Quickest Sequence Phase Detection

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Abstract—A phase detection sequence is a length-n cyclic sequence, such that the location of any length-k contiguous subsequence can be determined from a noisy observation of that subsequence. In this paper, we derive bounds on the minimal possible k in the limit of $n \to \infty$, and describe some sequence constructions. We further consider multiple phase detection sequences, where the location of any length-k contiguous subsequence of each sequence can be determined simultaneously from a noisy mixture of those subsequences. We study the optimal trade-offs between the lengths of the sequences, and describe some sequence constructions. We compare these phase detection problems to their natural channel coding counterparts, and show a strict separation between the fundamental limits in the multiple sequence case. Both adversarial and probabilistic noise models are addressed.

I. INTRODUCTION

A magician enters the room with a 32-card deck. He invites five volunteers to the stage and claims he will read their minds. Another volunteer is asked to cut the deck a few times and pass the top five cards to the volunteers, one for each. "Now I need you to think about your card and I will tell what it is," the magician says. Silence. "Please concentrate! Think harder." A long pause. "Okay, the weather is not good today. It is interfering with the brainwaves between us. I need you to work with me a bit," the magician begs. "Could the people with red cards move one step closer to me?" Another long pause. "Hmm, you have the six of clubs. You have the five of spades..." Sure enough, he gets them all!

This is Diaconis' mind-reading trick [1], [2]. The magic makes use of a binary de Bruijn sequence of order 5 [3], which is a length-32 circulant binary sequence such that every length-5 binary string occurs as a contiguous subsequence exactly once. The magician enters the room with the 32 cards prearranged such that their color (black/red) corresponds to the de Bruijn sequence. Cutting the deck only shifts the sequence cyclically. By the property of de Bruijn sequence, knowing the colors reveals the location (or *phase*) of the 5 contiguous cards inside the deck, hence uniquely determines their identities. More generally, this trick can be performed with k volunteers and a deck of size $n = 2^k$, by using a de Bruijn sequence of order k, which is a binary sequence such that every length-k binary string occurs as a contiguous subsequence exactly once [3].

Suppose now that some of the volunteers are not collaborative and may lie when asked about their card color. Can the magician still guess the cards correctly? In other words, can one design a length-n sequence such that the set of all length-k contiguous subsequences forms a good error-correcting code? Besides its appeal as a card trick, such a sequence can also be useful e.g. for phase detection in positioning systems. Imagine that a satellite sends the length-n sequence periodically. A user hearing a noisy chunk of the sequence would like to figure out the location of his chunk within the original sequence, so as to measure the transmission delay and compute his distance to the satellite. Fixing the sequence length n (which results in a given ambiguity of the distance estimation), it is clearly desirable to minimize k, as this results in the fastest positioning. Clearly, k cannot be smaller than $\log n$, and this lower bound can be achieved in case there is no noise, by using a de Bruijn sequence of order k. As we shall see, in the noisy case $k = O(\log n)$ is also sufficient, and we will in fact be interested in characterizing the exact constant $\frac{\log n}{k}$, which will be referred to as *rate*.

In reality, positioning systems typically employ multiple satellites, each transmitting its own length-n_i sequence. Sequences get combined through a multiple access channel (MAC) when reaching the user. Upon hearing a length-k chunk of the combined sequence, the user wishes to measure his distance to all of the satellites by locating the chunk within each one of the sequences. We note that existing techniques (such as GPS [4]) typically employ sequences (e.g. Gold codes [5]) that possess good autocorrelation and cross-correlation properties, and use $k = N \cdot n_1 = \cdots = N \cdot n_L$, for some repetition factor $N \geq 1$. From our perspective, these systems hence operate at zero rates. In fact, when the repetition factor $N > 1$, this does not precisely fall under our setup; we further remark on this in Example 1. In what follows, we focus on fast positioning at non-zero rates. We are interested in characterizing the optimal trade-offs among $\left(\frac{\log n_1}{k}, \cdots, \frac{\log n_L}{k} \right)$ that ensure successful detection, as well as in constructing sequences that achieve the optimal trade-offs.

In what follows, we refer to the first problem, which only involves a single-sequence design, as *point-to-point phase detection*. We refer to the second problem as *multiple access phase detection*. Different noise models are considered: the adversarial noise and the probabilistic noise. For the probabilistic noise, different error criteria are discussed: the vanishing error criterion and the zero error criterion. These models are defined formally in the sequel. We also compare the phase detection problems to their natural channel coding counterparts.

A. Point-to-Point Phase Detection

In Sections II, III, and IV, we consider point-to-point phase detection.

An (n, k) *point-to-point phase detection scheme* consists of

- a sequence $x^n \triangleq (x_1, x_2, \ldots, x_n) \in \mathcal{X}^n$, and
- a detector $\hat{m} \colon \mathcal{Y}^k \to [n] \cup e$, where $[n] \triangleq \{1, 2, \ldots, n\}$ and e is an error symbol.

We assume that the detector observes a noisy version y^k of the sequence x_m^{m+k-1}, and attempts to correctly identify the *phase* m. Clearly, any reliable scheme would require $k \geq \log_{|\mathcal{X}|} n$. Thus, it is natural to define the *efficiency* of a scheme as the excess multiplicative factor it uses over the minimal possible, i.e., $k/\log_{|\mathcal{X}|} n$. However, for comparison to channel coding, it would be more convenient to work with the inverse of this quantity and take logarithms in base 2, namely work with the *rate*

$$R \triangleq \frac{\log_2 n}{k}.$$

We note that any phase detection scheme induces a *codebook*[1] $\mathcal{C} = \{x_m^{m+k-1} \colon m \in [n]\} \subseteq \mathcal{X}^k$ of rate R. Here and throughout indices are taken cyclically, modulo the set $[n]$. Also, we assume throughout that $k \leq n$.

We discuss three distinct models: the adversarial noise model in Section II, the probabilistic noise with vanishing error in Section III, and the probabilistic noise with zero error in Section IV. For convenience, let the function $\phi(m; x^n)$ return the length-k contiguous subsequence of x^n starting at phase m, i.e., $\phi(m; x^n) = x_m^{m+k-1}$. We will typically omit the dependence on the sequence x^n, and simple write $\phi(m; x^n) = \phi(m)$.

For the adversarial noise model, we assume that $\mathcal{X} = \mathcal{Y} = \{0, 1\}$ and the observation sequence y^k is obtained from $\phi(m)$ by flipping at most pk bits, where m is the correct phase, and p is fixed and given. We define the *minimum distance* of a scheme as the minimum Hamming distance of its induced codebook. A rate R is said to be *achievable* in this setting if, for a divergent sequence of k's, there exist (n, k) schemes with $\frac{\log n}{k} \geq R$, such that m can be recovered from y^k without error. Namely, we require the scheme to have a minimum distance $d > 2pk$. The *capacity* of adversarial phase detection $C_{\mathrm{ad}}(p)$ is defined as the supremum over all achievable rates[2].

Several works have addressed this noise model in the literature. The trade-off between the rate and the minimum distance of the code was studied in [6], [7]. Kumar and Wei provided a lower bound on d in the regime of $d \leq \sqrt{k}$ for *m-sequences*, which are generated by linear feedback shift registers [6]. Some explicit sequence constructions were also provided in [8]–[12]. By a concatenation of an optimal binary channel code with the Reed–Solomon code, Berkowitz and Kopparty have recently constructed a phase detection scheme with nonzero rate and nonzero relative distance [12]. For

generalization to two dimensional phase detection, see [13]–[16].

In Section II, we focus on the tradeoff between the rate and the minimum distance in the asymptotic limit. We note that a codebook induced by any phase detection scheme can be used as a channel code in the standard binary adversarial channel model [17]. The capacity of the latter setup is unknown. Clearly however, any upper bound for that capacity, such as the MRRW upper bound [18], also serves as an upper bound for $C_{\mathrm{ad}}(p)$. The best known binary adversarial channel coding lower bound is given by Gilbert and Varshamov [19], [20]. Applying the Lovász local lemma [21], we show in Section II-A that this rate is also achievable for adversarial phase detection. In Section II-B, we characterize the family of *linear* phase detection schemes and study their performance.

For the probabilistic noise model with vanishing error criterion, we assume that the phase is uniformly distributed, i.e., $M \sim \mathrm{Unif}[n]$. We further assume that the noisy observation y^k is obtained from $\phi(m)$ via a discrete memoryless channel $p(y|x)$. The probability of error is defined as

$$P_e^{(k)} = \mathsf{P}\{M \neq \hat{m}(Y^k)\}.$$

A rate R is said to be *achievable* if, for a divergent sequence of k's, there exist (n, k) schemes with $\frac{\log n}{k} \geq R$ and $\lim_{k \to \infty} P_e^{(k)} = 0$. The *vanishing error capacity* of probabilistic phase detection C_{ve} is defined as the supremum over all achievable rates.

As before, the codebook induced by any phase detection scheme is also a channel code. Thus, the Shannon capacity of the channel $p(y|x)$ is an upper bound for C_{ve}. In Section III-A, we show that in fact C_{ve} equals the Shannon capacity. Moreover, we present in Section III-B a concatenated construction with $O(k \log k)$ complexity that achieves the capacity of probabilistic phase detection. As a consequence, this construction also establishes the equivalence between channel coding and phase detection for this noise model.

For the probabilistic noise model with zero error criterion, we again assume that the noisy observation y^k is obtained from $\phi(m)$ via a discrete memoryless channel $p(y|x)$. A rate R is said to be *achievable* if, for a divergent sequence of k's, there exist (n, k) schemes with $\frac{\log n}{k} \geq R$ such that the phase m can be recovered with *zero error* for any $m \in [n]$. Similar to Shannon's zero error channel coding [22], achievable rates can be equivalently defined on the *confusion graph* $G = (\mathcal{X}, E)$ associated with the channel $p(y|x)$. Here the vertex set is $\mathcal{X}$ and two distinct vertices are connected $(u, v) \in E$ if they may result in the same output, i.e., there exists a $y \in \mathcal{Y}$ such that $p_{Y|X}(y|u) > 0$ and $p_{Y|X}(y|v) > 0$. Let $G^k = (\mathcal{X}^k, E_k)$ be the k-fold *strong product* of G, where two distinct vertices are connected $(u^k, v^k) \in E_k$ if for all $i \in [k]$, either $u_i = v_i$ or $(u_i, v_i) \in E$. Then, a rate R is achievable if and only if, for a divergent sequence of k's, there exist (n, k) schemes with $\frac{\log n}{k} \geq R$ such that $(\phi(m), \phi(m')) \notin E_k$ for any two distinct phases $m, m' \in [n]$, or in other words, the induced codebook forms an independent set of G^k. The *zero error capacity* $C_{\mathrm{ze}}(G)$ is defined as the supremum over all achievable rates.

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[1]The codebook is treated as a multiset, namely there might be repetitions in its elements.

[2]Here we define capacity asymptotically. Note that similarly to adversarial channel coding, it is not guaranteed short sequences with rate above the capacity do not exist.

We note the distinction between phase detection and channel coding under the zero error criterion. For zero error channel coding (in contrast to vanishing error and adversarial channel coding) if a rate R is achievable at some length k, it is also achievable for all multiples of k (by concatenation) and thus for a divergent sequence of k's. However, this argument cannot be applied to the phase detection setting, since concatenating the codewords of two induced codebooks may not necessarily result in a new codebook that can be chained up into a single sequence. Nevertheless, and despite the fact that the zero error channel capacity is generally unknown, we show in Section IV that the zero error capacity for phase detection coincides with its channel coding counterpart.

