A Note on the Paper "Optimality Conditions for Vector Optimization Problems with Difference of Convex Maps"

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Abstract

In this work, some counterexamples are given to refute some results reported in the paper by Guo and Li [8] (J Optim Theory Appl 162,(2014), 821-844). We correct the faulty in some of their theorems and we present alternative proofs. Moreover, we extend the definition of approximately pseudo-dissipative in the setting of metrizable topological vector spaces.

Keywords: Convex mapping, Optimality condition, Local weak minimal solution, Subdifferential, Pareto minimal point.

1 Introduction

In several optimization problems nonlinear and nonconvex functions can be decomposed into the difference of convex (DC) functions (see [18]).
In the last decade, different kinds of DC programming have been investigated extensively and significant results have been achieved, see for example [1, 2, 3, 4, 5, 8, 10, 12, 16, 18, 19] and the references therein. Here, we briefly mention the results on duality and optimality in [1, 2, 6, 5, 8, 16, 17]. In [1, 2, 6] the authors consider optimization problems with objectives given as DC functions and constraints described by convex inequalities. For Banach spaces, they obtain necessary and sufficient optimality conditions for DC infinite and semi-infinite programs. Efficient upper estimates of certain subdifferentials of value functions for the DC optimization problem are given in [1]. In [2], the authors provide characterizations of the Farkas-Minkowski constraint qualification.
Fang and Zhao introduced the local and global KKT type conditions for the DC optimization problem in [3]. Using properties of the subdifferential, they provide some sufficient and necessary conditions for these optimality conditions. In the case of DC optimization, weak and strong duality assertions for extended Ky Fan inequalities are provided in [16]. The authors in [16] apply their dual problems also to a convex optimization problem and a generalized variational inequality problem. By using the properties of the epigraphs of the conjugate functions, Sun, et al. [17] introduced a closedness qualification condition. They then employed their condition to investigate duality and Farkas-type results for a DC infinite programming problem. Also in [11] established optimality conditions under convexity and continuity assumptions for set functions.

In [8], Guo and Li use the notions of strong subdifferential and epsilon subdifferential to obtain necessary and sufficient optimality conditions for an epsilon-weak Pareto minimal point and an epsilon-proper Pareto minimal point of a DC vector optimization problem.
In this article, we show that some theorems and results in [8] are not correct. Furthermore, we clarify an existence gap by providing some counterexamples. Finally we present corrected versions of their results.

2 Preliminaries

Let us briefly recall the notation used in this work. For the most part, we follow notations as in [8]. Throughout this paper, X is a metrizable topological vector space. Furthermore, Y and Z stand for topological vector spaces. We will denote the dual of Y and Z by Y^* and Z^* respectively, with duality pairing denoted by $\langle .,. \rangle$. The origins of the topological vector spaces are denoted by $0_X, 0_Y$, and 0_Z. As usual $L(X,Y)$ is the set of all linear continuous operators from X to Y. Moreover, let $K \subset Y$ and $D \subset Z$ be proper (i.e., $K \neq \{0_Y\} \neq Y$) convex cones with nonempty interior (i.e., $\mathrm{int}K \neq \emptyset$). Let $l(K) = K \cap -K$ be the linearity of K. The cone K determines an order relation on Y denoted in the sequel by $\preceq_K$. We recall the following definition of ordering relations:

$$\begin{aligned} y' \preceq_K y &\Leftrightarrow y - y' \in K, \\ y' \prec_K y &\Leftrightarrow y - y' \in \mathrm{int}K, \\ y' \npreceq_K y &\Leftrightarrow y - y' \notin K, \\ y' \nprec_K y &\Leftrightarrow y - y' \notin \mathrm{int}K. \end{aligned}$$

The negative polar cone(or dual cone) K^* of K and the strict polar cone $(K^*)^\circ$ of K are defined respectively by

$$K^* = \{y^* \in Y^* : \langle y^*, y \rangle \geq 0 \text{ for all } y \in K\},$$

and

$$(K^*)^\circ = \{y^* \in Y^* : \langle y^*, y \rangle > 0 \text{ for all } y \in K \setminus l(K)\}.$$

Clearly, $(K^*)^\circ \subset K^* \setminus \{0\}$ since $K + K \setminus l(K) = K \setminus l(K)$. For $A \subset X$ the indicator function $\delta_A : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is defined by

$$\delta_A(x) = \begin{cases} 0 & x \in A, \\ +\infty & x \notin A. \end{cases}$$

Remark 2.1. Note that in a locally convex space Y, always there exists a convex cone with nonempty interior. Indeed, if U be a convex neighborhood of zero and $y \notin Y$, then it is sufficient to consider $K = \mathrm{cone}(U - y) \subset Y$.

