

Network-Oblivious Algorithms*

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Abstract

A framework is proposed for the design and analysis of *network-oblivious algorithms*, namely, algorithms that can run unchanged, yet efficiently, on a variety of machines characterized by different degrees of parallelism and communication capabilities. The framework prescribes that a network-oblivious algorithm be specified on a parallel model of computation where the only parameter is the problem's input size, and then evaluated on a model with two parameters, capturing parallelism granularity and communication latency. It is shown that, for a wide class of network-oblivious algorithms, optimality in the latter model implies optimality in the Decomposable BSP model, which is known to effectively describe a wide and significant class of parallel platforms. The proposed framework can be regarded as an attempt to port the notion of obliviousness, well established in the context of cache hierarchies, to the realm of parallel computation. Its effectiveness is illustrated by providing optimal network-oblivious algorithms for a number of key problems. Some limitations of the oblivious approach are also discussed.

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1 Introduction

Communication plays a major role in determining the performance of algorithms on current computing systems and has a considerable impact on energy consumption. Since the relevance of communication increases with the size of the system, it is expected to play an even greater role in the future. Motivated by this scenario, a large body of results have been devised concerning the design and analysis of communication-efficient algorithms. While often useful and deep, these results do not yet provide a coherent and unified theory of the communication requirements of computations. One major obstacle toward such a theory lies in the fact that, *prima facie*, communication is defined only with respect to a specific mapping of a computation onto a specific machine structure. Furthermore, the impact of communication on performance depends on the latency and bandwidth properties of the channels connecting different parts of the target machine. Hence, the design, optimization, and analysis of algorithms can become highly machine-dependent, which is undesirable from the economical perspective of developing efficient and portable software. The outlined situation has been widely recognized, and a number of approaches have been proposed to solve it or to mitigate it.

On one end of the spectrum, we have the *parallel slackness* approach, based on the assumption that, as long as a sufficient amount of parallelism is exhibited, general and automatic latency-hiding techniques can be deployed to achieve an efficient execution. Broadly speaking, the required algorithmic parallelism should be proportional to the product of the number of processing units by the worst-case latency of the target machine [44]. Further assuming that this amount of parallelism is available in computations of practical interest, algorithm design can dispense altogether with communication concerns and focus on the maximization of parallelism. The functional/data-flow and the PRAM models of computations have often been supported with similar arguments. Unfortunately, as argued in [13, 14, 15], latency hiding is not a scalable technique, due to fundamental physical constraints. Hence, parallel slackness does not really solve the communication problem. (Nevertheless, functional and PRAM models are quite valuable and have significantly contributed to the understanding of other dimensions of computing.)

On the other end of the spectrum, we could place the *universality* approach, whose objective is the development of machines (nearly) as efficient as any other machine of (nearly) the same cost, at executing any computation (see, e.g., [38, 13, 7, 16]). To the extent that a universal machine with very small performance and cost gaps could be identified, one could adopt a model of computation sufficiently descriptive of such a machine, and focus most of the algorithmic effort on this model. As technology approaches the inherent physical limitations to information processing, storage, and transfer, the emergence of a universal architecture becomes more likely. Economy of scale can also be a force favoring convergence in the space of commercial machines. While this appears as a perspective worthy of investigation, at the present stage, neither the known theoretical results nor the trends of commercially available platforms indicate an imminent strong convergence.

In the middle of the spectrum, a variety of computational models proposed in the literature can be viewed as variants of an approach aiming at realizing an *efficiency/portability/design-complexity tradeoff* [9]. Well-known examples of these models are LPRAM [3], BSP [44] and its refinements (such as D-BSP [25, 11], BSP* [5], E-BSP [34], and BSPRAM [43]), LogP [24], QSM [30], and several others. These models aim at capturing features common to most (reasonable) machines, while ignoring features that differ. The hope is that performance of real machines be largely determined by the modeled features, so that optimal algorithms in the proposed model translate into near optimal ones on real machines. A drawback of these models is that they include parameters that

affect execution time. Then, in general, efficient algorithms are parameter-aware, since different algorithmic strategies can be more efficient for different values of the parameters. One parameter present in virtually all models is the number of processors. Most models also exhibit parameters describing the time required to route certain communication patterns. Increasing the number of parameters, from just a small constant to logarithmically many in the number of processors, can considerably increase the effectiveness of the model with respect to realistic architectures, such as point-to-point networks, as extensively discussed in [11]. A price is paid in the increased complexity of algorithm design necessary to gain greater efficiency across a larger class of machines. The complications further compound if the hierarchical nature of the memory is also taken into account, so that communication between processors and memories becomes an optimization target as well.

It is natural to wonder whether, at least for some problems, parallel algorithms can be designed that, while independent of any machine/model parameters, are nevertheless efficient for wide ranges of these parameters. In other words, we are interested in exploring the world of efficient *network-oblivious* algorithms with a spirit similar to the one that motivated the development of efficient *cache-oblivious* algorithms [27]. In this paper, we define the notion of network-oblivious algorithms and propose a framework for their design and analysis. Our framework is based on three models of computation, each with a different role, as briefly outlined below.

The three models are based on a common organization consisting of a set of CPU/memory nodes communicating through some interconnection. Inspired by the BSP model and its aforementioned variants, we assume that the computation proceeds as a sequence of supersteps, where in a superstep each node performs local computation and sends/receives messages to/from other nodes, which will be consumed in the subsequent superstep. Each message occupies a constant number of words.

The first model of our framework (specification model), is used to specify network-oblivious algorithms. In this model, the number of CPU/memory nodes, referred to as virtual processors, is a function $v(n)$ of the input size and captures the amount of parallelism exhibited by the algorithm. The second model (evaluation model) is the basis for analyzing the performance of network-oblivious algorithms on different machines. It is characterized by two parameters, independent of the input: the number p of CPU/memory nodes, simply referred to as processors in this context, and a fixed latency/synchronization cost σ per superstep. The communication complexity of an algorithm is defined in this model as a function of p and σ . Finally, the third model (execution machine model) enriches the evaluation model, by replacing parameter σ with two independent parameter vectors of size logarithmic in the number of processors, which represent, respectively, the inverse of the bandwidth and the latency costs of suitable nested subsets of processors. In this model, the communication time of an algorithm is analyzed as a function of p and of the two parameter vectors. In fact, the execution machine model of our framework coincides with the Decomposable Bulk Synchronous Parallel (D-BSP) model [25, 11], which is known to describe reasonably well the behavior of a large class of point-to-point networks by capturing their hierarchical structure [10].

A *network-oblivious algorithm* is designed in the specification model but can be run on the evaluation or execution machine models by letting each processor of these models carry out the work of a pre-specified set of virtual processors. The main contribution of this paper is an optimality theorem showing that for a wide and interesting class of network-oblivious algorithms, which satisfy some technical conditions and whose communication requirements depend only on the input size and not on the specific input instance, optimality in the evaluation model automatically translates into optimality in the D-BSP model, for suitable ranges of the models' parameters. It is this

circumstance that motivates the introduction of the intermediate evaluation model, which simplifies the analysis of network-oblivious algorithms, while effectively bridging the performance analysis to the more realistic D-BSP model.

In order to illustrate the potentiality of the framework we devise network-oblivious algorithms for several fundamental problems, such as matrix multiplication, Fast Fourier Transform, comparison-based sorting, and a class of stencil computations. In all cases, except for stencil computations, we show, through the optimality theorem, that these algorithms are optimal when executed on the D-BSP, for wide ranges of the parameters. Unfortunately, there exist problems for which optimality on the D-BSP cannot be attained in a network-oblivious fashion for wide ranges of parameters. We show that this is the case for the broadcast problem.

To help place our network-oblivious framework into perspective, it may be useful to compare it with the well-established sequential cache-oblivious framework [27]. In the latter, the specification model is the Random Access Machine; the evaluation model is the Ideal Cache Model $IC(\mathcal{M}, \mathcal{B})$, with only one level of cache of size \mathcal{M} and line length \mathcal{B} ; and the execution machine model is a machine with a hierarchy of caches, each with its own size and line length. In the cache-oblivious context, the simplification in the analysis arises from the fact that, under certain conditions, optimality on $IC(\mathcal{M}, \mathcal{B})$, for all values of \mathcal{M} and \mathcal{B} , translates into optimality on multilevel hierarchies.

The notion of obliviousness in parallel settings has been recently addressed by several papers. In a preliminary version of this work [12] (see also [31]), we proposed a framework similar to the one presented here, where messages are packed in blocks whose fixed size is a parameter of the evaluation and execution machine models. While blocked communication may be preferable for models where the memory and communication hierarchies are seamlessly integrated (e.g., multicores), latency-based models like the one used here are equivalent for that scenario and also capture the case when communication is accomplished through a point-to-point network. Later, Chowdhury et al. [20] introduced a multilevel hierarchical model for multicores and the notion of *multicore-oblivious algorithm* for this model. A multicore-oblivious algorithm is specified with no mention of any machine parameters, such as the number of cores, number of cache levels, cache sizes and block lengths. Cole and Ramachandran [21, 22, 23] presented parallel algorithms for a shared-memory multicore model featuring a two-level memory, which are oblivious to the number of processors and to the memory parameters. Finally, Blleloch et al. [17] introduced a parallel version of the cache-oblivious framework in [27], named the Parallel Cache-Oblivious model, and described a scheduler for oblivious irregular computations. In contrast to these approaches, Valiant [45] studies parallel algorithms for multicore architectures advocating a parameter-aware design of portable algorithms. In his paper, he presents optimal algorithms for the Multi-BSP, a bridging model for multicore architectures which exhibits a hierarchical structure akin to that of our execution machine model. In fact, for the latter model we are able to show that, for the same computational problems, optimal parallel algorithms do exist that are parameter-free.

The rest of the paper is organized as follows. In Section 2 we formally define the three models relevant to the framework and in Section 3 we prove the optimality theorem mentioned above. In Section 4, we present the network-oblivious algorithms for matrix multiplication, Fast Fourier Transform, comparison-based sorting, and stencil computations. We also discuss the impossibility result regarding the broadcast problem. Section 5 extends the optimality theorem by presenting a less powerful version which, however, applies to a wider class of algorithms. Section 6 concludes the paper with some final remarks.

2 The Framework

We begin by introducing a parallel machine model $M(v)$, which underlies the specification, the evaluation, and the execution components of our framework. Specifically, $M(v)$ consists of a set of v processing elements, denoted by P_0, P_1, \dots, P_{v-1} , each equipped with a CPU and an unbounded local memory, which communicate through some interconnection. (For simplicity, throughout this paper, v is assumed to be a power of two.) The instruction set of each CPU is essentially that of a standard Random Access Machine, augmented with the three primitives `sync(i)`, `send(m, q)`, and `receive()`. Furthermore, each P_r has access to its own index r and to the number v of processing elements. When P_r invokes primitive `sync(i)`, with i in the integer range $[0, \log v)$, then it waits that each processing element whose index shares with r the i most significant bits also perform a `sync(i)`, or terminate its execution.¹ In other words, `sync(i)` is a barrier which synchronizes the $v/2^i$ processing elements whose indices share the i most significant bits. When P_r invokes `send(m, q)`, with $0 \leq q < v$, a constant-size message m is sent to P_q ; the message will be available in P_q only after a `sync(k)`, where k is not bigger than the number of most significant bits shared by r and q . On the other hand, the function `receive()` returns an element in the set of messages received up to the preceding barrier and removes it from the set.

In this paper, we restrict our attention to algorithms where the sequence of labels of the sync operations is the same for all processing elements, and where the last operation executed by each processing element is a `sync`.² In this case, the execution of an algorithm can be viewed as a sequence of *supersteps*, where a superstep consists of all operations performed between two consecutive sync operations, including the second of these sync operations. Supersteps are labeled by the index of their terminating sync operation: namely, a superstep terminating with `sync(i)` will be referred to as an *i -superstep*, for $0 \leq i < \log v$. Furthermore, we make the reasonable assumption that in an i -superstep, each P_r can send messages only to processing elements whose index agrees with r in the i most significant bits, that is, message exchange occurs only between processors belonging to the same synchronization subset. We observe that the results of this paper would hold even if, in the various models considered, synchronizations were not explicitly labeled. However, explicit labels can help reduce synchronization costs. For instance, they become crucial for the efficient execution of the algorithms on point-to-point networks, especially those of large diameter.

Consider an $M(v)$ -algorithm \mathcal{A} satisfying the above restrictions. For a given input instance I , we use $L_{\mathcal{A}}^i(I)$ to denote the set of i -supersteps executed by \mathcal{A} on input I , and define $S_{\mathcal{A}}^i(I) = |L_{\mathcal{A}}^i(I)|$, for $0 \leq i < \log v$. Algorithm \mathcal{A} can be naturally and automatically adapted to execute on a smaller machine $M(p)$, with $p < v$, by stipulating that processing element P_j of $M(p)$ will carry out the operations of the v/p consecutively numbered processing elements of $M(v)$ starting with $P_{j(v/p)}$, for each $0 \leq j < p$. We call this adaptation *folding*. Under folding, supersteps with a label $i < \log p$ on $M(v)$ become supersteps with the same label on $M(p)$, while supersteps with label $i \geq \log p$ on $M(v)$ become local computation on $M(p)$. Hence, when considering the communication occurring in the execution of \mathcal{A} on $M(p)$, the set $L_{\mathcal{A}}^i(I)$ is relevant as long as $i < \log p$.

A *network-oblivious algorithm* \mathcal{A} for a given computational problem Π is designed on $M(v(n))$, referred to as *specification model*, where the number $v(n)$ of processing elements, which is a function

¹For notational convenience, throughout this paper we use $\log x$ to mean $\max\{1, \log_2 x\}$.

²As we will see in the paper, several algorithms naturally comply or can easily be adapted to comply with these restrictions. Nevertheless, a less restrictive family of algorithms for $M(v)$ can be defined, by allowing processing elements to feature different traces of labels of their sync operations, still ensuring termination. The exploration of the potentialities of these algorithms is left for future research.

of the input size, is selected as part of the algorithm design. The processing elements are called *virtual processors* and are denoted by $VP_0, VP_1, \dots, VP_{v(n)-1}$, in order to distinguish them from the processing elements of the other two models. Since the folding mechanism illustrated above enables \mathcal{A} to be executed on a smaller machine $M(p)$, the design effort can be kept focussed on just one convenient virtual machine size, oblivious to the actual number of processors on which the algorithm will be executed.

