

# New approximations for DQPSK transmission bit error rate

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## Abstract

In this correspondence our aim is to use some tight lower and upper bounds for the differential quaternary phase shift keying transmission bit error rate in order to deduce accurate approximations for the bit error rate by improving the known results in the literature. The computation of our new approximate expressions are significantly simpler than that of the exact expression.

## Index Terms

Bit error rate, Marcum  $Q$ -function, bounds.

## I. INTRODUCTION

Recently, Ferrari and Corazza [1] have studied the performance analysis of the differential quaternary phase shift keying (DQPSK) transmission with Gray coding over the additive white Gaussian noise (AWGN) channel. We note that the bit error rate (BER) in the case of Gray coding can be written in terms of the Marcum  $Q$ -function and the modified Bessel function of the first kind and zero order. In [1] the authors used some bounds for the Marcum  $Q$ -function in order to propose some simple, but accurate approximations for the BER. By using some other bounds for the Marcum  $Q$ -function (deduced by Wang and Wu [3], and Baricz and Sun [2]),

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very recently Sun et al. [4] proposed another approximation for BER. In this correspondence we make a contribution to the subject by using some new results on Marcum  $Q$ -function: we deduce a new tight upper bound for the BER and combining this with some existing tight bounds we construct some new accurate approximations. These results improve and complement the results of Ferrari and Corazza [1], and also of Sun et al. [4]. In Section II we discuss the bounds for BER of DQPSK, in Section III we present five new approximations for BER and we compare them with the similar ones deduced from inequalities for the Marcum  $Q$ -function. Finally, in Section IV the conclusion of the paper is given.

## II. BOUNDS FOR BER OF DQPSK

It is known that for the DQPSK transmission with Gray coding over an AWGN channel the BER can be expressed as follows [5]:

$$\text{BER} \triangleq Q(a, b) - \frac{1}{2} I_0(ab) e^{-\frac{a^2+b^2}{2}}, \quad (1)$$

where  $a = \sqrt{\gamma(2 - \sqrt{2})}$ ,  $b = \sqrt{\gamma(2 + \sqrt{2})}$  and  $\gamma$  is the bit signal-to-noise ratio (SNR). Here  $Q(a, b)$  stands for the Marcum  $Q$ -function, defined by

$$Q(a, b) \triangleq \int_b^\infty x e^{-\frac{x^2+a^2}{2}} I_0(ax) dx,$$

where  $b \geq 0$ ,  $a > 0$  and  $I_0$  is the modified Bessel function of the first kind and zero order. Note that the Marcum  $Q$ -function plays an important role in the studies of digital communications over fading channels [6].

Now, let us consider the complementary error function, defined by

$$\text{erfc}(x) \triangleq \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt,$$

and for shortness let us introduce the notation

$$e(a, b) \triangleq \text{erfc}\left(\frac{b-a}{\sqrt{2}}\right) = \frac{2}{\sqrt{\pi}} \int_{\frac{b-a}{\sqrt{2}}}^\infty e^{-t^2} dt.$$

Recently, Ferrari and Corazza [1] by using their bounds from [7] (see also [8]) for the Marcum  $Q$ -function

$$Q(a, b) \geq \sqrt{\frac{\pi}{2}} \frac{b I_0(ab)}{e^{ab}} e(a, b), \quad (2)$$

$$Q(a, b) \leq \frac{I_0(ab)}{e^{ab}} \left[ e^{-\frac{(b-a)^2}{2}} + a \sqrt{\frac{\pi}{2}} e(a, b) \right], \quad (3)$$

where  $b \geq a > 0$ , deduced the following bounds for BER of DQPSK with Gray coding:

$$\begin{aligned} \text{BER} &> I_0(ab) \left[ \sqrt{\frac{\pi}{2}} \frac{b}{e^{ab}} e(a, b) - \frac{1}{2} e^{-\frac{a^2+b^2}{2}} \right] \triangleq L_1, \\ \text{BER} &< I_0(ab) \left[ \sqrt{\frac{\pi}{2}} \frac{a}{e^{ab}} e(a, b) + \frac{1}{2} e^{-\frac{a^2+b^2}{2}} \right] \triangleq U_1. \end{aligned}$$