B. Multiple Access Phase Detection

In Sections V and VI, we consider multiple access phase detection. We only discuss the two-user case for simplicity. But all the results extend to more users.

An (n_1, n_2, k) *multiple access phase detection scheme* consists of

- two sequences $x_1^{n_1} = (x_{11}, x_{12}, \ldots, x_{1,n_1}) \in \mathcal{X}_1^n$ and $x_2^{n_2} = (x_{21}, x_{22}, \ldots, x_{2,n_2}) \in \mathcal{X}_2^n$, and
- a detector that declares two phase estimates $\hat{m}_1 \colon \mathcal{Y}^k \to [n_1] \cup \{e\}$ and $\hat{m}_2 \colon \mathcal{Y}^k \to [n_2] \cup \{e\}$.

We assume that the detector observes y^k, which is the output of a discrete memoryless multiple access channel $(\mathcal{X}_1 \times \mathcal{X}_2, p(y|x_1, x_2), \mathcal{Y})$ with the two inputs $\phi_1(m_1) = \phi_1(m_1; x_1^{n_1}) \triangleq (x_{1,m_1}, x_{1,m_1+1}, \ldots, x_{1,m_1+k-1})$ and $\phi_2(m_2) = \phi_2(m_2; x_2^{n_2}) \triangleq (x_{2,m_2}, x_{2,m_2+1}, \ldots, x_{2,m_2+k-1})$, and attempts to correctly identify the *phases* (m_1, m_2). Similar to the point-to-point case, we define the *rates* of the two sequences as

$$R_1 \triangleq \frac{\log_2 n_1}{k} \quad \text{and} \quad R_2 \triangleq \frac{\log_2 n_2}{k}.$$

We note that every multiple access phase detection scheme induces two (multiset) *codebooks*

$$\mathcal{C}_1 = \{\phi_1(m_1) \colon m_1 \in [n_1]\} \subseteq \mathcal{X}_1^k \tag{1}$$

and

$$\mathcal{C}_2 = \{\phi_2(m_2) \colon m_2 \in [n_2]\} \subseteq \mathcal{X}_2^k \tag{2}$$

of rates R_1 and R_2 respectively.

We discuss two different error criteria: the vanishing error criterion in Section V and the zero error criterion in Section VI.

Under the vanishing error criterion, we assume that the phase pair (M_1, M_2) is uniformly distributed over $[n_1] \times [n_2]$. The probability of error is defined as

$$P_e^{(k)} = \mathsf{P}\{(M_1, M_2) \neq (\hat{m}_1(Y^k), \hat{m}_2(Y^k))\}.$$

A rate pair (R_1, R_2) is said to be *achievable* if, for a divergent sequence of k's, there exist (n_1, n_2, k) schemes with $\frac{\log n_1}{k} \geq R_1$, $\frac{\log n_2}{k} \geq R_2$, and $\lim_{k \to \infty} P_e^{(k)} = 0$. The *vanishing error capacity region* $\mathscr{C}_{\text{ve}}$ is defined as the closure of the set of achievable rate pairs.

In Section V-A, we establish the vanishing error capacity region of multiple access phase detection. This region turns out to be *strictly* included, in general, in the capacity region of its channel coding counterpart. This is in contrast to all models in the point-to-point case, in which phase detection either achieves the same best known rate or shares the same capacity as its channel coding counterpart. Due to the lack of synchronization between sequences, a phase detection scheme achieves at best the usual MAC capacity region without the time-sharing random variable. In Section V-B, we provide a low-complexity ($O(k \log k)$) sequence construction that achieves any rate pair in the capacity region.

Under the zero error criterion, a rate pair (R_1, R_2) is said to be *achievable* if, for a divergent sequence of k's, there exist (n_1, n_2, k) schemes with $\frac{\log n_1}{k} \geq R_1$ and $\frac{\log n_2}{k} \geq R_2$ such that (m_1, m_2) can be recovered from y^k with *zero error* for any pair $(m_1, m_2) \in [n_1] \times [n_2]$. The *zero error capacity region* $\mathscr{C}_{\text{ze}}$ is defined as the closure of the set of achievable rate pairs. We note that the problem of zero error phase detection in MACs is generally very difficult, as it is at least as hard as the zero error MAC coding problem, which in turn is open even in the simplest cases, e.g., the binary adder channel [23]–[28]. Nevertheless, in Section VI-A, we demonstrate the distinction between the phase detection and the channel coding problems, by showing a separation between their capacity regions.

In Sections VI-B and VI-C, we restrict our attention to a simple channel model, the modulo-2 addition channel with $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{Y} = \{0, 1\}$ and $Y = X_1 \oplus X_2$. For this channel, a rate pair (R_1, R_2) is achievable if every element in the sumset

$$\mathcal{C}_{\text{sum}} \triangleq \{\phi_1(m_1) \oplus \phi_2(m_2) \colon m_1 \in [n_1], m_2 \in [n_2]\} \tag{3}$$

can be uniquely expressed as an element in the induced codebook $\mathcal{C}_1$ plus an element in the induced codebook $\mathcal{C}_2$. Note that $\mathcal{C}_{\text{sum}}$ is defined as a regular set with distinct elements (rather than a multiset). Hence, any $\mathcal{C}_1$ and $\mathcal{C}_2$ induced by a valid scheme must also have distinct elements.

Clearly, the zero-error channel coding capacity region $\{(R_1, R_2) \colon R_1 + R_2 \leq 1\}$ is an outer bound for that of phase detection. In Section VI-B, we establish the achievability of this region by a random construction that exploits properties of linear codes, in a way that resembles Wyner's linear Slepian–Wolf codes [29]. We further provide in Section VI-C an explicit sequence construction that achieves this region, by exploiting properties from finite field theory. As an consequence, the induced code from our phase detection sequences can be used for channel coding and achieve any rate pair in the zero-error capacity region, *without using time sharing*[3].

II. POINT-TO-POINT: ADVERSARIAL NOISE

In this section we discuss the adversarial noise model. We first examine whether the adversarial phase detection schemes achieve the best known rate for adversarial channel coding, namely the Gilbert–Varshamov (GV) bound [19], [20].

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[3] For other channel codes that achieve this region without using time sharing, see for example [30], [31]. For a channel code that achieves the rate pair $(1/2, 1/2)$ with the *same* codebook, see [23], [32] for a construction utilizing the parity check matrix of a BCH code.

A. Fundamental Limit

Theorem 1. *An (n, k) point-to-point phase detection scheme with minimum distance d exists if*

$$n \leq \frac{2^k}{16k \sum_{i=0}^{d} \binom{k}{i}}. \tag{4}$$

Corollary 1. *The capacity for adversarial phase detection is lower bounded by*

$$C_{ad}(p) \geq 1 - h(2p),$$

where $h(\cdot)$ is the binary entropy function.

We show the existence of a good sequence using the probabilistic method. We note that while several different proofs of the GV bound exist [19], [20], [33], none of them seem to directly extend to our setting. This is simply due to the fact that there is a dependence between the codewords in the induced codebook. To alleviate this technical difficulty, we need the following well-known lemma.

Lemma 1 (Lovász Local Lemma [21]). *Let $A_1, \ldots, A_N$ be a set of "bad" events with $\mathsf{P}(A_j) \leq q < 1$, where each event A_j is mutually independent of all but at most L of the other events. If $4qL \leq 1$, then*

$$\mathsf{P}\left\{\cap_{j=1}^{N} A_j^c\right\} > 0.$$

Proof of Theorem 1: We generate the phase detection sequence X^n i.i.d. $\sim \mathrm{Bern}(1/2)$ and apply minimum distance detection. Let $\{A_j\}$ be the collection of events where the Hamming distance between a pair of codewords $\mathrm{wt}\big(\phi(m_1) \oplus \phi(m_2)\big) \leq d$ where $m_1 < m_2$. We have

$$\mathsf{P}(A_j) \stackrel{(a)}{=} \mathsf{P}\{\mathrm{wt}(Z^k) \leq d, Z^k \text{ i.i.d.} \sim \mathrm{Bern}(1/2)\}$$
$$= \sum_{i=0}^{d} \binom{k}{i} \frac{1}{2^k},$$

where (a) follows since for any two distinct phases $m_1 \neq m_2$, the sum of the two codewords $\phi(m_1) \oplus \phi(m_2)$ is i.i.d. $\sim \mathrm{Bern}(1/2)$ even if they are overlapping subsequences of X^n. Now each A_j is mutually independent of all other events, except for a set of at most $4kn$ events. This is because the random variable $\phi(m_1) \oplus \phi(m_2)$ is mutually independent of all X_i's with $i \in [n] \setminus \{m_1 - k + 1, m_1 - k + 2, \ldots, m_1 + k - 1\} \setminus \{m_2 - k + 1, m_2 - k + 2, \ldots, m_2 + k - 1\}$, which excludes at most $4kn$ events. Applying Lemma 1, the phase detection sequence X^n has minimum distance greater than d with positive probability

$$\mathsf{P}\left\{\cap_{j=1}^{\frac{1}{2}n(n-1)} A_j^c\right\} > 0$$

if

$$16kn \sum_{i=0}^{d} \binom{k}{i} \frac{1}{2^k} \leq 1, \tag{5}$$

or equivalently the condition in (4). This completes the proof of Theorem 1. ∎

Proof of Corollary 1: Set $d = 2pk$ in (4). Applying the Hamming ball volume approximation

$$\sum_{i=0}^{2pk} \binom{k}{i} \leq 2^{kh(2p)}$$

and plugging $R = \frac{\log n}{k}$ in (5), we have

$$R \leq 1 - h(2p) - \frac{\log(16k)}{k}.$$

Letting $k \to \infty$, it follows that a rate R is achievable if $R < 1 - h(2p)$. ∎

Remark 1. In the standard channel coding setup, a random codebook attains the GV bound with high probability. In contrast, the probability of randomly drawing a good scheme for our setup is exponentially small. This is most obvious in the noiseless case ($p = 0$), where it is well known that the fraction occupied by de Bruijn sequences among all sequences vanishes exponentially fast [3].

B. Linear Phase Detection Schemes

Theorem 1 and Corollary 1 showed the existence of a good adversarial phase detection scheme. Now, we discuss explicit constructions of such schemes. First, we ask whether phase detection schemes are "equivalent" to error-correcting codes in a certain sense. Clearly, any adversarial phase detection scheme induces a codebook that can be used as an error-correcting code for the corresponding adversarial channel coding problem. The converse direction seems more challenging. Given an error-correcting code, is it possible to "chain up" all or a sizable fraction of its codewords to create a sequence, and use the decoding rule as the detector? If so, what structure should such a code possess? In the following, we answer these questions for the class of *linear* error-correcting codes.

First, we note that in order to induce any error-correcting code with minimum distance $d > 1$, the phase detection sequence x^n should not contain 0^k as a contiguous subsequence, for otherwise a shift by one from that position would create a codeword that is at distance 1 from 0^k. Following that, an (n, k) phase detection scheme is said to be *linear* if $\mathcal{C} \cup \{0^k\}$, namely its induced codebook together with the zero codeword, forms a linear code. Let r be the dimension of this linear code. Then, the length of the linear phase detection sequence is $n = 2^r - 1$.