While a network-oblivious algorithm is specified for a large virtual machine, it is useful to analyze its communication requirements on machines with reduced degrees of parallelism. For these purposes, we introduce the *evaluation model* $M(p, \sigma)$, where $p \geq 1$ is power of two and $\sigma \geq 0$, which is essentially an $M(p)$ where the additional parameter σ is used to account for the latency plus synchronization cost of each superstep. The processing elements of $M(p, \sigma)$ are called *processors* and are denoted by P_0, P_1, \dots, P_{p-1} . Consider the execution of an algorithm \mathcal{A} on $M(p, \sigma)$ for a given input I . For each superstep s , the metric of interest that we use to evaluate the communication requirements of the algorithm is the maximum number of messages $h_{\mathcal{A}}^s(I, p)$ sent/destined by/to any processor in that superstep. Thus, the set of messages exchanged in the superstep can be viewed as forming an $h_{\mathcal{A}}^s(I, p)$ -*relation*, where $h_{\mathcal{A}}^s(I, p)$ is often referred to as the *degree* of the relation. In the evaluation model, the communication cost of a superstep of degree h is defined as $h + \sigma$, and it is independent of the superstep's label. For our purposes, it is convenient to consider the cumulative degree of all i -supersteps, for $0 \leq i < \log p$:

$$F_{\mathcal{A}}^i(I, p) = \sum_{s \in L_{\mathcal{A}}^i(I)} h_{\mathcal{A}}^s(I, p).$$

Then, the *communication complexity* of \mathcal{A} on $M(p, \sigma)$ is defined as

$$H_{\mathcal{A}}(n, p, \sigma) = \max_{I: |I|=n} \left\{ \sum_{i=0}^{\log p - 1} (F_{\mathcal{A}}^i(I, p) + S_{\mathcal{A}}^i(I) \cdot \sigma) \right\}. \quad (1)$$

We observe that the evaluation model with this performance metric coincides with the BSP model [44] where the bandwidth parameter g is set to 1 and the latency/synchronization parameter ℓ is set to σ .

Next, we turn our attention to the last model used in the framework, called *execution machine model*, which represents the machines where network-oblivious algorithms are actually executed. We focus on parallel machines whose underlying interconnection exhibits a hierarchical structure, and use the *Decomposable BSP (D-BSP)* model [25, 11] as our execution machine model. A D-BSP($p, \mathbf{g}, \boldsymbol{\ell}$), with $\mathbf{g} = (g_0, \dots, g_{\log p - 1})$ and $\boldsymbol{\ell} = (\ell_0, \dots, \ell_{\log p - 1})$, is an $M(p)$ where the cost of an i -superstep depends on parameters g_i and ℓ_i , for $0 \leq i < \log p$. The processing elements, called processors and denoted by P_0, P_1, \dots, P_{p-1} as in the evaluation model, are partitioned into nested clusters: for $0 \leq i \leq \log p$, a set formed by all the $p/2^i$ processors whose indices share the most significant i bits is called an *i -cluster*. Hence, during an i -superstep each processor communicates only with processors of its i -cluster. For the communication within an i -cluster, parameter ℓ_i represents the latency plus synchronization cost (in time units), while g_i represents an inverse measure of bandwidth (in units of time per message). By importing the notation adopted in the evaluation model, we define the *communication time* of an algorithm \mathcal{A} on D-BSP($p, \mathbf{g}, \boldsymbol{\ell}$) as,

$$D_{\mathcal{A}}(n, p, \mathbf{g}, \boldsymbol{\ell}) = \max_{I: |I|=n} \left\{ \sum_{i=0}^{\log p - 1} (F_{\mathcal{A}}^i(I, p)g_i + S_{\mathcal{A}}^i(I)\ell_i) \right\}. \quad (2)$$

Through the folding mechanism discussed above, any network-oblivious algorithm \mathcal{A} specified on $M(v(n))$ can be transformed into an algorithm for $M(p)$ with $p < v(n)$, hence into an algorithm for $M(p, \sigma)$ or D-BSP(p, \mathbf{g}, ℓ). In this case, the quantities $H_{\mathcal{A}}(n, p, \sigma)$ and $D_{\mathcal{A}}(n, p, \mathbf{g}, \ell)$ denote, respectively, the communication complexity and communication time of the folded algorithm. Moreover, since algorithms designed on the evaluation model $M(p, \sigma)$ or on the execution machine model D-BSP(p, \mathbf{g}, ℓ) can be regarded as algorithms for $M(p)$, once the parameters σ or \mathbf{g} and ℓ are fixed, we can also analyze the communication complexities/times of their foldings on smaller machines. These relations among the models are crucial for the effective exploitation of our framework.

The following definitions establish useful notions of optimality for the two complexity measures introduced above relatively to the evaluation and execution machine models. For each measure, optimality is defined with respect to a class of algorithms, whose actual role will be made clear later in the paper. Let \mathcal{C} denote a class of algorithms, solving a given problem Π .

Definition 1. *Let $0 < \beta \leq 1$. An $M(p, \sigma)$ -algorithm $\mathcal{B} \in \mathcal{C}$ is β -optimal on $M(p, \sigma)$ with respect to \mathcal{C} if for each $M(p, \sigma)$ -algorithm $\mathcal{B}' \in \mathcal{C}$ and for each n ,*

$$H_{\mathcal{B}}(n, p, \sigma) \leq \frac{1}{\beta} H_{\mathcal{B}'}(n, p, \sigma).$$

Definition 2. *Let $0 < \beta \leq 1$. A D-BSP(p, \mathbf{g}, ℓ)-algorithm $\mathcal{B} \in \mathcal{C}$ is β -optimal on D-BSP(p, \mathbf{g}, ℓ) with respect to \mathcal{C} if for each D-BSP(p, \mathbf{g}, ℓ)-algorithm $\mathcal{B}' \in \mathcal{C}$ and for each n ,*

$$D_{\mathcal{B}}(n, p, \mathbf{g}, \ell) \leq \frac{1}{\beta} D_{\mathcal{B}'}(n, p, \mathbf{g}, \ell).$$

Note that the above definitions do not require β to be a constant: intuitively, larger values of β correspond to higher degrees of optimality.

3 Optimality Theorem for Static Algorithms

In this section, we show that, for a certain class of network-oblivious algorithms, β -optimality in the evaluation model, for suitable ranges of parameters p and σ , translates into β' -optimality in the execution machine model, for some $\beta' = \Theta(\beta)$ and suitable ranges of parameters p , \mathbf{g} , and ℓ . This result, which we refer to as optimality theorem, holds under a number of restrictive assumptions; nevertheless, it is applicable in a number of interesting case studies, as illustrated in the subsequent sections. The optimality theorem shows the usefulness of the intermediate evaluation model since it provides a form of “bootstrap,” whereby from a given degree of optimality on a family of machines we infer a related degree of optimality on a much larger family. It is important to remark that the class of algorithms for which the optimality theorem holds includes algorithms that are *network-aware*, that is, whose code can make explicit use of the architectural parameters of the model (p and σ , for the evaluation model, and p , \mathbf{g} , and ℓ , for the execution machine model) for optimization purposes.

In a nutshell, the approach we follow hinges on the fact that both communication complexity and communication time (Equations 1 and 2) are expressed in terms of quantities of the type $F_{\mathcal{A}}^i(I, p)$. If communication complexity is low then these quantities must be low, whence communication time must be low as well. Below, we discuss a number of obstacles to be faced when attempting to refine the outlined approach into a rigorous argument and how they can be handled.

A first obstacle arises whenever the performance functions are linear combinations of other auxiliary metrics. Unfortunately, worst-case optimality of these metrics does not imply optimality of their linear combinations (nor viceversa), since the worst case of different metrics could be realized by different input instances. In the cases of our interest, the “losses” incurred cannot be generally bounded by constant factors. To circumvent this obstacle, we restrict our attention to *static algorithms*, defined by the property that the following quantities are equal for all input instances of the same size n : (i) the number of supersteps; (ii) the sequence of labels of the various supersteps; and (iii) the set of source-destination pairs of the messages exchanged in any individual superstep. This restriction allows us to overload the notation, writing n instead of I in the argument of functions that become invariant for instances of the same size, namely $L_{\mathcal{A}}^i(n)$, $S_{\mathcal{A}}^i(n)$, $h_{\mathcal{A}}^s(n, p)$, and $F_{\mathcal{A}}^i(n, p)$. Likewise, the max operation becomes superfluous and can be omitted in Equation 1 and Equation 2. Static algorithms naturally arise in DAG (Directed Acyclic Graph) computations. In a *DAG algorithm*, for every instance size n there exists (at most) one DAG where each node with indegree 0 represents an input value, while each node with indegree greater than 0 represents a value produced by a unit-time operation whose operands are the values of the node’s predecessors (nodes with outdegree 0 are viewed as outputs). The computation requires the execution of all operations specified by the nodes, complying with the data dependencies imposed by the arcs.³

In order to prove the optimality theorem, we need a number of technical results and definitions. Recall that folding can be employed to transform an $M(p, \sigma)$ -algorithm into an $M(q, \sigma)$ -algorithm, for any $q < p$. The following lemma establishes a useful relation between the communication metrics when folding is applied.

Lemma 1. *Let \mathcal{B} be a static $M(p, \sigma)$ -algorithm. For every $1 \leq j \leq \log p$ and for every input size n , considering the folding of \mathcal{B} on $M(2^j, 0)$ we have*

$$\sum_{i=0}^{j-1} F_{\mathcal{B}}^i(n, 2^j) \leq \frac{p}{2^j} \sum_{i=0}^{j-1} F_{\mathcal{B}}^i(n, p).$$

Proof. The lemma follows by observing that in every i -superstep, with $i < j$, messages sent/destined by/to processor P_k of $M(2^j, 0)$, with $0 \leq k < 2^j$, are a subset of those sent/destined by/to the $p/2^j$ $M(p, \sigma)$ -processors whose computations are carried out by P_k . \square

It is easy to come up with algorithms where the bound stated in the above lemma is not tight. In fact, while in an i -superstep each message must be exchanged between processors whose indices share *at least* i most significant bits, some messages which contribute to $F_{\mathcal{B}}^i(n, p)$ may be exchanged between processors whose indices share $j > i$ most significant bits, thus not contributing to $F_{\mathcal{B}}^i(n, 2^j)$. Motivated by this observation, we define below a class of network-oblivious algorithms where a parameter α quantifies how tight the upper bound of Lemma 1 is, when considering their foldings on smaller machines. This parameter will be employed to control the extent to which an optimality guarantee in the evaluation model translates into an optimality guarantee in the execution model.

Definition 3. *A static network-oblivious algorithm \mathcal{A} specified on $M(v(n))$ is said to be (α, p) -wise, for some $0 < \alpha \leq 1$ and $1 < p \leq v(n)$, if considering the folding of \mathcal{A} on $M(2^j, 0)$ we have*

$$\sum_{i=0}^{j-1} F_{\mathcal{A}}^i(n, 2^j) \geq \alpha \frac{p}{2^j} \sum_{i=0}^{j-1} F_{\mathcal{A}}^i(n, p),$$

³In the literature, DAG problems have also been referred to as pebble games [39, Section 10.1].

for every $1 \leq j \leq \log p$ and every input size n .

(We remark that in the above definition parameter α is not necessarily a constant and can be made, for example, a function of p .) As an example, a network-oblivious algorithm for $M(v(n))$ where, for each i -superstep there is always at least one segment of $v(n)/2^{i+1}$ virtual processors consecutively numbered starting from $k \cdot (v(n)/2^{i+1})$, for some $k \geq 0$, each sending a number of messages equal to the superstep degree to processors outside the segment, is an (α, p) -wise algorithm for each $1 < p \leq v(n)$ and $\alpha = 1$. However, (α, p) -wiseness holds even if the aforementioned communication scenario is realized only in an average sense. Furthermore, consider a pair of values α' and p' such that $1 < p' \leq p$, and $1 < \alpha' \leq \alpha$. It is easy to see that $(p/p')F_{\mathcal{A}}^i(n, p) \geq F_{\mathcal{A}}^i(n, p')$, for every $0 \leq i < \log p'$, and this implies that a network-oblivious algorithm which is (α, p) -wise is also (α', p') -wise.

A final issue to consider is that the degrees of supersteps with different labels contribute with the same weight to the communication complexity while they contribute with different weights to the communication time. The following lemma will help in bridging this difference.

Lemma 2. *For $m \geq 1$, let $\langle X_0, \dots, X_{m-1} \rangle$ and $\langle Y_0, \dots, Y_{m-1} \rangle$ be two arbitrary sequences of real values, and let $\langle f_0, \dots, f_{m-1} \rangle$ be a non-increasing sequence of non-negative values. If $\sum_{i=0}^{k-1} X_i \leq \sum_{i=0}^{k-1} Y_i$, for every $1 \leq k \leq m$, then*

$$\sum_{i=0}^{m-1} X_i f_i \leq \sum_{i=0}^{m-1} Y_i f_i.$$

Proof. By defining $S_0 = 0$ and $S_k = \sum_{j=0}^{k-1} (Y_j - X_j) \geq 0$, for $1 \leq k \leq m$, we have:

$$\begin{aligned} \sum_{i=0}^{m-1} f_i (Y_i - X_i) &= \sum_{i=0}^{m-1} f_i (S_{i+1} - S_i) = \sum_{i=0}^{m-1} f_i S_{i+1} - \sum_{i=1}^{m-1} f_i S_i \geq \\ &\geq \sum_{i=0}^{m-1} f_i S_{i+1} - \sum_{i=1}^{m-1} f_{i-1} S_i = f_{m-1} S_m \geq 0. \end{aligned}$$

□

We are now ready to state and prove the optimality theorem. Let \mathcal{C} denote a class of static algorithms solving a problem Π , with the property that for any algorithm $\mathcal{A} \in \mathcal{C}$ for q processing elements, all of its foldings on q' processing elements, $2 \leq q' < q$, also belong to \mathcal{C} .