Definition 2.1. The vector-valued map $F : X \longrightarrow Y$ is said to be K-convex iff, for all $x_1, x_2 \in X$ and $0 \leq \lambda \leq 1$, the following inequality

$$F(\lambda x_1 + (1 - \lambda)x_2) \preceq_K \lambda F(x_1) + (1 - \lambda)F(x_2),$$

holds. Also F is said to be K-convexlike iff for all $x_1, x_2 \in X$ and $0 \leq \lambda \leq 1$ there exists $x_3 \in X$ such that

$$F(x_3) \preceq_K \lambda F(x_1) + (1 - \lambda)F(x_2).$$

It is worth to mention that F is K-convexlike on a convex subset $C \subset X$ iff $F(C) + K$ be convex.

Definition 2.2. Let X and Y be topological linear spaces, Y be ordered by a convex cone $K \subset Y$, and $F : X \longrightarrow Y$ be a given map. For an arbitrary $\bar{x} \in X$, the set

$$\partial F(\overline{x}) := \{T \in L(X,Y)|\ T(x-\overline{x}) \preceq_K F(x) - F(\overline{x}),\ \ \forall x \in X\}$$

is called the strong subdifferential of F at $\overline{x}$. Also let $\varepsilon \in K$, then ε-subdifferential of F at $\bar{x}$ is defined as following

$$\partial_\varepsilon F(\bar{x}) := \{T \in L(X,Y)|T(x-\overline{x}) \preceq_K F(x) - F(\bar{x}) + \varepsilon,\ \ \forall x \in X\}\,.$$

We consider the following cone-constrained vector optimization problem as in [8] sometimes called DC vector optimization where refers to difference of two cone convex functions:

$$(P)\quad \begin{cases} K - \text{Min}\ (F(x) - G(x))\,, \\ \text{subject to}\ x \in C\ \text{and}\ H(x) - S(x) \in -D, \end{cases}$$

where $F, G : X \longrightarrow Y$ are K-convex and $S, H : X \longrightarrow Z$ are D-convex maps and C is a convex subset of X.

Definition 2.3. [8] Suppose that $\Omega := \{x \in C : H(x) - S(x) \in -D\}$ and $\varepsilon \in K$. An element $\bar{x} \in \Omega$ is called an ε-weak local Pareto minimal solution of problem (P) iff there exists a neighborhood U of $\bar{x}$ such that

$$F(\bar{x}) - G(\bar{x}) \in \varepsilon\text{WMin}(F-G)(U \cap \Omega),$$

i.e.,

$$(F-G)(U \cap \Omega) \subset F(\bar{x}) - G(\bar{x}) - \epsilon + Y \setminus -\text{int}K,$$

where

$$(F-G)(U \cap \Omega) = \{F(x) - G(x) : x \in U \cap \Omega\}.$$

Similarly, $\bar{x}$ is said to be an ε-proper local Pareto minimal solution of problem (P) iff there exists a neighborhood U of $\bar{x}$ such that

$$F(\overline{x}) - G(\overline{x}) \in \varepsilon\text{PMin}(F-G)(U \cap \Omega),$$

i.e., there exists a convex cone $K^{'} \subset Y$ with $K \setminus l(K) \subseteq \text{int}K^{'}$ such that

$$(F-G)(U \cap \Omega) \subset F(\overline{x}) - G(\overline{x}) - \varepsilon + Y \setminus -\text{int}K^{'}.$$

In the sequel we use the following well-known property, see [9].

Lemma 2.1. *Let K be a convex cone in topological vector space Y. Then the following assertion holds*

$$y \in \text{int}K \Rightarrow \langle y^*, y\rangle > 0, \qquad \forall y^* \in K^* \setminus \{0\}. \tag{2.1}$$

The following definition is based on metrizable topological vector space which is slightly different from Definition 3.1 in [8]. We note that X with the topology generated by metric d is a topological vector space.

Definition 2.4. A set valued $M : X \rightrightarrows L(X,Y)$ is said to be approximately pseudo-dissipative at $\bar{x}$ iff, for every $\epsilon \in \text{int}K$, one can find a neighborhood U of $\bar{x}$ such that

$$\forall x \in U,\ \ \exists T \in M(x),\ \ T^* \in M(\overline{x})\ \ \text{s.t.}\ (T - T^*)\,(x - \overline{x}) \preceq_K \varepsilon d(x,\overline{x}). \tag{2.2}$$

3 Sufficient optimality condition

In this part, first we review the Theorems 3.1 and 3.2 stated in [8], then we give an example which demonstrates these theorems are not correct.