Theorem 2. *A phase detection scheme with $n = 2^r - 1$ is linear if and only if it is generated by a linear feedback shift register (LFSR) with a primitive characteristic polynomial $a(z) = \sum_{i=0}^{r-1} a_i z^i + z^r$ over $GF(2)$, i.e.,*

$$x_{r+j} = \sum_{i=0}^{r-1} a_i x_{i+j}, \quad j \in [n]. \tag{6}$$

Corollary 2. *The non-zero codewords of a linear code of dimension r can be chained up to a sequence of length $2^r - 1$ if*

and only if any r contiguous columns of the generator matrix $G_{r \times k} = [\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_k]$ are linearly independent, and

$$\mathbf{g}_{r+j} = \sum_{i=0}^{r-1} a_i \mathbf{g}_{i+j}, \quad j \in [k-r], \tag{7}$$

where a_i's are the coefficients of a primitive polynomial $a(z) = \sum_{i=0}^{r-1} a_i z^i + z^r$ over $GF(2)$.

Proof of Theorem 2: To prove sufficiency, suppose that x^n is generated by an LFSR with a primitive characteristic polynomial in (6) and a nonzero initial state vector $(x_1, x_2, \ldots, x_r)$. Then, every length-r string except 0^r occurs exactly once in x^n (see [34, Theorem 8.33]). It follows that for any distinct codewords $\phi(m_1) = c^k$ and $\phi(m_2) = d^k$, there exists $\phi(m_3) = e^k$ such that $c_j + d_j = e_j$ for $j \in [r]$. For $r < j \leq k$, $c_j + d_j = e_j$ follows since the sequence is generated by an LFSR of degree r.

For necessity, let x^n be a sequence associated with a linear phase detection scheme. We show that the first r columns $\mathbf{g}_1, \ldots, \mathbf{g}_r$ of the generator matrix $G_{r \times k} = [\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_k]$ must be linearly independent. Assuming that contrary, there exist $f_1, \ldots, f_r \in \{0, 1\}$ not all zero such that

$$\sum_{i=1}^{r} f_i \mathbf{g}_i = \mathbf{0}. \tag{8}$$

Let $[x_1, \ldots, x_k] = [u_1, \ldots, u_r] G_{r \times k}$. Multiplying both sides of (8) by $[u_1, \ldots, u_r]$, we have $\sum_{i=1}^{r} f_i x_i = 0$. Applying this to every codeword in $\mathcal{C}$, and recalling that the codewords are all contiguous subsequences of x^n, we have

$$\sum_{i=1}^{r} f_i x_{i+j} = 0, \quad j \in [n].$$

Let $i_0 = \max\{i \in [r] : f_i = 1\}$. If $i_0 = 1$, then x^n has to be 0^n, in contradiction. For $i_0 > 1$, we have

$$x_{j+i_0} = \sum_{i=1}^{i_0-1} f_i x_{i+j}, \quad j \in [n],$$

which implies x^n is generated by an LFSR of degree $i_0 - 1 < r$. But this contradicts the fact that x^n is of length $2^r - 1$ and all codewords $\phi(m), m \in [n]$, are distinct.

Now, since the first r columns of $G_{r \times k}$ are linearly independent, there exist $a_0, \ldots, a_{r-1}$ such that

$$\mathbf{g}_{r+1} = \sum_{i=0}^{r-1} a_i \mathbf{g}_{i+1}.$$

From this it follows that (6) holds and x^n is generated by an LFSR. Finally, an LFSR sequence is of maximum length if and only if the characteristic polynomial is primitive. ∎

Proof of Corollary 2: The sufficiency follows since for a linear code, the relation (7) implies (6). The necessity follows the same way as the necessity in Theorem 2. ∎

Remark 2. As an application of Theorem 2, we can design a card trick for adversarial crowds. Picking the primitive polynomial $a(z) = z^5 + z^4 + z^2 + z + 1$ and $k = 9$, we get a sequence of length $n = 31$ and minimum distance $d = 3$.

Ordering cards according to this sequence, the magician can now correct one lie out of 9 contiguous color reads.

Remark 3. When the characteristic polynomial of the LFSR is irreducible but not primitive, the sequence it generates has length t, which equals the order of the characteristic polynomial. Depending on the initial state x^r, the LFSR generates one out of $s = \frac{2^r - 1}{t}$ disjoint sequences $x^t(1), \ldots, x^t(s)$. The length-k contiguous subsequences of each sequence $\mathcal{C}^{(i)} = \{x_m^{m+k-1}(i) : m \in [t]\}$ together with the zero codeword form a linear code $\cup_{i=1}^{s} \mathcal{C}^{(i)} \cup \{0^k\}$. Conversely for a linear code, if the first r columns of its generator matrix are linearly independent and (7) holds with a_i's being the coefficients of an irreducible but not primitive polynomial of order t, then its nonzero codewords can be partitioned into s equal size subsets, each of which can be chained up to a phase detection sequence.

We now provide two results on the performance of linear phase detection schemes. In Theorem 3, we cite a known result from [34, Theorem 8.85] on asymptotic relative distance, which improves upon [6, Theorem 1]. Then, inspired by a linear programing bound for LDPC codes [35], we provide in Theorem 4 an upper bound on the sequence length of a linear phase detection scheme of a given minimum distance, using the linear programing method originated by Delsarte [36].

Theorem 3 (Theorem 8.85 [34]). *For every (n, k) linear phase detection scheme, for every $m \in [n]$,*

$$\left| \mathrm{wt}(x_m^{m+k-1}) - \frac{k}{2} \right| \leq \sqrt{n} \left(\frac{\log n}{\pi} + 1 \right).$$

In particular, for (n, k) such that $\lim_{k \to \infty} \frac{\sqrt{n} \log n}{k} = 0$, the relative distance of the induced code converges to

$$\lim_{k \to \infty} \frac{d}{k} = \frac{1}{2}. \tag{9}$$

Remark 4. We note a similar result in [6, Theorem 1], which claims (9) for every $0 < \mu \leq 1$ and $k = \mu n$. Theorem 3 improves upon [6] by allowing k to be sublinear in n.

For the next result, we need the following definitions. For $t \in [k]$ and $z \in \mathbb{R}$, let

$$K_t(z) = \sum_{j=0}^{t} (-1)^j \binom{z}{j} \binom{l-z}{t-j}$$

be the Krawtchouk polynomial [37, Ch. 5. § 2] [38], where the binomial coefficient for $z \in \mathbb{R}$ is defined as $\binom{z}{i} = \frac{z(z-1)(z-2)\cdots(z-i+1)}{i!}$. For large k, the exponent of $K_t(z)$ can be approximated as [35, Equation (40)]

$$\frac{1}{k} \log K_{\lfloor pk \rfloor}(\lfloor \lambda k \rfloor) = h(p) + \mathrm{Int}(p, \lambda) + o(1),$$

where

$$\mathrm{Int}(p, \lambda)$$
$$= \int_0^{\lambda} \log \left(\frac{1 - 2p + \sqrt{(1-2p)^2 - 4(1-y)y}}{2(1-y)} \right) dy. \tag{10}$$

Theorem 4. *Every (n, k) linear phase detection scheme with length $n = 2^r - 1$ and minimum distance $d = 2t + 1$ must satisfy*

$$2^r \cdot \frac{K_t^2(ic)\binom{(k-r)/c^2}{i}c^{2i}}{\binom{k}{t}} \leq 2^k$$

for every $i \in [k]$ such that $ic^2 \leq k - r$. Here c is the number of nonzero coefficients of the characteristic polynomial $a(z) = \sum_{j=0}^r a_j z^j$.

Remark 5. Compared to Delsarte's linear programing bound for channel codes [36], the bound in Theorem 4 can sometimes be better. For example, when $r = 20, t = 5$, and $c = 3$, the linear programing bound yields $k \geq 41$, while Theorem 4 requires $k \geq 42$. We note, however, that with further optimization for these specific parameters, the best known channel coding upper bound is $k \geq 43$ [39].

Remark 6. For low-complexity LFSR implementation, it may be desirable to choose a characteristic polynomial with low coefficient weight. According to a conjecture in finite field theory [40], [41], there are infinitely many primitive polynomials with coefficient weight $c = 3$. For this class of primitive polynomials, Theorem 4 implies that when the adversarial channel can flip at most a fraction p of the inputs, the rate of the linear phase detection scheme must satisfy

$$\max_{0 \leq \mu \leq \frac{1-R}{9}} \left\{ 2\mu \log 3 + h\left(\frac{9\mu}{1-R}\right)\frac{(1-R)}{9} + 2\operatorname{Int}(p, 3\mu) \right\}$$
$$\leq 1 - h(p) - R, \quad (11)$$

where $\operatorname{Int}(p, 3\mu)$ is given in (10). This bound can sometimes be better than the second MRRW bound [18], which is the best known asymptotic upper bound for binary channel codes. For example, when $p = 0.05$, the second MRRW bound requires $R \leq 0.6927$. However, $p = 0.05$ and $R = 0.6927$ violate condition (11) when $\mu = 0.03073$.

Proof of Theorem 4: Following the same line of reasoning as in Section II-C (29)–(36) and (48)–(49) of [35], we have for every $\alpha \in [k]$,

$$2^r \cdot \frac{K_t^2(\alpha)B_\alpha}{\binom{k}{t}} \leq 2^k, \quad (12)$$

where B_α is the number of codewords of weight α in the dual code of the linear code induced by the phase detection scheme. Now we show that when the coefficient weight of the characteristic polynomial is c, for every $i \in [k]$ such that $ic^2 \leq k - r$, we can lower bound

$$B_{ic} \geq \binom{(k-r)/c^2}{i}c^{2i}. \quad (13)$$

To that end, note that our $(k - r) \times k$ parity check matrix, which is also the generator matrix of the dual code, can be written in the following form

$$\begin{bmatrix} 1 & a_1 & \cdots & a_{r-1} & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & a_1 & \cdots & a_{r-1} & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & a_1 & \cdots & a_{r-1} & 1 \end{bmatrix}$$

A weight ic codeword of the dual code could come from the sum of i rows of H whose nonzero elements (the 1's) are in disjoint columns. We lower bound the number of such codewords. First, we select an arbitrary row from the $(k - r)$ rows. Since each row of H has weight c, the locations of the 1's in the chosen row overlap that of at most c^2 rows (including itself). Then a second row is chosen from the $(k - r - c^2)$ remaining non-overlapping rows. We continue in this manner until we obtain i rows (we will not exhaust all rows provided that $ic^2 \leq k - r$). Hence, the number of choices is lower bounded by

$$\frac{1}{i!}(k-r)(k-r-c^2)\cdots(k-r-(i-1)c^2)$$
$$= \binom{(k-r)/c^2}{i}c^{2i},$$

which establishes (13). Plugging (13) into (12) with $\alpha = ic$ completes the proof. ∎

III. POINT-TO-POINT: PROBABILISTIC NOISE, VANISHING ERROR

In this section we discuss the probabilistic noise model with a vanishing error criterion. We first show that the capacity in this case coincides with the Shannon capacity of the observation channel. We then proceed to describe a low-complexity coding construction, based on a concatenation of a channel code and a de Bruijn sequence, that approaches this fundamental limit.

