

# Correspondence

## On Linear $H^\infty$ Equalization of Communication Channels

Alper T. Erdogan, Babak Hassibi, and Thomas Kailath

**Abstract**—As an alternative to existing techniques and algorithms, we investigate the merit of the  $H^\infty$  approach to the linear equalization of communication channels. We first give the formulation of all causal  $H^\infty$  equalizers using the results of and then look at the finite delay case. We compare the risk-sensitive  $H^\infty$  equalizer with the MMSE equalizer with respect to both the average and the worst-case BER performances and illustrate the improvement due to the use of the  $H^\infty$  equalizer.

**Index Terms**— $H^\infty$  estimation, linear equalization, risk-sensitive estimation.

### I. INTRODUCTION

Equalization is a well-studied problem in the area of communications. The data model for the equalization is generally described by a linear model of the type shown in Fig. 1. The discrete data sequence  $\{b_i\}$  passes through the linear time-invariant channel  $H(z)$ , which causes inter-symbol interference (ISI). The observation sequence  $\{y_i\}$  is then formed by the addition of an unknown measurement disturbance  $\{v_i\}$  with the output of the communication channel  $H(z)$ . In many cases, in addition to the structural model given for the observations, it is also possible to give statistical descriptions of the input sequence  $\{b_i\}$ , the additive disturbance sequence  $\{v_i\}$ , and even the channel  $H(z)$  itself.

In equalization, the basic aim is to *invert* the effect of the channel to reduce the ISI so that symbol-by-symbol detection can be applied to the output of the equalizer. This is accomplished by estimating  $b_{i-d}$ , where  $d > 0$  represents some prescribed finite delay using the observations  $\{y_j, j \leq i\}$ . This is achieved via a causal linear time-invariant filter  $K(z)$ , which is known as the *equalizer*.

In this correspondence, we address the robustness against variations in the system parameters by approaching the equalization problem from the  $H^\infty$  estimation point of view. The richness of robust  $H^\infty$  theory, and especially its stochastic interpretation of risk-sensitive estimation, has been the basic motivation for our approach. Finally, and perhaps most importantly, the results obtained in this attempt provide us with a new and different perspective for the understanding and analysis of the equalization problem, as well as for  $H^\infty$  estimation itself.

In the next section of this correspondence, we pose the equalization problem from the  $H^\infty$  estimation perspective and its stochastic counterpart risk-sensitive estimation. Then, we will describe the equalizer formulations for the causal and finite delay cases in Section III. In Section IV, we compare the risk-sensitive and MMSE equalizers for the average and the worst-case BER performances. The conclusion is given in Section V.

### II. $H^\infty$ AND RISK SENSITIVE EQUALIZATION

The basic aim in  $H^\infty$  equalization is to minimize the maximum energy gain from the disturbances to the estimation errors. This property

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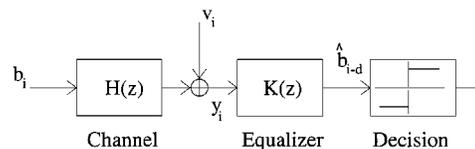


Fig. 1. Linear data model.

ensures the fact that if the disturbances are small (in energy), then the estimation errors will be as well.

The optimal  $H^\infty$  equalization problem can be formulated as follows.

**Problem 1 (Optimal  $H^\infty$  Equalization Problem):** Find a causal equalizer  $K(z)$  that satisfies

$$\inf_{K(z)} \sup_{\omega \in [0, 2\pi]} \left| e^{-j\omega d} - K(e^{j\omega})H(e^{j\omega}) \right|^2 q + \left| K(e^{j\omega}) \right|^2 r \quad (1)$$

$$= \inf_{K(z)} \sup_{b, v \in l^2, b, v \neq 0} \frac{\sum_{i=-\infty}^{\infty} |b_{i-d} - \hat{b}_{i-d}|^2}{q^{-1} \sum_{i=-\infty}^{\infty} |b_i|^2 + r^{-1} \sum_{i=-\infty}^{\infty} |v_i|^2} \triangleq \gamma_{opt}^2 \quad (2)$$

where  $l^2$  denotes the space of square-summable sequences, and  $q$  and  $r$  are positive weights. Moreover, find the min-max energy gain  $\gamma_{opt}^2$ .

