System-Level Optimization in Poisson Cellular Networks: An Approach Based on the Generalized Benders Decomposition

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Abstract—During the last decade, stochastic geometry has been widely employed for system-level analysis in cellular networks. The resulting analytical frameworks are, however, not always amenable for system-level optimization. This is due to three main reasons: (i) the performance metric of interest may not be formulated in closed-form; (ii) under some analytically tractable modeling assumptions, important system parameters may not explicitly appear in the analytical frameworks; and (iii) the optimization problem may not possess any structural properties, e.g., convexity, that facilitate the development of numerical algorithms for optimizing multiple (continuous- and discrete-valued) parameters at an affordable computational complexity and with performance-guarantee, e.g., the convergence to the global optimum is provable. In this letter, we leverage the new definition of coverage probability in [1], we show that it is suitable to formulate mixed-integer non-linear system-level resource allocation problems in Poisson cellular networks, and we prove that the global optimum can be efficiently calculated by applying the generalized Benders decomposition. Numerical results are illustrated in order to compare the proposed approach against brute-force and greedy-like optimization algorithms.

Index Terms—Cellular networks, stochastic geometry, system-level analysis, mixed-integer non-linear optimization.

I. INTRODUCTION

Stochastic geometry is a widely employed analytical tool for modeling and analyzing the performance of cellular networks [2]. The Poisson Point Process (PPP) is, in particular, the most analytically tractable abstraction for modeling cellular networks. For example, homogeneous and inhomogeneous PPPs have been used for modeling the locations of the Base Stations (BSs) in cellular networks [2], [3], and Boolean schemes of rectangles, whose centers are distributed according to a PPP, have been used for modeling blocking objects [4].

With the aid of stochastic geometry, relevant key performance metrics that quantify the performance of cellular networks, e.g., the spectral efficiency, can be formulated in computable analytical expressions. As recently elaborated in [1], however, the resulting analytical frameworks are not always conveniently formulated for system-level optimization. This originates, in particular, from two main reasons: (i) the performance metric of interest may not be formulated in closed-form, which may result in a difficult function to optimize; and (ii) under analytically tractable modeling assumptions, some system parameters may not explicitly appear in the analytical frameworks, which may lead to meaningless optimization problems. Motivated by these considerations, the authors of [1]

have introduced a new definition of coverage probability that is suitable to formulate meaningful optimization problems. In [1], for example, the optimal transmit power and deployment density of the BSs that maximize the energy efficiency are computed. More recently, the same authors have leveraged the framework in [1] to prove the impact of several design parameters on the coverage [5], and to compute the energy-spectral efficiency Pareto front in cellular networks [6].

In spite of these promising results, the authors of [1], [5], [6] have tackled the optimization of a single parameter, e.g., the transmit power or the density of the BSs, and have unveiled the complexity of solving multi-parameter optimization problems (e.g., to optimize the transmit power *and* the density of the BSs). Also, they have shown that it is often difficult to prove even the existence of an optimizer for every set of parameters. From these research works, we can conclude that it is difficult, in general, to prove structural properties (e.g., the convexity) of optimization problems that originate from stochastic geometry models and analysis. This is, however, needed in order to facilitate the development of numerical algorithms for optimizing multiple (continuous- and discrete-valued) parameters at an affordable computational complexity and with performance-guarantee, e.g., the convergence to the global optimum in a finite number of iterations is provable.

In this letter, we investigate this open issue and propose an efficient approach to tackle it. We consider a Poisson cellular network whose resources (spectrum and power) are to be optimally allocated among multiple services (called tenants) so as to maximize the aggregate spectral efficiency under individual performance-guarantee for each tenant. Also, we optimize the scheduling of tenants so as to maximize the aggregate spectral efficiency. By embracing the analytical formulation of the spectral efficiency in [1], we prove that the joint scheduling and resource allocation problem of interest is a Mixed-Integer Non-Linear (MINL) optimization problem, whose global optimum can be calculated by employing the Generalized Benders Decomposition (GBD) [7]. The efficiency of the proposed approach stems from three properties: (i) the spectral efficiency in [1] is a jointly concave function in the transmit power and the bandwidth of the BSs; (ii) the MINL resource allocation problem fulfills the separability conditions in [8]; (iii) the master problem in [7] is linear. Numerical results confirm that the proposed approach offers the same accuracy as brute-force optimization at a similar (or lower) complexity as (than) a sub-optimal greedy algorithm.

