

Convolutional Approximate Message-Passing

Keigo Takeuchi, *Member, IEEE*

Abstract—This letter proposes a novel message-passing algorithm for signal recovery in compressed sensing. The proposed algorithm solves the disadvantages of approximate message-passing (AMP) and orthogonal/vector AMP, and realizes their advantages. AMP converges only in a limited class of sensing matrices while it has low complexity. Orthogonal/vector AMP requires a high-complexity matrix inversion while it is applicable for a wide class of sensing matrices. The key feature of the proposed algorithm is the so-called Onsager correction via a convolution of messages in all preceding iterations while the conventional message-passing algorithms have correction terms that depend only on messages in the latest iteration. Thus, the proposed algorithm is called convolutional AMP (CAMP). Ill-conditioned sensing matrices are simulated as an example in which the convergence of AMP is not guaranteed. Numerical simulations show that CAMP can improve the convergence property of AMP and achieve high performance comparable to orthogonal/vector AMP in spite of low complexity comparable to AMP.

Index Terms—Compressed sensing, approximate message-passing, orthogonal invariance, state evolution.

I. INTRODUCTION

APPROXIMATE message-passing (AMP) [1] is a low-complexity algorithm of signal recovery in compressed sensing [2], [3]. When the sensing matrix has independent and identically distributed (i.i.d.) zero-mean sub-Gaussian elements [4], [5], AMP was proved to be asymptotically Bayes-optimal in a certain region of the compression rate. However, AMP fails to converge when the sensing matrix is ill-conditioned [6] or has non-zero mean [7].

To solve this convergence issue of AMP, orthogonal AMP (OAMP) [8] and vector AMP (VAMP) [9] were proposed. OAMP and VAMP are equivalent to each other. The Bayes-optimal version of OAMP/VAMP was originally proposed by Oppor and Winther [10]. OAMP/VAMP was proved to be asymptotically Bayes-optimal when the sensing matrix is orthogonally invariant [9], [11]. However, OAMP/VAMP has high complexity unless the singular-value decomposition (SVD) of the sensing matrix can be computed efficiently.

This letter proposes a novel message-passing (MP) algorithm that solves both the convergence issue of AMP and the complexity issue of OAMP/VAMP. The proposed MP uses the same matched filter as AMP while OAMP/VAMP utilizes a linear minimum mean-square error (LMMSE) filter. Furthermore, it performs the so-called Onsager correction via a convolution of messages in all preceding iterations while

AMP and OAMP/VAMP have correction terms that depend only on messages in the latest iteration. Thus, the proposed MP is called convolutional AMP (CAMP).

The tap coefficients in the convolution are determined so as to realize the asymptotic Gaussianity of the estimation errors of CAMP. For that purpose, they are defined such that a general error model proposed in [12] contains the error model of CAMP asymptotically. Since the asymptotic Gaussianity in the general error model has been proved for any orthogonally invariant sensing matrix [12], the estimation errors of CAMP are asymptotically Gaussian-distributed. Numerical simulations for ill-conditioned sensing matrices show that CAMP can achieve performance comparable to OAMP/VAMP in spite of complexity comparable to AMP.

II. MEASUREMENT MODEL

Consider the M -dimensional linear measurements $\mathbf{y} \in \mathbb{R}^M$ of an unknown N -dimensional sparse signal vector $\mathbf{x} \in \mathbb{R}^N$,

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{w}, \quad \mathbf{w} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I}_M). \quad (1)$$

In (1), $\mathbf{A} \in \mathbb{R}^{M \times N}$ denotes a known sensing matrix. The vector \mathbf{w} is additive white Gaussian noise (AWGN) with covariance $\sigma^2 \mathbf{I}_M$. The triple $(\mathbf{A}, \mathbf{x}, \mathbf{w})$ is independent random variables. For simplicity, the signal vector \mathbf{x} is assumed to have i.i.d. elements with zero mean and unit variance. Furthermore, the power normalization $N^{-1} \mathbb{E}[\|\mathbf{A}\|^2] = 1$ is assumed.

