

# UNIQUE CONTINUATION FOR A MEAN FIELD GAME SYSTEM

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ABSTRACT. For a mean field game system, we prove the unique continuation which asserts that if Cauchy data are zero on arbitrarily chosen lateral subboundary, then the solution identically vanishes.

## 1. INTRODUCTION AND KEY CARLEMAN ESTIMATE

Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain with smooth boundary  $\partial\Omega$ , let  $T > 0$ , and  $Q := \Omega \times (0, T)$ . In this article, we consider a system of the mean field game:

$$\begin{cases} \partial_t u(x, t) + a_1(x, t)\Delta u(x, t) - \frac{1}{2}\kappa(x, t)|\nabla u(x, t)|^2 - h(x, t)u = F, \\ \partial_t v(x, t) - \Delta(a_2(x, t)v) - \operatorname{div}(\kappa(x, t)v(x, t)\nabla u(x, t)) = G \quad \text{in } Q. \end{cases} \quad (1.1)$$

Throughout this article, we assume

$$a_1, a_2 \in C^2(\overline{Q}), > 0 \quad \text{on } \overline{Q}, \quad \kappa \in C^{1,0}(\overline{Q}), \quad h \in L^\infty(Q). \quad (1.2)$$

Here and henceforth we set  $C^{2,2}(\overline{Q}) := \{a \in C(\overline{Q}); \partial_t^k a, \partial_i \partial_j a, \partial_i a, a \in C(\overline{Q}) \text{ for } 0 \leq k \leq 2 \text{ and } 1 \leq i, j \leq n\}$ ,  $C^{1,0}(\overline{Q}) := \{a \in C(\overline{Q}); \partial_j a \in C(\overline{Q}), \quad 1 \leq j \leq n\}$  and  $H^{2,1}(Q) := \{w \in L^2(Q); w, \partial_t w, \partial_i w, \partial_i \partial_j w \in L^2(Q) \text{ for } 1 \leq i, j \leq n\}$ .

In (1.1),  $x$  and  $t$  are the state and the time variables, and  $u$  and  $v$  denote the value of the game and the population density of players respectively (e.g., Achdou, Cardaliaguet, Delarue, Porretta and Santambrogio [1], Lasry and Lions [10]).

The main purpose of this article is establish the unique continuation for (1.1):

**Theorem 1.** *Let  $\gamma \subset \partial\Omega$  be arbitrarily chosen non-empty relatively open subboundary. We*

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assume that  $(u, v), (\tilde{u}, \tilde{v}) \in H^{2,1}(Q) \times H^{2,1}(Q)$  satisfy (1.1) and

$$\begin{cases} u, \nabla u, \Delta u, \tilde{u}, \nabla \tilde{u}, \Delta \tilde{u} \in L^\infty(Q), \\ v, \nabla v, \tilde{v}, \nabla \tilde{v} \in L^\infty(Q), \quad \partial_t(u - \tilde{u}), \partial_t(v - \tilde{v}) \in L^2((\partial\Omega \setminus \Gamma) \times (0, T)). \end{cases} \quad (1.3)$$

Then  $u = \tilde{u}$ ,  $\nabla u = \nabla \tilde{u}$ ,  $v = \tilde{v}$  and  $\nabla v = \nabla \tilde{v}$  on  $\Gamma \times (0, T)$  implies  $u = \tilde{u}$  and  $v = \tilde{v}$  in  $Q$ .

The unique continuation for a single parabolic equation is well-known and as early works we can refer to Mizohata [15], Saut and Scheurer [16] among many other works.

The mean field game system is a mixed type of backward and forward parabolic equations for  $u$  and  $v$  respectively, and so the initial boundary value problem requires special cares. However, it is straightforward to establish a Carleman estimate which is applicable to the unique continuation and other problems such as inverse problems.