There are very few cases where a closed-form solution to the optimal  $H^\infty$  equalization problem can be found, and in general, one relaxes the minimization and settles for a suboptimal solution.

**Problem 2 (Suboptimal  $H^\infty$  Equalization Problem):** Given  $\gamma > 0$ , find, if possible, a causal equalizer  $K(z)$  that guarantees

$$\sup_{b, v \in l^2, b, v \neq 0} \frac{\sum_{i=-\infty}^{\infty} |b_{i-d} - \hat{b}_{i-d}|^2}{q^{-1} \sum_{i=-\infty}^{\infty} |b_i|^2 + r^{-1} \sum_{i=-\infty}^{\infty} |v_i|^2} < \gamma^2. \quad (3)$$

This clearly requires checking whether  $\gamma > \gamma_{opt}$ .

As shown in [1] and [2], one possible approach to solve this problem is based on  $J$ -spectral factorization. We need to first introduce the following so-called *Popov function*:

$$\begin{bmatrix} r + qH(z)H^*(z^{-*}) & -qH(z)z^d \\ -qz^{-d}H^*(z^{-*}) & -\gamma^2 + q \end{bmatrix}$$

which can be regarded as a certain indefinite generalization of the spectral density function  $r + qH(z)H^*(z^{-*})$ . Then a causal  $\gamma$ -level equalizer  $K(z)$  exists if, and only if, the Popov function admits a canonical factorization of the form

$$\begin{bmatrix} r + qH(z)H^*(z^{-*}) & -qH(z)z^d \\ -qz^{-d}H^*(z^{-*}) & q - \gamma^2 \end{bmatrix} = \underbrace{\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix}}_{\Lambda(z)} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} L_{11}^*(z^{-*}) & L_{21}^*(z^{-*}) \\ L_{12}^*(z^{-*}) & L_{22}^*(z^{-*}) \end{bmatrix} \quad (4)$$

with  $\Lambda(z)$  and  $L_{11}(z)$  causal and causally invertible, and  $L_{12}(z)$  strictly causal. If this is the case, then all possible  $H^\infty$  equalizers of level  $\gamma$  are given by

$$K(z) = (L_{22}(z)Q(z) - L_{21}(z))(L_{11}(z) - L_{12}(z)Q(z))^{-1} \quad (5)$$

where  $Q(z)$  is any causal and strictly contractive operator, i.e.,  $Q(z)$  is causal and is such that  $|Q(e^{j\omega})|^2 < 1$  for all  $\omega \in [0, 2\pi]$ .

An important choice results from taking  $Q = 0$  so that

$$K_{cen}(z) = -L_{21}(z)L_{11}^{-1}(z) \quad (6)$$

which is the so-called ‘‘central’’ filter. Although the  $H^\infty$  estimation formulation is a deterministic one, the central equalizer also has a nice stochastic interpretation: If we assume that the disturbances  $\{b_i\}$  and  $\{v_i\}$  are stationary independent Gaussian random processes with variances  $q$  and  $r$ , respectively, the central filter is the risk-sensitive equalizer that minimizes

$$\frac{2}{\theta} \log \left( E \exp \left( \frac{\theta}{2} \sum_{i=-\infty}^{\infty} |s_i - \hat{s}_i|^2 \right) \right) \quad (7)$$

where  $\theta = 1/\gamma^2$  is known as the risk-sensitivity parameter.

### III. $H^\infty$ EQUALIZERS

#### A. Casual Case

When  $d = 0$ , we constrain our equalizer to use only  $\{y_j; j \leq i\}$  to estimate the value of  $b_i$ . In [1], when the channel  $H(z)$  is causal and when  $d = 0$ , the  $J$ -spectral factorization (4) was explicitly obtained, and thereby, a characterization for all  $H^\infty$  equalizers was derived. The main results can be summarized as follows.