An important assumption is the right-orthogonal invariance of \mathbf{A} : In the SVD $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$, the $N \times N$ orthogonal matrix \mathbf{V} is independent of $\mathbf{U}\mathbf{\Sigma}$ and Haar-distributed [13]. This class of matrices contains zero-mean i.i.d. Gaussian matrices.

As an additional technical assumption, the empirical eigenvalue distribution of $\mathbf{A}^T \mathbf{A}$ converges almost surely to a deterministic distribution with a compact support in the large system limit, in which M and N tend to infinity while the compression rate $\delta = M/N$ is kept $\mathcal{O}(1)$. Let μ_k denote the k th moment of the empirical eigenvalue distribution,

$$\mu_k = \frac{1}{N} \text{Tr} \left(\mathbf{\Lambda}^k \right), \quad (2)$$

with $\mathbf{\Lambda} = \mathbf{\Sigma}^T \mathbf{\Sigma}$. The technical assumption implies that any moment μ_k converges almost surely in the large system limit. In particular, the power normalization $N^{-1} \mathbb{E}[\|\mathbf{A}\|^2] = 1$ implies $\mu_1 \xrightarrow{\text{a.s.}} 1$ in the large system limit.

III. CONVOLUTIONAL AMP

A. Algorithm

The so-called Onsager correction is used to guarantee the asymptotic Gaussianity of the estimation errors before thresholding in each iteration of MP. The Onsager correction in AMP

The author was in part supported by the Grant-in-Aid for Scientific Research (B) (JSPS KAKENHI Grant Number 18H01441), Japan.

K. Takeuchi is with the Department of Electrical and Electronic Information Engineering, Toyohashi University of Technology, Aichi 441-8580, Japan (e-mail: takeuchi@ee.tut.ac.jp).

TABLE I
COMPLEXITY IN $M \leq N$ AND THE NUMBER OF ITERATIONS t .

AMP	OAMP/VAMP	CAMP
$\mathcal{O}(tMN)$	$\mathcal{O}(M^2N + tMN)$	$\mathcal{O}(tMN + t^2M)$

depends only on a message in the latest iteration. While AMP is a low-complexity algorithm, the Onsager correction in AMP fails to guarantee the asymptotic Gaussianity, with the only exception of zero-mean i.i.d. sensing matrices [4], [5].

The proposed CAMP has Onsager correction applicable to all right-orthogonally invariant sensing matrices. The correction term is a convolution of messages in all preceding iterations. Thus, the proposed MP is called convolutional AMP.

Let $\mathbf{x}_t \in \mathbb{R}^N$ denote an estimator of \mathbf{x} in iteration t of CAMP. The estimator \mathbf{x}_t is recursively given by

$$\mathbf{x}_{t+1} = f_t(\mathbf{x}_t + \mathbf{A}^T \mathbf{z}_t), \quad (3)$$

$$\mathbf{z}_t = \mathbf{y} - \mathbf{A}\mathbf{x}_t + \sum_{\tau=0}^{t-1} \xi_\tau^{(t-1)} g_{t-\tau-1}^{(1)} \mathbf{z}_\tau, \quad (4)$$

$$\xi_t^{(t')} = \prod_{\tau=t}^{t'} \left\langle f'_\tau(\mathbf{x}_\tau + \mathbf{A}^T \mathbf{z}_\tau) \right\rangle, \quad (5)$$

with $\mathbf{x}_0 = \mathbf{0}$. In the CAMP, $\{f_t : \mathbb{R} \rightarrow \mathbb{R}\}$ are a sequence of Lipschitz-continuous thresholding functions. For any function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(\mathbf{v})$ represents the element-wise application of f to a vector \mathbf{v} , i.e. $[f(\mathbf{v})]_n = f([v]_n)$. The notional convention $\sum_{\tau=0}^{-1} \cdots = \mathbf{0}$ is used in (4). The notation $\langle \mathbf{v} \rangle = N^{-1} \sum_{n=1}^N v_n$ denotes the arithmetic mean of the elements of $\mathbf{v} = (v_1, \dots, v_N)^T$. The CAMP reduces to conventional AMP in the case of $g_0^{(1)} = \delta^{-1}$ and $g_t^{(1)} = 0$ for all $t > 0$.

