Asymptotic profile of solutions to the heat equation on thin plate with boundary heating

Eun-ho Lee[a], Woocheol Choi[b,*]

[a]*School of Mechanical Engineering, Sungkyunkwan University, Suwon, Gyeonggi-do, Republic of Korea*
[b]*Department of Mathematics, Sungkyunkwan University, Suwon, Gyeonggi-do, Republic of Korea*

Abstract

In this section, we consider the heat equation on a plate with thickness $h > 0$ being heated by a heat source on upper and lower faces of the plate. We obtain an asymptotic profile of the solution as the thickness $h > 0$ approaches to zero.

Keywords: Heat equation, Thin plate, Focused surface heating, Robin boundary condition
2010 MSC: 35K05, 35K08, 35K20

1. Introduction

Industries have used focused surface heating for engineering purposes. Food industry has employed infrared (IR) energy to heat surface of foods for blanching, drying, roasting, and thawing processes [17, 18]. Manufacturing industries also have done the IR surface heating for metal forming [9, 12] and soldering [1, 10] processes. Many of these industrial applications have based on trial and error experiences, study of mathematical analysis has not been sufficiently reported. A few studies of the focused IR heating analysis are about design of the reflectors in IR heating [11, 13], multiphysics simulation [14], and friction stir welding HSBS. Specically, this problem is modelled by the following heat equation with Robin boundary condition

$$\begin{aligned}
\partial_t U(\mathbf{x},t) - \Delta U(\mathbf{x},t) &= 0 && \text{in } \Omega_h \times [0,\infty) \\
U(\mathbf{x},0) &= G(\mathbf{x}) && \text{on } \Omega_h \\
\tfrac{\partial U}{\partial x_3}(\mathbf{x},t) &= F(\mathbf{x},t) + a[T_0 - U(\mathbf{x},t)] && \text{on } P \times \{h\} \times [0,\infty) \\
\tfrac{\partial U}{\partial x_3}(\mathbf{x},t) &= -F(\mathbf{x},t) - a[T_1 - U(\mathbf{x},t)] && \text{on } P \times \{0\} \times [0,\infty) \\
\tfrac{\partial U}{\partial \nu}(\mathbf{x},t) &= 0 && \text{on } \partial P \times [0,h] \times [0,\infty),
\end{aligned} \tag{1.1}$$

where $\Omega_h = P \times [0,h]$ is a plate with small thickness $h > 0$ and $P \subset \mathbb{R}^2$. In this paper, we are interested in asymptotic profile of solution $U(\mathbf{x},t)$ to (1.1) as the thickness $h > 0$ of the plate approaches to zero. In (1.1) no heat change is assumed on $\partial P \times [0,h]$ as the surface measure of $\partial P \times [0,h]$ is much smaller than that of the upper and lower boundary $P \times \{0,h\}$.

In the literature, there have been a lot of interest in studying a similar problem to (1.1), namely the reaction diffusion equation on thin domains with Neumann boundary condition

$$\begin{aligned}
\partial_t U(\mathbf{x},t) - \Delta U(\mathbf{x},t) &= f(U)(\mathbf{x},t) && \text{in } \Omega_h \times [0,\infty) \\
U(\mathbf{x},0) &= G(\mathbf{x}) && \text{on } \Omega_h \\
\tfrac{\partial U}{\partial \nu}(\mathbf{x},t) &= 0 && \text{on } \partial\Omega_h \times [0,\infty).
\end{aligned} \tag{1.2}$$

The mathematical analysis for the asymptotic limit $h \to 0$ of (1.2) was initiated by Hale and Raugel [7] where the authors raised a general question: *If we consider an evolution equation on a spatial domain* Ω

*Corresponding author.
Email addresses: e.h.lee@skku.edu (Eun-ho Lee), choiwc@skku.edu (Woocheol Choi)

such that Ω is small in a direction, is it possible to approximate the dynamics by an equation on a lower dimensional spatial domain? This question was answered affirmatively for problem (1.2) in [7] and extended to various settings [8, 16, 15, 5, 2]. More precisely, Hale and Raguel [7] showed that the solution $U(\mathbf{x}, t)$ to (1.2) is approximated by the two dimensional problem

$$\begin{aligned} \partial_t u(x,t) - \Delta u(x,t) &= f(u)(x,t) && \text{in } P \times [0,\infty) \\ u(x,0) &= g(x) && \text{on } P \\ \tfrac{\partial u}{\partial \nu}(x,t) &= 0 && \text{on } \partial P \times [0,\infty) \end{aligned} \tag{1.3}$$

as the thickness $h > 0$ get close to zero, where $g : P \to \mathbb{R}$ is properly chosen in terms of G.