- i) The channel  $H(z)$  is *nonminimum phase* (in other words,  $H^{-1}(z)$  is not causal). In this case

$$\gamma_{opt}^2 = q \quad (8)$$

which is the same energy gain obtained from  $K(z) = 0$ , i.e., not equalizing at all! To this end, note that when  $K(z) = 0$ , then the estimation error is  $b_i - \hat{b}_i = b_i - 0 = b_i$  so that the energy gain from the disturbances to the estimation errors becomes

$$\frac{\sum_{i=-\infty}^{\infty} |b_i|^2}{q^{-1} \sum_{i=-\infty}^{\infty} |b_i|^2 + r^{-1} \sum_{i=-\infty}^{\infty} |v_i|^2} \leq q. \quad (9)$$

Therefore, there is no hope for causally equalizing a nonminimum-phase channel.

- ii) The channel  $H(z)$  is *minimum phase* (in other words,  $H^{-1}(z)$  is causal). In this case

$$\gamma_{opt} = \gamma_{opt, smoothing} \quad (10)$$

where smoothing equalizer is the Wiener filter that is noncausal. This implies that from an  $H^\infty$  point of view, one can obtain

the same performance as the smoother by causally equalizing a minimum-phase channel and without using future observations. This means that one can expect to equalize a minimum-phase channel without incurring any delay.

In the minimum-phase case (where, for simplicity, we have taken  $q = 1$ ),  $\Lambda(z)$  in (4) takes the form shown in (11) at the bottom of the page, where the monic and minimum phase transfer function  $\Delta(z)$  and the scalar  $R_\Delta$  are found from the standard spectral factorization:

$$R_\Delta \Delta(z) \Delta^*(z^{-*}) = \frac{\gamma^2}{1 - \gamma^2} H(z) H^*(z^{-*}) - r \geq 0. \quad (12)$$

Moreover, the min–max energy gain is

$$\gamma_{opt}^2 = \gamma_{opt, smoothing}^2 = \sup_{\omega \in [0, 2\pi]} \frac{r}{r + |H(e^{j\omega})|^2}. \quad (13)$$

Finally, all  $H^\infty$  equalizers are given by (5), and the central equalizer, which is also the risk-sensitive equalizer, is given by

$$\begin{aligned} K_{central}(z) &= -L_{21}(z)L_{11}^{-1}(z) \\ &= \frac{h_0(1 - \gamma^2)}{h_0 H(z) - (1 - \gamma^2)R_\Delta \Delta(z)}. \end{aligned} \quad (14)$$

We should also remark that another revealing choice of the causal contraction  $Q(z)$  yields the following equalizer:

$$K(z) = \frac{1 - \gamma^2}{H(z)} \quad (15)$$

which is simply a scaled version of the zero-forcing equalizer. Thus, an appropriately scaled zero-forcing equalizer is  $H^\infty$ -optimal. However, although it has the optimum worst-case performance, due to its noise-enhancement properties, the zero-forcing equalizer has undesirable average performance compared with, say, the central  $H^\infty$ -optimal equalizer.

