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Dynamical Systems in Categories

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In this article we establish a bridge between dynamical systems, including topological and measurable dynamical systems as well as continuous skew product flows and nonautonomous dynamical systems; and coalgebras in categories having all finite products. We introduce a straightforward unifying definition of abstract dynamical system on finite product categories. Furthermore, we prove that such systems are in a unique correspondence with monadic algebras whose signature functor takes products with the time space. We substantiate that the categories of topological spaces, metrisable and uniformisable spaces have exponential objects w.r.t. locally compact Hausdorff, σ-compact or arbitrary time spaces as exponents, respectively. Exploiting the adjunction between taking products and exponential objects, we demonstrate a one-to-one correspondence between monadic algebras (given by dynamical systems) for the left-adjoint functor and comonadic coalgebras for the other. This, finally, provides a new, alternative perspective on dynamical systems.

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1 Introduction

Dynamical behaviour of state based systems is a central research topic common to both computer science and dynamical systems theory. The former discipline studies such structures in a multitude of manifestations, e.g. as infinite state transition systems ([Min67, Rab69, Tho90, Tho06, Tho09]), Kripke structures ([Kri63, BCG88]), Petri nets ([Esp97]), event systems ([Tor95, CL08]), finite state machines, various kinds of automata ([Kle56, RS59, Sch65, MP71, DKV09]), Turing machines ([Min61]), etc., the latter one devotes its main attention to the understanding and the description of different facets, such as long-time behaviour, of complex dynamical systems on topological, metric, measurable or probability spaces (see e.g. [GH55, Sel71, AG01, AHK03, Arn98]).

A problem common to both fields is to describe the transition of states, which may or may not be deterministic. For that the two fields have developed specific concepts and methods. Dynamical systems theory, for example, applies ideas from topology, functional analysis, measure theory or differential equations. Computer science exploits concepts from the theory of formal languages or various modal and timed logics to describe dynamical behaviour.

Already during the 1970ies the relationship between automata theory and linear control systems has been studied ([AM74, AM80]) in category theoretic language. Later, Rutten in [Rut00] proposed the theory of coalgebra, a branch of category theory already used for a uniform treatment of different sorts of state based systems in programming semantics, to investigate techniques and concepts of computer science, as well as dynamical systems theory, in a common setting.

In this paper we follow Rutten's suggestion and explore which types of dynamical systems are suitable to be modelled as coalgebras in appropriate categories. For those we give a detailed account of the translation process, involving certain monadic algebras as an intermediate step. Finally, we find out that our construction can be regarded as an instance of a quite general category theoretic result on the relationship of algebras and coalgebras. In the end the coalgebraic point of view offers one free parameter: the signature. With only small modifications dynamical behaviour containing observations or non-determinism can be described. Making such variability available for the study of classical dynamical systems constitutes one motivation for the origination of this paper.

The structure of the text is as follows: to ease readability for researchers from the areas of dynamical systems and computer science, and to keep the presentation of the material mostly self-contained, we gather in Section 2 basic concepts from topology, measure theory and category theory, as well as a number of variants of dynamical systems appearing in the literature. In Section 3 we pursue a straightforward modelling of these existing notions in so-called finite product categories. In this context we observe a very general connection between nonautonomous dynamical systems and dynamical systems on product spaces. In Section 4, finally, we translate the coined definitions into the language of coalgebra.

In this respect we first exhibit a connection to monadic algebras w.r.t. an endofunctor that takes products with a time space (Subsections 4.1 and 4.2). As a byproduct we recognise the notion of topological conjugacy as the natural category theoretic concept of isomorphism between algebras. In a similar way other category theoretic constructions can become meaningful for particular cases of dynamical systems.

Subsequently, we verify that for our main examples the mentioned endo-functor fulfils a specific property, known as left-adjointness. Based on this general assumption, we demonstrate how to transform monadic algebras in a one-to-one fashion into so-called comonadic coalgebras. Thereby we exhibit the particular transformation of monadic algebras arising from dynamical systems into coalgebras as a special case of a well-understood abstract result in category theory: if F, G is a pair of endo-functors where F is left- and G is right-adjoint, then monads for F and comonads for G, as well as monadic F-algebras and comonadic G-coalgebras, are in bijective correspondence (cf. Propositions 4.5.3, 4.5.4 and 4.5.5).

Due to the presence of adjointness, for the specific structures arising from dynamical systems it makes no difference if they are considered as algebras or as coalgebras. However, from the coalgebraic understanding of transition systems in computer science, there are coalgebras known that look quite similar to those stemming from our construction, yet which may fail to satisfy the adjointness condition. Thus, they lack a corresponding equivalent on the side of algebras, i.e. a definition in the standard sense of dynamical systems, but they still represent dynamical behaviour. We pose it as a problem for future investigations to further discover all benefits coming from the realm of coalgebra to e.g. topological or symbolic dynamics.

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2 Preliminaries and Notation

In this part we will make the reader familiar with some notation and conventions used throughout the following text. We start with recalling standard concepts from topology and measure theory. As our aim for later is to build a bridge from dynamical systems to coalgebra, the further necessary prerequisites are twofold: first we will introduce fundamental concepts from category theory needed for abstractly modelling dynamical systems and to understand the translation process from the standard definition of dynamical system to the field of coalgebra. Second, we present key definitions from the wide-spread theory of dynamical systems to see which examples fall under the scope of the method to be presented later on.

To summarise some basic notation, we write $\emptyset$ for the *empty set*, and $\mathfrak{P}(X)$ for the *powerset* of some set X. Moreover, we use $X \subseteq Y$ to express *set inclusion*, as opposed to $X \subset Y$ for *proper set inclusion.* If $f\colon X \to Y$ is a function from X to Y and $U \subseteq X$ and $V \subseteq Y$ are subsets, we write $f[U]$ for the *image* of U under f and $f^{-1}[V] := \{x \in X \mid f(x) \in V\}$ for the *preimage* of V w.r.t. f. Furthermore, we use $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ to denote the sets of *natural numbers including zero*, *integers* and *reals*, respectively. For $\mathbb{T} \in \{\mathbb{Z}, \mathbb{R}\}$ we abbreviate by $\mathbb{T}_{\geq 0}$ and $\mathbb{T}_{\leq 0}$ the respective non-negative and non-positive numbers.

2.1 Preliminaries related to topology and measure theory

Dynamical systems in topological spaces constitute an example of special importance in the following section. We therefore begin by recollecting some standard notions from topology. The more involved concepts occurring thereafter will mainly be needed in Subsection 4.4.

As usual a *topological space* is a pair $\boldsymbol{X} = (X, \tau)$ where X is a set and $\tau \subseteq \mathfrak{P}(X)$ is a *topology on* X, i.e. a collection of subsets of X that is closed under finite intersections and arbitrary unions. The members of τ are called *open sets* of $\boldsymbol{X}$, the elements of X are often referred to as *points*. If $\tau = \mathfrak{P}(X)$, i.e. the largest possible topology on X, the topology τ and the space $\boldsymbol{X}$ are said to be *discrete*. Contrarily, the least topology on X is $\tau = \{\emptyset, X\}$, which is called *indiscrete topology*. A subcollection $\mathcal{U} \subseteq \mathfrak{P}(X)$ is called a *base* for the topology τ if $\tau = \{\bigcup \mathcal{V} \mid \mathcal{V} \subseteq \mathcal{U}\}$. A set $\mathcal{V} \subseteq \mathfrak{P}(X)$ is a *subbase* of τ if $\{\bigcap \mathcal{V}' \mid \mathcal{V}' \subseteq \mathcal{V} \text{ finite}\}$ is a base of τ. A topological space for whose topology there exists a countable base is said to be *second-countable* or *completely separable*.

For a subset $V \subseteq X$ its *interior*, denoted by $\mathrm{int}_{\boldsymbol{X}}(V)$, is the largest open set contained in V, i.e. $\mathrm{int}_{\boldsymbol{X}}(V) = \bigcup \{U \in \tau \mid U \subseteq V\}$.

If $x \in X$ is a point, then a subset $V \subseteq X$ is called a *neighbourhood* of x if there is some open set $U \in \tau$ such that $x \in U \subseteq V$. A neighbourhood is said to be *open* if it belongs to τ itself. The collection of all neighbourhoods of a point $x \in X$ is denoted by $\mathcal{U}_x(\boldsymbol{X})$. A subcollection $\mathcal{V} \subseteq \mathcal{U}_x(\boldsymbol{X})$ is called a *neighbourhood base at the point* x if for every $U \in \mathcal{U}_x(\boldsymbol{X})$ there exists some $V \in \mathcal{V}$ such that $V \subseteq U$. Thus, $\mathcal{V} \subseteq \mathcal{U}_x(\boldsymbol{X})$ is a neighbourhood base at x if and only if $\mathcal{U}_x(\boldsymbol{X}) = \{U \subseteq X \mid \exists V \in \mathcal{V} \colon V \subseteq U\}$.

For topological spaces $\boldsymbol{X} = (X, \tau)$ and $\boldsymbol{Y} = (Y, \sigma)$, a map $f \colon X \to Y$ is said to be *continuous (w.r.t. $\boldsymbol{X}$ and $\boldsymbol{Y}$)*, or *τ-σ-continuous*, if $f^{-1}[U] \in \tau$ for any $U \in \sigma$. A map $f \colon X \to Y$ is said to be *continuous at a point* $x \in X$ if $f^{-1}[U] \in \mathcal{U}_x(\boldsymbol{X})$ for any $U \in \mathcal{U}_{f(x)}(\boldsymbol{Y})$. Clearly, $f \colon X \to Y$ is continuous (w.r.t. $\boldsymbol{X}$ and $\boldsymbol{Y}$) if and only if it is continuous at any point $x \in X$. We collect all τ-σ-continuous functions $f \colon X \to Y$ in the set $C(\boldsymbol{X}, \boldsymbol{Y})$. If $f \in C(\boldsymbol{X}, \boldsymbol{Y})$ is bijective and its inverse is continuous, too, then f is called a *homeomorphism between $\boldsymbol{X}$ and $\boldsymbol{Y}$*.

A topological space $\boldsymbol{X}$ is said to be *Hausdorff* if any two distinct points $x, y \in X$, $x \neq y$, can be separated by disjoint (open) neighbourhoods, i.e. if there exist $U \in \mathcal{U}_x(\boldsymbol{X})$ and $V \in \mathcal{U}_y(\boldsymbol{X})$ such that $U \cap V = \emptyset$. A subset $K \subseteq X$ is called *compact* if for any $\mathcal{U} \subseteq \tau$ such that $K \subseteq \bigcup \mathcal{U}$ there exists a finite subset $\mathcal{U}' \subseteq \mathcal{U}$ such that $K \subseteq \bigcup \mathcal{U}'$. We denote the set of all compact subsets of $\boldsymbol{X}$ by $K(\boldsymbol{X})$. The topological space $\boldsymbol{X}$ is called *compact* if $X \in K(\boldsymbol{X})$. Moreover, we call the space $\boldsymbol{X}$ *locally compact* if, for any point $x \in X$, it possesses a neighbourhood base $\mathcal{V} \subseteq \mathcal{U}_x(\boldsymbol{X})$ satisfying $\mathcal{V} \subseteq K(\boldsymbol{X})$.

A special case of locally compact spaces are *σ-compact spaces* introduced in the following definition.

2.1.1 Definition. A topological space $\boldsymbol{X}$ is said to be *σ-compact* if it admits a *countable exhaustion by compact subsets*. That is, there exists a sequence $(K_n)_{n \in \mathbb{N}}$ of subsets of X such that

(1) K_n is compact for each $n \in \mathbb{N}$,

(2) $K_n \subseteq \mathrm{int}_{\boldsymbol{X}}(K_{n+1})$ for each $n \in \mathbb{N}$,

(3) $X = \bigcup_{n\in\mathbb{N}} K_n$. $\Diamond$

A σ-compact Hausdorff space is necessarily locally compact. In fact, the following lemma holds (we refer to [vQ79, Satz 8.19(b), p. 111] for a proof):

2.1.2 Lemma. *For every Hausdorff space $\boldsymbol{X}$ the following are equivalent:*

(i) $\boldsymbol{X}$ *is σ-compact,*

(ii) $\boldsymbol{X}$ *is locally compact and there exists a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of X such that $X = \bigcup_{n\in\mathbb{N}} K_n$.*

One of the main constructions concerning topological spaces that we will use later is that of *product spaces*. If I is an index set and for each $i \in I$ the pair $\boldsymbol{X}_i = (X_i, \tau_i)$ is a topological space, then we may define a topology τ on the Cartesian product $X := \prod_{i\in I} X_i$ by the subbase $\bigcup_{i\in I} \left\{ \hat{U}_i \;\middle|\; U_i \in \tau_i \right\}$. Here $\hat{U}_i$ stands for the product $\prod_{j\in I} V_j$ where $V_j = U_i$ if $j = i$ and $V_j = X_j$, otherwise. In this way τ is the least topology on X such that all coordinate projections $p_i \colon X \to X_i$, $i \in I$, defined by $p_i\left((x_j)_{j\in I}\right) := x_i$ are continuous. We write $\prod_{i\in I} \boldsymbol{X}_i$ for the pair (X, τ) and call it *product space* of $(\boldsymbol{X}_i)_{i\in I}$. We mention that for $I = \emptyset$ the resulting space is the indiscrete space on the one-element set.

Moreover, in the following we need spaces with a richer structure than just a topology, namely metric and uniform spaces. A *metric space*, as usual, is a pair (X, d) where X is a set and $d \colon X^2 \to \mathbb{R}_{\geq 0}$ is a *metric*, i.e. a map satisfying $d(x, y) = d(y, x)$, $d(x, y) = 0$ exactly if $x = y$, and $d(x, z) \leq d(x, y) + d(y, z)$, each requirement for all $x, y, z \in X$. With every metric space we can associate an underlying topological space (X, τ) given by the base $\{U(x, \varepsilon) \mid x \in X, \varepsilon \in \mathbb{R}_{>0}\}$ where $U(x, \varepsilon) := \{y \in X \mid d(x, y) < \varepsilon\}$ denotes the *open ball* around $x \in X$ with radius $\varepsilon > 0$. Topological spaces arising in this way are called *metrisable*.

A slight generalisation of metric spaces are *uniform spaces*.

