GENERIC NEWTON POLYGONS FOR L-FUNCTIONS OF (A, B)-EXPONENTIAL SUMS

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ABSTRACT. In this paper, we consider the following (A, B)-polynomial f over finite field:
$$f(x_0, x_1, \cdots, x_n) = x_0^A h(x_1, \cdots, x_n) + g(x_1, \cdots, x_n) + P_B(1/x_0),$$
where h is a Deligne polynomial of degree d, g is an arbitrary polynomial of degree $< dB/(A + B)$ and $P_B(y)$ is a one-variable polynomial of degree $\leq B$. Let Δ be the Newton polyhedron of f at infinity. We show that Δ is generically ordinary if $p \equiv 1$ mod D, where D is a constant only determined by Δ. In other words, we prove that the Adolphson–Sperber conjecture is true for Δ.

1. INTRODUCTION

Let p be a prime number, q be a power of p, and $\mathbb{F}_q$ be the finite field of q elements. For each positive integer k, let $\mathbb{F}_{q^k}$ be the finite extension of $\mathbb{F}_q$ of degree k and $\mathbb{F}_{q^k}^*$ denote the set of non-zero elements in $\mathbb{F}_{q^k}$. Given a Laurent polynomial $f(x) \in \mathbb{F}_q[x_1^{\pm}, \cdots, x_n^{\pm}]$, we define the toric exponential sum as

$$S_k^*(f) := \sum_{x \in (\mathbb{F}_{q^k}^*)^n} \zeta_p^{\mathrm{Tr}_k f(x)},$$

where $\mathrm{Tr}_k : \mathbb{F}_{q^k} \to \mathbb{F}_p$ denotes the trace of the field extension. A well known theorem of Dwork–Bombieri–Grothendieck implies that the following generating L-function is rational:

$$L^*(f, T) := \exp\Big(\sum_{k=1}^{\infty} \frac{S_k^*(f)T^k}{k}\Big) = \frac{\prod_{i=1}^{d_1}(1 - \alpha_i T)}{\prod_{j=1}^{d_2}(1 - \beta_j T)},$$

where $\alpha_i (1 \leq i \leq d_1)$ and $\beta_j (1 \leq j \leq d_2)$ are non-zero algebraic integers. Equivalently,

$$S_k^*(f) = \beta_1^k + \cdots + \beta_{d_2}^k - \alpha_1^k - \cdots - \alpha_{d_1}^k.$$

We are interested in studying the exponential sum $S_k^*(f)$, which is reduced to understanding the reciprocal zeros and reciprocal poles. However, one does not even know the number of zeros and poles if there is no smooth condition on f.

Let us introduce a smooth condition first. Using multi-index notation, we write

$$f(x) = \sum_{\mathbf{v} \in A} a_{\mathbf{v}} x^{\mathbf{v}},$$

where A is a finite subset of $\mathbb{Z}^n$ and $a_{\mathbf{v}} \in \mathbb{F}_q^*$. The convex hull of A and the origin is called the *Newton polyhedron of f at infinity*, denoted by $\Delta_\infty(f)$. If δ is a face of $\Delta_\infty(f)$, we define the restriction of f to δ to be

$$f_\delta = \sum_{\mathbf{v} \in \delta} a_{\mathbf{v}} x^{\mathbf{v}}.$$

Let $\bar{\mathbb{F}}_q^*$ denote the nonzero elements of an algebraic closure of $\mathbb{F}_q$.

Definition 1.1. The Laurent polynomial f is called $\Delta_\infty(f)$-*nondegenerate* if for each closed face δ of $\Delta_\infty(f)$ which does not contain the origin, the system of equations

$$x_1 \frac{\partial f_\delta}{\partial x_1} = \cdots = x_n \frac{\partial f_\delta}{\partial x_n} = 0$$

has no common zeros in $\bar{\mathbb{F}}_q^*$.

We always assume that $\Delta_\infty(f)$ is n-dimensional. If f is $\Delta_\infty(f)$-nondegenerate, then Adolphson and Sperber [1] proved that the L-function $L^*(f,T)^{(-1)^{n-1}}$ is a polynomial of degree $n!\mathrm{Vol}(\Delta_\infty(f))$, where $\mathrm{Vol}(\Delta_\infty(f))$ is the volume of $\Delta_\infty(f)$. Using l-adic methods, Denef and Loeser [4] explicitly determined the complex absolute values of the zeros of $L^*(f,T)^{(-1)^{n-1}}$. Hence the remaining interesting question is to determine the p-adic absolute values of the zeros. This question equals to determine the q-adic Newton polygon of the polynomial

$$L^*(f,T)^{(-1)^{n-1}} = \sum_{i=0}^{n!\mathrm{Vol}(\Delta_\infty(f))} A_i(f)T^i, A_i(f) \in \mathbb{Z}[\zeta_p].$$

The q-adic *Newton polygon* of $L^*(f,T)^{(-1)^{n-1}}$, denoted by $\mathrm{NP}(f)$, is defined as the lower convex closure in $\mathbb{R}^2$ of the points

$$(i, \mathrm{ord}_q A_r(f)), \ i = 0, 1, ..., n!\mathrm{Vol}(\Delta_\infty(f)).$$

Unfortunately, there is no effective way to compute the Newton polygon since the Newton polygon varies greatly as f and p vary. However, when f is $\Delta_\infty(f)$-nondegenerate, Adolphson and Sperber [1] proved that $\mathrm{NP}(f)$ has a certain combinatorial lower bound, called the Hodge polygon (see Definition 1.2 below).

Let Δ be the n-dimensional integral convex polyhedron $\Delta_\infty(f)$ in $\mathbb{R}^n$, and $C(\Delta)$ be the cone in $\mathbb{R}^n$ generated by Δ. For a point $u \in \mathbb{R}^n$, the weight $w(u)$ is defined to be the smallest non-negative real number c such that $u \in c\Delta$. If there is no such c, then we define $w(u) = \infty$. Let δ be a co-dimensional 1 face of Δ which does not contain the origin. Let $C(\delta)$ be the cone in $\mathbb{R}^n$ over δ. Let $\sum_{i=1}^n e_i X_i = 1$ be the equation of the hyperplane δ in $\mathbb{R}^n$, where e_i $(i = 1, \cdots, n)$ are rational numbers. Then for $u = (u_1, \cdots, u_n) \in C(\delta)$, one has $w(u) = \sum_{i=1}^n e_i u_i$. Let $D(\delta)$ be the least common denominator of the rational numbers e_i $(1 \le i \le n)$, and $D(\Delta)$ be the least common multiple of all the $D(\delta)$ with δ running over all the co-dimensional 1 faces of Δ which do not contain the origin. Then $w(\mathbb{Z}^n) \subseteq \frac{1}{D(\Delta)}\mathbb{Z}_{\ge 0} \cup \{\infty\}$, where $\mathbb{Z}_{\ge 0}$ denotes the set of nonnegative integers. The integer $D := D(\Delta)$ is called the denominator of Δ.

For an integer k, let

$$W_\Delta(k) := \mathrm{card}\left\{ u \in \mathbb{Z}^n : w(u) = \frac{k}{D} \right\}$$

be the number of lattice points in $\mathbb{Z}^n$ with weight k/D. Let

$$H_\Delta(k) := \sum_{i=0}^{n} (-1)^i \binom{n}{i} W_\Delta(k - iD).$$

Definition 1.2. The *Hodge polygon* $\mathrm{HP}(\Delta)$ of Δ is defined to be the lower convex polygon in $\mathbb{R}^2$ with the origin and vertices

$$\Big(\sum_{k=0}^{m} H_\Delta(k), \frac{1}{D} \sum_{k=0}^{m} k H_\Delta(k) \Big), \quad m = 0, 1, 2, \cdots, nD.$$

In this paper, we are interested in the variation of $\mathrm{NP}(f)$ when f varies. Let $\mathcal{M}_p(\Delta)$ be the set of Δ-nondegenerate f over the algebraic closure of $\mathbb{F}_p$ with $\Delta_\infty(f) = \Delta$. By the Grothendieck specialization theorem, the lowest Newton polygon

$$\mathrm{GNP}(\Delta, p) := \inf_{f \in \mathcal{M}_p(\Delta)} \mathrm{NP}(f)$$

exists, and can be obtained for all f in some Zariski open dense subset of $\mathcal{M}_p(\Delta)$. The lowest polygon is called the *generic Newton polygon*, denoted by $\mathrm{GNP}(\Delta, p)$.

It follows from Adolphson and Sperber [1] that for $f \in \mathcal{M}_p(\Delta)$, one has $\mathrm{NP}(f) \geq \mathrm{HP}(\Delta)$. For every prime p and every $f \in \mathcal{M}_p(\Delta)$, we have

$$\mathrm{NP}(f) \geq \mathrm{GNP}(\Delta, p) \geq \mathrm{HP}(\Delta).$$

One intriguing question is to decide when the generic Newton polygon coincides with the Hodge polygon. The family $\mathcal{M}_p(\Delta)$ is called *generically ordinary* if $\mathrm{GNP}(\Delta, p) = \mathrm{HP}(\Delta)$. And a Laurent polynomial is called *ordinary* if $\mathrm{NP}(f) = \mathrm{HP}(\Delta)$. Based on results of Sperber [11, 12, 13] and Carpentier [2] on p-adic estimates of exponential sums, Adolphson and Sperber [1] conjectured that

Conjecture 1.3. (Adolphson–Sperber) If $p \equiv 1 \pmod{D}$, then $\mathrm{GNP}(\Delta, p) = \mathrm{HP}(\Delta)$.

