A new smoothness result for Caputo-type fractional ordinary differential equations *

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Abstract

We present a new smoothness result for Caputo-type fractional ordinary differential equations, which reveals that, subtracting a non-smooth function that can be obtained by the information available, a non-smooth solution belongs to C^m for some positive integer m.

Keywords: Caputo, fractional differential equation, smoothness.

1 Introduction

Let us consider the following model problem: seek $0 < h \leqslant a$ and

$$y \in \left\{ v \in C[0,h] : \|v - c_0\|_{C[0,h]} \leqslant b \right\}$$

such that

$$\begin{cases} D_*^\alpha y = f(x,y), & 0 \leqslant x \leqslant h, \\ y(0) = c_0, \end{cases} \tag{1.1}$$

where $a > 0$, $b > 0$, $0 < \alpha < 1$, $c_0 \in \mathbb{R}$, and

$$f \in C\left([0,a] \times [c_0 - b, c_0 + b]\right).$$

Above, the Caputo-type fractional differential operator $D_*^\alpha : C[0,h] \to C_0^\infty(0,h)'$ is given by

$$D_*^\alpha z := DJ^{1-\alpha}\big(z - z(0)\big) \tag{1.2}$$

for all $z \in C[0,h]$, where D denotes the well-known first order generalized differential operator, and the Riemann-Liouville fractional integral operator $J^{1-\alpha} : C[0,h] \to C[0,h]$ is defined by

$$J^{1-\alpha} z(x) := \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} z(t)\,\mathrm{d}t, \quad 0 \leqslant x \leqslant h,$$

for all $z \in C[0,h]$.

By [2, Lemma 2.1], the above problem is equivalent to seeking solutions of the following Volterra integration equation:

$$y(x) = c_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f\big(t, y(t)\big)\,\mathrm{d}t. \tag{1.3}$$

Diethelm and Ford [2] proved that, if f is continuous, then (1.3) has a solution $y \in C[0,h]$ for some $0 < h \leqslant a$, and this solution is unique if f is Lipschitz continuous. A natural question arises whether y can be smoother than being continuous. This is not only of theoretical value, but also of great importance in developing numerical methods for (1.3).

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To this question, Miller and Feldstein [5] gave the first answer: if f is analytic, then y is analytic in $(0, h)$ for some $0 < h \leqslant a$. Then Lubich [4] considered the behavior of the solution near 0. He showed that, if f is analytic at the origin, then there exists a function Y of two variables that is analytic at the origin such that

$$y(x) = Y(x, x^\alpha), \quad 0 \leqslant x \leqslant h,$$

for some $0 < h \leqslant a$. The above work suggests that non-smoothness of the solution to (1.1) is generally unavoidable. However, Diethelm [1] established a sufficient and necessary condition under which y is analytic on $[0, h]$ for some $0 < h \leqslant a$. But, since we have already seen that non-smoothness of y is generally unavoidable, it is not surprising that this condition is unrealistic. Recently, Deng [3] proposed two conditions: under the first condition the solution belongs to C^m for some positive integer m; under the second one the solution is a polynomial. It should be noted that, the second condition is just the one proposed in [1], and the first condition is also unrealistic.

The main result of this paper is that, although the solution y of (1.1) does not generally belong to C^m for some positive integer m, we can still construct a non-smooth function of the form

$$S(x) := c_0 + \sum_{j=1}^{n} c_j x^{\gamma_j},$$

such that

$$y - S \in C^m,$$

provided f is sufficiently smooth. Most importantly, given c_0 and f, we can obtain S by a simple computation. This is significant in the development of numerical methods for (1.1). In addition, we obtain a sufficient and necessary condition under which $y \in C^m$. We note that this condition is essentially the same as the first condition mentioned already in [3, Theorem 2.8], but the necessity was not considered therein.

The rest of this paper is organized as follows. In Section 2 we introduce some basic notation and preliminaries. In Section 3 we state the main results of this paper, and present their proofs in Section 4.

2 Notation and Preliminaries

Let $0 < h < \infty$. We use $C[0, h]$ to denote the space of all continuous real functions defined on $[0, h]$. For any $k \in \mathbb{N}_{>0}$ and $0 \leqslant \gamma \leqslant 1$, define

$$C^k[0, h] := \left\{ v \in C[0, h] : v^{(j)} \in C[0, h] \quad \text{for } j = 1, 2, \ldots, k \right\}, \tag{2.1}$$

$$C^{k,\gamma}[0, h] := \left\{ v \in C^k[0, h] : \max_{0 \leqslant x < y \leqslant h} |v|_{C^{k,\gamma}[0,h]} < \infty \right\}, \tag{2.2}$$

and endow the above two spaces with two norms respectively by

$$\|v\|_{C^k[0,h]} := \max_{0 \leqslant j \leqslant k} \max_{0 \leqslant x \leqslant h} \left| v^{(j)}(x) \right| \qquad \text{for all } v \in C^k[0, h], \tag{2.3}$$

$$\|v\|_{C^{k,\gamma}[0,h]} := \max \left\{ \|v\|_{C^k[0,h]}, |v|_{C^{k,\gamma}[0,h]} \right\} \qquad \text{for all } v \in C^{k,\gamma}[0, h]\,. \tag{2.4}$$

Here the semi-norm $|\cdot|_{C^{k,\gamma}[0,h]}$ is given by

$$|v|_{C^{k,\gamma}[0,h]} := \sup_{0 \leqslant x < y \leqslant h} \frac{\left| v^{(k)}(x) - v^{(k)}(y) \right|}{(y - x)^\gamma}$$

for all $v \in C^{k,\gamma}[0, h]$, and it is obvious that $C^k[0, h]$ coincides with $C^{k,0}[0, h]$.

For any $s \in \mathbb{N}_{>0}$, define

$$\Lambda_s := \left\{ \beta = (\beta_1, \beta_2, \ldots, \beta_s) \in \{1, 2\}^s \right\},$$

and, for any $\beta \in \Lambda_s$, we use the following notation:

$$\partial_\beta g := \frac{\partial}{\partial x_{\beta_s}} \frac{\partial}{\partial x_{\beta_{s-1}}} \cdots \frac{\partial}{\partial x_{\beta_1}} g(x_1, x_2),$$

where g is a real function of two variables. In addition, we define

$$\Lambda_0 := \{\emptyset\} \,,$$

and denote by $\partial_\emptyset$ the identity mapping.

3 Main Results

Let us first make the following assumption on f.

Assumption 1. *There exist a positive integer n, and a positive constant M such that*

$$f \in C^n \left([0,a] \times [c_0 - b, c_0 + b]\right),$$

$$\max_{(x,y)\in[0,a]\times[c_0-b,c_0+b]} \max_{\substack{0\leqslant i\leqslant n \\ 0\leqslant j\leqslant n \\ i+j\leqslant n}} \left| \frac{\partial^i}{\partial x^i}\frac{\partial^j}{\partial y^j} f(x,y) \right| \leqslant M.$$

Throughout this paper, we assume that the above assumption is fulfilled.