Theorem 1 (Optimality theorem). *Let $\mathcal{A} \in \mathcal{C}$ be network-oblivious and (α, p^*) -wise, for some $\alpha \in (0, 1]$ and a power of two p^* . Let also $(\sigma_0^m, \dots, \sigma_{\log p^* - 1}^m)$ and $(\sigma_0^M, \dots, \sigma_{\log p^* - 1}^M)$ be two vectors of non-negative values, with $\sigma_j^m \leq \sigma_j^M$, for every $0 \leq j < \log p^*$. If \mathcal{A} is β -optimal on $M(2^j, \sigma)$ w.r.t. \mathcal{C} , for $\sigma_{j-1}^m \leq \sigma \leq \sigma_{j-1}^M$ and $1 \leq j \leq \log p^*$, then, for every power of two $p \leq p^*$, \mathcal{A} is $\alpha\beta/(1 + \alpha)$ -optimal on $D\text{-BSP}(p, \mathbf{g}, \ell)$ w.r.t. \mathcal{C} as long as:*

- $g_i \geq g_{i+1}$ and $\ell_i/g_i \geq \ell_{i+1}/g_{i+1}$, for $0 \leq i < \log p - 1$;
- $\max_{1 \leq k \leq \log p} \{\sigma_{k-1}^m 2^k / p\} \leq \ell_i / g_i \leq \min_{1 \leq k \leq \log p} \{\sigma_{k-1}^M 2^k / p\}$, for $0 \leq i < \log p$.⁴

⁴Note that in order to allow for a nonempty range of values for the ratio ℓ_i/g_i , the σ^m and σ^M vectors must be such that $\max_{1 \leq k \leq \log p} \{\sigma_{k-1}^m 2^k / p\} \leq \min_{1 \leq k \leq \log p} \{\sigma_{k-1}^M 2^k / p\}$. This will always be the case for the applications discussed in the next section.

Proof. Fix the value p and the vectors \mathbf{g} and $\boldsymbol{\ell}$ so to satisfy the hypotheses of the theorem, and consider a D-BSP($p, \mathbf{g}, \boldsymbol{\ell}$)-algorithm $\mathcal{C} \in \mathcal{C}$. By the β -optimality of \mathcal{A} on the evaluation model $M(2^j, \psi p/2^j)$, for each $1 \leq j \leq \log p$ and ψ such that $\sigma_{j-1}^m \leq \psi p/2^j \leq \sigma_{j-1}^M$, we have

$$H_{\mathcal{A}} \left(n, 2^j, \frac{\psi p}{2^j} \right) \leq \frac{1}{\beta} H_{\mathcal{C}} \left(n, 2^j, \frac{\psi p}{2^j} \right)$$

since \mathcal{C} can be folded into an algorithm for $M(2^j, \psi p/2^j)$, still belonging to \mathcal{C} . By the definition of communication complexity it follows that

$$\sum_{i=0}^{j-1} \left(F_{\mathcal{A}}^i(n, 2^j) + S_{\mathcal{A}}^i(n) \frac{\psi p}{2^j} \right) \leq \frac{1}{\beta} \sum_{i=0}^{j-1} \left(F_{\mathcal{C}}^i(n, 2^j) + S_{\mathcal{C}}^i(n) \frac{\psi p}{2^j} \right),$$

and then, by applying Lemma 1 to the right side of the above inequality, we obtain

$$\sum_{i=0}^{j-1} \left(F_{\mathcal{A}}^i(n, 2^j) + S_{\mathcal{A}}^i(n) \frac{\psi p}{2^j} \right) \leq \frac{1}{\beta} \sum_{i=0}^{j-1} \left(\frac{p}{2^j} F_{\mathcal{C}}^i(n, p) + S_{\mathcal{C}}^i(n) \frac{\psi p}{2^j} \right). \quad (3)$$

Define $\psi_p^m = \max_{1 \leq k \leq \log p} \{\sigma_{k-1}^m 2^k/p\}$ and $\psi_p^M = \min_{1 \leq k \leq \log p} \{\sigma_{k-1}^M 2^k/p\}$. The condition imposed by the theorem on the ratio ℓ_i/g_i implies that $\psi_p^m \leq \psi_p^M$, hence, by definition of these two quantities, we have that $\sigma_{j-1}^m 2^j/p \leq \psi_p^m$, $\psi_p^M \leq \sigma_{j-1}^M 2^j/p$.

Let us first set $\psi = \psi_p^M$ in Inequality 3, and note that, by the above observation, $\sigma_{j-1}^m \leq \psi_p^M p/2^j \leq \sigma_{j-1}^M$. By multiplying both terms of the inequality by $2^j/(\psi_p^M p)$, and by exploiting the non-negativeness of the $F_{\mathcal{A}}^i(n, 2^j)$ terms, we obtain

$$\sum_{i=0}^{j-1} S_{\mathcal{A}}^i(n) \leq \frac{1}{\beta} \sum_{i=0}^{j-1} \left(\frac{F_{\mathcal{C}}^i(n, p)}{\psi_p^M} + S_{\mathcal{C}}^i(n) \right).$$

Next, we make $\log p$ applications of Lemma 2, one for each $j = 1, 2, \dots, \log p$, by setting $m = j$, $X_i = S_{\mathcal{A}}^i(n)$, $Y_i = (1/\beta) (F_{\mathcal{C}}^i(n, p)/\psi_p^M + S_{\mathcal{C}}^i(n))$, and $f_i = \ell_i/g_i$. This gives

$$\sum_{i=0}^{j-1} S_{\mathcal{A}}^i(n) \frac{\ell_i}{g_i} \leq \frac{1}{\beta} \sum_{i=0}^{j-1} \left(F_{\mathcal{C}}^i(n, p) \frac{\ell_i}{\psi_p^M g_i} + S_{\mathcal{C}}^i(n) \frac{\ell_i}{g_i} \right),$$

for $1 \leq j \leq \log p$. Since, by hypothesis, $\ell_i/g_i \leq \psi_p^M$, for each $0 \leq i < \log p$, we have $\ell_i/\psi_p^M g_i \leq 1$, hence we can write

$$\sum_{i=0}^{j-1} S_{\mathcal{A}}^i(n) \frac{\ell_i}{g_i} \leq \frac{1}{\beta} \sum_{i=0}^{j-1} \left(F_{\mathcal{C}}^i(n, p) + S_{\mathcal{C}}^i(n) \frac{\ell_i}{g_i} \right), \quad (4)$$

for $1 \leq j \leq \log p$.

Now, let us set $\psi = \psi_p^m$ in Inequality 3, which, again, guarantees $\sigma_{j-1}^m \leq \psi_p^m p/2^j \leq \sigma_{j-1}^M$. By exploiting the wiseness of \mathcal{A} in the left side and the non-negativeness of $S_{\mathcal{A}}^i(n)$, we obtain

$$\sum_{i=0}^{j-1} \alpha \frac{p}{2^j} F_{\mathcal{A}}^i(n, p) \leq \frac{1}{\beta} \sum_{i=0}^{j-1} \left(\frac{p}{2^j} F_{\mathcal{C}}^i(n, p) + S_{\mathcal{C}}^i(n) \frac{\psi_p^m p}{2^j} \right).$$

By multiplying both terms by $2^j/(p\alpha)$ and observing that, by hypothesis, $\psi_p^m \leq \ell_i/g_i$, for each $0 \leq i < \log p$, we get

$$\sum_{i=0}^{j-1} F_{\mathcal{A}}^i(n, p) \leq \frac{1}{\alpha\beta} \sum_{i=0}^{j-1} \left(F_{\mathcal{C}}^i(n, p) + S_{\mathcal{C}}^i(n) \frac{\ell_i}{g_i} \right). \quad (5)$$

Summing Inequality 4 with Inequality 5 yields

$$\sum_{i=0}^{j-1} \left(F_{\mathcal{A}}^i(n, p) + S_{\mathcal{A}}^i(n) \frac{\ell_i}{g_i} \right) \leq \frac{1+\alpha}{\alpha\beta} \sum_{i=0}^{j-1} \left(F_{\mathcal{C}}^i(n, p) + S_{\mathcal{C}}^i(n) \frac{\ell_i}{g_i} \right),$$

for $1 \leq j \leq \log p$. Applying Lemma 2 with $m = \log p$, $X_i = F_{\mathcal{A}}^i(n, p) + S_{\mathcal{A}}^i(n)\ell_i/g_i$, $Y_i = (1 + \alpha)/(\alpha\beta) (F_{\mathcal{C}}^i(n, p) + S_{\mathcal{C}}^i(n)\ell_i/g_i)$, and $f_i = g_i$ yields

$$\sum_{i=0}^{\log p - 1} (F_{\mathcal{A}}^i(n, p)g_i + S_{\mathcal{A}}^i(n)\ell_i) \leq \frac{1+\alpha}{\alpha\beta} \sum_{i=0}^{\log p - 1} (F_{\mathcal{C}}^i(n, p)g_i + S_{\mathcal{C}}^i(n)\ell_i).$$

Then, by definition of communication time we have

$$D_{\mathcal{A}}(n, p, \mathbf{g}, \boldsymbol{\ell}) \leq \frac{1+\alpha}{\alpha\beta} D_{\mathcal{C}}(n, p, \mathbf{g}, \boldsymbol{\ell}),$$

and the theorem follows. \square

A few remarks are in order regarding the above optimality theorem. The complexity metrics adopted in this paper target exclusively interprocessor communication, thus a (sequential) network-oblivious algorithm specified on $M(v)$ but using only one of the virtual processors would clearly be optimal with respect to these metrics. For meaningful applications of the theorem, the class \mathcal{C} must be suitably defined to exclude such degenerate cases and to contain algorithms where the work is sufficiently well balanced among the processing elements.

Note that the theorem requires that both the g_i 's and ℓ_i/g_i 's form non-increasing sequences. The assumption is rather natural since it reflects the fact that larger submachines exhibit more expensive communication (hence, a larger g parameter) and larger network capacity (hence, a larger ℓ/g ratio).

Some of the issues encountered in establishing the optimality theorem have an analog in the context of memory hierarchies. For example, time in the Hierarchical Memory Model (HMM) can be linked to I/O complexity as discussed in [1], so that optimality of the latter for different cache sizes implies the optimality of the former for wide classes of functions describing the access time to different memory locations. Although, to the best of our knowledge, the question has not been explicitly addressed in the literature, a careful inspection of the arguments of [1] shows that some restriction to the class of algorithms is required, to guarantee that the maximum value of the I/O complexity for different cache sizes is simultaneously reached for the same input instance. (For example, the optimality of HMM time does not follow for the class of arbitrary comparison-based sorting algorithms, since the known I/O complexity lower bound for this problem [4] may not be simultaneously reachable for all relevant cache sizes.) Moreover, the monotonicity we have assumed for the g_i and the ℓ_i/g_i sequences has an analog in the assumption that the function used in [1] to model the memory access time is polynomially bounded.

In the cache-oblivious framework the equivalent of our optimality theorem requires algorithms to satisfy the *regularity condition* [27, Lemma 6.4], which requires that the number of cache misses decreases by a constant factor when the cache size is doubled. On the other hand, our optimality theorem gives the best bound when the network-oblivious algorithm is $(\Theta(1), p)$ -wise, that is, when the communication complexity decreases by a constant factor when the number of processors is doubled. Although the regularity condition and wiseness cannot be formalized in a similar fashion due to the significant differences between the cache- and network-oblivious frameworks, we observe that both assumptions require the oblivious algorithms to react seamlessly and smoothly to small changes of the machine parameters.

4 Algorithms for Key Problems

In this section, we illustrate the use of the proposed framework by developing efficient network-oblivious algorithms for a number of fundamental computational problems: matrix multiplication (Subsection 4.1), Fast Fourier Transform (Subsection 4.2), and sorting (Subsection 4.3). All of our algorithms exhibit $\Theta(1)$ -optimality on the D-BSP for wide ranges of the machine parameters. In Subsection 4.4, we also present network-oblivious algorithms for stencil computations. These latter algorithms run efficiently on the D-BSP although they do not achieve $\Theta(1)$ -optimality, which appears to be a hard challenge in this case. In Subsection 4.5, we also establish a negative result by proving that there cannot exist a network-oblivious algorithm for broadcasting which is simultaneously $\Theta(1)$ -optimal on two sufficiently different $M(p, \sigma)$ machines.

As prescribed by our framework, the performance of the network-oblivious algorithms on the D-BSP is derived by analyzing their performance on the evaluation model. Optimality is assessed with respect to classes of algorithms where the computation is not excessively unbalanced among the processors, namely, algorithms where an individual processor cannot perform more than a constant fraction of the total minimum work for the problem. For this purpose, we exploit some recent lower bounds which rely on mild assumptions on work distributions and strengthen previous bounds based on stronger assumptions [40]. Finally, we want to stress that all of our algorithms are also work optimal.

4.1 Matrix Multiplication

The n -MM problem consists of multiplying two $\sqrt{n} \times \sqrt{n}$ matrices, A and B , using only semiring operations. For convenience, we assume that n is an even power of 2 (the general case requires minor yet tedious modifications). A result in [35] shows that any static algorithm for the n -MM problem which uses only semiring operations must compute all $n^{3/2}$ *multiplicative terms*, that is the products $A[i, k] \cdot B[k, j]$, with $0 \leq i, j, k \leq \sqrt{n}$.

Let \mathcal{C} denote the class of static algorithms for the n -MM problem such that any $\mathcal{A} \in \mathcal{C}$ for q processing elements satisfies the following properties: (i) no entry of A or B is initially replicated (however, the entries of A and B are allowed to be initially distributed among the processing elements in an arbitrary fashion); (ii) no processing element computes more than $n^{3/2} / \min\{q, 11^3\}$ multiplicative terms; (iii) all of the foldings of \mathcal{A} on q' processing elements, $2 \leq q' < q$, also belong to \mathcal{C} . The following lemma establishes a lower bound on the communication complexity of the algorithms in \mathcal{C} .

