ON THE NUMBER OF DIOPHANTINE m-TUPLES IN FINITE FIELDS

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ABSTRACT. We use a new argument to improve the error term in the asymptotic formula for the number of Diophantine m -tuples in finite fields, which is due to A. Dujella and M. Kazalicki (2021) and N. Mani and S. Rubinstein-Salzedo (2021).

1. INTRODUCTION

1.1. Motivation and set-up. We recall the classical definition of a *Diophantine* m-tuple as a vector $(a_1, \ldots, a_m) \in \mathbb{N}^m$ such that all shifted products $a_i a_j + 1$, $1 \leq i \leq j \leq m$, are perfect squares.

The long-standing conjecture on the finiteness of the set of Diophantine quintuples, after a series of intermediate results by various authors, has been established in a striking work of Dujella [\[3\]](#page-4-0), who has also shown the the non-existence of Diophantine sextuples. More recently, He, Togbé and Ziegler $[8]$ have show the non-existence of Diophantine quintuples is shown, see also [\[1\]](#page-4-1). Quite naturally, these results have suggested to study the generalisation of this notion to other algebraic domains such as, for example, the set of rational numbers or points on curves, as well as in many other directions, see, for example, [\[2,](#page-4-2)[5,](#page-5-1)[6,](#page-5-2)[11\]](#page-5-3) and references therein. The notion also readily extends to the setting of finite fields, see $\left[4, 7, 13\right]$ $\left[4, 7, 13\right]$ $\left[4, 7, 13\right]$.

Let q be an odd prime power and let \mathbb{F}_q be the finite field of q elements.

For $r \in \mathbb{F}_q^*$, we say that an *m*-tuple $(a_1, \ldots, a_m) \in \mathbb{F}_q^m$ form a Diophantine m-tuple in \mathbb{F}_q with a shift r if all $m(m-1)/2$ shifted products $a_i a_j + r$ are perfect squares in \mathbb{F}_q .

Remark 1.1. We note that it is customary to exclude zero values from the domain from which a_1, \ldots, a_m are drawn. However in the counting results below this makes no difference, while this simplifies the notation. In particular, the total number of such m-tuples over \mathbb{F}_q with a zero

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2 I. E. SHPARLINSKI

entry (which is at most mq^{m-1}) can be absorbed in the error term of our asymptotic formula.

Let $N_r(m, q)$ be the number of distinct Diophantine m-tuple in \mathbb{F}_q with a shift r. It has been shown by Dujella and Kazalicki $|4|$ that for $r = 1$ and a prime p we have

(1.1)
$$
N_1(m,p) = 2^{-m(m-1)/2}p^m + o(p^m).
$$

Using some ideas of Dujella and Kazalicki [\[4\]](#page-5-4), Mani and Rubinstein-Salzedo [\[13,](#page-5-6) Theorem 5.1] have given explicit formulas for $N_r(2, q)$ and $N_r(3, q)$ and for $m \geq 4$ presented a more precise than (1.1) asymptotic formula

(1.2)
$$
N_r(m,q) = 2^{-m(m-1)/2}q^m + O(q^{m-1/2}),
$$

where the implied constant may depend on m , which also holds for any $r \in \mathbb{F}_q^*$ (and it is also easy to see that for any odd prime power q rather than just for a prime $q = p$ as in [\[13\]](#page-5-6)). We also observe that the bound [\(1.2\)](#page-1-1) can be derived within the initial approach of Dujella and Kazalicki $|4|$ if one appeals to a version of the Lang-Weil bound $|12|$.

1.2. New bound. Here we show that using some simple arguments the bound on the error term in [\(1.2\)](#page-1-1) can be improved.

Theorem 1.2. For a fixed $m \geq 4$, uniformly over $r \in \mathbb{F}_q^*$, we have

$$
N(m, q) = 2^{-m(m-1)/2} q^m + O(q^{m-1}),
$$

where the implied constant may depend on m.

As in [\[13\]](#page-5-6) our proof is based on an application of the *Weil bound* for multiplicative character sums with polynomials, see, for example, [\[9,](#page-5-8) Theorem 11.23].

