

Bounds for the chromatic index of signed multigraphs

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Abstract

The paper studies edge-coloring of signed multigraphs and extends classical Theorems of Shannon [6] and König [4] to signed multigraphs.

We prove that the chromatic index of a signed multigraph (G, σ_G) is at most $\lfloor \frac{3}{2}\Delta(G) \rfloor$. Furthermore, the chromatic index of a balanced signed multigraph (H, σ_H) is at most $\Delta(H) + 1$ and the balanced signed multigraphs with chromatic index $\Delta(H)$ are characterized.

1 Introduction

Edge-coloring of graphs is a classical topic in graph theory and early results for upper bound of the chromatic index of graphs are proved by König and Shannon. König [4] proved that the chromatic index of a bipartite graph is equal to its maximum vertex degree Δ , and Shannon [6] proved that it is bounded by $\lfloor \frac{3}{2}\Delta \rfloor$ for arbitrary graphs.

There are several notions of (vertex-) coloring of signed graphs, which had been discussed intensively, see [7] for a survey. Recently, a very natural notion of edge-coloring signed graphs had been introduced in [1, 9]. This notion generalizes edge-coloring of graphs. Indeed, for all-negative signed graphs it coincides with edge-coloring (unsigned) graphs. Motivated by the aforementioned results of König and Shannon, we prove corresponding bounds for signed graphs. We show that the chromatic index of a balanced signed graph is at most $\Delta + 1$, and it is at most Δ , if the graph has a matching, whose removal results in a graph with even maximum degree (Theorem 3.1). We further prove that Shannon's bound also applies to signed graphs (Theorem 3.2).

2 Definitions and basic results

For simplicity, we use the term graph instead of multigraph in this paper. Thus, graphs may contain parallel edges and loops. The vertex set of a graph G is denoted by $V(G)$ and

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its edge set by $E(G)$. Let $X \subseteq V(G)$ or $X \subseteq E(G)$, the graph induced by X is denoted by $G[X]$.

Let e be an edge that is incident to vertices v and w . If there is no harm of confusion, then we will also use the term vw to denote e . Edge vw consists of two half-edges one of which is incident to v and the other to w . The half-edges are denoted by $h_e(v)$ and $h_e(w)$. If $v = w$, then e is a loop. The set of half-edges of G is denoted by $H(G)$ and $H_G(v)$ denotes the set of half-edges that are incident to v . The cardinality of $H_G(v)$ is the degree of v and is denoted by $d_G(v)$. Furthermore, $\Delta(G) = \max\{d_G(v) : v \in V(G)\}$ is the maximum degree of G . The set of vertices of maximum degree in G is denoted by $V_{\Delta(G)}$. A circuit is a connected 2-regular graph.

For the definition of an edge-coloring of signed graphs we will use a natural correspondence between bidirectional graphs and signed graphs. Let G be a graph and $\tau_G : H(G) \rightarrow \{\pm 1\}$ be a function. This function defines a bidirection of the edges of G . If $e = vw$ is an edge, then half-edge $h_e(v)$ is oriented away from v if $\tau_G(h_e(v)) = 1$, and towards v if $\tau_G(h_e(v)) = -1$. Edge e is introverted if $\tau_G(h_e(v)) = \tau_G(h_e(w)) = 1$ and extroverted if $\tau_G(h_e(v)) = \tau_G(h_e(w)) = -1$.

Let G be a graph with bidirection τ . The signed graph (G, σ) is the graph G together with a function $\sigma : E(G) \rightarrow \{\pm 1\}$ with $\sigma(e) = -(\tau(h_e(v))\tau(h_e(w)))$ for each edge $e = vw$. The function σ is called a signature of G . An edge e is negative if $\sigma(e) = -1$, otherwise it is positive. The set of negative edges of (G, σ) is denoted by N_σ . The set $E(G) - N_\sigma$ is the set of positive edges. By definition, N_σ consists of those edges which are introverted or extroverted in the bidirected graph.

A circuit is positive if the product of the signs of its edges is positive, otherwise it is negative. A subgraph $(H, \sigma|_{E(H)})$ of (G, σ) is balanced if all circuits in $(H, \sigma|_{E(H)})$ are positive, otherwise it is unbalanced. If $\sigma(e) = 1$ for all $e \in E(G)$, then σ is the all positive signature and it is denoted by $\mathbf{1}$, and if $\sigma(e) = -1$ for all $e \in E(G)$, then σ is the all negative signature and it is denoted by $-\mathbf{1}$.