Lemma 3. *The communication complexity of any n -MM algorithm in \mathcal{C} when executed on $M(p, \sigma)$ is $\Omega(n/p^{2/3} + \sigma)$.*

Proof. The bound for $\sigma = 0$ is proved in [40, Theorem 2], and it clearly extends to the case $\sigma > 0$. The additive σ term follows since at least one message is sent by some processing element. \square

We now describe a static network-oblivious algorithm for the n -MM problem, which follows from the parallelization of the respective cache-oblivious algorithm [27]. Then, we prove its optimality in the evaluation model, for wide ranges of the parameters, and in the execution model through the optimality theorem. The algorithm is specified on $M(n)$, and requires that the input and output matrices be evenly distributed among the n VPs. We denote with A , B and C the two input matrices and the output matrix respectively, and with A_{hk} , B_{hk} and C_{hk} , with $0 \leq h, k \leq 1$, their four quadrants. The network-oblivious algorithm adopts the following recursive strategy:

1. Partition the VPs into eight segments $S_{hk\ell}$, with $0 \leq h, k, \ell \leq 1$, containing the same number of consecutively numbered VPs. Replicate and distribute the inputs so that the entries of $A_{h\ell}$ and $B_{\ell k}$ are evenly spread among the VPs in $S_{hk\ell}$.
2. In parallel, for each $0 \leq h, k, \ell \leq 1$, compute recursively the product $M_{hk\ell} = A_{h\ell} \cdot B_{\ell k}$ within $S_{hk\ell}$.
3. In parallel, for each $0 \leq i, j < \sqrt{n}$, the VP responsible for $C[i, j]$ collects $M_{hk0}[i', j']$ and $M_{hk1}[i', j']$, with $h = \lfloor 2i/\sqrt{n} \rfloor$, $k = \lfloor 2j/\sqrt{n} \rfloor$, $i' = i \bmod (\sqrt{n}/2)$ and $j' = j \bmod (\sqrt{n}/2)$, and computes $C[i, j] = M_{hk0}[i', j'] + M_{hk1}[i', j']$.

At the i -th recursion level, with $0 \leq i \leq (\log n)/3$, 8^i ($n/4^i$)-MM subproblems are solved by distinct $M(n/8^i)$'s formed by distinct segments of VPs. The recursion stops at $i = (\log n)/3$ when each VP solves sequentially an $n^{1/3}$ -MM subproblem. It is easy to see, by unfolding the recursion, that the algorithm comprises a constant number of $3i$ -supersteps for every $0 \leq i < (\log n)/3$, where each VP sends/receives $O(2^i)$ messages. In order to enforce wiseness, without worsening the asymptotic communication complexity and communication time exhibited by the algorithm in the evaluation and execution machine models, we may assume that, in each $3i$ -superstep, VP_j sends 2^i dummy messages to $VP_{j+n/2^{3i+1}}$, for $0 \leq j < n/2^{3i+1}$.

Theorem 2. *The communication complexity of the above n -MM network-oblivious algorithm when executed on $M(p, \sigma)$ is*

$$H_{\text{MM}}(n, p, \sigma) = O\left(\frac{n}{p^{2/3}} + \sigma \log p\right),$$

for every $1 < p \leq n$ and $\sigma \geq 0$. The algorithm is $(\Theta(1), n)$ -wise and $\Theta(1)$ -optimal with respect to \mathcal{C} on any $M(p, \sigma)$ with $1 < p \leq n$ and $\sigma = O(n/(p^{2/3} \log p))$.

Proof. From the previous discussion, we conclude that

$$H_{\text{MM}}(n, p, \sigma) = O\left(\sum_{i=0}^{(\log p)/3} \left(\frac{n2^i}{p} + \sigma\right)\right) = O\left(\frac{n}{p^{2/3}} + \sigma \log p\right).$$

As anticipated, the wiseness is guaranteed by the dummy messages introduced in each superstep. Finally, it is easy to see that the algorithm satisfies the three requirements for belonging to \mathcal{C} , hence its optimality follows from Lemma 3. \square

Corollary 1. *The above n -MM network-oblivious algorithm is $\Theta(1)$ -optimal with respect to \mathcal{C} on any D-BSP(p, \mathbf{g}, ℓ) machine with $1 < p \leq n$, non-increasing g_i 's and ℓ_i/g_i 's, and $\ell_0/g_0 = O(n/p)$.*

Proof. Since the network-oblivious algorithm is $(\Theta(1), n)$ -wise and belongs to \mathcal{C} , the corollary follows by plugging $p^* = n$, $\sigma_i^m = 0$, and $\sigma_i^M = \Theta(n/((i+1)2^{2i/3}))$ into Theorem 1. \square

4.1.1 Space-Efficient Matrix Multiplication

Observe that the network-oblivious algorithm described above incurs an $O(n^{1/3})$ memory blow-up per VP. As described below, the recursive strategy can be modified so to incur only a constant memory blow-up, at the expense of an increased communication complexity. The resulting network-oblivious algorithm turns out to be $\Theta(1)$ -optimal with respect to the class of algorithms featuring constant memory blow-up.

We assume, as before, that the entries of A, B and C be evenly distributed among the VPs. The VPs are (recursively) divided into four segments which solve the eight $(n/4)$ -MM subproblems in two rounds: in the first round, the segments compute $A_{00} \cdot B_{00}$, $A_{01} \cdot B_{11}$, $A_{11} \cdot B_{10}$ and $A_{10} \cdot B_{01}$ (one product per segment), while in the second round they compute $A_{01} \cdot B_{10}$, $A_{00} \cdot B_{01}$, $A_{10} \cdot B_{00}$ and $A_{11} \cdot B_{11}$ (again, one product per segment). The recursion ends when each VP solves sequentially an 1-MM subproblem. By unfolding the recursion, we get that for every $0 \leq i < \log n/2$, the algorithm executes $\Theta(2^i)$ $2i$ -supersteps where each VP sends/receives $\Theta(1)$ messages. At any time each VP contains only $O(1)$ matrix entries, but the recursion requires it to handle a stack of $O(\log n)$ entries. However, it is easy to see that only a constant number of bits are needed for each stack entry, hence, under the natural assumption that each matrix entry occupies a constant number of $\Theta(\log n)$ -bit words, the entire stack at each VP requires storage proportional to $O(1)$ matrix entries. Therefore, the algorithm incurs only a constant memory blow-up. As before, the algorithm can be easily made $(\Theta(1), n)$ -wise by adding suitable dummy messages and, when executed on $M(p, \sigma)$, its communication complexity becomes $O(n/\sqrt{p} + \sigma\sqrt{p})$.

Let \mathcal{C}' denote the class of static algorithms for the n -MM problem such that any $\mathcal{A} \in \mathcal{C}'$ for q processing elements satisfies the following properties: (i) the local storage required at each processing element is $O(n/q)$; (ii) all of the foldings of \mathcal{A} on q' processing elements, $2 \leq q' < q$, also belong to \mathcal{C}' . Since it is proved in [32] that any n -MM algorithm in \mathcal{C}' when running on $M(p, 0)$ must exhibit an $\Omega(n/\sqrt{p})$ communication complexity, the above network-oblivious algorithm is $\Theta(1)$ -optimal with respect to \mathcal{C}' on any $M(p, \sigma)$ with $1 < p \leq n$ and $\sigma = O(n/p)$. Consequently, Theorem 1 yields optimality of the algorithm on any D-BSP(p, \mathbf{g}, ℓ) machine with $1 < p \leq n$, non-increasing g_i 's and ℓ_i/g_i 's, and $\ell_0/g_0 = O(n/p)$.

4.2 Fast Fourier Transform

The n -FFT problem consists of computing the Discrete Fourier Transform of n values using the n -input FFT DAG, where a vertex is a pair $\langle w, l \rangle$, with $0 \leq w < n$ and $0 \leq l < \log n$, and there exists an arc between two vertices $\langle w, l \rangle$ and $\langle w', l' \rangle$ if $l' = l + 1$, and either w and w' are identical or their binary representations differ exactly in the l -th bit [37].

Let \mathcal{C} denote the class of static algorithms for the n -FFT problem such that any $\mathcal{A} \in \mathcal{C}$ for q processing elements satisfies the following properties: (i) each DAG node is evaluated exactly once (i.e., recomputation is not allowed); (ii) no input value is initially replicated; (iii) no processing element computes more than $\epsilon n \log n$ DAG nodes, for some constant $0 < \epsilon < 1$; (iv) all of the foldings of \mathcal{A} on q' processing elements, $2 \leq q' < q$, also belong to \mathcal{C} . Note that, as in the preceding

subsection, the class of algorithms we are considering makes no assumptions on the input and output distributions. we make no assumptions on the input and output distributions. The following lemma establishes a lower bound on the communication complexity of the algorithms in \mathcal{C} .

Lemma 4. *The communication complexity of any n -FFT algorithm in \mathcal{C} when executed on $M(p, \sigma)$ is $\Omega((n \log n)/(p \log(n/p)) + \sigma)$.*

Proof. The bound for $\sigma = 0$ is proved in [40, Theorem 11], and it clearly extends to the case $\sigma > 0$. The additive σ term follows since at least one message is sent by some processing element. \square

We now describe a static network-oblivious algorithm for the n -FFT problem, and then prove its optimality in the evaluation and execution models. The algorithm is specified on $M(n)$, and exploits the well-known decomposition of the FFT DAG into two sets of \sqrt{n} -input FFT subDAGs, with each set containing \sqrt{n} such subDAGs [2]. For simplicity, in order to ensure integrality of the quantities involved, we assume $n = 2^{2^k}$ for some integer $k \geq 0$. We assume that at the beginning the n inputs are evenly distributed among the n VPs. In parallel, each of the \sqrt{n} segments of \sqrt{n} consecutively numbered VPs computes recursively the assigned subDAG. Then, the outputs of the first set of subDAGs are permuted in a 0-superstep so to distribute the inputs of each subDAGs of the second set among the VPs of a distinct segment. The permutation pattern is equivalent to the transposition of a $\sqrt{n} \times \sqrt{n}$ matrix. Finally, each segment computes recursively the assigned subDAG.

At the i -th recursion level, with $i \geq 0$, $n^{1-1/2^i}$ $n^{1/2^i}$ -FFT subproblems are solved by $n^{1-1/2^i}$ $M(n^{1/2^i})$ models formed by distinct segments of VPs. The recurrence stops at $i = \log \log n$ when each segment of two VPs computes a 2-input subDAG. It is easy to see, by unfolding the recursion, that the algorithm comprises $O(2^i)$ supersteps with label $(1 - 1/2^i) \log n$, for each $0 \leq i < \log \log n$, where each VP sends/receives $O(1)$ messages. As before, in order to enforce wiseness without affecting the algorithm's asymptotic performance, we assume that in each $(1 - 1/2^i) \log n$ -superstep, VP $_j$ sends a dummy message to VP $_{j+n^{1/2^i}/2}$, for each $0 \leq j < n^{1/2^i}/2$.

Theorem 3. *The communication complexity of the above n -FFT network-oblivious algorithm when executed on $M(p, \sigma)$ is*

$$H_{\text{FFT}}(n, p, \sigma) = O\left(\left(\frac{n}{p} + \sigma\right) \frac{\log n}{\log(n/p)}\right),$$

for every $1 < p \leq n$ and $\sigma \geq 0$. The algorithm is $(\Theta(1), n)$ -wise and $\Theta(1)$ -optimal with respect to \mathcal{C} on any $M(p, \sigma)$ with $1 < p \leq n$ and $\sigma = O(n/p)$.

Proof. From the previous discussion, we conclude that

$$H_{\text{FFT}}(n, p, \sigma) = O\left(\sum_{i=0}^{\log(\log n / \log(n/p))} 2^i \left(\frac{n}{p} + \sigma\right)\right) = O\left(\left(\frac{n}{p} + \sigma\right) \frac{\log n}{\log(n/p)}\right).$$

The wiseness is ensured by the dummy messages, and since the algorithm satisfies the requirements for belonging to \mathcal{C} , its optimality follows from Lemma 4. \square

We now apply Theorem 1 to show that the network-oblivious algorithm is $\Theta(1)$ -optimal on the D-BSP for wide ranges of the machine parameters.

Corollary 2. *The above n -FFT network-oblivious algorithm is $\Theta(1)$ -optimal with respect to \mathcal{C} on any D -BSP(p, \mathbf{g}, ℓ) machine with $1 < p \leq n$, non-increasing g_i 's and ℓ_i/g_i 's, and $\ell_0/g_0 = O(n/p)$.*

Proof. Since the network-oblivious algorithm is $(\Theta(1), n)$ -wise and belongs to \mathcal{C} , we get the claim by plugging $p^* = n$, $\sigma_i^m = 0$, and $\sigma_i^M = \Theta(n/2^i)$ in Theorem 1. \square

We observe that although we described the network-oblivious algorithm assuming $n = 2^{2^k}$, in order to ensure integrality of the quantities involved, the above results can be generalized to the case of n arbitrary power of two. In this case, the FFT DAG is recursively decomposed into a set of $2^{\lfloor \log \sqrt{n} \rfloor}$ -input FFT subDAGs and a set of $n/2^{\lfloor \log \sqrt{n} \rfloor}$ -input FFT subDAGs. The optimality of the resulting algorithm in both the evaluation and execution machine models can be proved in a similar fashion as before.

4.3 Sorting

The n -sort problem requires to label n (distinct) input keys with their ranks, using only comparisons, where the *rank* of a key is the number of smaller keys in the input sequence.

Let \mathcal{C} denote the class of static algorithms for the n -sort problem such that any $\mathcal{A} \in \mathcal{C}$ for q processing elements satisfies the following properties: (i) initially, no input key is replicated and, during the course of the algorithm, only a constant number of copies per key are allowed at any time; (ii) no processing element performs more than $\epsilon n \log n$ comparisons, for an arbitrary constant $0 < \epsilon < 1$; (iii) all of the foldings of \mathcal{A} on q' processing elements, $2 \leq q' < q$, also belong to \mathcal{C} . We make no assumptions on how the keys are distributed among the processing elements at the beginning and at the end of the algorithm. The following lemma establishes a lower bound on the communication complexity of the algorithms in \mathcal{C} .