II. CELLULAR NETWORKS – THE POISSON ABSTRACTION

We consider a single operator downlink cellular network. The locations of the BSs are assumed to constitute a homogeneous PPP of density λ_{BS}. Each BS transmits with a constant power P_{tot} over a spectrum of bandwidth B_{tot}. The

Telecommunications Operator (TO) offers T different services to its Mobile Terminals (MTs). An MT requests the ith, for $i = 1, 2, \ldots, T$, service if it wishes to obtain an average data rate at least equal to τ_i (bit/s). The MTs that request the ith service are collectively referred to as the ith tenant.

The locations of the MTs of the ith tenant are assumed to follow a homogeneous PPP of density λ_{MT_i}. For example, a tenant may be constituted by the MTs that need to use the network for mobile broadband applications (i.e., τ_i and λ_{MT_i} are likely to be large and small, respectively), while another tenant may be constituted by the MTs that need to use the network for sensing applications (i.e., τ_i and λ_{MT_i} are likely to be small and large, respectively). In this letter, due to space limitations, we do not consider the case studies of multi-operator cellular networks and services that necessitate other than a minimum guaranteed data rate, e.g., a maximum transmission delay or a minimum number of MTs to be served. These generalizations are postponed to future research.

The locations of the BSs are denoted by $x \in \Psi_{\mathrm{BS}} \subseteq \mathbb{R}^2$. The channels of the T tenants and MTs are assumed to be mutually independent and identically distributed. As far as the channels are concerned, therefore, the index i can be omitted. An unbounded path-loss model is considered, i.e., $l(x) = \kappa r^\alpha$, where $r = \|x\|$ is the distance of the BS at x from the typical MT, and κ and $\alpha > 2$ denote the path-loss constant and the path-loss slope, respectively. The power gain of the fast-fading is assumed to be an exponential random variable of unit mean. A cell association based on the highest average received power is assumed, which, for the considered path-loss model, is equivalent to the shortest distance cell association.

Let us denote by P_i and B_i the transmit power and bandwidth that the TO allocates to the ith tenant, respectively. Thus, P_i and B_i are the power and spectrum that each BS has at its disposal to serve the MTs of the ith tenant associated to it. We assume that the tenants are served over disjoint bands of the spectrum owned by the TO, i.e., B_{tot}, so that no inter-tenant interference is present. For each tenant, we assume that the BSs equally allocate the bandwidth B_i among all the MTs associated to them, and that their transmit power P_i is evenly spread across the available bandwidth B_i. This implies that the considered network model is subject to inter-cell interference, but is not subject to intra-cell interference [1], [5].

Under these modeling assumptions, the resource allocation problem that we are interested in analyzing jointly encompasses: (i) the scheduling of the tenants; and (ii) the optimization of their transmit power, P_i, and bandwidth, B_i. In fact, it is not guaranteed that, given P_{tot} and B_{tot}, the T tenants can achieve the requested minimum rates τ_i, for $i = 1, 2, \ldots, T$, simultaneously. Thus, the optimal sub-set of tenants ($\leq T$) that maximizes the performance metric of interest (the network's spectral efficiency) needs to be identified. The optimization problem is formulated in the next section, by using recent results on the spectral efficiency and average rate [1].

III. PROBLEM STATEMENT – AN MINL FORMULATION

Let $\mathcal{S}_i = \mathcal{S}(P_i, B_i, \lambda_{\mathrm{MT}_i}; \lambda_{\mathrm{BS}})$ and $\mathcal{R}_i = \mathcal{R}(P_i, B_i, \lambda_{\mathrm{MT}_i}; \lambda_{\mathrm{BS}})$ be the network's spectral efficiency and the user's average rate of the ith tenant, respectively. $\mathcal{S}_i$

and $\mathcal{R}_i$ yield the bit/s/m and the bit/s that the TO is capable of delivering to the ith tenant throughout the network and to the typical MT, respectively. Let $\beta_i \in \{0, 1\}$ be 0-1 variables for $i = 1, 2, \ldots, T$. We assume, by definition, the following relation among β_i, P_i, and B_i for $i = 1, 2, \ldots, T$:

$$\mathcal{S}(P_i = 0, B_i = 0, \lambda_{\mathrm{MT}_i}; \lambda_{\mathrm{BS}}) = 0 \quad \forall i = 1, 2, \ldots, T \quad (1)$$

$$\mathcal{R}(P_i = 0, B_i = 0, \lambda_{\mathrm{MT}_i}; \lambda_{\mathrm{BS}}) = 0 \quad \forall i = 1, 2, \ldots, T \quad (2)$$

$$\beta_i = 0 \Leftrightarrow P_i = 0, \ B_i = 0 \quad \forall i = 1, 2, \ldots, T \quad (3)$$

The conditions in (1) and (2) imply the physical constraint that $\mathcal{S}_i$ and $\mathcal{R}_i$ need to be zero if no power and spectrum are allocated to the ith tenant. The condition in (3) allows us to interpret β_i as an indicator variable that is equal to zero if and only if the ith tenant is not scheduled (or admitted), and, hence, the TO allocates no power and spectrum to its MTs.