It is known that a relevant Carleman estimate can derive the conditional stability in determining  $u$  and  $v$  by Cauchy data on  $\Gamma \times (0, T)$ . For example we can refer to Klibanov and Timonov [9] under geometric constraints on  $\Gamma$ . We can establish the conditional stability from arbitrary subboundary by the same way as Huang, Imanuvilov and Yamamoto [3], but here we omit the details.

Our key Carleman estimate is stated as Theorem 2 in Section 2, and is derived directly thanks to that the second-order coupling terms of  $u$  appear in the equation in  $v$  as  $\Delta u$ . For general cases, such coupling should be a linear combination of  $\partial_i \partial_j u$ ,  $1 \leq i, j \leq n$ . The derivation of a relevant Carleman estimate is more complicated and in a forthcoming work we will pursue.

As for inverse problems, we refer to Klibanov [5], Klibanov and Averboukh [6], Klibanov, Li and Liu [7], [8], Liu, Mou and Zhang [11], Liu and Yamamoto [12], Liu and Zhang [13], [14]

This article is composed of three sections. In Section 2, we prove a key Carleman estimate for (1.1) and Section 3 is devoted to the completion of the proof of Theorem 1.

## 2. KEY CARLEMAN ESTIMATE

For subboundary  $\Gamma \subset \partial\Omega$ , we see that there exists  $d \in C^2(\overline{\Omega})$  such that

$$d > 0 \quad \text{in } \Omega, \quad |\nabla d| > 0 \quad \text{on } \overline{\Omega}, \quad d = 0 \quad \text{on } \partial\Omega \setminus \Gamma, \quad \nabla d \cdot \nu \leq 0 \quad \text{on } \partial\Omega \setminus \Gamma \quad (2.1)$$

(e.g, Imanuvilov [4]). Here  $\nu$  denotes the unit outward normal vector to  $\partial\Omega$ .

For arbitrarily fixed  $t_0 \in (0, T)$  and  $\delta > 0$  such that  $0 < t_0 - \delta \leq t_0 + \delta < T$ , we set

$$I = (t_0 - \delta, t_0 + \delta), \quad Q_I = \Omega \times I.$$

We set

$$P_k v(x, t) := \partial_t v + (-1)^k a(x, t) \Delta v + R(x, t, v), \quad k = 1, 2,$$

where  $a \in C^2(\overline{Q_I})$ ,  $> 0$  on  $\overline{Q_I}$ , and

$$|R(x, t, v)| \leq C_0(|v(x, t)| + |\nabla v(x, t)|), \quad (x, t) \in Q_I. \quad (2.2)$$

Moreover, let

$$\varphi(x, t) = e^{\lambda(d(x) - \beta(t - t_0)^2)},$$

where  $\lambda > 0$  is a sufficiently large parameter and  $\beta > 0$  is arbitrarily given. Henceforth  $C > 0$  denote generic constants which independent of  $s > 0$ , but depends on  $\lambda, \beta, C_0$  in (2.2). Then

**Lemma 1.** *There exist constants  $s_0 > 0$  and  $C > 0$  such that*

$$\int_{Q_I} \left( \frac{1}{s} (|\partial_t v|^2 + |\Delta v|^2) + s |\nabla v|^2 + s^3 |v|^2 \right) e^{2s\varphi} dx dt \leq C s^4 \int_{Q_I} |P_k v|^2 e^{2s\varphi} dx dt + C \mathcal{B}(v), \quad k = 1, 2 \quad (2.3)$$

for all  $s > s_0$  and  $v \in H^{2,1}(Q_I)$  satisfying  $v \in H^1(\partial\Omega \times I)$ . Here and henceforth we set

$$\begin{aligned} \mathcal{B}(v) &:= e^{Cs} \|v\|_{H^1(\Gamma \times I)}^2 + s^3 \int_{(\partial\Omega \setminus \Gamma) \times I} (|v|^2 + |\nabla_{x,t} v|^2) e^{2s} dS dt \\ &+ s^2 \int_{\Omega} (|v(x, t_0 - \delta)|^2 + |\nabla v(x, t_0 - \delta)|^2 + |v(x, t_0 + \delta)|^2 + |\nabla v(x, t_0 + \delta)|^2) e^{2s\varphi(x, t_0 - \delta)} dx. \end{aligned}$$