A. Fundamental Limit

Theorem 5. *The vanishing error capacity for probabilistic phase detection over a channel $p(y|x)$ is*

$$C_{ve} = \max_{p(x)} I(X; Y).$$

Before we proceed to the proof, we need a technical lemma. We denote the typical set of length-k vectors corresponding to (X, Y) by

$$\mathcal{T}_\epsilon^{(k)}(X, Y)$$
$$:= \left\{ (x^k, y^k): \left| \frac{\#\{i: (x_i, y_i) = (x, y)\}}{n} - p(x, y) \right| \right.$$
$$\left. \leq \epsilon p(x, y) \text{ for all } x \in \mathcal{X}, y \in \mathcal{Y} \right\}.$$

Lemma 2 (Lemma 24.2 [42]). *Let $(X, Y) \sim p(x, y) \neq p(x)p(y)$ and $(X^n, Y^n) \sim \prod_{i=1}^n p_{X,Y}(x_i, y_i)$. If $\epsilon > 0$ is sufficiently small, then there exists $\gamma(\epsilon) > 0$ that depends only on $p(x, y)$ such that*

$$P\{(X_m^{m+k-1}, Y^k) \in \mathcal{T}_\epsilon^{(k)}(X, Y)\} \leq 2^{-k\gamma(\epsilon)} \quad (14)$$

for every $m > 1$. Moreover, for non-overlapping sequences, i.e., for $k + 1 \leq m \leq n - k + 1$,

$$P\{(X_m^{m+k-1}, Y^k) \in \mathcal{T}_\epsilon^{(k)}(X, Y)\} \leq 2^{-k(I(X;Y)-\delta(\epsilon))}, \quad (15)$$

where $\delta(\epsilon)$ tends to zero as $\epsilon \to 0$.

Proof of Theorem 5: Clearly, any phase detection sequence is also a channel code. Thus, the above rate cannot be exceeded. We proceed to prove the achievability. Recall $\phi(m) = x_m^{m+k-1}$.

Phase detection sequence generation. We generate the sequence X^n i.i.d. $\sim p(x)$.

Detection. Upon receiving y^k, the detector declares $\hat{m}$ is the phase estimate if it is the unique phase such that $(\phi(\hat{m}), y^k) \in \mathcal{T}_\epsilon^{(k)}(X, Y)$; otherwise—if there is none or more than one—it declares an error.

Analysis of the probability of error. Without loss of generality, we assume the phase $M = 1$. The detector makes an error only if one or more of the following events occurs:

$$\mathcal{E}_1 = \{(\phi(1), Y^k) \notin \mathcal{T}_\epsilon^{(k)}(X, Y)\},$$
$$\mathcal{E}_2 = \{(\phi(m), Y^k) \in \mathcal{T}_\epsilon^{(k)}(X, Y) \text{ for some } m \neq 1\}.$$

By the law of large numbers, $\mathsf{P}(\mathcal{E}_1)$ tends to zero as $k \to \infty$. For the second term, we have

$$\mathsf{P}(\mathcal{E}_2)$$
$$\leq \left(\sum_{m=2}^{k} + \sum_{m=n-k+2}^{n} \right) \mathsf{P}\{(\phi(m), Y^k) \in \mathcal{T}_\epsilon^{(k)}(X, Y)\}$$
$$+ \sum_{m=k+1}^{n-k+1} \mathsf{P}\{(\phi(m), Y^k) \in \mathcal{T}_\epsilon^{(k)}(X, Y)\}$$
$$\overset{(a)}{=} 2(k-1)2^{-k\gamma(\epsilon)} + (2^{kR} - 2k + 1)2^{-k(I(X;Y)-\delta(\epsilon))},$$

which tends to zero as $k \to \infty$ if $R < I(X; Y) - \delta(\epsilon)$. Here the first and the second terms in (a) follow from (14) and (15) respectively. Letting $\epsilon \to 0$ completes the proof. ∎

Example 1. Consider the case of GPS signaling. For GPS, the binary (BPSK) symbol duration is about 1μsec, and the length of the underlying Gold code sequence is $n = 1023$. Consider a typical observation time of 1 second, which corresponds to a repetition factor $N \approx 1000$ and $k \approx 1e6$ binary observations. A correlator receiver can thus increase the SNR by about 60dB by coherently integrating over this sequence (assuming symbol timing has been recovered). Due to the good autocorrelation structure of the Gold code, an SNR of 30dB is typically sufficient in order to distinguish the correct phase (out of the 1023 possibilities, and typically also over several Doppler hypotheses), with a small enough error probability. Namely, one can operate at an SNR of -30dB, and provide positioning with uncertainty of $1023\,\mu$sec; multiplied by the speed of light, this yields a positioning modulo $\approx 30,000$ km, which is sufficient as it is of the same order of the distance to the satellites.

Let us now show that one can significantly improve sensitivity using a more general phase detection sequence. Using the same observation period of 1 second, let us assume a much lower SNR of -44dB. Using the Gaussian capacity formula and Theorem 5, we have that

$$\frac{\log n}{k} \approx \frac{1}{2} \log_2(1 + SNR)$$

can be asymptotically achieved. Using our $k = 1e6$ and solving for n, we get that the largest n that can be supported is $n \approx 4e8$. Since this large n is also (much) larger than k, we can in principle design a phase detection sequence with roughly these parameters that attains a low error probability. This will reliably find our distance to the satellite with an uncertainty of about 120 billion km, a huge overkill, but saves 14dB in the SNR relative to the competing GPS solution operating with the same observation time. To make the comparison more precise, one should look more carefully at many important details such as the exact error probability performance, the effect of multiple Doppler hypotheses, complexity of detection, and accounting for multiple satellites. Most of these issues are beyond the scope of this paper. In the next subsection and in Secion VI we discuss the issues of complexity and multiple sequences.

B. A Low-Complexity Construction

Now we present a sequence construction with low-complexity detection that achieves the capacity asymptotically. The construction consists of three main ingredients:

1) a de Bruijn sequence with an efficient decoding algorithm [43],
2) a capacity achieving low-complexity code, e.g. a polar code [44], that protects the de Bruijn sequence against noise, and
3) an i.i.d. synchronization sequence, which is known at the detector *a priori*, that allows the detector to find the block boundary.

The details are as follows.

Phase detection sequence design. We design a de Bruijn sequence u^{2^r} of order r according the method in [43]. To encode it to a phase detection sequence x^n, we let $r = sl$, where s and l are integers. The de Bruijn sequence is chopped up into length-s chunks, each of which is encoded into a length-t codeword using a channel code of rate $R_{\text{ch}} = s/t$. Then, a synchronization sequence $b^{3\tau}$ is generated i.i.d. $\sim p(x)$, where the parameter τ is a linear function of t, i.e., $\tau = c_1 t + c_2$ for some constants $c_1 > 0$ and c_2. Below we use $\tau = t$, but $\tau \neq t$ will prove useful later in Section V-B. This sequence $b^{3\tau}$ is inserted every l blocks. The middle chunk of the synchronization sequence $b_{\tau+1}^{2\tau}$ is given to the detector. The chunks b^τ and $b_{2\tau+1}^{3\tau}$ play the role of "guarding bits" between codewords and the middle chunk $b_{\tau+1}^{2\tau}$. Their purpose is to simplify the analysis of the error probability event associated with the synchronization detection (later denoted $\mathcal{E}_1$), as will become clear in the sequel. This is illustrated in Figure 1.

Detection. We choose the length of the detection window to be

$$k = lt + 3\tau + \max\{t, \tau\}. \tag{16}$$

The extra $\max\{t, \tau\}$ symbols are the margin to ensure there are l complete channel code blocks and a complete synchronization sequence in the received sequence. Upon receiving y^k, the detector first finds an $\hat{w}_1 \in \{0\} \cup [k - \tau]$ such that

Fig. 1. Construction for probabilistic phase detection.

$(b_{\tau+1}^{2\tau}, y_{\hat{w}_1+1}^{\hat{w}_1+\tau}) \in \mathcal{T}_\epsilon^{(\tau)}(X,Y)$. If there are more than one, it chooses the smallest index. It declares an error if there is none. This determines the block boundary of the channel code blocks, i.e., a complete block starts from index $(\hat{w}_1 - \tau \mod t) + 1$ in y^k. By design, there are at least l complete channel code blocks in y^k (the dashed-line parts in y^k of Figure 1). The detector then applies the channel decoder to recover l blocks of messages. This corresponds to $ls = r$ contiguous bits in the de Bruijn sequence (the dashed-line parts in the u sequence of Figure 1), which uniquely determines the location of these bits $u_{s\hat{w}_2+1}^{s\hat{w}_2+r}$ via the de Bruijn decoder of [43]. The phase estimate is then declared as

$$\hat{m} = \hat{w}_2 t + \left\lceil \frac{\hat{w}_2}{l} \right\rceil 3\tau + 1 - (\hat{w}_1 - \tau \mod t).$$

Analysis of the probability of error. For clarity of notation, we set $\tau = t$ in the following analysis. Similar analysis can be done for other linear functions of t. Let W_1 be the actual index of the noisy version of B_τ in Y^k. The detector makes

an error only if at least one of the following events occurs:

$$\mathcal{E}_1 = \{W_1 \neq \hat{W}_1\},$$
$$\mathcal{E}_2 = \{\text{an error in channel decoding}\}.$$

Given $\mathcal{E}_1^c \cap \mathcal{E}_2^c$, the de Bruijn decoder can figure out the phase of the decoded r bits with zero error. Since we are using a good channel code, we have $\mathsf{P}(\mathcal{E}_2 \cap \mathcal{E}_1^c) \to 0$ as $t \to \infty$. To bound $\mathsf{P}(\mathcal{E}_1)$, assume for convenience and without loss of generality that $W_1 = t - 1$. We have

$$\mathsf{P}(\mathcal{E}_1) = \mathsf{P}\{(B_{t+1}^{2t}, Y_{w_1+1}^{w_1+t}) \in \mathcal{T}_\epsilon^{(t)} \text{ for some } w_1 \neq t-1\}$$
$$\leq \sum_{w_1=0}^{t-2} \mathsf{P}\{(B_{t+1}^{2t}, Y_{w_1+1}^{w_1+t}) \in \mathcal{T}_\epsilon^{(t)}\}$$
$$+ \sum_{w_1=t}^{2t-2} \mathsf{P}\{(B_{t+1}^{2t}, Y_{w_1+1}^{w_1+t}) \in \mathcal{T}_\epsilon^{(t)}\}$$
$$+ \sum_{w_1=2t-1}^{k-t} \mathsf{P}\{(B_{t+1}^{2t}, Y_{w_1+1}^{w_1+t}) \in \mathcal{T}_\epsilon^{(t)}\}$$
$$\overset{(a)}{\leq} (t-1)2^{-t\gamma(\epsilon)} + (t-1)2^{-t\gamma(\epsilon)}$$

$$+ ((l+1)t + 2)2^{-t(I(X;Y)-\delta(\epsilon))},$$

which, for fixed l, tends to zero as $t \to \infty$. Here, the first term in (a) follows from Lemma 2 and the fact that B_{t+1}^{2t} and its preceding guarding block B^t are i.i.d.$\sim p(x)$. The second term follows since B_{t+1}^{2t} and its succeeding guarding block B_{2t+1}^{3t} are i.i.d.$\sim p(x)$. The third term follows by virtue of the packing lemma [42, Lemma 3.1], since any length-t chunk from two channel code blocks is independent of B_{t+1}^{2t}. Note the role of the "guarding bits" here is to make sure $Y_{w_1+1}^{w_1+t}$ never overlaps with both B_{t+1}^{2t} and a codeword, as we cannot generally assume too much about the statistics of a specific codeword. Therefore, the probability of error averaged over all possible realizations of B^{3t} tends to zero as $t \to \infty$. It follows that a good deterministic sequence b^{3t} exists (in fact, most choices are good).