The problem (**P1**) of jointly scheduling the tenants and allocating their resources (power, spectrum) is formulated as:

$$\mathbf{P1}: \quad \underset{P_i, B_i, \beta_i}{\text{maximize}} \quad \sum_{i=1}^{T} \ln\left[(1 + \beta_i)(1 + \mathcal{S}_i)\right] \quad (4)$$

$$\text{subject to} \quad \mathcal{R}_i \geq \beta_i \tau_i \quad (5)$$

$$0 \leq B_i \leq \beta_i B_{\mathrm{tot}} \quad (6)$$

$$0 \leq P_i \leq \beta_i P_{\mathrm{tot}} \quad (7)$$

$$\sum_{i=1}^{T} B_i \leq B_{\mathrm{tot}} \quad (8)$$

$$\sum_{i=1}^{T} P_i \leq P_{\mathrm{tot}} \quad (9)$$

$$\beta_i \in \{0, 1\} \quad (10)$$

P1 can be interpreted as follows. The objective function in (4) is the aggregate spectral efficiency of all the tenants that can be scheduled. The logarithmic function is employed in order to ensure a certain level of fairness among the tenants [9]. Based on (1)-(3), the objective function is appropriately chosen to be non-negative, i.e., if $\beta_i = 0$, then $\ln(1) = 0$. This is the reason of the presence of the "+1" terms. If $\beta_i = 0$, in particular, the ith tenant does not contribute to the objective function. The constraint in (5) ensures that the minimum rate requirement is fulfilled. The constraints in (6) and (7) ensure that the bandwidth and power are non-negative, and the constraints in (8) and (9) that the total allocated spectrum and power do not exceed B_{tot} and P_{tot}, respectively. In (5)-(7) the presence of the 0-1 variables β_i ensures that the constraints are effective only if the ith tenant is scheduled ($\beta_i = 1$). Also, by virtue of (3), and the constraints in (6) and (7), there is no need to add the 0-1 variables β_i in (8) and (9).

P1 requires an analytical expression for $\mathcal{S}_i$ and $\mathcal{R}_i$. For the considered network model, $\mathcal{S}_i$ and $\mathcal{R}_i$ are defined and formulated in [1, Eq. (7)] and [1, Eq. (9)], and, for completeness, they are reported in the next two lemmas, where: (i) γ_D is the minimum signal-to-interference ratio in order for the typical MT to decode the data from the serving BS; (ii) γ_A is the minimum average signal-to-noise ratio in order for the typical MT to detect the pilot signals during the cell association phase; (iii) N_0 is the receiver's noise spectral density at the typical MT; (vi) $L(x) = 1 - (1 + x/3.5)^{-3.5}$; and (v) $\Upsilon(y, z) = {}_2F_1(-2/z, 1, 1 - 2/z, -y) - 1$, where ${}_2F_1(\cdot)$ is the Gaussian hypergeometric function.

Lemma 1: Consider the network model in Section II. Define $\Phi\left(\lambda_{\mathrm{BS}}, \lambda_{\mathrm{MT}_i}, \gamma_D, \alpha\right) = 1 + L\left(\lambda_{\mathrm{MT}_i}/\lambda_{\mathrm{BS}}\right)\Upsilon\left(\gamma_D, \alpha\right)$. The spectral efficiency of the ith tenant can be formulated as:

$$\mathcal{S}_i = B_i \log_2\left(1 + \gamma_D\right)\frac{\lambda_{\mathrm{BS}}L\left(\lambda_{\mathrm{MT}_i}/\lambda_{\mathrm{BS}}\right)}{\Phi\left(\lambda_{\mathrm{BS}}, \lambda_{\mathrm{MT}_i}, \gamma_D, \alpha\right)} \quad (11)$$

$$\times\left(1 - \exp\left(-\pi\lambda_{\mathrm{BS}}\left(\frac{P_i/B_i}{\kappa\gamma_A N_0}\right)^{2/\alpha}\Phi\left(\lambda_{\mathrm{BS}}, \lambda_{\mathrm{MT}_i}, \gamma_D, \alpha\right)\right)\right)$$

Proof: It follows from the definition of $\mathcal{S}_i$ in [1, Eq. (7)] and the analytical steps reported in [1, Appendix A]. $\square$

Lemma 2: Consider the network model in Section II. The user's average rate of the ith tenant is equal to $\mathcal{R}_i = \mathcal{S}_i/\lambda_{\mathrm{MT}_i}$, where $\mathcal{S}_i$ is formulated in closed-form in (11).