Lemma 5. *The communication complexity of any n -sort algorithm in \mathcal{C} when executed on $M(p, \sigma)$ is $\Omega((n \log n)/(p \log(n/p)) + \sigma)$.*

Proof. The bound for $\sigma = 0$ is proved in [40, Theorem 8], and it clearly extends to the case $\sigma > 0$. The additive σ term follows since at least one message is sent by some processing element. \square

We now present a static network-oblivious algorithm for the n -sort problem, and then prove its optimality in the evaluation and execution models. The algorithm implements a recursive version of the *Columnsort* strategy, as described in [36]. Consider the n input keys as an $r \times s$ matrix, with $r \cdot s = n$ and $r \geq s^2$. Columnsort is organized into seven *phases* numbered from 1 to 7. During Phases 1, 3, 5 and 7 the keys in each column are sorted recursively (in Phase 5 adjacent columns are sorted in reverse order). During Phases 2, 4, 6 and 8 the keys of the matrix are permuted: in Phase 2 (resp., 4) a transposition (resp., diagonalizing permutation [36]) of the $r \times s$ matrix is performed maintaining the $r \times s$ shape; in Phase 6 (resp., 8) an $r/2$ -cyclic shift (resp., the reverse of the $r/2$ -cyclic shift) is done.⁵ Columnsort can be implemented on $M(n)$ as follows. For convenience, assume that $n = 2^{(3/2)^d}$ for some integer $d \geq 0$, and set $r = n^{2/3}$ and $s = n/r$ (the more general case is discussed later). The algorithm starts with the input keys evenly distributed among the n VPs. In the odd phases the keys of each column are evenly

⁵In the original paper [36], the shift in Phase 6 is not cyclic: a new column is added containing the $r/2$ overflowing keys and $r/2$ large dummy keys, while the first column is filled with $r/2$ small dummy keys. However, it is easy to see that a cyclic shift suffices if the first $r/2$ keys in the first column are considered smaller than the last $r/2$ keys.

distributed among the VPs of a distinct segment of r consecutively numbered VPs, which form an independent $M(r)$. Then, each segment solves recursively the subproblem corresponding to the column it received. The even phases entail a constant number of 0-supersteps of constant degree. At the i -th recursion level, with $0 \leq i \leq \log_{3/2} \log n$, each segment of $n^{(2/3)^i}$ consecutively numbered VPs forming an independent $M(n^{(2/3)^i})$ solves 4^i subproblems of size $n^{(2/3)^i}$. The recurrence stops at $i = \log_{3/2} \log n$ when each VP solves, sequentially, a subproblem of constant size. It is easy to see, by unfolding the recursion, that the algorithm consists of $\Theta(4^i)$ supersteps with label $(1 - 2/3^i) \log n$, for each $0 \leq i < \log_{3/2} \log n$, where each VP sends/receives $O(1)$ messages. As before, in order to enforce wiseness, without affecting the algorithm's asymptotic performance, we assume that in each $(1 - (2/3)^i) \log n$ -superstep, VP_j sends a dummy message to $\text{VP}_{j+n^{(2/3)^i}/2}$, for each $0 \leq j < n^{(2/3)^i}/2$.

Theorem 4. *The communication complexity of the above network-oblivious algorithm for n -sort when executed on $M(p, \sigma)$ is*

$$H_{\text{sort}}(n, p, \sigma) = O\left(\left(\frac{n}{p} + \sigma\right) \left(\frac{\log n}{\log(n/p)}\right)^{\log_{3/2} 4}\right),$$

for every $1 < p \leq n$ and $\sigma \geq 0$. The algorithm is $(\Theta(1), n)$ -wise and it is $\Theta(1)$ -optimal with respect to \mathcal{C} on any $M(p, \sigma)$ with $p = O(n^{1-\delta})$, for any arbitrary constant $\delta \in (0, 1)$, and $\sigma \geq 0$.

Proof. By the previous description of the algorithm, we conclude that

$$H_{\text{sort}}(n, p, \sigma) = O\left(\sum_{i=0}^{\log_{3/2}(\log n / \log(n/p))} 4^i \left(\frac{n}{p} + \sigma\right)\right) = O\left(\left(\frac{n}{p} + \sigma\right) \left(\frac{\log n}{\log(n/p)}\right)^{\log_{3/2} 4}\right).$$

The wiseness is guaranteed by the dummy messages. Since the algorithm satisfies the three requirements to be in \mathcal{C} , its optimality follows from Lemma 5. \square

Corollary 3. *The above n -sort network-oblivious algorithm is $\Theta(1)$ -optimal with respect to \mathcal{C} on any D-BSP(p, \mathbf{g}, ℓ) machine with $p = O(n^{1-\delta})$, for some arbitrary constant $\delta \in (0, 1)$, and non-increasing g_i 's and ℓ_i/g_i 's.*

Proof. Since the network-oblivious algorithm is $(\Theta(1), n)$ -wise and belongs to \mathcal{C} , we get the claim by plugging $p^* = n$, $\sigma_i^m = 0$, and $\sigma_i^M = +\infty$ in Theorem 1. \square

Consider now the more general case when n is an arbitrary power of two. Now, the input keys must be regarded as the entries of an $r \times s$ matrix, where r is the smallest power of two greater than or equal to $n^{2/3}$. Simple, yet tedious, calculations show that the results stated in Theorem 4 and Corollary 3 continue to hold in this case.

Finally, we remark that the above network-oblivious sorting algorithm turns out to be $\Theta(1)$ -optimal on any D-BSP(p, \mathbf{g}, ℓ), as long as $p = O(n^{1-\delta})$ for constant δ , with respect to a wider class of algorithms which satisfy the requirements (i), (ii), and (iii), specified above for \mathcal{C} , but need not be static. By applying the lower bound for sorting in [40] on two processors, it is easy to show that $\Omega(n)$ messages must cross the bisection for this class of algorithms. Therefore, we get an $\Omega(g_0 n/p)$ lower bound on the communication time on D-BSP(p, \mathbf{g}, ℓ), which is matched by our network-oblivious algorithm.

4.4 Stencil Computations

A *stencil* defines the computation of any element in a d -dimensional spatial grid at time t as a function of neighboring grid elements at time $t-1, t-2, \dots, t-\tau$, for some integers $\tau \geq 1$ and $d \geq 1$. Stencil computations arise in many contexts, ranging from iterative finite-difference methods for the numerical solution of partial differential equations, to algorithms for the simulation of cellular automata, as well as in dynamic programming algorithms and in image-processing applications. Also, the simulation of a d -dimensional mesh [14] can be envisioned as a stencil computation.

In this subsection, we restrict our attention to stencil computations with $\tau = 1$. To this purpose, we define the (n, d) -*stencil* problem which represents a wide class of stencil computations (see, e.g., [28]). Specifically, the problem consists in evaluating all nodes of a DAG of n^{d+1} nodes, each represented by a distinct tuple $\langle i_0, i_1, \dots, i_d \rangle$, with $0 \leq i_0, i_1, \dots, i_d < n$, where each node $\langle i_0, i_1, \dots, i_d \rangle$ is connected, through an outgoing arc, to (at most) 3^d neighbors, namely $\langle i_0 + \delta_0, i_1 + \delta_1, \dots, i_{d-1} + \delta_{d-1}, i_d + 1 \rangle$ for each $\delta_0, \delta_1, \dots, \delta_{d-1} \in \{0, \pm 1\}$ (whenever such nodes exist). We suppose n to be a power of two. Intuitively, the (n, d) -stencil problem consists of n time steps of a stencil computation on a d -dimensional spatial grid of side n , where each DAG node corresponds to a grid element (first d coordinates) at a given time step (coordinate i_d).

Let \mathcal{C}_d denote the class of static algorithms for the (n, d) -stencil problem such that any $\mathcal{A} \in \mathcal{C}_d$ for q processing elements satisfies the following properties: (i) each DAG node is evaluated once (i.e., recomputation is not allowed); (ii) no processing element computes more than ϵn^{d+1} DAG nodes, for some constant $0 < \epsilon < 1$; (iii) all of the foldings of \mathcal{A} on q' processing elements, $2 \leq q' < q$, also belong to \mathcal{C}_d . Note that, as before, this class of algorithms makes no assumptions on the input and output distributions. The following lemma establishes a lower bound on the communication complexity of the algorithms in \mathcal{C}_d .

Lemma 6. *The communication complexity of any (n, d) -stencil algorithm in \mathcal{C}_d when executed on $M(p, \sigma)$ is $\Omega(n^d/p^{(d-1)/d} + \sigma)$.*

Proof. The bound for $\sigma = 0$ is proved in [40, Theorem 5], and it clearly extends to the case $\sigma > 0$. The additive σ term follows since at least one message is sent by some processing element. \square

In what follows, we develop efficient network-oblivious algorithms for the (n, d) -stencil problem, for the special cases of $d = \{1, 2\}$. The generalization to values $d > 2$, and to other types of stencils, is left as an open problem.

4.4.1 The $(n, 1)$ -Stencil Problem

The $(n, 1)$ -stencil problem consists of the evaluation of a DAG shaped as a 2-dimensional array of side n . We reduce the solution of the stencil problem to the computation of a diamond DAG. Specifically, we define a *diamond DAG* of side n as the intersection of a $(2n-1, 1)$ -stencil DAG with the following four half-planes: $i_0 + i_1 \geq (n-1)$, $i_0 - i_1 \leq (n-1)$, $i_0 - i_1 \geq -(n-1)$, and $i_0 + i_1 \leq 3(n-1)$ (i.e., the largest diamond included in the stencil).⁶ It follows that an $(n, 1)$ -stencil DAG can be partitioned into five full or truncated diamond DAGs of side less than n which can be evaluated in a suitable order, with the outputs of one DAG evaluation providing the inputs for subsequent DAG evaluations.

⁶We observe that our definition of diamond DAG is consistent with the one in [14], whose edges are a superset of those of the diamond DAG defined in [3].

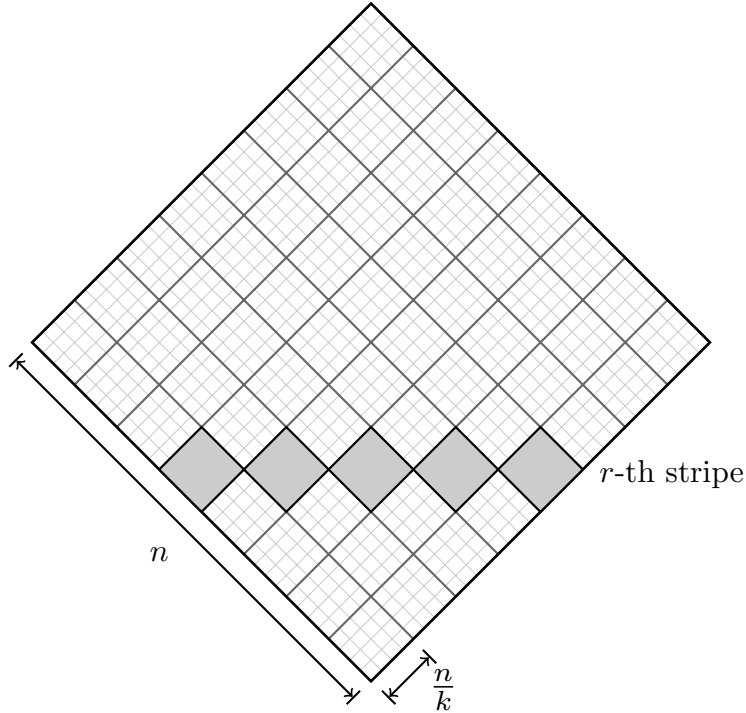


Figure 1: The decomposition of the diamond DAG performed by our algorithm.

Our network-oblivious algorithm for the $(n, 1)$ -stencil is specified on $M(n)$ and consists of five *stages*, where in each stage the whole $M(n)$ machine takes care of the evaluation of a distinct diamond DAG (full or truncated) according to the aforementioned partition. We require that all of the $O(n)$ inputs necessary for the evaluation of a diamond DAG are evenly distributed among the n VPs at the start of the stage in charge of the DAG. No matter how the inputs are assigned to the VPs at the beginning of the algorithm, the data movement required to guarantee the correct input distribution at the various stages can be accomplished in $O(1)$ 0-supersteps where each VP sends/receives $O(n)$ messages.

We now focus on the evaluation of the individual diamond DAGs. For ease of presentation, we consider the evaluation of a full diamond DAG of side n on $M(n)$. Simple yet tedious modifications are required for dealing with truncated or smaller diamond DAGs. We exploit the fact that the DAG can be decomposed recursively into smaller diamonds. (Oblivious parallel algorithms adopting this strategy have been studied in [19, 29], but these algorithms, specified for a different computational framework, are not oblivious to the number of processors p , hence they cannot be directly compared to ours.)

Let $k = 2^{\lceil \sqrt{\log n} \rceil}$. The diamond DAG is partitioned into $2k - 1$ horizontal stripes, each containing up to k diamonds of side n/k , as depicted in Figure 1. The DAG evaluation is accomplished into $2k - 1$ non-overlapping *phases*. In the r -th such phase, with $0 \leq r < 2k - 1$, the diamonds in the r -th stripe are evaluated in parallel by distinct $M(n/k)$ submachines formed by disjoint segments

of consecutively numbered VPs.⁷ At the beginning of each phase, a 0-superstep is executed so to provide the VPs of each $M(n/k)$ submachine with the appropriate input, that is, the immediate predecessors (if any) of the diamond assigned to the submachine. In this superstep each VP sends/receives $O(1)$ messages. In each phase, the diamonds of side n/k are evaluated recursively.

In general, at the i -th recursive level, with $i \geq 1$, a total of $(2k-1)^i$ non-overlapping phases are executed where diamonds of side $n_i = n/k^i$ are evaluated in parallel by distinct $M(n_i)$ submachines. Each such phase starts with a superstep of label $(i-1) \cdot \log k$ in order to provide each $M(n_i)$ with the appropriate input. In turn, the evaluation of a diamond of side n_i within an $M(n_i)$ submachine is performed recursively by partitioning its nodes into $2k-1$ horizontal stripes of diamonds of side $n_{i+1} = n/k^{i+1}$ which are evaluated in $2k-1$ non-overlapping phases by $M(n_{i+1})$ submachines, with each phase starting with a superstep of label $i \cdot \log k$ where each VP sends/receives $O(1)$ messages (and thus each processor sends/receives $O(n/p)$ messages). The recursion ends at level $\tau = \lfloor \log_k n \rfloor$, which is the first level where the diamond of side n_τ becomes smaller than k . If $n_\tau > 1$, each diamond of side n_τ assigned to an $M(n_\tau)$ submachine is evaluated straightforwardly in $2n_\tau - 1$ supersteps of label $\tau \cdot \log k$. Instead, if $n_\tau = 1$, at recursion level τ each VP independently evaluates a 1-node diamond, and no communication is required.