Rate. By design, the rate of the sequence is

$$
\begin{aligned}
R &= \frac{\log n}{k} \\
&= \frac{\log\left[2^{r \frac{lt + 3\tau}{ls}}\right]}{lt + 3\tau + \max\{t, \tau\}} \qquad (17) \\
&\stackrel{(a)}{=} \frac{\log\left[2^{r \frac{(l+3)t}{ls}}\right]}{(l+4)t} \\
&= R_{\text{code}}\left(1 - \frac{4}{l+4}\right) \\
&\quad + \frac{R_{\text{code}}}{s(l+4)} \log\left[\frac{1}{R_{\text{code}}}\left(1 + \frac{3}{l}\right)\right],
\end{aligned}
$$

which, for fixed R_{code} and l, tends to $R_{\text{code}}\left(1 - \frac{4}{l+4}\right)$ as $s \to \infty$. Choosing a large l and a capacity achieving code for the underlying channel $p(y|x)$ ensures the rate of the phase detection sequence can be as close to C_{prob} as desired. Note that in step (a), we set $\tau = t$. But one can verify that the rate approaches capacity C_{prob} for other choices of τ.

Complexity. Finding the block boundary is $O(k)$ complexity. Recalling that r is linear in k and using the method of [43], decoding the de Bruijn sequence is $O(k \log k)$ complexity. There exist capacity achieving channel codes with $O(k \log k)$ decoding complexity, e.g., polar codes [44]. Therefore, the overall detection complexity is $O(k \log k)$.

Remark 7. For future reference, we refer to the above construction an (R_{ch}, l, t, τ) *point-to-point phase detection sequence.* Once these four parameters are given, $s = tR_{\text{ch}}, r = ls$, and both k and R can be expressed as in (16) and (17). As shown above, an $(R_{\text{ch}}, l, t, \tau)$ point-to-point phase detection sequence has detection complexity $O(k \log k)$. Moreover, for $\tau = c_1 t + c_2$ with some constants $c_1 > 0$ and c_2, the achievable rate of the sequence satisfies

$$\lim_{l \to \infty} \lim_{t \to \infty} R(R_{\text{ch}}, l, t, \tau) = R_{\text{ch}}.$$

This construction will also prove useful in Section V-B.

Remark 8. It appears plausible that the synchronization sequence could be discarded, and that the codeword boundary could be determined as part of the detection process. This

coding scheme, in a sense, shows the equivalence between error-correcting codes and phase detection schemes for the probabilistic setting.

Remark 9. Our analysis for the point-to-point phase detection problem in the probabilistic noise model assumed a uniformly distributed phase, which in channel coding terms corresponds to an average error probability criterion. In channel coding, the capacity under a more stringent maximal error probability criterion remains the same; this is easily shown by throwing away the worse half of a good average error probability codebook. In the sequence phase detection problem however, it is not immediately clear whether the capacity remains the same, as throwing bad codewords can significantly shorten the sequence. However, using our specific construction above and using a maximal error capacity achieving channel code (which may increase the detection complexity), we can show that the resulting phase detection sequence is capacity achieving under maximal error probability criterion.

IV. POINT-TO-POINT: PROBABILISTIC NOISE, ZERO ERROR

In this section, we consider zero error phase detection. Let $\alpha(G)$ denote the *independence number* of a graph G, i.e., the cardinality of a maximum independent set of G. Then, the *Shannon capacity of a graph G* can be defined as [22]

$$C(G) \triangleq \sup_k \frac{\log \alpha(G^k)}{k} = \lim_{k \to \infty} \frac{\log \alpha(G^k)}{k},$$

where G^k is the k-fold strong product of G (see definition in Section I-A). It is well known that $C(G)$ is the zero error capacity of any channel $p(y|x)$ with confusion graph G. An explicit expression for $C(G)$ is unknown. Nevertheless, the following theorem shows that $C(G)$ is also the fundamental limit in the zero error phase detection setting.

Theorem 6. *The zero error capacity for phase detection in a channel with confusion graph G coincides with the Shannon capacity of this graph, i.e.,*

$$C_{ze}(G) = C(G).$$

Proof: Again, the induced codebook of every phase detection scheme is also a good channel code for the same confusion graph, and thus $C_{\text{ze}}(G) \leq C(G)$. For the other direction, we show that every channel code of rate R can be used to construct a phase detection scheme with the same rate in the asymptotic limit.

To this end, we first note that the rate $\log \alpha(G)$ can be readily achieved. This can be done by employing a one-shot zero error channel code of the same rate (which exists by definition), and using it to construct a de Bruijn sequence of alphabet size $\alpha(G)$ and order k (cf. the existence of de Buijn sequences of any alphabet size and any order [45]). When $C(G) > \log \alpha(G)$, which means that the graph capacity can be achieved only by block coding over the product graph, then concatenating a length-k zero error channel code according to a de Bruijn sequence of alphabet size $\alpha(G^k)$ (a naive extension

of the one-shot approach above) does not immediately work (see also Remark 10). This is because the phase detector may not always know where a complete codeword starts or ends, which may result in detection errors. In what follows, we design a novel zero error synchronization sequence that enables the detector to determine the block boundary without error, and with vanishing loss in rate.

Augmented codebook and synchronization sequence. For any G with $C(G) > 0$, there exist two distinct vertices $\beta, \gamma \in \mathcal{X}$ such that $(\beta, \gamma) \notin E$. Let $\mathcal{C}^{(t)}$ be a zero error channel code of length t and rate $R_{\mathrm{ch}} = \frac{1}{t} \log |\mathcal{C}^{(t)}|$. We create an augmented codebook by sandwiching each codeword between two guarding γ's, i.e.,

$$\tilde{\mathcal{C}}^{(t+2)} = \{(\gamma, c^t, \gamma) : c^t \in \mathcal{C}^{(t)}\}.$$

The sequence

$$\beta^{t+2} = (\underbrace{\beta, \ldots, \beta}_{t+2})$$

will be used as the synchronization sequence.

Phase detection sequence design. We take a de Bruijn sequence with alphabet size $|\mathcal{C}^{(t)}|$ and order r. We associate each symbol in the de Bruijn alphabet with a different codeword in the augmented codebook $\tilde{\mathcal{C}}^{(t+2)}$ (note that $|\tilde{\mathcal{C}}^{(t+2)}| = |\mathcal{C}^{(t)}|$ by design). Then, similar to the concatenated structure in Figure 1, we concatenate (in a sequential manner) the augmented codewords according to the de Bruijn sequence. Between every r consecutive blocks of augmented codewords, we insert a synchronization block β^{t+2}. This way, the de Bruijn sequence of length $|\mathcal{C}^{(t)}|^r$ is mapped to a phase detection sequence x^n of length $n = \frac{(r+1)(t+2)}{r} |\mathcal{C}^{(t)}|^r$.

Detection. We choose the length of the detection window to be

$$k = (r+2)(t+2).$$

This ensures that the window will contain r complete codeword blocks and one complete synchronization block. For each $w_1 \in \{0\} \cup [k - t - 2]$, define

$$\mathcal{S}_y(w_1) \triangleq \left\{ u^{t+2} \in \mathcal{X}^{t+2} : \prod_{i=1}^{t+2} p_{Y|X}(y_{w_1+i}|u_i) > 0 \right\}$$

as the set of input sequences that may result in the output sequence $y_{w_1+1}^{w_1+t+2}$. The detector finds a $\hat{w}_1 \in \{0\} \cup [k-t-2]$ such that $\beta^{t+2} \in \mathcal{S}_y(w_1)$. If there are more than one, it chooses the smallest index. It declares an error if there is none. If $\hat{w}_1$ is found, then the first complete block starts from index $(\hat{w}_1 \mod t+2)+1$ of y^k. Knowing the block boundary, the detector can then decode the r codewords from $\mathcal{C}^{(t)}$. This corresponds to r contiguous symbols in the de Bruijn sequence, which uniquely determine the starting position $\hat{w}_2+1$ in the de Bruijn sequence. Then, the phase estimate is declared as

$$\hat{m} = \left(\hat{w}_2 + \left\lceil \frac{\hat{w}_2}{r} \right\rceil \right)(t+2) + 1 - (\hat{w}_1 \mod t+2).$$

Error Analysis. The crucial part of the error analysis is to show the synchronization sequence β^{t+2} can be detected with zero error. Once the block boundary is found, the r

codewords can be decoded with zero error, and the location of the corresponding r symbols in the de Bruijn sequence can also be found with zero error.

To see the scheme ensures zero error detection of β^{t+2}, we show that for all $m \in [n]$ such that $x_m^{m+t+1} \neq \beta^{t+2}$, $(x_m^{m+t+1}, \beta^{t+2}) \notin E_{t+2}$, where E_{t+2} is the edge set of G^{t+2}. There are two cases. When x_m^{m+t+1} is a complete block, we have $x_m = x_{m+t+1} = \gamma$ and hence $(x_m^{m+t+1}, \beta^{t+2}) \notin E_{t+2}$ since $(\beta, \gamma) \notin E$. When x_m^{m+t+1} consists of two (partial) blocks, we know at least one block must be a codeword block. Thus, considering the last symbol of the first block and the first symbol of the second block, we know at least one is γ. This implies $(x_m^{m+t+1}, \beta^{t+2}) \notin E_{t+2}$. In summary, β^{t+2} is never confusable with other x_m^{m+t+1} at the detector.

Rate. By design, the rate of the scheme is

$$
\begin{aligned}
R &= \frac{\log \left(\frac{(r+1)(t+2)}{r} |\mathcal{C}^{(t)}|^r \right)}{(r+2)(t+2)} \\
&= \frac{r \log |\mathcal{C}^{(t)}|}{(r+2)(t+2)} + \frac{\log \left(\frac{(r+1)(t+2)}{r} \right)}{(r+2)(t+2)} \\
&= \frac{r\,t}{(r+2)(t+2)} R_{\mathrm{ch}} + \frac{\log \left(\frac{(r+1)(t+2)}{r} \right)}{(r+2)(t+2)},
\end{aligned}
$$

which tends to R_{ch} as $r \to \infty$ and $t \to \infty$. Therefore, by choosing a zero error capacity achieving channel code, the phase detection scheme achieves $C(G)$. ∎

Remark 10. Suppose $C(G)$ is achieved by a finite block code of length $s > 1$ (e.g., for the pentagon graph [46]). In this case, generating a phase detection sequence using a de Bruijn sequence with codewords of the capacity achieving code as symbols, cannot work. To see this, recall that the induced codebook associated with any zero error phase detection sequence forms a zero error channel code of the same rate. However, a simple calculation shows that this rate is equal to $\frac{\log n}{\log (n/s)} \cdot C(G)$, which exceeds the capacity $C(G)$ for $s > 1$.

