangled state (23) into the problem of achieving asymptotic stability of the subspace $\mathcal{H}_S$. Suppose that the following generalized measurement is available

$$\mathcal{T}[\rho] = \sum_{k=1}^{3} M_k \rho M_k^{\dagger}$$

with operators (represented in the computational basis):

$$M_1 = \frac{1}{\sqrt{4}} (\sigma_+ \otimes I), \quad M_2 = \frac{1}{\sqrt{4}} (I \otimes \sigma_+),$$
$$M_3 = \sqrt{I - M_1^{\dagger} M_1 - M_2^{\dagger} M_2}.$$

where $\sigma_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. These Kraus operators may be used to describe a discrete-time spontaneous emission process, where the event associated to $M_{1,2}$ corresponds to the decay of one qubit (with probability $\frac{1}{4}$ each), and we neglect the event of the two qubits decaying in the same time interval. In the Bell basis, the operators take the form

$$M_1^B = \begin{bmatrix} 0 & 0 & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ \frac{1}{4} & -\frac{1}{4} & 0 & 0 \end{bmatrix}, \quad M_2^B = \begin{bmatrix} 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 & 0 \end{bmatrix},$$
$$M_3^B = \begin{bmatrix} 0.8536 & 0.1464 & 0 & 0 \\ 0.1464 & 0.8536 & 0 & 0 \\ 0 & 0 & 0.8660 & 0 \\ 0 & 0 & 0 & 0.8660 \end{bmatrix}.$$

Let us then apply the algorithm developed in section VI. The canonical QR decomposition of the matrices M_k^B returns the following triangular factors (we do not report here the corresponding orthogonal matrices Q_k, see (24) for the final form of the controls):

$$R_1 = \begin{bmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
$$R_3 = \begin{bmatrix} 0.8660 & 0.2887 & 0 & 0 \\ 0 & 0.8165 & 0 & 0 \\ 0 & 0 & 0.8660 & 0 \\ 0 & 0 & 0 & 0.8660 \end{bmatrix}.$$

According to the proposed approach, by inspection of the upper triangular factors R_i we can decide about the feasibility of the stabilization task. Indeed, as the blocks $R_{P,k}$, $k = 1, \ldots, 3$ are non-zero blocks, namely

$$R_{P,1} = \begin{bmatrix} -\frac{\sqrt{2}}{4} & 0 & 0 \end{bmatrix}, \quad R_{P,2} = \begin{bmatrix} -\frac{\sqrt{2}}{4} & 0 & 0 \end{bmatrix}, \quad R_{P,3} = \begin{bmatrix} 0.2887 & 0 & 0 \end{bmatrix},$$

then the stabilization problem is feasible.

Moreover, notice that at this step no further transformation is needed on the matrices, as the obtained R factors are already decomposed according to

$$\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_S^{(1)} \oplus \mathcal{H}_R^{(1)}.$$

where $\mathcal{H}_R^{(1)} = \bigcap_k \ker R_{P,k}$. Continuing with the iteration, we have then to determine the subspace $\mathcal{H}_R^{(2)} = \bigcap_k \ker R_{P,k}^{(1)}$. By inspection one can see that this space is empty, and therefore the iteration stops successfully. The set of unitary controls that have to be applied when the corresponding outcome k is measured is then

$$U_k = BQ_k^\dagger B^\dagger,$$

that is:

$$U_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & 0 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad U_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

$$U_3 = \begin{bmatrix} 0.9856 & 0 & 0 & 0.1691 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -0.1691 & 0 & 0 & 0.9856 \end{bmatrix}. \tag{24}$$

It can be shown by direct computation that the Hamiltonians needed to implement these unitary transformation (using ideally unbounded control pulses in order to make the dissipation effect negligible on when the control is acting) form a 3-dimensional control algebra [8].

VIII. Conclusions

Completely positive, trace-preserving maps represent general quantum dynamics for open systems, and if the environment is memoryless, also represent generators of discrete-time quantum Markov semigroups. Theorem 2 provides a characterization of the semigroup dynamics that render a certain pure state, or the set of states with support on a subspace, attractive, by employing LaSalle's Invariance Principle. In order to exploit this result for constructive design of stabilizing unitary feedback control strategies, we developed a suitable linear algebraic tool, which holds some interest *per se*. We proved that a *canonical* QR decomposition can be derived by specializing the well-known orthonormalization approach, and that it is key to study the potential of the feedback control scheme presented in Section VI. In fact, we determined which quantum generalized measurements can be simulated controlling a given one, and which pure states or subspaces can be rendered globally asymptotically stable. Theorem 5 gives a constructive procedure to build the controls, and also a simple test on the existence of such controls: If the algorithm does not stop on the first step, then the control problem has a solution. We believe that the provided results also represent a mathematical standpoint from which interesting, and more challenging, control problems can be tackled, in particular when the control choice is constrained by a multipartite structure of the system of interest.