This conjecture generalizes a conjecture of Dwork [5] and Mazur [9]. Wan [14, 15] established several decomposition theorems for the generic Newton polygon and proved that the Adolphson–Sperber conjecture is true for $n \leq 3$ but false for every higher dimension $n \geq 4$. He also proved that a weaker form of the A-S conjecture is true when we replace D by another effectively computable positive integer D^*.

Le [8] generalized Wan's theory by giving the regular decomposition theorem and showing that the star decomposition, hyperplane decomposition and collapsing decomposition raised by Wan are regular decompositions. In [8], Le claimed the following result without proof: Let $\Delta_\infty(f) = \bigcup_i \Delta_i$ be a complete regular integral decomposition of $\Delta_\infty(f)$ and f_i be the restriction of f on Δ_i. If each f_i is generically Δ_i-nondegenerate, then f is also generically $\Delta_\infty(f)$-nondegenerate. It seems that this result is not evident. In Section 2, we will give the explicit proof by using some results of Gelfand, Kapranov and Zelevinsky.

A polynomial $h \in \mathbb{F}_q[x_1, \cdots, x_n]$ of degree $d > 0$ is called a *Deligne polynomial* if its degree d prime to p, and its highest degree term defines a smooth projective hypersurface in $\mathbb{P}^{n-1}$. Let A, B, d be positive integers relatively prime to p and f be an (A, B)-polynomial of the following form:

(1.1)
$$f(x_0, x_1, \cdots, x_n) = x_0^A h(x_1, \cdots, x_n) + g(x_1, \cdots, x_n) + P_B(1/x_0) \in \mathbb{F}_q[x_0^\pm, x_1, \ldots, x_n],$$

where h is a Deligne polynomial of degree d, g is an arbitrary polynomial of degree $< dB/(A + B)$ and $P_B(y)$ is a one-variable polynomial of degree $\leq B$. Set $\Delta = \Delta_\infty(f)$. Let

$\widetilde{\mathcal{M}}_p(\Delta)$ be the set of nondegenerate (A, B)-polynomials of form (1.1) and

$$\widetilde{\text{GNP}}(\Delta, p) := \inf_{f \in \widetilde{\mathcal{M}}_p(\Delta)} \text{NP}(f).$$

Fu and Wan [6] proved that if $p \equiv 1 \mod [A, dB]$, then $\widetilde{\text{GNP}}(\Delta, p) = \text{HP}(\Delta)$. In [6, Remark 5.3], Fu and Wan calculated that $D(\Delta) = [A, B, dB/(A+B, dB)]$ and conjectured that the Adolphson–Sperber conjecture is true for Δ. For $A = B = 1$, Le [8] conjectured that

Conjecture 1.4. $p \equiv 1 \pmod{D}$ if and only if $\widetilde{\text{GNP}}(\Delta, p) = \text{HP}(\Delta)$.

Moreover, Le proved that Conjecture 1.4 holds for odd d. Recently, Li [10] gave a counterexample of Conjecture 1.4 for even d. Since the family considered by Le is slightly smaller than $\mathcal{M}_p(\Delta)$, Li [10] asked whether Conjecture 1.4 is true if the family f becomes larger, i.e., with $\deg g \leq \frac{d}{2}$ instead of $\deg g < \frac{d}{2}$. For such f, we have $\widetilde{\mathcal{M}}_p(\Delta) = \mathcal{M}_p(\Delta)$. Hence $\widetilde{\text{GNP}}(\Delta, p) = \text{GNP}(\Delta, p)$. It is well known that if $\text{GNP}(\Delta, p) = \text{HP}(\Delta)$, then $p \equiv 1 \pmod{D}$ (see [15, Section 1.3]). Hence Li's question equals to ask that whether the Adolphson–Sperber conjecture holds for Δ when $A = B = 1$.

Using Wan's facial decomposition theorem (Theorem 2.3 below) and the regular decomposition theorem (Corollary 2.14 below) generalized by Le, we prove the following:

Theorem 1.5. *Suppose f is given by (1.1) and $\Delta = \Delta_\infty(f)$. Then the Adolphson–Sperber conjecture is true for Δ, i.e., if $p \equiv 1 \mod [A, B, dB/(A + B, dB)]$, then $\text{GNP}(\Delta, p) = \text{HP}(\Delta)$.*

This paper is organized as follows. In Section 2, we will introduce Wan's decomposition theory and the regular decomposition theorem generalized by Le. In Section 3, we present the proof of Theorem 1.5.

2. DECOMPOSITION THEORY

Essentially, the idea of decomposition theory is to reduce from the harder non-diagonal case to the easier diagonal one. So in this section we introduce the diagonal local theory first. A more complete treatment of the theory can be found in [15].

A Laurent polynomial $f \in \mathbb{F}_q[x_1^\pm, \cdots, x_n^\pm]$ is called *diagonal* if f has exactly n non-constant terms and $\Delta := \Delta_\infty(f)$ is n-dimensional. Write $f(x) = \sum_{j=1}^n a_j x^{\mathbf{v}_j}$, where $a_j \in \mathbb{F}_q^*$ and each $\mathbf{v}_j$ is written as a column vector. Then the square matrix of Δ is defined to be

$$\mathbf{M}(\Delta) = (\mathbf{v}_1, \cdots, \mathbf{v}_n).$$

If f is diagonal, then $\det \mathbf{M}(\Delta) \neq 0$ in $\mathbb{Z}$. We have

Proposition 2.1. [15, Section 2.1] *f is $\Delta_\infty(f)$-nondegenerate if and only if $\gcd(p, \det \mathbf{M}(\Delta)) = 1$.*

Let $S(\Delta)$ be the solution set of the following linear system

$$\mathbf{M}(\Delta) \cdot (r_1, r_2, ..., r_n)^t \equiv 0 \pmod{1}, \quad r_i \in \mathbb{Q} \cap [0, 1),$$

where $(r_1, r_2, ..., r_n)^t$ means the transpose of $(r_1, r_2, ..., r_n)$. Then $S(\Delta)$ is an abelian group and its order is given by $|\det \mathbf{M}(\Delta)|$. By the fundamental structure of finite abelian group, $S(\Delta)$ can be decomposed into a direct product of invariant factors, i.e.,

$$S(\Delta) = \bigoplus_{i=1}^n \mathbb{Z}/s_i(\Delta)\mathbb{Z},$$

where $s_i(\Delta)|s_{i+1}(\Delta)$ for $i = 1, 2, ..., n-1$. Wan [15] proved the following criterion for ordinarity.

Proposition 2.2. *Suppose that $f \in \mathbb{F}_q[x_1^{\pm}, \cdots, x_n^{\pm}]$ is $\Delta_\infty(f)$-nondegenerate and diagonal. Let $s_n(\Delta)$ be the largest invariant factor of $S(\Delta)$. If $p \equiv 1 \mod s_n(\Delta)$, then f is ordinary.*

Note that $s_n(\Delta)|\det \mathbf{M}(\Delta)$. We conclude that if $p \equiv 1 \mod (\det \mathbf{M}(\Delta))$, then f is ordinary.

For the general case, we will describe the facial decomposition for the Newton polygon [14] and the regular decomposition theorem for the generic Newton polygon [8].

Theorem 2.3. [14, Facial decomposition theorem] *Let f be a Laurent polynomial with n variables over $\mathbb{F}_q$. Assume $\Delta_\infty(f)$ is n-dimensional and $\delta_1, \cdots, \delta_h$ are all the co-dimensional 1 faces of $\Delta_\infty(f)$ which do not contain the origin. For $1 \le i \le h$, let Δ_i be the convex polytope of lattice points in δ_i and the origin. Then f is $\Delta_\infty(f)$-nondegenerate and ordinary if and only if f_{δ_i} is Δ_i-nondegenerate and ordinary for $1 \le i \le h$.*

By facial decomposition theorem, to study the ordinarity of f, we may reduce to the case when $\Delta_\infty(f)$ has only one co-dimensional 1 face δ not containing the origin. A decomposition of δ then gives a decomposition of $\Delta_\infty(f)$ by connecting with the origin.

Definition 2.4. Let δ be an $(n-1)$-dimensional convex polytope with A being a finite subset of δ including all vertices of δ. A *subdivision* of (δ, A) is a family $\mathcal{F} = \{(\delta_i, A_i) : i \in I\}$ of polytopes such that

- each A_i is a subset of $A \cap \delta_i$, A_i contains all vertices of δ_i and $\dim \delta_i = n-1$.
- any $\delta_i \cap \delta_j$ is a face (possibly empty) of both δ_i and δ_j, and
$$A_i \cap (\delta_i \cap \delta_j) = A_j \cap (\delta_i \cap \delta_j).$$
- the union of all δ_i coincides with δ.