V. MULTIPLE ACCESS: PROBABILISTIC NOISE, VANISHING ERROR

So far in all the models we have discussed, phase detection either achieves the best known achievable rate of its channel coding counterpart, or shares the same capacity as that of channel coding. In this section, we encounter the first model, the multiple access phase detection with vanishing error, whose capacity region is strictly included in that of its channel coding counterpart.

A. Fundamental Limit

Theorem 7. *The vanishing error capacity region $\mathscr{C}_{ve}$ for phase detection over the channel $p(y|x_1, x_2)$ is the set of all rate pairs (R_1, R_2) such that*

$$
\begin{aligned}
R_1 &\leq I(X_1; Y | X_2), \\
R_2 &\leq I(X_2; Y | X_1), \quad\quad (18) \\
R_1 + R_2 &\leq I(X_1, X_2; Y)
\end{aligned}
$$

for some $p(x_1)p(x_2)$.

Remark 11. We note that this region is not convex in general. Compared to the usual MAC capacity region, which is the convex hull of $\mathscr{C}_{\text{ve}}$, this region can be a strict subset (see, for example, the *push-to-talk* MAC with binary inputs and output, given by $p(0|0,0) = p(1|0,1) = p(1|1,0) = 1$ and $p(0|1,1) = 1/2$ [47, Problem 3.2.6]).

Proof of Theorem 7: We prove the achievability through random sequence generation and joint typicality detection.

Sequence generation. Fix a pmf $p(x_1)p(x_2)$. Let $n_1 = 2^{kR_1}$ and $n_2 = 2^{kR_2}$. We generate the two sequences $X_1^{n_1}$ i.i.d. $\sim p(x_1)$ and $X_2^{n_2}$ i.i.d. $\sim p(x_2)$.

Detection. Upon receiving y^k, the detector declares the phase estimate $(\hat{m}_1, \hat{m}_2) \in [n_1] \times [n_2]$ if it is the unique pair such that $(\phi_1(\hat{m}_1), \phi_2(\hat{m}_2), y^k) \in \mathcal{T}_\epsilon^{(k)}(X_1, X_2, Y)$; if there is none or more than one, it declares an error.

Analysis of the probability of error. Without loss of generality, we assume that the correct phase pair is $(M_1, M_2) = (1, M_2)$. the detector makes an error only if one or more of the following events occur:

$$\mathcal{E}_1 = \{(\phi_1(1), \phi_2(M_2), Y^k) \notin \mathcal{T}_\epsilon^{(k)}\},$$
$$\mathcal{E}_2 = \{(\phi_1(m_1), \phi_2(M_2), Y^k) \in \mathcal{T}_\epsilon^{(k)} \text{ for some } m_1 \neq 1\},$$
$$\mathcal{E}_3 = \{(\phi_1(1), \phi_2(m_2), Y^k) \in \mathcal{T}_\epsilon^{(k)} \text{ for some } m_2 \neq M_2\},$$
$$\mathcal{E}_4 = \{(\phi_1(m_1), \phi_2(m_2), Y^k) \in \mathcal{T}_\epsilon^{(k)}$$
$$\text{for some } m_1 \neq 1 \text{ and } m_2 \neq M_2\}.$$

By the law of large number, $\mathsf{P}(\mathcal{E}_1)$ tends to zero as $k \to \infty$. For $\mathcal{E}_2$, we have

$$\mathsf{P}(\mathcal{E}_2) \leq \sum_{m_1=k+1}^{n_1-k+1} \mathsf{P}\{(\phi_1(m_1), \phi_2(M_2), Y^k) \in \mathcal{T}_\epsilon^{(k)}\}$$
$$+ \sum_{m_1=2}^{k} \mathsf{P}\{(\phi_1(m_1), \phi_2(M_2), Y^k) \in \mathcal{T}_\epsilon^{(k)}\}$$
$$+ \sum_{m_1=n_1-k+2}^{n_1} \mathsf{P}\{(\phi_1(m_1), \phi_2(M_2), Y^k) \in \mathcal{T}_\epsilon^{(k)}\}$$
$$\overset{(a)}{\leq} (2^{kR_1} - 2k + 1)2^{-kI(X_1;Y|X_2)} + 2(k-1)2^{-k\gamma_1(\epsilon)},$$

which tends to zero as $k \to \infty$ if $R_1 < I(X_1; Y|X_2) - \delta(\epsilon)$. Here the first term in (a) follows since $\phi_1(m_1)$ in that range does not overlap with the right chunk $\phi_1(1)$ and the probability can be bounded by the packing lemma [42, Lemma 3.1]. The second term in (a) corresponds to the overlapping chunks and the probability is bounded by Lemma 2 with $X_m^{m+k-1} \leftarrow \phi_1(m_1), Y^k \leftarrow (\phi_2(M_2), Y^k)$. We can similarly show that $\mathsf{P}(\mathcal{E}_3)$ tends to zero as $k \to \infty$ if $R_2 < I(X_2; Y|X_1) - \delta(\epsilon)$. For $\mathcal{E}_4$, there are four different cases:

- There are $(2^{kR_1} - 2k+1)(2^{kR_2} - 2k+1)$ pairs (m_1, m_2) such that neither $\phi_1(m_1)$ nor $\phi_2(m_2)$ overlaps with the right chunks. By the packing lemma, we can bound

$$\mathsf{P}\{(\phi_1(m_1), \phi_2(m_2), Y^k) \in \mathcal{T}_\epsilon^{(k)}\} \leq 2^{-kI(X_1, X_2; Y)}.$$

- There are $2(k-1)(2^{kR_2}-2k+1)$ pairs (m_1, m_2) such that $\phi_1(m_1)$ overlaps with the right chunk while $\phi_2(m_2)$ does not. Applying Lemma 2 first and then the packing lemma (note the independence between $\phi_1(m_1)$ and $\phi_2(m_2)$ for any m_1 and m_2), we have

$$\mathsf{P}\{(\phi_1(m_1), \phi_2(m_2), Y^k) \in \mathcal{T}_\epsilon^{(k)}\}$$
$$= \mathsf{P}\{(\phi_1(m_1), Y^k) \in \mathcal{T}_\epsilon^{(k)}\}$$
$$\cdot \mathsf{P}\{(\phi_1(m_1), \phi_2(m_2), Y^k) \in \mathcal{T}_\epsilon^{(k)} | (\phi_1(m_1), Y^k) \in \mathcal{T}_\epsilon^{(k)}\}$$
$$\leq 2^{-k(\gamma_2(\epsilon)+I(X_2;Y|X_1))}.$$

- There are $2(k-1)(2^{kR_1} - 2k + 1)$ pairs (m_1, m_2) such that $\phi_2(m_2)$ overlaps with the right chunk while $\phi_1(m_1)$ does not. Similarly, we have

$$\mathsf{P}\{(\phi_1(m_1), \phi_2(m_2), Y^k) \in \mathcal{T}_\epsilon^{(k)}\}$$
$$\leq 2^{-k(\gamma_3(\epsilon)+I(X_1;Y|X_2))}.$$

- The rest $4(k-1)^2$ pairs are such that both $\phi_1(m_1)$ and $\phi_2(m_2)$ overlap with the right chunks. We note that the event $\{(\phi_1(m_1), \phi_2(m_2), Y^k) \in \mathcal{T}_\epsilon^{(k)}(X_1, X_2, Y)\}$ implies $\{(\phi_1(m_1), Y^k) \in \mathcal{T}_\epsilon^{(k)}(X_1; Y)\}$. Thus, we can bound the error as

$$\mathsf{P}\{(\phi_1(m_1), \phi_2(m_2), Y^k) \in \mathcal{T}_\epsilon^{(k)}\}$$
$$\leq \mathsf{P}\{(\phi_1(m_1), Y^k) \in \mathcal{T}_\epsilon^{(k)}\}$$
$$\leq 2^{-k\gamma_2(\epsilon)}.$$

Combining all four cases, we have $\mathsf{P}(\mathcal{E}_4)$ tends to zeros as $k \to \infty$ if $R_1 + R_2 < I(X_1, X_2; Y) - \delta(\epsilon)$, $R_1 < I(X_1; Y|X_2) - \delta(\epsilon)$, and $R_2 < I(X_2; Y|X_1) - \delta(\epsilon)$. Letting $\epsilon \to 0$ completes the proof of the achievability.

For the converse, we wish to show for any $(2^{kR_1}, 2^{kR_2}, k)$ multiple access phase detection scheme with vanishing probability of error $\lim_{k\to\infty} P_e^{(k)} = 0$, the rate pair $(R_1, R_2) \in \mathscr{C}_{\text{ve}}$. Given the two sequences $x_1^{n_1}$ and $x_2^{n_2}$, the joint distribution of (M_1, M_2, Y^k) is

$$\frac{1}{2^{k(R_1+R_2)}} \prod_{i=0}^{k-1} p_{Y|X_1,X_2}(y_{1+i}|x_{1,m_1+i}, x_{2,m_2+i}).$$

By Fano's inequality, we have $H(M_1, M_2|Y^k) \leq k(R_1 + R_2)P_e^{(k)} + 1 \leq k\epsilon_k$, where ϵ_k tends to zero as $k \to \infty$. We bound the sum rate as follows

$$k(R_1 + R_2)$$
$$= H(M_1, M_2)$$
$$\overset{(a)}{\leq} I(M_1, M_2; Y^k) + k\epsilon_k$$
$$= \sum_{i=0}^{k-1} I(M_1, M_2; Y_{1+i}|Y^i) + k\epsilon_k$$
$$\overset{(b)}{\leq} \sum_{i=0}^{k-1} I(M_1, M_2, Y^i, x_{1,M_1+i}, x_{2,M_2+i}; Y_{1+i}) + k\epsilon_k$$
$$\overset{(c)}{=} \sum_{i=0}^{k-1} I(x_{1,M_1+i}, x_{2,M_2+i}; Y_{1+i}) + k\epsilon_k,$$

where (a) follows from Fano's inequality and (c) follows since $(M_1, M_2, Y^i) \to (x_{1,M_1+i}, x_{2,M_2+i}) \to Y_{1+i}$ form a Markov chain due to the memorylessness of the channel. Note here in both (b) and (c), x_{j,M_j+i} is a function of M_j that takes value x_{j,m_j+i} when $M_j = m_j$ for $j = 1, 2$. Now we bound the individual rate as follows

$$
\begin{aligned}
kR_1 &= H(M_1|M_2) \\
&\overset{(c)}{\leq} I(M_1; Y^k|M_2) + k\epsilon_k \\
&= \sum_{i=0}^{k-1} I(M_1; Y_{1+i}|M_2, Y^i) + k\epsilon_k \\
&\leq \sum_{i=0}^{k-1} I(Y^i, M_1, M_2, x_{1,M_1+i}; Y_{1+i}|x_{2,M_2+i}) + k\epsilon_k \\
&= \sum_{i=0}^{k-1} I(x_{1,M_1+i}; Y_{1+i}|x_{2,M_2+i}) + k\epsilon_k,
\end{aligned}
$$

where (c) follows since $H(M_1|Y^k, M_2) \leq H(M_1, M_2|Y^k) \leq k\epsilon_k$. Now flipping the role of 1 and 2, we have

$$
kR_2 \leq \sum_{i=0}^{k-1} I(x_{2,M_2+i}; Y_{1+i}|x_{1,M_1+i}) + k\epsilon_k.
$$