The problem (1.1) models a thin plate with an heat source F on boundary and the main concern is the effect of the heat source F on temperature of plate when $h > 0$ is very small. Therefore it is admissible to consider the case that the initial state G is static, i.e.,

$$\begin{aligned} -\Delta G(\mathbf{x}) &= 0 && \text{in } \Omega_h \\ \tfrac{\partial G}{\partial x_3}(\mathbf{x}) &= a[T_0 - G(\mathbf{x})] && \text{on } P \times \{h\} \\ \tfrac{\partial G}{\partial x_3}(\mathbf{x}) &= -a[T_1 - G(\mathbf{x})] && \text{on } P \times \{0\} \\ \tfrac{\partial G}{\partial \nu}(\mathbf{x}) &= 0 && \text{on } \partial P \times [0,h]. \end{aligned}$$

As far as we know, there has been no results on the asymptotic profile for the heat equation with the Robin boundary condition on thin plate. We observe in problem (1.1) that the termperature on plate interacts with the outside termperature since the convection coefficient $a > 0$ is nonzero. Therefore the asymptotic behavior as $h \to 0$ should be different from the case $a = 0$ because the interaction could effect more the temperature inside of the plate if the thickness of plate is more thin. Now we state the main result of this paper.

Theorem 1.1. *Let $U \in C^2([0,\infty); \Omega_h)$ be a solution to* (1.1) *with $F \in L^\infty([0,\infty); \Omega_h)$. We assume that $h \in (0, 1/3a)$ and let $\alpha_1 = \alpha_1(h)$ be the smallest positive solution of*

$$\tan(hq) = \frac{2aq}{q^2 - a^2}. \tag{1.4}$$

Then for each $(x, x_3) \in P \times [0,h]$ and $t \geq 0$, the solution U satisfies

$$\begin{aligned} &\left| U((x,x_3),t) - \left(G(x,x_3) + \frac{\alpha_1^2}{2a} \int_0^t \int_{P\times\{0,h\}} e^{-\alpha_1^2(t-s)} W(x,t,y,s) F(y,y_3,s) dS_y ds \right) \right| \\ &\quad \leq \frac{19h}{3} \|F\|_{L^\infty([0,t])} \end{aligned}$$

where $W(x,t,y,s)$ denotes the Green's function of the heat equation on the two dimensional domain P with Neumann boundary condition

$$\begin{aligned} \partial_t u(x,t) - \Delta u(x,t) &= 0 && \textit{in } P \times [0,\infty) \\ u(x,0) &= g(x) && \textit{on } P \\ \tfrac{\partial u(x,t)}{\partial \nu} &= 0 && \textit{on } \partial P \times [0,\infty). \end{aligned}$$

Here we denoted by $\int_{P\times\{0,h\}} f(y,y_3) dS_y$ the sum $\int_P f(y,0)dy + \int_P f(y,h)dy$ for integrable function $f : P \times \{0,h\} \to \mathbb{R}$ and $\|F\|_{L^\infty([0,t])} := \sup_{(y,y_3,s)\in P\times\{0,h\}\times[0,t]} |F(y,y_3,s)|$.

For the value $\alpha_1 = \alpha_1(h)$ defined in Theorem 1.1, we will prove that it satisfies $\lim_{h\to 0} \frac{\alpha_1(h)}{\sqrt{2a/h}} = 1$ (see Lemma 3.3). Based on this property and the integral represntation of Theorem 1.1, we will investigate the effect of the thickness $h > 0$ and the material property of the plate on the focused IR heating.

In order to prove Theorem 1.1, we consider the Green's function K_h associated to (1.1) and write the solution $U(\mathbf{x}, t)$ to (1.1) in terms of K_h. The Green's function K_h is known to admits a series expansion of which term depends on $h > 0$ and the proof of of Theorem 1.1 is reduced to study the asymptotic profile of the series expansion of K_h when $h > 0$ is small. We will show that the first term of the series is dominant and the contribution of the other terms can be estimates as $O(h)$.