Proof: It is similar to the proof of Lemma 1 [1]. $\square$

It is worth mentioning, as detailed in [1, Remark 7], that $\mathcal{S}_i$ in Lemma 1 is defined from the viewpoint of the typical MT rather than from the viewpoint of the typical BS. Thus, it is an approximated formulation of the network's spectral efficiency. Besides the proof in [1, Appendix A], this remark justifies the relation between $\mathcal{R}_i$ and $\mathcal{S}_i$ in Lemma 2 as well.

P1 is an MINL program and, hence, is NP-hard [8]. Tackling **P1** via global optimization methods based on the branch-and-bound algorithm is possible, but subject to identifying suitable bounds for its application [8]. To devise an algorithm that is computationally efficient and provably convergent to the global optimum, we need the following two preliminary results.

Proposition 1: The network's spectral efficiency, $\mathcal{S}_i$, and the user's average rate, $\mathcal{R}_i$, in Lemma 1 and Lemma 2, respectively, are jointly concave functions in P_i and B_i.

Proof: The proof follows by computing the Hessian matrix of $\mathcal{S}_i$ (or $\mathcal{R}_i$) as a function of P_i and B_i, and observing that the corresponding quadratic form is negative semidefinite. $\square$

Lemma 3: By conditioning upon the 0-1 variables β_i, **P1** is a convex optimization problem.

Proof: The proof follows by noting that, conditioning on the variables β_i, the objective in (4) is a concave function since it is the sum of concave functions by virtue of Proposition 1, the constraint in (5) is a concave function by virtue of Proposition 1, and the constraints in (6)-(9) are affine functions. $\square$

Based on Proposition 1 and Lemma 3, the following conclusions about **P1** can be drawn.

- If no scheduling is considered, **P1** is a convex optimization problem that can be efficiently solved in polynomial time regardless of the number of tenants. This feature originates from the mathematical tractability of the recently proposed definition of coverage probability in [1].
- The complexity of **P1** originates from jointly optimizing the scheduling of the tenants, and the power and spectrum of the tenants admitted into the network.
- By virtue of Proposition 1, **P1** may be tackled by employing an exhaustive search over the T binary variables β_i, and by applying convex optimization methods to compute, for each possible case, the optimal power and spectrum to be allocated to the corresponding tenants. Even for modest values of the number of services that the TO needs to support, e.g., $T = 20$, the computational complexity of such a brute-force method may be pro-

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Algorithm 1 Proposed algorithm for solving **P2**

1: Set an arbitrary $\beta_i = \bar{\beta}_i$ $(i = 1, 2, \ldots, T)$ that makes **P3** feasible. Solve **P3**, and denote its global solution by $\left(P_i^*, B_i^*\right)$ and the vector of Lagrange multipliers by $\boldsymbol{\mu}^*$. Set $\varepsilon \geq 0$, $K_1 = 1$, $K_2 = 0$, $\boldsymbol{\mu}^{(1)} = \boldsymbol{\mu}^*$, UBD $= f\left(P_i^*, B_i^*, \bar{\beta}_i\right)$, and compute $L^*\left(\beta_i, \boldsymbol{\mu}^{(1)}\right)$ in (21).
2: Solve **P4** and denote its global solution by $\left(\beta_0^*, \beta_i^*\right)$. Set LBD $= \beta_0^*$.
3: **while** UBD $-$ LBD $\geqslant \varepsilon$ **do**
4:     Set $\bar{\beta}_i = \beta_i^*$. Solve **P3**.
5:     **if P3** is feasible **then**
6:         Compute the global solution, $\left(P_i^*, B_i^*\right)$, and the Lagrange multipliers, $\boldsymbol{\mu}^*$, of **P3**. Set $K_1 = K_1 + 1$, $\boldsymbol{\mu}^{(K_1)} = \boldsymbol{\mu}^*$, compute $L^*\left(\bar{\beta}_i, \boldsymbol{\mu}^{(K_1)}\right)$ in (21). If $f\left(P_i^*, B_i^*, \bar{\beta}_i\right) <$ UBD, set UBD $= f\left(P_i^*, B_i^*, \bar{\beta}_i\right)$. Return to Step 2.
7:     **else**
8:         Compute the Lagrange multipliers, $\boldsymbol{\rho}_*$, corresponding to the global solution of **P5**. Set $K_2 = K_2 + 1$, $\boldsymbol{\rho}^{(K_2)} = \boldsymbol{\rho}_*$, and compute $L_*\left(\bar{\beta}_i, \boldsymbol{\rho}^{(K_2)}\right)$ in (24). Return to Step 2.
9:     **end if**
10: **end while**
11: The solution of **P2** is $P_i = P_i^*$ and $B_i = B_i^*$ from the solution of **P3**, and $\beta_i = \beta_i^*$ from the solution of **P4**.