#### B. Finite Delay Case

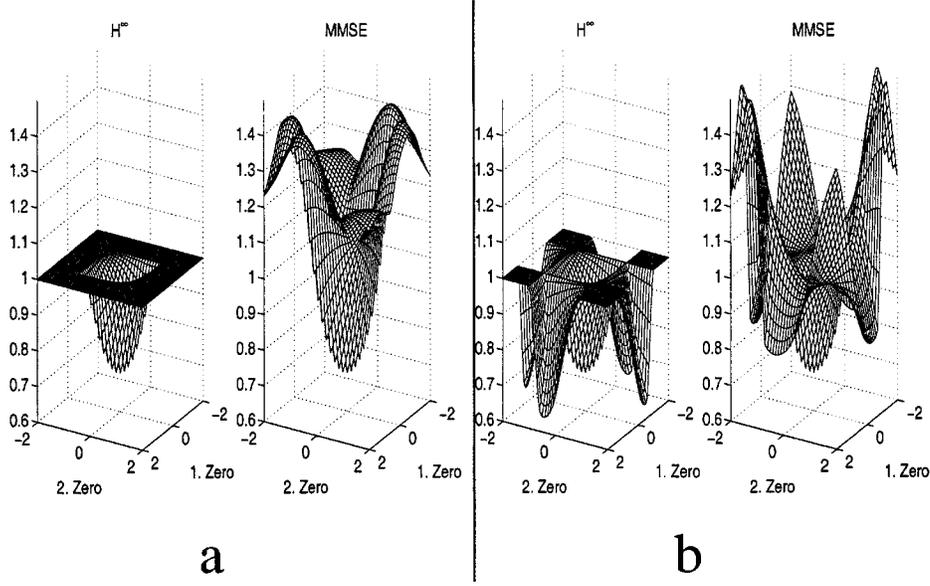
1) *Improvement Due to Delay*: In the previous section, we constrained the equalizer to be causal by choosing  $d = 0$ . We can relax the causality constraint by allowing the equalizer to use a finite number of future observations. This case would be equivalent to choosing  $d > 0$ . With this relaxation, it will be possible to equalize nonminimum-phase channels by an appropriate choice of  $d$ . Moreover, one can also expect an improvement in equalizing minimum-phase channels with respect to other criteria (such as the  $H^2$  or risk-sensitive criteria).

In order to illustrate the effect of delay, it will be instructive to look at the special case of equalizing the single zero channel  $H(z) = 1 + az^{-1}$ .

*Lemma 1 (Equalization of Single-Zero Channel)*: Consider the scalar single-zero FIR channel  $H(z) = 1 + az^{-1}$ , and suppose we want to solve the problem

$$\begin{aligned} \min_{\text{causal } K(z)} \left\| \left[ z^{-d} - K(z)H(z) - K(z) \right] \right\|_\infty \\ = \gamma_{opt, filtering} \end{aligned} \quad (16)$$

$$\Lambda(z) = \begin{bmatrix} \frac{h_0 H(z) - (1 - \gamma^2)R_\Delta \Delta(z)}{\sqrt{(h_0^2 - (1 - \gamma^2)R_\Delta)(1 - \gamma^2)}} & \frac{(-H(z) + h_0 \Delta(z))\sqrt{(R_\Delta)}}{\sqrt{h_0^2 - (1 - \gamma^2)R_\Delta}} \\ -h_0 \sqrt{\frac{1 - \gamma^2}{h_0^2 - (1 - \gamma^2)R_\Delta}} & \frac{\sqrt{R_\Delta}(1 - \gamma^2)}{\sqrt{h_0^2 - (1 - \gamma^2)R_\Delta}} \end{bmatrix} \quad (11)$$


 Fig. 2. Optimal  $\gamma$  plot for two-(real) zero channel as a function of zero locations.

for  $d \geq 0$ . Then, we have the following result.

1) If  $d = 0$ , then

$$\gamma_{opt, filtering}^2 = \begin{cases} \max_{w \in [0, 2\pi]} \frac{1}{1 + |H(e^{jw})|^2}, & \text{if } |a| < 1 \\ 1, & \text{if } |a| \geq 1. \end{cases} \quad (17)$$

2) If  $d = 1$ , then

$$\gamma_{opt, filtering}^2 = \begin{cases} \max_{w \in [0, 2\pi]} \frac{1}{1 + |H(e^{jw})|^2}, & \text{if } |a| < 2 \\ \frac{2}{|a|^2}, & \text{if } |a| \geq 2. \end{cases} \quad (18)$$

3) If  $d \geq 2$ , then

$$\gamma_{opt, filtering}^2 = \max_{w \in [0, 2\pi]} \frac{1}{1 + |H(e^{jw})|^2}. \quad (19)$$

This example shows that for the channel  $H(z) = 1 + az^{-1}$  with  $r = 1$ , allowing a single delay in equalization results in extending the region for which  $\gamma_{opt} = \gamma_{smoothing}$  from inside the unit circle to inside a circle of radius two. Outside this region, although the intersymbol interference component begins to dominate,  $\gamma$  does not stay constant and decreases with increasing value of  $a$  at a rate slower than the smoothing case. Furthermore, for  $d \geq 2$ ,  $\gamma_{opt}^2 = \gamma_{opt, smoothing}^2$ , i.e., a delay of two units is sufficient to obtain the same  $H^\infty$  performance as the smoother in equalizing a single-zero channel.