(Theorem 3.1 [8]) Let $\overline{x} \in \Omega$. Assume that the set-valued maps $\partial_\epsilon G$ and ∂S are both approximately pseudo-dissipative at $\overline{x}$. If in addition, for any $T \in \partial_\epsilon G(\bar{x})$ and $L \in \partial S(\bar{x})$, there exist $y^* \in K^* \setminus \{0\}$ and $z^* \in D^*$ such that

$$\begin{cases} y^*oT + z^*oL \in \partial(y^*oF + z^*oH)(\overline{x}), \\ \langle z^*, H(\overline{x}) - S(\overline{x})\rangle = 0, \end{cases}$$

then $\bar{x}$ is an ϵ-weak local Pareto minimal solution of problem (P).

(Theorem 3.2, [8]) Let $\bar{x} \in \Omega$. Assume that the set-valued maps $\partial_\varepsilon G$ and ∂S are both approximately pseudo-dissipative at $\bar{x}$. If in addition, for any $T \in \partial_\varepsilon G(\overline{x})$ and $L \in \partial S(\overline{x})$ there exist $y^* \in (K^*)^\circ$ and $z^* \in D^*$ such that

$$\begin{cases} y^*oT + z^*oL \in \partial(y^*oF + z^*oH)(\overline{x}), \\ \langle z^*, H(\overline{x}) - S(\overline{x})\rangle = 0, \end{cases}$$

then $\bar{x}$ is an ε-proper local Pareto minimal solution of problem (P).

The following example shows that Theorems 3.1 and 3.2 and subsequent corollaries 3.1, 3.2, 3.3, 3.4, 3.5, 3.6 in [8] are not correct, and need several corrections.

Example 3.1. Let $X = \mathbb{R}, Y = Z = \mathbb{R}^2, C = [-1,1], K = D = [0,+\infty) \times [0,+\infty), \bar{x} = 0, \varepsilon = (0,0)$. Define $F, G, H, S : \mathbb{R} \to \mathbb{R}^2$ by

$$\begin{cases} F(x) = (x^4, x^2) \\ G(x) = (x^2, 2x^2) \\ H(x) = (x, -1) \\ S(x) = (x+1, 0). \end{cases}$$

Clearly F, G are K-convex and H, S are D-convex and

$$\Omega = \{x \in C : \ H(x) - S(x) \in -D\} = [-1, 1].$$

Also we have

$$\partial G_\varepsilon(x) = \{(2x, 4x)\} \quad \text{and} \quad \partial S(x) = \{(1,0)\}.$$

Since $\partial G_\varepsilon, \partial S$ are continuous then by Lemma 3 in [15] are approximately pseudo-dissipative at $\bar{x} = 0$. For given $T \in \partial G_\varepsilon(\bar{x})$ and $L \in \partial S(\bar{x})$, we let $z^* = 0, y^* \in K^* \setminus \{0\}$. One can easily check that

$$\begin{cases} \langle z^*, H(\bar{x}) - S(\bar{x})\rangle = \langle z^*, (-1,-1)\rangle = 0, \\ y^*oT + z^*oL = 0 \in \partial(y^*oF + z^*oH)\,(\bar{x}) = \partial\big(y^*o(x^4, x^2)\big)\,(0). \end{cases}$$

Observe that all hypotheses of Theorem 3.1 in [8] are satisfied, but $\bar{x}$ is not an ε-weak local Pareto minimal solution of problem (P). Indeed, for any neighborhood U of $\bar{x} = 0$ and $x \in U \cap \Omega$, one has

$$F(x) - G(x) - (F(\overline{x}) - G(\overline{x})) = (x^4 - x^2, -x^2) \in -\mathrm{int}K = (-\infty, 0) \times (-\infty, 0).$$

The following theorems are modifications of Theorems 3.1 and 3.2 in [8] respectively.

Theorem 3.1. *Let $\bar{x} \in \Omega$. Assume that the set-valued maps $\partial_\varepsilon G$ and ∂S are both approximate pseudo-dissipative at $\bar{x}$. If in addition, for any $(T, L) \in \partial_\varepsilon G(\overline{x}) \times \partial S(\overline{x})$ and $(\alpha, \beta) \in \mathrm{int}K \times \mathrm{int}D$ there exist $(y^*, z^*) \in K^*\backslash\{0\} \times D^*$ such that*

$$\begin{cases} \ y^* o(T-\alpha) + \ z^* o(L-\beta) \in \partial(y^* oF + \ z^* oH)(\overline{x}), \\ \langle z^*, H(\overline{x}) - S(\overline{x}) \rangle = 0, \end{cases}$$

then $\bar{x}$ is an ε-weak local Pareto minimal solution of problem (P).