Each δ_i is called a cell of the subdivision $\mathcal{F}$. In particular, if each δ_i is a simplex, then we call that $\mathcal{F} = \{(\delta_i, A_i) : i \in I\}$ is a *triangulation* of (δ, A). We say that a cell δ_i is *indecomposable* if it is a simplex and contains no point in A other than vertices. An integral polytope is a convex polytope whose vertices are lattice points. A subdivision $\mathcal{F}$ is called *integral* if each δ_i is an integral polytope. An integral subdivision $\mathcal{F}$ is called *complete* if each δ_i is indecomposable. Clearly, a complete subdivision should be a triangulation.

In [8], Le studied the regular subdivision which is defined as follows.

Definition 2.5. ([8, Section 4.2.2] or [16, Section 10]) A subdivision $\mathcal{F} = \{(\delta_i, A_i) : i \in I\}$ of (δ, A) is called *regular* if there is a piecewise linear function $\phi : \delta \to \mathbb{R}$ such that

(1) ϕ is concave, i.e. $\phi(\lambda x + (1-\lambda)x') \ge \lambda\phi(x) + (1-\lambda)\phi(x')$, for all $x, x' \in \delta$ and $0 \le \lambda \le 1$;
(2) the domains of linearity of ϕ are precisely the δ_i for $i \in I$.

Definition 2.6. [3, Chapter 2] Let $\mathcal{F} = \{(\delta_i, A_i) : i \in I\}$ and $\mathcal{F}' = \{(\delta_j', A_j') : j \in J\}$ be two subdivisions of (δ, A). We say that $\mathcal{F}$ is a *refinement* (resp. *regular refinement*) of $\mathcal{F}'$ if the collection of (δ_i, A_i) such that $\delta_i \subseteq \delta_j'$ forms a subdivision (resp. regular subdivision) of (δ_j', A_j') for any $j \in J$.

We have the following property for regular subdivision.

Lemma 2.7. [3, Lemma 2.3.16] *Let $\mathcal{F}$ and $\mathcal{F}'$ be two subdivisions of (δ, A) such that $\mathcal{F}$ is regular and $\mathcal{F}'$ is a regular refinement of $\mathcal{F}$. Then $\mathcal{F}'$ is also regular.*

In the following, let B be a set of lattice points in $\mathbb{R}^{n-1}$ and let $g = \sum_{\mathbf{w} \in B} b_{\mathbf{w}} y^{\mathbf{w}} \in \mathbb{F}_q[y_1^{\pm}, \ldots, y_{n-1}^{\pm}]$ be the generic Laurent polynomial associated to B. The Newton polyhedron $\Delta(g)$ of g is defined to be the convex hull of B in $\mathbb{R}^{n-1}$. We assume $\dim \Delta(g) = n-1$. The *principle B-determinant* $E_B(g)$ of g is defined to be the resultant of

$$\left\{ y_1 \frac{\partial g}{\partial y_1}, \cdots, y_{n-1} \frac{\partial g}{\partial y_{n-1}}, g \right\}.$$

Recall that (see [7, Chapter 10]) $E_B(g)$ is a polynomial in variables $\{b_{\mathbf{w}}\}_{\mathbf{w} \in B}$. Let

$$E_B(g) = \sum_{k \in \mathbb{Z}^{|B|}} c_k \mathbf{b}^k,$$

where $|B|$ denotes the number of points in B and $\mathbf{b} = (b_{\mathbf{w}})_{\mathbf{w} \in B}$. The Newton polyhedron of $E_B(g)$ is defined to be the convex hull of lattice points $k \in \mathbb{Z}^{|B|}$ such that $c_k \neq 0$. When $B = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is the set of vertices of an $(n-1)$-dimensional simplex in $\mathbb{R}^{n-1}$ and $g = \sum_{i=1}^n b_i y^{\mathbf{v}_i}$. Then

$$E_B(g) = \pm \left(\widetilde{\mathrm{Vol}}(\Delta(g)) b_1 \cdots b_n \right)^{\widetilde{\mathrm{Vol}}(\Delta(g))},$$

where $\widetilde{\mathrm{Vol}}(\Delta(g))$ is $(n-1)!$ times the volume of $\Delta(g)$.

Definition 2.8. The Laurent polynomial g is called $\Delta(g)$-*nondegenerate* if for each closed face δ of $\Delta(g)$, the system of polynomial equations

$$g_\delta = y_1 \frac{\partial g_\delta}{\partial y_1} = \ldots = y_{n-1} \frac{\partial g_\delta}{\partial y_{n-1}} = 0$$

have no solutions in $\bar{\mathbb{F}}_q^*$, where $g_\delta = \sum_{\mathbf{w} \in B \cap \delta} b_{\mathbf{w}} y^{\mathbf{w}}$.

One has the following explicit criterion to determine the $\Delta(g)$-nondegeneracy.

Theorem 2.9. [7, Chapter 10] *The generic Laurent polynomial g is $\Delta(g)$-nondegenerate if and only if $E_B(g) \neq 0$.*

Gelfand, Kapranov and Zelevinsky showed the following result:

Theorem 2.10. [7, Chapter 10] *There is a one-to-one correspondence between the set of all regular triangulations of $(\Delta(g), B)$ and the set of vertices of the Newton polyhedron of $E_B(g)$. If $\mathcal{F} = \{(\delta_i, B_i) : i \in I\}$ is a complete regular triangulation of $(\Delta(g), B)$, then the corresponding term in $E_B(g)$ has the form*

$$\prod_{\delta_i} E_{B_i}(g_{\delta_i}),$$

where $g_{\delta_i} = \sum_{\mathbf{w} \in B_i} b_{\mathbf{w}} y^{\mathbf{w}}$ is the restriction of g to (δ_i, B_i).

Let Δ be an n-dimensional, integral, convex polytope in $\mathbb{R}^n$ which has only one co-dimensional 1 face δ not containing the origin. Let A be the set of all the lattice points on face δ,

$$f(x) = \sum_{\mathbf{v} \in A} a_{\mathbf{v}} x^{\mathbf{v}} \in \mathbb{F}_q[x_1^{\pm}, x_2^{\pm}, \cdots, x_n^{\pm}]$$

be the generic Laurent polynomial associated to A. We have $\Delta_\infty(f) = \Delta$. For a subdivision $\mathcal{F} = \{(\delta_i, A_i) : i \in I\}$ of δ, we always assume $A_i = \delta_i \cap A$. Thus we say $\{\delta_i\}_{i \in I}$ is a subdivision of δ for simplicity.

One can choose an affine transformation $T \in \mathrm{GL}_n(\mathbb{Z})$ on $\mathbb{R}^n$ such that $T(A)$ lies on the hyperplane $v_n = e$ for some positive integer e. Let

$$f'(x) := \sum_{\mathbf{v} \in A} a_{\mathbf{v}} x^{T(\mathbf{v})} = x_n^e g'(x_1, \ldots, x_{n-1}),$$

where $g' \in \mathbb{F}_q[x_1^{\pm}, \ldots, x_{n-1}^{\pm}]$. Then f is $\Delta_{\infty}(f)$-nondegenerate if and only if f' is $\Delta_{\infty}(f')$-nondegenerate. When $p \nmid e$, one checks by definition that f' is $\Delta_{\infty}(f')$-nondegenerate if and only if g' is $\Delta(g')$-nondegenerate.

Let $\{\delta_i\}_{i \in I}$ be a complete regular triangulation of δ. Under the affine transformation T, one derives a complete regular triangulation $\{\delta_i'\}_{i \in I}$ (resp. $\{\delta_i''\}_{i \in I}$) of $T(\delta)$ (resp. $\Delta(g')$). Denote the restriction of f (resp. f', resp. g') to δ_i (resp. δ_i', resp. δ_i'') by f_i (resp. f_i', resp. g_i') . Denote the set of lattice points in $\Delta(g')$ by B'. Let $B_i' := B' \cap \delta_i''$.

If each f_i is $\Delta_{\infty}(f_i)$-nondegenerate, then each f_i' is $\Delta_{\infty}(f_i')$-nondegenerate. Since δ_i' is indecomposable, one has f_i' is diagonal. By Proposition 2.1, one deduces $p \nmid e$. Hence g_i' is $\Delta(g_i')$-nondegenerate. By Theorem 2.9, one has $E_{B_i'}(g_i') \neq 0$. It follows from Theorem 2.10 that $E_{B'}(g') \neq 0$. That is, g' is $\Delta(g')$-nondegenerate. Hence f (resp. f') is $\Delta_{\infty}(f)$ (resp. $\Delta_{\infty}(f')$)-nondegenerate. So we obtain:

Corollary 2.11. *If $\mathcal{F} = \{(\delta_i, A_i) : i \in I\}$ is a complete regular triangulation of $(\Delta(f), A)$ such that each f_i is $\Delta_{\infty}(f_i)$-nondegenerate, then f is $\Delta_{\infty}(f)$-nondegenerate.*

Let $\phi : \delta \to \mathbb{R}$ be a continuous function. We extend the domain of ϕ to $C(\Delta)$ by setting

$$\phi(\mathbf{r}) = w(\mathbf{r})\phi\Big(\frac{\mathbf{r}}{w(\mathbf{r})}\Big).$$

Definition 2.12. For $\mathbf{r} \in C(\Delta)$, we define

$$m(\phi, A; \mathbf{r}) := \sup \Big\{ \sum_{\mathbf{v} \in A} l_{\mathbf{v}} \phi(\mathbf{v}) : \sum_{\mathbf{v} \in A} l_{\mathbf{v}} \mathbf{v} = \mathbf{r}, l_{\mathbf{v}} \in \mathbb{R}_{\geq 0} \Big\}.$$

We call ϕ is homogeneous with respect to A if

$$m(\phi, A; \mathbf{r}) = \inf \Big\{ \sum_{\mathbf{v} \in A} l_{\mathbf{v}} \phi(\mathbf{v}) : \sum_{\mathbf{v} \in A} l_{\mathbf{v}} \mathbf{v} = \mathbf{r}, l_{\mathbf{v}} \in \mathbb{R}_{\geq 0} \Big\}$$

for all $\mathbf{r} \in C(\Delta)$.