A resigning of a signed graph (G, σ) at a vertex v defines a signed graph (G, σ') with $\tau'_G(h) = -\tau_G(h)$ if $h \in H_G(v)$, $\tau'_G(h) = \tau_G(h)$ otherwise, and $\sigma'(e) = -(\tau'_G(h_e(u))\tau'_G(h_e(w)))$ for each edge $e = uw$. Note that resigning at v changes the sign of each edge incident to v but the sign of a loop incident to v is unchanged. Thus, resigning does not change the parity of the number of negative edges in an eulerian subgraph of a signed graph. Resigning defines an equivalence relation on the set of signatures of a graph G . We say that (G, σ_1) and (G, σ_2) are equivalent if they can be obtained from each other by a sequence of resignings. We also say that σ_1 and σ_2 are equivalent signatures of G . In [8] it is proved that two signed graphs are equivalent if and only if they have the same set of negative circuits.

We frequently will use the following well known fact. Let (G, σ) be a signed graph and

$S \subseteq E(G)$. If $G[S]$ is a forest, then there is a signature σ' that is equivalent to σ and $S \subseteq N_{\sigma'}$.

It suffices to consider edge-cuts for resigning since resigning at every vertex of a set $X \subseteq V(G)$ changes the sign of every edge of the edge-cut $\partial_G(X)$ and the signs of all other edges are unchanged. Thus, Harary's characterization of balanced signed graphs [3] has the following consequence.

Observation 2.1. *A signed graph (G, σ) is balanced if and only if it is equivalent to $(G, \mathbf{1})$.*

We define a signed graph (G, σ) to be antibalanced if it is equivalent to $(G, -\mathbf{1})$. Clearly, (G, σ) is antibalanced if and only if the sign product of every even circuit is 1 and it is -1 for every odd circuit.

2.1 Coloring signed graphs

Basically we follow the approach of [1, 9] for edge-coloring signed graphs. For simplicity we use the word coloring instead of edge-coloring since we only consider edge-coloring in this paper. For technical reasons we will use elements of symmetric sets for coloring as it is introduced in [2]. A set S together with a sign “-” is a symmetric set if it satisfies the following conditions:

1. $s \in S$ if and only if $-s \in S$.
2. If $s = s'$, then $-s = -s'$.
3. $s = -(-s)$.

An element s of a symmetric set S is self-inverse if $s = -s$. A symmetric set with self-inverse elements $0_1, \dots, 0_t$ and non-self-inverse elements $\pm s_1, \dots, \pm s_k$ is denoted by S_{2k}^t . Clearly, $|S_{2k}^t| = t + 2k$.

Let (G, σ) be a signed graph with orientation τ and t, k be two non-negative integers not both equal to 0. A function $c: E(G) \rightarrow S_{2k}^t$ is an S_{2k}^t -coloring of (G, σ) (with orientation τ) if

$$\tau(h_{e_1}(v))c(e_1) \neq \tau(h_{e_2}(v))c(e_2)$$

for every vertex $v \in V(G)$ and any two different edges $e_1, e_2 \in E_G(v)$. An example is given in Figure 1.

By definition, if (G, σ) and (G, σ') are equivalent, then (G, σ) admits an S_{2k}^t -coloring if and only if (G, σ') admits an S_{2k}^t -coloring. We will often (implicitly) use the following statements.

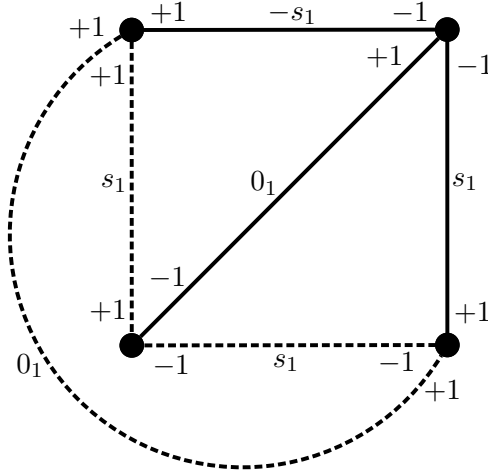


Figure 1: An example of an S_2^1 -coloring. Negative edges are drawn as dashed lines.