Proof. By approximately pseudo-dissipativity of $\partial_\varepsilon G$ and ∂S at $\bar{x}$, for given $\alpha \in \mathrm{int}K$ and $\beta \in \mathrm{int}D$ there exist neighborhoods V_α and V_β of $\bar{x}$ such that (2.2) holds for $\partial_\varepsilon G$ and ∂S. Let $V = V_\alpha \cap V_\beta$. Hence

$$\begin{aligned} &\forall x \in V, \ \exists \left(T^{'}, T\right) \in \partial_\varepsilon G(x) \times \partial G(\overline{x}), \ \left(L^{'}, L\right) \in \partial S(x) \times \partial S(\overline{x}) \\ &\text{such that} \begin{Bmatrix} \left(T^{'} - T\right)(x - \overline{x}) \preceq_K \alpha d(x, \overline{x}), \\ \left(L^{'} - L\right)(x - \overline{x}) \preceq_D \beta d(x, \overline{x}). \end{Bmatrix}. \end{aligned} \tag{3.1}$$

We claim that for all $x \in V \cap \Omega$, there exist $y^* \in K^* \setminus \{0\}$ and $z^* \in D^*$ such that

$$\begin{aligned} &\langle y^*, F(x) - G(x) - (F(\overline{x}) - G(\overline{x})) + \varepsilon \rangle \\ &+ \langle y^*, \alpha(d(x, \overline{x}) - 1) \rangle + \langle z^*, \beta(d(x, \overline{x}) - 1) \rangle \geq 0. \end{aligned} \tag{3.2}$$

Fix $x \in V \cap \Omega$. Then by (3.1) there exists $T^{'} \in \partial_\varepsilon G(x)$ and $L^{'} \in \partial S(x)$, such that $\forall y \in X$ the following hold

$$\begin{cases} G(y) - G(x) - T^{'}(y - x) + \varepsilon \in K, \\ S(y) - S(x) - L^{'}(y - x) \in D. \end{cases} \tag{3.3}$$

Next let $y = \bar{x}$, we get

$$\begin{cases} G(\overline{x}) - G(x) - T^{'}(\overline{x} - x) + \varepsilon \in K, \\ S(\overline{x}) - S(x) - L^{'}(\overline{x} - x) \in D. \end{cases} \tag{3.4}$$

Since $T \in \partial_\varepsilon G(\bar{x})$ and $L \in \partial S(\bar{x})$, by the assumption there exists $(y^*, z^*) \in K^*\backslash\{0\} \times D^*$ such that

$$\begin{cases} \ y^* o(T-\alpha) + \ z^* o(L-\beta) \in \partial(y^* oF + \ z^* oH)(\overline{x}), \\ \langle z^*, H(\overline{x}) - S(\overline{x}) \rangle = 0. \end{cases} \tag{3.5}$$

Therefore

$$\begin{aligned} &\langle y^*, F(x) - F(\bar{x}) - T(x - \bar{x}) \rangle + \langle z^*, H(x) - H(\bar{x}) - T(x - \bar{x}) \rangle \\ &- \langle y^*, \alpha \rangle - \langle z^*, \beta \rangle \geq 0. \end{aligned} \tag{3.6}$$

By using the fact that $y^* \in K^*, z^* \in D^*$, and (3.4) we deduce that

$$\begin{cases} \langle y^*, G(\overline{x}) - G(x) - T^{'}(\overline{x} - x) + \varepsilon \rangle \geq 0, \\ \langle z^*, S(\overline{x}) - S(x) - L^{'}(\overline{x} - x) + \varepsilon \rangle \geq 0. \end{cases} \tag{3.7}$$

From (3.6) and (3.7) we obtain that

$$\begin{aligned} &\langle y^*, F(x) - G(x) - (F(\overline{x}) - G(\overline{x})) - (T^{'} - T)(x - \overline{x}) + \varepsilon \rangle \\ &+ \langle z^*, H(x) - S(x) - (H(\overline{x}) - S(\overline{x})) - (L^{'} - L)(x - \overline{x}) \rangle \\ &- \langle y^*, \alpha \rangle - \langle z^*, \beta \rangle \geq 0. \end{aligned} \tag{3.8}$$