Let $\{\delta_i\}_{i \in I}$ be a regular integral subdivision of δ and ϕ be a piecewise linear function on δ corresponding to $\{\delta_i\}_{i \in I}$. Let $A_i = A \cap \delta_i$. One has that

$$\phi(\lambda \mathbf{u} + (1 - \lambda)\mathbf{u}') = \lambda \phi(\mathbf{u}) + (1 - \lambda)\phi(\mathbf{u}') \text{ for } \mathbf{u}, \mathbf{u}' \in \delta_i, 0 \leq \lambda \leq 1.$$

For $i \in I$, denote by $C(\delta_i)$ the cone generated by δ_i. For $\mathbf{r}, \mathbf{r}' \in C(\delta_i)$, $a \in \mathbb{R}_{\geq 0}$, using the property of weight function that $w(\mathbf{r} + \mathbf{r}') = w(\mathbf{r}) + w(\mathbf{r}')$, $w(a\mathbf{r}) = aw(\mathbf{r})$, one has

$$\phi(\mathbf{r} + \mathbf{r}') = \phi\Big(\frac{\mathbf{r} + \mathbf{r}'}{w(\mathbf{r} + \mathbf{r}')}\Big) \cdot w(\mathbf{r} + \mathbf{r}')$$

$$= \phi\Big(\frac{w(\mathbf{r})}{w(\mathbf{r} + \mathbf{r}')} \frac{\mathbf{r}}{w(\mathbf{r})} + \frac{w(\mathbf{r}')}{w(\mathbf{r} + \mathbf{r}')} \frac{\mathbf{r}'}{w(\mathbf{r}')}\Big) \cdot w(\mathbf{r} + \mathbf{r}')$$

$$= w(\mathbf{r})\phi\Big(\frac{\mathbf{r}}{w(\mathbf{r})}\Big) + w(\mathbf{r}')\phi\Big(\frac{\mathbf{r}'}{w(\mathbf{r}')}\Big) = \phi(\mathbf{r}) + \phi(\mathbf{r}'),$$

$$\phi(a\mathbf{r}) = w(a\mathbf{r})\phi\Big(\frac{a\mathbf{r}}{w(a\mathbf{r})}\Big) = aw(r)\phi\Big(\frac{\mathbf{r}}{w(\mathbf{r})}\Big) = a\phi(\mathbf{r}).$$

We conclude that $C(\delta_i)(i \in I)$ are precisely the domains of linearity of ϕ. Hence $m(\phi, A_i; \mathbf{r}) = \phi(\mathbf{r})$ for any $\mathbf{r} \in C(\delta_i)$. That is, $\phi|_{\delta_i}$ is homogeneous with respect to A_i.

Combining with [8, Proposition 4.9, Theorem 5.1], we obtain that

Theorem 2.13. *Let $\{\delta_i\}_{i \in I}$ be a regular integral triangulatin of δ. If f is generically $\Delta_\infty(f)$-nondegenerate, f_i is generically $\Delta_\infty(f_i)$-nondegenerate and ordinary for each i, then f is generically ordinary.*

By Corollary 2.11 and Theorem 2.13, we have

Corollary 2.14. *(Regular subdivision theorem, [8, Theorem 4.3]) Let $\{\delta_i\}_{i \in I}$ be a complete regular integral triangulatin of δ. If each f_i is generically $\Delta_\infty(f_i)$-nondegenerate and ordinary, then f is also generically $\Delta_\infty(f)$-nondegenerate and ordinary.*

Let H be a hyperplane passing through the origin such that the intersection of δ and H is an integral polytope of co-dimension 2. This hyperplane cuts the polytope Δ (resp. the face δ) into two pieces. In [14, Section 7], such decomposition of Δ is called a *hyperplane decomposition* . In Section 3, we will use a series of hyperplane decompositions to obtain a complete regular integral triangulation of (A, B)-polytope.

3. Proof of Theorem 1.5

In this section, we will give a complete regular integral triangulation of Δ associated to the (A, B)-polynomials. In the end, using Corollary 2.14, we prove that Theorem 1.5 is true.

Let f be an (A, B)-polynomial of the form (1.1). Let $\mathbf{e}'_0, \mathbf{e}'_1, \cdots, \mathbf{e}'_n$ be the standard basis for $\mathbb{R}^{n+1}$. Then $\Delta = \Delta_\infty(f)$ is the polytope spanned by the origin and the vectors $-B\mathbf{e}'_0, A\mathbf{e}'_0, A\mathbf{e}'_0 + d\mathbf{e}'_1, \cdots, A\mathbf{e}'_0 + d\mathbf{e}'_n$.

There are two co-dimensional 1 faces of Δ not containing the origin: the face δ_d spanned by $-B\mathbf{e}'_0, A\mathbf{e}'_0 + d\mathbf{e}'_1, \cdots, A\mathbf{e}'_0 + d\mathbf{e}'_n$ and the face δ'_d spanned by $A\mathbf{e}'_0, A\mathbf{e}'_0 + d\mathbf{e}'_1, \cdots, A\mathbf{e}'_0 + d\mathbf{e}'_n$. The facial decomposition allows us to study the nondegeneracy and ordinarity on the faces δ_d and δ'_d. Let Δ_d (resp. Δ'_d) be the convex hull of $\delta_d \cup 0$ (resp. $\delta'_d \cup 0$) in $\mathbb{R}^{n+1}$.

It follows from [6, Section 5] that

$$D(\Delta'_d) \quad = \quad A$$

and

$$D(\Delta_d) \quad = \quad [B, dB/(A+B, dB)] = dB/(A+B, d).$$

Let $s := (A + B, d)$. If $s = 1$, then $D(\Delta_d) = dB$. It follows from [6, Theorem 5.1] that the Adolphson–Sperber conjecture is true for Δ. Hence we always assume that $s \geq 2$ in the following. Wan [14, Section 7] has proved the Adolphson–Sperber conjecture holds for Δ'_d. By the facial decomposition theorem, it remains to show the Adolphson–Sperber conjecture is also true for Δ_d.

Let $\mathcal{V}$ be the set of all the lattice points on δ_d. For $t = 0, 1, \ldots, s$, let

$$\mathcal{V}_t := \left\{ (v_0, v_1, \cdots, v_n) \in \mathbb{Z}^{n+1}_{\geq 0} : v_0 = \frac{(A+B)t}{s} - B, \sum_{i=1}^{n} v_i = \frac{dt}{s} \right\}.$$

Then

$$\mathcal{V} = \bigcup_{t=0}^{s} \mathcal{V}_t.$$

Clearly, $\mathcal{V}_0 = \{-B\mathbf{e}_0'\}$. For $t = 1, \ldots, s-1$, let $\mathcal{H}_t$ be the hyperplane passing through the origin and $\mathcal{V}_t$. Then $\mathcal{H}_t \cap \delta_d$ is a convex polytope of co-dimension 2 spanned by $\mathcal{V}_t$. This gives a hyperplane decomposition of Δ_d. One has that $\{\mathcal{H}_t\}_{t=1}^{s-1}$ cuts δ_d into s closed pieces $\delta_d(1), \ldots, \delta_d(s)$, where $\delta_d(t)$ is the convex hull of of $\mathcal{V}_{t-1} \cup \mathcal{V}_t$ for $t = 1, \ldots, s-1$. Let ϕ_t be a concave piecewise linear function associated to $\mathcal{H}_t$. Then the function $\phi := \sum_{t=1}^{s-1} \phi_t$ is concave and the domains of linearity of ϕ are exactly $\delta_d(t)$ for $t = 1, \ldots, s$. It follows from that [7, Example 2.2 in Chapter 7] that $\{\mathcal{H}_t\}_{t=1}^{s-1}$ gives a regular subdivision of δ_d:

$$(3.1) \qquad \delta_d = \bigcup_{t=1}^{s} \delta_d(t).$$

If $n = 1$, then $\delta_d(t)$ is a line segment containing only two lattice points $((A+B)(t-1) - sB)/s, d(t-1)/s)$ and $((A+B)t - sB)/s, dt/s)$ for $t \in \{1, \ldots, s\}$. Thus $\delta_d = \bigcup_{t=1}^{s} \delta_d(t)$ is a complete regular triangulation of δ_d. Moreover,

$$\left| \begin{matrix} \frac{(A+B)(t-1)-sB}{s} & \frac{(A+B)t-sB}{s} \\[2mm] \frac{d(t-1)}{s} & \frac{dt}{s} \end{matrix} \right| = \pm \frac{dB}{s}.$$

By Proposition 2.1 and Corollary 2.14, the Adolphson–Sperber conjecture is true for Δ_d. We assume $n \geq 2$ in what follows.