Define $J \in \mathbb{N}$ and a strictly increasing sequence $\{\gamma_i\}_{i=1}^J$ by

$$\{\gamma_j : 1 \leqslant j \leqslant J\} = \{i + j\alpha : i,j \in \mathbb{N},\ 0 < i + j\alpha < m\}, \tag{3.1}$$

where

$$m := \max\{j \in \mathbb{N} : j < n\alpha\}. \tag{3.2}$$

Define $c_1,\, c_2,\, \ldots,\, c_J \in \mathbb{R}$ by

$$Q(x) - S(x) + c_0 \in \operatorname{span}\left\{x^{i+j\alpha} : i,j \in \mathbb{N},\ i + j\alpha \geqslant m\right\}, \tag{3.3}$$

where

$$Q(x) := \sum_{s=0}^{n-1} \sum_{\beta\in\Lambda_s} \frac{\partial_\beta f(0,c_0)}{\Gamma(\alpha)} \int_0^x (x-t_0)^{\alpha-1}\,\mathrm{d}t_0 \prod_{k=1}^{s} \int_0^{t_{k-1}} \frac{1+(-1)^{\beta_k+1}}{2} + \frac{1+(-1)^{\beta_k}}{2} \sum_{j=1}^{J} \gamma_j c_j t_k^{\gamma_j - 1}\,\mathrm{d}t_k, \tag{3.4}$$

and

$$S(x) := c_0 + \sum_{j=1}^{J} c_j x^{\gamma_j}. \tag{3.5}$$

Above and throughout, a product of a sequence of integrals should be understood in expanded form. For example, (3.4) is understood by

$$\begin{aligned} Q(x) := \sum_{s=0}^{n-1} \sum_{\beta\in\Lambda_s} \frac{\partial_\beta f(0,c_0)}{\Gamma(\alpha)} \int_0^x (x-t_0)^{\alpha-1}\,\mathrm{d}t_0 & \int_0^{t_0} \frac{1+(-1)^{\beta_1+1}}{2} + \frac{1+(-1)^{\beta_1}}{2} \sum_{j=1}^{J} \gamma_j c_j t_1^{\gamma_j-1}\,\mathrm{d}t_1 \\ & \int_0^{t_1} \frac{1+(-1)^{\beta_2+1}}{2} + \frac{1+(-1)^{\beta_2}}{2} \sum_{j=1}^{J} \gamma_j c_j t_2^{\gamma_j-1}\,\mathrm{d}t_2 \\ & \cdots \\ & \int_0^{t_{s-1}} \frac{1+(-1)^{\beta_s+1}}{2} + \frac{1+(-1)^{\beta_s}}{2} \sum_{j=1}^{J} \gamma_j c_j t_s^{\gamma_j-1}\,\mathrm{d}t_s. \end{aligned}$$

Remark 3.1. *It is easy to see that we can express Q in the form*

$$Q(x) = \sum_{j=1}^{L} d_j x^{\gamma_j},$$

where $\{\gamma_j\}_{j=J+1}^{L}$ is a strictly increasing sequence such that $\gamma_J < \gamma_{J+1}$ and

$$\{\gamma_j : 1 \leqslant j \leqslant L\} = \{i + j\alpha :\ i,j \in \mathbb{N},\, i \leqslant n-1,\, 1 \leqslant j \leqslant 1 + (n-1)\gamma_J\}.$$

Moreover, for $1 \leqslant j \leqslant J$, the value of d_j only depends on c_0, c_1, ..., c_{j-1}, and f (more precisely, $\partial_\beta f(0, c_0)$, $\beta \in \Lambda_s$, $1 \leqslant s \leqslant n-1$). Obviously, there exist(s) uniquely c_1, c_2, ..., c_J such that (3.3) holds, and hence c_1, c_2, ..., c_J are/is well-defined. Furthermore, if $\gamma_J + \alpha - m > 0$, then

$$Q - S \in C^{m,\gamma_J+\alpha-m}[0, a];$$

and if $\gamma_J + \alpha - m = 0$, then

$$Q - S \in C^{m,\alpha}[0, a].$$

Remark 3.2. *Note that, S only depends on c_0 and*

$$\{\partial_\beta f(0, c_0) : \beta \in \Lambda_s, 0 \leqslant s < n\}.$$

Since c_0 and f are already available, we can obtain S by a simple calculation.

Define

$$h^* := \min\left\{a, \left(\frac{b\Gamma(1+\alpha)}{M}\right)^{\frac{1}{\alpha}}\right\}.$$

By [2, Theorem 2.2] we know that there exists a unique solution $y^* \in C[0, h^*]$ to (1.1). Now we state the most important result of this paper in the following theorem.

Theorem 3.1. *There exist two positive constant C_0 and C_1 that only depends on a, α and M, such that, for any $0 < h \leqslant h^*$ and $K > 0$ such that*

$$\|(Q - S)'\|_{C^{m-1}[0,h]} + C_1 h^\alpha + C_0 h^\alpha \sum_{j=1}^{m} K^j \leqslant K,$$

we have $y^* - S \in C^m[0, h]$ and

$$\|(y^* - S)'\|_{C^{m-1}[0,h]} \leqslant K. \tag{3.6}$$

Corollary 3.1. *There exists $0 < h \leqslant h^*$ such that $y^* \in C^m[0, h]$ if, and only if,*

$$\frac{\partial^i}{\partial x^i} f(0, c_0) = 0 \quad \text{for all } 0 \leqslant i < m. \tag{3.7}$$

Remark 3.3. *Corollary 3.1 states that $y^* \in C^1[0, h]$ for some $0 < h \leqslant h^*$ if and only if $f(0, c_0) = 0$. So we only have $y^* \in C[0, h] \setminus C^1[0, h]$, if $f(0, c_0) \neq 0$. This yields great difficulty in developing high order numerical methods for (1.1), although $y^* \in C^m(0, h]$. Many numerical methods for (1.1) may not even converge theoretically, since they require that $y^* \in C^m[0, h]$ for some positive integer m. However, we can obtain the numerical values of y^* at some left-most nodes by solving the following problem ($y^* = y + S$): seek $y \in C^m[0, \widetilde{h}]$ such that*

$$y(x) = c_0 - S(x) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f\big(t, y(t) + S(t)\big)\, \mathrm{d}t, \quad 0 \leqslant x \leqslant \widetilde{h},$$

where $\widetilde{h} \ll h$. Then we start the numerical methods for (1.1).

Remark 3.4. *Assuming that f satisfies $f(x, c_0) = 0$ for all $0 \leqslant x \leqslant a$, it is easy to see that*

$$c_i = 0 \quad \text{for all } 1 \leqslant i \leqslant J,$$

and hence $S = c_0$. Then Theorem 3.1 implies $y^* \in C^m[0, h]$. Actually, in this case, it is easy to see that $y^* = c_0$.

Remark 3.5. *Put*

$$\Theta := \{1 \leqslant j \leqslant J : \gamma_j \notin \mathbb{N}\}.$$

Obviously,

$$\sum_{j\in\Theta} c_j x^{\gamma_j}$$

is the singular part (compared to the C^m regularity) in S, and thus the singular part in y^*. Corollary 3.1 essentially claims that (3.7) holds if and only if $c_j = 0$ for all $j \in \Theta$. Since (3.7) is rare, we can consider singularity as an intrinsic property of solutions to fractional differential equations. In addition, we have the following result: that $c_j = 0$ for all $1 \leqslant j \leqslant J$ is equivalent to that $c_j = 0$ for all $j \in \Theta$. This is contained in the proof of Corollary 3.1 in Section 4.3.