2.1.3 Definition. A *uniform space* (X, Θ) is a set X equipped with a non-empty family Θ of subsets of the Cartesian product $X \times X$ (Θ is called the *uniform structure* or *uniformity* of X and its elements *entourages*) that satisfies the following axioms:

(1) Every entourage $U \in \Theta$ is *reflexive*, i.e. $U \supseteq \{(x, x) \mid x \in X\}$.

(2) Θ is *upwards closed*, i.e. if $U \in \Theta$ and $U \subseteq V \subseteq X \times X$, then also $V \in \Theta$.

(3) Θ is *closed w.r.t. finite intersections*, i.e. $U, V \in \Theta$ always implies $U \cap V \in \Theta$.

(4) If $U \in \Theta$, then there exists $V \in \Theta$ such that[1], whenever $(x,y),(y,z) \in V$, then $(x,z) \in U$.[2]

(5) Θ is *closed under inverses (transposes)*: for every $U \in \Theta$, always the *inverse entourage* $U^{-1} = \{(y,x) \mid (x,y) \in U\}$ is a member of Θ, as well.

For $x \in X$ and $U \in \Theta$, we write $U[x]$ to indicate $\{y \in X \mid (x,y) \in U\}$. $\Diamond$

Every uniform space (X,Θ) gives rise to a topological space on X, by defining a subset $U \subseteq X$ to be open if and only if for every $x \in U$ there exists an entourage $V \in \Theta$ such that $V[x] \subseteq U$.

In this topology, the neighbourhood filter of a point x is $\{V[x] \mid V \in \Theta\}$. The topology defined by a uniform structure is said to be *generated by the uniformity*. Topological spaces whose topology is induced by a uniformity are said to be *uniformisable*.

Another important class of structures are *measurable spaces*, i.e. pairs (X,Σ) where X is a set and Σ is a *σ-algebra* on X, which is a non-empty collection $\Sigma \subseteq \mathfrak{P}(X)$ being closed w.r.t. countable unions and intersections, and complementation. Clearly, arbitrary intersections of σ-algebras on X form again a σ-algebra wherefore there always exists a least σ-algebra on X containing a given collection of subsets $\mathcal{U} \subseteq \mathfrak{P}(X)$, said to be *generated by $\mathcal{U}$*. Especially, $\mathcal{U} = \emptyset$ generates the least possible σ-algebra on X, namely $\{\emptyset, X\}$.

A map $f\colon X \to Y$ between the carrier sets of two measurable spaces $\mathbb{X} = (X,\Sigma)$ and $\mathbb{Y} = (Y,\Omega)$ is said to be *measurable* if we have $f^{-1}[U] \in \Sigma$ for all $U \in \Omega$.

If $\boldsymbol{X} = (X,\tau)$ is a topological space, then the σ-algebra generated by the topology τ is called *Borel σ-algebra* belonging to $\boldsymbol{X}$.

Furthermore, as for topological spaces we need to deal with products of measurable spaces $\mathbb{X}_i = (X_i,\Sigma_i)$, $i \in I$. We simply put $\prod_{i\in I}\mathbb{X}_i := (\prod_{i\in I} X_i, \Sigma)$ where Σ is generated by the collection $\bigcup_{i\in I}\left\{\hat{U}_i \,\middle|\, U_i \in \Sigma_i\right\}$ and the sets $\hat{U}_i$ are defined analogously as for products of topological spaces (cf. page 6). We call Σ *product σ-algebra* and $\prod_{i\in I}\mathbb{X}_i$ *product space* of $(\mathbb{X}_i)_{i\in I}$. The definition above ensures that all projection maps $p_i\colon \prod_{j\in I} X_j \to X_i$, $i \in I$, are indeed measurable.

Measurable spaces (X,Σ) form the basis to define *measures*, which are mappings $\mu\colon \Sigma \to \overline{\mathbb{R}}_{\geq 0}$ into the set $\overline{\mathbb{R}}_{\geq 0} = [0,\infty]$ of *affinely extended* non-negative real numbers satisfying $\mu(\emptyset) = 0$ and the axiom of *σ-additivity*: for every countable sequence $(U_i)_{i\in I} \in \Sigma^{\mathbb{N}}$ of pairwise disjoint measurable sets, one requires the equality $\mu(\bigcup_{i\in\mathbb{N}} U_i) = \sum_{i\in\mathbb{N}}\mu(U_i)$ to hold. A triple (X,Σ,μ) such that (X,Σ) is a measurable space and μ is a measure on (X,Σ) constitutes a *measure space*. If $\mu(X) = 1$, the map μ is called a *probability measure* and (X,Σ,μ) a *probability space*.

[1] It follows from reflexivity of V (see condition (1)) that $V \subseteq U$.

[2] Condition (4) may also be rewritten as follows: for every $U \in \Theta$ there is some $V \in \Theta$ such that $V \subseteq V \circ V \subseteq U$, where $V \circ V = \{(x,z) \in X \times X \mid \exists y \in X\colon (x,y),(y,z) \in V\}$ denotes the binary relational product of V with itself.

Particularly simple examples of (probability) measures are so-called *Dirac measures*: for a given element $x \in X$ the *Dirac measure* δ_x *centred in* x maps a measurable set $U \in \Sigma$ to 1 if $x \in U$ and to 0, otherwise.

If (X, Σ) and (Y, Ω) are measurable spaces and $f\colon (X, \Sigma) \to (Y, \Omega)$ is a measurable map between them, then every measure μ on (X, Σ) induces one on (Y, Ω), the *push-forward measure* $\mu \circ f^{-1}$. By definition it satisfies $(\mu \circ f^{-1})(V) := \mu\left(f^{-1}[V]\right)$ for every $V \in \Omega$. For measure spaces (X, Σ, μ) and (Y, Ω, ν) a measurable map $f\colon (X, \Sigma) \to (Y, \Omega)$ is called *measure preserving* if $\nu = \mu \circ f^{-1}$.

2.2 Basic notions from category theory

Driven by the wish to keep the presentation of the material as self-contained as possible, we outline here a collection of fundamental concepts from category theory, always with a view on applications to dynamical systems. Of course, this cannot replace a look in a standard introductory monograph on category theory such as [AHS06] or [Awo10]. In the following we will cover concepts such as category, monomorphism, epimorphism, isomorphism, terminal object, product, functor, natural transformation, natural equivalence, adjunction, monads, comonads.

A category can be seen as an abstraction of a number of different things. The most intuitive for our purposes is the one coming from sets (as objects), together with functions between them (as morphisms) and composition of functions (as composition).

2.2.1 Definition. A category is given by a class $\mathcal{C}$ of *objects* together with a class of *morphisms* (or *arrows*, or *maps*) and a notion of *composition* between morphisms satisfying the following axioms:

(1) Every morphism f belonging to $\mathcal{C}$ is uniquely associated with two objects from $\mathcal{C}$, representing a unique starting point $\mathrm{dom}(f)$ and an end point $\mathrm{codom}(f)$. Denoting $\mathrm{dom}(f)$ by A and $\mathrm{codom}(f)$ by B, then f is often written as $A \xrightarrow{f} B$. It is part of the definition that for any two objects A, B in $\mathcal{C}$, the collection of morphisms f satisfying $\mathrm{dom}(f) = A$ and $\mathrm{codom}(f) = B$ forms a set as opposed to a proper class. This set is usually written as $\mathcal{C}(A, B)$ or $\mathrm{Hom}(A, B)$.

(2) Every object A in $\mathcal{C}$ is associated with a distinguished *identity morphism* $A \xrightarrow{1_A} A$.

(3) Whenever A, B, C are objects of $\mathcal{C}$ and $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ are morphisms, then there is a unique morphism $A \xrightarrow{h} C$, called *composition of* f *and* g. We will denote the composite h just by juxtaposition of both factors, i.e. $A \xrightarrow{h} C = A \xrightarrow{fg} C$.

(4) The composition rule has to obey two laws: for all objects A, B, C, D in $\mathcal{C}$ and morphisms $A \xrightarrow{f} B$, $B \xrightarrow{g} C$ and $C \xrightarrow{h} D$, we have

$$A \xrightarrow{(fg)h} D = A \xrightarrow{f(gh)} D \qquad \text{(associativity)}$$

$$A \xrightarrow{f1_B} B = A \xrightarrow{f} B \qquad \text{(right neutrality)}$$

$$A \xrightarrow{1_A f} B = A \xrightarrow{f} B \quad \Diamond \qquad \text{(left neutrality)}$$

The definition of category enables a strong duality principle that allows to transform many concepts or statements into dual ones:

2.2.2 Remark. From every category $\mathcal{C}$ one can naturally derive the so-called *opposite category* $\mathcal{C}^\partial$ by reversing the direction of morphisms and swapping the order of composition. By definition, the object class of $\mathcal{C}^\partial$ coincides with that of $\mathcal{C}$, and so does the class of all morphisms. However, the role of domain and codomain is swapped: if A and B are objects of $\mathcal{C}$ and $A \xrightarrow{f} B$ is a morphism in $\mathcal{C}$, then (and only then) $B \xrightarrow{f} A$ is a morphism in $\mathcal{C}^\partial$. This is to say more precisely, that $\operatorname{dom}^\partial(f) := \operatorname{codom}(f)$ and $\operatorname{codom}^\partial(f) := \operatorname{dom}(f)$ for any morphism of $\mathcal{C}$, i.e. $\mathcal{C}^\partial(A, B) := \mathcal{C}(B, A)$ for all objects A and B of $\mathcal{C}$. The identical morphisms 1_A for A in $\mathcal{C}$ remain the distinguished identical morphisms of $\mathcal{C}^\partial$. Yet, the composition operation of $\mathcal{C}^\partial$ now needs to swap factors, in order to be well-defined: whenever A, B and C are objects of $\mathcal{C}$ and $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ are morphisms in $\mathcal{C}^\partial$, then, according to the definition, $C \xrightarrow{g} B$ and $B \xrightarrow{f} A$ are morphisms of $\mathcal{C}$, such that $C \xrightarrow{gf} A$ is again a morphism of $\mathcal{C}$. Therefore, $A \xrightarrow{gf} C$ is a morphism of $\mathcal{C}^\partial$, and this is the one that one defines as the composition of f with g in $\mathcal{C}^\partial$. If one would not use juxtaposition for the product of morphisms and write more exactly $g *_{\mathcal{C}} f$ and $f *_{\mathcal{C}^\partial} g$ for the composition in $\mathcal{C}$ and $\mathcal{C}^\partial$, respectively, then the previous definition can simply be given by $f *_{\mathcal{C}^\partial} g := g *_{\mathcal{C}} f$. It is straightforward to verify that $\mathcal{C}^\partial$ defined in this way yields again a category.

Thereby, now any statement or concept that is purely written in the axioms of category theory, can be transformed into a dual one. Namely, one instantiates the definition or statement in $\mathcal{C}^\partial$ and reinterprets the meaning in $\mathcal{C}$. Dual definitions arising in this way often receive the prefix *co* in their names, e.g. *product* and *coproduct*, *algebra* and *coalgebra* etc.

To create some intuition for categories, we present a few examples. The conditions from Definition 2.2.1 are verified without any difficulties.

2.2.3 Example. (a) The category $\mathcal{S}et$ consists of all sets (as objects), functions as morphisms, ordinary composition of functions, and identical maps as identity morphisms.

(b) Taking all topological spaces as objects, all continuous maps[3] between them, together with ordinary composition of functions and identical functions, then this structure forms the category $\mathcal{Top}$ of topological spaces.

(c) Similarly, all measurable spaces with measurable maps, standard composition and identity maps form a category $\mathcal{Measrbl}$, namely that of measurable spaces.

(d) Instead of topological spaces one may also take just all metric spaces as objects, and continuous maps between them. That is, besides changing the class of objects, we keep everything as it is defined in $\mathcal{Top}$. In this way the category $\mathcal{Met}_0$ of metric spaces with continuous maps is obtained. If we forget the information about the metric, and view each of these spaces just as a topological space, we get the category $\mathcal{Met}$ of metrisable spaces (with continuous mappings).

Similarly, one can restrict the structure of $\mathcal{Top}$ to all locally compact Hausdorff spaces, yielding the category $\mathcal{LComp}$.

Another popular example which is interesting for studying dynamics are uniform spaces, that, together with continuous maps[4], form the category $\mathcal{Unif}_0$. As with metric spaces, we may also look at the underlying topological spaces of these, which yields the category $\mathcal{Unif}$ of uniformisable topological spaces.

The idea used in the last mentioned example is part of a general scheme:

2.2.4 Definition. A category $\mathcal{D}$ is a *subcategory* of a category $\mathcal{C}$, if the objects and morphisms of $\mathcal{D}$ form subclasses of those of $\mathcal{C}$ and the composition rule and identical morphisms of $\mathcal{D}$ are given by restriction of the respective concepts from $\mathcal{C}$.

If for all A, B from $\mathcal{D}$ we have $\mathcal{D}(A, B) = \mathcal{C}(A, B)$, then $\mathcal{D}$ is called a *full subcategory* of $\mathcal{C}$. $\diamondsuit$

Clearly, full subcategories are uniquely given their class of objects, as seen e.g. in the case of locally compact Hausdorff spaces in relation to $\mathcal{Top}$. A second example of full subcategories are those of metrisable spaces $\mathcal{Met}$ or uniformisable spaces $\mathcal{Unif}$ in $\mathcal{Top}$.

One advantage of category theory is that it allows to formally speak about aspects that are “almost the same” or “very similar” in very different settings. With its abstract view, category theory does not only provide a language for these kinds of observations, it also enables an axiomatic treatment of certain properties, and to

[3] To mention a technical fact, one cannot use just functions. In order to have a unique domain and codomain associated with each morphism, it is formally necessary to use triples (A, f, B) consisting of the continuous function and the two topological spaces A and B specifying the topologies w.r.t. which f is continuous. This kind of formalisation is tacitly assumed in all our examples without explicitly mentioning it.

[4] One would have a choice for uniformly continuous maps here, too, yielding a different category.

transport knowledge between different fields. Examples for this are the following notions:

2.2.5 Definition. Let $\mathcal{C}$ be a category, A, B objects in $\mathcal{C}$ and $A \xrightarrow{f} B$ a morphism from A to B.

(1) f is called *monic* or a *monomorphism* if for all objects C of $\mathcal{C}$ and morphisms $C \xrightarrow{g} A$, $C \xrightarrow{h} A$, the equality $gf = hf$ implies $g = h$.

(2) Dually, f is called *epi* or an *epimorphism* if for all objects C of $\mathcal{C}$ and morphisms $B \xrightarrow{g} C$, $B \xrightarrow{h} C$, the equality $fg = fh$ implies $g = h$.