Proposition 3.1. *The polytope δ_d has a complete regular triangulation.*

Proof. Let $L_d \subseteq \mathbb{R}^{n+1}$ be the hyperplane passing through δ_d, and $\rho : L_d \to \mathbb{R}^n$ be the projection of L_d onto its last n coordinates, which is linear and one-to-one. We will first give complete regular triangulations of $\delta_d(1), \ldots, \delta_d(s)$ separately.

For $t = 1, \cdots, s$, denote by $\tilde{\delta}_d(t)$ the image of $\delta_d(t)$ under the projection ρ. Given a subdivision (resp. regular subdivision) $\{\delta_{i,t}\}_{i \in I}$ of $\tilde{\delta}_d(t)$, if each $\rho^{-1}(\delta_{i,t})$ is an integral polytope, then $\{\rho^{-1}(\delta_{i,t})\}_{i \in I}$ is a subdivision (resp. regular subdivision) of $\delta_d(t)$. This is because that if $\phi_t : \tilde{\delta}_d(t) \mapsto \mathbb{R}$ is a piecewise linear function whose linearity domains are exactly $\{\delta_{i,t}\}_{i \in I}$, then define $\hat{\phi}_t : \delta_d(t) \to \mathbb{R}$ by $\hat{\phi}_t(v_0, \mathbf{v}) = \phi(\mathbf{v})$. The domains of linearity of $\hat{\phi}_t$ are exactly $\{\rho^{-1}(\delta_{i,t})\}_{i \in I}$. Moreover, if ϕ_t is concave, then $\hat{\phi}_t$ is concave. It follows from [7, Example 2.2 in Chapter 7] that $\{\rho^{-1}(\delta_{i,t})\}_{i \in I}$ is a subdivision (resp. regular subdivision) of $\delta_d(t)$. In the following, for subsets $X, Y \subset \mathbb{R}^n$, let $P(X, Y)$ denote the convex hull of $X \cup Y$ in $\mathbb{R}^n$, $\hat{X}$ denote the set $\rho^{-1}(X) \in L^d$, and $P(\hat{X}, \hat{Y})$ denote the convex hull of $\hat{X} \cup \hat{Y}$ in $\mathbb{R}^{n+1}$.

For $t = 0, 1, \ldots, s$, define

$$\mathcal{S}_t := \{(v_1, \cdots, v_n) \in \mathbb{R}_{\geq 0}^n : v_1 + \cdots + v_n = dt/s\}.$$

Then $\tilde{\delta}_d(t)$ is the convex hull of $\mathcal{S}_t \cup \mathcal{S}_{t-1}$ for $t = 1, \ldots, s$. Lattice points in $\mathcal{S}_t$ are projections of elements in $\mathcal{V}_t$. Let $\mathbf{e}_0, \mathbf{e}_1, \cdots, \mathbf{e}_n$ be the standard basis for $\mathbb{R}^n$. For $1 \leq i \leq n-1$, we define

$$\Gamma_t^i := \{(v_1, \cdots, v_n) \in \mathcal{S}_t : v_k = 0 \text{ for each } 1 \leq k \leq i\}.$$

It is an $(n-i-1)$-dimensional simplex with vertices $\frac{dt}{s}\mathbf{e}_{i+1}, \frac{dt}{s}\mathbf{e}_{i+2}, \ldots, \frac{dt}{s}\mathbf{e}_n$. We have a flag of $\mathcal{S}_t$:

$$\Gamma_t^{n-1} \subseteq \cdots \subseteq \Gamma_t^1 \subseteq \Gamma_t^0 := \mathcal{S}_t.$$

For $1 \leq j \leq n-1$, we define

$$\Lambda_t^j := \{(v_1, \cdots, v_n) \in \mathcal{S}_t : v_k = 0 \text{ for } n-j+1 \leq k \leq n\}.$$

It is an $(n-j-1)$-dimensional simplex with vertices $\frac{d(t-1)}{s}\mathbf{e}_1, \frac{d(t-1)}{s}\mathbf{e}_2, \ldots, \frac{d(t-1)}{s}\mathbf{e}_{n-j}$. We have a flag of $\mathcal{S}_t$:

$$\Lambda_t^{n-1} \subseteq \cdots \subseteq \Lambda_t^1 \subseteq \Lambda_t^0 := \mathcal{S}_t.$$

First, we will give a regular subdivision of $\tilde{\delta}_d(t) = P(\mathcal{S}_t, \mathcal{S}_{t-1})$ for $1 \le t \le s$, which will be lifted to a regular subdivision of $\delta_d(t)$. When $t = 1$, we take the trivial subdivision $\{P(\Gamma_1^0, \Lambda_0^{n-1})\}$ of $\tilde{\delta}_d(1)$. When $t \ge 2$, for $1 \le i \le n-1$, let $\mathcal{H}_t^i$ be the affine subspace generated by $\Gamma_t^i \cup \Lambda_{t-1}^{n-i}$. By inputting the coordinates of vertices of Γ_t^i and Λ_{t-1}^{n-i}, we calculate that the equation of $\mathcal{H}_t^i$ is given by

$$(3.2) \qquad \frac{s}{d(t-1)}\sum_{j=1}^{i} x_j + \frac{s}{dt}\sum_{j=i+1}^{n} x_j = 1.$$

Lemma 3.2. *We have $\mathcal{H}_t^i \cap P(\mathcal{S}_t, \mathcal{S}_{t-1}) = P(\Gamma_t^i, \Lambda_{t-1}^{n-i})$.*

Proof. By definition, we have

$$P(\mathcal{S}_t, \mathcal{S}_{t-1}) = \left\{\mathbf{v} = (v_1, \cdots, v_n) \in \mathbb{R}^n_{\ge 0} : \frac{d(t-1)}{s} \le v_1 + \cdots + v_n \le \frac{dt}{s}\right\}.$$

Given $\mathbf{x} = (x_1, \cdots, x_n) \in P(\mathcal{S}_t, \mathcal{S}_{t-1})$ satisfying (3.2), let

$$\lambda_j = \frac{sx_j}{d(t-1)} \text{ for } j = 1, \ldots, i, \text{ and } \lambda_j = \frac{sx_j}{dt} \text{ for } j = i+1, \ldots, n.$$

We have $\lambda_j \ge 0$ for $j = 1, \cdots, n$, $\sum_{j=1}^{n} \lambda_j = 1$, and

$$\mathbf{x} = \lambda_1 \frac{d(t-1)}{s}\mathbf{e}_1 + \ldots + \lambda_i \frac{d(t-1)}{s}\mathbf{e}_i + \lambda_{i+1}\frac{dt}{s}\mathbf{e}_{i+1} + \ldots + \lambda_n \frac{dt}{s}\mathbf{e}_n.$$

Hence $\mathbf{x} \in P(\Gamma_t^i, \Lambda_{t-1}^{n-i})$. So $\mathcal{H}_t^i \cap P(\mathcal{S}_t, \mathcal{S}_{t-1}) \subseteq P(\Gamma_t^i, \Lambda_{t-1}^{n-i})$. Another direction is obvious. This proves Lemma 3.2. $\qquad\qquad\square$

It follows from Lemma 3.2 that the intersection of $\mathcal{H}_t^i$ and $P(\mathcal{S}_t, \mathcal{S}_{t-1})$ is an integral polytope of codimensional 1. Hence $\mathcal{H}_t^i$ gives a hyperplane decomposition of $P(\mathcal{S}_t, \mathcal{S}_{t-1})$. Moreover, one verifies that $\mathcal{S}_t$ and $\mathcal{S}_{t-1}$ lie on the opposite side of $\mathcal{H}_t^i$. Hence $\mathcal{H}_t^i$ cuts $P(\mathcal{S}_t, \mathcal{S}_{t-1})$ into two polytopes:

$$P(\mathcal{S}_t, \Gamma_t^i \cup \Lambda_{t-1}^{n-i}) = P(\Gamma_t^0, \Lambda_{t-1}^{n-i}), \text{ and } P(\Gamma_t^i \cup \Lambda_{t-1}^{n-i}, \mathcal{S}_{t-1}) = P(\Gamma_t^i, \Lambda_{t-1}^0).$$

Suppose $i \ge 2$ and the hyperplanes $\{\mathcal{H}_t^j\}_{j=1}^{i-1}$ cut $P(\mathcal{S}_t, \mathcal{S}_{t-1})$ into i polytopes: $P(\Gamma_t^{i-1}, \Lambda_{t-1}^0)$ and $P(\Gamma_t^j, \Lambda_{t-1}^{n-j-1})$ for $j = 0, \ldots, i-2$. Note that

$$P(\Gamma_t^i, \Lambda_{t-1}^{n-i}) \subseteq \mathcal{H}_t^i \cap P(\Gamma_t^{i-1}, \Lambda_{t-1}^0) \subseteq \mathcal{H}_t^i \cap P(\mathcal{S}_t, \mathcal{S}_{t-1}) = P(\Gamma_t^i, \Lambda_{t-1}^{n-i}).$$