4 Proofs

Let $0 < h < \infty$. For any $k \in \mathbb{N}$ and $\gamma \in [0,1]$, define

$$\mathcal{C}^{k,\gamma}[0,h] := \Big\{ v \in C^{k,\gamma}[0,h] : v^{(j)}(0) = 0, \quad j = 0,1,2,\dots,k \Big\}, \tag{4.1}$$

$$\widehat{\mathcal{C}}^{k,\gamma}[0,h] := \Big\{ v \in \mathcal{C}^{k,\gamma}[0,h] : \|v + S - c_0\|_{C[0,h]} \leqslant b \Big\}. \tag{4.2}$$

In particular, we use $\mathcal{C}^k[0,h]$ and $\widehat{\mathcal{C}}^k[0,h]$ to abbreviate $\mathcal{C}^{k,0}[0,h]$ and $\widehat{\mathcal{C}}^{k,0}[0,h]$ respectively for $k \in \mathbb{N}_{>0}$, and use $\mathcal{C}[0,h]$ and $\widehat{\mathcal{C}}[0,h]$ to abbreviate $\mathcal{C}^0[0,h]$ and $\widehat{\mathcal{C}}^0[0,h]$ respectively. In addition, for a function v defined on $(0,h]$ with $h > 0$, by $v \in \mathcal{C}^{k,\gamma}[0,h]$ we mean that, setting $v(0) := 0$, the function v belongs to $\mathcal{C}^{k,\gamma}[0,h]$.

In the remainder of this paper, unless otherwise specified, we use C to denote a positive constant that only depends on α, a and M, and its value may differ at each occurrence. By the definitions of c_1, c_2, $\dots$, c_J, it is easy to see that $|c_j| \leqslant C$ for all $1 \leqslant j \leqslant J$, and we use this implicitly in the forthcoming analysis.

4.1 Some Auxiliary Results

We start by introducing some operators. For $0 < h \leqslant a$, define $\mathcal{P}_{1,h} : \widehat{\mathcal{C}}^m[0,h] \to \mathcal{C}[0,h]$, $\mathcal{P}_{2,h} : \widehat{\mathcal{C}}^m[0,h] \to \mathcal{C}[0,h]$, and $\mathcal{P}_{3,h} : \widehat{\mathcal{C}}^m[0,h] \to \mathcal{C}[0,h]$, respectively, by

$$\mathcal{P}_{1,h} z(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \mathcal{G}_{1,h} z(t)\, \mathrm{d}t, \tag{4.3}$$

$$\mathcal{P}_{2,h} z(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \mathcal{G}_{2,h} z(t)\, \mathrm{d}t, \tag{4.4}$$

$$\mathcal{P}_{3,h} z(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \mathcal{G}_{3,h} z(t)\, \mathrm{d}t, \tag{4.5}$$

for all $z \in \widehat{\mathcal{C}}^m[0,h]$, where $\mathcal{G}_{1,h}z$, $\mathcal{G}_{2,h}z$, $\mathcal{G}_{3,h}z \in \mathcal{C}[0,h]$ are given respectively by

$$\mathcal{G}_{1,h} z(t_0) := \sum_{s=1}^{n} \sum_{\substack{\beta \in \Lambda_s \\ \beta_s = 2}} \prod_{k=1}^{s-1} \int_0^{t_{k-1}} \frac{1 + (-1)^{\beta_k + 1}}{2} + \frac{1 + (-1)^{\beta_k}}{2} \sum_{j=1}^{J} \gamma_j c_j t_k^{\gamma_j - 1}\, \mathrm{d}t_k$$
$$\int_0^{t_{s-1}} z'(t_s) \partial_\beta f\big(t_s, z(t_s) + S(t_s)\big)\, \mathrm{d}t_s, \tag{4.6}$$

$$\mathcal{G}_{2,h} z(t_0) := \sum_{\substack{\beta \in \Lambda_n \\ \beta_n = 2}} \prod_{k=1}^{n-1} \int_0^{t_{k-1}} \frac{1 + (-1)^{\beta_k + 1}}{2} + \frac{1 + (-1)^{\beta_k}}{2} \sum_{j=1}^{J} \gamma_j c_j t_k^{\gamma_j - 1}\, \mathrm{d}t_k$$
$$\int_0^{t_{n-1}} \partial_\beta f\big(t_n, z(t_n) + S(t_n)\big) \sum_{j=1}^{J} \gamma_j c_j t_n^{\gamma_j - 1}\, \mathrm{d}t_n, \tag{4.7}$$

$$\mathcal{G}_{3,h} z(t_0) := \sum_{\substack{\beta \in \Lambda_n \\ \beta_n = 1}} \prod_{k=1}^{n-1} \int_0^{t_{k-1}} \frac{1 + (-1)^{\beta_k + 1}}{2} + \frac{1 + (-1)^{\beta_k}}{2} \sum_{j=1}^{J} \gamma_j c_j t_k^{\gamma_j - 1}\, \mathrm{d}t_k$$
$$\int_0^{t_{n-1}} \partial_\beta f\big(t_n, z(t_n) + S(t_n)\big)\, \mathrm{d}t_n, \tag{4.8}$$

for all $0 \leqslant t_0 \leqslant h$.

Then let us present the following important results for the above operators.

Lemma 4.1. *Let $0 < h \leqslant a$. For any $z \in \widehat{\mathcal{C}}^m[0,h]$, we have*

$$\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f\big(t, z(t) + S(t)\big)\, \mathrm{d}t = Q(x) + \mathcal{P}_{1,h} z(x) + \mathcal{P}_{2,h} z(x) + \mathcal{P}_{3,h} z(x) \tag{4.9}$$

for all $0 \leqslant x \leqslant h$.

Proof. Let $\beta \in \Lambda_s$ with $1 \leqslant s < n$. For any $0 < t_s \leqslant h$, applying the fundamental theorem of calculus yields

$$\partial_\beta f\big(t_s, z(t_s) + S(t_s)\big) = \partial_\beta f\big(\epsilon, z(\epsilon) + S(\epsilon)\big) + \int_\epsilon^{t_s} \partial_{\widetilde{\beta}} f\big(t_{s+1}, z(t_{s+1}) + S(t_{s+1})\big)\,\mathrm{d}t_{s+1} + \int_\epsilon^{t_s} \left(z'(t_{s+1}) + \sum_{j=1}^J \gamma_j c_j t_{s+1}^{\gamma_j - 1} \right) \partial_{\widetilde{\widetilde{\beta}}} f\big(t_{s+1}, z(t_{s+1}) + S(t_{s+1})\big)\,\mathrm{d}t_{s+1}$$

for all $0 < \epsilon \leqslant t_s$, where $\widetilde{\beta} := (\beta_1, \beta_2, \dots, \beta_s, 1)$ and $\widetilde{\widetilde{\beta}} := (\beta_1, \beta_2, \dots, \beta_s, 2)$. Taking limits on both sides of the above equation as ϵ approaches $0+$, we obtain

$$\partial_\beta f\big(t_s, z(t_s) + S(t_s)\big) = \partial_\beta f\big(0, c_0)\big) + \int_0^{t_s} \partial_{\widetilde{\beta}} f\big(t_{s+1}, z(t_{s+1}) + S(t_{s+1})\big)\,\mathrm{d}t_{s+1} + \int_0^{t_s} \left(z'(t_{s+1}) + \sum_{j=1}^J \gamma_j c_j t_{s+1}^{\gamma_j - 1} \right) \partial_{\widetilde{\widetilde{\beta}}} f\big(t_{s+1}, z(t_{s+1}) + S(t_{s+1})\big)\,\mathrm{d}t_{s+1}.$$

Using this equality repeatedly, we easily obtain (4.9). This completes the proof. ■

Lemma 4.2. *Let $0 < h \leqslant a$. For any $z \in \widehat{\mathcal{C}}^m[0,h]$, we have $\mathcal{P}_{1,h} z \in \mathcal{C}^{m,\alpha}[0,h]$ and*

$$\left\| (\mathcal{P}_{1,h} z)' \right\|_{C^{m-1}[0,h]} \leqslant C h^\alpha \sum_{j=1}^m \left\| z' \right\|_{C^{m-1}[0,h]}^j, \tag{4.10}$$

$$\left| (\mathcal{P}_{1,h} z)^{(m)} \right|_{C^{0,\alpha}[0,h]} \leqslant C \sum_{j=1}^m \left\| z' \right\|_{C^{m-1}[0,h]}^j. \tag{4.11}$$