Form $H(x) - S(x) \in -D$ we have

$$\langle z^*, H(x) - S(x) \rangle \leq 0.$$

In addition, using $\langle z^*, H(\bar{x}) - S(\bar{x}) \rangle = 0$ we get

$$\begin{aligned} &\langle y^*, F(x) - G(x) - (F(\overline{x}) - G(\overline{x})) + \varepsilon \rangle - \langle y^*, (T^{'} - T)(x - \overline{x}) \rangle \\ &- \langle z^*, (L^{'} - L)(x - \overline{x}) \rangle - \langle y^*, \alpha \rangle - \langle z^*, \beta \rangle \geq 0, \end{aligned} \tag{3.9}$$

by using (3.1) and $(y^*, z^*) \in K^* \setminus \{0\} \times D^*$, we obtain that

$$\begin{cases} \langle y^*, \alpha d(x, \overline{x}) - (T^{'} - T)(x - \overline{x}) \rangle \geq 0, \\ \langle z^*, \beta d(x, \overline{x}) - (L^{'} - L)(x - \overline{x}) \rangle \geq 0. \end{cases} \tag{3.10}$$

Next by combining (3.10) and (3.9) the following holds

$$\begin{aligned} &\langle y^*, F(x) - G(x) - (F(\overline{x}) - G(\overline{x})) + \varepsilon \rangle \\ &+ \langle y^*, \alpha(d(x, \overline{x}) - 1) \rangle + \langle z^*, \beta(d(x, \overline{x}) - 1) \rangle \geq 0. \end{aligned} \tag{3.11}$$

This completes the proof of (3.2). Next, X is metrizable, so there exists a neighborhood $U \subseteq V$ of $\bar{x}$ such that for all $y \in U$ we have $d(y, \bar{x}) \leq 1$. Assume that $y \in U \cap \Omega \subseteq V \cap \Omega$ be given, so there exists $y^* \in K^* \setminus \{0\}, z^* \in D^*$ such that (3.2) holds for $x = y$. On the other hand, using $\alpha \in \text{int}K, \beta \in \text{int}D$ follows that

$$\langle y^*, \alpha(d(y, \overline{x}) - 1) \rangle \leq 0 \quad \text{and} \quad \langle z^*, \beta(d(y, \overline{x}) - 1) \rangle \leq 0. \tag{3.12}$$

Combining (3.2) with (3.12), yields

$$\langle y^*, F(y) - G(y) - (F(\overline{x}) - G(\overline{x})) + \varepsilon \rangle \geq 0. \tag{3.13}$$

Finally by Lemma 2.1 one has

$$F(y) - G(y) - (F(\overline{x}) - G(\overline{x})) + \varepsilon \notin -\text{int}K,$$

but since $y \in U \cap \Omega$ was arbitrary, thus $\bar{x}$ is a $\varepsilon-$weak local Pareto minimal solution of problem (P). This complete the proof. □

By similar argument as the previous theorem, we can obtain the following theorem for sufficient optimality condition.

Theorem 3.2. *Let $\bar{x} \in \Omega$. Assume that the set-valued maps $\partial_\varepsilon G$ and ∂S are both approximately pseudo-dissipative at $\bar{x}$. If in addition, for any $(T, L) \in \partial_\varepsilon G(\overline{x}) \times \partial S(\overline{x})$ and $(\alpha, \beta) \in \text{int}K \times \text{int}D$ there exist $(y^*, z^*) \in (K^*)^\circ \times D^*$ such that*

$$\begin{cases} y^* o(T - \alpha) + z^* o(L - \beta) \in \partial(y^* oF + z^* oH)(\overline{x}), \\ \langle z^*, H(\overline{x}) - S(\overline{x}) \rangle = 0, \end{cases}$$

then $\bar{x}$ is an ε-proper local Pareto minimal solution of problem (P).

4 Necessary Optimality Conditions

In this section, we provide sufficient optimality conditions for an ε-weak Pareto minimal solution and an ε-proper Pareto minimal solution for the cone-constrained vector optimization problem (P). Here the objective function and constraint set are given as differences of two vector-valued maps. Our results are corrections of Theorems 4.1 and 4.2 in [8].