Then $P(\Gamma_t^i, \Lambda_{t-1}^{n-i}) = \mathcal{H}_t^i \cap P(\Gamma_t^{i-1}, \Lambda_{t-1}^0)$. Since Γ_t^{i-1} and Λ_{t-1}^0 lie on the opposite side of $\mathcal{H}_t^i$, one has that $\mathcal{H}_t^i$ cuts $P(\Gamma_t^{i-1}, \Lambda_{t-1}^0)$ into two parts:

$$P(\Gamma_t^i \cup \Lambda_{t-1}^{n-i}, \Gamma_t^{i-1}) = P(\Gamma_t^{i-1}, \Lambda_{t-1}^{n-i}), \text{ and } P(\Gamma_t^i \cup \Lambda_{t-1}^{n-i}, \Lambda_{t-1}^0) = P(\Gamma_t^i, \Lambda_{t-1}^0).$$

For $j = 0, \ldots, i-2$, $P(\Gamma_t^j, \Lambda_{t-1}^{n-j-1})$ is contained in $P(\Gamma_t^0, \Lambda_{t-1}^{n-i})$. Then one has that the hyperplanes $\{\mathcal{H}_t^j\}_{j=1}^{i}$ cut $P(\mathcal{S}_t, \mathcal{S}_{t-1})$ into $i+1$ polytopes: $P(\Gamma_t^i, \Lambda_{t-1}^0)$ and $P(\Gamma_t^j, \Lambda_{t-1}^{n-j-1})$ for $i = 0, \ldots, i-1$. By induction we can prove the series of hyperplanes $\{\mathcal{H}_t^i\}_{i=1}^{n-1}$ divide $\tilde{\delta}_d(t)$ into n parts:

$$(3.3) \qquad \tilde{\delta}_d(t) = \bigcup_{i=0}^{n-1} P(\Gamma_t^i, \Lambda_{t-1}^{n-i-1}),$$

which is a regular subdivision by the previous paragraph.

We now show that (3.3) can be lifted to a regular subdivision of $\delta_d(t)$. For $i = 0, 1, \cdots, n - 1$, let

$$\hat{\Gamma}_t^i := \rho^{-1}(\Gamma_t^i) = \{((A + B)t/s - B, \mathbf{v}) \in \mathbb{R}^{n+1} : \mathbf{v} \in \Gamma_t^i\}$$

and

$$\hat{\Lambda}_{t-1}^{n-i-1} := \rho^{-1}(\Lambda^{n-i-1}) = \{((A + B)(t - 1)/s - B, \mathbf{v}) \in \mathbb{R}^{n+1} : \mathbf{v} \in \Lambda_{t-1}^{n-i-1}\}.$$

Then

$$P(\hat{\Gamma}_t^i, \hat{\Lambda}_{t-1}^{n-i-1}) = \rho^{-1}\left(P(\Gamma_t^i, \Lambda_{t-1}^{n-i-1})\right)$$

is an integral polytope. Hence we get a regular subdivision of $\delta_d(t)$:

(3.4)
$$\delta_d(t) = \bigcup_{i=0}^{n-1} P(\hat{\Gamma}_t^i, \hat{\Lambda}_{t-1}^{n-i-1}).$$

Next, for $1 \le t \le s$ and $0 \le i \le n - 1$, we will give a regular triangulation for each $P(\Gamma_t^i, \Lambda_{t-1}^{n-i-1})$, which will be lifted to a regular triangulation of $P(\hat{\Gamma}_t^i, \hat{\Lambda}_{t-1}^{n-i-1})$. For $i + 1 \le k \le n, 1 \le j \le \frac{td}{s} - 1$, let $\mathcal{H}_t^i(k, j)$ be the affine subspace generated by Λ_{t-1}^{n-i-1} and points on Γ_t^i whose k-th coordinates are j. We will show that each $\mathcal{H}_t^i(k, j)$ is indeed a hyperplane by calculating its equation. Suppose that one equation of $\mathcal{H}_t^i(k, j)$ is given by

(3.5)
$$a_1^{(t,i)} x_1 + \cdots + a_n^{(t,i)} x_n = 1.$$

By definition, the vertices $\frac{(t-1)d}{s}\mathbf{e}_1, \ldots, \frac{(t-1)d}{s}\mathbf{e}_{i+1}$ of Λ_{t-1}^{n-i-1} lie on $\mathcal{H}_t^i(k, j)$. We have

$$a_1^{(t,i)} = \cdots = a_{i+1}^{(t,i)} = \frac{s}{(t - 1)d}.$$

If $k = i + 1$, then the following $(n - i - 1)$ points lie on $\mathcal{H}_t^i(k, j)$:

$$(0, \cdots, 0, j, \frac{td}{s} - j, 0, \cdots, 0), \ldots, (0, \cdots, 0, j, 0, \cdots, 0, \frac{td}{s} - j),$$

where j is located at each $(i + 1)$-th coordinate. We have that

$$\frac{sj}{(t - 1)d} + a_{i+2}^{(t,i)}(\frac{td}{s} - j) = \cdots = \frac{sj}{(t - 1)d} + a_n^{(t,i)}(\frac{td}{s} - j) = 1.$$

Hence

$$a_{i+2}^{(t,i)} = \cdots = a_n^{(t,i)} = \frac{s((t - 1)d - sj)}{d(t - 1)(td - sj)}.$$

Suppose $k > i + 1$. For $\tilde{k} \in \{i + 2, \cdots, n\} \setminus \{k\}$, inputting the coordinates of points whose k-th coordinate is j, $\tilde{k}$-th coordinate is $\frac{td}{s} - j$ and other coordinates are zeros, we solve that

$$a_j^{(t,i)} = \frac{s}{(t - 1)d} \text{ for } j \ne k \text{ and } a_k = \frac{sj - d}{(t - 1)dj}.$$

The above calculation shows that $\mathcal{H}_t^i(k, j)$ is only determined by one linear equation. In other words, $\mathcal{H}_t^i(k, j)$ is a hyperplane in $\mathbb{R}^n$.

For $1 \le l \le i + 1, 1 \le m \le \frac{(t-1)d}{s} - 1$, let $\mathcal{P}_t^i(l, m)$ be the affine subspace generated by Γ_t^i and points in Λ_{t-1}^{n-i-1} whose l-th coordinate is m. One calculates that $\mathcal{P}_t^i(l, m)$ is only determined by one equation

(3.6)
$$b_1^{(t,i)} x_1 + \cdots + b_n^{(t,i)} x_n = 1,$$

whose coefficients are given as follows. If $l = i + 1$, then

$$b_{i+1}^{(t,i)} = \cdots = b_n^{(t,i)} = \frac{s}{td} \text{ and } b_1^{(t,i)} = \cdots = b_i^{(t,i)} = \frac{s(td - sm)}{td((t-1)d - sm)} > b_l^{(t,i)}.$$

If $l < i + 1$, then

$$b_j^{(t,i)} = \frac{s}{td} \text{ for } j \neq l \text{ and } b_l^{(t,i)} = \frac{s}{td} + \frac{1}{tm} > b_j^{(t,i)}.$$

The series of hyperplanes

$$\left\{ v_k = j \right\}_{i+1 \leq k \leq n; 1 \leq j \leq \frac{td}{s} - 1}$$

divide Γ_t^i into $(\frac{td}{s})^{n-i-1}$ unit simplices of dimension $(n - i - 1)$. Denote the set of these $(\frac{td}{s})^{n-i-1}$ unit simplices by Υ_t^i. The series of hyperplanes

$$\left\{ v_l = m \right\}_{1 \leq l \leq i+1; 1 \leq m \leq \frac{(t-1)d}{s} - 1}$$

divide Λ_{t-1}^{n-i-1} into $(\frac{(t-1)d}{s})^i$ unit simplices of dimension $(n - i - 1)$. Denote the set of these $(\frac{(t-1)d}{s})^i$ unit simplices by Φ_{t-1}^{n-i-1}. We have

Lemma 3.3.

$$P(\Gamma_t^i, \Lambda_{t-1}^{n-i-1}) = \bigcup_{\alpha \in \Upsilon_t^i, \beta \in \Phi_{t-1}^{n-i-1}} P(\alpha, \beta).$$

Proof. Suppose that $\mathbf{v}_1, \cdots, \mathbf{v}_{n-i}$ are vertices of Γ_t^i, and $\mathbf{v}_{n-i+1}, \cdots, \mathbf{v}_{n+1}$ are vertices of Λ_{t-1}^{n-i-1}. For any $\mathbf{x} \in P(\Gamma_t^i, \Lambda_{t-1}^{n-i-1})$, we have

$$\mathbf{x} = \sum_{j=1}^{n-i} \lambda_j \mathbf{v}_j + \sum_{j=n-i+1}^{n+1} \lambda_j \mathbf{v}_j$$

for some $\lambda_j \geq 0$ such that $\sum_{j=1}^{n+1} \lambda_j = 1$. Let $\lambda = \lambda_1 + \cdots + \lambda_{n-i}$, $\mathbf{x}_1 = \frac{1}{\lambda} \sum_{j=1}^{n-i} \lambda_j \mathbf{v}_j$ and $\mathbf{x}_2 = \frac{1}{1-\lambda} \sum_{j=n-i+1}^{n+1} \lambda_j \mathbf{v}_j$. We have $\mathbf{x}_1 \in \Gamma_t^i$, $\mathbf{x}_2 \in \Lambda_{t-1}^{n-i-1}$ and $\mathbf{x} = \lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2$. Choose $\alpha \in \Upsilon_t^i$, $\beta \in \Phi_{t-1}^{n-i-1}$ such that $\mathbf{x}_1 \in \alpha$, $\mathbf{x}_2 \in \beta$, we get $\mathbf{x} \in P(\alpha, \beta)$. Hence $P(\Gamma_t^i, \Lambda_{t-1}^{n-i-1}) \subseteq \bigcup_{\alpha \in \Upsilon_t^i, \beta \in \Phi_{t-1}^{n-i-1}} P(\alpha, \beta)$. The other direction is evident. This proves Lemma 3.3. $\square$