The proof of the lemma with  $k = 1$  is done similarly to Lemma 7.1 (p.186) in Bellassoued and Yamamoto [2] or Theorem 3.2 in Yamamoto [17] by keeping all the boundary integral terms  $v|_{\partial Q}$  which are produced by integration by parts and using  $d|_{\partial\Omega \setminus \Gamma} = 0$  in (2.1). The proof for  $k = 2$  follows directly from the case  $k = 1$  by setting  $V(x, t) := v(x, 2t_0 - t)$  and using  $\varphi(x, t) = \varphi(x, 2t_0 - t)$  for  $(x, t) \in Q_I$ .

We emphasize that the backward parabolic Carleman estimate is the same as the forward parabolic Carleman estimate thanks to the symmetry of the weight  $\varphi(x, t)$  with respect to  $t$  centered at  $t_0$ .

Using the Carleman estimate (2.3) we prove a Carleman estimate for a mean field game system. Setting  $y := u - \tilde{u}$  and  $z := v - \tilde{v}$  and subtracting the system (1.1) with  $(\tilde{u}, \tilde{v}, \tilde{F}, \tilde{G})$  from (1.1) with  $(u, v, F, G)$ , we reach

$$\begin{cases} \partial_t y + a_1(x, t) \Delta y + R_1(x, t, y) = h(x, t) z + F - \tilde{F}, \\ \partial_t z - a_2(x, t) \Delta z + R_2(x, t, z) = \kappa v \Delta y + R_3(x, t, y) + G - \tilde{G} \quad \text{in } Q_I. \end{cases} \quad (2.4)$$

Here by (1.2) and (1.3), we can verify

$$|R_j(x, t, y)| \leq C_0 \sum_{k=0}^1 |\nabla^k y(x, t)|, \quad j = 1, 3, \quad |R_2(x, t, z)| \leq C_0 \sum_{k=0}^1 |\nabla^k z(x, t)|, \quad (x, t) \in Q_I. \quad (2.5)$$

We apply Carleman estimate (2.3) to the first equation in (2.4), and multiply the resulting equality by  $s$ :  $y$  and obtain

$$\begin{aligned} & \int_{Q_I} (|\partial_t y|^2 + |\Delta y|^2 + s^2 |\nabla y|^2 + s^4 |y|^2) e^{2s\varphi} dxdt \\ & \leq C \int_{Q_I} s |hz|^2 e^{2s\varphi} dxdt + C \int_{Q_I} s |F - \tilde{F}|^2 e^{2s\varphi} dxdt + Cs\mathcal{B}(y) \end{aligned} \quad (2.6)$$

for all  $s > s_0$ . In terms of (2.5), application of (2.3) with  $k = 1$  to  $z$  yields

$$\begin{aligned} & \int_{Q_I} \left( \frac{1}{s} (|\partial_t z|^2 + |\Delta z|^2) + s |\nabla z|^2 + s^3 |z|^2 \right) e^{2s\varphi} dxdt \\ & \leq C \int_{Q_I} (|\kappa \Delta y|^2 + |y|^2 + |\nabla y|^2) e^{2s\varphi} dxdt + C \int_{Q_I} |G - \tilde{G}|^2 e^{2s\varphi} dxdt + C\mathcal{B}(z) \end{aligned} \quad (2.7)$$

for all  $s > s_0$ .