Appendix

A. Quantum Measurements

1) Projective Measurements: In quantum mechanics, observable quantities are associated to Hermitian operators, with their spectrum associated to the possible outcomes. Suppose that we are interested in measuring the *observable* $C = \sum_i c_i \Pi_i$. The basic postulates that describe the quantum (strong, projective, or von Neumann's) measurements are the following:

(i') The probability of obtaining c_i as the outcome of a measure on a system described by the density operator ρ is $p_i = \text{Tr}(\rho \Pi_i)$.

(ii') [Lüders's Postulate] Immediately after a measurement that gives c_i as an outcome the system state becomes: $\rho|_i = \frac{1}{\text{trace}(\Pi_i \rho \Pi_i)} \Pi_i \rho \Pi_i$.

Notice that the spectrum of the observable does not play any role in the computation of the probabilities.

2) Generalized Measurements: If we get information about a quantum system by measuring another system which is correlated to the former, the projective measurement formalism is not enough, but it can be used to derive a more general one. A typical procedure to obtain generalized measurements on a quantum system of interest is the following:

- The system of interest $\mathcal{A}$ is augmented by adding another subsystem $\mathcal{B}$, initially decoupled from $\mathcal{A}$. Let $\rho_\mathcal{A} \otimes \rho_\mathcal{B}$, with $\rho_\mathcal{B} = |\phi\rangle\langle\phi|$, be the joint state;
- The two systems are coupled through a joint unitary evolution $U_{\mathcal{AB}}$;
- A direct, von Neumann measurement of an observable $X_\mathcal{B} = \sum_j x_j \Pi_j$, $\Pi_j = |\xi_j\rangle\langle\xi_j|$, is performed on $\mathcal{B}$.
- The conditioned state of the joint system after the measurement is then of the form

$$\rho_{\mathcal{AB}}|_j = \frac{1}{p_j}(I_\mathcal{A} \otimes \Pi_j)U_{\mathcal{AB}}(\rho_\mathcal{A} \otimes \rho_\mathcal{B})U_{\mathcal{AB}}^\dagger(I_\mathcal{A} \otimes \Pi_j) = \rho'_{\mathcal{A},j} \otimes \Pi_j,$$

with p_j the probability of obtaining the j-th outcome.

- One can compute the effect of the measurement on $\mathcal{A}$ alone, which is nontrivial if $U_{\mathcal{AB}}$ entangled the two subsystems, i.e. $U_{\mathcal{AB}}(\rho_\mathcal{A} \otimes \rho_\mathcal{B})U_{\mathcal{AB}}^\dagger$ cannot be written in factorized form. One then gets that $\rho'_{\mathcal{A},j} = \frac{1}{p_j}M_j \rho_\mathcal{A} M_j^\dagger$, with $M_j = \langle\xi_j|U_{\mathcal{AB}}|\phi\rangle$.

If now the average over the possible outcomes is taken, we obtain a state transformation in Kraus form. This construction is actually general, in the sense that if the dimension of $\mathcal{B}$ corresponds (at least) to the necessary number of outcomes, *any* Kraus map can be actually generated this way.

References

[1] R. Alicki and K. Lendi. *Quantum Dynamical Semigroups and Applications*. Springer-Verlag, Berlin, 1987.

[2] C. Altafini. Feedback stabilization of isospectral control systems on complex ßag manifolds: application to quantum ensembles. *IEEE Transactions on Automatic Control*, 11(52):2019–2028, 2007.

[3] L. Bouten, R. van Handel, and M. R. James. A discrete invitation to quantum filtering and feedback control. *SIAM Review*, 51(2):239–316, 2009.

[4] D. Bouwmeester, A. Ekert, and A. Zeilinger, editors. *The Physics of Quantum Information: Quantum Cryptography, Quantum Teleportation, Quantum Computation*. Springer-Verlag, 2000.

[5] H. P. Breuer and F. Petruccione. *The Theory of Open Quantum Systems*. Oxford University Press, UK, 2006.

[6] D. Burgarth and V. Giovannetti. The generalized lyapunov theorem and its application to quantum channels. *New Journal of Physics*, (9), 2007.

[7] A. R. R. Carvalho, P. Milman, R. L. de Matos Filho, and L. Davidovich. Decoherence, pointer engineering, and quantum state protection. *Physical Review Letters*, 86(22):4988–4991, 2001.

[8] D. D'Alessandro. *Introduction to Quantum Control and Dynamics*. Applied Mathematics & Nonlinear Science. Chapman & Hall/CRC, 2007.

[9] E. B. Davies. *Quantum Theory of Open Systems*. Academic Press, USA, 1976.

[10] R. L. de Matos Filho and W. Vogel. Engineering the hamiltonian of a trapped atom. *Physical Review A*, 58(3):R1661–R1664, 1998.