To define the tap coefficients $\{g_t^{(1)}\}$ in the CAMP, consider a discrete-time dynamical system $\{g_t^{(k)} : k, t = 0, 1, \dots\}$,

$$g_0^{(k)} = \mu_{k+1} - \mu_k, \quad (6)$$

$$g_1^{(k)} = g_0^{(k)} - g_0^{(k+1)} + g_0^{(1)} \mu_{k+1}, \quad (7)$$

$$g_t^{(k)} = g_{t-1}^{(k)} - g_{t-1}^{(k+1)} + \sum_{\tau=1}^{t-1} g_{t-\tau-1}^{(1)} \left(g_\tau^{(k)} - g_{\tau-1}^{(k)} \right) + g_{t-1}^{(1)} \mu_{k+1} \quad \text{for } t \geq 2, \quad (8)$$

where μ_k denotes the k th moment (2) of the empirical eigenvalue distribution of $\mathbf{A}^T \mathbf{A}$.

In a practical implementation, the moment sequence should be replaced by the asymptotic one in the large system limit. This replacement implies that the complexity to compute $\{g_t^{(1)}\}$ can be independent of the system size if the asymptotic eigenvalue distribution of $\mathbf{A}^T \mathbf{A}$ has a closed-form expression.

The computational complexity of the CAMP, AMP, and OAMP/VAMP is compared in Table I. The complexity of AMP is dominated by matrix-vector multiplication. The first term for OAMP/VAMP is the worst-case complexity of the SVD of \mathbf{A} . The second term for the CAMP is due to computation of the Onsager correction term. As long as the number of iterations t is much smaller than M and N , the complexity of the CAMP is comparable to that of AMP.

B. State Evolution

The tap coefficients in the CAMP have been determined so as to guarantee the asymptotic Gaussianity of the estimation errors. The author [12] proposed a general error model and used state evolution (SE) to prove that the estimation error before thresholding is asymptotically Gaussian-distributed in the general error model. To prove the asymptotic Gaussianity of the estimation error $\mathbf{h}_t = \mathbf{x}_t + \mathbf{A}^T \mathbf{z}_t - \mathbf{x}$ before the thresholding f_t , thus, it is sufficient to show that the error model of the CAMP is included into the general error model.

Let $\mathbf{q}_{t+1} = f_t(\mathbf{x} + \mathbf{h}_t) - \mathbf{x}$ denote the estimation error after the thresholding. According to the definition of the general error model [12], define $\mathbf{b}_t = \mathbf{V}^T \tilde{\mathbf{q}}_t$, $\mathbf{m}_t = \mathbf{V}^T \mathbf{h}_t$, and

$$\tilde{\mathbf{q}}_0 = \mathbf{q}_0, \quad \tilde{\mathbf{q}}_t = \mathbf{q}_t - \xi_{t-1} \mathbf{h}_{t-1} \quad (9)$$

for $t > 0$, where ξ_t is an abbreviation of $\xi_t^{(t)}$ given in (5). Then, \mathbf{m}_t satisfies the following equation:

$$\begin{aligned} \mathbf{m}_t = & (\mathbf{I}_N - \mathbf{\Lambda})(\mathbf{b}_t + \xi_{t-1} \mathbf{m}_{t-1}) + \mathbf{\Sigma}^T \mathbf{U}^T \mathbf{w} \\ & + \sum_{\tau=0}^{t-1} \xi_\tau^{(t-1)} g_{t-\tau-1}^{(1)} (\mathbf{m}_\tau - \mathbf{b}_\tau - \xi_{\tau-1} \mathbf{m}_{\tau-1}), \end{aligned} \quad (10)$$

with $\mathbf{m}_t = \mathbf{0}$ for all $t < 0$.