In Fig. 2, the  $\gamma$  values for linear  $H^\infty$  and MMSE equalizers are compared for two-(real) zero channels. Fig. 2(a) compares the  $d = 0$  case, and Fig. 2(b) compares the  $d = 1$  case.

Unfortunately, there is no known explicit formulation for arbitrary  $d > 0$  and general nonminimum-phase channels. However, it has been shown that in order to get an improvement over  $\gamma_{opt} = 1$ , the delay  $d$  should be chosen greater than the number of nonminimum-phase zeros of the channel, i.e., the number of zeros outside of the unit circle.

2) *J-Spectral Factorization for Finite Delay Case*: We obtained the explicit  $J$ -spectral factorization for the zero delay case. For the finite delay case, unfortunately, we cannot obtain the  $J$ -spectral factorization as explicit as the zero delay case. However, we can still systematically carry on the factorization.

- 1) We first calculate optimal value of  $\gamma_{opt, filtering}$ , for example, using the bisection method [4].
- 2) For a  $\gamma > \gamma_{opt, filtering}$ , the Popov function can be written as

$$\Sigma(z) = \begin{bmatrix} I + H(z)H^*(z^{-*}) & -z^d H(z) \\ -z^{-d} H^*(z^{-*}) & (1 - \gamma^2)I \end{bmatrix}. \quad (20)$$

Note that  $\Sigma(z)$  is not unimodular. In fact, its determinant is

$$\begin{aligned} \det(\Sigma(z)) &= \det((1 - \gamma^2)(I + H(z)H^*(z^{-*})) - H(z)H^*(z^{-*})) \\ &= (1 - \gamma^2) \det\left(I - \frac{\gamma^2}{1 - \gamma^2} H(z)H^*(z^{-*})\right) \end{aligned}$$

which is independent of  $d$ .

- 3) Since  $\gamma > \gamma_{smoothing}$ , we again define

$$\Delta(z)R_\Delta\Delta^*(z^{-*}) = \frac{\gamma^2}{1 - \gamma^2} H(z)H^*(z^{-*}) - I \quad (21)$$

so that

$$\det(\Sigma(z)) = -(1 - \gamma^2) \det(\Delta(z)R_\Delta\Delta^*(z^{-*})). \quad (22)$$

- 4) We first extract (23), shown at the bottom of the next page, so that  $\Sigma_1(z)$  is unimodular with determinant

$$\det(\Sigma_1(z)) = -(1 - \gamma^2). \quad (24)$$

- 5) Since  $\Sigma_1(z)$  is unimodular, one can apply standard spectral factorization techniques outlined in [5] to obtain

$$\Sigma_1(z) = P'(z) \underbrace{\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}}_J P'^*(z^{-*}) \quad (25)$$

where  $P'(z)$  is causal and causally invertible. If we summarize the steps in obtaining  $P'(z)$ , we first decompose

$$z^d \Delta^{-1}(z)H(z) = z^d W_1(z) + W_2(z) \quad (26)$$

where  $W_1(z) = w_{1,0} + w_{1,1}z^{-1} + \dots + w_{1,d-1}z^{-(d-1)}$ , and  $W_2(z) = w_{2,0} + w_{2,1}z^{-1} + \dots$ . Using this substitution, we apply the following factorization to  $\Sigma_1(z)$ :

$$\Sigma_1(z) = \underbrace{\begin{bmatrix} I & W_2(z) \\ 0 & I \end{bmatrix}}_{P_1(z)} \underbrace{\begin{bmatrix} -I + \frac{W_1(z)W_1^*(z^{-*})}{1-\gamma^2} & -z^d W_1(z) \\ -z^{-d} W_1^*(z^{-*}) & (1-\gamma^2)I \end{bmatrix}}_{\Sigma_2(z)} \cdot \underbrace{\begin{bmatrix} I & 0 \\ W_2^*(z^{-*}) & I \end{bmatrix}}_{P_1^*(z^{-*})} \quad (27)$$