This paper is organized as follows. In Section 2, we recall the Green's function associated to (1.1) and study its series expansion. In Section 3 we obtain estimates on the terms in the expansion. This will enable us to prove Theorem 1.1.

2. Green's formula

In this section, we study the Green's function $K_h(\mathbf{x}, t, \mathbf{y}, s) : \Pi_h \to \mathbb{R}^+$ associated to (1.1) defined on

$$\Pi_h = \{(\mathbf{x}, t, \mathbf{y}, s) \in (\Omega_h \times [0, \infty))^2 \ : \ t > s\}$$

satisfying for each $(\mathbf{x}, t) \in \Omega_h \times [0, \infty)$ that

$$\begin{aligned} \partial_s K_h(\mathbf{x}, t, \mathbf{y}, s) - \Delta_y K_h(\mathbf{x}, t, \mathbf{y}, s) &= 0 \quad \text{in } \Omega_h \times [0, T] \\ \frac{\partial K_h}{\partial n_y}(\mathbf{x}, t, \mathbf{y}, s) + aK_h(\mathbf{x}, t, \mathbf{y}, s) &= 0 \quad \text{on } (y, s) \in P \times \{0, h\} \\ \frac{\partial K_h}{\partial n_y}(\mathbf{x}, t, \mathbf{y}, s) &= 0 \quad \text{on } \partial P \times [0, h] \times [0, T] \end{aligned}$$

and that

$$\lim_{t \to s} K_h(\mathbf{x}, t, \mathbf{y}, s) = \delta_{\mathbf{x}}(\mathbf{y}).$$

The solution $U(\mathbf{x}, t)$ to (1.1) is then written as

$$\begin{aligned} U(\mathbf{x}, t) = &\int_{\Omega_h} K_h(\mathbf{x}, t, \mathbf{y}, 0) G(\mathbf{y}) d\mathbf{y} \\ &+ \int_0^t \int_{\partial P \times \{h\}} K_h(\mathbf{x}, t, \mathbf{y}, s)[F(\mathbf{y}, s) + aT_0] dS_y ds \\ &+ \int_0^t \int_{\partial P \times \{0\}} K_h(\mathbf{x}, t, \mathbf{y}, s)[F(\mathbf{y}, s) + aT_1] dS_y ds. \end{aligned}$$

Since $G(\mathbf{x})$ is a static state of (1.1), we have

$$U(\mathbf{x}, t) = G(\mathbf{x}) + \int_0^t \int_{\partial P \times \{0, h\}} K_h(\mathbf{x}, t, \mathbf{y}, s)[F(\mathbf{y}, s)] dS_y ds. \tag{2.1}$$

As the domain Ω_h equals to the product $P \times [0, h]$, the Green's function K_h is also a product of two Green's functions corresponding to P and $[0, h]$ described as follows.

Let $\Psi = \{(x, t, y, s) \in (P \times [0, T])^2 \ : \ t > s\}$ and $W : \Psi \to \mathbb{R}^+$ be the Green's function to the problem

$$\begin{aligned} \partial_t u(x, t) - \Delta u(x, t) &= 0 && (x, t) \in P \times (0, T) \\ u(x, 0) &= g(x) && x \in P \\ \frac{\partial u}{\partial n}(x, t) &= 0 && \text{on } \partial P \times [0, T]. \end{aligned} \tag{2.2}$$

The existence of the Green's function for the above problem was proved in [4] for any smooth domain $P \subset \mathbb{R}^2$. Next we consider $\Phi_h := \{(z, t, w, s) \in ([0, h] \times [0, \infty))^2 : t > s\}$ and the Green's function $G_h : \Phi_h \to \mathbb{R}^+$ to the problem

$$\begin{aligned} \partial_t u(z, t) - \Delta u(z, t) &= 0 && (z, t) \in [0, h] \times [0, T] \\ u(z, 0) &= g(z) && z \in [0, h] \\ \frac{\partial u}{\partial \nu}(z, t) + au(z, t) &= f(z, t) && (z, t) \in \{0, h\} \times [0, T]. \end{aligned}$$