Note that each $\alpha \in \Upsilon_t^i$ (resp. $\beta \in \Phi_{t-1}^{n-i-1}$) lies on one side of each hyperplane $\{v_k = j\}$ (resp. $\{v_l = m\}$). Hence $P(\alpha, \beta)$ lies on one side of each hyperplane $\mathcal{H}_t^i(k, j)$ (resp. $\mathcal{P}_t^i(l, m)$). For different pairs (α, β) and (α', β'), if $\alpha \neq \alpha'$, then we choose (k, j) such that α and α' lie on the opposite sides of hyperplane $\{v_k = j\}$ in Γ_t^i, which means that $P(\alpha, \beta)$ and $P(\alpha', \beta')$ lie on the opposite sides of $\mathcal{H}_t^i(k, j)$. If $\beta \neq \beta'$, then we can similarly choose $\mathcal{P}_t^i(l, m)$ such that $P(\alpha, \beta)$ and $P(\alpha', \beta')$ lie on the opposite sides of $\mathcal{P}_t^i(l, m)$.

Hence the hyperplanes

$$\left\{ \mathcal{H}_t^i(k, j), \mathcal{P}_t^i(l, m) \right\}_{\substack{i+1 \leq k \leq n; 1 \leq j \leq td/s - 1 \\ 1 \leq l \leq i+1; 1 \leq m \leq (t-1)d/s - 1}}$$

divide $P(\Gamma_t^i, \Lambda_{t-1}^{n-i-1})$ into a regular subdivision (see Figure 1. for the case $A = B = 1$, $d = 2$ and $n = 3$):

$$P(\Gamma_t^i, \Lambda_{t-1}^{n-i-1}) = \bigcup_{\alpha \in \Upsilon_t^i, \beta \in \Phi_{t-1}^{n-i-1}} P(\alpha, \beta).$$
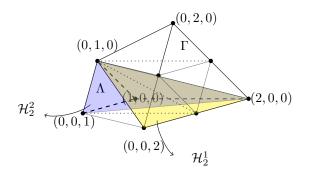
FIGURE 1.   A complete regular subdivision of $\tilde{\delta}_d(2)$ when $A = B = 1$, $d = 2$ and $n = 3$

For $0 \le i \le n-1, \alpha \in \Upsilon_t^i, \beta \in \Phi_{t-1}^{n-i-1}$, let

$$\hat{\alpha} := \rho^{-1}(\alpha) = \left\{ \left(\frac{t(A+B)}{s} - B, \mathbf{v} \right) : \mathbf{v} \in \alpha \right\}$$

and

$$\hat{\beta} := \rho^{-1}(\beta) = \left\{ \left(\frac{(t-1)(A+B)}{s} - B, \mathbf{v} \right) : \mathbf{v} \in \beta \right\}.$$

Let $P(\hat{\alpha}, \hat{\beta})$ be the convex hull of $\hat{\alpha}$ and $\hat{\beta}$. Then $P(\hat{\alpha}, \hat{\beta}) = \rho^{-1}\big(P(\alpha, \beta)\big)$ is an integral polytope. Hence we get a regular subdivision (see Figure 2. for the case $A = 2$, $B = 1$, $d = 6$ and $n = 2$)

$$(3.7) \qquad P(\hat{\Gamma}_t^i, \hat{\Lambda}_{t-1}^{n-i-1}) = \bigcup_{\alpha \in \Upsilon_t^i, \beta \in \Phi_{t-1}^{n-i-1}} P(\hat{\alpha}, \hat{\beta}).$$

Combining (3.1), (3.4) and (3.7), we have a decomposition:

$$(3.8) \qquad \delta_d = \bigcup_{t=1}^{s} \bigcup_{i=0}^{n-1} \bigcup_{\alpha \in \Upsilon_t^i, \beta \in \Phi_{t-1}^{n-i-1}} P(\hat{\alpha}, \hat{\beta}).$$

We have proved that (3.1) is a regular subdivision of δ_d, (3.4) is a regular subdivision of $\delta_d(t)$ and (3.7) is a regular subdivision of $P(\hat{\Gamma}_t^i, \hat{\Lambda}_{t-1}^{n-i-1})$. It follows from Lemma 2.7 that to prove (3.8) is a regular subdivision, it suffices to prove (3.8) is a subdivision, and this follows from Lemma 3.5. $\square$

Using the same method as the proof of Lemma 3.3, we can prove

Lemma 3.4. *For any* $\mathbf{u} \in P(\hat{\alpha}, \hat{\beta})$, *there exist points* $\mathbf{a} \in \hat{\alpha}$ *and* $\mathbf{b} \in \hat{\beta}$ *such that* $\mathbf{u} = \lambda \mathbf{a} + (1 - \lambda)\mathbf{b}$ *for some real number* $\lambda \in [0, 1]$.

It follows from Lemma 3.4 that any lattice point in $P(\hat{\alpha}, \hat{\beta})$ must lie in $\hat{\alpha}$ or $\hat{\beta}$ by considering its first coordinate. Since simplices α and β contain no lattice points other than vertices, one has $P(\hat{\alpha}, \hat{\beta})$ is an indecomposable simplex.

Lemma 3.5. *Choose two pieces* $P(\hat{\alpha}_1, \hat{\beta}_1)$ *and* $P(\hat{\alpha}_2, \hat{\beta}_2)$ *of (3.8) such that* $P(\hat{\alpha}_1, \hat{\beta}_1) \subseteq P(\hat{\Gamma}_{t_1}^i, \hat{\Lambda}_{t_1-1}^{n-i-1})$ *and* $P(\hat{\alpha}_2, \hat{\beta}_2) \subseteq P(\hat{\Gamma}_{t_2}^j, \hat{\Lambda}_{t_2-1}^{n-j-1})$. *If* $t_1 = t_2$, *then*

$$P(\hat{\alpha}_1, \hat{\beta}_1) \cap P(\hat{\alpha}_2, \hat{\beta}_2) = P(\hat{\alpha}_{12}, \hat{\beta}_{12}),$$
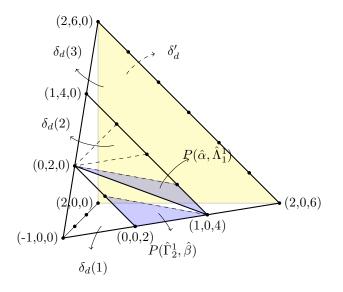
FIGURE 2.   A complete regular subdivision of $\delta_d(2)$ when $A = 2, B = 1$, $d = 6$ and $n = 2$

where $\alpha_{12} = \alpha_1 \cap \alpha_2$, $\beta_{12} = \beta_1 \cap \beta_2$. If $|t_1 - t_2| = 1$, then $P(\hat{\alpha}_1, \hat{\beta}_1) \cap P(\hat{\alpha}_2, \hat{\beta}_2) = \hat{\beta}_1 \cap \hat{\alpha}_2$ or $\hat{\beta}_2 \cap \hat{\alpha}_1$. If $|t_1 - t_2| > 1$, then $P(\hat{\alpha}_1, \hat{\beta}_1) \cap P(\hat{\alpha}_2, \hat{\beta}_2) = \emptyset$. In particular, $P(\hat{\alpha}_1, \hat{\beta}_1) \cap P(\hat{\alpha}_2, \hat{\beta}_2)$ is a common face of $P(\hat{\alpha}_1, \hat{\beta}_1)$ and $P(\hat{\alpha}_2, \hat{\beta}_2)$.

Proof. We only prove the case $t_1 = t_2$, the others are similar and much simpler. It is obvious that $P(\hat{\alpha}_{12}, \hat{\beta}_{12}) \subseteq P(\hat{\alpha}_1, \hat{\beta}_1) \cap P(\hat{\alpha}_2, \hat{\beta}_2)$. It remains to show the other direction. Let $\mathbf{u} \in P(\hat{\alpha}_1, \hat{\beta}_1) \cap P(\hat{\alpha}_2, \hat{\beta}_2)$. By Lemma 3.4, there exist $\mathbf{a}_1 \in \hat{\alpha}_1$, $\mathbf{b}_1 \in \hat{\beta}_1$, $\mathbf{a}_2 \in \hat{\alpha}_2$, $\mathbf{b}_2 \in \hat{\beta}_2$, and $0 \le \lambda, \mu \le 1$ satisfying that

$$(3.9) \qquad \mathbf{u} = \lambda \mathbf{a}_1 + (1 - \lambda)\mathbf{b}_1 = \mu \mathbf{a}_2 + (1 - \mu)\mathbf{b}_2.$$

Comparing the first coordinate of elements of (3.9), we have $\lambda = \mu$. Hence

$$\lambda(\mathbf{a}_1 - \mathbf{a}_2) = (1 - \lambda)(\mathbf{b}_2 - \mathbf{b}_1).$$

We divide the proof into two cases.