Theorem 4.1. *[8] Let $\varepsilon \in K$ and $\bar{x} \in \Omega$. If the vector-valued map $F : X \longrightarrow Y$ is a K-convex map, the vector-valued map $H : X \longrightarrow Z$ is a D-convex map, and $\bar{x}$ is an ε-proper local minimal solution of (P), then there exist $y^* \in (K^*)^\circ \cup \{0\}$ and $z^* \in D^*$ and $(y^*, z^*) \neq (0_{Y^*}, 0_{Z^*})$ such that*

$$\begin{cases} (y^* o\partial G + z^* o\partial H)(\bar{x}) \cap \partial_{\langle y^*,\varepsilon\rangle}(y^* oF + z^* oH + \delta_{U\cap C})(\bar{x}), \\ \langle z^*, H(\overline{x}) - S(\overline{x})\rangle = 0, \end{cases}$$

where U is a neighborhood of $\bar{x}$.

The following example shows that Theorems 4.1 and 4.2 and subsequent corollaries in [8] are not correct.

Example 4.1. Take $X = \mathbb{R}, Y = Z = \mathbb{R}, C = [-1, 1], K = D = [0, +\infty), \bar{x} = 0, \varepsilon = 0$. Consider $F, G, H, S : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} -1 & x \neq 0, \\ 0 & x = 0. \end{cases} \qquad G(x) = \begin{cases} -2 & x \neq 0, \\ 0 & x = 0. \end{cases} \qquad H(x) = x - 1, \ \ S(x) = x.$$

Clearly F, G are K-convex and H, S are D-convex, with $\partial G(\overline{x}) = \{0\}, \partial H(\overline{x}) = \{1\}$. One can verify that

$$\Omega = \{x \in C : \ H(x) - S(x) \in -D\} = [-1, 1].$$

Hence for a neighborhood U of $\bar{x} = 0$ we have

$$F(x) - G(x) - (F(\overline{x}) - G(\overline{x})) + \varepsilon \notin -\operatorname{int} K, \quad \forall x \in U \cap \Omega,$$

which implies that $\bar{x} = 0$ is ε-weak local minimal solution of (P). If

$$\begin{cases} (y^* o\partial G + z^* o\partial H)(\bar{x}) \subset \partial_{\langle y^*,\varepsilon\rangle}(y^* oF + z^* oH + \delta_{U\cap C})(\bar{x}), \\ \langle z^*, H(\bar{x}) - S(\bar{x})\rangle = 0, \end{cases}$$

then

$$\langle z^*, H(\overline{x}) - S(\overline{x})\rangle = \langle z^*, -1\rangle = 0,$$

which implies that $z^* = 0$. Therefore one has

$$(y^* o\partial G + z^* o\partial H)(\bar{x}) = 0 \in \partial (y^* oF + \delta_{U\cap C})(\bar{x}),$$

which gives $F(x) \geq 0$ for all $x \in U \cap \Omega$, that is contradiction.

We generalize the result (Theorem 3.3 in [13]) Farkas-Minskowski for D-convexlike single value functions.

Lemma 4.2. *Let C be a convex subset of X. If the map $F : C \longrightarrow Y$ is K-convexlike and $G : C \longrightarrow Z$ is D-convexlike and the system*

$$\begin{cases} F(x) \in -\text{int} K, \\ G(x) \in -\text{int} D, \end{cases}$$

has no solution in C, then there exist $(y^*, z^*) \in K^* \times D^*$ with $(y^*, z^*) \neq (0, 0)$, such that

$$\langle y^*, F(x)\rangle + \langle z^*, G(x)\rangle \geq 0 \qquad \forall x \in C.$$

Proof. We can easily prove that $F(C)+K$ and $G(C)+D$ are convex sets. Define the set valued map $g: C \rightrightarrows X \times Y$ by

$$g(x) = (F(x)+K) \times (G(x)+D).$$

One can check that $g(C) \cap \mathrm{int}((-K) \times (-D)) = \emptyset$. Next, $g(C)$ is convex set hence, by the separation theorem, there exist a non zero $(y^*, z^*) \in K^* \times D^*$ and $\alpha \in \mathbb{R}$ such that for all $(k, d) \in (K, D), x \in C$ we have

$$\langle y^*, -k \rangle + \langle z^*, -d \rangle \leq \alpha \leq \langle y^*, F(x)+k \rangle + \langle z^*, G(x)+d \rangle .$$

Choosing $k = d = 0$ yields

$$\langle y^*, F(x) \rangle + \langle z^*, G(x) \rangle \geq 0, \qquad \forall x \in C.$$

□

In the rest of this section, we present modification of Theorem 4.1 and 4.2 (Necessary optimality conditions) in [8] by assuming convex-like condition which is weaker than convexity.