The explicit formula of G_h was obtained in [3] as in (2.7). Now we can state the product formula of K_h appeared in [6]:

$$K_h(x, x_3, t, y, y_3, s) = W(x, t, y, s) G_h(x_3, t, y_3, s). \tag{2.3}$$

Here $x, y \in \mathbb{R}^2$ and $x_3, y_3 \in \mathbb{R}$. In order to study the asymptotic behavior of $U(\mathbf{x}, t)$ with the formula (2.1), we shall investigate the asymptotic behavior of K_h. In view of (2.3), it is reduced to study the behavior of G_h for small $h > 0$. By using the formula of G_h obtained in [3] we have the following lemma.

Lemma 2.1. *We have*

$$G_h(z, t, w, s) = \sum_{m=1}^{\infty} P_m(z, t, w, s), \tag{2.4}$$

where for each $m \in \mathbb{N}$,

$$P_m(z, t, w, s) = \frac{2e^{-\alpha_m^2(t-s)}[\alpha_m \cos(\alpha_m z) + a\sin(\alpha_m z)][\alpha_m \cos(\alpha_m w) + a\sin(\alpha_m w)]}{2a + h(a^2 + \alpha_m^2)}, \tag{2.5}$$

and α_m *is the* m*-th positive solution* $q > 0$ *of equation*

$$\tan(hq) = \frac{2aq}{q^2 - a^2}, \tag{2.6}$$

arranged in increasing order.

Proof. We recall from [3, 605 page] the formula of G_h given as

$$G_h(z, t, w, s) = \sum_{m=1}^{\infty} P_m(z, t, w, s), \tag{2.7}$$

where

$$\begin{aligned} P_m(z, t, w, s) =& \frac{2}{h} e^{-\beta_m^2(t-s)/h^2}[\beta_m \cos(\beta_m z/h) + B\sin(\beta_m z/h)] \\ &\times \frac{[\beta_m \cos(\beta_m w/h) + B\sin(\beta_m w/h)]}{(\beta_m^2 + B^2)[1 + B/(\beta_m^2 + B^2)] + B}. \end{aligned} \tag{2.8}$$

Here β_m are positive solutions to

$$\tan\beta_m = \frac{2\beta_m B}{\beta_m^2 - B^2} \quad \text{with } B = ah, \tag{2.9}$$

arranged in increasing order for $m \in \mathbb{N}$. Letting $\alpha_m = \beta_m/h$ in (2.8), we find that

$$\begin{aligned} &P_m(z, t, w, s) \\ &= \frac{2he^{-\alpha_m^2(t-s)}[\alpha_m \cos(\alpha_m z) + a\sin(\alpha_m z)][\alpha_m \cos(\alpha_m w) + a\sin(\alpha_m w)]}{2ah + h^2(a^2 + \alpha_m^2)} \\ &= \frac{2e^{-\alpha_m^2(t-s)}[\alpha_m \cos(\alpha_m z) + a\sin(\alpha_m z)][\alpha_m \cos(\alpha_m w) + a\sin(\alpha_m w)]}{2a + h(a^2 + \alpha_m^2)}. \end{aligned}$$

Also equation (2.9) is written as

$$\tan(h\alpha_m) = \frac{2h^2 a\alpha_m}{h^2\alpha_m^2 - h^2 a^2} = \frac{2a\alpha_m}{\alpha_m^2 - a^2}.$$

The proof is finished. □

3. Estimates for the asymptotic formula

In this section, we obtain the estimates for the terms in the expansion of G_h given by (2.7). First we shall show that the effect of P_m in (2.7) with $m \geq 2$ to the solution U of (1.1) are relatively very small when $h > 0$ is close to zero. For this aim we begin with the following lemma.