2. Proof of Theorem [1.2](#page-1-2)

2.1. Preliminary transformations. Since there are $O(q^{m-1})$ choices of m-tuples $(a_1, \ldots, a_m) \subseteq \mathbb{F}_q^m$ for which $a_i a_j + r = 0$ for some $1 \leq i \leq j \leq m$, or with $a_i = 0$ for some $1 \leq i \leq m$, following the argument of [\[13\]](#page-5-6), we write

$$
N_r(m,q) = 2^{-m(m-1)/2} \sum_{a_1,\dots,a_m \in \mathbb{F}_q^*} \prod_{1 \le i < j \le m} (1 + \chi(a_i a_j + r)) + O(q^{m-1}),
$$

where χ is the quadratic character of \mathbb{F}_q , we refer to [\[9,](#page-5-8) Chapter 3] for a background on characters. Therefore

(2.1)
$$
N_r(m, q) = 2^{-m(m-1)/2} q^m + \sum_{\substack{\epsilon \in \{0,1\}^{m(m-1)/2} \\ \epsilon \neq 0}} R(\epsilon) + O(q^{m-1}),
$$

where for $\boldsymbol{\varepsilon} = (\varepsilon_{i,j})_{1 \leq i < j \leq m} \in \{0,1\}^m$

(2.2)
$$
R(\varepsilon) = \sum_{a_1,\dots,a_m \in \mathbb{F}_q^*} \prod_{1 \leq i < j \leq m} \chi(a_i a_j + r)^{\varepsilon_{i,j}}.
$$

We now fix $\boldsymbol{\varepsilon} \in \{0,1\}^{m(m-1)/2}$ with $\boldsymbol{\varepsilon} \neq \boldsymbol{0}$ and estimate $R(\boldsymbol{\varepsilon})$. Renumbering the variables a_1, \ldots, a_m , we see that without loss of generality, we can assume that

$$
(2.3) \t\t \varepsilon_{1,2} = 1.
$$

We now consider the following two cases depending on vanishing and non-vanishing of the exponents $\varepsilon_{i,j}$ with $2 \leq i < j \leq m$.

2.2. Vanishing exponents $\varepsilon_{i,j}$ with $2 \leq i < j \leq m$. Assume that

(2.4)
$$
\varepsilon_{i,j} = 0
$$
, for all $2 \leq i < j \leq m$.

Then we see that under the conditions [\(2.4\)](#page-2-0) the expression for $R(\epsilon)$ in [\(2.2\)](#page-2-1) simplifies as

$$
R(\varepsilon) = \sum_{a_1,\dots,a_m \in \mathbb{F}_q^*} \prod_{2 \leq j \leq m} \chi (a_1 a_j + r)^{\varepsilon_{1,j}}
$$

=
$$
\sum_{a_1 \in \mathbb{F}_q^*} \prod_{2 \leq j \leq m} \sum_{a_j \in \mathbb{F}_q^*} \chi (a_1 a_j + r)^{\varepsilon_{1,j}}.
$$

Hence, estimating the sums over a_3, \ldots, a_m trivially as $q-1$ and recalling our assumption (2.3) , we obtain

$$
|R(\varepsilon)| \leqslant (q-1)^{m-2} \sum_{a_1 \in \mathbb{F}_q^*} \left| \sum_{a_2 \in \mathbb{F}_q^*} \chi(a_1 a_2 + r) \right|.
$$

Clearly, for every $a_1 \in \mathbb{F}_q^*$ we have

$$
\sum_{a_2 \in \mathbb{F}_q^*} \chi(a_1 a_2 + r) = \sum_{a \in \mathbb{F}_q^*} \chi(a + r)
$$

=
$$
\sum_{a \in \mathbb{F}_q} \chi(a) - \chi(1) = \sum_{a \in \mathbb{F}_q} \chi(a) - 1 = -1.
$$

Hence we obtain

$$
(2.5) \t\t |R(\varepsilon)| \leqslant (q-1)^{m-1}
$$

in this case.

2.3. Non-vanishing exponents $\varepsilon_{i,j}$ with $2 \leq i \leq j \leq m$. We now assume that

(2.6)
$$
\varepsilon_{i,j} \neq 0
$$
, for some $2 \leq i < j \leq m$.