By unfolding the recursion, one can easily see that the evaluation of a diamond DAG of side n entails, overall, $(2k-1)^i$ supersteps of label $(i-1) \cdot \log k$, for $1 \leq i \leq \tau$, and, if $n_\tau > 1$, $(2k-1)^\tau n_\tau$ supersteps of label $\tau \cdot \log k$. In each of these supersteps, every VP sends/receives $O(1)$ messages.

In order to guarantee $(\Theta(1), n)$ -wiseness of our algorithm, we assume that suitable dummy messages are added in the each superstep so to make each VP exchange the same number of messages.

Theorem 5. *The communication complexity of the above network-oblivious algorithm for the $(n, 1)$ -stencil problem when executed on $M(p, \sigma)$ is*

$$H_{1\text{-stencil}}(n, p, \sigma) = O\left(n4^{\sqrt{\log n}}\right),$$

for every $1 < p \leq n$ and $0 \leq \sigma = O(n/p)$. The algorithm is $(\Theta(1), n)$ -wise and $\Omega\left(1/4^{\sqrt{\log n}}\right)$ -optimal with respect to \mathcal{C}_1 on any $M(p, \sigma)$ with $1 < p \leq n$ and $\sigma = O(n/p)$.

Proof. As observed before, the communication required at the beginning of each of the five stages contributes an additive factor $O(n)$ to the communication complexity, hence it is negligible. Let us then concentrate on the communication complexity for one diamond DAG evaluation. Recall that $\tau = \lfloor \log_k n \rfloor$. First suppose that $p \leq k^\tau$. Observe that at every recursion level i , with $0 \leq i < \lfloor \log_k p \rfloor$, the evaluation of each diamond of side $n_i = n/k^i$ is performed by $p/k^i > 1$ processors, while at every recursion level i , with $\lfloor \log_k p \rfloor \leq i \leq \tau$, each diamond of side n/k^i is evaluated by a single processor of $M(p, \sigma)$ and no communication takes place. Then, by the

⁷We observe that some $M(n/k)$ submachines may not be assigned to subproblems, since the number of diamonds in a stripe could be smaller than k . In order to comply with the requirement that in the algorithm execution the sequence of superstep labels is the same at each processing element, we assume that idle $M(n/k)$ submachines are assigned dummy diamonds of side n/k to be evaluated.

previous discussion it follows that

$$\begin{aligned}
H_{1\text{-stencil}}(n, p, \sigma) &= O\left(\sum_{i=0}^{\lceil \log_k p \rceil - 1} (2k-1)^{i+1} \left(\frac{n}{p} + \sigma\right)\right) \\
&= O\left((2k)^{\log_k p + 1} \frac{n}{p}\right) \\
&= O\left(n 2^{\log_k p} k\right) \\
&= O\left(n 4^{\sqrt{\log n}}\right),
\end{aligned}$$

where we exploited the upper bound on σ . Instead, if $k^\tau < p \leq n$, we have that at every recursion level i , with $0 \leq i \leq \tau$, the evaluation of each diamond of side $n_i = n/k^i$ is performed by $p/k^i > 1$ processors. Then, by the above discussion and recalling that for $i = \tau$, diamonds of side $n_\tau = n/k^\tau$ are evaluated straightforwardly in $2n_\tau - 1$ supersteps, we obtain

$$\begin{aligned}
H_{1\text{-stencil}}(n, p, \sigma) &= O\left(\sum_{i=0}^{\tau-1} (2k-1)^{i+1} \left(\frac{n}{p} + \sigma\right)\right) + O\left((2k-1)^\tau \frac{n}{k^\tau} \left(\frac{n}{p} + \sigma\right)\right) \\
&= O\left((2k)^\tau \frac{n}{k^\tau} \frac{n}{p}\right) \\
&= O(n 2^\tau k) \\
&= O\left(n 4^{\sqrt{\log n}}\right),
\end{aligned}$$

where we exploited the upper bound on σ and the fact that $p > k^\tau$ hence, by definition of τ , $n/p < k$. The wiseness is ensured by the dummy messages. It is easy to see that the algorithm complies with the requirements for belonging to \mathcal{C}_1 , hence the claimed optimality is a consequence of Lemma 6, and the theorem follows. \square

Finally, we show that the network-oblivious algorithm for the $(n, 1)$ -stencil problem achieves $\Omega\left(1/4^{\sqrt{\log n}}\right)$ -optimality on the D-BSP as well, for wide ranges of machine parameters.

Corollary 4. *The above network-oblivious algorithm for the $(n, 1)$ -stencil problem is $\Omega\left(1/4^{\sqrt{\log n}}\right)$ -optimal with respect to \mathcal{C}_1 on any D-BSP(p, \mathbf{g}, ℓ) machine with $1 < p \leq n$, non-increasing g_i 's and ℓ_i/g_i 's, and $\ell_0/g_0 = O(n/p)$.*

Proof. The corollary follows by Theorem 5 and by applying Theorem 1 with $p^* = n$, $\sigma_i^m = 0$, and $\sigma_i^M = \Theta(n/2^i)$. \square

We remark that a tighter analysis of the algorithm and/or the adoption of different values for the recursion degree k , still independent of p and σ , may yield slightly better efficiency. However, it is an open problem to devise a network-oblivious algorithm which is $\Theta(1)$ -optimal on the D-BSP for wide ranges of the machine parameters.

4.4.2 The $(n, 2)$ -Stencil Problem

In this subsection we present a network-oblivious algorithm for the $(n, 2)$ -stencil problem, which requires the evaluation of a DAG shaped as a 3-dimensional array of side n . Both the algorithm and its analysis are a suitable adaptation of the ones for the $(n, 1)$ -stencil problem. In order to evaluate a 3-dimensional domain we make use of two types of subdomains which, intuitively, play the same role as the diamond for the $(n, 1)$ -stencil: the octahedron and the tetrahedron. An octahedron of side n is the intersection of a $(2n-1, 2)$ -stencil with the following eight half-spaces: $i_0 + i_2 \geq (n-1)$, $i_0 - i_2 \leq (n-1)$, $i_0 - i_2 \geq -(n-1)$, $i_0 + i_2 \leq 3(n-1)$, $i_0 + i_1 \geq (n-1)$, $i_0 - i_1 \leq (n-1)$, $i_0 - i_1 \geq -(n-1)$, and $i_0 + i_1 \leq 3(n-1)$; a tetrahedron of side n is the intersection of a $(2n-1, 2)$ -stencil with the following four half-spaces: $i_0 + i_1 \geq (n-1)$, $i_0 - i_1 \geq (n-1)$, $i_1 + i_2 \leq 2(n-1)$, and $i_1 - i_2 \leq 0$.

As shown in [14], a 3-dimensional array of side n can be partitioned into 17 instances of (possibly truncated) octahedra or tetrahedra of side n (see Figure 6 of [14]). Our network-oblivious algorithm exploits this partition and is specified on $M(n^2)$. It consists of 17 stages, where in each stage the VPs take care of the evaluation of one polyhedra of the partition. We assume that at the beginning of the algorithm the inputs are evenly distributed among the n^2 VPs, and also impose that the inputs of each stage be evenly distributed among the VPs. The data movement required to guarantee the correct input distribution for each stage can be accomplished in $O(1)$ 0-supersteps, where each VP sends/receives $O(1)$ messages.

Let $k = 2^{\lceil \sqrt{\log n} \rceil}$. An octahedron of side n can be partitioned into octahedra and tetrahedra of side n/k in $\log k$ steps, where the i -th such step, with $1 \leq i \leq \log k$, refines a partition of the initial octahedron into octahedra or tetrahedra of side $n/2^{i-1}$ by decomposing each of these polyhedra into smaller ones of side $n/2^i$, according to the scheme depicted in [14, Figure 5]. The final partition is obtained at the end of the $\log k$ -th step. The octahedra and tetrahedra of the final partition can be grouped in horizontal stripes in such a way that the polyhedra of each stripe can be evaluated in parallel. Consider first the set of octahedra of the partition. It can be seen that the projection of these octahedra on the (i_0, i_2) -plane coincides with the decomposition of the diamond DAG depicted in Figure 1. As a consequence, we can identify $2k-1$ horizontal stripes of octahedra, where each stripe contains up to k^2 octahedra of side n/k . Moreover, the interleaving of octahedra and tetrahedra in the basic decompositions of [14, Figure 5], implies that there is a stripe of tetrahedra between each pair of consecutive stripes of octahedra. Hence, there are also $(2k-1)-1$ horizontal stripes of tetrahedra, each containing up to k^2 tetrahedra of side n/k . Overall, the octahedron of side n is partitioned into $4k-3$ horizontal stripes of at most k^2 polyhedra of side n/k each, where stripes of octahedra are interleaved with stripes of tetrahedra. With a similar argument one can derive a partition of a tetrahedron of side n into $2k-1 \leq 4k-3$ horizontal stripes of at most k^2 polyhedra of side n/k each, where stripes of octahedra are interleaved with stripes of tetrahedra.

Once established the above preliminaries, the network-oblivious algorithm to evaluate a 3-dimensional array of side n on $M(n^2)$ follows closely from the recursive strategy used for the $(n, 1)$ -stencil problem: the evaluation of an octahedron is accomplished in $4k-3$ non-overlapping phases, in each of which the polyhedra (either octahedra or tetrahedra) of side n/k in one horizontal stripe of the partition described above are recursively evaluated in parallel by distinct $M(n^2/k^2)$ submachines formed by disjoint segments of consecutively numbered VPs; a tetrahedron of side n can be evaluated through a recursive strategy similar to the one for the octahedron within the same complexity bounds. As usual, we add to each superstep $O(1)$ dummy messages per VP to

guarantee $(\Theta(1), n^2)$ -wiseness.

Theorem 6. *The communication complexity of the above network-oblivious algorithm for the $(n, 2)$ -stencil problem when executed on $M(p, \sigma)$ is*

$$H_{2\text{-stencil}}(n, p, \sigma) = O\left(\frac{n^2}{\sqrt{p}} 8^{\sqrt{\log n}}\right),$$

for every $1 < p \leq n^2$ and $0 \leq \sigma = O(n^2/p)$. The algorithm is $(\Theta(1), n^2)$ -wise and $\Omega(1/8^{\sqrt{\log n}})$ -optimal with respect to \mathcal{C}_2 on any $M(p, \sigma)$ with $1 < p \leq n^2$ and $\sigma = O(n^2/p)$.

Proof. Let $H_{\text{octahedron}}(n, p, \sigma)$ be the communication complexity required by the recursive strategy presented above for the evaluation of an octahedron of side n , when executed on $M(p, \sigma)$. The recursion depth of that strategy is $\tau = \lfloor \log_k n \rfloor$. First suppose that $p \leq k^{2\tau}$. At every recursion level i , with $0 \leq i < \lceil (\log_k p)/2 \rceil$, the evaluation of each polyhedron of side $n_i = n/k^i$ is performed by $p/k^{2i} > 1$ processors, while at every recursion level i , with $\lceil (\log_k p)/2 \rceil \leq i \leq \tau$, each polyhedron of side n/k^i is evaluated by a single processor of $M(p, \sigma)$ and no communication takes place. Then, by the previous discussion it follows that

$$\begin{aligned} H_{\text{octahedron}}(n, p, \sigma) &= O\left(\sum_{i=0}^{\lceil (\log_k p)/2 \rceil - 1} (4k - 3)^{i+1} \left(\frac{n^2}{p} + \sigma\right)\right) \\ &= O\left((4k)^{(\log_k p)/2 + 1} \frac{n^2}{p}\right) \\ &= O\left(\frac{n^2}{\sqrt{p}} 2^{\log_k p} k\right) \\ &= O\left(\frac{n^2}{\sqrt{p}} 4^{\sqrt{\log n}}\right), \end{aligned}$$

where we used the hypothesis $\sigma = O(n^2/p)$. Instead, when $k^{2\tau} < p \leq n^2$, we have that at every recursion level i , with $0 \leq i \leq \tau$, the evaluation of each polyhedron of side $n_i = n/k^i$ is performed by $p/k^{2i} > 1$ processors. Then, since for $i = \tau$, polyhedra of side $n_\tau = n/k^\tau$ are evaluated straightforwardly in $\Theta(n_\tau)$ supersteps, we obtain

$$\begin{aligned} H_{\text{octahedron}}(n, p, \sigma) &= O\left(\sum_{i=0}^{\tau-1} (4k - 3)^{i+1} \left(\frac{n^2}{p} + \sigma\right)\right) + O\left((4k - 3)^\tau \frac{n}{k^\tau} \left(\frac{n^2}{p} + \sigma\right)\right) \\ &= O\left((4k)^\tau \frac{n}{k^\tau} \frac{n^2}{p}\right) \\ &= O\left(\frac{n^2}{\sqrt{p}} 4^\tau k\right) \\ &= O\left(\frac{n^2}{\sqrt{p}} 8^{\sqrt{\log n}}\right), \end{aligned}$$

where we used the hypothesis $\sigma = O(n^2/p)$ and the inequalities $n/k^\tau < k$ and $k^{2\tau} < p$. Similar upper bounds on the communication complexity can be proved for the evaluation of a tetrahedron of side n and for the evaluation of truncated octahedra or tetrahedra.