Lemma 3.1. *For $m \geq 2$ we have*

$$\int_0^t \int_{P\times\{0,h\}} W(x,t,y,s)P_m(x_3,t,y_3,s)F(y,y_3,s)dS_y ds \leq \frac{8}{h\alpha_m^2}\|F\|_{L^\infty([0,t])}.$$

Proof. We estimate (2.5) as follows

$$\begin{aligned} P_m(x_3,t,y_3,s) &\leq \frac{2e^{-\alpha_m^2(t-s)}(\alpha_m+a)^2}{2a+h(a^2+\alpha_m^2)} \\ &\leq \frac{4e^{-\alpha_m^2(t-s)}(\alpha_m^2+a^2)}{h(a^2+\alpha_m^2)} = \frac{4}{h}e^{-\alpha_m^2(t-s)}. \end{aligned} \tag{3.1}$$

Using this we obtain

$$\begin{aligned} A &:= \int_0^t \int_{P\times\{0,h\}} W(x,t,y,s)P_m(x_3,t,y_3,s)F(y,y_3,s)dS_y ds \\ &\leq \|F\|_{L^\infty([0,t])}\int_0^t \int_{P\times\{0,h\}} W(x,t,y,s)\frac{4}{h}e^{-\alpha_m^2(t-s)}dS_y ds \end{aligned} \tag{3.2}$$

In view of the fact that taking $g \equiv 1$ in (2.2) implies $u(x,t) \equiv 1$, one has

$$\int_P W(x,t,y,s)dy = 1. \tag{3.3}$$

Using this in (3.2) we obtain

$$\begin{aligned} A &\leq 2\|F\|_{L^\infty([0,t])}\int_0^t \frac{4}{h}e^{-\alpha_m^2 s}ds \\ &\leq 2\|F\|_{L^\infty([0,t])}\int_0^t \frac{4}{h}e^{-\alpha_m^2 s}ds = \frac{8\|F\|_{L^\infty([0,t])}}{h\alpha_m^2}\int_0^{\alpha_m^2 t} e^{-s}ds \\ &\leq \frac{8\|F\|_{L^\infty([0,t])}}{h\alpha_m^2}. \end{aligned}$$

The proof is finished. □

Next we find the following estimates on α_m for $m \geq 2$.

Lemma 3.2. *Assume that $h < \frac{\pi}{2a}$. For $m \geq 2$, we have $\alpha_m \in \left[\frac{(m-1)\pi}{h}, \frac{(m-1)\pi}{h} + \frac{\pi}{2h}\right]$.*

Proof. For $q \geq 0$ we let $\Phi(q) = \frac{2aq}{q^2-a^2}$. From (2.6) we see that α_m is the m-th positive solution of

$$\tan(hq) = \Phi(q).$$

Using an elementary calculus, we find that

- For $q \in (0,a)$, the function $\Phi(q)$ is negative and decreasing function with

$$\Phi(0) = 0 \quad \text{and} \quad \lim_{q\to a^-}\Phi(q) = -\infty. \tag{3.4}$$

- For $q \in (a, \infty)$, the function $\Phi(q)$ is positive and decreasing function with

$$\lim_{q \to a^+} \Phi(q) = \infty \quad \text{and} \quad \lim_{x \to \infty} \Phi(q) = 0. \tag{3.5}$$

By the way the funcction $q \to \tan(hq)$ is a periodic function with period π/h. Taking this account with (3.4) and (3.5), we find that

$$\alpha_1 \in \left(0, \frac{\pi}{2h}\right)$$

and for $m \geq 2$,

$$\alpha_m \in \left(\frac{(m-1)\pi}{h}, \frac{(m-1)\pi}{h} + \frac{\pi}{2h}\right).$$

The proof is done. □

For the first solution α_1 to (2.6), we have the following result.

Lemma 3.3. *Assume that $ha \leq 1$. Then we have*

$$\alpha_1 \leq \sqrt{a^2 + \frac{2a}{h}} \leq \frac{\sqrt{3a}}{\sqrt{h}}. \tag{3.6}$$

In addition, if we further assume that $ha \leq 1/3$, then we have

$$\sqrt{a^2 + \frac{2a}{h + 2ah^2}} \leq \alpha_1$$

From the above estimates, we find that $\lim_{h \to 0} \frac{\alpha_1}{\sqrt{2a/h}} = 1$.