Using  $\kappa \in L^\infty(Q_I)$  and substituting (2.6) into the terms including  $\Delta y, \nabla y, y$  on the right-hand side of (2.7), we have

$$\begin{aligned} & \int_{Q_I} \left( \frac{1}{s} (|\partial_t z|^2 + |\Delta z|^2) + s |\nabla z|^2 + s^3 |z|^2 \right) e^{2s\varphi} dxdt \\ & \leq C \int_{Q_I} s |z|^2 e^{2s\varphi} dxdt + C \int_{Q_I} (s |F - \tilde{F}|^2 + |G - \tilde{G}|^2) e^{2s\varphi} dxdt + Cs(\mathcal{B}(y) + \mathcal{B}(z)) \end{aligned}$$

for all large  $s > 0$ . Absorbing the first term on the right-hand side into the left-hand side by choosing  $s > 0$  sufficiently large, we see

$$\begin{aligned} & \int_{Q_I} \left( \frac{1}{s} (|\partial_t z|^2 + |\Delta z|^2) + s |\nabla z|^2 + s^3 |z|^2 \right) e^{2s\varphi} dxdt \\ & \leq C \int_{Q_I} (s |F - \tilde{F}|^2 + |G - \tilde{G}|^2) e^{2s\varphi} dxdt + Cs(\mathcal{B}(y) + \mathcal{B}(z)) \end{aligned} \quad (2.8)$$

for all  $s > s_0$ . Adding (2.8) and (2.6) and choosing  $s > 0$  large again to absorb the term  $\int_{Q_I} s |hz|^2 e^{2s\varphi} dxdt$  on the right-hand side into the left-hand side, we obtain

**Theorem 2 (Carleman estimate for a mean field game).** *There exist constants  $s_0 > 0$  and  $C > 0$  such that*

$$\begin{aligned} & \int_{Q_I} \left( |\partial_t(u - \tilde{u})|^2 + |(\Delta(u - \tilde{u}))|^2 + s^2 |\nabla(u - \tilde{u})|^2 + s^4 |u - \tilde{u}|^2 + \frac{1}{s} (|\partial_t(v - \tilde{v})|^2 + |\Delta(v - \tilde{v})|^2) \right. \\ & \left. + s |\nabla(v - \tilde{v})|^2 + s^3 |v - \tilde{v}|^2 \right) e^{2s\varphi} dxdt \leq C \int_{Q_I} (s |F - \tilde{F}|^2 + |G - \tilde{G}|^2) e^{2s\varphi} dxdt \\ & + Cs(\mathcal{B}(u - \tilde{u}) + \mathcal{B}(v - \tilde{v})) \quad \text{for all } s > s_0. \end{aligned}$$

### 3. PROOF OF THEOREM 1

We arbitrarily choose  $t_0 \in (0, T)$  and  $\delta > 0$  such that  $0 < t_0 - \delta < t_0 + \delta < T$ . We define

$$d_0 := \min_{x \in \bar{\Omega}} d(x), \quad d_1 := \max_{x \in \bar{\Omega}} d(x), \quad 0 < r < \left( \frac{d_0}{d_1} \right)^{\frac{1}{2}} < 1. \quad (3.1)$$

We note that  $0 < r < 1$ .

We now show

**Lemma 2.** *Under regularity condition (1.3), if  $u = \tilde{u}$ ,  $v = \tilde{v}$ ,  $\nabla u = \nabla \tilde{u}$  and  $\nabla v = \nabla \tilde{v}$  on  $\Gamma \times (t_0 - \delta, t_0 + \delta)$  imply  $u = \tilde{u}$  and  $v = \tilde{v}$  in  $\Omega \times (t_0 - r\delta, t_0 + r\delta)$ .*

For the proof of Theorem 1, it suffices to prove Lemma 2. Indeed, since  $t_0 \in (0, T)$  and  $\delta > 0$  can be arbitrarily chosen and the Carleman estimate is invariant with respect to  $t_0$  provided that  $0 < t_0 - \delta < t_0 + \delta < T$ , we can apply Lemma 2 by changing  $t_0$  over  $(\delta, T - \delta)$  to obtain  $u = \tilde{u}$  and  $v = \tilde{v}$  in  $\Omega \times ((1 - r)\delta, T - (1 - r)\delta)$ . Since  $\delta > 0$  can be arbitrary, this means that  $u = \tilde{u}$  and  $v = \tilde{v}$  in  $\Omega \times (0, T)$ .