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hibitive. Given the need of future networks to overcome the "one-size-fits-all" design paradigm, and to be highly adaptable to support multiple and disparate services, a computationally-efficient algorithm with provable convergence to the global optimum is needed.

In Section IV, we capitalize on Proposition 1 and the structure of the objective and the constraints in (4)-(10) in order to develop an algorithm, based on the GBD, that is provably convergent to the global solution of **P1**. This cannot be guaranteed, in general, by using general-purpose solvers for MINL problems. The proposed approach has, in general, exponential complexity, but numerical results show that it is faster than a (sub-optimal) greedy algorithm (see Section V).

IV. ALGORITHMIC SOLUTION – GBD-BASED APPROACH

The proposed approach for solving **P1** is based on the GBD originally introduced in [7]. We show, in particular, that the specific properties of $\mathcal{S}_i$ and $\mathcal{R}_i$ proved in Proposition 1 allows us to obtain an efficient iterative implementation that necessitates the solution of a few convex optimization problems and Mixed-Integer Linear (MIL) optimization problems, whose global solution can be provably guaranteed.

Define the column-vector of constraints $\mathbf{G}\left(P_i, B_i, \beta_i\right) = [G_1; G_2; \ldots; G_7]$, whose seven rows are: $G_1 = \beta_i \tau_i - \mathcal{R}_i$; $G_2 = B_i - \beta_i B_{\mathrm{tot}}$; $G_3 = P_i - \beta_i P_{\mathrm{tot}}$; $G_4 = -B_i$; $G_5 = -P_i$; $G_6 = \sum_{i=1}^{T} B_i - B_{\mathrm{tot}}$; and $G_7 = \sum_{i=1}^{T} P_i - P_{\mathrm{tot}}$. Define $f\left(P_i, B_i, \beta_i\right) = f_1\left(\beta_i\right) + f_2\left(P_i, B_i\right)$, where $f_1\left(\beta_i\right) = -\sum_{i=1}^{T}\ln\left(1 + \beta_i\right)$ and $f_2\left(P_i, B_i\right) = -\sum_{i=1}^{T}\ln\left(1 + \mathcal{S}_i\right)$. **P1** can be re-written, in canonical form, as follows:

$$\mathbf{P2}: \quad \underset{P_i, B_i, \beta_i}{\text{minimize}} \quad f\left(P_i, B_i, \beta_i\right) \quad (12)$$

$$\text{subject to} \quad \mathbf{G}\left(P_i, B_i, \beta_i\right) \leq 0 \quad (13)$$

$$\beta_i \in \{0, 1\} \quad (14)$$

Algorithm 1 reports the proposed approach for solving **P2**. Algorithm 1 is an iterative algorithm that recursively solves the three optimization problems, **P3**, **P4**, and **P5** defined in the following paragraphs. The optimality and rational of Algorithm 1 are discussed at the end of this section.

a) Problem P3:
a) Problem P3: **P3** is defined as follows:

$$\textbf{P3}: \quad \underset{P_i, B_i}{\text{minimize}} \quad f\left(P_i, B_i, \bar{\beta}_i\right) \tag{15}$$

$$\text{subject to} \quad \mathbf{G}\left(P_i, B_i, \bar{\beta}_i\right) \leq 0 \tag{16}$$

where $\bar{\beta}_i$ denotes a given instance of β_i $(i = 1, 2, \ldots, T)$, as stated in Algorithm 1. Since the 0-1 variables are not subject to optimization, **P3** is a convex problem by virtue of Lemma 3. Thus, its global optimum is obtained in polynomial time.