Proof. Recall from Lemma 2.1 that $\alpha_1 > 0$ is the smallest positive solution $q > 0$ to

$$\tan(hq) = \frac{2aq}{q^2 - a^2}. \tag{3.7}$$

By an elementary calculus, the function $z \to \tan z$ has the following estimate

$$z + \frac{z^3}{3} \leq \tan z \leq z + \frac{2z^3}{3} \quad \text{for } z \in [0, 1]. \tag{3.8}$$

We see that $z \to \tan(hz)$ is increasing for $z \in (0, \pi/2h)$. Let us set $z_0 = \sqrt{\frac{2a}{h} + a^2}$. If $z_0 < \frac{\pi}{2h}$ we have

$$\tan(hz_0) \geq hz_0 = \frac{2az_0}{z_0^2 - a^2}. \tag{3.9}$$

Combining this with (3.4) and (3.5) we deduce $\alpha_1 \leq z_0$. In the case $z_0 \geq \frac{\pi}{2h}$, we have $\alpha_1 \leq z_0$ by Lemma 3.2. Hence the first inequality of (3.6) holds true. This also implies $\alpha_1 \leq \frac{\sqrt{3a}}{\sqrt{h}}$ because we have $a^2 \leq \frac{a}{h}$ from the condition $ha \leq 1$.

Assume that $ha \leq 1/3$. Then we have $h\alpha_1 \leq \sqrt{3ah} \leq 1$. Combining this with the second inequality of (3.8), we deduce

$$\frac{2a\alpha_1}{\alpha_1^2 - a^2} = \tan(h\alpha_1) \leq h\alpha_1 + \frac{2(h\alpha_1)^3}{3} \leq h\alpha_1 + 2h^2 a\alpha_1,$$

where we used $h\alpha_1 \leq \sqrt{3ah}$ in the second inequality. Rearranging this, we get

$$a^2 + \frac{2a}{h + 2h^2 a} \leq \alpha_1^2.$$

This completes the proof of this lemma. □

Lemma 3.4. *For $0 \leq z, w \leq h$ with $h \leq \frac{1}{3a}$ we have*

$$\left| P_1(z,t,w,s) - \frac{\alpha_1^2}{2a} e^{-\alpha_1^2(t-s)} \right| \leq \frac{5}{2}\, h\alpha_1^2 e^{-\alpha_1^2(t-s)}. \tag{3.10}$$

Proof. We recall from (2.5) that $P_1(z,t,w,s)$ is given by

$$P_1(z,t,w,s) = \frac{2e^{-\alpha_1^2(t-s)}[\alpha_1\cos(\alpha_1 z) + a\sin(\alpha_1 z)][\alpha_1\cos(\alpha_1 w) + a\sin(\alpha_1 w)]}{2a + h(a^2+\alpha_1^2)}.$$

Throughout the proof, we keep in mind that $\alpha_1^2 z^2 \leq \alpha_1^2 h^2 \leq 3ha \leq 1$ from Lemma 3.3. Since $1 - \frac{v^2}{2} \leq \cos v \leq 1$ for $v \in \mathbb{R}$, we have

$$1 - \frac{\alpha_1^2 z^2}{2} \leq \cos(\alpha_1 z) \leq 1.$$

It then follows using $\alpha_1^2 z^2 \leq 3ha$ that

$$\alpha_1\left(1 - \frac{3ha}{2}\right) \leq \alpha_1\cos(\alpha_1 z) \leq \alpha_1,$$

which gives

$$\alpha_1\left(1 - \frac{3ha}{2}\right) \leq \alpha_1\cos(\alpha_1 z) + a\sin(\alpha_1 z) \leq \alpha_1(1+ha),$$

where we used $0 \leq \alpha_1 z \leq h\alpha_1 \leq 1$. From this we obtain

$$\alpha_1^2\left(1 - \frac{3ha}{2}\right)^2 \leq [\alpha_1\cos(\alpha_1 z) + a\sin(\alpha_1 z)][\alpha_1\cos(\alpha_1 w) + a\sin(\alpha_1 w)] \leq \alpha_1^2(1+ha)^2. \tag{3.11}$$

From Lemma 3.3 we find

$$\frac{2a}{1+2ah} + a^2 h \leq h\alpha_1^2 \leq 2a + a^2 h. \tag{3.12}$$

Combining this with (3.12) we find that $D := 2a + h(a^2+\alpha_1^2)$ satisfies

$$D \geq \left(\frac{2a}{1+2ah} + a^2 h\right) + 2a + ha^2 \geq 4a - 2a^2 h \tag{3.13}$$

and

$$D \leq \left(2a + a^2 h\right) + 2a + ha^2 \leq 4a + 2a^2 h. \tag{3.14}$$

Combining (3.11) with (3.13) and (3.14) we deduce

$$2e^{-\alpha_1^2(t-s)}\frac{\alpha_1^2(1-2ha)^2}{4a+2a^2h} \leq P_1(z,t,w,s) \leq 2e^{-\alpha_1^2(t-s)}\frac{\alpha_1^2(1+3ah)}{4a-2a^2h}. \tag{3.15}$$