(3) f is an *isomorphism* if there exists a morphism $B \xrightarrow{f'} A$ such that $ff' = 1_A$ and $f'f = 1_B$. $\Diamond$

Shortly speaking, monomorphisms are those morphisms which can be cancelled from the right, and epimorphisms are those which can be cancelled from the left w.r.t. composition. We remark that monomorphisms and epimorphisms form an instance of the duality principle described in Remark 2.2.2: epimorphisms in a category $\mathcal{C}$ are precisely those morphisms that are monomorphisms in $\mathcal{C}^{\partial}$, and vice versa, of course. It is for historic reasons and for their fundamental role, that they are not just called *co-monomorphisms*. Isomorphisms are those morphisms having an "inverse" morphism (which is necessarily unique).

Let us now see, what is encoded in these notions in concrete examples. In the category of sets, monomorphisms are exactly the injective maps, and epimorphisms are the surjective ones. An isomorphism in $\mathcal{Set}$ is of course nothing but a bijection, thus an epimorphism and a monomorphism. This is a fact that only generalises in that every isomorphism must be monic and epi, but not conversely. For instance, in the category of topological spaces, the identical map 1_X from a set X equipped with the discrete topology to X equipped with the indiscrete topology is an isomorphism precisely if X has at most one element. This is so because the inverse map, which is again the identical mapping, fails to be continuous for $|X| > 1$. Nevertheless, the mentioned map is both epi and monic.

It is easy to see that the isomorphisms in $\mathcal{Top}$ are exactly the homeomorphisms. Moreover, using the discrete topology on the two-element topological space, one can show that monomorphisms in $\mathcal{Top}$ are exactly those continuous maps, that are injective (as maps in $\mathcal{Set}$). Analogously, with the help of the indiscrete two-element topological space, one can prove that epimorphisms in $\mathcal{Top}$ are precisely those continuous maps having underlying surjective functions.

With almost the same arguments, it can be seen that for the category $\mathcal{Measrbl}$ of measurable spaces with measurable maps, epimorphisms and monomorphisms are exactly the measurable maps being surjective, and injective, respectively. Isomorphisms are such bijective maps where images and preimages of measurable sets are measurable.

Characteristic properties that occur in a very similar fashion in different places are not limited to morphisms. They may also be found w.r.t. to objects, or objects and morphisms. Examples for this are terminal objects or products which are presented next.

2.2.6 Definition. An object I in a category $\mathcal{C}$ is said to be *terminal* if for every other object X of $\mathcal{C}$ there exists *exactly* one morphism from X to I. Assuming that the terminal object is fixed, we denote this unique morphism here by $X \xrightarrow{!_X} I$. $\Diamond$

It easily follows from the definition that terminal objects, if they exist, are uniquely determined up to isomorphism. Therefore, one usually picks a canonical representative and speaks about *the* terminal object of a category $\mathcal{C}$. This also motivates why we have suppressed the terminal object in the notation for the unique morphisms into terminal objects.

Again, it is good to have some examples for terminal objects. In $\mathcal{Set}$ every one-element set is a terminal object, in $\mathcal{Top}$ the one-element topological space with the indiscrete topology is terminal, and in $\mathcal{Measrbl}$ the one-element measurable space with the σ-algebra consisting of the full and the empty set is terminal.

It turns out that terminal objects can also be seen as products with no factors. The corresponding definition of a product is as follows.

2.2.7 Definition. Let $\mathcal{C}$ be a category and $(X_i)_{i\in I}$ be a set-indexed family of objects from $\mathcal{C}$. An object P of $\mathcal{C}$ together with a family of morphisms $\left(P \xrightarrow{p_i} X_i\right)_{i\in I}$ is called a *product* of $(X_i)_{i\in I}$ if for any (other) object Q of $\mathcal{C}$ together with morphisms $\left(Q \xrightarrow{q_i} X_i\right)_{i\in I}$ there exists *exactly* one morphism $Q \xrightarrow{h} P$ such that $q_i = hp_i$ holds for all $i \in I$. This unique morphism h is called *tupling* of the morphisms $(q_i)_{i\in I}$ and, considering the product as fixed, denoted here by $\langle q_i\rangle_{i\in I}$. The members of the family $\left(P \xrightarrow{p_i} X_i\right)_{i\in I}$ are usually named *projection morphisms* or simply *projections*.

If we have $X_i = X$ for all $i \in I$ and one object X, then a product of $(X_i)_{i\in I}$ is usually called *I-th power of X*. $\Diamond$

Again it is routine to verify that any two products of a family $(X_i)_{i\in I}$ are isomorphic. So one commonly chooses a certain construction of a product and calls it *the* product $\prod_{i\in I} X_i$ of $(X_i)_{i\in I}$. Moreover, also the corresponding projection morphisms are then usually left implicit, although they are technically important to distinguish the product.

In the case of finite index sets $I = \{\nu_1, \dots, \nu_n\}$, we also write $X_{\nu_1} \times \cdots \times X_{\nu_n}$ instead of $\prod_{i=1}^{n} X_{\nu_i}$, and often I-th powers are abbreviated as X^I, e.g. X^2 is written for $X \times X$. In this article we try to avoid the notation X^I for powers since it clashes with the also common notation X^Y for exponential objects occurring in Subsections 4.3 et seqq.

For completeness we also mention that the dual notion of product and power is that of a *coproduct* and *copower*. Since these only appear in a short side-remark in this paper, we refer the reader to the literature, e.g. [AHS06, Awo10], for further details.

In the familiar categories we mentioned earlier, products exist and are given by the constructions that one expects. In $\mathcal{Set}$ the Cartesian product $\prod_{i\in I} X_i$ of sets $(X_i)_{i\in I}$ together with the maps $p_i\colon \prod_{j\in I} X_j \to X_i$, $(x_j)_{j\in I} \mapsto x_i$ is indeed a product of $(X_i)_{i\in I}$ in the sense of category theory. The tupling of mappings $q_i\colon Q \to X_i$, $i \in I$, is given by $h(q) := (q_i(q))_{i\in I}$ for $q \in Q$. The defining property from Definition 2.2.7 can now readily be checked.

In the category $\mathcal{Top}$ of topological spaces the product of topological spaces $(X_i)_{i\in I}$ is given by the topological space on the Cartesian product of the carrier sets, carrying the product topology. Similarly, in $\mathcal{Measrbl}$ the product is the measurable space on the product of the carrier sets of the factors (as in $\mathcal{Set}$), equipped with the product σ-algebra. In both cases the choice of the product topology (and the product σ-algebra, respectively) ensures that the tupling as calculated in $\mathcal{Set}$ is actually continuous (measurable, respectively) such that it can act as a tupling in $\mathcal{Top}$ (and $\mathcal{Measrbl}$), too.

Next, we show a simple observation how the uniqueness property of a product can be exploited to prove that two morphisms from one object into a product are identical. We shall use this fundamental relationship several times in later proofs.

2.2.8 Remark. Suppose that I is any index set, A and $(P_i)_{i\in I}$ are objects in a category $\mathcal{C}$ such that a product $\prod_{i\in I} P_i$ with projections $\left(\prod_{j\in I} P_j \xrightarrow{p_i} P_i\right)_{i\in I}$ exists. Then for any two morphisms $A \xrightarrow{f,g} \prod_{i\in I} P_i$, checking the equality $f = g$ is equivalent to verifying $fp_i = gp_i$ for all $i \in I$.

Certainly, if f equals g, then the described condition follows by composition with the projections. Thus, we only need to explain the converse direction. If we have $fp_i = gp_i$ for $i \in I$, we simply put $q_i := fp_i = gp_i$, and hence the object A together with $\left(A \xrightarrow{q_i} P_i\right)_{i\in I}$ plays the role of the object Q in Definition 2.2.7 w.r.t. the product $\prod_{i\in I} P_i$. By definition of the product, there now exists a unique morphism $A \xrightarrow{h} \prod_{i\in I} P_i$ such that $hp_i = q_i$ for all $i \in I$. By our assumption we already have two candidates fulfilling this requirement, namely f and g. Thus, by uniqueness, these two morphisms must be equal (to h).

With the following definition, we simply introduce a bit of jargon for categories where we can construct finite products at will. We have already seen that concrete instances of this definition are given, for example, by the categories $\mathcal{Set}$, $\mathcal{Top}$ and $\mathcal{Measrbl}$.

2.2.9 Definition. We say that a category $\mathcal{C}$ *has binary products* if for any two objects X, Y from $\mathcal{C}$ a product $X \times Y$ (with corresponding projection morphisms) exists.

Moreover, we speak of a category *having finite products* (or of a *finite product category*, also called *Cartesian (monoidal) category* by some authors) if it has binary products and a terminal object. ◊

By iterating the binary product construction and using the terminal object as the product with no factors, it is clear that in a finite product category, indeed, products $(\cdots((X_1 \times X_2) \times X_3) \times \cdots) \times X_n$ of any finite number $n \geq 0$ of objects $X_1, \ldots, X_n$ exist.

So far we have introduced very basic category theoretic notions, like special sorts of objects, morphisms or combinations thereof. In the next step we touch a source of much deeper theoretic results, namely "morphisms between categories" (functors) and "morphisms between those" (natural transformations). This lays the foundations for the definition of more interesting notions, such as algebras, coalgebras, monads, comonads, and monadic algebras and comonadic coalgebras, respectively. Furthermore, it paves the way for speaking about powerful concepts such as adjointness.

2.2.10 Definition. If $\mathcal{C}$ and $\mathcal{D}$ are categories, then a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ associates with every object X of $\mathcal{C}$ an object $F(X)$ belonging to $\mathcal{D}$ and with every morphism $X \xrightarrow{f} Y$ between objects X, Y of $\mathcal{C}$ a morphism $F(X) \xrightarrow{F(f)} F(Y)$ in $\mathcal{D}$ such that the following axioms are satisfied:

(1) $F(1_X) = 1_{F(X)}$ holds for all X from $\mathcal{C}$.

(2) $F(fg) = F(f)F(g)$ for all morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ between objects X, Y and Z belonging to $\mathcal{C}$. Here the composition on the left-hand side is done in $\mathcal{C}$ and the one between $F(f)$ and $F(g)$ is carried out in $\mathcal{D}$.

It is customary to agree on omission of brackets for $F(f)$ and $F(X)$ if the argument consists of just one symbol.

Moreover, if $\mathcal{C} = \mathcal{D}$, then the functor F is said to be an *endo-functor* of the category $\mathcal{C}$. We write $\operatorname{End}\mathcal{C}$ for the class of all endo-functors of $\mathcal{C}$. ◊

Thus, a functor is like a mapping between categories that is structurally compatible: the first condition ensures compatibility with the identical morphisms and the second one compatibility with composition. Intuitively, functors should be viewed as morphisms between the categories $\mathcal{C}$ and $\mathcal{D}$. We mention that this intuition can even be made precise by forming the class $\mathcal{Cat}$ of small categories, i.e. those, whose object class is a set rather than a proper class. Equipping $\mathcal{Cat}$ with the functors as morphisms and the canonical composition of functors as explained in Remark 2.2.11, $\mathcal{Cat}$ indeed forms a category.

2.2.11 Remark. If $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ are categories and $\mathcal{C} \xrightarrow{F} \mathcal{D}$ and $\mathcal{D} \xrightarrow{G} \mathcal{E}$ are functors, one can easily check that putting $GFX := G(F(X))$ and $GFf := G(F(f))$ for a morphism $X \xrightarrow{f} Y$ and objects X and Y of $\mathcal{C}$ defines a functor $\mathcal{C} \xrightarrow{GF} \mathcal{E}$. We admit that viewing functors as morphisms between categories, it would have been more natural to write FG for this functor (cf. our notation for composition of morphisms in categories 2.2.1). Yet, in later sections, we will be concerned with quite a few object-wise calculations involving functors, which motivates the slightly inconsistent notation we have chosen here.

Many important constructions in mathematics are in fact functors. For instance associating with any Lie group its Lie algebra is functorial, sending a group to its abelianization is a functor from groups to the category of Abelian groups, Stone-Čech-compactification can be viewed as a functor from $\mathcal{Top}$ to the category of compact Hausdorff spaces, taking the fundamental group at a certain base point is a functor from $\mathcal{Top}$ to the category of groups, and many more.

In the following example, we present simpler cases, which at the same have greater relevance for our topic.

2.2.12 Example. (a) There is a trivial endo-functor associated with every category $\mathcal{C}$. The *identical functor* $\mathcal{C} \xrightarrow{1_{\mathcal{C}}} \mathcal{C}$ maps objects and morphisms of $\mathcal{C}$ identically.

(b) Similarly obvious are *constant functors*: if T is an object of category $\mathcal{D}$, then mapping any object X of another category $\mathcal{C}$ to T and any $\mathcal{C}$-morphism $X \xrightarrow{f} Y$ to $T \xrightarrow{1_T} T$ certainly yields a functor $\mathcal{C} \xrightarrow{T} \mathcal{D}$ that is usually denoted with the same symbol as the object uniquely determining it. In the special case that $\mathcal{C} = \mathcal{D}$, one has, of course, a constant endo-functor.

(c) Another easy, but useful instance of functors are *forgetful functors*. These simply forget some structure of the objects and morphisms. For instance, with every topological space $\boldsymbol{X} = (X, \tau)$ we may associate the underlying carrier set $U(\boldsymbol{X}) := X$, and with every $\mathcal{Top}$-morphism (continuous map) $(X, \tau) \xrightarrow{f} (Y, \sigma)$ the underlying map $X \xrightarrow{f} Y$ in $\mathcal{Set}$. It is evident that this definition yields a functor $\mathcal{Top} \xrightarrow{U} \mathcal{Set}$ as $\mathcal{Top}$-morphisms are composed in the same way as mappings and the identical $\mathcal{Top}$-morphisms map elements identically.