Lemma 4.3. *Let $0 < h \leqslant a$. For any $z \in \widehat{\mathcal{C}}^m[0,h]$, we have $\mathcal{P}_{2,h} z$, $\mathcal{P}_{3,h} z \in \mathcal{C}^{m,\alpha}[0,h]$ and*

$$\left\| (\mathcal{P}_{2,h} z)' \right\|_{C^{m-1}[0,h]} + \left\| (\mathcal{P}_{3,h} z)' \right\|_{C^{m-1}[0,h]} \leqslant C h^\alpha, \tag{4.12}$$

$$\left| (\mathcal{P}_{2,h} z)^{(m)} \right|_{C^{0,\alpha}[0,h]} + \left| (\mathcal{P}_{3,h} z)^{(m)} \right|_{C^{0,\alpha}[0,h]} \leqslant C. \tag{4.13}$$

To prove the above two lemmas, we need several lemmas below.

Lemma 4.4. *Let $0 < h \leqslant a$ and $g \in \mathcal{C}^m[0,h]$. We have $w \in \mathcal{C}^{m,\alpha}[0,h]$ and*

$$\left\| w' \right\|_{C^{m-1}[0,h]} \leqslant C h^\alpha \left\| g' \right\|_{C^{m-1}[0,h]}, \tag{4.14}$$

$$\left| w^{(m)} \right|_{C^{0,\alpha}[0,h]} \leqslant C \left\| g^{(m)} \right\|_{C[0,h]}, \tag{4.15}$$

where

$$w(x) := \int_0^x (x-t)^{\alpha-1} g(t)\,\mathrm{d}t, \quad 0 \leqslant x \leqslant h.$$

Proof. Since $g \in \mathcal{C}^m[0,h]$ we have

$$w^{(i)}(x) = \int_0^x (x-t)^{\alpha-1} g^{(i)}(t)\,\mathrm{d}t, \quad 1 \leqslant i \leqslant m.$$

Then $w \in \mathcal{C}^m[0,h]$ and (4.14) follow, and (4.15) follows from [6, Theorem 3.1]. This completes the proof. ■

Lemma 4.5. *Let $0 < h \leqslant a$, and k, $l \in \mathbb{N}$ such that $k \leqslant m$ and $l\alpha \leqslant 1$. For any $g \in \mathcal{C}^{k,l\alpha}[0,h]$, define*

$$w(x) := \int_0^x \sum_{j=1}^J \gamma_j c_j t^{\gamma_j - 1} g(t)\,\mathrm{d}t, \quad 0 < x \leqslant h.$$

Then we have the following results:

- *If* $(l+1)\alpha \leqslant 1$*, then we have* $w \in \mathcal{C}^{k,(l+1)\alpha}[0,a]$ *and*

$$\|w\|_{\mathcal{C}^{k,(l+1)\alpha}} \leqslant C \|g\|_{\mathcal{C}^{k,l\alpha}} .$$

- *If* $(l+1)\alpha > 1$*, then we have* $w \in \mathcal{C}^{k+1,(l+1)\alpha-1}[0,a]$ *and*

$$\|w\|_{\mathcal{C}^{k+1,(l+1)\alpha-1}} \leqslant C \|g\|_{\mathcal{C}^{k,l\alpha}} .$$

For any $0 < h \leqslant a$, $w \in \mathcal{C}[0,h]$, and $\beta \in \Lambda_s$ with $1 \leqslant s \leqslant n$, define $\mathcal{T}_{w,\beta,h} : \widehat{\mathcal{C}}^m[0,h] \to \mathcal{C}[0,h]$ by

$$\mathcal{T}_{w,\beta,h} z(x) := w(x)\partial_\beta f\big(x, z(x) + S(x)\big),$$

for all $z \in \widehat{\mathcal{C}}^m[0,h]$.

Lemma 4.6. *For* $0 \leqslant k \leqslant m$*, we have* $\mathcal{T}_{w,\beta,h} z \in \mathcal{C}^{\min\{k,n-s\}}[0,h]$ *and*

$$\|\mathcal{T}_{w,\beta,h} z\|_{C^{\min\{k,n-s\}}[0,h]} \leqslant C \|w\|_{C^k[0,h]} \sum_{j=0}^{\min\{k,n-s\}} \|z'\|^j_{C^{m-1}[0,h]} \tag{4.16}$$

for all $0 < h \leqslant a$*,* $w \in \mathcal{C}^k[0,h]$*,* $\beta \in \Lambda_s$ *with* $1 \leqslant s \leqslant n$*, and* $z \in \widehat{\mathcal{C}}^m[0,h]$*.*

The proofs of Lemmas 4.5 and 4.6 are presented in Appendix A. In the rest of this subsection, we give the proofs of Lemmas 4.2 and 4.3.

Proof of Lemma 4.2. By (4.3), (4.6), and Lemma 4.4, it suffices to show that, for each $\beta \in \Lambda_s$ with $\beta_s = 2$, we have $g_0 \in \mathcal{C}^m[0,h]$ and

$$\|g_0\|_{C^m[0,h]} \leqslant C \sum_{j=1}^{\min\{m,n-s+1\}} \|z'\|^j_{C^{m-1}[0,h]} , \tag{4.17}$$

where, if $s = 1$, then

$$g_0(x) := \int_0^x z'(t)\partial_2 f\big(t, z(t) + S(t)\big)\, \mathrm{d}t;$$

if $2 \leqslant s \leqslant n$, then

$$\begin{aligned}
g_0(x) &:= \int_0^x \left(\frac{1+(-1)^{\beta_1+1}}{2} + \frac{1+(-1)^{\beta_1}}{2} \sum_{j=1}^J \gamma_j c_j t^{\gamma_j - 1} \right) g_1(t)\, \mathrm{d}t,\\
g_1(x) &:= \int_0^x \left(\frac{1+(-1)^{\beta_2+1}}{2} + \frac{1+(-1)^{\beta_2}}{2} \sum_{j=1}^J \gamma_j c_j t^{\gamma_j - 1} \right) g_2(t)\, \mathrm{d}t,\\
&\vdots\\
g_{s-2}(x) &:= \int_0^x \left(\frac{1+(-1)^{\beta_{s-1}+1}}{2} + \frac{1+(-1)^{\beta_{s-1}}}{2} \sum_{j=1}^J \gamma_j c_j t^{\gamma_j - 1} \right) g_{s-1}(t)\, \mathrm{d}t,\\
g_{s-1}(x) &:= \int_0^x z'(t)\partial_\beta f\big(t, z(t) + S(t)\big)\, \mathrm{d}t.
\end{aligned}$$

To do so, we proceed as follows. If $s = 1$, then by Lemma 4.6 we obtain $g_0 \in \mathcal{C}^m[0,h]$ and (4.17). Let us suppose that $2 \leqslant s \leqslant n$. By Lemma 4.6 it follows $g_{s-1} \in \mathcal{C}^{\min\{m,n-s+1\}}[0,h]$ and

$$\|g_{s-1}\|_{C^{\min\{m,n-s+1\}}[0,h]} \leqslant C \sum_{j=1}^{\min\{m,n-s+1\}} \|z'\|^j_{C^{m-1}[0,h]} .$$

Then, by the simple estimate

$$(n-s+1) + (s-1)\alpha > m,$$

applying Lemma 4.5 to g_{s-2}, g_{s-3}, $\ldots$, g_0 successively yields $g_0 \in \mathcal{C}^m[0,h]$ and (4.17). This completes the proof of Lemma 4.2. ■