Therefore, contrary to  $\Sigma_1(z)$ ,  $\Sigma_2(z)$  is FIR besides being unimodular. Now, it is easy to reduce the degree of  $\Sigma_2(z)$  by applying Gaussian elimination iteratively to the highest power of  $z^{-1}$  terms at four blocks. Following this procedure, we obtain

$$\Sigma_2(z) = P_2(z)JP_2^*(z^{-*}) \quad (28)$$

where  $P_2(z)$  is causal and unimodular.

Consequently, we can write

$$\begin{aligned} \Sigma(z) &= F_1(z)P_1(z)P_2(z)JP_2^*(z^{-*})P_1^*(z^{-*})F_1^*(z^{-*}) \\ &= P(z)JP^*(z^{-*}) \end{aligned} \quad (29)$$

where  $P(z) = F_1(z)P_1(z)P_2(z)$  is also causal and causally invertible.

- 6) Finally, we obtain the desired spectral factor  $L(z) = \begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix}$  by multiplying  $P(z)$  with a  $J$ -unitary matrix  $\Theta$  from right, i.e.,  $L(z) = P(z)\Theta$  so that  $L_{11}(z)$  is minimum phase, and  $L_{12}(z)$  is strictly causal. Here, we obtain the  $J$ -unitary matrix using the same procedure we used in the zero delay case.

#### IV. COMPARATIVE PERFORMANCE OF EQUALIZATION

In this section of the paper, we will compare the performance of the central  $H^\infty$  equalizers with MMSE equalizers. The most reasonable criterion for the comparison is BER, especially when all bits have the same significance.

In general, for the various channels that we have studied, the risk-sensitive equalizer and the MMSE equalizer have either similar average BER performances or the MMSE equalizer slightly outperforms the risk-sensitive equalizer. It thus appears that in terms of average BER performance, there is no gain in using central  $H^\infty$  equalizers, compared with MMSE ones in the ideal setup.

Another criterion of interest, rather than the average behavior, is how the equalizers will behave for individual paths of the noise process, particularly for the worst-case noise disturbance. For the infinite horizon, worst-case noise disturbance refers to the singletone noise located at the frequency at which error spectrum takes its maximum value. In the finite horizon case, we can obtain the worst-case noise disturbance by calculating the singular vector of the error transfer matrix, which maps

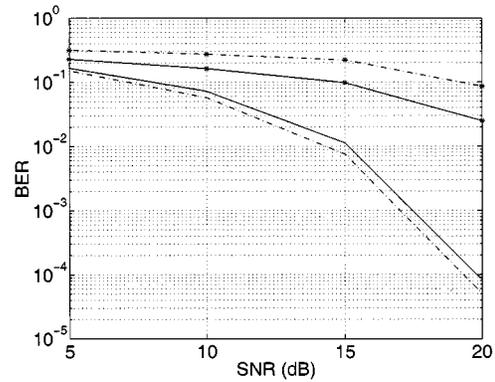


Fig. 3. Worst-case and average BER performances for the channel  $H(z) = 1 + 0.65z^{-1} - 0.52z^{-2} - 0.2975z^{-3}$ .

error disturbances to the equalization errors to the equalization errors, corresponding to the maximum singular value.

For time-invariant systems, the resulting noise waveform is a windowed cosine function located at the frequency where error spectrum takes its maximum value. Typically, both MMSE and risk-sensitive equalizers have their maximum error spectrum values located at the same frequency, which is determined by the zero locations of the channel, especially if the noise spectral density is assumed to be flat. Since the maximum error spectrum value for the risk-sensitive equalizer is less than the MMSE equalizer, we expect a better worst-case performance for the risk-sensitive equalizer.