It is important to mention here that motivated by the above results of Ferrari and Corazza, very recently Wang and Wu [3] proved that

$$Q(a, b) \geq \sqrt{\frac{\pi}{2}} \frac{b I_0(ab)}{e^{ab} - e^{-ab}} E(a, b),$$

while Baricz and Sun [2] proved that

$$Q(a, b) \leq \frac{I_0(ab)}{e^{ab} + e^{-ab}} \left[ e^{-\frac{(b-a)^2}{2}} + e^{-\frac{(b+a)^2}{2}} + a \sqrt{\frac{\pi}{2}} E(a, b) \right],$$

where in both of the inequalities  $b \geq a > 0$  and

$$E(a, b) \triangleq e(a, b) - e(-a, b) = \frac{2}{\sqrt{\pi}} \int_{\frac{b-a}{\sqrt{2}}}^{\frac{b+a}{\sqrt{2}}} e^{-t^2} dt.$$

Now, by using the above lower and upper bounds for the Marcum  $Q$ -function, Sun et al. [4] deduced the following:

$$\begin{aligned} \text{BER} &> I_0(ab) \left[ \sqrt{\frac{\pi}{2}} \frac{b}{e^{ab} - e^{-ab}} E(a, b) - \frac{1}{2} e^{-\frac{a^2+b^2}{2}} \right] \triangleq L_2, \\ \text{BER} &< I_0(ab) \left[ \sqrt{\frac{\pi}{2}} \frac{a}{e^{ab} + e^{-ab}} E(a, b) + \frac{1}{2} e^{-\frac{a^2+b^2}{2}} \right] \triangleq U_2. \end{aligned}$$

We note that the lower and upper bounds for the Marcum  $Q$ -function of Wang and Wu [3] and of Baricz and Sun [2], mentioned above, are tighter than (2) and (3), as it was pointed out in [2], [3]. This in turn implies that the bounds  $L_2$  and  $U_2$  are tighter than the bounds  $L_1$  and  $U_1$ , that is, we have  $L_2 > L_1$  and  $U_2 < U_1$ .

Now, let us consider the following inequality

$$Q(a, b) \leq \frac{I_0(ab)}{e^{ab} + \lambda_0} \left[ a \sqrt{\frac{\pi}{2}} e(a, b) + (e^{ab} + \lambda_0) e^{-\frac{a^2+b^2}{2}} \right],$$

which holds for all  $b \geq a > 0$  and it was deduced by Baricz [9]. Here  $\lambda_0 = e^{\rho_0} (I_0(\rho_0) / I_1(\rho_0) - 1) \simeq 3.03442206626763$ ,  $\rho_0 \simeq 1.54512596391949$  is the unique simple positive root of the equation  $(x+1)I_1(x) = xI_0(x)$ , and  $I_1$  stands for the modified Bessel function of order 1. We

note that in the above inequality the constant  $\lambda_0$  is best possible, that is, cannot be replaced by any larger constant. Since the above inequality is an improvement of (3), its equivalent form

$$\text{BER} < I_0(ab) \left[ \sqrt{\frac{\pi}{2}} \frac{a}{e^{ab} + \lambda_0} e(a, b) + \frac{1}{2} e^{-\frac{a^2+b^2}{2}} \right] \triangleq U_3$$

clearly provides a better upper bound than (3), that is, we have  $U_3 < U_1$ . The lower and upper bounds  $L_1, L_2, U_1, U_2$  and  $U_3$  are shown in Fig. 1 together with the exact BER as functions of the SNR on the interval  $(0, 1.5)$ . Surprisingly, all of these bounds are extremely tight.