Proof of (10): From the definitions of \mathbf{m}_t and \mathbf{h}_t , we use the SVD $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$ to have

$$\mathbf{m}_t = \mathbf{V}^T \mathbf{q}_t + \mathbf{\Sigma}^T \mathbf{U}^T \mathbf{z}_t. \quad (11)$$

Left-multiplying (4) by $\mathbf{\Sigma}^T \mathbf{U}^T$ and substituting (1) and (11), we obtain

$$\begin{aligned} \mathbf{m}_t = & (\mathbf{I}_N - \mathbf{\Lambda}) \mathbf{V}^T \mathbf{q}_t + \mathbf{\Sigma}^T \mathbf{U}^T \mathbf{w} \\ & + \sum_{\tau=0}^{t-1} \xi_\tau^{(t-1)} g_{t-\tau-1}^{(1)} \left(\mathbf{m}_\tau - \mathbf{V}^T \mathbf{q}_\tau \right). \end{aligned} \quad (12)$$

Using (9) and the definitions of \mathbf{b}_t and \mathbf{m}_t , we arrive at (10). \blacksquare

For $\tau = 0, 1, \dots$ and $\tau' = 0, \dots, \tau$, define

$$g_{\tau', \tau}^{(k)} = \frac{1}{N} \sum_{n=1}^N \frac{\partial [\mathbf{\Lambda}^k \mathbf{m}_\tau]_n}{\partial [\mathbf{b}_{\tau'}]_n}. \quad (13)$$

When $g_{\tau', \tau}^{(0)} = 0$ holds for all τ' and τ , the general error model in [12] includes the error model of the CAMP. The following theorem implies that the inclusion is correct in the large system limit. Thus, the asymptotic Gaussianity of the estimation errors is guaranteed in the CAMP.

Theorem 1: For all $\tau = 0, 1, \dots$ and $\tau' = 0, \dots, \tau$, the almost sure convergence $g_{\tau', \tau}^{(0)} \xrightarrow{\text{a.s.}} 0$ holds in the large system limit.

Proof: The proof is by induction to show

- 1) $g_{\tau', \tau}^{(0)} \xrightarrow{\text{a.s.}} 0$,
- 2) the almost sure convergence of ξ_τ to a constant,
- 3) Let $\tilde{g}_0^{(k)} = g_{\tau, \tau}^{(k)}$ and $\tilde{g}_{\tau', \tau}^{(k)} = g_{\tau', \tau}^{(k)} / \xi_{\tau'}^{(\tau-1)}$ for $\tau' < \tau$. $\tilde{g}_{\tau', \tau}^{(k)}$ depends on τ and τ' only through $\tau - \tau'$.

According to [12, Theorem 1], the statement 2) follows from the statement 1). Thus, we only focus on the first and last statements. For $\tau = 0$, we use (10) to obtain $g_{0,0}^{(0)} = \mu_0 - \mu_1 \xrightarrow{\text{a.s.}} 0$, because of $\mu_0 = 1$ and $\mu_1 \xrightarrow{\text{a.s.}} 1$.

For some t , assume the three statements for all $\tau < t$ and $\tau' \leq \tau$. We shall prove the first and last statements for $\tau = t$.

We first prove the statement 3). For $t' = t$ and $t' = t - 1$, we use (10) to obtain

$$g_{t,t}^{(k)} = \mu_k - \mu_{k+1}, \quad (14)$$

$$g_{t-1,t}^{(k)} = \xi_{t-1}(g_{t-1,t-1}^{(k)} - g_{t-1,t-1}^{(k+1)}) + \xi_{t-1}g_0^{(1)}(g_{t-1,t-1}^{(k)} - \mu_k), \quad (15)$$

where we have used the second induction hypothesis. Similarly, for $t' \leq t - 2$ we have

$$\begin{aligned} g_{t',t}^{(k)} &= \xi_{t-1}(g_{t',t-1}^{(k)} - g_{t',t-1}^{(k+1)}) + \sum_{\tau=t'}^{t-1} \xi_{\tau}^{(t-1)} g_{t-\tau-1}^{(1)} g_{t',\tau}^{(k)} \\ &\quad - \xi_{t'}^{(t-1)} g_{t-t'-1}^{(1)} \mu_k - \sum_{\tau=t'+1}^{t-1} \xi_{\tau-1}^{(t-1)} g_{t-\tau-1}^{(1)} g_{t',\tau-1}^{(k)}. \end{aligned} \quad (16)$$