We illustrate this fact for the example channel

$$H(z) = 1 + 0.65z^{-1} - 0.52z^{-2} - 0.2975z^{-3}. \quad (30)$$

The lower two lines in Fig. 3 correspond to the average BER behavior for two equalizers, whereas the upper two lines correspond to the worst-case BER. It is clear from this figure that average BER performance of the MMSE equalizer (dashed line) is slightly better, but they are actually very close. However, if we look at the worst-case BER performances, the risk-sensitive equalizer has considerably better performance than the MMSE equalizer. Since the performance of the risk-sensitive equalizer is less sensitive to the worst-case noise disturbance, for individual noise paths, the maximum deviation from average performance for risk-sensitive equalizer is smaller than MMSE equalizer, and therefore, it is more robust in this sense. We can consider the effect of modeling errors in channel and statistics of the disturbances in the system to be the additional noise injected to the system. Since risk-sensitive equalizers have better worst-case performance, they will be more robust against modeling errors.

#### V. CONCLUSION

In this correspondence, we introduced the  $H^\infty$  criterion as an alternative method for the equalization of communication channels. All previous methods and algorithms mainly concentrate on the average behavior (e.g., BER) of the equalizer without being concerned with the worst-case performance. Moreover, from [1], we know that using

$$\Sigma(z) = \underbrace{\begin{bmatrix} \Delta(z)R_\Delta^{1/2} & 0 \\ 0 & I \end{bmatrix}}_{F_1(z)} \underbrace{\begin{bmatrix} \Delta^{-1}(z)(I + H(z)H^*(z^{-*}))\Delta^{-*}(z^{-*}) & -z^d \Delta^{-1}(z)H(z) \\ -z^{-d} H^*(z^{-*})\Delta^{-*}(z^{-*}) & (1-\gamma^2)I \end{bmatrix}}_{\Sigma_1(z)} \underbrace{\begin{bmatrix} R_\Delta^{*/2} \Delta^*(z^{-*}) & 0 \\ 0 & I \end{bmatrix}}_{F_1^*(z^{-*})} \quad (23)$$

causal  $H^\infty$  equalizers, it is possible to achieve the same worst-case performance as the noncausal smoothing filter in equalizing minimum-phase channels. For nonminimum-phase channels, we need a sufficient number of delays (at least equal to the number of nonminimum-phase zeros of the channel) to achieve the desired worst-case performance. Therefore, study of  $H^\infty$  estimation provides us with a rigorous basis for the importance of the concepts of minimum-phase channels and delay in the equalization problem.

As in the case of  $H^\infty$ -optimal zero-forcing equalization, having a good (indeed optimal) worst-case performance alone is not sufficient for having acceptable performance in real applications. Fortunately, the  $H^\infty$  criterion provides a family of filters that achieve the same worst-case performance and have different performances with respect to other criteria (such as MSE or BER). Therefore, it is a design problem to choose a filter among this family that also has a good average performance. In this correspondence, we illustrated that the central  $H^\infty$ , or risk-sensitive, filter is such a possible choice. We showed that it has good average properties due to its stochastic interpretation, although it does not appear to be better than the MMSE equalizer in terms of BER in the ideal case. Another possible choice, besides the risk-sensitive filter, is the mixed  $H^2/H^\infty$  solution, i.e., an  $H^\infty$  filter with the best MSE behavior, which is an area of active research. In the last section of the paper, we introduced BER performance for the worst-case noise disturbance as an alternative performance measure, and we illustrated that the central  $H^\infty$  equalizer is better and, therefore, more robust than the MMSE equalizer.

#### REFERENCES

- [1] B. Hassibi, A. T. Erdogan, and T. Kailath, "Equalization with an  $H^\infty$  criterion," in *Proc. Int. Symp. Inform. Theory*, 1998.
- [2] B. Hassibi, "Indefinite metric spaces in estimation control and adaptive filtering," Ph.D. dissertation, Aug. 1996.
- [3] K. Glover and J. Doyle, "State-space formulae for all stabilizing controllers that satisfy an  $H^\infty$  norm bound and relations to risk sensitivity," *Syst. Contr. Lett.*, vol. 11, pp. 167–172, 1988.
- [4] S. Boyd, V. Balakrishnan, and P. Kabamba, "A bisection method for computing the  $H^\infty$  norm of a transfer matrix and related problems," *Math. Contr., Signals, Syst.*, vol. 2, pp. 207–219, 1989.
- [5] F. Callier, "On polynomial matrix spectral factorization by symmetric extraction," *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 453–464, 1985.