Recall that the algorithm for the $(n, 2)$ -stencil problem consists of 17 stages, where in each stage the VPs take care of the evaluation of one (possibly truncated) octahedron or tetrahedron of side n , and that the data movement which ensures the correct input distribution for each stage can be accomplished in $O(1)$ 0-supersteps, where each VP sends/receives $O(1)$ messages. This implies that

$$H_{2\text{-stencil}}(n, p, \sigma) = O\left(\frac{n^2}{\sqrt{p}} 8^{\sqrt{\log n}}\right).$$

Since the strategies for the evaluation of (possibly truncated) octahedra or tetrahedra can be made $(\Theta(1), n^2)$ -wise, through the introduction of suitable dummy messages, the overall algorithm is also $(\Theta(1), n^2)$ -wise. Moreover, the algorithm complies with the requirements for belonging to \mathcal{C}_2 , hence the claimed optimality is a consequence of Lemma 6. \square

Corollary 5. *The above network-oblivious algorithm for the $(n, 2)$ -stencil problem is $\Omega\left(1/8^{\sqrt{\log n}}\right)$ -optimal with respect to \mathcal{C}_2 on any $D\text{-BSP}(p, \mathbf{g}, \ell)$ machine with $1 < p \leq n^2$, non-increasing g_i 's and ℓ_i/g_i 's, and $\ell_0/g_0 = O(n^2/p)$.*

Proof. The corollary follows by Theorem 6 and by applying Theorem 1 with $p^* = n^2$, $\sigma_i^m = 0$, and $\sigma_i^M = \Theta(n^2/2^i)$. \square

4.5 Limitations of the Oblivious Approach

In this subsection, we establish a negative result by showing that for the broadcast problem, defined below, a network-oblivious algorithm can achieve $O(1)$ -optimality on $M(p, \sigma)$ only for very limited ranges of σ . Let $V[0, \dots, n-1]$ be a vector of n entries. The n -broadcast problem requires to copy the value $V[0]$ into all other $V[i]$'s. Let \mathcal{C} denote the class of static algorithms for the n -broadcast problem such that any $\mathcal{A} \in \mathcal{C}$ for q processing elements satisfies the following properties: (i) at least ϵq processing elements hold entries of V , for some constant $0 < \epsilon \leq 1$, and the distribution of the entries of V among the processing elements cannot change during the execution of the algorithm; (ii) all of the foldings of \mathcal{A} on q' processing elements, $2 \leq q' < q$, also belong to \mathcal{C} . The following lemma establishes a lower bound on the communication complexity of the algorithms in \mathcal{C}

Theorem 7. *The communication complexity of any n -broadcast algorithm in \mathcal{C} when executed on $M(p, \sigma)$, with $1 < p \leq n$ and $\sigma \geq 0$, is $\Omega\left(\max\{2, \sigma\} \log_{\max\{2, \sigma\}} p\right)$.*

Proof. Let \mathcal{A} be an algorithm in \mathcal{C} . Suppose that the execution of \mathcal{A} on $M(p, \sigma)$ requires t supersteps, and let p_i denote the number of processors that “know” the value $V[0]$ by the end of the i -th superstep, for $1 \leq i \leq t$. Clearly, $p_0 = 1$ and $p_t \geq \epsilon p$, since by definition of \mathcal{C} , at least ϵp processors hold entries of V to be updated with the value $V[0]$. During the i -th superstep, $p_i - p_{i-1}$ new processors get to know $V[0]$. Since at the beginning of this superstep only p_{i-1} processors know the value, we conclude that the superstep involves an h -relation with $h \geq \lceil (p_i - p_{i-1})/p_{i-1} \rceil$. Therefore, the communication complexity of \mathcal{A} is

$$H_{\mathcal{A}}(n, p, \sigma) \geq \sum_{i=1}^t \left(\left\lceil \frac{p_i - p_{i-1}}{p_{i-1}} \right\rceil + \sigma \right) = \sum_{i=1}^t \left(\left\lceil \frac{p_i}{p_{i-1}} \right\rceil - 1 + \sigma \right).$$

Assuming, without loss of generality, that the p_i 's are strictly increasing, we obtain

$$H_{\mathcal{A}}(n, p, \sigma) = \Omega\left(t \max\{2, \sigma\} + \sum_{i=1}^t \frac{p_i}{p_{i-1}}\right).$$

Since $\prod_{i=1}^t p_i/p_{i-1} = p_t$, it follows that $\sum_{i=1}^t p_i/p_{i-1}$ is minimized for $p_i/p_{i-1} = (p_t)^{1/t} \geq (\epsilon p)^{1/t}$, for $1 \leq i \leq t$. Hence,

$$H_{\mathcal{A}}(n, p, \sigma) = \Omega\left(t \left(\max\{2, \sigma\} + p^{1/t}\right)\right). \quad (6)$$

Standard calculus shows that the right-hand side is minimized (to within a constant factor) by choosing $t = \Theta\left(\log_{\max\{2, \sigma\}} p\right)$, and the claim follows. \square

The above lower bound is tight. Consider the following $M(p, \sigma)$ algorithm for n -broadcast. Let the entries of V be evenly distributed among the processors, with $V[0]$ held by processor P_0 . For convenience we assume n is a power of two. Let κ be the smallest power of 2 greater than or equal to $\max\{2, \sigma\}$. The algorithm consists of $\lceil \log_{\kappa} p \rceil$ supersteps: in the i -th superstep, with $0 \leq i < \lceil \log_{\kappa} p \rceil$, each P_{jp/κ^i} , with $0 \leq j < \kappa^i$, sends the value $V[0]$ to $P_{(\kappa j + \ell)p/\kappa^{i+1}}$, for each $0 \leq \ell < \kappa$. (When $\log_{\kappa} p$ is not an integer value, in the last superstep only values of ℓ that are multiples of κ^{i+1}/p are used.) It is immediate to see that the algorithm belongs to \mathcal{C} and that its communication complexity on $M(p, \sigma)$ is

$$H_{\text{broad}_{\kappa}}(n, p, \sigma) = O((\kappa + \sigma) \log_{\kappa} p) = O\left(\max\{2, \sigma\} \log_{\max\{2, \sigma\}} p\right).$$

Therefore, the algorithm is $O(1)$ -optimal. Observe that the algorithm is aware of parameter σ and, in fact, this knowledge is crucial to achieve optimality. In order to see this, we prove that *any* network-oblivious algorithm for n -broadcast can be $\Theta(1)$ -optimal on $M(p, \sigma)$, only for limited ranges of σ . Let $H^*(n, p, \sigma)$ denote the best communication complexity achievable on $M(p, \sigma)$ by an algorithm for n -broadcast belonging to \mathcal{C} . By the above discussion we know that $H^*(n, p, \sigma) = \Theta\left(\max\{2, \sigma\} \log_{\max\{2, \sigma\}} p\right)$. Let $\mathcal{A} \in \mathcal{C}$ be a network-oblivious algorithm for n -broadcast specified on $M(v(n))$. For every $1 < p \leq v(n)$ and $0 \leq \sigma_1 \leq \sigma_2$, we define the maximum slowdown incurred by \mathcal{A} with respect to the best $M(p, \sigma)$ -algorithm in \mathcal{C} , for $\sigma \in [\sigma_1, \sigma_2]$, as

$$\text{GAP}_{\mathcal{A}}(n, p, \sigma_1, \sigma_2) = \max_{\sigma_1 \leq \sigma \leq \sigma_2} \left\{ \frac{H_{\mathcal{A}}(n, p, \sigma)}{H^*(n, p, \sigma)} \right\}.$$

Theorem 8. *Let $\mathcal{A} \in \mathcal{C}$ be a network-oblivious algorithm for n -broadcast specified on $M(v(n))$. For every $1 < p \leq v(n)$ and $0 \leq \sigma_1 \leq \sigma_2$, we have*

$$\text{GAP}_{\mathcal{A}}(n, p, \sigma_1, \sigma_2) = \Omega\left(\frac{\log \max\{2, \sigma_2\}}{\log \max\{2, \sigma_1\} + \log \log \max\{2, \sigma_2\}}\right).$$

Proof. The definition of function GAP implies that

$$\text{GAP}_{\mathcal{A}}(n, p, \sigma_1, \sigma_2) = \Omega\left(\frac{H_{\mathcal{A}}(n, p, \sigma_1)}{H^*(n, p, \sigma_1)} + \frac{H_{\mathcal{A}}(n, p, \sigma_2)}{H^*(n, p, \sigma_2)}\right).$$

Let t be the number of supersteps executed by the folding of \mathcal{A} on $M(p, \sigma)$, and note that, since \mathcal{A} is network-oblivious, this number cannot depend on σ . By arguing as in the proof of Theorem 7 (see

Inequality 6) we get that $H_{\mathcal{A}}(n, p, \sigma) = \Omega(t (\max\{2, \sigma\} + p^{1/t}))$, for any σ , hence $\text{GAP}_{\mathcal{A}}(n, p, \sigma_1, \sigma_2)$ is bounded from below by

$$\Omega\left(\frac{t (\max\{2, \sigma_1\} + p^{1/t})}{\max\{2, \sigma_1\} \log_{\max\{2, \sigma_1\}} p} + \frac{t (\max\{2, \sigma_2\} + p^{1/t})}{\max\{2, \sigma_2\} \log_{\max\{2, \sigma_2\}} p}\right),$$

which is minimized for $t = \Theta(\log p / (\log \max\{2, \sigma_1\} + \log \log \max\{2, \sigma_2\}))$. Substituting this value of t in the above formula yields the stated result. \square

An immediate consequence of the above theorem is that if a network-oblivious algorithm for n -broadcast is $\Theta(1)$ -optimal on $M(p, \sigma)$ it cannot be simultaneously $\Theta(1)$ -optimal on an $M(p, \sigma')$, for any σ' sufficiently larger than σ . A similar limitation of the optimality of a network-oblivious algorithm for n -broadcast can be argued with respect to its execution on $\text{D-BSP}(p, \mathbf{g}, \ell)$.

5 Extension to the Optimality Theorem

The optimality theorem of Section 3 makes crucial use of the wiseness property. Broadly speaking, a network-oblivious algorithm is $(\Theta(1), p)$ -wise when the communication performed in the various supersteps is somewhat balanced in the sense that the maximum number of messages sent/received by a virtual processor does not differ significantly from the average number of messages sent/received by other virtual processors belonging to the same region of suitable size. While there exist $(\Theta(1), p)$ -wise network-oblivious algorithms for a number of important problems, as shown in Section 4, there are cases where wiseness may not be guaranteed.

As a simple example of poor wiseness, consider a network-oblivious algorithm \mathcal{A} for $M(n)$ consisting of one 0-superstep where VP_0 sends n messages to $\text{VP}_{n/2}$. Fix p with $2 \leq p \leq n$. Clearly, for each $1 \leq j \leq \log p$ we have that $H_{\mathcal{A}}(n, 2^j, 0) = n$, hence the algorithm is (α, p) -wise only for $\alpha = O(1/p)$. When executed on a $\text{D-BSP}(p, \mathbf{g}, \mathbf{0})$, the communication time of the algorithm is ng_0 . However, as already observed in [11], under reasonable assumptions the communication time of the algorithm's execution on the D-BSP can be improved by first evenly spreading the n messages among clusters of increasingly larger size which include the sender, and then gathering the messages within clusters of increasingly smaller size which include the receiver. Motivated by this observation, we introduce a more effective protocol to execute network-oblivious algorithms on the D-BSP. By employing this protocol we are able to prove an alternative optimality theorem which requires a much weaker property than wiseness, at the expense of a slight (polylogarithmic) loss of efficiency.

Let \mathcal{A} be a network-oblivious algorithm specified on $M(v(n))$, and consider its execution on a $\text{D-BSP}(p, \mathbf{g}, \ell)$, with $1 \leq p \leq v(n)$. As before, each D-BSP processor P_j , with $0 \leq j < p$, carries out the operations of the $v(n)/p$ consecutively numbered VPs of $M(v(n))$ starting with $\text{VP}_{j(v(n)/p)}$. However, the communication required by each superstep is now performed on D-BSP more effectively by enforcing a suitable balancing. More precisely, each i -superstep s of \mathcal{A} , with $0 \leq i < \log p$, is executed on the D-BSP through the following protocol, which we will refer to as *novel protocol*:

1. (*Computation phase*) Each D-BSP processor performs the local computations of its assigned virtual processors.

2. (*Ascend phase*) For $k = \log p - 1$ down to $i + 1$: within each k -cluster Γ_k , the messages which originate in Γ_k but are destined outside Γ_k , are evenly distributed among the $p/2^k$ processors of Γ_k .
3. (*Descend phase*) For $k = i$ to $\log p - 1$: within each k -cluster Γ_k , the messages currently residing in Γ_k are evenly distributed among the processors of the $(k + 1)$ -clusters inside Γ_k which contain their final destinations.

Observe that each iteration of the ascend/descend phases requires a prefix-like computation to assign suitable intermediate destinations to the messages in order to guarantee their even distribution in the appropriate clusters.

Lemma 7. *Let \mathcal{A} be a network-oblivious algorithm specified on $M(v(n))$ and consider its execution on $D\text{-BSP}(p, \mathbf{g}, \ell)$, with $1 < p \leq v(n)$, using the novel protocol. Let ξ_s be the sequence of supersteps employed by the protocol for executing an i -superstep s of \mathcal{A} , where $0 \leq i < \log p$. Then, for every $i < k < \log p$, ξ_s comprises $O(1)$ k -supersteps of degree $O(2^k h_{\mathcal{A}}^s(n, 2^k)/p)$, and $O(\log p)$ k -supersteps each of constant degree.*

Proof. Consider iteration k of the ascend phase of the protocol, with $i + 1 \leq k \leq \log p - 1$, and a k -cluster Γ_k . As invariant at the beginning of the iteration, we have that the at most $h_{\mathcal{A}}^s(n, 2^{k+1})$ messages originating in each $k + 1$ -cluster Γ' included in Γ_k and destined outside Γ_k are evenly distributed among the processors of Γ' . Hence, the even distribution of these messages among the $p/2^k$ processors of Γ_k requires a prefix-like computation and an $O(\lceil 2^{k+1} h_{\mathcal{A}}^s(n, 2^{k+1})/p \rceil)$ -relation within Γ_k . Consider now iteration k of the descend phase of the protocol, with $i \leq k \leq \log p - 1$, and a k -cluster Γ_k . As invariant at the beginning of the iteration, we have that the at most $2h_{\mathcal{A}}^s(n, 2^{k+1})$ messages to be moved in the iteration are evenly distributed among the processors of Γ_k . Since each $(k + 1)$ -cluster included in Γ_k receives at most $h_{\mathcal{A}}^s(n, 2^{k+1})$ messages, the iteration requires a prefix-like computation and an $O(\lceil 2^{k+1} h_{\mathcal{A}}^s(n, 2^{k+1})/p \rceil)$ -relation within Γ_k . The lemma follows, since each prefix-like computation in a k -cluster can be performed in $O(\log p)$ k -supersteps of constant degree (e.g., using a straightforward tree-based strategy [33]). \square

We now define the notion of fullness which is weaker than wiseness but which still allows us to port the optimality of network-oblivious algorithms with respect to the evaluation model onto the execution machine model, although incurring some loss of efficiency.