### III. APPROXIMATIONS FOR BER OF DQPSK

Observe that there is an interesting formal symmetry between the bounds  $L_1$  and  $U_1$ , as well as between the bounds  $L_2$  and  $U_2$ . Based on the tight bounds  $L_1$  and  $U_1$ , and in view of the formal symmetry, Ferrari and Corazza [1] pointed out that it is natural to approximate the BER simply by considering the arithmetic mean of the quantities  $L_1$  and  $U_1$ . In this spirit, they derived the following approximate expression for BER:

$$\text{BER}_1 \triangleq \mathcal{A}(L_1, U_1; 0.5),$$

i.e.

$$\text{BER}_1 = \sqrt{\frac{\pi}{8}} \frac{a+b}{e^{ab}} I_0(ab) e(a, b),$$

where  $\mathcal{A}(x, y; \omega) = \omega x + (1 - \omega)y$  is the weighted arithmetic mean of  $x$  and  $y$  with weights  $\omega, 1 - \omega > 0$ . By using the same idea, it is natural to propose the following new approximate expressions for BER:

$$\text{BER}_2 \triangleq \mathcal{A}(L_2, U_2; 0.5)$$

and

$$\text{BER}_3 \triangleq \mathcal{A}(L_2, U_3; 0.5),$$

that is,

$$\text{BER}_2 = \sqrt{\frac{\pi}{8}} \frac{(a+b)e^{ab} - (a-b)e^{-ab}}{e^{2ab} - e^{-2ab}} I_0(ab) E(a, b)$$

and

$$\text{BER}_3 = \sqrt{\frac{\pi}{8}} I_0(ab) \left[ \frac{bE(a, b)}{e^{ab} - e^{-ab}} + \frac{ae(a, b)}{e^{ab} + \lambda_0} \right].$$

We note that these approximations are better than  $\text{BER}_1$ . A similar approximation to that of  $\text{BER}_2$  was proposed recently by Sun et al. [4] in the form

$$\text{BER}_4 = \frac{e^{-\frac{(b+a)^2}{2}}}{\sqrt{8\pi ab}} + \frac{1}{4} \left( \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} \right) E(a, b),$$

however, this is better than  $\text{BER}_1$  only for  $\gamma \in (0.9, 1.4)$ . Motivated by the above results, in this correspondence our aim is to propose the following approximations based on better weight functions  $\omega_i$ ,  $i \in \{5, 6, 7\}$  :

$$\text{BER}_5 \triangleq \mathcal{A}(\text{L}_1, \text{U}_1; \omega_5),$$

$$\text{BER}_6 \triangleq \mathcal{A}(\text{L}_2, \text{U}_2; \omega_6),$$

$$\text{BER}_7 \triangleq \mathcal{A}(\text{L}_2, \text{U}_3; \omega_7),$$

where

$$\omega_5(\gamma) = \begin{cases} 0.65\sqrt[4]{\gamma}, & \text{if } \gamma < 1 \\ 0.5 + 1.1 \cdot \frac{e^{-\frac{\pi}{2\sqrt{\gamma}}}}{\gamma^{1.5}} \cdot \sqrt{\frac{1}{2}}, & \text{if } 1 \leq \gamma \end{cases},$$

$$\omega_6(\gamma) = \begin{cases} e^{-\frac{\gamma^2}{2.9}} \cdot 0.25 + 0.5, & \text{if } 0 \leq \gamma < 1 \\ \frac{e^{-\frac{1}{2\gamma+1}}}{(\gamma+0.5)^{1.5}} \cdot \sqrt{\frac{1}{2\pi}} \cdot 1.15 + 0.5, & \text{if } 1 \leq \gamma < 5 \\ \frac{1}{\pi} (1 + \gamma)^{-1} \cdot 0.65 + 0.5, & \text{if } 5 \leq \gamma \end{cases},$$

$$\omega_7(\gamma) = \begin{cases} (1 - \gamma)^2 \cdot 0.95, & \text{if } 0 \leq \gamma < 1 \\ 0.5 - 1.4 \cdot e^{-\gamma^{1.2}} + 0.02, & \text{if } 1 \leq \gamma \leq 8 \\ \frac{1}{5.2\gamma} + 0.5, & \text{if } 8 < \gamma \end{cases}.$$