Now we introduce a time-sharing random variable $Q \sim \mathrm{Unif}[k]$, which is independent of (M_1, M_2, Y^k). We can write

$$
\begin{aligned}
R_1 + R_2 &\leq I(x_{1,M_1+Q}, x_{2,M_2+Q}; Y_{1+Q}|Q) + \epsilon_k, \\
R_1 &\leq I(x_{1,M_1+Q}; Y_{1+Q}|x_{2,M_2+Q}, Q) + \epsilon_k, \\
R_2 &\leq I(x_{2,M_2+Q}; Y_{1+Q}|x_{1,M_1+Q}, Q) + \epsilon_k.
\end{aligned}
$$

Note that $\mathsf{P}\{Y_{1+Q} = y|x_{1,M_1+Q} = x_1, x_{2,M_2+Q} = x_2\} = p(y|x_1, x_2)$, which is distributed according to the channel conditional pmf. Hence, we identify $X_1 = x_{1,M_1+Q}$, $X_2 = x_{2,M_2+Q}$, and $Y = Y_{1+Q}$ to obtain

$$
\begin{aligned}
R_1 + R_2 &\leq I(X_1, X_2; Y|Q) + \epsilon_k \\
&\leq I(Q, X_1, X_2; Y) + \epsilon_k \\
&\overset{(d)}{=} I(X_1, X_2; Y) + \epsilon_k,
\end{aligned}
$$

where (d) follows since $Q \to (X_1, X_2) \to Y$ form a Markov chain. We similarly obtain

$$
\begin{aligned}
R_1 &\leq I(X_1; Y|X_2) + \epsilon_k, \\
R_2 &\leq I(X_2; Y|X_1) + \epsilon_k.
\end{aligned}
$$

Note that since M_1 and M_2 are independent and uniform over $[n_1]$ and $[n_2]$, $M_1 + Q$ and $M_2 + Q$ are independent, and so are x_{1,M_1+Q} and x_{2,M_2+Q}. Therefore, we can restrict the inputs to independent distribution $p(x_1)p(x_2)$. Letting $k \to \infty$ completes the proof of the converse. $\blacksquare$

Remark 12. We note the connection between the above converse proof, and that of the totally asynchronous MAC [48], [49]. Unlike channel coding in the usual (synchronous) MAC setting, where the two inputs can be correlated through the time-sharing random variable Q, the two inputs x_{1,M_1+Q} and x_{2,M_2+Q} in the phase detection setting are independent even with the time-sharing random variable. Therefore, while the input pmf for the channel coding problem is $p(q)p(x_1|q)p(x_2|q)$,

it is $p(x_1)p(x_2)$ in the phase detection setting. This essentially results in the strict gap between the capacity regions of the two problems (see also Remark 11).

Remark 13. One can similarly show that the vanishing error capacity region for phase detection in the L-user MAC $p(y|x_1, \ldots, x_L)$ is the set of rate tuples $(R_1, \ldots, R_L)$ such that

$$
\sum_{i \in \mathcal{J}} R_i \leq I(X_{\mathcal{J}}; Y|X_{\mathcal{J}^c}) \quad \text{for every } \mathcal{J} \subseteq [L] \tag{19}
$$

for some $\prod_{i=1}^{L} p(x_i)$. Here $X_{\mathcal{J}} = \{X_i : i \in \mathcal{J}\}$.

B. A Low-Complexity Construction

In this section, we build on the point-to-point phase detection sequence construction in Section III-B and provide an $O(k \log k)$ complexity sequence construction that achieves any rate pairs $(R_1, R_2) \in \mathscr{C}_{\mathrm{ve}}$. The construction consists of several ingredients:

1) the vanishing error capacity achieving phase detection sequence for a point-to-point channel $p(y|x)$, as given in Section III-B,
2) the rate-splitting [50] technique, which is a point-to-point channel coding technique for achieving arbitrary rate pairs in the MAC region without time sharing, and
3) a novel symbol-by-symbol mapping that enables rate-splitting in the phase detection setting.

Details are as follows.

Rate splitting. In the random coding scheme, we simultaneously detect the phases m_1 and m_2 by checking typicality of all possible pairs via brute force. In practice, it is unclear whether simultaneous detection can be implemented at low complexity. In our design, we circumvent this difficulty by employing the rate splitting technique of [50], which transforms the MAC coding problem into three point-to-point channel coding problems. Fix a pmf $p(u)p(v)p(x_2)$ and a function $x_1(u, v)$. We target the rate pair

$$
\begin{aligned}
R_1 &= I(U; Y) + I(V; Y|X_2, U), \\
R_2 &= I(X_2; Y|U).
\end{aligned} \tag{20}
$$

It is known [50] that for any rate point $(I_1, I_2) \in \mathscr{C}_{\mathrm{ve}}$, there exists a pmf $p(u)p(v)p(x_2)$ and a function $x_1(u, v)$ such that $I_1 = R_1$ and $I_2 = R_2$.

Sequence construction. We design three vanishing error phase detection sequences for three point-to-point channels $U \to Y$, $X_2 \to (Y, U)$, and $V \to (Y, U, X_2)$ respectively, each according to the construction in Section III-B. Specifically, u^{n_u}, $x_2^{n_2}$, and v^{n_v} are $(I(U; Y), l, t, t_u)$, $(I(X_2; Y|U), l, t, t_2)$, and $(I(V; Y|U, X_2), l, t, t_v)$ point-to-point phase detection sequences, respectively (see Remark 7 for the definition of an $(R_{\mathrm{ch}}, l, t, \tau)$ point-to-point phase detection sequence).

Given u^{n_u} and v^{n_v}, we form an x_1 sequence of length $n_u n_v$ through the symbol-by-symbol mapping

$$
x_{1,m_1} = x_1(u_{m_u}, v_{m_v}) \quad \text{for } m_1 \in [n_u n_v], \tag{21}
$$

where

$$m_u = m_1 \pmod{n_u},$$
$$m_v = m_1 \pmod{n_v}.$$

Note that when n_u and n_v are relatively prime, each phase $m_1 \in [n_u n_v]$ corresponds to a *distinct* phase pair (m_u, m_v). Moreover, the way the u and the v sequences are ordered ensures that any length-k chunk of the x_1 sequence is formed from a length-k chunk of the u sequence and a length-k chunk of the v sequence. Such an $x_1^{n_u n_v}$ sequence simulates the channel output when the two phase detection sequences u^{n_u} and v^{n_v} go through a deterministic MAC $x_1(u, v)$. Finally, recall that any t_u (and t_v) that is a linear function of t results in the same asymptotic rate of the phase detection sequence (cf. Section III-B). Hence, by adjusting the parameters t_u and t_v, it is always possible to make n_u and n_v relatively prime.

Detection. The way the u, x_2, v sequences are designed allows multiple access phase detection through successive point-to-point phase detection in the channels $U \to Y$, $X_2 \to (Y, U)$, and $V \to (Y, U, X_2)$. We choose the length of the detection window to be

$$k = lt + 3 \max\{t_u, t_v, t_2\} + \max\{t, t_u, t_v, t_2\}$$

and successively detect the phases in the order $\hat{m}_u \to \hat{m}_2 \to \hat{m}_v$. The phase of the x_1 sequence is declared to be the unique $\hat{m}_1 \in [n_u n_v]$ such that $\hat{m}_1 \pmod{n_u} = \hat{m}_u$ and $\hat{m}_1 \pmod{n_v} = \hat{m}_v$.

Analysis of the probability of error. By the analysis in the point-to-point case, the probability of error for detecting each sequence $\mathsf{P}(\mathcal{E}_j)$, $j = 1, 2, 3$, tends to zero as $t \to \infty$. By successive cancellation, the total probability of error $\mathsf{P}(\mathcal{E}) \leq \mathsf{P}(\mathcal{E}_1) + \mathsf{P}(\mathcal{E}_2) + \mathsf{P}(\mathcal{E}_3)$, which tends to zero as $t \to \infty$.

Rate. Letting $t \to \infty$ and then $l \to \infty$, the rates of the u, x_2, and v phase detection sequences approach, respectively,

$$R_u = I(U; Y),$$
$$R_2 = I(X_2; Y|U),$$
$$R_v = I(V; Y|U, X_2).$$

Moreover, we have

$$
\begin{aligned}
R_1 &= \frac{\log n_1}{k} \\
&= \frac{\log n_u n_v}{k} \\
&= \frac{\log n_u + \log n_v}{k} \\
&= R_u + R_v \\
&= I(U; Y) + I(V; Y|U, X_2),
\end{aligned}
$$

which, together with R_2, is exactly our target rate pair (20).

Complexity. Each of the three point-to-point phase detection sequence has detection complexity $O(k \log k)$. Therefore, the total complexity of the multiple access detection complexity is also $O(k \log k)$.

Remark 14. The original symbol-by-symbol mapping in the channel coding setting, which maps

$$x_{1i} = x_1(u_i, v_i),$$

does not provide the desired relation $R_1 = R_u + R_v$ in the sequence setting. This is because knowing the phase m_u simultaneously reveals the phase m_v. In contrast, the mapping in (21) ensures that for each phase $m_u \in [n_u]$, all possible phases $m_v \in [n_v]$ appear in the x_1 sequence. This creates the independence between the two phases M_u and M_v.

Remark 15. Rate splitting can be generalized to L-user MACs. More precisely, one can split X_j, $j \in [L-1]$, into two auxiliary layers U_j and V_j, and keep X_L unsplit. It is shown [50] that there exists a successive decoding order that achieves any rate tuple in the L-user MAC region (19). Together with the symbol-by-symbol mappings $x_j(u_j, v_j)$ applied as in (21), we can design a rate-optimal low-complexity phase detection scheme for an L-user MAC.

VI. MULTIPLE ACCESS: PROBABILISTIC NOISE, ZERO ERROR

In this section, we consider zero error phase detection in multiple access channels. We first demonstrate a strict separation between the channel coding setting and the phase detection setting in Section VI-A. Then, we restrict our attention to zero error phase detection in the modulo-2 addition MAC in Sections VI-B and VI-C. We note that for channel coding in the modulo-2 addition MAC, any rate pair in the zero-error capacity region $\{(R_1, R_2): R_1 + R_2 \leq 1\}$ can be achieved by time sharing between two rate-one codes. However, time sharing is not applicable in the phase detection scenario. Thus, our sequence design requires different ideas.

A. Separation Between Phase Detection and Channel Coding

Let us consider again the push-to-talk MAC (see definition in Remark 11). The zero error capacity region for channel coding is the set of rate pairs (R_1, R_2) such that

$$R_1 + R_2 \leq 1. \tag{22}$$

To see this, first note that the two corner points $(0, 1)$ and $(1, 0)$ can be achieved with zero error using any channel code of rate 1, and other points are achievable by time sharing. Moreover, since the output alphabet is binary, the rate region (22) is also an outer bound.