Using that $ah \leq 1/3$ we have

$$\frac{(1-2ha)^2}{4a+2a^2h} \geq (1-4ha)\frac{1}{4a(1+ah/2)} \geq \frac{1}{4a}(1-4ha)(1-ah/2) \geq \frac{1}{4a} - \frac{9h}{16}.$$

Similarly,

$$\frac{1+3ah}{4a-2a^2h} = \frac{1+3ah}{4a(1-ah/2)} \leq \frac{1}{4a}\left[(1+3ah)(1+ah)\right] \leq \frac{1}{4a}\left(1+5ah\right) = \frac{1}{4a} + \frac{5h}{4}.$$

Gathering the above two estimates in (3.15), we obtain

$$2e^{-\alpha_1^2(t-s)}\alpha_1^2\left(\frac{1}{4a} - \frac{9h}{16}\right) \leq P_1(z,t,w,s) \leq 2e^{-\alpha_1^2(t-s)}\alpha_1^2\left(\frac{1}{4a} + \frac{5h}{4}\right).$$

From this we find

$$\left|P_1(z,t,w,s) - \frac{\alpha_1^2}{2a}e^{-\alpha_1^2(t-s)}\right| \leq \frac{5}{2}\alpha_1^2 e^{-\alpha_1^2(t-s)}h.$$

The proof is finished. □

Lemma 3.5. *Assume that $ah \leq 1/3$. Then we have*

$$\left|\int_0^t \int_{P\times\{0,h\}} W(x,t,y,s)P_1(x_3,t,y_3,s)F(y,y_3,s)dS_y ds \right.$$
$$\left. - \int_0^t \int_{P\times\{0,h\}} 2e^{-\alpha_1^2(t-s)}\frac{\alpha_1^2}{4a}W(x,t,y,s)F(y,y_3,s)dS_y ds\right| \leq 5h\|F\|_{L^\infty([0,t])}.$$

Proof. Using (3.10) and (3.3) we deduce

$$\begin{aligned}
&\left|\int_0^t \int_{P\times\{0,h\}} W(x,t,y,s)P_1(x_3,y,y_3,s)F(y,y_3,s)dS_y ds \right.\\
&\left. - \int_0^t \int_{P\times\{0,h\}} \frac{\alpha_1^2}{2a}e^{-\alpha_1^2(t-s)}W(x,t,y,s)F(y,y_3,s)dS_y ds\right|\\
&\leq \frac{5h}{2}\left|\int_0^t \int_{P\times\{0,h\}} \alpha_1^2 e^{-\alpha_1^2(t-s)}W(x,t,y,s)F(y,y_3,s)dS_y ds\right|\\
&\leq 5h\|F\|_{L^\infty([0,t])}\left(\int_0^\infty e^{-s}ds\right)\\
&= 5h\|F\|_{L^\infty([0,t])}.
\end{aligned}$$

The proof is finished. □

Now we give the proof of Theorem 1.1.

Proof. From (2.1), (2.3), and (2.7) we have

$$\begin{aligned}
&U((x,x_3),t)\\
&= G(x,x_3) + \sum_{m=1}^\infty \int_0^t \int_{P\times\{0,h\}]} W(x,t,y,s)P_m(x_3,t,y_3,s)[F(y,y_3,s)]dS_y ds.
\end{aligned} \tag{3.16}$$

Using Lemma 3.1 we deduce

$$\begin{aligned}
&\sum_{m=2}^\infty \int_0^t \int_{P\times\{0,h\}]} W(x,t,y,s)P_m(x_3,t,y_3,s)[F(y,y_3,s)]dS_y ds\\
&\leq \sum_{m=2}^\infty \frac{8}{h\alpha_m^2}\|F\|_{L^\infty([0,t])},
\end{aligned} \tag{3.17}$$

and the estimate of Lemma 3.2 enables us to obtain

$$\sum_{m=2}^\infty \frac{1}{h\alpha_m^2} \leq \sum_{m=2}^\infty \frac{h}{(m-1)^2\pi^2} = \frac{h}{6}.$$