[11] D. P. DiVincenzo. Quantum computation. *Science*, 270:255–261, October 1995.

[12] H. O. Everitt, editor. *Experimental Aspects of Quantum Computing*. Springer, Berlin, 2005.

[13] A. Ferrante, M. Pavon, and G. Raccanelli. Driving the propagator of a spin system: a feedback approach. In *Proceedings of the 41st IEEE Conference on Decision and Control*, volume 1, pages 46–50, Dec. 2002.

[14] R. P. Feynman. Simulating physics with computers. *International Journal of Theoretical Physics*, 21, 1982.

[15] S. Grivopoulos and B. Bamieh. Lyapunov-based control of quantum systems. In *42th IEEE Conference on Decision and Control*, pages 434–438, Dec. 2003.

[16] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, New York, 1990.

[17] E. Knill. On protected realization of quantum information. *Physical Review A*, 74(4):042301:1–11, 2006.

[18] E. Knill, R. Laflamme, and L. Viola. Theory of quantum error correction for general noise. *Physical Review Letters*, 84(11):2525–2528, 2000.

[19] K. Kraus. *States, Effects, and Operations: Fundamental Notions of Quantaum Theory*. Lecture notes in Physics. Springer-Verlag, Berlin, 1983.

[20] B. Kummerer. Quantum markov processes and application in physics. In *Quantum Independent Increment Processes*, volume II of *Lecture Notes in Mathemtics 1866*. Springer Berlin - Heidelberg, 2006.

[21] J. P. LaSalle. *The Stability and Control of Discrete Processes*, volume 62 of *Applied mathematical sciences*. Springer Verlag New York, 1980.

[22] D. A. Lidar, I. L. Chuang, and K. B. Whaley. Decoherence-free subspaces for quantum computation. *Physical Review Letters*, 81(12):2594–2597, 1997.

[23] S. Lloyd and L. Viola. Engineering quantum dynamics. *Physical Review A*, 65:010101:1–4, 2001.

[24] M. Mirrahimi, P. Rouchon, and G. Turinici. Lyapunov control of bilinear schroedinger equations. *Automatica*, (41):1987–1994, 2005.

[25] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Information*. Cambridge University Press, Cambridge, 2002.

[26] J. F. Poyatos, J. I. Cirac, and P. Zoller. Quantum reservoir engineering with laser cooled trapped ions. *Physical Review Letters*, 77(23):4728–4731, 1996.

[27] J. J. Sakurai. *Modern Quantum Mechanics*. Addison-Wesley, New York, 1994.

[28] A. Shabani and D. A. Lidar. Theory of initialization-free decoherence-free subspaces and subsystems. *Physical Review A*, 72(4):042303:1–14, 2005.

[29] F. Ticozzi and L. Viola. Single-bit feedback and quantum dynamical decoupling. *Physical Review A*, 74(5):052328:1–11, 2006.

[30] F. Ticozzi and L. Viola. Quantum Markovian subsystems: Invariance, attractivity and control. *IEEE Transaction on Automatic Control*, 53(9):2048–2063, 2008.

[31] F. Ticozzi and L. Viola. Analysis and synthesis of attractive quantum Markovian dynamics. *Automatica*, page doi:10.1016/j.automatica.2009.05.005, 2009.

[32] R. van Handel, J. K. Stockton, and H. Mabuchi. Feedback control of quantum state reduction. *IEEE Transaction on Automatic Control*, 50(6):768–780, 2005.

[33] Frank Verstraete, Michael M. Wolf, and J. Ignacio Cirac. Quantum computation, quantum state engineering, and quantum phase transitions driven by dissipation. *On-line preprint: http://arxiv.org/abs/0803.1447*, 2008.

[34] L. Viola, E. Knill, and R. Laflamme. Constructing qubit in physical systems. *Journal of Physics A*, (34):7067–7079, 2001.

[35] L. Viola, E. Knill, and S. Lloyd. Dynamical decoupling of open quantum system. *Physical Review Letters*, 82(12):2417–2421, 1999.

[36] X. Wang and S. Schirmer. Analysis of lyapunov method for control of quantum states: generic case. *On-line preprint: http://www.arXiv.org/abs/0901.4515*, 2009.

[37] X. Wang and S. Schirmer. Analysis of lyapunov method for control of quantum states: non–generic case. *On-line preprint: http://www.arXiv.org/abs/0901.4522*, 2009.

[38] H. M. Wiseman. Quantum theory of continuous feedback. *Physical Review A*, 49(3):2133–2150, 1994.

[39] R. Wu, A. Pechen, C. Brif, and H. Rabitz. Controllability of open quantum systems with kraus-map dynamics. *Journal of Physics A: Mathematical and Theoretical*, 40(21):5681–5693, 2007.

[40] E. Zeidler. *Applied Functional Analysis: Applications to Mathematical Physics*, volume 108 of *Applied mathematical sciences*. Springer Verlag New York, 1999.