Observation 2.2. *Let (G, σ) be a signed graph with orientation τ . If (G, σ) has an S_{2k}^t -coloring c , then (G, σ) has no negative loop, and for each edge $e = vw$: $\tau(h_e(v))c(e) = \tau(h_e(w))c(e)$ if and only if e is negative or $c(e)$ is self-inverse.*

Let c be an S_{2k}^t -coloring of (G, σ) and $v \in V(G)$. We say that color a appears at v , if $\tau(h_e(v))c(e) = a$ for an edge $e = vw$. Otherwise, we say that v misses a . Indeed, c induces an S_{2k}^t -coloring c_H of the half-edges of (G, σ) , where $c_H(h_e(v)) = \tau(h_e(v))c(e)$.

For S_{2k}^0 and S_{2k}^1 our definition coincides with the definition of an edge-coloring of simple signed graphs given in [1, 9]. We will need a slight extension of some basic results which are observed by Behr [1].

- Lemma 2.3.**
1. *A signed path has an S_2^0 -coloring. [1]*
 2. *A signed circuit has an S_2^0 -coloring if and only if it is positive. [1]*
 3. *A negative signed circuit has an S_2^1 -coloring that colors precisely one negative edge with color 0.*

Proof. We prove item 3. A negative circuit (C, σ) has a negative edge e . Then $(C - e, \sigma|_{E(C-e)})$ is a path and therefore, it has an S_2^0 -coloring by statement 1. This coloring can be extended to a S_2^1 -coloring of (C, σ) by coloring e with color 0. \square

Let c be an S_{2k}^t -coloring of (G, σ) . A color class of a self-inverse color is called a self-inverse color class and a color class of a non-self-inverse color is called non-self-inverse. By Observation 2.2, a self-inverse color class is a matching in G and a non-self-inverse color class $c^{-1}(\pm s)$ consist of paths and balanced circuits, by Lemma 2.3. This is due to the fact that the two sets $c^{-1}(s)$ and $c^{-1}(-s)$ cannot not separated from each other in view of the half-edges. Note that the induced S_{2k}^t -coloring c_H of the half-edges may use color $-s$ even

for edges of $c^{-1}(s)$. This fact allows a simple variant of Kempe-switching in signed graphs. Let D be a component of a color class. Let $c'(e) = -c(e)$ if $e \in D$ and $c'(e) = c(e)$ otherwise. Then c' is an S_{2k}^t -coloring of (G, σ) . We say that c' is obtained from c by resigning a color at D .

We will also follow the approach of [1, 9] for defining the signed chromatic index of a signed graph. Coloring an edge with a self-inverse color annuls the sign of the edge. Thus, it is appropriate to minimize the number of self-inverse elements in the definition of the chromatic index of a signed graph. Since the number of self-inverse elements determines the parity of a symmetric set we need two sets for coloring. For $i \in \{0, 1\}$ let $\chi'_i(G, \sigma)$ be the minimum $2k + i$ such that (G, σ) admits an S_{2k}^i -coloring. The chromatic index of (G, σ) is $\min\{\chi'_0(G, \sigma), \chi'_1(G, \sigma)\}$ and it is denoted by $\chi'(G, \sigma)$. Note that by Observation 2.2, $\chi'(G, -\mathbf{1}) = \chi'(G)$.

Let L be a spanning subgraph of a graph G with $\Delta(L) \leq 2$. The edge set of L is called a layer of G . The following result is folklore and can easily be deduced from Petersen's 2-factorization theorem for even regular graphs [5].

Lemma 2.4. *The edge set of any graph G can be decomposed into $\lceil \frac{\Delta(G)}{2} \rceil$ layers.*

We also say that G is decomposable into $\lceil \frac{\Delta(G)}{2} \rceil$ layers.

3 Upper bounds for the chromatic index

The chromatic index of an antibalanced signed graph is equal to the chromatic index of its underlying unsigned graph. Therefore, the case $\sigma = -\mathbf{1}$ of Theorem 3.2 is Shannon's Theorem [6]. Theorem 3.1 extends König's Theorem [4] that the chromatic index of a bipartite graph is equal to its maximum degree to balanced signed graphs.

Our proofs are based on a decomposition of the edge set of (G, σ) into signed layers. Shannon [6] used this idea, which he attributed to Foster, for the proof of his theorem for graphs with even maximum degree. Our proof then also gives a proof by this method for Shannon's theorem for graphs with odd maximum degree.