For zero error phase detection, a simple outer bound of $\mathscr{C}_{\text{ze}}$ is its vanishing error counterpart $\mathscr{C}_{\text{ve}}$, which is shown to be the rate region (18) in Theorem 7. For any rate pair (R_1, R_2) in the rate region (18),

$$
\begin{aligned}
R_1 + R_2 &\leq I(X_1, X_2; Y) \\
&= H(Y) - H(Y|X_1, X_2) \\
&\overset{(a)}{\leq} 1 - p_{X_1}(1) p_{X_2}(1) \\
&\overset{(b)}{\leq} 1.
\end{aligned}
$$

For equalities in both (a) and (b) to hold, we must have $p_{X_1}(1) = 0, p_{X_2}(1) = 1/2$ or $p_{X_1}(1) = 1/2, p_{X_2}(1) = 0$, which correspond to the two corner points $(0,1)$ and $(1,0)$ respectively. Any other input pmf $p(x_1)p(x_2)$ results in a sum rate strictly less than 1. Therefore, other than the two corner points, the rate pair (R_1, R_2) along the line $R_1 + R_2 = 1$ is not achievable in the phase detection setting, which establishes the separation.

B. Fundamental Limit for Modulo-2 Addition MAC

Theorem 8. *The zero error capacity region $\mathscr{C}_{ze}$ for multiple access phase detection over the channel $Y = X_1 \oplus X_2$ is the set of rate pairs (R_1, R_2) such that*

$$R_1 + R_2 \leq 1.$$

Proof: As argued above, this rate region is an outer bound since any codebooks induced by the zero error phase detection scheme can be used for the zero error channel coding problem. In what follows, we prove the achievability of this region using properties of linear codes, in a way that resembles Wyner's linear code for the Slapian–Wolf problem [29].

We choose the first sequence $x_1^{n_1}$ to be a linear sequence generated by LFSR with a primitive characteristic polynomial $a(z) = \sum_{i=0}^{r-1} a_i z^i + z^r$ over $GF(2)$ (cf. Section II-B). Then, for $r \leq k \leq n_1$, the induced codebook together with the all-zero codeword $\mathcal{C}_1 \cup \{0^k\}$ form a linear code. Let $H_{(k-r) \times k}$ be a parity check matrix of this linear code. This allows us to define 2^{k-r} *cosets*

$$\mathcal{C}(s^{k-r}) = \{a^k \colon Ha^k = s^{k-r}\} \subseteq \{0,1\}^k.$$

Clearly, the linear code belongs to the zero coset $\mathcal{C}_1 \cup \{0^k\} = \mathcal{C}(0^{k-r})$.

Now, suppose there exist a sequence $x_2^{n_2}$ such that each length-k chunk $\phi_2(m_2)$ of the sequence belongs to a different non-zero coset $\mathcal{C}(s^{k-r})$ with $s^{k-r} \neq 0^{k-r}$. Then, the phase pair (m_1, m_2) can be recovered with zero-error from their sum through successive cancellation detection as follows. We take $H(\phi_1(m_1) \oplus \phi_2(m_2)) = 0^{k-r} \oplus H\phi_2(m_2) \triangleq s^{k-r}$. By design, there is only one chunk of $x_2^{n_2}$ that belongs to the coset $\mathcal{C}(s^{k-r})$. This uniquely determines the phase m_2. Once $\phi_2(m_2)$ is recovered, we know $\phi_1(m_1) = y^k \oplus \phi_2(m_2)$. Then by design, m_1 can be uniquely determined by its first r bits.

We now proceed to show the existence of such a sequence $x_2^{n_2}$, using Lovász local lemma (Lemma 1). We generate $X_2^{n_2}$ i.i.d. uniform. Let the "bad" events be $A_j = \{H\phi_2(m_2) = H\phi_2(m_2')$ for some $m_2 \neq m_2'\}$. Since $\phi_2(m_2) \oplus \phi_2(m_2')$ is i.i.d. uniform whether or not the two chunks overlap, the probability that the sum falls in the null space of H is

$$\mathsf{P}(A_j) = \mathsf{P}\{H(\phi_2(m_2) \oplus \phi_2(m_2')) = 0^{k-r}\} = \frac{1}{2^{k-r}}.$$

Now each A_j is mutually independent of all other events, except for a set of at most $4kn_2$ events. This is because the random variable $\phi_2(m_2) \oplus \phi_2(m_2')$ is mutually independent of all X_{2i}'s with $i \in [n_2] \setminus \{m_2-k+1, m_2-k+2, \ldots, m_2+k-1\} \setminus \{m_2'-k+1, m_2'-k+2, \ldots, m_2'+k-1\}$, which excludes

at most $4kn_2$ events. Applying Lemma 1, the sequence $X_2^{n_2}$ exists with positive probability

$$\mathsf{P}\left\{\cap_{j=1}^{\frac{1}{2}n_2(n_2-1)} A_j^c\right\} > 0$$

if

$$16kn_2 2^{-(k-r)} \leq 1,$$

or equivalently

$$\frac{\log n_2}{k} + \frac{r}{k} \leq 1 - \frac{\log(16k)}{k}.$$

By the definition of the rates, $R_2 = \frac{\log n_2}{k}$ and $R_1 = \frac{\log n_1}{k} = \frac{\log(2^r-1)}{k} \approx \frac{r}{k}$. Letting $k \to \infty$, we conclude that a good sequence $x_2^{n_2}$ exists if $R_1 + R_2 < 1$. $\blacksquare$

C. Sequence Construction for Modulo-2 Addition MAC

In this Section, we show that not only does the sequence $x_2^{n_2}$ from the previous section exists, but it can also be a linear sequence. Moreover, we provide an explicit sequence construction that achieves any rate pair $(R_1, R_2) \in \mathscr{C}_{ze}$.

Sequence Construction. Let $a_1(z), a_2(z) \in \mathbb{F}_2(z)$ be two distinct primitive polynomials of degree r_1 and r_2 respectively. Let $x_1^{n_1}$ and $x_2^{n_2}$ be the two linear sequences generated by the $a_1(z)$ and $a_2(z)$ respectively (cf. Section II-B). Letting $k \geq r_1 + r_2$, the rates of the two sequences are

$$R_1 = \frac{\log(2^{r_1} - 1)}{k} \approx \frac{r_1}{k}$$

and

$$R_2 = \frac{\log(2^{r_2} - 1)}{k} \approx \frac{r_2}{k}.$$

Analysis of Detectability. We need to show that every element in $\mathcal{C}_{\text{sum}}$ can be uniquely expressed as an element in $\mathcal{C}_1$ plus an element in $\mathcal{C}_2$ (see definitions of $\mathcal{C}_1, \mathcal{C}_2$, and $\mathcal{C}_{\text{sum}}$ in (1), (2) and (3) respectively). Let us first recall some definitions and facts from the LFSR theory [34].

A one-sided infinite binary sequence $\mathbf{x} = \{x_i\}_{i \in \mathbb{N}}$ is said to be an *LFSR sequence* if it satisfies the recursion $x_{r+j} = \sum_{i=0}^{r-1} a_i x_{i+j}$, for all $j \in \mathbb{N}$. The polynomial $a(z) = z^r + \sum_{i=0}^{r-1} a_i z^i$ is a *characteristic polynomial* of $\mathbf{x}$. The first r bits $(x_1, \ldots, x_r)$ is the *initial state* of $\mathbf{x}$.

Let $\mathcal{S}(a(z)) = \{\mathbf{x} \colon a(z) \text{ is a characteristic polynomial of } \mathbf{x}\}$. Then $\mathcal{S}(a(z))$ contains 2^r sequences, each corresponding to an r-bit initial state. One can check that $\mathcal{S}(a(z))$ is an r dimensional vector space over $\mathbb{F}_2$. Define $\mathcal{S}(a_1(z)) + \mathcal{S}(a_2(z)) = \{\mathbf{x} \oplus \mathbf{y} \colon \mathbf{x} \in \mathcal{S}(a_1(z)), \mathbf{y} \in \mathcal{S}(a_2(z))\}$. When $a_1(z)$ and $a_2(z)$ are relatively prime, we have [34, Theorems 8.54, 8.55]

$$\mathcal{S}(a_1(z)) \cap \mathcal{S}(a_2(z)) = \{\mathbf{0}\}, \tag{23}$$

$$\mathcal{S}(a_1(z)) + \mathcal{S}(a_2(z)) = \mathcal{S}(a_1(z)a_2(z)). \tag{24}$$

This is exactly the case for our construction, since $a_1(z)$ and $a_2(z)$ are distinct primitive polynomials and hence relatively prime.

Now suppose that there exists a nonzero $c^k \in \mathcal{C}_1 \cap \mathcal{C}_2$. One can find an $\mathbf{x} \in \mathcal{S}(a_1(z))$ such that $x^k = c^k$, since

the first r_1 bits of $\mathbf{x}$ can be arbitrary and the rest $k - r_1$ bits are generated by the same polynomial $a_1(z)$. Similarly there is a $\mathbf{y} \in \mathcal{S}(a_2(z))$ such that $y^k = c^k$. Note that $\mathbf{x} \oplus \mathbf{y} \in \mathcal{S}(a_1(z)a_2(z))$ by (24) and that $a_1(z)a_2(z)$ is a polynomial of degree $r_1 + r_2 \leq k$. The first k bits $x^k \oplus y^k = 0^k$ fully determines the whole sequence, hence $\mathbf{x} \oplus \mathbf{y} = \mathbf{0}$. It follows that $\mathbf{x} = \mathbf{y}$ and $\mathbf{x}, \mathbf{y} \in \mathcal{S}(a_1(z)) \cap \mathcal{S}(a_2(z))$, which contradicts (23) since $\mathbf{x}$ and $\mathbf{y}$ are nonzero sequences starting with $c^k \neq 0^k$. This proves that $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. Notice that $\mathcal{C}_i \cup \{0^k\}$ is a linear code, and thus a vector space over $\mathbb{F}_2$ (cf. Theorem 2). For two vector spaces A and B, we know $\dim(A) + \dim(B) = \dim(A \cap B) + \dim(A + B)$. Therefore every element in $\mathcal{C}_{\text{sum}}$ can be uniquely expressed as $c_1^k \oplus c_2^k$, where $c_i^k \in \mathcal{C}_i, i = 1, 2$.

Remark 16. Here the crucial property is that $a_1(z)$ and $a_2(z)$ are relatively prime. In order to generalize to more than two users, one can choose L distinct primitive polynomials $a_1(z), \ldots, a_L(z)$. This ensures they are relative prime. Thus (23) and (24) generalize as

$$\mathcal{S}(a_1(z)) \cap \cdots \cap \mathcal{S}(a_L(z)) = \{\mathbf{0}\},$$
$$\mathcal{S}(a_1(z)) + \cdots + \mathcal{S}(a_L(z)) = \mathcal{S}(a_1(z) \cdots a_L(z)).$$

Constructing the phase detection sequences from these polynomials and following a similar analysis, one can show that the zero error capacity region for phase detection in the L-user modulo-2 addition MAC is the set of rate tuples $(R_1, \ldots, R_L)$ such that $\sum_{i=1}^{L} R_i \leq 1$.

VII. FUTURE RESEARCH

There are several remaining questions. In adversarial point-to-point channel coding, it is easy to show that the GV bound can be attained using linear codes. Is it also true that a linear phase detection scheme can achieve that bound? Furthermore, since sequence design is more difficult than codebook design, can we obtain upper bounds on $C_{\text{ad}}(p)$ that are tighter than the ones obtained for the adversarial channel coding setup?

In the zero error point-to-point setup, we have shown how to achieve $C(G)$ in the limit of long sequences. However, when $C(G) = \alpha(G)$, this can be achieved in finite length. Suppose that $C(G) > \alpha(G)$ and is achieved by a finite length channel block code. Does there exists a finite length phase detection sequence of rate exactly $C(G)$?

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