(d) The fourth example will play a central role in Subsection 4.1 et seqq. We assume a category $\mathcal{C}$ such that a product $X \times Y$ exists for any two objects X and Y from $\mathcal{C}$. Since products are only unique up to isomorphism, we consider now one particular choice for $X \times Y$ (together with corresponding projection morphisms) as fixed for any X, Y in $\mathcal{C}$. Furthermore, for objects X, Y, U, V and $\mathcal{C}$-morphisms $X \xrightarrow{f} U$ and $Y \xrightarrow{g} V$, we define $X \times Y \xrightarrow{f \times g} U \times V$

by $f \times g := \langle \mathrm{pr}_X\, f, \mathrm{pr}_Y\, g\rangle$ where $X \times Y \xrightarrow{\mathrm{pr}_X} X$ and $X \times Y \xrightarrow{\mathrm{pr}_Y} Y$ are the projection morphisms coming with $X \times Y$. Hence, $f \times g$ is the unique morphism $X \times Y \xrightarrow{h} U \times V$ making the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] & U \\
X \times Y \arrow[u, "\mathrm{pr}_X"] \arrow[d, "\mathrm{pr}_Y"'] \arrow[r, dashed, "h"] & U \times V \arrow[u, "\mathrm{pr}'_U"'] \arrow[d, "\mathrm{pr}'_V"] \\
Y \arrow[r, "g"] & V
\end{tikzcd}$$

commute, in which $U \times V \xrightarrow{\mathrm{pr}'_U} U$ and $U \times V \xrightarrow{\mathrm{pr}'_V} V$ are the projections of $U \times V$.

Since the morphism h is unique with respect to this property, it is evident, that $1_X \times 1_Y = 1_{X\times Y}$, as the latter indeed ensures commutativity of the corresponding diagram. Moreover, given morphisms $X \xrightarrow{f_1} U$, $U \xrightarrow{f_2} W$ and $Y \xrightarrow{g_1} V$, $V \xrightarrow{g_2} Z$, putting the two commutative diagrams for $f_1 \times g_1$ and $f_2 \times g_2$ together, it is clear that $(f_1 \times g_1)\,(f_2 \times g_2)$ makes the whole diagram

$$\begin{tikzcd}
X \arrow[r, "f_1"] \arrow[rr, bend left, "f_1 f_2"] & U \arrow[r, "f_2"] & W \\
X \times Y \arrow[u, "\mathrm{pr}_X"] \arrow[d, "\mathrm{pr}_Y"'] \arrow[r, "f_1 \times g_1"] & U \times V \arrow[u, "\mathrm{pr}'_U"'] \arrow[d, "\mathrm{pr}'_V"] \arrow[r, "f_2 \times g_2"] & W \times Z \arrow[u, "\mathrm{pr}''_W"'] \arrow[d, "\mathrm{pr}''_Z"] \\
Y \arrow[r, "g_1"] \arrow[rr, bend right, "g_1 g_2"'] & V \arrow[r, "g_2"] & Z
\end{tikzcd}$$

commute. Thus, by uniqueness, it follows that $(f_1 f_2) \times (g_1 g_2)$ is equal to $(f_1 \times g_1)\,(f_2 \times g_2)$.

Hence, we have established that $\mathcal{C} \times \mathcal{C} \xrightarrow{-_1 \times -_2} \mathcal{C}$ is functorial in both arguments, so it is a so-called *bifunctor* into $\mathcal{C}$.

(e) The next example will also play an important role in Subsection 4.1 et seqq. If a category $\mathcal{C}$ has binary products, and T is an object of $\mathcal{C}$, then we may certainly plug in the constant endo-functor T (see (b)) into the first coordinate of the bifunctor $\times$ given in (d). This yields an endo-functor $\mathcal{C} \xrightarrow{T \times -} \mathcal{C}$ mapping every object X to the chosen product $T \times X$ and morphisms $X \xrightarrow{f} Y$ to

$T \times X \xrightarrow{T \times f} T \times Y := T \times X \xrightarrow{1_T \times f} T \times Y = T \times X \xrightarrow{\langle \mathrm{pr}_T, \mathrm{pr}_X f \rangle} T \times Y$. The latter is the unique morphism $T \times X \xrightarrow{h} T \times Y$ making the diagram

$$\begin{tikzcd}
 & T \\
T \times X \arrow[ur, "\mathrm{pr}_T"] \arrow[r, dashed, "h"] \arrow[d, "\mathrm{pr}_X"'] & T \times Y \arrow[u, "\mathrm{pr}'_T"'] \arrow[d, "\mathrm{pr}'_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

commute, in which $T \times X \xrightarrow{\mathrm{pr}_T} T$ and $T \times X \xrightarrow{\mathrm{pr}_X} X$ are the projections of $T \times X$, and $T \times Y \xrightarrow{\mathrm{pr}'_T} T$ and $T \times Y \xrightarrow{\mathrm{pr}'_Y} Y$ are the projections of $T \times Y$. In a similar way, also a functor $- \times T$ may be defined.

A more subtle analysis of the situation makes clear, of course, that it is not necessary to require that all binary products exist in $\mathcal{C}$ in order to define the functor $T \times -$. It is sufficient if for every X in $\mathcal{C}$ a product $T \times X$ exists, and making a specific choice for it, one can explicitly define $T \times -$ along the lines of item (d).

(f) The last example is also a bifunctor, i.e. an assignment that is functorial in both its input arguments: if $\mathcal{C}$ is any category, the so-called *hom-functor* is a bifunctor from $\mathcal{C}^\partial \times \mathcal{C}$ to the category of sets. For every pair of objects A, B from $\mathcal{C}$, the hom-functor associates the set of morphisms $\mathcal{C}(A, B)$. Moreover, if C, D are further objects of $\mathcal{C}$ and $A \xrightarrow{f} C$ is a morphism in $\mathcal{C}^\partial$ and $B \xrightarrow{g} D$ is one in $\mathcal{C}$, then $\mathcal{C}(f, g) : \mathcal{C}(A, B) \to \mathcal{C}(C, D)$ is given by composition in $\mathcal{C}$, i.e. $\mathcal{C}(f, g)(h) := fhg$ for any $h \in \mathcal{C}(A, B)$. It is not difficult to check that this assignment is indeed functorial. We remark that sometimes, in particular if the category $\mathcal{C}$ is clear from the context, $\mathrm{Hom}(-, -)$ is written instead of $\mathcal{C}(-, -)$.

Going one step further, we now also consider "morphisms between functors". These are called *natural transformations*.

2.2.13 Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $\mathcal{C} \xrightarrow{F,G} \mathcal{D}$ be functors. A *natural transformation* $F \xrightarrow{\eta} G$ is a $\mathcal{C}$-indexed family of $\mathcal{D}$-morphisms $\left(FX \xrightarrow{\eta_X} GX\right)_{X \in \mathcal{C}}$ such that for all X, Y from $\mathcal{C}$ and all $\mathcal{C}$-morphisms $X \xrightarrow{f} Y$, the following square

$$\begin{tikzcd}
FX \arrow[r, "\eta_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\
FY \arrow[r, "\eta_Y"] & GY
\end{tikzcd}$$

commutes, i.e. $\eta_X Gf = Ff\eta_Y$.

A natural transformation $F \xrightarrow{\eta} G$ is called *natural equivalence* if for every fibre the morphism $FX \xrightarrow{\eta_X} GX$ is an isomorphism. ◊

There are canonical ways of composing natural transformations with each other and with functors. The details and the notation we shall apply later for these compositions are contained in the following remark.

2.2.14 Remark. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories, $\mathcal{C} \xrightarrow{F,G} \mathcal{D}$ be functors and $F \xrightarrow{\eta} G$ be a natural transformation.

(a) For any functor $\mathcal{C} \xrightarrow{H} \mathcal{D}$ and a transformation $G \xrightarrow{\varepsilon} H$, we can define the morphism $(\eta\varepsilon)_X := \eta_X \varepsilon_X$ for any object X in $\mathcal{C}$. Then $F \xrightarrow{\eta\varepsilon} H$, given by $\left(FX \xrightarrow{(\eta\varepsilon)_X} HX\right)_{X\in\mathcal{C}}$ is again a natural transformation due to the commutativity of the diagram

$$\begin{tikzcd}
FX \arrow[r, "\eta_X"] \arrow[d, "Ff"'] \arrow[rr, bend left, dashed, "(\eta\varepsilon)_X"] & GX \arrow[r, "\varepsilon_X"] \arrow[d, "Gf"] & HX \arrow[d, "Hf"] \\
FY \arrow[r, "\eta_Y"] \arrow[rr, bend right, dashed, "(\eta\varepsilon)_Y"'] & GY \arrow[r, "\varepsilon_Y"] & HY.
\end{tikzcd}$$

Let us note that w.r.t. this composition there also exists a neutral element, namely the *identical natural transformation* $F \xrightarrow{1_F} F$, given as $FX \xrightarrow{1_{FX}} FX$ for X in $\mathcal{C}$.

(b) For any functor $\mathcal{D} \xrightarrow{H} \mathcal{E}$ we put $HFX \xrightarrow{(H\eta)_X := H(\eta_X)} HGX$ for all X in $\mathcal{C}$, thus obtaining a natural transformation $HF \xrightarrow{H\eta} HG$. This follows since the functor H turns the commutative square belonging to a $\mathcal{C}$-morphism $X \xrightarrow{f} Y$ and the transformation η into the commuting square

$$\begin{tikzcd}
HFX \arrow[r, "H\eta_X"] \arrow[d, "HFf"'] & HGX \arrow[d, "HGf"] \\
HFY \arrow[r, "H\eta_Y"] & HGY.
\end{tikzcd}$$

(c) For any functor $\mathcal{E} \xrightarrow{H} \mathcal{C}$ we put $FHX \xrightarrow{(\eta_H)_X := \eta_{HX}} GHX$ for X in $\mathcal{E}$, yielding a natural transformation $FH \xrightarrow{\eta_H} GH$ since the diagram

$$\begin{tikzcd}
FHX \arrow[r, "\eta_{HX}"] \arrow[d, "FHf"'] & GHX \arrow[d, "GHf"] \\
FHY \arrow[r, "\eta_{HY}"] & GHY
\end{tikzcd}$$

commutes for all X, Y in $\mathcal{E}$ and $f \in \mathcal{E}(X, Y)$.

If we have a finite product category $\mathcal{C}$, we can also use functors and natural transformations to agree on some notation concerning the terminal object and the morphisms into it.

2.2.15 Remark. Assume that $\mathcal{C}$ is a finite product category. Since any terminal object has the property of a product with no factors, we use the following notation for the constant functor yielding a fixed terminal object I. As I is the zeroth power of any object X from $\mathcal{C}$, we write $\mathcal{C} \xrightarrow{-^0} \mathcal{C}$ for the constant endo-functor with value $I = X^0$. We already know from Example 2.2.12(b) that this functor maps any morphism $X \xrightarrow{f} Y$ to $f^0 = 1_I = !_I$.

Besides, the unique morphisms $X \xrightarrow{!_X} I$ into the terminal object can be grouped together in a natural transformation $1_\mathcal{C} \xrightarrow{!} -^0$ as $!_X f^0 = !_X = f!_Y$ holds for every morphism $X \xrightarrow{f} Y$.

Moreover, whenever we write $X \times Y$ in a finite product category $\mathcal{C}$, we agree to mean by this the result of the bifunctor $\mathcal{C} \times \mathcal{C} \xrightarrow{-_1 \times -_2} \mathcal{C}$ given by one particular (implicit or explicit) choice of the product (see Example 2.2.12(d)). This choice is naturally accompanied by a choice of projections for each product. However, instead of capturing these by two additional natural transformations, we leave them implicit and use ad-hoc notation as needed.

Knowing now about functors and natural transformations, we can introduce the notion of adjointness of functors. This is a very pervasive concept in category theory, which can be motivated as a weak form of categorical equivalence: one says that two functors $\mathcal{C} \xrightarrow{F} \mathcal{D}$ and $\mathcal{D} \xrightarrow{G} \mathcal{C}$ constitute an equivalence of two categories $\mathcal{C}$ and $\mathcal{D}$ if there exist natural transformations $1_\mathcal{C} \xrightarrow{\vartheta} GF$ and $FG \xrightarrow{\varepsilon} 1_\mathcal{D}$ that are natural equivalences (i.e. consist of isomorphisms). Adjointness of functors F and G weakens this setting in such that two natural transformations $1_\mathcal{C} \xrightarrow{\vartheta} GF$ and $FG \xrightarrow{\varepsilon} 1_\mathcal{D}$ must exist, but it is not required any more that these are natural equivalences. However, one asks for two conditions to be satisfied, which easily follow in case of a categorical equivalence, but not conversely.

2.2.16 Definition. Let $\mathcal{C}$, $\mathcal{D}$ be categories and $\mathcal{C} \xrightarrow{F} \mathcal{D}$, $\mathcal{D} \xrightarrow{G} \mathcal{C}$ be functors. One says that F is *left-adjoint* to G (and that G is *right-adjoint* to F), written as $F \dashv G$, if there exist natural transformations $1_\mathcal{C} \xrightarrow{\vartheta} GF$ (called *unit* of the adjunction) and $FG \xrightarrow{\varepsilon} 1_\mathcal{D}$ (called *co-unit* of the adjunction) such that for all objects X from $\mathcal{C}$ and Y from $\mathcal{D}$ the following two axioms, known as *co-unit-unit equations*, hold:

$$\begin{aligned} 1_{FX} &= F\vartheta_X \varepsilon_{FX} \\ 1_{GY} &= \vartheta_{GY} G\varepsilon_Y. \end{aligned} \qquad \Diamond$$

Let us note that using Remark 2.2.14, the co-unit-unit equations can be compactly stated as $1_F = F\vartheta\varepsilon_F$ and $1_G = \vartheta_G G\varepsilon$.

The following relationship between adjointness of functors described by unit and co-unit, and natural equivalence of hom-functors is well-known (see e.g. [AHS06, 19.3, 19.10, 19.11, 19.A] and [Awo10, 9.4, 9.5, 9.6], giving a few more details).

2.2.17 Proposition. *For categories $\mathcal{C}$, $\mathcal{D}$ and functors $\mathcal{C} \xrightarrow{F} \mathcal{D}$ and $\mathcal{D} \xrightarrow{G} \mathcal{C}$ the following are equivalent:*

(a) $F \dashv G$

(b) There exists a natural equivalence between the hom-bifunctors $\mathcal{D}(F, -)$ and $\mathcal{C}(-, G)$.

More precisely, if $1_\mathcal{C} \xrightarrow{\vartheta} GF$ and $FG \xrightarrow{\varepsilon} 1_\mathcal{D}$ are the unit and co-unit of the adjunction $F \dashv G$, then one defines the natural equivalence $\mathcal{D}(F, -) \xrightarrow{\nu} \mathcal{C}(-, G)$ by $\nu_{X,Y}\left(FX \xrightarrow{g} Y\right) := \vartheta_X Gg$ for X in $\mathcal{C}$, Y in $\mathcal{D}$ and $g \in \mathcal{D}(FX, Y)$. Its inverse $\nu^{-1}_{X,Y}$ is given by $\nu^{-1}_{X,Y}\left(X \xrightarrow{f} GY\right) := Ff\varepsilon_Y$ for X in $\mathcal{C}$, Y in $\mathcal{D}$ and $f \in \mathcal{C}(X, GY)$.