These approximations can be rewritten as follows

$$\text{BER}_5 = \omega_5(\gamma)\text{L}_1 + (1 - \omega_5(\gamma))\text{U}_1,$$

$$\text{BER}_6 = \omega_6(\gamma)\text{L}_2 + (1 - \omega_6(\gamma))\text{U}_2,$$

$$\text{BER}_7 = \omega_7(\gamma)\text{L}_2 + (1 - \omega_7(\gamma))\text{U}_3.$$

It is important to note here that the computations of approximate BER expressions  $\text{BER}_i$ , where  $i \in \{1, 2, \dots, 7\}$ , completely avoid the computation of the Marcum  $Q$ -function. Tables I and II contain an estimate of BER by using the Matlab function `marcumq` (BER) and the estimates based upon  $\text{BER}_i$ , where  $i \in \{1, \dots, 7\}$ . As we can see the approximations  $\text{BER}_i$ , where  $i \in$

$\{5, 6, 7\}$ , are better than the approximations  $\text{BER}_i$ , where  $i \in \{1, 2, 3, 4\}$ . Moreover, the novel approximate expressions  $\text{BER}_i$ , where  $i \in \{5, 6, 7\}$ , are consistent with the results of Weinberg [10], obtained by using of Monte-Carlo estimators. We note that in our further calculations we considered the values obtained by the `marcumq` function as the “exact” values for BER.

We give a hint about the construction of the weight functions. By using the values of BER,  $L_1$ ,  $L_2$ ,  $U_1$ ,  $U_2$  and  $U_3$  first we calculated numerically the values of the weight functions. The representation of the initial numerical values for  $\omega_6$  can be seen in Fig. 2. This looks like the probability density function of the normal distribution but having a heavy tail, so we started to search for expression involved in probability density functions with heavy tails (eg. Student, Lévy,  $\alpha$ -stable). By using the same idea the parameters in the expression of the functions  $\omega_i$ ,  $i \in \{5, 6, 7\}$ , were obtained by numerical experimentations.

Now, in order to estimate the tightness of the new approximate expression and to compare with the tightness of the known approximate expressions, for  $i \in \{5, 6, 7\}$  let us consider the relative errors  $\varepsilon_i \triangleq (\text{BER}_i - \text{BER}) / \text{BER}$ . Table III contains the relative errors  $\varepsilon_i$ , where  $i \in \{5, 6, 7\}$ . Finally, in Fig. 3 and 4 the exact BER and the approximate expressions  $\text{BER}_i$ , where  $i \in \{1, \dots, 7\}$ , are shown as functions of the SNR on the intervals  $(0, 2]$  and  $[2, 4]$ . We find that our new approximate expressions are quite accurate for the whole region of SNR.

#### IV. CONCLUSION

In this correspondence we deduced a new tight upper bound for the BER of DQPSK with Gray coding over the AWGN channel by using a new tight bound deduced for the Marcum  $Q$ -function. This and the bounds from [2] show that approximations of BER for  $\gamma = 1$  obtained by using adaptive Simpson quadrature (see [10]) and some previous approximations are not in the interval determined by the lower and upper bounds. Although the estimations from [1] and [4] are the best possible estimations for large values of SNR, by using constants in the convex combinations of the lower and upper bounds, they can be improved by considering weight functions in the calculation of approximations. The advantage of this technique is that we can obtain better approximations also for small values of SNR. Based on numerical experiments we proposed five new approximate BER expressions for the AWGN channel, which use the best known bounds and which are accurate in the whole region of SNR.