Theorem 3.1. *If (G, σ) is a balanced signed graph, then $\chi'(G, \sigma) \leq \Delta(G) + 1$. In particular, $\chi'(G, \sigma) = \Delta(G)$ if and only if G has a (possibly empty) matching M such that $\Delta(G - M)$ is even. The bound is best possible.*

Proof. By Lemma 2.4, (G, σ) decomposes into $k = \lceil \frac{\Delta(G)}{2} \rceil$ layers. Each layer is balanced and therefore, S_2^0 -colorable by Lemma 2.3. The S_2^0 -colorings of the k layers easily combine to an S_{2k}^0 -coloring of (G, σ) and the first statement is proved.

If G has a matching M such that $\Delta(G - M)$ is even, say $\Delta(G - M) = 2t$, then it follows as above that $(G - M, \sigma|_{E(G-M)})$ has an S_{2t}^0 -coloring c . If $M \neq \emptyset$, then c can be extended to an S_{2t}^1 -coloring of (G, σ) by coloring the elements of M with color 0.

If $\chi'(G, \sigma) = \Delta(G)$, then we consider first the case when $\Delta(G) = 2k + 1$. Thus, (G, σ) has an S_{2k}^1 -coloring c and therefore, $c^{-1}(0)$ is a matching in G that covers every vertex of maximum degree. Thus, $\Delta(G - c^{-1}(0)) = \Delta(G) - 1$ is even and the statement is proved. The case $\Delta(G)$ even is trivial.

Let $r > 0$ be an odd integer. Then $\chi'(G, \mathbf{1}) = r + 1$, for every r -regular graph G , which does not have a 1-factor. \square

Next we will prove a Shannon-type theorem for signed graphs.

Theorem 3.2. *For each signed graph $(G, \sigma) : \chi'(G, \sigma) \leq \lfloor \frac{3}{2}\Delta(G) \rfloor$.*

Proof. Let (G, σ) be a signed graph. By Lemma 2.3, a layer has an S_2^1 -coloring that colors precisely one negative edge in each negative circuit (if there is any) with the self-inverse color and all other edges with the non-self-inverse color. We assume that a layer L_i has such a coloring c_i which uses colors 0_i and $\pm s_i$.

Claim 1: If $\Delta(G) = 2t$, then (G, σ) has an S_{2t}^t -coloring c with $\bigcup_{i=1}^t c^{-1}(0_i) \subseteq N_\sigma$.

Proof: By Lemma 2.4, $E(G)$ decomposes into t layers L_1, \dots, L_t . By our assumptions, layer L_i has an S_2^1 -coloring c_i which uses the colors 0_i and $\pm s_i$ and each edge colored with 0_i is negative. Now, the colorings c_1, \dots, c_t easily combine to an S_{2t}^t -coloring c of (G, σ) with $\bigcup_{i=1}^t c^{-1}(0_i) \subseteq N_\sigma$. \blacksquare

Claim 2: If $\Delta(G) = 2t + 1$ and V_Δ is an independent set, then there is an equivalent signature σ' such that (G, σ') has an S_{2t}^t -coloring c with $\bigcup_{i=1}^t c^{-1}(0_i) \subseteq N_{\sigma'}$.

Proof: By Hall's Theorem, there is a matching M of m edges e_1, \dots, e_m that covers all vertices of $V_\Delta = \{x_1, \dots, x_m\}$. Let $e_j = x_j y_j$, then $d_G(y_j) \leq 2t$. Since M is a matching, there is a signature σ' equivalent to σ such that $M \subseteq N_{\sigma'}$. By Lemma 2.4, $G - M$ decomposes into t layers L_1, \dots, L_t . For each $j \in \{1, \dots, m\}$, add e_j to a layer $L \in \{L_1, \dots, L_t\}$ such that y_j is incident to at most one half-edge of L , which is possible since $d_G(y_j) \leq 2t$. We obtain a decomposition of G into t sub-cubic graphs L'_1, \dots, L'_t . We claim that for each of these subgraphs there is an S_2^1 -coloring in which every edge colored with the self-inverse color belongs to $N_{\sigma'}$.

For $i \in \{1, \dots, t\}$ let $L'_i = L_i + M_i$, i.e. $M_i \subseteq M$ is the set of edges added to L_i .

By our assumption, there is an S_2^1 -coloring c_i of $(L_i, \sigma'|_{E(L_i)})$ that colors precisely one negative edge with color 0_i in each negative circuit and all other edges with $\pm s_i$. Note that, if $x_j y_j \in M_i$, then y_j misses color 0_i and at least one of $\pm s_i$. Furthermore, since V_Δ is an independent set, every edge colored with 0_i is adjacent to at most one edge of M_i . Thus, by

resigning colors s_i and $-s_i$ at some unbalanced circuits of L_i if necessary, we can transform c_i to an S_2^1 -coloring c'_i of $(L_i, \sigma'|_{E(L_i)})$ such that

1. $(c'_i)^{-1}(0_i) \subseteq N_{\sigma'}$,
2. if $x_j y_j \in M_i$ is incident with an edge colored with 0_i , then x_j and y_j either both miss color s_i or both miss color $-s_i$.