Theorem 4.3. *Let $\bar{x} \in \Omega$. If the vector-valued map $F : X \longrightarrow Y$ is a K-convexlike map, the vector-valued map $H : X \longrightarrow Z$ is a D-convexlike map, and $\bar{x}$ is an ε-weak local minimal solution of (P), then there exist $(y^*, z^*) \in K^* \times D^*$ and $(y^*, z^*) \neq (0_{Y^*}, 0_{Z^*})$ such that*

$$(y^* o \partial G + z^* o \partial H)(\bar{x}) \cap \partial_{\langle y^*, \varepsilon \rangle}(y^* o F + z^* o H + \delta_{U \cap C})(\bar{x}) \neq \emptyset, \tag{4.1}$$

where U is a neighborhood of $\bar{x}$.

Proof. Let $\bar{x} \in \Omega$ and $\varepsilon \in K$. Since $\bar{x}$ is an ε-weak local minimal solution of (P), there exists a neighborhood U of $\bar{x}$ such that for all $x \in U \cap C$,

$$F(x) - G(x) - (F(\bar{x}) - G(\bar{x})) + \varepsilon \notin -\mathrm{int}K.$$

Now suppose that $T \in \partial G(\bar{x})$ and $L \in \partial H(\bar{x})$ be arbitrary elements. Note that F is K-convexlike and G is D-convexlike mapping, thus $F(\cdot) - F(\bar{x}) - T(\cdot - \bar{x}) + \varepsilon$ is K-convexlike mapping and $H(\cdot) - H(\bar{x}) - L(\cdot - \bar{x})$ is D-convexlike mapping. We prove that the system

$$\begin{cases} F(x) - F(\bar{x}) - T(x - \bar{x}) + \varepsilon \in -\mathrm{int}K \\ H(x) - H(\bar{x}) - L(x - \bar{x}) \in -\mathrm{int}D, \end{cases} \tag{4.2}$$

has no solution in $U \cap C$. Arguing by contradiction, assume that there exists a solution $x_0 \in U \cap C$ of (4.2). Thus

$$\begin{cases} F(x_0) - F(\bar{x}) - T(x_0 - \bar{x}) + \varepsilon \in -\mathrm{int}K, \\ H(x_0) - H(\bar{x}) - L(x_0 - \bar{x}) \in -\mathrm{int}D. \end{cases} \tag{4.3}$$

Since $T \in \partial G(\bar{x})$ and $L \in \partial S(\bar{x})$, we have

$$G(x) - G(\bar{x}) - T(x - \bar{x}) \in K \quad \forall x \in X,$$

and

$$S(x) - S(\bar{x}) - L(x - \bar{x}) \in D \quad \forall x \in X.$$

Let $x = x_0$,thus one has

$$\begin{cases} -G(x_0) + G(\bar{x}) + T(x_0 - \bar{x}) \in -K, \\ -S(x_0) + S(\bar{x}) + L(x_0 - \bar{x}) \in -D. \end{cases} \tag{4.4}$$

Next, note that

$$-K - \text{int}K = -\text{int}K, \quad -D - \text{int}D = -\text{int}D, \quad H(\bar{x}) - S(\bar{x}) \in -D.$$

Then combining (4.3) with (4.4), gives us that

$$\begin{cases} F(x_0) - G(x_0) - (F(\bar{x}) - G(\bar{x})) + \varepsilon \in -\text{int}K, \\ H(x_0) - S(x_0) \in -\text{int}D, \end{cases}$$

this contradicts the assumption $\bar{x}$ is an ε-weak local minimal solution of (P). Hence, the system (4.2) has no solution. Now by Lemma (4.2) there exists $(y^*, z^*) \neq (0, 0)$ such that for all $x \in U \cap C$,

$$\langle y^*, F(x) - F(\bar{x}) - T(x - \bar{x}) + \varepsilon\rangle + \langle z^*, H(x) - H(\bar{x}) - L(x - \bar{x})\rangle \geq 0.$$

Consequently,

$$(y^* oF + z^* oH)(x) - (y^* oF + z^* oH)(\bar{x}) + \langle y^*, \varepsilon\rangle - (y^* oT + z^* oL)(x - \bar{x}) \geq 0.$$

Hence it follows that

$$(y^* oT + z^* oL)(\bar{x}) \in \partial_{\langle y^*, \varepsilon\rangle}(y^* oF + z^* oH + \delta_{U\cap C})(\bar{x}).$$

This completes the proof. □

By similar proof of the previous theorem we can obtain the following theorem for necessary optimality condition.