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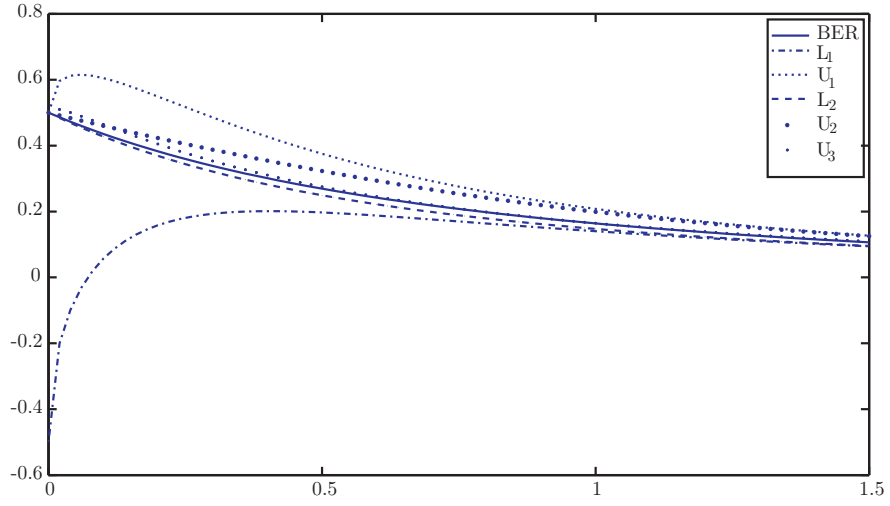