Conversely, if the natural equivalence $\mathcal{D}(F, -) \xrightarrow{\nu} \mathcal{C}(-, G)$ is given, then one puts $\vartheta_X := \nu_{X,FX}(1_{FX})$ for X in $\mathcal{C}$ and $\varepsilon_Y := \nu^{-1}_{GY,Y}(1_{GY})$ for Y in $\mathcal{D}$.

Next, we define the concept of an *algebra for an endo-functor*, of a *monad* and of an *algebra for a monad*, which has a richer structure than just an algebra for a functor. Using duality (see Remark 2.2.2) these notions have duals, known as *coalgebra*, *comonad* and *coalgebra for a comonad*.

2.2.18 Definition. Let $\mathcal{C}$ be a category and $F \in \operatorname{End}\mathcal{C}$ an endo-functor, called *signature functor*. Then an *algebra* for the endo-functor F (also called *algebra of signature* F or F*-algebra*) is any pair $\left(A, FA \xrightarrow{\varphi} A\right)$ where A is an object of $\mathcal{C}$ and $\varphi \in \mathcal{C}(FA, A)$ is a morphism.

Dually, a *coalgebra* for F (or F*-coalgebra*) is a pair, which is an algebra for F considered as an endo-functor of $\mathcal{C}^\partial$, i.e. a pair $\left(A, A \xrightarrow{\varphi} FA\right)$ where A belongs to $\mathcal{C}$ and $\varphi \in \mathcal{C}(A, FA)$. ◊

To give an intuition in what sense this definition describes algebraic structures, we present two examples:

2.2.19 Example. Consider a category $\mathcal{C}$, in which for any object X the product $X \times X$ exists, and fix one particular choice for this product as $\Delta(X) := X \times X$ with projections $\Delta(X) \xrightarrow{\mathrm{pr}^X_i} X$ $(i \in \{1, 2\})$. This setting can be extended to an endo-functor if we define for any $X \xrightarrow{f} Y$ from $\mathcal{C}$ the morphism $X \times X \xrightarrow{\Delta(f)} Y \times Y$ to be $\Delta(f) := f \times f = \left\langle \mathrm{pr}^X_1 f, \mathrm{pr}^X_2 f \right\rangle$ (cf. Example 2.2.12(d)).

Now an algebra of signature Δ is a pair consisting of an object A and a morphism $\Delta(A) = A \times A \xrightarrow{\varphi} A$. This morphism can be seen as a binary operation on A.

More concretely, if $\mathcal{C} = \mathcal{Set}$, then a Δ-algebra is any structure (A, f), where $f\colon A \times A \to A$ is an arbitrary binary operation on A. For example, it can be a semigroup, or a loop or a trivial structure with a projection operation etc. However, we do not know precisely what sort of structure it is: the concept of Δ-algebra is not powerful enough to encode information about possible identities that may hold for the function f. It just encodes that f is binary.

Similarly, if we let $\mathcal{C} = \mathcal{Top}$, then a Δ-algebra is just any pair consisting of a topological space $\boldsymbol{A}$ together with a continuous binary operation $f\colon \boldsymbol{A} \times \boldsymbol{A} \to \boldsymbol{A}$.

Second, we give a concrete example in the category of sets, which already prepares the central idea to be used in Subsection 4.3. There, however, we will have a bit more structural information at our disposal than just an algebra, namely monadicity which is discussed subsequently.

2.2.20 Example. We consider $\mathcal{C} = \mathcal{Set}$ and the endo-functor $T \times -$ for some fixed set T (cf. Example 2.2.12(e)). An algebra for $T \times -$ is a simply a pair (A, φ), where A is a set and $\varphi\colon T \times A \to A$ is a mapping. Of course, this encodes the same information as a structure with many unary operations on A, one for each $t \in T$: $\big(A, (\varphi(t, -))_{t\in T}\big)$.

We can actually store the same amount of information also in a coalgebraic structure. Yet, we need to use a different functor: instead of $T \times -$ we use the endo-functor $\mathcal{Set}\,(T, -) = -^T$. It maps any set X to the set of morphisms $\mathcal{Set}\,(T, X)$, which is nothing but the set of mappings X^T (or T-sequences in X). A coalgebra for $\mathcal{Set}\,(T, -)$ is now a pair consisting of a set A together with a map $\psi\colon A \to A^T$, which associates with every element $x \in A$ a sequence $\psi(x) \in A^T$.

If now a $(T \times -)$-algebra (A, φ) as above is given, then we may put, for instance, $\psi(x) := \varphi(-, x) \in A^T$ and obtain a $\mathcal{Set}\,(T, -)$-coalgebra without losing any information. Fortunately, we can even reverse this process: if a coalgebra (A, ψ) for $\mathcal{Set}\,(T, -)$ is given, then we can define (A, φ) by $\varphi(t, x) := (\psi(x))\,(t)$ and keep all the information that was stored in the coalgebra also in the algebra.

We saw in Example 2.2.19 that only having an algebra or coalgebra for a certain signature functor does not give us a lot of structure to work with. To amend this we introduce now the notion of *monad* (and its dual), which will be used to define monadic algebras (and comonadic coalgebras).

2.2.21 Definition. (1) A triple (T, δ, η), in which $\mathcal{C} \xrightarrow{T} \mathcal{C}$ is an endo-functor, and $TT \xrightarrow{\delta} T$ and $1_{\mathcal{C}} \xrightarrow{\eta} T$ are natural transformations, is called a *monad* (originally called *standard construction*, [God58], later also *triple*, see e.g. [EM65]) if the following two diagrams commute

$$\begin{tikzcd} T(T(TX)) \arrow[r, "T(\delta_X)"] \arrow[d, "\delta_{TX}"'] & T(TX) \arrow[d, "\delta_X"] \\ T(TX) \arrow[r, "\delta_X"] & TX \end{tikzcd} \tag{2.1a}$$

$$\begin{tikzcd} TX \arrow[r, "\eta_{TX}"] \arrow[d, "T\eta_X"'] \arrow[dr, "1_{TX}"] & TTX \arrow[d, "\delta_X"] \\ TTX \arrow[r, "\delta_X"] & TX \end{tikzcd} \tag{2.1b}$$

for every object X in $\mathcal{C}$. Using the composition notions from Remark 2.2.14 these can also be stated more compactly as $T(\delta)\delta = \delta_T\delta$ and $\eta_T\delta = 1_T = T(\eta)\delta$.

(2) The dual notion is that of a *comonad*, i.e. a triple (T, δ, η), where $\mathcal{C} \xrightarrow{T} \mathcal{C}$ is an endo-functor, and $T \xrightarrow{\delta} TT$ and $T \xrightarrow{\eta} 1_T$ are natural transformations satisfying that the diagrams

$$\begin{tikzcd} T(T(TX)) & T(TX) \arrow[l, "T(\delta_X)"'] \\ T(TX) \arrow[u, "\delta_{TX}"] & TX \arrow[l, "\delta_X"] \arrow[u, "\delta_X"'] \end{tikzcd} \tag{2.2a}$$

$$\begin{tikzcd} TX & TTX \arrow[l, "\eta_{TX}"'] \\ TTX \arrow[u, "T\eta_X"] & TX \arrow[l, "\delta_X"] \arrow[u, "\delta_X"'] \arrow[ul, "1_{TX}"'] \end{tikzcd} \tag{2.2b}$$

commute for every X from $\mathcal{C}$, i.e. $\delta T(\delta) = \delta\delta_T$ and $\delta\eta_T = 1_T = \delta T(\eta)$. ◊

Monads are used to encode extra structure about algebras for endo-functors, for instance, identities that hold between operations of algebras in the sense of universal algebra. All such structures can indeed be interpreted as algebras for specific endo-functors. The mentioned extra information in so-called monadic algebras is expressed in the additional commuting diagrams in the following definition:

2.2.22 Definition. Let $\mathcal{C}$ be a category and $\mathcal{C} \xrightarrow{T} \mathcal{C}$ be an endo-functor.

(1) If (T, δ, η) is a monad, then a T-algebra $\left(A, TA \xrightarrow{\varphi} A\right)$ is said to be *monadic w.r.t.* (T, δ, η) (or a *T-algebra for the monad* (T, δ, η)) if the following two diagrams

$$\begin{tikzcd} TTX \arrow[r, "T\varphi"] \arrow[d, "\delta_X"'] & TX \arrow[d, "\varphi"] \\ TX \arrow[r, "\varphi"] & X \end{tikzcd} \tag{2.3a}$$

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "1_X"'] & TX \arrow[d, "\varphi"] \\ & X \end{tikzcd} \tag{2.3b}$$

commute for every X in $\mathcal{C}$.

(2) If (T, δ, η) is a comonad, then a T-coalgebra $\left(A, TA \xrightarrow{\varphi} A\right)$ is said to be *comonadic w.r.t.* (T, δ, η) (or a *T-coalgebra for the comonad* (T, δ, η)) if the following two diagrams

$$\begin{tikzcd} TTX & TX \arrow[l, "T\varphi"'] \\ TX \arrow[u, "\delta_X"] & X \arrow[l, "\varphi"] \arrow[u, "\varphi"'] \end{tikzcd} \tag{2.4a}$$

$$\begin{tikzcd} X & TX \arrow[l, "\eta_X"'] \\ & X \arrow[u, "\varphi"'] \arrow[ul, "1_X"] \end{tikzcd} \tag{2.4b}$$

commute for every X in $\mathcal{C}$. $\Diamond$

2.3 Classical dynamical systems theory

A central problem studied in classical dynamical systems theory is the following: given a set T (whose elements are to be interpreted as points in *time*) a set X, the *state* (or *phase*) *space*, and an indexed family $(\varphi_t)_{t\in T}$ of mappings from X to X, called the *evolution rule* of the dynamical system, one is interested in the time behaviour of states $x \in X$ under the evolution rule.

The most important cases for the time set T are the integers, reals, and their subsets of non-negative numbers. This implies that one also has an addition structure on the time space, which usually at least satisfies the axioms of a *monoid*, i.e. an associative binary operation with a (two-sided) neutral element. Often also the state space carries some extra structure such as a topology, a uniformity, a metric, a differentiable structure, a σ-algebra, a measure etc. The functions describing the evolution rule are then required to be structure preserving w.r.t. X, i.e. continuous (if X is a topological, uniform or metric space, e.g. a subspace of $\mathbb{R}^n$), differentiable (if X is a geometric manifold), measurable (if X is a measurable space), measure preserving (if X is a measure space, in particular a probability space) etc. Accordingly, there is a large variety of literature studying different types of dynamical systems depending on the setting that is assumed for X and T.

It is the aim of this paper to present a unifying framework that extends the foundations of the classical theory. The initial step towards this goal is the following simple observation: clearly, the evolution rule can also be specified more compactly by just one map

$$\begin{array}{rccc} \varphi\colon & T \times X & \longrightarrow & X \\ & (t,x) & \longmapsto & \varphi(t,x) := \varphi_t(x). \end{array}$$

This function is then usually assumed to fulfil the compatibility conditions from above concerning structure that $T \times X$ inherits from T and X by a canonical product construction in the respective settings. In general, the constraint that φ has to be structure preserving is a stronger condition than just requiring it for the individual mappings φ_t, $t \in T$. However, in special cases both assumptions can be equivalent, as mentioned, for instance, for the discrete time case of measurable dynamical systems on p. 536 of [Arn98].

This can be considered as a motivation to study the more restrictive form of an evolution rule given by a structure preserving map φ instead of an indexed family $(\varphi_t)_{t\in T}$. The following paragraph demonstrates that we thereby do not lose an important class of examples from where the notion of dynamical system originates.

Simple, and at the same time, prototypical representatives of so-called *discrete time* dynamical systems arise in the following way: one starts with a topological space X and any continuous function $f\colon X \to X$. Often, X is a subspace of $\mathbb{R}^n$

for some $n \in \mathbb{N} \setminus \{0\}$ with the usual topology inherited from the Euclidean metric. The states are the points of the topological space X. The evolution rule of the dynamical system is given by iterating the function f. That is, the time space is the set of natural numbers, $\mathbb{N}$, which clearly can be equipped with a monoid structure $\langle \mathbb{N}; +, 0 \rangle$. In Definition 2.3.1, we will understand this monoid more generally as a topological monoid by considering the set of natural numbers as the carrier of a discrete topological space, hence the name *discrete time dynamical system*. The dynamics is then given by

$$\begin{array}{rccc} \varphi\colon & \mathbb{N} \times X & \longrightarrow & X \\ & (n, x) & \longmapsto & f^n(x), \end{array}$$

where $f^0 := \mathrm{id}_X$ is the identical mapping and $f^{n+1} := f \circ f^n$ for $n \in \mathbb{N}$. This evolution rule fulfils the properties of a dynamical system as described in the following paragraphs.

At the same time, using the axioms below, one can see that every discrete time dynamical system $\varphi\colon \mathbb{N} \times X \to X$ over a topological space X is given by iteration of a continuous self-map, namely $f := \varphi(1, \cdot)\colon X \to X$. This also explains why one frequently encounters definitions of dynamical systems just as a pair of a space X and a structurally compatible self-mapping $f\colon X \to X$, e.g. a topological space and a continuous map, or a measurable space and a measurable map etc. These kinds of definitions are subsumed by the discrete cases considered here.

However, existing variants of dynamical systems do not only differ in the type of time space used (discrete vs. continuous time), also in the sort of (state) space (topological, measurable, etc.) or mappings (continuous, measurable etc.). Therefore, one of the aims of this paper is to give a definition of dynamical system (see Definition 3.2.1) that encompasses many of the competing notions that can be found throughout the literature. This is possible using the language of category theory. In this formulation we shall then see that dynamical systems are in fact a special instance of a well-known concept in algebra and theoretical computer science, namely that of a monadic algebra.