Proof of Lemma 4.3. Let us first show that $\mathcal{G}_{2,h}z \in \mathcal{C}^m[0,h]$ and

$$\|\mathcal{G}_{2,h}z\|_{C^m[0,h]} \leqslant C. \tag{4.18}$$

By (4.7) it suffices to show that, for any $\beta \in \Lambda_n$ with $\beta_n = 2$, we have $g_0 \in \mathcal{C}^m[0,h]$ and

$$\|g_0\|_{C^m[0,h]} \leqslant C, \tag{4.19}$$

where

$$\begin{aligned}
g_0(x) &:= \int_0^x \left(\frac{1+(-1)^{\beta_1+1}}{2} + \frac{1+(-1)^{\beta_1}}{2} \sum_{j=1}^J \gamma_j c_j t^{\gamma_j - 1} \right) g_1(t)\,\mathrm{d}t, \\
g_1(x) &:= \int_0^x \left(\frac{1+(-1)^{\beta_2+1}}{2} + \frac{1+(-1)^{\beta_2}}{2} \sum_{j=1}^J \gamma_j c_j t^{\gamma_j - 1} \right) g_2(t)\,\mathrm{d}t, \\
&\vdots \\
g_{n-2}(x) &:= \int_0^x \left(\frac{1+(-1)^{\beta_{s-1}+1}}{2} + \frac{1+(-1)^{\beta_{s-1}}}{2} \sum_{j=1}^J \gamma_j c_j t^{\gamma_j - 1} \right) g_{s-1}(t)\,\mathrm{d}t, \\
g_{n-1}(x) &:= \int_0^x \partial_\beta f\big(t, z(t) + S(t)\big) \sum_{j=1}^J \gamma_j c_j t^{\gamma_j - 1}\,\mathrm{d}t,
\end{aligned}$$

for all $0 \leqslant x \leqslant h$. Noting the fact that

$$\partial_\beta f\big(\cdot, z(\cdot) + S(\cdot)\big) \in C[0,h],$$

and $\gamma_j \geqslant \alpha$ for all $1 \leqslant j \leqslant J$, we easily obtain $g_{n-1} \in \mathcal{C}^{0,\alpha}[0,h]$ and

$$\|g_{n-1}\|_{C^{0,\alpha}[0,h]} \leqslant C.$$

Then, applying Lemma 4.5 to g_{n-2}, g_{n-3}, $\ldots$, g_0 successively, and using the fact $n\alpha > m$, we obtain $g_0 \in \mathcal{C}^m[0,h]$ and (4.19). Thus we have showed $\mathcal{G}_{2,h}z \in \mathcal{C}^m[0,h]$ and (4.18).

Similarly, we can show that $\mathcal{G}_{3,h}z \in \mathcal{C}^m[0,h]$ and $\|\mathcal{G}_{3,h}z\|_{C^m[0,h]} \leqslant C$. Consequently, by (4.4), (4.5), and Lemma 4.4, we infer that $\mathcal{P}_{2,h}z$, $\mathcal{P}_{3,h}z \in \mathcal{C}^{m,\alpha}[0,h]$, and (4.12) and (4.13) hold. This completes the proof. ■

4.2 Proof of Theorem 3.1

By Lemmas 4.2 and 4.3 there exist two positive constants C_0 and C_1 that only depend on a, α and M, such that

$$\|(\mathcal{P}_{1,h}z)'\|_{C^{m-1}[0,h]} \leqslant C_0 h^\alpha \sum_{j=1}^m \|z'\|^j_{C^{m-1}[0,h]}\,, \tag{4.20}$$

$$\|(\mathcal{P}_{2,h}z)'\|_{C^{m-1}[0,h]} + \|(\mathcal{P}_{3,h}z)'\|_{C^{m-1}[0,h]} \leqslant C_1 h^\alpha, \tag{4.21}$$

for all $0 < h \leqslant a$ and $z \in \widehat{\mathcal{C}}^m[0,h]$. Let $0 < h \leqslant h^*$ and $K > 0$ such that

$$\|(Q-S)'\|_{C^{m-1}[0,h]} + C_1 h^\alpha + C_0 h^\alpha \sum_{j=1}^m K^j \leqslant K. \tag{4.22}$$

Define $\mathcal{J} : V \to \mathcal{C}[0,h]$ by

$$\mathcal{J}z(x) := c_0 - S(x) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f\big(t, z(t) + S(t)\big)\,\mathrm{d}t, \tag{4.23}$$

for all $z \in V$ and $x \in [0,h]$, where

$$V := \left\{ v \in \widehat{\mathcal{C}}^m[0,h] : \|v'\|_{C^{m-1}[0,h]} \leqslant K \right\}. \tag{4.24}$$

Remark 4.1. *It is clear that V is a bounded, closed, convex subset of $C^m[0,h]$.*

Remark 4.2. *Let $\delta > 0$. If we put*

$$K := \|(Q-S)'\|_{C^{m-1}[0,h]} + C_1 a^\alpha + \delta,$$
$$h := \min\left\{ h^*, \left(\delta^{-1} C_0 \sum_{j=1}^m K^j\right)^{-\frac{1}{\alpha}} \right\},$$

then (4.22) holds.

For the operator $\mathcal{J}$, we have the following key result.

Lemma 4.7. *For each $z \in V$, we have $\mathcal{J}z \in V$ and*

$$\left|(\mathcal{J}z)^{(m)}\right|_{C^{0,\gamma}[0,h]} \leqslant \left|(Q-S)^{(m)}\right|_{C^{0,\gamma}[0,h]} + C\sum_{j=0}^m K^j, \tag{4.25}$$

where $\gamma := \alpha$ if $\gamma_J + \alpha = m$, and $\gamma := \gamma_J + \alpha - m$ if $\gamma_J + \alpha > m$.

Proof. Let us first show $\mathcal{J}z \in V$. Using (4.23) and the fact $h \leqslant \left(\frac{b\Gamma(1+\alpha)}{M}\right)^{\frac{1}{\alpha}}$, we have

$$|\mathcal{J}z(x) + S(x) - c_0| = \frac{1}{\Gamma(\alpha)}\left|\int_0^x (x-t)^{\alpha-1} f\big(t, z(t)+S(t)\big)\,\mathrm{d}t\right| \leqslant \frac{Mh^\alpha}{\Gamma(1+\alpha)} \leqslant b$$

for all $x \in [0,h]$, and so

$$\|\mathcal{J}z + S - c_0\|_{C[0,h]} \leqslant b.$$

By Lemma 4.1 we have

$$\mathcal{J}z(x) = c_0 - S(x) + Q(x) + \mathcal{P}_{1,h}z(x) + \mathcal{P}_{2,h}z(x) + \mathcal{P}_{3,h}z(x), \tag{4.26}$$

and then, by Lemmas 4.2 and 4.3, and the fact $c_0 - S + Q \in \mathcal{C}^m[0,h]$, we obtain $\mathcal{J}z \in \mathcal{C}^m[0,h]$. It remains, therefore, to show that

$$\|(\mathcal{J}z)'\|_{C^{m-1}[0,h]} \leqslant K. \tag{4.27}$$

To this end, note that, by (4.26), (4.20) and (4.21) we obtain

$$\|(\mathcal{J}z)'\|_{C^{m-1}[0,h]} \leqslant \|(Q-S)'\|_{C^{m-1}[0,h]} + C_1 h^\alpha + C_0 h^\alpha \sum_{j=1}^m K^j,$$

and then (4.27) follows from (4.22). We have thus showed $\mathcal{J}z \in V$.