b) Problem P5: Let $\bar{\beta}_i$ be an instance of β_i that makes **P3** feasible. Let θ be a real variable. **P5** is defined as:

$$\textbf{P5}: \quad \underset{P_i, B_i, \theta}{\text{minimize}} \quad \theta \tag{17}$$

$$\text{subject to} \quad \mathbf{G}\left(P_i, B_i, \bar{\beta}_i\right) - \theta \mathbf{1} \leq 0 \tag{18}$$

Since the 0-1 variables β_i are not to be optimized, **P5** is a convex optimization problem by virtue of Proposition 1. Let $\boldsymbol{\rho}_* = [\rho_i; \rho_{0B_i}; \rho_{0P_i}; \rho_{00B_i}; \rho_{00P_i}; \rho_B; \rho_P]$ be the column-vector of Lagrange multipliers of **P5**, whose eth entry corresponds to G_e $(e = 1, 2, \ldots, 7)$ in (18). $\boldsymbol{\rho}_*$ is needed to formulate **P4**. It is worth mentioning that **P5** is an equivalent (convex) formulation of [7, Eq. (6)], as elaborated in [10].

c) Problem P4: Before formulating **P4**, we introduce the auxiliary optimization problems **P4*** and **P4**$_*$.

Let $\boldsymbol{\mu}^* = [\mu_i; \mu_{0B_i}; \mu_{0P_i}; \mu_{00B_i}; \mu_{00P_i}; \mu_B; \mu_P]$ be the column-vector of Lagrange multipliers of **P3**, whose eth entry corresponds to G_e $(e = 1, 2, \ldots, 7)$ in (16). Let $\bar{b}_i = \mu_{0B_i} - \mu_{00B_i} + \mu_B$ and $\bar{p}_i = \mu_{0P_i} - \mu_{00P_i} + \mu_P$, **P4*** is defined as:

$$\textbf{P4}^*: \quad \underset{P_i, B_i}{\text{minimize}} \sum_{i=1}^{T} \left(-\ln\left(1 + \mathcal{S}_i\right) - \mu_i \mathcal{S}_i + \bar{b}_i B_i + \bar{p}_i P_i \right) \tag{19}$$

$$\text{subject to (8) and (9)} \tag{20}$$

Let f_{P4^*} be the objective function evaluated at the optimal solution of **P4*** in (19). Define the following function[1]:

$$L^*\left(\beta_i, \boldsymbol{\mu}^*\right) = f_{\text{P4}^*} - \mu_B B_{\text{tot}} - \mu_P P_{\text{tot}} \tag{21}$$
$$+ \sum_{i=1}^{T} \beta_i \left(\mu_i \lambda_{\text{MT}_i} \tau_i - \ln(2) - \mu_{0B_i} B_{\text{tot}} - \mu_{0P_i} P_{\text{tot}} \right)$$

Let $\boldsymbol{\rho}_*$ be the column-vector defined in **P5**. Let $\underline{b}_i = \rho_{0B_i} - \rho_{00B_i} + \rho_B$, $\underline{p}_i = \rho_{0P_i} - \rho_{00P_i} + \rho_P$. **P4**$_*$ is defined as follows:

$$\textbf{P4}_*: \quad \underset{P_i, B_i}{\text{minimize}} \sum_{i=1}^{T} \left(-\rho_i \mathcal{S}_i + \underline{b}_i B_i + \underline{p}_i P_i \right) \tag{22}$$

$$\text{subject to (8) and (9)} \tag{23}$$

Let f_{P4_*} be the objective function evaluated at the optimal solution of **P4**$_*$ in (22). Define the following function:

$$L_*\left(\beta_i, \boldsymbol{\rho}_*\right) = f_{\text{P4}_*} - \rho_B B_{\text{tot}} - \rho_P P_{\text{tot}} \tag{24}$$
$$+ \sum_{i=1}^{T} \beta_i \left(\rho_i \lambda_{\text{MT}_i} \tau_i - \rho_{0B_i} B_{\text{tot}} - \rho_{0P_i} P_{\text{tot}} \right)$$

By virtue of Proposition 1, **P4*** and **P4**$_*$ are convex and their global solution can be computed in polynomial time.

Let β_0 be a real number. **P4** is defined as follows:

$$\textbf{P4}: \quad \underset{\beta_0, \beta_i}{\text{minimize}} \quad \beta_0 \tag{25}$$

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[1]The $\ln(2)$ term is obtained by noting that $\ln(1 + \beta_i) = \beta_i \ln(2)$.