From the last induction hypothesis, we can define $\tilde{g}_{\tau-\tau'}^{(k)} = g_{\tau',\tau}^{(k)} / \xi_{\tau'}^{\tau-1}$ for $\tau < t$. Using (14) and this change of variables yields

$$\tilde{g}_{t-1,t}^{(k)} = \tilde{g}_0^{(k)} - \tilde{g}_0^{(k+1)} - g_0^{(1)} \mu_{k+1}, \quad (17)$$

$$\begin{aligned} \tilde{g}_{t',t}^{(k)} &= \tilde{g}_{t-t'-1}^{(k)} - \tilde{g}_{t-t'-1}^{(k+1)} + \sum_{\tau=1}^{t-t'-1} g_{t-t'-\tau-1}^{(1)} (\tilde{g}_{\tau}^{(k)} - \tilde{g}_{\tau-1}^{(k)}) \\ &\quad - g_{t-t'-1}^{(1)} \mu_{k+1} \end{aligned} \quad (18)$$

for $t' \leq t - 2$, with $\tilde{g}_0^{(k)} = g_{t,t}^{(k)}$. Since the right-hand sides (RHSs) depend on t' and t only through $t - t'$, we find that the statement 3) holds for $\tau = t$, and can re-write the left-hand sides of (17) and (18) as $\tilde{g}_1^{(k)}$ and $\tilde{g}_{t-t'}^{(k)}$, respectively.

Finally, we prove the statement 1). It is sufficient to prove $\tilde{g}_{t'}^{(0)} \xrightarrow{\text{a.s.}} 0$ and $\tilde{g}_{t'}^{(k)} + g_{t'}^{(k)} \xrightarrow{\text{a.s.}} 0$ for all $t' = 0, \dots, t$ and k . The proof is by induction. For $t' = 0$, we have $\tilde{g}_0^{(0)} = \mu_0 - \mu_1 \xrightarrow{\text{a.s.}} 0$. Comparing (6) and $\tilde{g}_0^{(k)} = \mu_k - \mu_{k+1}$ yields $\tilde{g}_0^{(k)} = -g_0^{(k)}$.

For $t' = 1$, we use (17), $\tilde{g}_0^{(0)} \xrightarrow{\text{a.s.}} 0$, and $\tilde{g}_0^{(1)} = -g_0^{(1)}$ to obtain $\tilde{g}_1^{(0)} \xrightarrow{\text{a.s.}} -\tilde{g}_0^{(1)} - g_0^{(1)} + o(1) \xrightarrow{\text{a.s.}} 0$. Furthermore, we use (7), (17), and $\tilde{g}_0^{(k)} = -g_0^{(k)}$ to find $\tilde{g}_1^{(k)} + g_1^{(k)} \xrightarrow{\text{a.s.}} 0$.

Assume $\tilde{g}_{t'}^{(0)} \xrightarrow{\text{a.s.}} 0$ and $\tilde{g}_{t'}^{(k)} + g_{t'}^{(k)} \xrightarrow{\text{a.s.}} 0$ for all $t' < \tau \in \{2, \dots, t\}$, and prove $\tilde{g}_{\tau}^{(0)} \xrightarrow{\text{a.s.}} 0$ and $\tilde{g}_{\tau}^{(k)} + g_{\tau}^{(k)} \xrightarrow{\text{a.s.}} 0$. For the former statement, we use (18) and the induction hypotheses $\tilde{g}_{t'}^{(0)} \xrightarrow{\text{a.s.}} 0$ and $\tilde{g}_{t'-1}^{(1)} + g_{t'-1}^{(1)} \xrightarrow{\text{a.s.}} 0$ to obtain

$$\tilde{g}_{\tau}^{(0)} \xrightarrow{\text{a.s.}} -\tilde{g}_{\tau-1}^{(1)} - g_{\tau-1}^{(1)} + o(1) \xrightarrow{\text{a.s.}} 0. \quad (19)$$

For the latter statement, we use (8), (18), and the induction hypothesis $\tilde{g}_{t'}^{(k)} + g_{t'}^{(k)} \xrightarrow{\text{a.s.}} 0$ to find $\tilde{g}_{\tau}^{(k)} + g_{\tau}^{(k)} \xrightarrow{\text{a.s.}} 0$. Thus, $\tilde{g}_{t'}^{(0)} \xrightarrow{\text{a.s.}} 0$ and $\tilde{g}_{t'}^{(k)} + g_{t'}^{(k)} \xrightarrow{\text{a.s.}} 0$ hold for all $t' = 0, \dots, t$ and k . In other words, we have proved the statement 1). ■