Gathering this together with (3.17) and Lemma 3.5, we finally deduce from (3.16) the following estimate

$$\left| U((x,x_3),t) - \left(G(x,x_3) + \int_0^t \int_{P\times\{0\}} 2e^{-\alpha_1^2(t-s)} \frac{\alpha_1^2}{4a} W(x,t,y,s) F(y,y_3,s) dS_y ds \right) \right| \leq \left(5 + \frac{4}{3}\right) h \|F\|_{L^\infty([0,t])}.$$

The proof is finished. □

References

[1] C. Anguiano, M. Flix, A. Medel, M. Bravo, D. Salazar, H. Mrquez, H. Study of heating capacity of focused IR light soldering systems. Optics Express, 21(20) (2013), 23851–23865.

[2] F. Antoci, M. Prizzi, Reaction-diffusion equations on unbounded thin domains. Topol. Methods Nonlinear Anal. 18 (2001), 283–302.

[3] J.V. Beck, K.D. Cole, A. Haji-Sheikh, B. Litkouhi, Heat Conduction Using Greens Functions, Hemisphere, New York, 2011. Second edition.

[4] J. Choi, S. Kim, Green's function for second order parabolic systems with Neumann boundary condition. J. Differential Equations 254 (2013), 2834–2860.

[5] I. Ciuperca, Reaction-diffusion equations on thin domains with varying order of thinness. J. Differential Equations 126 (1996), 244–291.

[6] M. Haghpanahi, S. Salimi, P. Bahemmat, S. Sima, A spectral model for heat transfer with friction heat gain in geothermal borehole heat exchangers. Appl. Math. Model. 37 (2013), 9865–9884.

[7] J. Hale, G. Raugel, Reaction-diffusion equation on thin domains. J. Math. Pures Appl. (9) 71 (1992), 33–95.

[8] J. Hale, G. Raugel, A reaction-diffusion equation on a thin L -shaped domain. Proc. Roy. Soc. Edinburgh Sect. A 125 (1995), 283–327.

[9] P. B. Kadolkar, H. Lu, C.A. Blue, T. Ando, R. Mayer, Application of rapid Infrared heating to aluminum forgings. 25th Forging Industry Technical Conference (2004), 1–12.

[10] V. L. Lanin, Infrared heating in the technology of soldering components in electronics. Surface Engineering and Applied Electrochemistry, 435 (2007), 381–386.

[11] E.H. Lee, D.Y. Yang, W.H. Yang, Numerical modeling and experimental validation of focused surface heating using near-infrared rays with an elliptical reflector. International Journal of Heat and Mass Transfer, 78 (2014), 240–250.

[12] E.H. Lee, J.W. Yoon, D.Y. Yang, Study on springback from thermal-mechanical boundary condition imposed to V-bending and L-bending processes coupled with infrared rays local heating. Int. J. Mater. Form, 11 (2018), 417–433.

[13] E.H. Lee, D.Y. Yang, Experimental and numerical analysis of a parabolic reflector with a radiant heat source. International Journal of Heat and Mass Transfer, 85 (2015), 860–864.

[14] E.H. Lee, W. Kim, ElectricalThermalMechanical Analysis of Focused Infrared Heating Process. Int. J. of Precis. Eng. and Manuf.-Green Tech. (2020) (In press, online available).

[15] D. Li, B. Wang, X. Wang, Limiting behavior of non-autonomous stochastic reactiondiffusion equations on thin domains. J. Differential Equations 262 (2017), 1575–1602.

[16] F. Li, Y. Li, R. Wang, Limiting dynamics for stochastic reaction-diffusion equations on the Sobolev space with thin domains. Comput. Math. Appl. 79 (2020), 457–475.

[17] N.K. Rastogi, Recent trends and developments in infrared heating in food processing. Critical Reviews in Food Science and Nutrition, 52(9) (2012), 737–760.

[18] C. Sandu, Infrared radiative drying in food engineering: a process analysis. Biotechnology Progress, 2(3) (1986) ,109–119 .