The following informal definition of a dynamical system seems to be the core of all the different formulations that one encounters. Given a monoid $\mathbf{T} = \langle T; +, 0 \rangle$ and a mapping $\varphi\colon T \times X \to X$, we say that $(\mathbf{T}, \varphi)$ is a dynamical system provided the following compatibility conditions hold and all involved mappings are structure preserving w.r.t. the framework assumed for X and T:

(1) For all $x \in X$ we have $\varphi(0, x) = x$. (initial condition)

(2) For all $x \in X$ and all $s, t \in T$, it is $\varphi(s, \varphi(t, x)) = \varphi(s + t, x)$. (semigroup property)

We remark that in case X and T are just sets, i.e. no additional structure needs to be preserved, conditions (1) and (2) express that $\varphi\colon T \times X \to X$ is a so-called *monoid action* of $\mathbf{T}$ on X.

In the next step we are going to put this into a formal definition for the setting of topological spaces. On the one hand, this will be the basis for a straightforward generalisation to arbitrary abstract categories. On the other hand, the case of dynamical systems in a topological environment will receive the highest level of emphasis among all types of dynamical systems considered in this paper. After that we collect further variants from the literature to outline the scope of our general modelling: we either use them, with marginal modifications, as examples, or we discuss why they are not fitting in our framework. First, we will briefly focus on special cases such as dynamical systems in metric spaces. Subsequently, we introduce measurable dynamical systems and discuss measure preserving systems. Then we consider nonautonomous dynamical systems and their continuous and measurable variants generalising, for example, skew product flows.

We recall that for a topological space T, a monoid $\langle T; +, 0\rangle$ is called *topological monoid* if the addition operation $+\colon T \times T \to T$ is continuous w.r.t. the product topology. The constant $0\colon T^0 \to T$ is automatically continuous w.r.t. the unique topology on the one-element (terminal) topological space.

2.3.1 Definition. Let T, X be topological spaces. A *topological dynamical system over a monoid* is a triple $(\mathbf{T} = \langle T; +, 0\rangle, X, \varphi\colon T \times X \to X)$ where

(1) $\mathbf{T}$ is a *topological monoid*,

(2) $\varphi\colon T \times X \to X$ is a *topological monoid action*, i.e. it is continuous w.r.t. the product topology and satisfies the equalities (1) and (2) from above. $\Diamond$

As stated here our definition of topological dynamical system is a slight generalisation of the same concept defined by E. Glasner in Section 1 of [Gla07]. There, Glasner studies the special case of our definition where the state space is compact Hausdorff and the continuously acting topological monoid is actually a topological group. Similarly, in [Ner85, Section 1(i)] the notion of topological dynamical system is defined as a locally compact separable topological group acting continuously (on the right) on a compact metric space.

Our basic definition of topological dynamical system over a monoid subsumes both existing notions, and it will be the starting point for modelling and therefore generalising dynamical systems in any abstract category in Section 3.

Of course, the previous definition can also be given in settings that can be interpreted as prominent full subcategories of the category of topological spaces, such as, for instance, Hausdorff topological spaces, compact Hausdorff spaces, metrisable spaces etc. Then one requires that all involved spaces, namely T and X, belong to this subcategory and that all morphisms are continuous w.r.t. the topologies on the spaces that are induced by the interpretation. In this sense, we can, for example, define the notions of *metric dynamical system*, *(compact) Hausdorff topological dynamical system* etc. in complete analogy to Definition 2.3.1.

Another variant of dynamical system, which comes up via iterating measurable maps in the same way as explained earlier for continuous maps, is a *measurable dynamical system*, see e.g. [Arn98, p. 536]. Comparing the following definition, which we take from the mentioned monograph, with Definition 2.3.1, we have essentially replaced the notion of topological space by measurable space and that of continuity by measurability. In particular, we call a monoid $\langle T;\, +, 0\rangle$ on a measurable space T a *measurable monoid* if addition $+\colon T \times T \to T$ is measurable w.r.t. the product σ-algebra (generated by all binary Cartesian products of measurable sets from the σ-algebra on T). The constant $0\colon T^0 \to T$ is automatically measurable w.r.t. the unique full σ-algebra on the one-element (terminal) measurable space.

2.3.2 Definition. Let T, X be measurable spaces. A *measurable dynamical system over a monoid* is a triple $(\mathbf{T} = \langle T; +, 0\rangle, X, \varphi\colon T \times X \to X)$ where

(1) $\mathbf{T}$ is a *measurable monoid*,

(2) $\varphi\colon T \times X \to X$ is a *measurable monoid action*, i.e. it is measurable w.r.t. the product σ-algebra and satisfies the equalities (1) and (2) from above. $\lozenge$

We mention that again the prototype of this definition, to be found on p. 536 of [Arn98], is not as general as our version. Arnold only defines these dynamical systems for monoids $\mathbf{T}$ belonging to the set $\{\mathbb{R}, \mathbb{R}_{\geq 0}, \mathbb{R}_{\leq 0}, \mathbb{Z}, \mathbb{N}, \mathbb{Z}_{\leq 0}\}$, each to be understood with the usual addition operation as monoid operation and zero as the neutral element. Since he just considers these special cases, he does not mention the condition for $\mathbf{T}$ to be a measurable monoid. This is a requirement we have added to the definition in order to get a homogeneous general setting. Moreover, it is implicitly fulfilled by all the time monoids $\mathbf{T}$ listed as examples in [Arn98] (w.r.t. the Borel σ-algebra on T given by the standard metric topology on the uncountable monoids and the discrete topology on the countable monoids, respectively). This observation follows from continuity of the monoid operations and the fact that the Borel σ-algebra of the product of two topological Hausdorff spaces, one of which is second-countable, i.e. has a countable base, equals the product σ-algebra of the Borel σ-algebras given by the individual spaces (cf. [Bog07, Lemma 6.4.2(i), p. 525]).

In [Arn98, p. 537] the yet stronger notion of *measure preserving dynamical system* (or *metric dynamical system*, a historical term that we wish to avoid for clarity) is defined. The definition relies upon the concept of a measure preserving map, which was introduced on page 8. We recall that a self-map $f\colon X \to X$ of a measure space X carrying a measure μ is measure preserving if $\mu \circ f^{-1} = \mu$.

2.3.3 Definition. Suppose that $\mathbf{T} = \langle T; +, 0\rangle$ is a measurable monoid and $\mathbb{X}$ is a measure space with measure μ and underlying measurable space X. A measurable dynamical system $(\mathbf{T}, X, \varphi\colon T \times X \to X)$ is called *measure preserving* if the self-map $\varphi(t, \cdot)\colon \mathbb{X} \to \mathbb{X}$ is measure preserving for every point in time $t \in T$. $\lozenge$

Unfortunately, this definition is not suitable to be modelled within just one category: the requirement on the self-mappings $\varphi(t,\cdot)\colon \mathbb{X}\to\mathbb{X}$ to be measure preserving suggests to choose the category of measure spaces together with measure preserving mappings as morphisms. For a categorical modelling it would be desirable if not only the individual mappings $(\varphi(t,\cdot))_{t\in T}$ were measure preserving, but if the evolution rule $\varphi\colon T\times X\to X$ were measure preserving as a whole w.r.t. the product measure on $T\times X$ given by the measure μ on X and some measure m on T. Solving for simple cases (e.g. $\mathbf{T}$ being the reals with addition, and $\mathbb{X}$ being the reals with Lebesgue measure) the question if such a measure m exists at all, shows that it will often be uniquely determined by μ if it exists. So there is not much choice left for $\mathbf{T}$ once $\mathbb{X}$ is fixed. On the other hand, for $\mathbf{T}$ being in accordance with category chosen for $\mathbb{X}$, the measure on T should also be such that $\mathbf{T}$ is a measure preserving monoid. This means that the addition of $\mathbf{T}$ is measure preserving and, moreover that $0\colon T^0\to T$ is measure preserving w.r.t. the unique one-element measure space on T^0. This is equivalent to saying that the measure m on T is the Dirac measure centred in the point 0. This is a rather strong condition, prescribing a possibly different measure space on T than the requirement coming from φ.

In view of these arguments it seems that a reasonable categorical modelling of measure preserving dynamical systems should be done with the option to choose the time space and the state space from different categories. As this contradicts our original intention, we will not consider measure preserving dynamical systems further in this paper.

A different sort of dynamical system arises when studying dynamical behaviour of a system under an external influence, which itself is modelled by a dynamical system. This kind of constellation entails a so-called *skew product*; its study is subject to the field of nonautonomous dynamics.

In this context we refer to the article [BS03], where the concepts of *nonautonomous dynamical system (NDS)* and *continuous skew product flow* are defined. While the notion of continuous skew product flow is suitable for categorical modelling (cf. Subsection 3.3), nonautonomous dynamical systems as defined in [BS03], do not fit into an easy generalisation using just one category, unless they essentially form a special case of a continuous skew product flow with a discrete driving system. This is so because the driving system of an NDS as in [BS03, Definition 2.1] simply consists of sets and mappings without any topological structure, whereas the second part of an NDS is assumed to satisfy continuity requirements w.r.t. a metric. With the following definition we remove this asymmetry from [BS03, Definition 2.1] and emphasise its purely algebraic, non-topological aspect.

2.3.4 Definition. For monoids $\mathbf{S}$ and $\mathbf{T}$, such that $\langle S;+,0\rangle=\mathbf{S}\leq\mathbf{T}$ is a submonoid, and sets X and Y, we call a pair (θ,φ) of mappings $\theta\colon T\times X\to X$ and $\varphi\colon S\times X\times Y\to Y$ a *nonautonomous dynamical system with times* $\mathbf{S}\leq\mathbf{T}$ *on* Y

with base X if

(1) $(\mathbf{T}, X, \theta)$ is a (non-topological) dynamical system over the monoid $\mathbf{T}$ (i.e. θ is a left-monoid action of $\mathbf{T}$ on X), called *driving system*,

(2) φ is a *cocycle over* θ, that is, the following two equations, known as *cocycle property*,

$$\varphi(0, x, y) = y, \tag{2.5}$$
$$\varphi(t + s, x, y) = \varphi(t, \theta(s, x), \varphi(s, x, y)) \tag{2.6}$$

hold for all $s, t \in S$, $x \in X$ and $y \in Y$. ◊

It is not only with respect to continuity of the cocycle, where our Definition 2.3.4 slightly differs from the one given in [BS03]. Also w.r.t. to the time spaces, we are marginally more general here than what was written in [BS03]. The only time monoids $\mathbf{T}$ considered there are explicitly $\mathbb{R}$ (continuous time) and $\mathbb{Z}$ (discrete time). Furthermore, the submonoids $\mathbf{S}$ were always chosen as the non-negative points in time, i.e. $\mathbb{R}_{\geq 0}$ and $\mathbb{N}$, respectively. Since we assume an arbitrary monoid $\mathbf{T}$ in our definition, which does not need to have a compatible order relation, we can in general not speak of "non-negative" points in time. Hence, we replace the role of this special submonoid by allowing any submonoid $\mathbf{S}$.

The following is the companion definition to 2.3.4, focussing on the continuous aspect of nonautonomous dynamics and extending Definition 2.6 in [BS03] to general topological monoid actions.

2.3.5 Definition. Suppose $\mathbf{S} = \langle S; +, 0\rangle$ and $\mathbf{T}$ are topological monoids, $\mathbf{S} \leq \mathbf{T}$ being a submonoid, and X and Y be topological spaces. A pair (θ, φ) of mappings $\theta\colon T \times X \to X$ and $\varphi\colon S \times X \times Y \to Y$ is called *continuous skew product system*[5] *with times* $\mathbf{S} \leq \mathbf{T}$ *on* Y *with base* X if

(1) $(\mathbf{T}, X, \theta)$ is a topological dynamical system over the monoid $\mathbf{T}$ (as in Definition 2.3.1), called *driving system*,

(2) φ is a *continuous cocycle over* θ, that is, φ is continuous and fulfils the cocycle property as introduced in Definition 2.3.4. ◊

Comparing again to [BS03], we have generalised the role of a continuous time monoid $(\langle \mathbb{R}; +, 0\rangle)$ and its submonoid of non-negative time points $\langle \mathbb{R}_{\geq 0}; +, 0\rangle$ and that of a discrete time monoid $\langle \mathbb{Z}; +, 0\rangle$ with the submonoid $\langle \mathbb{N}; +, 0\rangle$, respectively, to any topological time monoid $\mathbf{T}$ with an arbitrary submonoid $\mathbf{S}$. Additionally, all spaces in Definition 2.6 of [BS03] were assumed to be metric spaces. However,

[5] We have replaced the word "flow" by "system" since "flow" explicitly refers to $\mathbf{T}$ being the reals with addition, and we allow arbitrary topological monoids, instead.

since the notion of continuity is a purely topological concept, and we have widened the time monoid to allow arbitrary continuous monoids, and groups in particular, it seems natural to formulate the whole definition within the setting of topological spaces. In this way we make sure not to lose interesting topological groups or monoids which are not metrisable. An example would be, for instance, the topological monoid $\mathbf{T} := \langle \mathbb{R}; +, 0\rangle$ consisting of the real line with the usual addition operation, carrying not the standard metric topology, but the Sorgenfrey topology, which fails to be metrisable. This structure is famous in topology and often used as a counterexample. At the same time, this choice for $\mathbf{T}$ can serve as a motivation for not restricting our Definitions 2.3.1 and 2.3.5 to topological groups. It is an example for a topological monoid, which is not a topological group, since the inverse operation is not continuous (see e.g. [Nyi81, Example 3, p. 799]).

We continue to comment a little bit more on the notion of *nonautonomous dynamical system* as stated in [BS03]. This concept is a weakening of a special case of continuous skew product flows where the base X is a discrete space, i.e. essentially a set. Then the condition that φ is continuous can be expressed equivalently[6] by the fact that the mappings $\varphi(\cdot, x, \cdot)\colon S \times Y \to Y$ are continuous for all $x \in X$, which is the formulation used in Definition 2.1 of [BS03]. Moreover, it follows from Definition 2.3.5 that the action θ is continuous. In case that X is discrete, this is again equivalent to $\theta(\cdot, x)\colon T \to X$, $t \mapsto \theta\,(t, x)$ being continuous for every point $x \in X$. This is the part of our definition that is dropped in [BS03, Definition 2.1], making it weaker, i.e. more general than 2.3.5 for discrete base X. Still it is less general than our notion of nonautonomous dynamical system (Definition 2.3.4), which simply requires sets and no topology at all. At the same time, mixing topological spaces and just sets in Definition 2.1 of [BS03] makes their construct more inhomogeneous. Similarly as for measure preserving dynamical systems, it is this inhomogeneity between continuous actions and simply group actions on a set, which renders Berger and Siegmund's definition unfit for a generalisation using just one category as we intend it in the following section.