We write $R(\epsilon)$ as

$$
R(\boldsymbol{\varepsilon}) = \sum_{a_1,\dots,a_m \in \mathbb{F}_q^*} \prod_{2 \leqslant j \leqslant m} \chi(a_1 a_j + r)^{\varepsilon_{1,j}} \prod_{2 \leqslant i < j \leqslant m} \chi(a_i a_j + r)^{\varepsilon_{i,j}}.
$$

Observe that for any $b \in \mathbb{F}_q^*$ the map

$$
(a_1, a_2, \ldots, a_m) \mapsto (a_1/b, a_2b, \ldots, a_mb)
$$

is a permutation on \mathbb{F}_p^m . Hence

$$
R(\varepsilon) = (p-1)^{-1} \sum_{b \in \mathbb{F}_q^*} \sum_{a_1, \dots, a_m \in \mathbb{F}_q^*} \prod_{2 \leq j \leq m} \chi(a_1 a_j + r)^{\varepsilon_{1,j}} \prod_{2 \leq i < j \leq m} \chi(a_i a_j b^2 + r)^{\varepsilon_{i,j}},
$$

which we now rearrange as

$$
R(\boldsymbol{\varepsilon}) = (q-1)^{-1} \sum_{a_2,\ldots,a_m \in \mathbb{F}_q^*} S(a_2,\ldots,a_m) T(a_2,\ldots,a_m),
$$

where

$$
S(a_2,\ldots,a_m) = \sum_{a_1 \in \mathbb{F}_q^*} \chi \left(\prod_{2 \leq j \leq m} (a_1 a_j + r)^{\varepsilon_{1,j}} \right),
$$

$$
T(a_2,\ldots,a_m) = \sum_{b \in \mathbb{F}_q^*} \chi \left(\prod_{2 \leq i < j \leq m} (a_i a_j b^2 + r)^{\varepsilon_{i,j}} \right).
$$

We now examine the polynomials

$$
F_{a_2,...,a_m}(X) = \prod_{2 \leq j \leq m} (a_j X + r)^{\varepsilon_{1,j}},
$$

$$
G_{a_2,...,a_m}(X) = \prod_{2 \leq i < j \leq m} (a_i a_j X^2 + r)^{\varepsilon_{i,j}}.
$$

Because of our assumptions (2.3) and (2.6) both these polynomials are of positive degree.

Furthermore, it is clear that there are at most $O(q^{m-2})$ choices for $(m-1)$ -tuples $(a_2, \ldots, a_m) \in \mathbb{F}_q^{m-1}$ for which at least one of the polynomials $F_{a_2,\ldots,a_m}(X)$ and $G_{a_2,\ldots,a_m}(X)$ is a perfect square in the algebraic closure of \mathbb{F}_q . In this case we estimate both sums $S(a_2, \ldots, a_m)$

and $T(a_2, \ldots, a_m)$ trivially as $q-1$. Hence, the contribution to $R(\epsilon)$ from such sums is

(2.7)
$$
A = O((q-1)^{-1}q^{m-2}(q-1)^2) = O(q^{m-1}).
$$

For other choices of $(a_2, \ldots, a_m) \in \mathbb{F}_q^{m-1}$, by the Weil bound, see, for example, [\[9,](#page-5-8) Theorem 11.23], we have

 $S(a_2, \ldots, a_m)$, $T(a_2, \ldots, a_m) = O(q^{1/2})$.

Hence, the contribution to $R(\epsilon)$ from such sums is

(2.8)
$$
B = O\left((q-1)^{-1}q^{m-1} (q^{1/2})^2\right) = O(q^{m-1}).
$$

Combining (2.7) and (2.8) we arrive to

(2.9)
$$
R(\varepsilon) = A + B = O(q^{m-1})
$$

in this case.

2.4. Concluding the proof. Substituting the bounds (2.5) and (2.9) in [\(2.1\)](#page-2-4) we immediately obtain the desired result.

3. COMMENTS

It is easy to see that all implied constants can be evaluated explicitly. Hence one can use our argument to estimate the smallest q (in terms of m) for which $N_r(m,q) > 0$ for all $r \in \mathbb{F}_q^*$. However the inductive approach of Dujella and Kazalicki [\[4,](#page-5-4) Theorem 17] seems to be more effective for this question.

Since we have multivariate character sums, it is also natural to try improve Theorem [1.2](#page-1-2) via the use of some version of the Deligne bound, see, for example [\[10\]](#page-5-9). Unfortunately, our polynomials have a high dimensional singularity locus, which seems to prevent this approach.

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6 I. E. SHPARLINSKI

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