Fig. 1. BER with lower and upper bounds

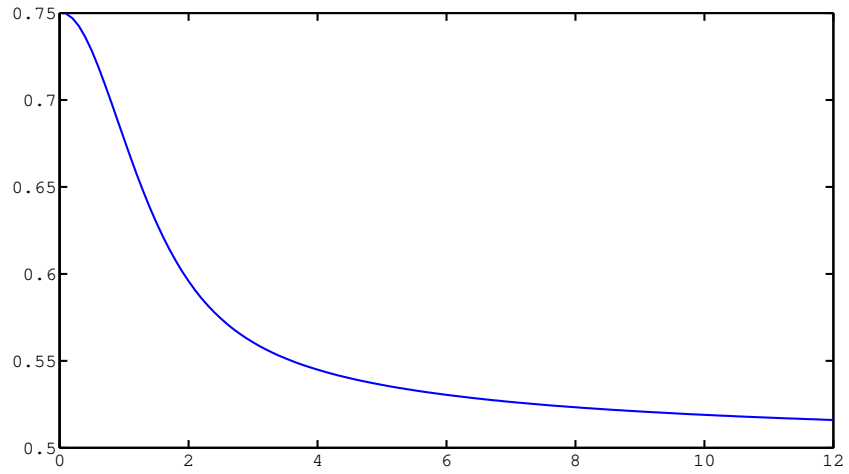


Fig. 2. The graph of the weight function  $\omega_6$



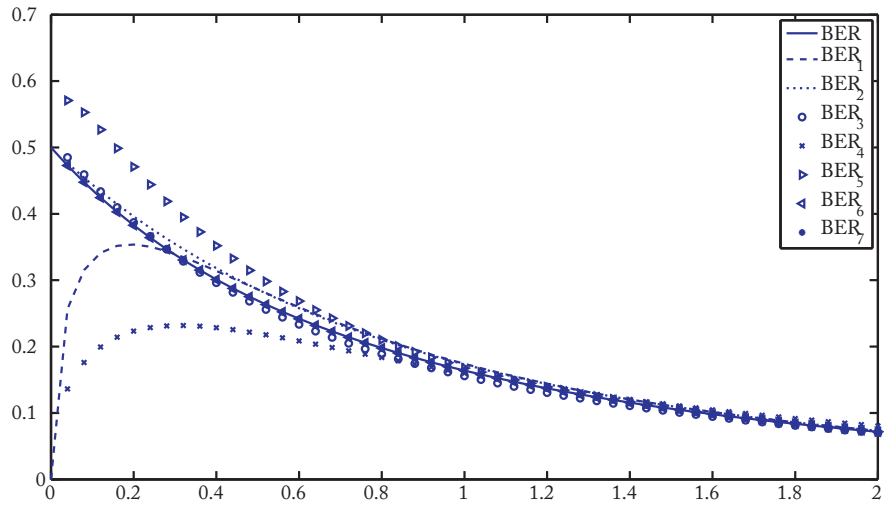


Fig. 3. Approximations for BER on  $[0, 2]$

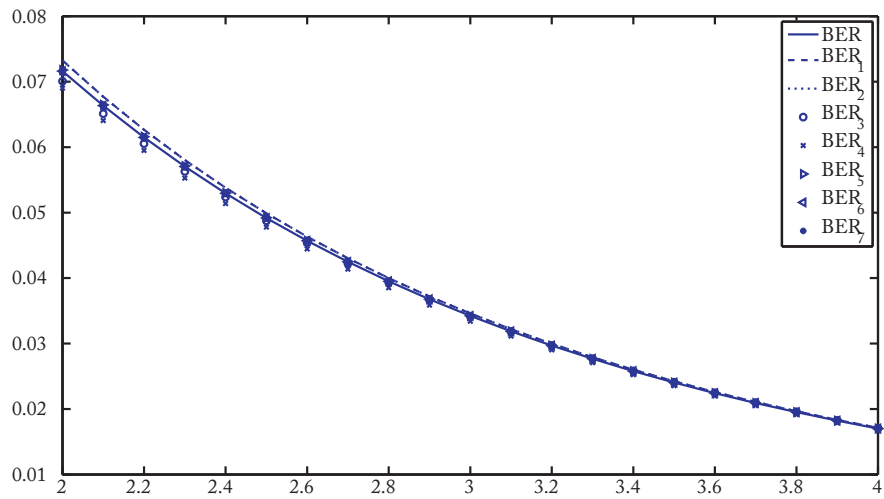


Fig. 4. Approximations for BER on  $[2, 4]$

$\gamma$ (dB)	BER	BER <sub>1</sub>	BER <sub>2</sub>	BER <sub>3</sub>
1	$1.639 \times 10^{-1}$	$1.739 \times 10^{-1}$	$1.731 \times 10^{-1}$	$1.556 \times 10^{-1}$
2	$7.161 \times 10^{-2}$	$7.324 \times 10^{-1}$	$7.322 \times 10^{-2}$	$7.007 \times 10^{-2}$
3	$3.422 \times 10^{-2}$	$3.458 \times 10^{-2}$	$3.458 \times 10^{-2}$	$3.416 \times 10^{-2}$
4	$1.701 \times 10^{-2}$	$1.711 \times 10^{-2}$	$1.711 \times 10^{-2}$	$1.706 \times 10^{-2}$
5	$8.648 \times 10^{-3}$	$8.683 \times 10^{-3}$	$8.683 \times 10^{-3}$	$8.677 \times 10^{-3}$
6	$4.461 \times 10^{-3}$	$4.474 \times 10^{-3}$	$4.4745 \times 10^{-3}$	$4.473 \times 10^{-3}$
7	$2.325 \times 10^{-3}$	$2.330 \times 10^{-3}$	$2.3308 \times 10^{-3}$	$2.33072 \times 10^{-3}$
8	$1.221 \times 10^{-3}$	$1.224 \times 10^{-3}$	$1.22405 \times 10^{-3}$	$1.22404 \times 10^{-3}$
9	$6.459 \times 10^{-4}$	$6.468 \times 10^{-4}$	$6.46883 \times 10^{-4}$	$6.46881 \times 10^{-4}$
10	$3.431 \times 10^{-4}$	$3.435 \times 10^{-4}$	$3.43588 \times 10^{-4}$	$3.43588 \times 10^{-4}$
11	$1.830 \times 10^{-4}$	$1.832 \times 10^{-4}$	$1.83249 \times 10^{-4}$	$1.83249 \times 10^{-4}$
12	$9.798 \times 10^{-5}$	$9.807 \times 10^{-5}$	$9.80723 \times 10^{-5}$	$9.80723 \times 10^{-5}$

TABLE I  
BER AND THE ESTIMATES BER<sub>1</sub>, BER<sub>2</sub> AND BER<sub>3</sub>