Finally, let us show (4.25). By Lemmas 4.2 and 4.3 we obtain

$$\left|(\mathcal{P}_{1,h}z)^{(m)}\right|_{C^{0,\alpha}[0,h]} + \left|(\mathcal{P}_{2,h}z)^{(m)}\right|_{C^{0,\alpha}[0,h]} + \left|(\mathcal{P}_{3,h}z)^{(m)}\right|_{C^{0,\alpha}[0,h]} \leqslant C\sum_{j=0}^m \|z'\|^j_{C^{m-1}[0,h]} \leqslant C\sum_{j=0}^m K^j.$$

From the fact $\gamma \leqslant \alpha$ it follows

$$\left|(\mathcal{P}_{1,h}z)^{(m)}\right|_{C^{0,\gamma}[0,h]} + \left|(\mathcal{P}_{2,h}z)^{(m)}\right|_{C^{0,\gamma}[0,h]} + \left|(\mathcal{P}_{3,h}z)^{(m)}\right|_{C^{0,\gamma}[0,h]} \leqslant C\sum_{j=0}^m K^j.$$

Using this estimate and the fact that $(Q-S)^{(m)} \in C^{0,\gamma}$ by the definitions of Q and S, the desired estimate (4.25) follows from (4.26). This completes the proof. ■

By the famous Arzelà-Ascoli Theorem and Lemma 4.7, it is evident that $\mathcal{J} : V \to V$ is a compact operator, where V is endowed with norm $\|\cdot\|_{C^m[0,h]}$. Therefore, since V is a bounded, closed, convex subset of $C^m[0,h]$, using the Schauder Fixed-Point Theorem gives that there exists $z \in V$ such that

$$\mathcal{J}z = z.$$

Putting

$$y(x) := z(x) + S(x), \quad 0 \leqslant x \leqslant h,$$

we obtain

$$y(x) = c_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f\big(t, y(t)\big)\, \mathrm{d}t, \quad 0 \leqslant x \leqslant h.$$

By [2, Lemma 2.1], the above y is a solution of (1.1), and then, since y^* is the unique solution of (1.1) on $[0, h^*]$, we have $y^* = y$ on $[0, h]$. Therefore, it is obvious that $y^* - S \in C^m[0, h]$ and (3.6) hold. This completes the proof of Theorem 3.1.

4.3 Proof of Corollary 3.1

Let us first state the following fact. For each $1 \leqslant j \leqslant J$, by the definition of c_j, a straightforward computing yields

$$c_j = \sum_{t \in \Upsilon_{j,1} \cup \Upsilon_{j,2}} t, \tag{4.28}$$

where

$$\Upsilon_{j,1} := \bigcup_{\substack{1 \leqslant s < n \\ s+\alpha=\gamma_j}} \left\{ \frac{\mathrm{B}(\alpha, 1+s) \partial_1^s f(0, c_0)}{\Gamma(\alpha)} \right\}, \tag{4.29}$$

$$\Upsilon_{j,2} := \bigcup_{s=1}^{n-1} \bigcup_{k=1}^{s} \bigcup_{\substack{\beta \in \Lambda_s \\ \#\beta = k \\ \Gamma_\beta \neq \emptyset}} \left\{ \frac{\mathrm{B}\left(\alpha, 1+s-k+\sum_{l=1}^k \gamma_{i_l}\right) \partial_\beta f(0, c_0)}{\Gamma(\alpha) \prod_{l=1}^k \gamma_{i_l}} \prod_{l=1}^k c_{i_l} \gamma_{i_l} : (i_1, i_2, \dots, i_k) \in \Xi_{\beta,j} \right\}. \tag{4.30}$$

Above, $\mathrm{B}(\cdot,\cdot)$ denotes the standard beta function, and

$$\#\beta := \sum_{\substack{1 \leqslant i \leqslant s \\ \beta_i = 2}} 1,$$

$$\Xi_{\beta,j} := \left\{ (i_1, i_2, \dots, i_{\#\beta}) : \alpha + s - \#\beta + \sum_{j=1}^{\#\beta} \gamma_{i_j} = \gamma_j \right\},$$

for all $1 \leqslant s < n$ and $\beta \in \Lambda_s$.

To prove Corollary 3.1, by Theorem 3.1 it suffices to show that (3.7) is equivalent to

$$c_j = 0 \quad \text{for all } j \in \Theta, \tag{4.31}$$

where

$$\Theta := \{1 \leqslant j \leqslant J : \gamma_j \notin \mathbb{N}\}.$$

But, by (4.28), (4.29) and (4.30), an obvious induction gives

$$c_j = 0 \quad \text{for all } 1 \leqslant j \leqslant J, \tag{4.32}$$

if (3.7) holds. Therefore, it remains to show that (4.31) implies (3.7).

To this end, let us assume that (4.31) holds. Note that we have (4.32). If this statement was false, then let

$$j_0 := \min\{1 \leqslant j \leqslant J : c_j \neq 0\}.$$

Obviously, we have $j_0 > 1$ and $\gamma_{j_0} \in \mathbb{N}$, and in this case, $\Upsilon_{j_0,1}$ is empty. Thus, by (4.28) we have

$$c_{j_0} = \sum_{t \in \Upsilon_{j_0,2}} t.$$

But, by the definition of $\Upsilon_{j_0,2}$ and the fact that $c_j = 0$ for all $1 \leqslant j < j_0$, it is straightforward that $c_{j_0} = 0$, which is contrary to the definition of j_0. Therefore (4.32) holds indeed. Using this result, from (4.28) and (4.30) it follows

$$c_j = \sum_{t \in \Upsilon_{j,1}} t \quad \text{for all } 1 \leqslant j \leqslant J,$$

and then, using (4.32) again, we obtain (3.7). This completes the proof of Corollary 3.1.

Appendix A Proofs of Lemmas 4.5 and 4.6

To prove Lemma 4.5, we need the following two lemmas.

Lemma A.1. *Let $h > 0$, $\gamma > 0$ and $g \in \mathcal{C}^1[0,h]$. We have $w \in \mathcal{C}^1[0,h]$ and*

$$w'(x) = \int_0^x t^{\gamma-1} g'(t)\,\mathrm{d}t. \tag{A.1}$$

where

$$w(x) := \int_0^x t^{\gamma-1} g(t)\,\mathrm{d}t, \quad 0 \leqslant x \leqslant h.$$

Since the proof of this lemma is straightforward, it is omitted.