Algorithm 2 Greedy algorithm

1: Initialize $\bar{\beta}_i = 1$ for $i = 1, 2, \ldots, T$.
2: Solve **P3** for $\beta_i = \bar{\beta}_i$.
3: **if** The convex problem **P3** is not feasible **then**
4:      Set $\bar{\beta}_{\hat{i}} = 0$, where $\hat{i} = \arg\max_i \{\tau_i\}$.
5:      Return to Step 2.
6: **else**
7:      Return the solution, $(P_i^*, B_i^*, \beta_i^*)$, of **P3**.
8: **end if**

$$\text{subject to} \quad L^*\left(\beta_i, \boldsymbol{\mu}^{(k_1)}\right) \leq \beta_0 \quad k_1 = 1, 2, \ldots, K_1 \tag{26}$$

$$L_*\left(\beta_i, \boldsymbol{\rho}^{(k_2)}\right) \leq 0 \quad k_2 = 1, 2, \ldots, K_2 \tag{27}$$

where $\boldsymbol{\mu}^{(k_1)} = \boldsymbol{\mu}^*$ and $\boldsymbol{\rho}^{(k_2)} = \boldsymbol{\rho}_*$ as defined in Algorithm 1.

P4 is referred to as the *master problem*. The master problem in (25)-(27) is, in particular, an MIL optimization problem, whose global solution can be computed by using the branch-and-cut algorithm based on Gomory cuts [11]. More specifically, a sequence of continuous relaxations of **P4** are solved by using standard linear programming methods [8, Chapter 5]. New Gomory cuts are added each time that the solution of **P4** results in some variables that are not integer numbers. It is proved in [11, Theorem 9.13] that the branch-and-cut algorithm based on Gomory cuts applied to an MIL converges to its global solution in a finite number of iterations.

Therefore, three remarks about Algorithm 1 can be made.

- In contrast to **P2**, the computation of the global solution of **P4** can be guaranteed in a finite number of iterations. The reason is that **P4** is a linear optimization problem and a systematic procedure to compute the cuts, which assures the convergence to the global optimum, exists.
- Algorithm 1 consists of iteratively solving convex and MIL optimization problems. Since all the considered optimization problems can be solved globally, Algorithm 1 provably converges to the optimum of **P2** in a finite number of iterations [7, Theorem 2.5]. In particular, Algorithm 1 is ε-optimal (with $\varepsilon > 0$).
- Since **P4** is a non-convex optimization problem, it is not possible to ensure that the global optimum of Algorithm 1 is computed in polynomial time. In Section V, we show, however, that Algorithm 1 is more efficient than other competing algorithms, including solving **P1** by using branch-and-bound methods applied to MINL problems.

V. NUMERICAL RESULTS

In this section, we substantiate the global optimality and analyze the complexity of Algorithm 1. With the aid of the OPTI toolbox integrated in Matlab [12], **P3**, **P5**, **P4***, and **P4**$_*$ are solved by using the IPOPT solver, which employs the primal-dual interior point method, and **P4** is solved by using the SCIP solver, which employs the branch-and-cut method.

We compare Algorithm 1 against three benchmark algorithms: (i) the exhaustive search (brute-force) approach described in Section III, which consists of solving 2^T convex optimization problems and then computing the best solution among them. The convex problems are solved by using the IPOPT solver of the OPTI toolbox; (ii) the branch-and-bound algorithm, which is directly applied to the MINL optimization