## A Class of Subspace Tracking Algorithms Based on Approximation of the Noise-Subspace

Tony Gustafsson and Craig S. MacInnes

**Abstract**—This correspondence introduces a novel class of so-called subspace tracking algorithms applicable to, for example, sensor array signal processing. The basic idea pursued in this correspondence is to reduce the amount of computations required for an exact SVD update, applying a perturbation-like strategy, which is interpreted as an approximation of a noise subspace. An interesting property of the derived algorithms is that they can be applied to SVD updating of both auto- and cross-covariance matrices.

**Index Terms**—Sensor array signal processing, subspace tracking.

### I. INTRODUCTION

The general mathematical problem considered in this contribution is that of designing computationally efficient methods for computing a low-rank approximation of the matrices.

- 1) *Autocorrelation (AC) case*:  $\hat{\mathbf{R}}_{xx}(t) = \mu \hat{\mathbf{R}}_{xx}(t-1) + (1 - \mu) \mathbf{x}(t) \mathbf{x}^H(t)$ . Here, it is implicitly assumed that  $E\{\mathbf{x}(t) \mathbf{x}^H(t)\}$  is of type "low rank" plus a scaled identity matrix, where  $E\{\cdot\}$  denotes expectation, and  $(\cdot)^H$  denotes conjugate transpose.
- 2) *Cross-correlation (CC) case*:  $\hat{\mathbf{R}}_{x\xi}(t) = \mu \hat{\mathbf{R}}_{x\xi}(t-1) + (1 - \mu) \mathbf{x}(t) \xi^H(t)$ . Here, it is implicitly assumed that  $\mathbf{R}_{x\xi}$  is of type "low rank."

The scalar  $\mu$  denotes the so-called forgetting factor  $0 < \mu \leq 1$ . The above problem is frequently encountered in adaptive direction-of-arrival (DOA) estimation using sensor arrays; see, e.g., [1], [5], and [8]. The reason for the interest in subspace tracking algorithms is due to the fact that computing the singular value decomposition (SVD) of  $\hat{\mathbf{R}}_{xx}(t)$  at every time instant is computationally prohibitive.

### II. PROBLEM FORMULATION

Even though the application of subspace tracking is not limited to sensor array processing, we formulate the problem in this context. Let the  $m$ -dimensional vector  $\mathbf{x}(t)$  contain the observed samples of an antenna array, where it is assumed that  $n < m$  narrowband plane waves impinge on the array. Hence, the following data model is assumed to be applicable; see, for example, [6]:

$$\mathbf{x}(t) = \mathbf{A}(\boldsymbol{\theta}_0(t)) \mathbf{s}(t) + \mathbf{e}(t) \quad (1)$$

where the  $m \times n$  matrix  $\mathbf{A}(\cdot)$  is the so-called steering matrix. The vector  $\boldsymbol{\theta}_0(t)$  contains the possibly time-varying DOA's. To simplify the notation, the arguments of  $\mathbf{A}$  are suppressed. In the reminder, we will make frequent use of the notation  $\mathbf{R}_{\alpha\beta} = E\{\boldsymbol{\alpha}(t) \boldsymbol{\beta}^H(t)\}$  for two arbitrary stationary random processes  $\boldsymbol{\alpha}(t)$  and  $\boldsymbol{\beta}(t)$ . The unmeasurable signal  $\mathbf{s}(t) \in \mathbb{C}^{n \times y}$  is assumed to be a stationary zero mean random process with covariance matrix  $\mathbf{R}_{ss}$ . The noise vectors  $\mathbf{e}(t)$  are assumed to

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