Lemma A.2. *Let $0 < h \leqslant a$, and $l \in \mathbb{N}_{>0}$ such that $l\alpha \leqslant 1 < (l+1)\alpha$. For any $g \in \mathcal{C}^{0,l\alpha}[0,h]$, we have $w \in \mathcal{C}^{0,(l+1)\alpha-1}[0,h]$ and*

$$\|w\|_{C^{0,(l+1)\alpha-1}[0,h]} \leqslant C\,\|g\|_{C^{0,l\alpha}[0,h]}\,,$$

where

$$w(x) := \sum_{j=1}^{J} \gamma_j c_j x^{\gamma_j-1} g(x), \quad 0 < x \leqslant h.$$

Proof. It suffices to prove that, for any $1 \leqslant j \leqslant J$, we have $v \in \mathcal{C}[0,h]$ and

$$\|v\|_{C^{0,(l+1)\alpha-1}[0,h]} \leqslant C\,\|g\|_{C^{0,l\alpha}[0,h]}\,,$$

where $v(x) := x^{\gamma_j-1} g(x)$, $0 < x \leqslant h$. Noting the fact that $l\alpha + \gamma_j > 1$ and $g \in \mathcal{C}^{0,l\alpha}[0,h]$, we easily obtain $v \in \mathcal{C}[0,h]$ and

$$\|v\|_{C[0,h]} \leqslant C\,\|g\|_{C^{0,l\alpha}[0,h]}\,.$$

It remains, therefore, to prove that

$$|v(y) - v(x)| \leqslant C(y-x)^{(l+1)\alpha-1}\,\|g\|_{C^{0,l\alpha}[0,h]}$$

for all $0 < x < y \leqslant h$. Moreover, since it holds

$$\begin{aligned} &|v(y) - v(x)| = \left|y^{\gamma_j-1}g(y) - x^{\gamma_j-1}g(x)\right| \\ =\ &\left|y^{\gamma_j-1}\big(g(y)-g(x)\big) + (y^{\gamma_j-1} - x^{\gamma_j-1})g(x)\right| \\ \leqslant\ &\left(y^{\gamma_j-1}(y-x)^{l\alpha} + \left|y^{\gamma_j-1} - x^{\gamma_j-1}\right| x^{l\alpha}\right)\|g\|_{C^{0,l\alpha}[0,h]}\,, \end{aligned}$$

by the fact $g \in \mathcal{C}^{0,l\alpha}[0,h]$, we only need to prove that

$$y^{\gamma_j-1}(y-x)^{l\alpha} + \left|y^{\gamma_j-1} - x^{\gamma_j-1}\right| x^{l\alpha} \leqslant C(y-x)^{(l+1)\alpha-1} \tag{A.2}$$

for all $0 < x < y \leqslant h$.

Let us first consider the case of $\gamma_j < 1$. A simple algebraic calculation gives

$$(x^{\gamma_j-1} - y^{\gamma_j-1})x^{l\alpha} = (y-x)^{l\alpha+\gamma_j-1}\big(A^{\gamma_j-1} - (1+A)^{\gamma_j-1}\big)A^{l\alpha},$$

where $A := \frac{x}{y-x}$. If $0 \leqslant A \leqslant 1$, then by the fact $l\alpha + \gamma_j - 1 > 0$ we have

$$\big(A^{\gamma_j-1} - (1+A)^{\gamma_j-1}\big)A^{l\alpha} < A^{l\alpha+\gamma_j-1} \leqslant 1.$$

If $A > 1$, then using the Mean Value Theorem and the fact $l\alpha + \gamma_j - 2 < 0$ gives

$$\big(A^{\gamma_j-1} - (1+A)^{\gamma_j-1}\big)A^{l\alpha} < (1-\gamma_j)A^{l\alpha+\gamma_j-2} < (1-\gamma_j) < 1.$$

Consequently, we obtain

$$(x^{\gamma_j-1} - y^{\gamma_j-1})x^{l\alpha} < (y-x)^{l\alpha+\gamma_j-1},$$

which, together with the trivial estimate

$$y^{\gamma_j-1}(y-x)^{l\alpha} < (y-x)^{\gamma_j-1}(y-x)^{l\alpha} = (y-x)^{l\alpha+\gamma_j-1},$$

yields (A.2).

Then, since (A.2) is evident in the case of $\gamma_j = 1$, let us consider the case of $1 < \gamma_j < 2$. Since $0 < \gamma_j - 1 < 1$, we have

$$y^{\gamma_j-1} - x^{\gamma_j-1} < (y-x)^{\gamma_j-1}.$$

By the definition of γ_j it is clear that

$$\gamma_j - 1 \geqslant (l+1)\alpha - 1.$$

Using the above two estimates, we obtain

$$\left|y^{\gamma_j-1} - x^{\gamma_j-1}\right| x^{l\alpha} \leqslant C(y^{\gamma_j-1} - x^{\gamma_j-1}) \leqslant C(y-x)^{(l+1)\alpha-1},$$

which, together with the estimate

$$y^{\gamma_j-1}(y-x)^{l\alpha} \leqslant C(y-x)^{l\alpha} \leqslant C(y-x)^{(l+1)\alpha-1},$$

indicates (A.2).

Finally, let us consider the case of $\gamma_j \geqslant 2$. Using the Mean Value Theorem gives

$$\left|y^{\gamma_j-1} - x^{\gamma_j-1}\right| x^{l\alpha} \leqslant C(y-x)^{(l+1)\alpha-1}.$$

and then, by the obvious estimate

$$y^{\gamma_j-1}(y-x)^{l\alpha} \leqslant C(y-x)^{(l+1)\alpha-1},$$

we obtain (A.2). This completes the proof. ■

Proof of Lemma 4.5 Since $g \in \mathcal{C}^{k,l\alpha}[0,h]$, by Lemma A.1 we have $w \in \mathcal{C}^k[0,h]$ and

$$w^{(i)}(x) = \int_0^x \sum_{j=1}^J \gamma_j c_j t^{\gamma_j-1} g^{(i)}(t)\,\mathrm{d}t, \quad i = 0,1,2,\dots,k. \tag{A.3}$$

It follows

$$\|w\|_{C^k[0,h]} \leqslant C\,\|g\|_{C^k[0,h]}\,.$$

Therefore, it remains to prove that

$$\left|w^{(k)}\right|_{C^{0,(l+1)\alpha}[0,h]} \leqslant C\,\|g\|_{C^{k,l\alpha}[0,h]} \tag{A.4}$$

if $(l+1)\alpha \leqslant 1$; and that $w^{(k+1)} \in \mathcal{C}^{0,(l+1)\alpha-1}[0,h]$ and

$$\left\|w^{(k+1)}\right\|_{C^{0,(l+1)\alpha-1}[0,h]} \leqslant C\,\|g\|_{C^{k,l\alpha}[0,h]} \tag{A.5}$$

if $(l+1)\alpha > 1$.

Let us first consider (A.4). Noting the fact that $g^{(k)} \in \mathcal{C}^{0,l\alpha}[0,h]$ and $\gamma_j \geqslant \alpha$ for all $1 \leqslant j \leqslant J$, by (A.3) a simple computing gives that

$$\left|w^{(k)}(y) - w^{(k)}(x)\right| \leqslant C\left|g^{(k)}\right|_{C^{0,l\alpha}[0,h]} (y-x)^{(l+1)\alpha}$$

for all $0 \leqslant x < y \leqslant h$, which implies (A.4).

Then let us consider (A.5). Since $g^{(k)} \in \mathcal{C}^{0,l\alpha}$, by Lemma A.2 we have $v \in \mathcal{C}^{0,(l+1)\alpha-1}[0,h]$ and

$$\|v\|_{C^{0,(l+1)\alpha-1}[0,h]} \leqslant C\left\|g^{(k)}\right\|_{C^{0,l\alpha}[0,h]},$$

where

$$v(x) := \sum_{j=1}^J \gamma_j c_j x^{\gamma_j-1} g^{(k)}(x), \quad 0 < x \leqslant h.$$

Then, by (A.3) we readily obtain $w^{(k+1)} \in \mathcal{C}^{0,(l+1)\alpha-1}$ and (A.5), and thus complete the proof of this lemma. ■

Before proving Lemma 4.6, let us introduce the following lemma.

Lemma A.3. *Let $0 < h \leqslant a$ and $\gamma > 0$. For any $g \in \mathcal{C}^k[0,h]$ with $1 \leqslant k \leqslant m$, we have $w \in \mathcal{C}^{k-1}[0,h]$ and*

$$\|w\|_{C^{k-1}[0,h]} \leqslant C \left\| g^{(k)} \right\|_{C[0,h]},$$

where

$$w(x) := g(x)x^{\gamma-1}, \quad 0 < x \leqslant h,$$

and C is a positive constant that only depends on a, k and γ.