CASE 1. $j = i$. The first i coordinates of the left hand side and the last $(n - i - 1)$ coordinates of the right hand side is 0. Thus $\mathbf{a}_1 = \mathbf{a}_2 \in \hat{\alpha}_1 \cap \hat{\alpha}_2 = \hat{\alpha}_{12}$ and $\mathbf{b}_1 = \mathbf{b}_2 \in \hat{\beta}_1 \cap \hat{\beta}_2 = \hat{\beta}_{12}$. Then $\mathbf{u} \in P(\hat{\alpha}_{12}, \hat{\beta}_{12})$. Thus $P(\hat{\alpha}_1, \hat{\beta}_1) \cap P(\hat{\alpha}_2, \hat{\beta}_2) \subseteq P(\hat{\alpha}_{12}, \hat{\beta}_{12})$.

CASE 2. $j \ne i$. Without loss of generality, we assume $j > i$. Note that the hyperplane $\hat{\mathcal{H}}_{t_1}^j$ passing through $\hat{\Gamma}_{t_1}^j$ and $\hat{\Lambda}_{t_1-1}^{n-j}$ separates $P(\hat{\Gamma}_{t_1}^i, \hat{\Lambda}_{t_1-1}^{n-i-1})$ and $P(\hat{\Gamma}_{t_1}^j, \hat{\Lambda}_{t_1-1}^{n-j-1})$. Then $\mathbf{b}_1 \in \hat{\Lambda}_{t_1-1}^{n-i-1} \subseteq \hat{\Lambda}_{t_1-1}^{n-j} \subseteq \hat{\mathcal{H}}_{t_1}^j$. Note that $\mathbf{u} \in \hat{\mathcal{H}}_{t_1}^j$. Then $\mathbf{a}_1 \in \hat{\mathcal{H}}_{t_1}^j$. It follows from Lemma 3.2 that $\mathbf{a}_1 \in \hat{\Gamma}_{t_1}^j$. On the other hand, $\hat{\mathcal{H}}_{t_1}^{i+1}$ passing through $\hat{\Gamma}_{t_1}^{i+1}$ and $\hat{\Lambda}_{t_1-1}^{n-i-1}$ also separates $P(\hat{\Gamma}_{t_1}^i, \hat{\Lambda}_{t_1-1}^{n-i-1})$ and $P(\hat{\Gamma}_{t_1}^j, \hat{\Lambda}_{t_1-1}^{n-j-1})$. Note that $\mathbf{u} \in \hat{\mathcal{H}}_{t_1}^{i+1}$. Since $\mathbf{a}_2 \in \hat{\Gamma}_{t_1}^j \subseteq \hat{\Gamma}_{t_1}^{i+1} \subseteq \hat{\mathcal{H}}_{t_1}^{i+1}$, one has $\mathbf{b}_2 \in \hat{\Lambda}_{t_1-1}^{n-i-1}$. We conclude that $\mathbf{a}_1, \mathbf{a}_2 \in \hat{\Gamma}_{t_1}^j$ and $\mathbf{b}_1, \mathbf{b}_2 \in \hat{\Lambda}_{t_1-1}^{n-i-1}$. Then using the same argument as Case 1, we conclude that $\mathbf{u} \in P(\hat{\alpha}_{12}, \hat{\beta}_{12})$. Then $P(\hat{\alpha}_1, \hat{\beta}_1) \cap P(\hat{\alpha}_2, \hat{\beta}_2) \subseteq P(\hat{\alpha}_{12}, \hat{\beta}_{12})$. $\qquad\square$

Proposition 3.6. *Let $n \geq 2$. The Adolphson–Sperber conjecture is true for Δ_d.*

Proof. Let $\alpha \in \Upsilon_t^i, \beta \in \Phi_{t-1}^{n-i-1}$, and $P^0(\hat{\alpha}, \hat{\beta})$ be the convex closure in $\mathbb{R}^{n+1}$ generated by the origin and lattice points in $P(\hat{\alpha}, \hat{\beta})$. Denote the vertices of $\hat{\beta}$ by

$$\mathbf{v}_j = \left(\frac{(A + B)(t - 1) - sB}{s}, v_{j,1}, \cdots, v_{j,i+1}, 0, \cdots, 0 \right)$$

for $j = 1, \ldots, i + 1$, the vertices of $\hat{\alpha}$ by

$$\mathbf{v}_j = \left(\frac{(A + B)t - sB}{s}, 0, \cdots, 0, v_{j,i+1}, \cdots, v_{j,n} \right)$$

for $j = i + 2, \ldots, n + 1$. Using matrix operations, we have

$$(n + 1)!\mathrm{Vol}(P^0(\hat{\alpha}, \hat{\beta})) = \pm \left| \mathbf{v}_1, \cdots, \mathbf{v}_{i+1}, \mathbf{v}_{i+2}, \cdots, \mathbf{v}_{n+1} \right|$$

$$= \pm \frac{Bs}{t(t - 1)d} \begin{vmatrix} v_{1,1} & \cdots & v_{i+1,1} \\ \vdots & \cdots & \vdots \\ v_{1,i+1} & \cdots & v_{i+1,i+1} \end{vmatrix} \cdot \begin{vmatrix} v_{i+2,i+1} & \cdots & v_{n+1,i+1} \\ \vdots & \vdots & \vdots \\ v_{i+2,n} & \cdots & v_{n+1,n} \end{vmatrix},$$

where each $\mathbf{v}_j$ is written as a column vector. Denote by $p_{i+1} : \mathbb{R}^n \to \mathbb{R}^{i+1}$ be the projection of $\mathbb{R}^n$ onto its first $i + 1$ coordinates. Let $\rho_{i+1} = p_{i+1} \circ \rho : L_d \to \mathbb{R}^{i+1}$ be the composition of ρ and p_{i+1}. Then $\rho_{i+1}(\hat{\Lambda}_{t-1}^i) = p_{i+1}(\Lambda_{t-1}^i) \subseteq \mathbb{R}^{i+1}$ is the simplex with vertices

$$\left(\frac{(t - 1)d}{s}, 0, \cdots, 0 \right), \left(0, \frac{(t - 1)d}{s}, 0, \cdots, 0 \right), \ldots, \left(0, 0, \cdots, 0, \frac{(t - 1)d}{s} \right).$$

Then $\rho_{i+1}(\hat{\beta}) = p_{i+1}(\beta) \in \mathbb{R}^{i+1}$ is the simplex with vertices

$$(v_{1,1}, v_{1,2}, \cdots, v_{1,i+1}), (v_{2,1}, v_{2,2}, \cdots, v_{2,i+1}), \ldots, (v_{i+1,1}, v_{i+1,2}, \cdots, v_{i+1,i+1}),$$

which is a cell of the hyperplane decompositions of $\rho_{i+1}(\hat{\Lambda}^i)$ cut by $\{v_l = m\}$ for $l = 1, \ldots, i + 1$ and $m = 1, \ldots, \frac{(t-1)d}{s} - 1$. Calculating the volume of $\rho_{i+1}(\hat{\beta})$, we have

$$\begin{vmatrix} v_{1,1} & \cdots & v_{i+1,1} \\ \vdots & \cdots & \vdots \\ v_{1,i+1} & \cdots & v_{i+1,i+1} \end{vmatrix} = \pm(i + 1)!\mathrm{Vol}(\rho_{i+1}(\hat{\beta})) = \pm \frac{(t - 1)d}{s}.$$

Similarly, we can prove

$$\begin{vmatrix} v_{i+2,i+1} & \cdots & v_{n+1,i+1} \\ \vdots & \cdots & \vdots \\ v_{i+2,n} & \cdots & v_{n+1,n} \end{vmatrix} = \pm \frac{td}{s}.$$

Hence

$$(n + 1)!\mathrm{Vol}(P^0(\hat{\alpha}, \hat{\beta})) = \frac{dB}{s}.$$

It follows from Proposition 2.2, Corollary 2.14 that if $p \equiv 1 \mod \frac{dB}{s}$, then Δ_d is generically ordinary. This finishes the proof of Proposition 3.6. $\qquad \square$

Proof of Theorem 1.5. If $s = (A + B, d) = 1$, it follows from [6, Theorem 5.1] that Theorem 1.5 holds. If $s \geq 2$, it then follows from Proposition 2.2 and 3.6 that Δ_d is generically ordinary if $p \equiv 1 \mod \frac{dB}{s}$. It follows from [14, Theorem 7.5] that the Adolphson–Sperber conjecture is true for Δ_d'. The face δ_d' lies on the hyperplane $x_0 = A$. Then $D(\Delta_d') = A$ and $\mathrm{GNP}(\Delta_d', p) = \mathrm{HP}(\Delta_d')$ if $p \equiv 1 \mod A$. Note that $\Delta = \Delta_d \cup \Delta_d'$. Then Theorem 1.5 follows from the facial decomposition theorem. This finishes the proof of Theorem 1.5. $\quad \square$

Acknowledgements

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