TABLE I: Simulation setup

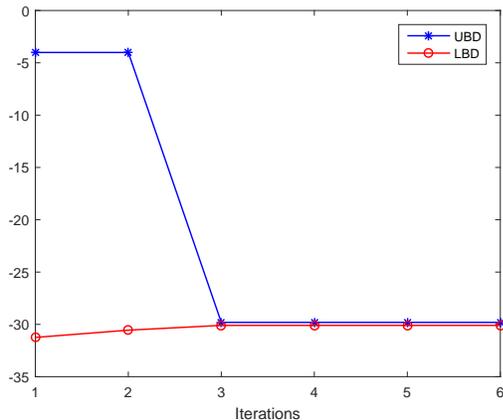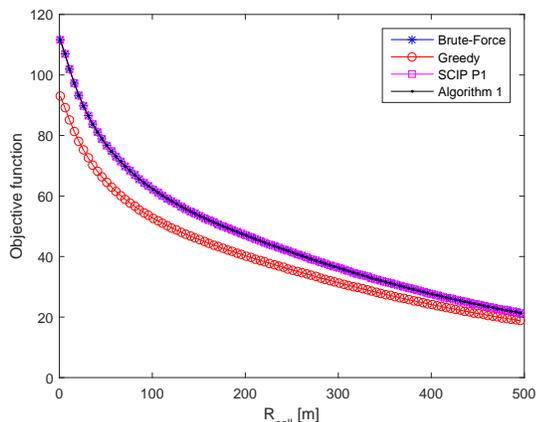| |
| --- |
| $\lambda_{\mathrm{BS}} = 1/\left(\pi\mathrm{R}_{\mathrm{cell}}^2\right)$, $\lambda_{\mathrm{MT}} = 1/\left(\pi\mathrm{R}_{\mathrm{MT}}^2\right)$, $R_{\mathrm{cell}} = 500$ m, |
| $\gamma_I = \gamma_A = 10$ dB, $\kappa = 38.9$ dB, $\alpha = 3.5$, $T = 20$, |
| $B_{\mathrm{tot}} = 100$ MHz, $P_{\mathrm{tot}} = 30$ dBm, $N_0 = -174$ dBm/Hz, |
| $\tau_{1:7} = 37.74, 885.17, 913.29, 796.18, 98.71, 261.87, 335.36$ |
| $\tau_{8:14} = 679.73, 136.55, 721.23, 106.76, 653.76, 494.17, 779.05$ |
| $\tau_{15:20} = 715.04, 903.72, 890.92, 334.16, 698.75, 197.81$ |
| $\mathrm{R}_{\mathrm{MT}_{1:7}} = 5.13, 15.75, 48.06, 1.75, 55.73, 43.82, 29.32$ |
| $\mathrm{R}_{\mathrm{MT}_{8:14}} = 34.71, 14.24, 27.53, 57.79, 32.81, 31.27, 13.90$ |
| $\mathrm{R}_{\mathrm{MT}_{15:20}} = 29.33, 37.44, 40.75, 23.73, 22.05, 59.28$ |

Fig. 1: Convergence of Algorithm 1.

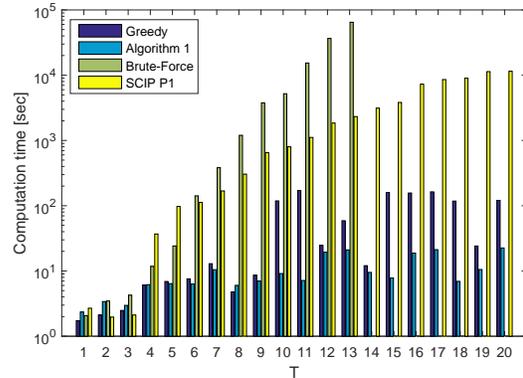Fig. 2: Comparison with benchmarks: Objective function.

Fig. 3: Comparison with benchmarks: Computation time.

The advantage of Algorithm 1 compared with brute-force and SCIP algorithms is apparent in Fig. 3. Since Algorithm 1 iteratively solves convex and MIL optimization problems, its computation time is lower than the considered benchmark algorithms. In the considered setup, Algorithm 1 may be even faster, for large value of T, than the greedy algorithm. This is due to the small number of iterations that are needed for convergence (around 3, as shown in Fig. 1). The computation time of Algorithm 1 does not increase exponentially with T.

VI. CONCLUSION

We have formulated a mixed-integer non-linear optimization problem based on utility functions obtained from stochastic geometry. By capitalizing on the generalized Benders decomposition, we have devised an efficient numerical algorithm that is provably convergent to the optimum in a finite number of iterations. Numerical simulations confirm the convergence to the same solution as a brute-force search but with reduced computational complexity. The proposed approach constitutes a viable and promising method for solving scheduling and resource allocation problems in Poisson cellular networks.

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problem **P1** (equivalently **P2**). In this case, for a fair comparison with Algorithm 1, the SCIP solver of the OPTI toolbox is applied; and (iii) the greedy-like Algorithm 2.

The simulation setup is given in Table I. The numerical results are obtained on a Lenovo Thinkpad T440 platform (Intel Core i7-4500U, CPU 1.80-2.40 GHz, RAM 8.00 GB).

In Fig. 1, the convergence of Algorithm 1 is analyzed. In particular, the UBD and LBD functions defined in Algorithm 1 are plotted as a function of the iterations. The gap between the UBD and LBD functions decreases with the number of iterations, which confirms the convergence of Algorithm 1.

In Fig. 2, the objective function in (4) that corresponds to the solution of Algorithm 1 is compared against the other three benchmark algorithms. Algorithm 1 returns the same objective function as the brute-force search and the SCIP general-purpose solver applied to **P2**, while Algorithm 2 yields worse performance due to its sub-optimality. Therefore, it is confirmed that algorithm 1 returns the global optimum.