Proof. If $k = 1$, then, by the Mean Value Theorem and the fact $g(0) = 0$, this lemma is evident. Thus, below we assume that $2 \leqslant k \leqslant m$. In the rest of this proof, for ease of notation, the symbol C denotes a positive constant that only depends on a, k and γ, and its value may differ at each occurrence.

Let us first show that, for $0 \leqslant i < k$, we have $w_i \in \mathcal{C}[0,h]$ and

$$\|w_i\|_{C[0,h]} \leqslant C \left\| g^{(k)} \right\|_{C[0,h]}, \tag{A.6}$$

where

$$w_i(x) := w^{(i)}(x), \quad 0 < x \leqslant h.$$

To this end, let $0 \leqslant i < k$, and note that an elementary computing gives

$$w_i(x) = \sum_{j=0}^{i} c_{ij} g^{(j)}(x) x^{\gamma-1-i+j}, \quad 0 < x \leqslant h, \tag{A.7}$$

where c_{ij} is a constant that only depends on γ, i and j, for all $0 \leqslant j \leqslant i$. Since $g \in \mathcal{C}^k[0,h]$, we have $g^{(j)} \in \mathcal{C}^{k-j}[0,h]$, and then, applying Taylor's formula with integral remainder yields

$$g^{(j)}(x) = \frac{1}{(k-j-1)!} \int_0^x (x-t)^{k-j-1} g^{(k)}(t) \, \mathrm{d}t, \quad 0 \leqslant x \leqslant h.$$

It follows that

$$\left| g^{(j)}(x) x^{\gamma-1-i+j} \right| \leqslant \frac{\|g^{(k)}\|_{C[0,h]}}{(k-j)!} x^{\gamma+k-(i+1)}, \quad 0 < x \leqslant h. \tag{A.8}$$

Since $\gamma + k - (i+1) \geqslant \gamma > 0$, this implies $g^{(j)}(x)x^{\gamma-i-1+j} \in \mathcal{C}[0,h]$ and

$$\left\| g^{(j)}(\cdot)(\cdot)^{\gamma-i-1+j} \right\|_{C[0,h]} \leqslant C \left\| g^{(k)} \right\|_{C[0,h]}.$$

Therefore, by (A.7) it follows $w_i \in \mathcal{C}[0,h]$ and (A.6).

Then let us proceed to prove this lemma. Let $i < k-1$. Note that by (A.7) we have

$$w_i'(x) = w_{i+1}(x), \quad 0 < x \leqslant h.$$

Since we have already proved that w_i, $w_{i+1} \in \mathcal{C}[0,h]$, by the Mean Value Theorem it is evident that $w_i \in \mathcal{C}^1[0,h]$ and

$$w_i'(x) = w_{i+1}(x), \quad 0 \leqslant x \leqslant h.$$

It follows $w_0 \in \mathcal{C}^{k-1}[0,h]$ and

$$w_0^{(i)} = w_i, \quad 0 \leqslant i < k,$$

and hence, by (A.6) we have

$$\|w_0\|_{C^{k-1}[0,h]} \leqslant C \left\| g^{(k)} \right\|_{C[0,h]}.$$

Noting the fact $w = w_0$, this completes the proof. ∎

Proof of Lemma 4.6 Below we employ the well-known principle of mathematical induction to prove this lemma. Firstly, it is clear that (4.16) holds in the case $k = 0$. Secondly, assuming that (4.16) holds for $k = l$ where $0 \leqslant l < m-1$, let us prove that (4.16) holds for $k = l+1$. To this end, a straightforward computing gives

$$(\mathcal{T}_{w,\beta,h} z)'(x) = \mathcal{T}_{w',\beta,h} z(x) + \mathcal{T}_{w,\widetilde{\beta},h} z(x) + \mathcal{T}_{\widetilde{w},\widetilde{\widetilde{\beta}},h} z(x) \tag{A.9}$$

for all $0 < x \leqslant h$, where $\widetilde{\beta} := (\beta_1, \beta_2, \ldots, \beta_s, 1)$, $\approx{\beta}, := (\beta_1, \beta_2, \ldots, \beta_s, 2)$, and

$$\widetilde{w}(x) := w(x) \left(z'(x) + \sum_{j=1}^{J} \gamma_j c_j x^{\gamma_j - 1} \right).$$

Since $w \in \mathcal{C}^k[0,h]$, we have $w' \in \mathcal{C}^{k-1}[0,h]$, and by Lemma A.3 we have $\widetilde{w} \in \mathcal{C}^{k-1}[0,h]$; consequently, $\mathcal{T}_{w',\beta,h}z$ and $\mathcal{T}_{\widetilde{w},\approx{\beta},h}z$ are well-defined, and they both belong to $\mathcal{C}[0,h]$. Therefore, by the Mean Value Theorem, and the fact $\mathcal{T}_{w,\beta,h}z \in \mathcal{C}[0,h]$, it follows that $\mathcal{T}_{w,\beta,h}z \in \mathcal{C}^1[0,h]$, and (A.9) holds for all $0 \leqslant x \leqslant h$. By our assumption, we have the following results: $\mathcal{T}_{w',\beta,h}z \in \mathcal{C}^{\min\{k-1,n-s\}}[0,h]$ and

$$\|\mathcal{T}_{w',\beta,h}z\|_{C^{\min\{k-1,n-s\}}[0,h]} \leqslant C \|w'\|_{C^{k-1}[0,h]} \sum_{j=0}^{\min\{k-1,n-s\}} \|z'\|^j_{C^{m-1}[0,h]};$$

$\mathcal{T}_{w,\widetilde{\beta},h}z \in \mathcal{C}^{\min\{k-1,n-s-1\}}[0,h]$ and

$$\left\|\mathcal{T}_{w,\widetilde{\beta},h}z\right\|_{C^{\min\{k-1,n-s-1\}}[0,h]} \leqslant C \|w\|_{C^{k-1}[0,h]} \sum_{j=0}^{\min\{k-1,n-s-1\}} \|z'\|^j_{C^{m-1}[0,h]};$$

$\mathcal{T}_{\widetilde{w},\approx{\beta},h}z \in \mathcal{C}^{\min\{k-1,n-s-1\}}[0,h]$ and

$$\left\|\mathcal{T}_{\widetilde{w},\approx{\beta},h}z\right\|_{C^{\min\{k-1,n-s-1\}}[0,h]} \leqslant C \|\widetilde{w}\|_{C^{k-1}[0,h]} \sum_{j=0}^{\min\{k-1,n-s-1\}} \|z'\|^j_{C^{m-1}[0,h]}.$$

In addition, by Lemma A.3 we easily obtain

$$\|\widetilde{w}\|_{C^{k-1}[0,h]} \leqslant C \|w\|_{C^k[0,h]} \left(1 + \|z'\|_{C^{k-1}[0,h]}\right).$$

As a consequence, we obtain $\mathcal{T}_{w,\beta,h}z \in \mathcal{C}^{\min\{k,n-s\}}$ and

$$\|(\mathcal{T}_{w,\beta,h}z)'\|_{C^{\min\{k-1,n-s-1\}}[0,h]} \leqslant C \|w\|_{C^k[0,h]} \sum_{j=0}^{\min\{k,n-s\}} \|z'\|^j_{C^{m-1}[0,h]}.$$

Then (4.16) follows from the obvious estimate

$$\|\mathcal{T}_{w,\beta,h}z\|_{C[0,h]} \leqslant C \|w\|_{C[0,h]}.$$

This completes the proof of Lemma 4.6. ∎

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