C. Closed-Form Solution

The sequence $\{g_t^{(k)}\}$ may be computed by solving the discrete-time dynamical systems (6)–(8) numerically when the moment sequence $\{\mu_k\}$ is given. However, it is possible to obtain a closed-form solution of the tap coefficients $\{g_t^{(k)}\}$ via

the η -transform $\eta(z)$ of the asymptotic eigenvalue distribution of $\mathbf{A}^T \mathbf{A}$ [14], given by

$$\eta(z) = \lim_{M=\delta N \rightarrow \infty} \sum_{k=0}^{\infty} \mu_k (-z)^k. \quad (20)$$

Theorem 2: Let $G_k(y)$ denote the generating function of $\{g_t^{(k)}\}$ with respect to $t = 0, 1, \dots$, defined as

$$G_k(y) = \sum_{t=0}^{\infty} y^t g_t^{(k)}. \quad (21)$$

Then, $G_1(y)$ is implicitly given by $\eta(x_s) = 1 - y$ in the large system limit, with

$$x_s = \frac{y}{(1-y)\{1 - yG_1(y)\}}. \quad (22)$$

Proof: Define the generating function of $\{g_t^{(k)}\}$ as

$$G(x, y) = \sum_{k=0}^{\infty} x^k G_k(y). \quad (23)$$

Theorem 2 follows from the following closed-form expression of $G(x, y)$:

$$G(x, y) \stackrel{\text{a.s.}}{=} \frac{x\eta(-x)\{yG_1(y) - 1\} + \eta(-x) - 1}{(1-y)\{1 - yG_1(y)\}x + y} + o(1). \quad (24)$$

By definition, $G(x, y)$ is a polynomial of x and y . Thus, the numerator of (24) must be zero when the denominator is zero. The point $-x_s$ given in (22) is a zero of the denominator for any y . Thus, we let the numerator at $x = -x_s$ be zero to obtain $\eta(x_s) = 1 - y$. Thus, we arrive at Theorem 2.

To complete the proof of Theorem 2, we shall prove (24). We first derive a closed-form expression of (21), given by

$$G_k(y) = g_0^{(k)} + g_1^{(k)}y + \sum_{t=2}^{\infty} y^t g_t^{(k)}. \quad (25)$$

Substituting (8) into the last term on the RHS of (25) yields

$$\begin{aligned} \sum_{t=2}^{\infty} y^t g_t^{(k)} &= \sum_{t=2}^{\infty} y^t g_{t-1}^{(k)} - \sum_{t=2}^{\infty} y^t g_{t-1}^{(k+1)} + \sum_{t=2}^{\infty} y^t g_{t-1}^{(1)} \mu_{k+1} \\ &\quad + \sum_{t=2}^{\infty} y^t \sum_{\tau=1}^{t-1} g_{t-\tau-1}^{(1)} g_{\tau}^{(k)} - \sum_{t=2}^{\infty} y^t \sum_{\tau=1}^{t-1} g_{t-\tau-1}^{(1)} g_{\tau-1}^{(k)}. \end{aligned} \quad (26)$$

For the first three terms, we have

$$\sum_{t=2}^{\infty} y^t g_{t-1}^{(k')} = y \sum_{t=1}^{\infty} y^t g_t^{(k')} = yG_{k'}(y) - g_0^{(k')}y \quad (27)$$

for $k' = 1, k, k + 1$. Since the Z-transform of convolution is the product of Z-transforms, the last term reduces to

$$\begin{aligned} \sum_{t=2}^{\infty} y^t \sum_{\tau=1}^{t-1} g_{t-\tau-1}^{(1)} g_{\tau}^{(k)} &= y^2 \sum_{t=0}^{\infty} y^t \sum_{\tau=0}^t g_{t-\tau}^{(1)} g_{\tau}^{(k)} \\ &= y^2 G_1(y) G_k(y). \end{aligned} \quad (28)$$