Definition 4. *A static network-oblivious algorithm \mathcal{A} specified on $M(v(n))$ is said to be (γ, p) -full, for some $\gamma > 0$ and $1 < p \leq v(n)$, if considering the folding of \mathcal{A} on $M(2^j, 0)$ we have*

$$\sum_{i=0}^{j-1} F_{\mathcal{A}}^i(n, 2^j) \geq \gamma \frac{p}{2^j} \sum_{i=0}^{j-1} S_{\mathcal{A}}^i(n),$$

for every $1 \leq j \leq \log p$ and input size n .

It is easy to see that a $(\Theta(1), p)$ -wise network-oblivious algorithm \mathcal{A} is also $(\Theta(1), p)$ -full as long as $h_{\mathcal{A}}^s(n, p) \geq 1$, for every i -superstep s of \mathcal{A} and every $1 < p \leq v(n)$. On the other hand, a $(\Theta(1), p)$ -full algorithm is not necessarily $(\Theta(1), p)$ -wise, as witnessed by the previously mentioned network-oblivious algorithm consisting of a single 0-superstep where VP_0 sends n messages to $\text{VP}_{n/2}$, which is $(\Theta(1), p)$ -full but not $(\Theta(1), p)$ -wise, for any $2 \leq p \leq n$. In this sense, (γ, p) -fullness is a weaker condition than $(\Theta(1), p)$ -wiseness.

The following theorem shows that when (γ, p) -full algorithms are executed on the D-BSP using the novel protocol, optimality in the evaluation model is preserved on the D-BSP within a polylogarithmic factor. As in Section 3, let \mathcal{C} denote a given class of static algorithms solving a problem Π , with the property that for any algorithm $\mathcal{A} \in \mathcal{C}$, all its foldings on $p \geq 2$ processing elements also belong to \mathcal{C} .

Theorem 9. *Let $\mathcal{A} \in \mathcal{C}$ be a (γ, p^*) -full network-oblivious algorithm for some $\gamma > 0$ and a power of two p^* . Let also $\{\sigma_0^m, \dots, \sigma_{\log p^* - 1}^m\}$ and $\{\sigma_0^M, \dots, \sigma_{\log p^* - 1}^M\}$ be two vectors of non-negative values, with $\sigma_j^m \leq \sigma_j^M$, for every $0 \leq j < \log p^*$. If \mathcal{A} is β -optimal on $M(2^j, \sigma)$ w.r.t. \mathcal{C} , for $\sigma_{j-1}^m \leq \sigma \leq \sigma_{j-1}^M$ and $1 \leq j \leq \log p^*$, then, for every power of two $p \leq p^*$, \mathcal{A} is $\Theta(\beta / ((1 + 1/\gamma) \log^2 p))$ -optimal on D-BSP(p, \mathbf{g}, ℓ) w.r.t. \mathcal{C} when executed with the above protocol, as long as:*

- the execution of \mathcal{A} on D-BSP(p, \mathbf{g}, ℓ) using the above protocol is in \mathcal{C} ;
- $g_i \geq g_{i+1}$ and $\ell_i/g_i \geq \ell_{i+1}/g_{i+1}$, for $0 \leq i < \log p - 1$;
- $\max_{1 \leq k \leq \log p} \{\sigma_{k-1}^m 2^k/p\} \leq \ell_i/g_i \leq \min_{1 \leq k \leq \log p} \{\sigma_{k-1}^M 2^k/p\}$, for $0 \leq i < \log p$.

Proof. Consider the execution of \mathcal{A} on a D-BSP(p, \mathbf{g}, ℓ) using the novel protocol. Let $\tilde{\mathcal{A}}$ denote the actual sequence of supersteps performed on the D-BSP in this execution of \mathcal{A} . Note that once the D-BSP parameters are fixed, $\tilde{\mathcal{A}}$ can be regarded as a network-oblivious algorithm specified on $M(p)$. Clearly, any optimality considerations on the communication time of the execution of $\tilde{\mathcal{A}}$ (regarded as a network-oblivious algorithm) on D-BSP(p, \mathbf{g}, ℓ) using the standard protocol, will also apply to the communication time of the execution of \mathcal{A} on D-BSP(p, \mathbf{g}, ℓ) using the novel protocol, being the two communication times the same.

We will assess the degree of optimality of the communication time of $\tilde{\mathcal{A}}$ by resorting to Theorem 1. This entails analyzing the communication complexity of $\tilde{\mathcal{A}}$ on $M(2^j, \sigma)$, for any $1 \leq j \leq \log p$, and determining its wiseness. Focus on $M(2^j, \sigma)$ for some $1 \leq j \leq \log p$, and consider an arbitrary i -superstep s of \mathcal{A} , for some $0 \leq i < j$. Let ξ_s be the sequence of supersteps in $\tilde{\mathcal{A}}$ executed in the ascend and descend phases associated with superstep s . From Lemma 7, we know that for every $i < k < \log p$, ξ_s comprises $O(1)$ k -supersteps of degree $O(2^k h_{\mathcal{A}}^s(n, 2^k)/p)$, and $O(\log p)$ k -supersteps each of constant degree. Now, in the execution on $M(2^j, \sigma)$ a k -superstep with $k \geq j$ becomes local to the processors and does not contribute to the communication complexity. Since each processor of $M(2^j, \sigma)$ corresponds to $p/2^j$ processors of $M(p)$, the communication complexity on $M(2^j, \sigma)$ contributed by the sequence ξ_s is

$$O\left(\sum_{k=i+1}^{j-1} \left(\frac{p}{2^j} \left(\frac{2^k}{p} h_{\mathcal{A}}^s(n, 2^k) + \log p\right) + \sigma \log p\right)\right).$$

Therefore, since $h_{\mathcal{A}}^s(n, 2^k) \leq 2^{j-k} h_{\mathcal{A}}^s(n, 2^j)$, the above summation is upper bounded by

$$O\left(\sum_{k=i+1}^{j-1} \left(h_{\mathcal{A}}^s(n, 2^j) + \frac{p \log p}{2^j} + \sigma \log p\right)\right) = O\left(\left(h_{\mathcal{A}}^s(n, 2^j) + \frac{p}{2^j} + \sigma\right) \log^2 p\right).$$

Recall that $L_{\mathcal{A}}^i(n)$ denotes the set of i -supersteps executed by \mathcal{A} , and $S_{\mathcal{A}}^i(n) = |L_{\mathcal{A}}^i(n)|$. Thus, the

communication complexity of $\tilde{\mathcal{A}}$ on $M(2^j, \sigma)$ can be written as

$$\begin{aligned}
H_{\tilde{\mathcal{A}}}(n, 2^j, \sigma) &= O\left(\sum_{i=0}^{j-1} \sum_{s \in L_{\mathcal{A}}^i(n)} \left(h_{\mathcal{A}}^s(n, 2^j) + \frac{p}{2^j} + \sigma\right) \log^2 p\right) \\
&= O\left(\log^2 p \left(\sum_{i=0}^{j-1} \sum_{s \in L_{\mathcal{A}}^i(n)} (h_{\mathcal{A}}^s(n, 2^j) + \sigma) + \sum_{i=0}^{j-1} \sum_{s \in L_{\mathcal{A}}^i(n)} \frac{p}{2^j}\right)\right) \\
&= O\left(\log^2 p \left(H_{\mathcal{A}}(n, 2^j, \sigma) + \frac{p}{2^j} \sum_{i=0}^{j-1} S_{\mathcal{A}}^i(n)\right)\right) \\
&= O((1 + 1/\gamma) \log^2 p \cdot H_{\mathcal{A}}(n, 2^j, \sigma)),
\end{aligned}$$

where the last inequality follows by the (γ, p^*) -fullness of \mathcal{A} .

The above inequality shows that algorithm $\tilde{\mathcal{A}}$ is $\beta/((1 + 1/\gamma) \log^2 p)$ -optimal as a consequence of the β -optimality of \mathcal{A} . Let us now assess the wiseness of $\tilde{\mathcal{A}}$. Consider again the sequence ξ_s of supersteps of $\tilde{\mathcal{A}}$ associated with an arbitrary i -superstep s of \mathcal{A} , for some $0 \leq i < \log p$. We know that for every $i < k < \log p$, ξ_s comprises $O(1)$ k -supersteps of degree $O(2^k h_{\mathcal{A}}^s(n, 2^k)/p)$, and $O(\log p)$ k -supersteps each of constant degree. Moreover, we can assume that suitable dummy messages are added so that in a k -superstep of degree $O(2^k h_{\mathcal{A}}^s(n, 2^k)/p)$ (resp., degree $O(1)$) all processors of a $(k + 1)$ -cluster send $\Theta(2^k h_{\mathcal{A}}^s(n, 2^k)/p)$ (resp., $\Theta(1)$) messages to the sibling $(k + 1)$ -cluster included in the same k -cluster. It is easy to see that the above considerations about the optimality of $\tilde{\mathcal{A}}$ remain unchanged, while $\tilde{\mathcal{A}}$ becomes $(\Theta(1), p)$ -wise. Finally, we recall that $\tilde{\mathcal{A}}$ belongs to class \mathcal{E} by hypothesis, and this is so even forcing it into being wise. Therefore, by applying Theorem 1 to $\tilde{\mathcal{A}}$, we can conclude that $\tilde{\mathcal{A}}$, hence \mathcal{A} , is $\Theta(\beta/((1 + 1/\gamma) \log^2 p))$ -optimal on a D-BSP(p, \mathbf{g}, ℓ) with parameters satisfying the initial hypotheses. \square

We conclude this section by observing that the relation stated by the above theorem between optimality in the evaluation model and optimality in D-BSP can be tighten when the g_i and ℓ_i parameters of the D-BSP decrease geometrically. In this case, it is known that a prefix-like computation within a k -cluster, for $0 \leq k < \log p$, can be performed in $O(g_k + \ell_k)$ communication time (e.g., see [11, Proposition 2.2.2]). Then, by a similar argument used to prove Theorem 9 it can be shown that a (γ, p) -full algorithm \mathcal{A} which is β -optimal in the evaluation model becomes $\Theta(\beta/((1 + 1/\gamma) \log p))$ -optimal when executed on the D-BSP, thus reducing by a factor $\log p$ the gap between the two optimality factors.

6 Conclusions

We introduced a framework to explore the design of network-oblivious algorithms, that is, algorithms which run efficiently on machines with different processing power and different bandwidth/latency characteristics, without making explicit use of architectural parameters for tuning performance. In the framework, a network-oblivious algorithm is written for $v(n)$ virtual processors (specification model), where n is the input size and $v(\cdot)$ a suitable function. Then, the performance of the algorithm is analyzed in a simple model (evaluation model) consisting of $p \leq v(n)$ processors and where the impact of the network topology on communication costs is accounted for by a latency parameter σ . Finally, the algorithm is executed on the D-BSP model [25, 11] (execution

machine model), which describes reasonably well the behavior of a large class of point-to-point networks by capturing their hierarchical structures. A D-BSP consists of $p \leq v(n)$ processors and its network topology is described by the $\log p$ -size vectors \mathbf{g} and $\mathbf{\ell}$, which account for bandwidth and latency costs within nested clusters, respectively. We have shown that for static network-oblivious algorithms, where the communication requirements depend only on the input size and not on the specific input instance (e.g., algorithms arising in DAG computations), the optimality on the evaluation model for certain ranges of p and σ translates into optimality on the D-BSP model for corresponding ranges of the model’s parameters. This result justifies the introduction of the evaluation model that allows for a simple analysis of network-oblivious algorithms while effectively bridging the performance analysis to D-BSP which more accurately models the communication infrastructure of parallel platforms through a logarithmic number of parameters.

We devised $\Theta(1)$ -optimal static network-oblivious algorithms for prominent problems such as matrix multiplication, FFT, and sorting, although in the case of sorting optimality is achieved only when the available parallelism is polynomially sublinear in the input size. Also, we devised suboptimal, yet efficient, network-oblivious algorithms for stencil computations, and we explored limitations of the oblivious approach by showing that for the broadcasting problem optimality in D-BSP can be achieved by a network-oblivious algorithm only for rather limited ranges of the parameters. Similar negative results were also proved in the realm of cache-oblivious algorithms (e.g., see [8, 18, 41, 42]). Despite these limitations, the pursuit of oblivious algorithms appears worthwhile even when the outcome is a proof that no such algorithm can be $\Theta(1)$ -optimal on an ample class of target machines. Indeed, the analysis behind such a result is likely to reveal what kind of adaptivity to the target machine is necessary to obtain optimal performance.

The present work can be naturally extended in several directions, some of which are briefly outlined below. First, it would be useful to further assess the effectiveness of our framework by developing novel efficient network-oblivious algorithms for prominent problems beyond the ones of this paper. Some progress in this direction has been done in [20, 26]. For the problems considered here, in particular sorting and stencil computations, it would be very interesting to investigate the potentiality of the network-oblivious approach at a fuller degree. More generally, it would be interesting to develop lower-bound techniques to limit the level of optimality that network-oblivious algorithms can reach on certain classes of target platforms. Another challenging goal concerns the generalization of the results of Theorems 1 and 9 to algorithms for a wider class of problems, and the weakening of the assumptions (wiseness or fullness) required to prove these results. It would be also useful to identify other classes of machines, for which network-oblivious algorithms can be effective. Another natural question to investigate is how to determine an efficient virtual-to-physical processor mapping for a network with arbitrary topology. Again, it would be very interesting to generalize our work to apply to computing scenarios, such as traditional time-shared systems as well as emerging global computing environments, where the amount of resources devoted to a specific application can itself vary dynamically over time, in the same spirit as [6] generalized the cache-oblivious framework to environments in which the amount of memory available to an algorithm can fluctuate. Finally, a somewhat ultimate goal is represented by the integration of cache- and network-obliviousness in a unified framework for the development of *machine*-oblivious computations, as recently done in [20] for shared-memory platforms.

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