$\gamma$ (dB)	BER <sub>4</sub>	BER <sub>5</sub>	BER <sub>6</sub>	BER <sub>7</sub>
1	$1.484 \times 10^{-1}$	$1.677 \times 10^{-1}$	$1.6383 \times 10^{-1}$	$1.645 \times 10^{-1}$
2	$6.908 \times 10^{-2}$	$7.179 \times 10^{-2}$	$7.162 \times 10^{-2}$	$7.133 \times 10^{-2}$
3	$3.348 \times 10^{-2}$	$3.423 \times 10^{-2}$	$3.4226 \times 10^{-2}$	$3.4233 \times 10^{-2}$
4	$1.677 \times 10^{-2}$	$1.7014 \times 10^{-2}$	$1.7017 \times 10^{-2}$	$1.7036 \times 10^{-2}$
5	$8.5931 \times 10^{-3}$	$8.6500 \times 10^{-3}$	$8.6500 \times 10^{-3}$	$8.6593 \times 10^{-3}$
6	$4.4788 \times 10^{-3}$	$4.4624 \times 10^{-3}$	$4.4616 \times 10^{-3}$	$4.4651 \times 10^{-3}$
7	$2.3741 \times 10^{-3}$	$2.3262 \times 10^{-3}$	$2.3257 \times 10^{-3}$	$2.3267 \times 10^{-3}$
8	$1.2841 \times 10^{-3}$	$1.2222 \times 10^{-3}$	$1.2219 \times 10^{-3}$	$1.2218 \times 10^{-3}$
9	$7.1447 \times 10^{-4}$	$6.4613 \times 10^{-4}$	$6.4597 \times 10^{-4}$	$6.4594 \times 10^{-4}$
10	$4.1463 \times 10^{-4}$	$3.4326 \times 10^{-4}$	$3.4318 \times 10^{-4}$	$3.4317 \times 10^{-4}$
11	$2.5591 \times 10^{-4}$	$1.8311 \times 10^{-4}$	$1.8307 \times 10^{-4}$	$1.8306 \times 10^{-4}$
12	$1.7151 \times 10^{-4}$	$9.8011 \times 10^{-5}$	$9.7990 \times 10^{-5}$	$9.7989 \times 10^{-5}$

TABLE II  
ESTIMATES OF BER BY USING BER<sub>4</sub>, BER<sub>5</sub>, BER<sub>6</sub> AND BER<sub>7</sub>

$\gamma(\text{dB})$	$\varepsilon_5$	$\varepsilon_6$	$\varepsilon_7$
1	$2.31 \times 10^{-2}$	$-4.51 \times 10^{-4}$	$3.86 \times 10^{-3}$
2	$2.52 \times 10^{-3}$	$1.96 \times 10^{-4}$	$-3.79 \times 10^{-3}$
3	$1.17 \times 10^{-4}$	$-1.36 \times 10^{-5}$	$1.94 \times 10^{-4}$
4	$9.01 \times 10^{-5}$	$2.66 \times 10^{-4}$	$1.38 \times 10^{-3}$
5	$1.96 \times 10^{-4}$	$1.93 \times 10^{-4}$	$1.26 \times 10^{-3}$
6	$2.52 \times 10^{-4}$	$8.90 \times 10^{-5}$	$8.65 \times 10^{-4}$
7	$2.72 \times 10^{-4}$	$4.55 \times 10^{-5}$	$5.04 \times 10^{-4}$
8	$2.72 \times 10^{-4}$	$2.54 \times 10^{-5}$	$-6.42 \times 10^{-5}$
9	$2.62 \times 10^{-4}$	$1.53 \times 10^{-5}$	$-3.32 \times 10^{-5}$
10	$2.48 \times 10^{-4}$	$1.00 \times 10^{-5}$	$-1.66 \times 10^{-5}$
11	$2.32 \times 10^{-4}$	$7.13 \times 10^{-6}$	$-6.71 \times 10^{-6}$
12	$2.17 \times 10^{-4}$	$5.44 \times 10^{-6}$	$-3.87 \times 10^{-7}$

TABLE III

RELATIVE ERRORS  $\varepsilon_i$  OF THE APPROXIMATE EXPRESSIONS  $\text{BER}_i$ , WHERE  $i \in \{5, 6, 7\}$ .