Similarly, for the fourth term we have

$$\begin{aligned} \sum_{t=2}^{\infty} y^t \sum_{\tau=1}^{t-1} g_{t-\tau-1}^{(1)} g_{\tau}^{(k)} &= \sum_{t=1}^{\infty} y^{t+1} \sum_{\tau=1}^t g_{t-\tau}^{(1)} g_{\tau}^{(k)} \\ &= yG_1(y) \{G_k(y) - g_0^{(k)}\}. \end{aligned} \quad (29)$$

Using these results, as well as (6) and (7), we obtain the closed-form expression

$$G_k(y) = \frac{\mu_k y G_1(y) - y G_{k+1}(y) + g_0^{(k)}}{(1-y)\{1 - y G_1(y)\}}. \quad (30)$$

The closed-form expression (24) follows from (30). Using the η -transform (20) yields

$$G(x, y) \stackrel{\text{a.s.}}{=} \frac{\eta(-x)yG_1(y) - x^{-1}yG(x, y) + G(x, 0)}{(1-y)\{1 - yG_1(y)\}} + o(1), \quad (31)$$

where we have used $G_0(y) \stackrel{\text{a.s.}}{\rightarrow} 0$ obtained from Theorem 1. Applying $G(x, 0) = x^{-1}\{\eta(-x) - 1\} - \eta(-x)$ obtained from (6) and solving $G(x, y)$, we arrive at (24). ■

The following corollary implies that the CAMP reduces to conventional AMP when the sensing matrix has i.i.d. Gaussian elements with mean proportional to $M^{-1/2}$. Thus, the CAMP has no ability to handle this non-zero mean case.

Corollary 1: If \mathbf{A} has independent Gaussian elements with mean $\sqrt{\gamma/M}$ and variance $(1-\gamma)/M$ for any $\gamma \in [0, 1)$, the CAMP is equivalent to conventional AMP.

Proof: The R-transform $R(z)$ [14, Section 2.4.2] of the asymptotic eigenvalue distribution of $\mathbf{A}^T \mathbf{A}$ is given by

$$R(z) = \frac{\delta}{\delta - z}. \quad (32)$$

Using Theorem 2 and the following relationship between the η and R transforms:

$$\eta(z) = \frac{1}{1 + zR(-z\eta(z))}, \quad (33)$$

we obtain

$$1 - y = \frac{1}{1 + \delta x_s \{\delta + x_s(1 - y)\}^{-1}}, \quad (34)$$

where x_s is given by (22). Substituting (22) and solving $G_1(y)$, we arrive at $G_1(y) = \delta^{-1}$.

From the definition (21), we find $g_0^{(1)} = \delta^{-1}$ and $g_t^{(1)} = 0$ for all $t > 0$. This implies that the update rule (4) reduces to that corresponding to conventional AMP. ■

The following corollary is utilized in numerical simulations.

Corollary 2: If \mathbf{A} is orthogonally invariant and has non-zero singular values $\sigma_0 \geq \dots \geq \sigma_{M-1} > 0$ satisfying condition number $\kappa = \sigma_0/\sigma_{M-1} \geq 1$, $\sigma_m/\sigma_{m-1} = \kappa^{-1/(M-1)}$, and $\sigma_0^2 = N(1 - \kappa^{-2/(M-1)})/(1 - \kappa^{-2M/(M-1)})$, then $g_t^{(1)} = g_t + C/(\kappa^2 - 1)$ holds for all t , with

$$g_t = \sum_{\tau=0}^{t-1} h_{t-\tau} g_\tau - h_{t+1}, \quad g_0 = -h_1, \quad (35)$$

$$h_t = \frac{C^{t-1}}{t!} - \frac{C^t}{(t+1)!}, \quad C = \frac{2}{\delta} \ln \kappa. \quad (36)$$