Now, c'_i can be extended to an S_2^1 -coloring of $(L'_i, \sigma'|_{E(L'_i)})$ by coloring every edge of M_i adjacent to a 0_i -colored edge with either s_i or $-s_i$ and all other edges of M_i with 0_i .

Thus, for every $i \in \{1, \dots, t\}$ there is an S_2^1 -coloring of L'_i in which every edge colored with the self-inverse color is negative. These colorings easily combine to an S_{2t}^t -coloring c of (G, σ') with $\bigcup_{i=1}^t c^{-1}(0_i) \subseteq N_{\sigma'}$. ■

Claim 3: If $\Delta(G) = 2t + 1$ and V_Δ is not an independent set, then there is an equivalent signature σ' such that (G, σ') has an S_{2t}^{t+1} -coloring c with $\bigcup_{i=1}^{t+1} c^{-1}(0_i) \subseteq N_{\sigma'}$.

Proof: Case 1: G has a matching M that covers all vertices of degree $2t + 1$. Since M is a matching, there is an equivalent signature σ' with $M \subseteq N_{\sigma'}$. By Claim 1, $(G - M, \sigma'|_{E(G - M)})$ has an S_{2t}^t -coloring c with $\bigcup_{i=1}^t c^{-1}(0_i) \subseteq N_{\sigma'}$. Coloring the edges of M with a new self-inverse color 0_{t+1} gives the desired coloring.

Case 2: G does not have a matching that covers all vertices of degree $2t + 1$. Let M_1 be a matching of G that covers the maximum number of vertices of degree $2t + 1$ in G . Thus, the vertices of degree $2t + 1$ form an independent set in $H_1 = G - M_1$. Let M_2 be a matching in H_1 that covers all vertices of degree $2t + 1$ in H_1 . The components of $G[M_1 \cup M_2]$ are isomorphic to K_2 or $K_{1,2}$. Thus, there is an equivalent signature σ' with $M_1 \cup M_2 \subseteq N_{\sigma'}$. Since all edges of M_2 are negative, there is an S_{2t}^t -coloring c of $(H_1, \sigma'|_{E(H_1)})$ with $\bigcup_{i=1}^t c^{-1}(0_i) \subseteq N_{\sigma'}$ by the same argumentation as in the proof of Claim 2. Coloring the edges of M_1 with a new self-inverse color 0_{t+1} gives the desired coloring. ■

Claims 1-3 show that in any case, there is an equivalent signature σ' such that (G, σ') has an S_{2t}^t -coloring or an S_{2t}^{t+1} -coloring (only if $\Delta = 2t + 1$) where every edge that is colored by a self-inverse color is negative.

Note that every self-inverse color class is a matching and therefore, the union of two self-inverse color classes consists of paths and even circuits. Since all these edges are negative, the circuits are balanced. Thus, the union of two self-inverse color classes can be colored with two (new) non-self-inverse colors $\pm a$.

By recoloring pairs of self-inverse color classes of the above colorings we obtain the appropriate colorings of (G, σ') . If t is odd, then an S_{2t}^t -coloring transforms to an S_{3t-1}^1 -coloring and an S_{2t}^{t+1} -coloring to an S_{3t+1}^0 -coloring. If t is even, then an S_{2t}^t -coloring transforms to an S_{3t}^0 -coloring and an S_{2t}^{t+1} to an S_{3t}^1 -coloring.

Thus, if $\Delta(G) = 2t$, then $\chi'(G, \sigma') \leq 3t \leq \lfloor \frac{3}{2}\Delta(G) \rfloor$, and if $\Delta(G) = 2t + 1$, then $\chi'(G, \sigma') \leq 3t + 1 \leq \lfloor \frac{3}{2}\Delta(G) \rfloor$. Since σ and σ' are equivalent, Theorem 3.2 is proved. \square

The bound of Theorem 3.2 is sharp. Since $\chi'(G, \mathbf{-1}) = \chi'(G)$, the bound is attained by the fat triangles, where any two vertices are connected by the same number of parallel edges.

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