**Proof of Lemma 2.** Once we derived the relevant Carleman estimate in Theorem 2, the proof of Lemma 2 is done similarly to Proposition 2 in [3] as follows. First we determine the constant  $\beta > 0$  in the weight of the Carleman estimate such that

$$\frac{d_1 - d_0}{\delta^2 - r^2\delta^2} < \beta < \frac{d_0}{r^2\delta^2}. \quad (3.2)$$

Here we note that (3.1) verifies  $0 < \frac{d_1 - d_0}{\delta^2 - r^2\delta^2} < \frac{d_0}{r^2\delta^2}$ , which allows us to choose  $\beta$  satisfying (3.2).

For short descriptions, we set

$$M_1 := \sum_{k=0}^1 (\|\nabla_{x,t}^k (u - \tilde{u})\|_{L^2((\partial\Omega \setminus \Gamma) \times I)}^2 + \|\nabla_{x,t}^k (v - \tilde{v})\|_{L^2((\partial\Omega \setminus \Gamma) \times I)}^2),$$

$$M_2 := \sum_{k=0}^1 (\|(u - \tilde{u})(\cdot, t_0 + (-1)^k \delta)\|_{H^1(\Omega)}^2 + \|(v - \tilde{v})(\cdot, t_0 + (-1)^k \delta)\|_{H^1(\Omega)}^2)$$

and  $\mu_1 := e^{\lambda(d_1 - \beta\delta^2)}$ . Since  $u = \tilde{u}$  and  $v = \tilde{v}$  on  $\Gamma \times I$ , Theorem 2 yields

$$s^3 \int_{Q_I} (|u - \tilde{u}|^2 + |v - \tilde{v}|^2) e^{2s\varphi} dx dt \leq C s^5 M_1 e^{2s} + C s^5 M_2 e^{2s\mu_1}$$

for all large  $s > 0$ . We shrink the integration region of the left-hand side to  $\Omega \times (t_0 - r\delta, t_0 + r\delta)$ .

Then, since  $\varphi(x, t) = e^{\lambda(d(x) - \beta(t - t_0)^2)} \geq e^{\lambda(d_0 - \beta r^2 \delta^2)} =: \mu_2$  in  $\Omega \times (t_0 - r\delta, t_0 + r\delta)$ , we obtain

$$e^{2s\mu_2} \int_{\Omega \times (t_0 - r\delta, t_0 + r\delta)} (|u - \tilde{u}|^2 + |v - \tilde{v}|^2) dx dt \leq C s^2 M_1 e^{2s} + C s^2 M_2 e^{2s\mu_1},$$

that is,

$$\|u - \tilde{u}\|_{L^2(\Omega \times (t_0 - r\delta, t_0 + r\delta))}^2 + \|v - \tilde{v}\|_{L^2(\Omega \times (t_0 - r\delta, t_0 + r\delta))}^2 \leq Cs^2 M_1 e^{-2s(\mu_2 - 1)} + Cs^2 M_2 e^{-2s(\mu_2 - \mu_1)} \quad (3.3)$$

for all large  $s > 0$ . Here, by (3.2), we see that  $\mu_2 > \max\{1, \mu_1\}$ , and so we let  $s \rightarrow \infty$  in (3.3), so that  $u = \tilde{u}$  and  $v = \tilde{v}$  in  $\Omega \times (t_0 - r\delta, t_0 + r\delta)$ . Thus the proof of Lemma 2, and so Theorem 1 are complete. ■

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