Proof: Since $\mu_k = N^{-1} \sigma_0^{2k} (1 - \kappa^{-2kM/(M-1)}) / (1 - \kappa^{-2k/(M-1)})$ holds for all $k > 0$, we use (20) and $N(1 - \kappa^{-a/(M-1)}) \rightarrow \delta^{-1} a \ln \kappa$ for any $a \in \mathbb{R}$ to find

$$\begin{aligned} \eta(z) &= 1 + \sum_{k=1}^{\infty} (-z)^k \left\{ \frac{C}{(1 - \kappa^{-2})} \right\}^k \frac{(1 - \kappa^{-2k})}{kC} \\ &= 1 - \frac{1}{C} \ln \left\{ \frac{\delta(\kappa^2 - 1) + 2\kappa^2 z \ln \kappa}{\delta(\kappa^2 - 1) + 2z \ln \kappa} \right\}, \end{aligned} \quad (37)$$

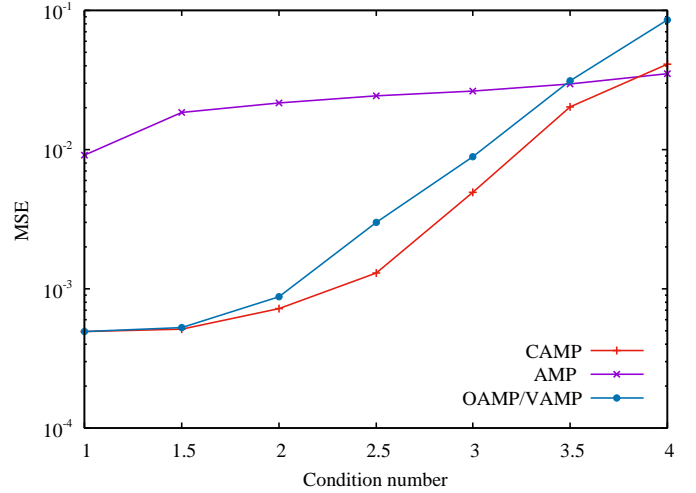


Fig. 1. MSE versus the condition number σ_0/σ_{M-1} for signal density $\rho = 0.1$, 100 iterations, $M = 614$, $N = 2^{10}$, and $1/\sigma^2 = 30$ dB.

where the second equality follows from $\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k-1} k^{-1} x^k$ for all $|x| < 1$. Using Theorem 2 yields

$$G_1(y) = \frac{C}{(\kappa^2 - 1)(1 - y)} + \frac{1}{y} + \frac{C}{(1 - y)(1 - e^{Cy})}. \quad (38)$$

It is an exercise to confirm that the generating function of g_t in Corollary 2 is equal to the sum of the second and last terms. Thus, Corollary 2 holds. ■

IV. NUMERICAL SIMULATION

The CAMP is compared to AMP and OAMP/VAMP in terms of the mean-square error (MSE) in signal recovery. As an example of ill-conditioned sensing matrices in Corollary 2, $\mathbf{A} = \text{diag}\{\sigma_0, \dots, \sigma_{M-1}\} \mathbf{H}$ is considered for $M \leq N$, with σ_m denoting the m th singular value in Corollary 2. The M rows of $\mathbf{H} \in \mathbb{R}^{M \times N}$ are selected uniformly and randomly from the rows of the $N \times N$ Hadamard orthogonal matrix.

We assume the Bernoulli-Gaussian (BG) prior: Each signal element takes 0 with probability $1 - \rho$. Otherwise, it is sampled from the zero-mean Gaussian distribution with variance ρ^{-1} . We use the soft thresholding [1]

$$f_t(x) = \begin{cases} x - \theta_t & \text{for } x \geq \theta_t, \\ 0 & \text{for } x \in (-\theta_t, \theta_t), \\ x + \theta_t & \text{for } x \leq -\theta_t. \end{cases} \quad (39)$$

For the sensing matrix in Corollary 2, we have no SE results of the CAMP or AMP for designing the threshold θ_t . Thus, the threshold θ_t is fixed to a constant θ over all iterations, which was optimized via an exhaustive search.

Figure 1 shows the MSEs of the CAMP, AMP, and OAMP/VAMP estimated from 10^5 independent trials. The CAMP outperforms AMP and achieves the MSEs comparable to OAMP/VAMP. The inferior performance of AMP is due to a bad convergence property of AMP. Using a large threshold θ improves the convergence property. Exhaustive search of θ implied that larger thresholds are required for AMP to converge than for the other algorithms. Thus, we conclude that CAMP improves the convergence property of AMP.

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