A third important notion in nonautonomous dynamics is that of *random dynamical system (RDS)*. A definition can, for instance, be found in [BS03, Definition 2.3] and in [Arn98, Definition 1.1.1]. Rephrasing Arnold's definition in our language, an RDS is an NDS (θ, φ), where φ is measurable, and the driving system θ is measure preserving w.r.t. a probability space on the base space X as in Definition 2.3.3. Berger and Siegmund's definition additionally requires that θ be ergodic. Due to the concerns in connection with modelling measure preserving dynamical systems, we will not consider RDS in this article.

However, relaxing the conditions on the driving system, we can introduce the following notion of *measurable nonautonomous dynamical system*.

[6] This follows because $S \times X \times Y$ then equals the copower $\coprod_{x\in X} S \times Y$, and φ is the cotupling of all the continuous maps $(\varphi(\cdot, x, \cdot))_{x\in X}$.

2.3.6 Definition. Given measurable monoids $\mathbf{S}$ and $\mathbf{T}$ such that $\mathbf{S} \leq \mathbf{T}$ is a submonoid, and measurable spaces X, Y, a pair (θ, φ) of mappings $\theta \colon T \times X \to X$ and $\varphi \colon S \times X \times Y \to Y$ is called *measurable nonautonomous dynamical system with times* $\mathbf{S} \leq \mathbf{T}$ *on* Y *with base* X if

(1) $(\mathbf{T}, X, \theta)$ is a measurable dynamical system over the monoid $\mathbf{T}$ (as in Definition 2.3.2), called *driving system*,

(2) φ is a *measurable cocycle over* θ, that is, φ is measurable and fulfils the cocycle property as introduced in Definition 2.3.4. ◊

In the subsequent section we will now see how to understand the notions presented so far from an abstract, categorical point of view.

3 Dynamical Systems in Abstract Categories

To model dynamical systems as coalgebras in abstract categories we are going to pursue the following strategy. We will move all the additional conditions that mappings involved in the definition of a dynamical system have to satisfy (e.g. continuity) into the definition of a suitably chosen category $\mathcal{C}$. For the examples we will be primarily interested in within this paper, this will mostly be the categories $\mathcal{Top}$, $\mathcal{Measrbl}$ and $\mathcal{Set}$ as introduced in Example 2.2.3(a)–(c).

Then we are going to explore what the two conditions (1) and (2) mentioned at the beginning of Subsection 2.3 (see page 24) mean in our abstract context. To this end we will first define straightforward generalisations of monoids and monoid actions in abstract categories. This will allow us to state a very general definition of dynamical system, which will comprise different variations of dynamical systems found in the literature. Further, we will generalise the definition of nonautonomous dynamical system and show that on the abstract level these can be understood as a special instance of our general category theoretic formulation of dynamical system.

In our definitions and results we will need certain requirements on the given category $\mathcal{C}$. For convenience we will most frequently suppose that $\mathcal{C}$ is a finite product category (cf. Definition 2.2.9). Regarding this assumption, we will always consider a particular product construction to be fixed in advance as explained in Remark 2.2.15. Moreover, in some cases we are going to need that certain projection morphisms are epimorphisms. This is not a condition that we can require to hold universally as this would exclude our main test cases: clearly, in $\mathcal{Set}$, $\mathcal{Top}$, $\mathcal{Measrbl}$ projections onto a non-empty factor of a product containing an empty factor fail to be epimorphisms (because the product has an empty carrier set and the map underlying the projection morphism is not surjective). Nevertheless, in the mentioned categories, having an empty factor in a product is basically the only case, when projections fail to be epi. Thus the assumption that some projections are epimorphisms is indeed a very mild condition.

As topological spaces, measurable spaces etc. are, in the context of this section, just objects of some abstract category, we will denote them here with standard, non-boldface symbols as in Subsection 2.2.

3.1 Monoids and monoid actions in abstract categories

The purpose of this subsection is to lift the notions of monoid and monoid action to any abstract category.

3.1.1 Definition. Let $\mathcal{C}$ be a finite product category, T be an object of $\mathcal{C}$, and $T \times T \xrightarrow{+} T$ and $T^0 \xrightarrow{e} T$ be morphisms.

(1) We call the triple $(T, +, e)$ a *$\mathcal{C}$-monoid setting.*

(2) A $\mathcal{C}$-monoid setting $(T, +, e)$ is called a *$\mathcal{C}$-monoid* if the following three diagrams commute:

$$\begin{tikzcd} T \arrow[rr, dotted, bend left, "\langle !_T e, 1_T\rangle"] \arrow[drr, "1_T"'] & \cong T^0 \times T \arrow[r, "e\times T"] & T \times T \arrow[d, "+"] \\ & & T \end{tikzcd} \tag{3.1a}$$

$$\begin{tikzcd} T \arrow[rr, dotted, bend left, "\langle 1_T, !_T e\rangle"] \arrow[drr, "1_T"'] & \cong T \times T^0 \arrow[r, "T\times e"] & T \times T \arrow[d, "+"] \\ & & T \end{tikzcd} \tag{3.1b}$$

$$\begin{tikzcd} (T \times T) \times T \arrow[r, "+\times T"] & T \times T \arrow[dr, "+"] & \\ & & T. \\ T \times (T \times T) \arrow[uu, "\cong (a_{12})"] \arrow[r, "T\times +"] & T \times T \arrow[ur, "+"'] & \end{tikzcd} \tag{3.1c}$$

The dotted arrows have just been added to make the morphism explicit and add nothing to the commutativity condition. The isomorphism a_{12} will be defined in the proof of Lemma 4.1.1. ♢

To model nonautonomous dynamics we need to generalise the concept of submonoid. The category theoretical answer to this task is, of course, to use embeddings which are a certain kind of monic (homo)morphisms. However, category theory does not give a satisfactory one-and-only answer to the question what an embedding should be. There are various notions of embedding occurring in specific categories, and most of them represent a category theoretic concept, in fact a certain type of monomorphism. However, not all of these specific concepts can be modelled by the same kind of monomorphism; sometimes just monomorphisms are the right choice (e.g. in the categories of sets, semigroups or rings, respectively), sometimes additional properties like extremality, strongness or regularity of the monomorphism (e.g. in the categories of topological, Hausdorff or metric spaces, respectively, cp. Examples 7.58 on p. 116 of [AHS06]) need to be assumed (cf. the introductory paragraphs to the subsections "Regular and extremal monomorphisms" on p. 114, "Subobjects" on p. 122 and "Embeddings" on p. 133 of [AHS06], respectively; see Remark 7.7.6(2) on p. 121 of the same monograph for a list of different kinds of monomorphisms used in different prominent categories). So each category comes with its own natural concept of embedding, which is why we do not define this term in abstract categories apart from requiring that it must be a monomorphism. We emphasise however, that in all of our applications we are considering concrete categories (over the category of sets), where we may use the embedding concept defined as an initial monomorphism (cp. [AHS06, Definition 8.6, p. 134]; see also Examples 8.8 there, for a list of appropriate embedding notions in familiar categories).

3.1.2 Definition. Suppose that $\mathbf{S} = \left\langle S; +^{\mathbf{S}}, e^{\mathbf{S}} \right\rangle$ and $\mathbf{T} = \left\langle T; +^{\mathbf{T}}, e^{\mathbf{T}} \right\rangle$ are $\mathcal{C}$-monoid settings in a finite product category $\mathcal{C}$. We call a morphism $S \xrightarrow{h} T$

(1) a *homomorphism* between the $\mathcal{C}$-monoid settings if the following two diagrams commute:

$$\begin{tikzcd} T \times T \arrow[r, "+^{\mathbf{T}}"] & T \\ S \times S \arrow[u, "h \times h"] \arrow[r, "+^{\mathbf{S}}"] & S \arrow[u, "h"'] \end{tikzcd} \tag{3.2a}$$

$$\begin{tikzcd} T^0 \arrow[r, "e^{\mathbf{T}}"] & T \\ S^0 \arrow[u, "h^0 = !_{S^0}"] \arrow[r, "e^{\mathbf{S}}"] & S \arrow[u, "h"'] \end{tikzcd} \tag{3.2b}$$

(2) an *embedding* if it is a homomorphism and an embedding in $\mathcal{C}$. We denote this by $\mathbf{S} \overset{h}{\hookrightarrow} \mathbf{T}$ or simply $\mathbf{S} \hookrightarrow \mathbf{T}$ if the particular embedding morphism is not interesting or is merely given by an existence condition. We then say that $\mathbf{S}$ *embeds as a submonoid setting* into $\mathbf{T}$. ◊

Having dealt with the generalisation of monoids in abstract categories, we can now turn towards their actions.

3.1.3 Definition. Let $\mathcal{C}$ be a finite product category and $(T, +, e)$ be a $\mathcal{C}$-monoid. Furthermore, let X be an object in $\mathcal{C}$ and $T \times X \xrightarrow{\varphi} X$ be a morphism. We call the pair $\left(X, T \times X \xrightarrow{\varphi} X\right)$ a *$\mathcal{C}$-monoid action* (of $(T, +, e)$ on X) if the following two diagrams commute

$$\begin{tikzcd}
X \arrow[r, phantom, "\cong"] \arrow[rr, dotted, bend left, "\langle !_X e, 1_X \rangle"] \arrow[drr, "1_X"'] & T^0 \times X \arrow[r, "e \times X"] & T \times X \arrow[d, "\varphi"] \\
& & X
\end{tikzcd} \tag{3.3a}$$

$$\begin{tikzcd}
(T \times T) \times X \arrow[r, "+\times X"] & T \times X \arrow[dr, "\varphi"] & \\
& & X. \\
T \times (T \times X) \arrow[uu, "\cong (a_{15})"] \arrow[r, "T \times \varphi"] & T \times X \arrow[ur, "\varphi"'] &
\end{tikzcd} \tag{3.3b}$$

Again the dotted arrow has been added for making the morphism explicit, and the canonical isomorphism a_{15} will properly be defined in the proof of Lemma 4.1.1. $\Diamond$

3.2 Abstract dynamical systems

We will define abstract dynamical systems as a straightforward generalisation of Definition 2.3.1 using the notions of $\mathcal{C}$-monoid and $\mathcal{C}$-monoid action from above.

In Section 4 we are going to establish a characterisation of such general dynamical systems as certain monadic algebras for the endo-functor $T \times -$ exploiting that products of the time space with every object in the considered category exist.

3.2.1 Definition. A *dynamical system on a finite product category* $\mathcal{C}$ is a triple $\left((T, +, e), X, T \times X \xrightarrow{\varphi} X\right)$, where $X \in \mathcal{C}$ is an object, $(T, +, e)$ is a $\mathcal{C}$-monoid and $\left(X, T \times X \xrightarrow{\varphi} X\right)$ is a $\mathcal{C}$-monoid action of $(T, +, e)$ on X.

To add some interpretative terminology to this definition, the object X will occasionally be called *state space*, T *time space*, $(T, +, e)$ *time structure* and the pair $\left(X, T \times X \xrightarrow{\varphi} X\right)$ or simply $T \times X \xrightarrow{\varphi} X$ *transition structure* of the dynamical system. $\Diamond$

The following result makes sure that this is indeed a generalisation of the notions developed in Subsection 2.3.

3.2.2 Corollary. *The notions of topological dynamical system over a monoid (as in Definition 2.3.1) and dynamical system on $\mathcal{Top}$ (as in Definition 3.2.1) coincide.*

Likewise, the concepts of measurable dynamical system over a monoid (as in Definition 2.3.2) and of dynamical system on the category of measurable spaces $\mathcal{Measrbl}$ (as in Definition 3.2.1) are the same.

Of course, in the same way we can use the categories $\mathcal{Met}_0$ of metric spaces or $\mathcal{Unif}_0$ of uniform spaces, each with continuous mappings as morphisms, instead of $\mathcal{Top}$ to study metric dynamics or uniform dynamics, respectively. With regard to basic aspects this is essentially the same as equipping the space with its underlying topology and forgetting about the metric or uniform structure, i.e. studying dynamical systems on the full subcategories $\mathcal{Met}$ and $\mathcal{Unif}$ of $\mathcal{Top}$, given by metrisable and uniformisable spaces, respectively.

One advantage of our abstract view on dynamical systems is that we now have a very simple way to translate a given system into others (in possibly different categories). Indeed, whenever we have a finite product preserving[7] functor between two categories and a dynamical system on one of them, we also get one on the other category.

3.2.3 Remark. For a finite product category $\mathcal{C}$, a finite product preserving functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ into some category $\mathcal{D}$ every dynamical system

$$\left((T,+,e)\,, X, T\times X \xrightarrow{\varphi} X\right)$$

on $\mathcal{C}$ gives rise to a dynamical system

$$\left((FT, F+, Fe)\,, FX, FT\times FX \xrightarrow{F\varphi} FX\right)$$

on $\mathcal{D}$.

Proof: It is clear that the functor F transforms the defining commutative diagrams for the dynamical system $\left((T,+,e)\,, X, T\times X \xrightarrow{\varphi} X\right)$ into commutative diagrams in the category $\mathcal{D}$. Since it preserves finite products, the resulting diagrams also have the correct form to describe a dynamical system on $\mathcal{D}$. We remark that these diagrams can be adjusted to use any other product bifunctor on $\mathcal{D}$ via the natural isomorphisms between different products of the same factors. This can be necessary if one has agreed on a particular product construction in $\mathcal{D}$ beforehand. □

3.3 Nonautonomous dynamics

Here we shall give an example how the notion of skew product system can also be lifted to our abstract setting. In fact, continuous skew product systems as

[7] This means for a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ that whenever P is a product of X and Y in $\mathcal{C}$ with projections pr_X and pr_Y, then $F(P)$ together with $F(\mathrm{pr}_X)$ and $F(\mathrm{pr}_Y)$ is a product of $F(X)$ and $F(Y)$ in $\mathcal{D}$, and that $F(I)$ is a terminal object in $\mathcal{D}$ whenever I is one in $\mathcal{C}$.

defined in 2.3.5 have a straightforward generalisation in abstract categories, called *abstract NDS*, and Definition 2.3.5 is the specialisation of the general concept for the category of topological spaces. Subsequently, we will see that for coinciding time monoids, abstract NDS are in turn a special instance of our abstract dynamical systems as given in Definition 3.2.1, namely in the case where the state space is a product of two spaces. This is a purely algebraic fact being true in the abstract categorical setting, no matter what category we choose.

The defining equations for continuous skew product systems clearly translate into commutative diagrams as we can see in the following definition.