Theorem 4.4. *Let $\bar{x} \in \Omega$. If the vector-valued map $F : X \longrightarrow Y$ is a K-convexlike map, the vector-valued map $H : X \longrightarrow Z$ is a D-convexlike map, and $\bar{x}$ is an ε-proper local minimal solution of (P), then there exist $y^* \in (K^*)^\circ \cup \{0\}$ and $z^* \in D^*$ and $(y^*, z^*) \neq (0_{Y^*}, 0_{Z^*})$ such that*

$$(y^* o\partial G + z^* o\partial H)(\bar{x}) \cap \partial_{\langle y^*, \varepsilon\rangle}(y^* oF + z^* oH + \delta_{U\cap C})(\bar{x}) \neq \emptyset. \tag{4.5}$$

Remark 4.1. To the best of our knowledge, there is no result on the existence of necessary optimality conditions of problem (P) under K-convexlike assumption. Therefore, Theorems 4.3 and 4.4 are new in the literature.

5 Acknowledgment

F. Bozorgnia was supported by the Portuguese National Science Foundation through FCT fellowship SFRH/BPD/33962/2009.

The authors are very grateful to anonymous referees for carefully reading their manuscript and for several comments and suggestions which helped them to improve the paper.

References

[1] N. Dinh, B. Mordukhovich, T. T. A. Nghia, *Sufficient optimality condition for vector optimization problems under D.C data.* J. Glob. Optim., 28 (2010), 101–138.

[2] N. Dinh, B. S. Mordukhovich, T. T. A. Nghia,*Qualification and optimality conditions for DC programs with infinite constraints.* Acta Math. Vietnam., 34 (2009), 125–155.

[3] D. H. Fang, C. Li, K. F. Ng, *Constraint qualifications for optimality conditions and total Lagrange dualities in convex ifinite programming.* Nonlinear Anal., 73 (2010), 1143–1159.

[4] D. H. Fang, M. D. Wang, X. P. Zhao, *The strong duality for DC optimization problems with composite convex functions.* J. Nonlinear Convex Anal., 16 (2015), 1337–1352.

[5] D. H. Fang, X. P. Zhao, *Local and global optimality conditions for DC infinite optimization problems.* Taiwanese J. Math., 18 (2014), 817–834.

[6] N. Gadhi, A. Metrane, *Sufficient optimality condition for vector optimization problems under D.C. data.* J. Glob. Optim., 28 (2004), 55–66.

[7] N. Gadhi, S. Dempe, *Necessary Optimality Conditions and a New Approach to Multiobjective Bilevel Optimization Problems.* J. Optim. Theory Appl 17, (2012), 1-15.

[8] X. L. Guo, S. J. Li, *Optimality conditions for vector optimization problems with difference of convex maps.* J. Optim. Theory Appl., 162 (2014), 821–844.

[9] J. Johannes *Vector optimization theory and application, and Extensions.* Springer, 2010.

[10] V. Jeyakumar, *Asymptotic dual conditions characterizing of optimality for convex programs.* J. Optim. Theory Appl., 93 (1997), 153–165.

[11] M. Laghdir, *Optimality conditions in DC constrained optimization.* Acta Math. Vietnam. **30** (2), (2005), 169-179.

[12] G. Li, X. Q. Yang, Y. Y. Zhou, *Stable strong and total parametrized dualities for DC optimization problems in locally convex spaces.* J. Ind. Manag. Optim., 9 (2013), 671–687.

[13] L.J. Lin, *Optimization of set-valued functions.* J. Math. Anal. Appl. 186 (1994), 30-51.

[14] J. E. Martinez-Legaz, M. Volle, *Duality in D.C. programming: the case of several D.C. constraints.* J. Math. Anal. Appl., 237 (1999), 657–671.

[15] J.P. Penot, *The directional subdifferential of the difference of two convex functions.* J. Glob. Optim. 49 (2011), 505-519.

[16] X. K. Sun, X. L. Guo, Y. Zhang, *Fenchel-Lagrange duality for DC programs with composite functions.* J. Nonlinear Convex Anal., 16 (2015), 1607–1618.

[17] X. K. Sun, S. J. Li, D. Zhao, *Duality and Farkas-type results for DC infinite programming with inequality constraints.* Taiwanese J. Math., 17 (2013), 1227–1244.

[18] H. Tuy, *Convex Analysis and Global Optimization.* Kluwer, Dordrecht (1998).

[19] M. Volle, *Duality principles for optimization problems dealing with the difference of vector-valued convex mappings.* J. Optim. Theory Appl., 114 (2002), 223– 241.