3.3.1 Definition. Suppose $\mathbf{S} = \langle S; +, e\rangle$ and $\mathbf{T}$ are $\mathcal{C}$-monoids in a finite product category $\mathcal{C}$ such that $\mathbf{S}$ embeds as a submonoid into $\mathbf{T}$. Let X and Y be objects in $\mathcal{C}$ and $T \times X \xrightarrow{\theta} X$ and $S \times X \times Y \xrightarrow{\varphi} Y$ be morphisms. The pair (θ, φ) of morphisms is called *abstract nonautonomous dynamical system (abstract NDS) in $\mathcal{C}$ with times* $\mathbf{S} \hookrightarrow \mathbf{T}$ *on* Y *with base* X if

(1) $(\mathbf{T}, X, \theta)$ is a dynamical system on $\mathcal{C}$ over the monoid $\mathbf{T}$ (as in Definition 3.2.1), called *driving system*,

(2) φ is an *abstract cocycle over* θ, that is, the following two diagrams, called *abstract cocycle property*,

$$\begin{tikzcd}
X \times Y \arrow[r, phantom, "\cong"] \arrow[rr, bend left, dotted, "\langle !_{X\times Y} e, 1_{X\times Y}\rangle"] \arrow[drr, "\mathrm{pr}'_Y"'] & S^0 \times (X \times Y) \arrow[r, "e\times(X\times Y)"] & S \times (X \times Y) \arrow[d, "\varphi"] \\
 & & Y
\end{tikzcd} \tag{3.4a}$$

$$\begin{tikzcd}
(S \times S) \times (X \times Y) \arrow[r, "+\times(X\times Y)"] & S \times (X \times Y) \arrow[dr, "\varphi"] & \\
 & & Y \\
S \times (S \times (X \times Y)) \arrow[uu, "\cong"] \arrow[r, "S\times\langle\langle \mathrm{pr}_S, \mathrm{pr}_X\rangle\theta, \varphi\rangle"] & S \times (X \times Y) \arrow[ur, "\varphi"'] &
\end{tikzcd} \tag{3.4b}$$

commute. Here pr_S and pr_X denote the first and second projection morphism of the product $S \times (X \times Y)$, and pr'_X and pr'_Y are the projections belonging to $X \times Y$. $\Diamond$

Evidently, we have the following corollary, which shows that our definition is sound, i.e. that it indeed entails the special case of continuous skew product system we started from.

3.3.2 Corollary. *Every continuous skew product system on a topological space with times $\mathbf{S} \leq \mathbf{T}$ (as in Definition 2.3.5) is an abstract NDS on $\mathcal{Top}$ with times $\mathbf{S} \hookrightarrow \mathbf{T}$ (as in Definition 3.3.1).*

Furthermore, every abstract NDS on $\mathcal{Top}$ with times $\mathbf{S} \overset{\varepsilon}{\hookrightarrow} \mathbf{T}$ is a continuous skew product system with times $\varepsilon(\mathbf{S}) \leq \mathbf{T}$, where $\varepsilon(\mathbf{S})$ denotes the image of the topological monoid $\mathbf{S}$ under the embedding ε, which is isomorphic to $\mathbf{S}$.

Similarly, interpreting Definition 3.3.1 in the category of sets, we obtain nonautonomous dynamical systems.

3.3.3 Corollary. *Every nonautonomous dynamical system on a set with times $\mathbf{S} \leq \mathbf{T}$ (as in Definition 2.3.4) is an abstract NDS on $\mathcal{Set}$ with times $\mathbf{S} \hookrightarrow \mathbf{T}$ (as in Definition 3.3.1).*

Furthermore, every abstract NDS on $\mathcal{Set}$ with times $\mathbf{S} \overset{\varepsilon}{\hookrightarrow} \mathbf{T}$ is a nonautonomous dynamical system with times $\varepsilon(\mathbf{S}) \leq \mathbf{T}$, where $\varepsilon(\mathbf{S})$ denotes the image of the monoid $\mathbf{S}$ under the embedding ε, which is isomorphic to $\mathbf{S}$.

Likewise, abstract NDS on $\mathcal{Measrbl}$ correspond to measurable NDS as in Definition 2.3.6. An explicit corollary is omitted for brevity.

Next we prove that for two equal time monoids abstract NDS can be understood as a special kind of abstract dynamical system. This fact has been known for concrete cases of dynamical systems, e.g. continuous flows (topological dynamical systems as in Definition 2.3.1 where time is given by the real numbers with addition) arising as solutions of nonautonomous ordinary differential equations, cf. Chapter IV of [Sel71], especially IV.A, IV.F and Theorem IV.11. Our lemma shows that this result only depends on the algebraic structure behind dynamical systems, not on the analytic or measure theoretic framework in which it is placed.

Since nonautonomous dynamics does not lie in the main focus of this article, we keep the proof sketchy and leave some details for the reader to work out.

3.3.4 Lemma. *Let X, Y belong to a finite product category $\mathcal{C}$, let $\mathbf{T} = \langle T; +, e\rangle$ be a $\mathcal{C}$-monoid setting and $T \times (X \times Y) \xrightarrow{\Phi} X \times Y$, $T \times X \xrightarrow{\theta} X$ be morphisms in $\mathcal{C}$ satisfying the condition[8] $\Phi\,\mathrm{pr}'_X = \langle \mathrm{pr}_T, \mathrm{pr}_X\rangle\,\theta$, i.e. the X-component of Φ does not depend on Y and is given by θ. Furthermore, we require that the morphisms[9] $T \times (T \times (X \times Y)) \xrightarrow{\langle \mathrm{pr}_{T_1}, \langle \mathrm{pr}_{T_2}, \mathrm{pr}_X\rangle\rangle} T \times (T \times X)$ and $X \times Y \xrightarrow{\mathrm{pr}'_X} X$ be epi. Then the following statements are equivalent:*

(a) The triple $\left(\mathbf{T}, X \times Y, T \times (X \times Y) \xrightarrow{\Phi} X \times Y\right)$ is a dynamical system on $\mathcal{C}$.

[8] As in Definition 3.3.1, pr'_X and pr'_Y denote the projection morphisms belonging to the product $X \times Y$.

[9] Here pr_{T_1}, pr_{T_2} and pr_X denote the projection morphisms of $T \times (T \times (X \times Y))$ onto the first factor T, onto the second factor T and on the X component of the product.

(b) *The pair* $(\theta, \Phi\, \mathrm{pr}'_Y)$ *is an abstract NDS in* $\mathcal{C}$ *with times* $\mathbf{S} = \mathbf{T}$ *on* Y *with base* X*.*

Proof: The argument is based on transforming the defining condition for the dynamical system $\left((T, +, e), X \times Y, T \times (X \times Y) \xrightarrow{\Phi} X \times Y\right)$. According to Definition 3.2.1, the notion of dynamical system is built upon $(T, +, e)$ being a $\mathcal{C}$-monoid and $\left(X \times Y, T \times (X \times Y) \xrightarrow{\Phi} X \times Y\right)$ being a $\mathcal{C}$-monoid action. The latter fact is a conjunction of two commuting diagrams, (3.3a) and (3.3b). These have the form

$$\begin{tikzcd}
X \times Y \arrow[rr, dotted, bend left, "\langle !_{X\times Y} e, 1_{X\times Y}\rangle"] \arrow[drr, "1_{X\times Y}"'] & \cong T^0 \times (X \times Y) \arrow[r, "e\times(X\times Y)"] & T \times (X \times Y) \arrow[d, "\Phi"] \\
 & & X \times Y
\end{tikzcd}$$

and

$$\begin{tikzcd}
(T \times T) \times (X \times Y) \arrow[r, "+\times(X\times Y)"] & T \times (X \times Y) \arrow[dr, "\Phi"] & \\
 & & X \times Y. \\
T \times (T \times (X \times Y)) \arrow[uu, "\cong"] \arrow[r, "T\times\Phi"] & T \times (X \times Y) \arrow[ur, "\Phi"'] &
\end{tikzcd}$$

Both diagrams express that two certain morphisms f, g, starting in the same object Z (either $X \times Y$ or $T \times (T \times (X \times Y))$) and ending in the product $X \times Y$, are identical. By definition of the product this is equivalent to the fact that the equalities $f\, \mathrm{pr}'_X = g\, \mathrm{pr}'_X$ and $f\, \mathrm{pr}'_Y = g\, \mathrm{pr}'_Y$ hold. This means that we can equivalently replace each of the two diagrams by a conjunction of two commutative diagrams.

Taking into account the assumption that $\Phi\, \mathrm{pr}'_X = \langle \mathrm{pr}_T, \mathrm{pr}_X \rangle\, \theta$, we get

$$T \times \langle \langle \mathrm{pr}_T, \mathrm{pr}_X \rangle\, \theta, \Phi\, \mathrm{pr}'_Y \rangle = T \times \langle \Phi\, \mathrm{pr}'_X, \Phi\, \mathrm{pr}'_Y \rangle = T \times \Phi,$$

and thus we see that the two diagrams arising from composition with pr'_Y are precisely the ones occurring in Definition 3.3.1. The other two ones, coming from composition with pr'_X, are equivalent to the two defining diagrams of the dynamical system $\left((T, +, e), X, T \times X \xrightarrow{\theta} X\right)$. This can be seen from a short calculation using again the assumption $\Phi\, \mathrm{pr}'_X = \langle \mathrm{pr}_T, \mathrm{pr}_X \rangle\, \theta$.

We show this exemplarily for the second diagram. Denoting the canonical isomorphism between $T \times (T \times (X \times Y))$ and $(T \times T) \times (X \times Y)$ by a, the equality of interest is

$$a\, (+ \times (X \times Y))\, \Phi\, \mathrm{pr}'_X = (T \times \Phi)\, \Phi\, \mathrm{pr}'_X\,.$$

Using the projection morphisms pr_{T_1}, pr_{T_2}, pr_X and pr_Y belonging to the product $T \times (T \times (X \times Y))$ in the order of the factors read from left to right, one can rewrite the left-hand side as

$$\begin{aligned} a\,(+ \times (X \times Y))\,\Phi\,\mathrm{pr}'_X &= a \left\langle \left\langle \mathrm{pr}_{T_1}, \mathrm{pr}_{T_2} \right\rangle +, \mathrm{pr}_X, \mathrm{pr}_Y \right\rangle \Phi\,\mathrm{pr}'_X \\ &= a \left\langle \left\langle \mathrm{pr}_{T_1}, \mathrm{pr}_{T_2} \right\rangle +, \mathrm{pr}_X, \mathrm{pr}_Y \right\rangle \langle \mathrm{pr}_T, \mathrm{pr}_X \rangle\,\theta \\ &= a \left\langle \left\langle \mathrm{pr}_{T_1}, \mathrm{pr}_{T_2} \right\rangle +, \mathrm{pr}_X \right\rangle \theta \\ &= \left\langle \mathrm{pr}_{T_1}, \left\langle \mathrm{pr}_{T_2}, \mathrm{pr}_X \right\rangle \right\rangle b\,(+ \times X)\,\theta, \end{aligned}$$

where $\left\langle \mathrm{pr}_{T_1}, \left\langle \mathrm{pr}_{T_2}, \mathrm{pr}_X \right\rangle \right\rangle$ is the projection on $T \times (T \times X)$ and b is the isomorphism in diagram (3.3b) belonging to θ. Similarly, we have for the other side

$$\begin{aligned} (T \times \Phi)\,\Phi\,\mathrm{pr}'_X &= \left\langle \mathrm{pr}_{T_1}, \left\langle \mathrm{pr}_{T_2}, \langle \mathrm{pr}_X, \mathrm{pr}_Y \rangle \right\rangle \Phi \right\rangle \Phi\,\mathrm{pr}'_X \\ &= \left\langle \mathrm{pr}_{T_1}, \left\langle \mathrm{pr}_{T_2}, \langle \mathrm{pr}_X, \mathrm{pr}_Y \rangle \right\rangle \Phi \right\rangle \langle \mathrm{pr}_T, \mathrm{pr}_X \rangle\,\theta \\ &= \left\langle \mathrm{pr}_{T_1}, \left\langle \mathrm{pr}_{T_2}, \langle \mathrm{pr}_X, \mathrm{pr}_Y \rangle \right\rangle \Phi\,\mathrm{pr}'_X \right\rangle \theta \\ &= \left\langle \mathrm{pr}_{T_1}, \left\langle \mathrm{pr}_{T_2}, \langle \mathrm{pr}_X, \mathrm{pr}_Y \rangle \right\rangle \langle \mathrm{pr}_T, \mathrm{pr}_X \rangle\,\theta \right\rangle \theta \\ &= \left\langle \mathrm{pr}_{T_1}, \left\langle \mathrm{pr}_{T_2}, \mathrm{pr}_X \right\rangle \theta \right\rangle \theta \\ &= \left\langle \mathrm{pr}_{T_1}, \left\langle \mathrm{pr}_{T_2}, \mathrm{pr}_X \right\rangle \right\rangle (T \times \theta)\,\theta. \end{aligned}$$

Hence, if $\left((T, +, e), X, T \times X \xrightarrow{\theta} X\right)$ is a dynamical system, then the desired equality follows. Since $\left\langle \mathrm{pr}_{T_1}, \left\langle \mathrm{pr}_{T_2}, \mathrm{pr}_X \right\rangle \right\rangle$ is an epimorphism, also the converse implication holds. □

4 Dynamical Systems as Algebras and Coalgebras

4.1 From monoids to monads

Here we are going to explore the connection of $\mathcal{C}$-monoids and their actions to monads and monadic algebras. For this we need to assume that for some object T from $\mathcal{C}$ all products $T \times X$ exist for X in $\mathcal{C}$. This allows us to define the endofunctor $T \times -$ on $\mathcal{C}$. Of course in finite product categories, this assumption is certainly valid.

The main and only result of this subsection is a lemma connecting the commutativity conditions from the definitions of a $\mathcal{C}$-monoid and a $\mathcal{C}$-monoid action with certain commutative diagrams for two derived natural transformations. In the next subsection we shall use this lemma to translate abstract $\mathcal{C}$-monoids into monads and $\mathcal{C}$-monoid actions into monadic algebras for the monad associated with the $\mathcal{C}$-monoid.

4.1.1 Lemma. *Suppose $\mathcal{C}$ is a finite product category with the endo-functor $T \times -$ for some object T and let $(T, +, e)$ be a $\mathcal{C}$-monoid setting. For every object X from $\mathcal{C}$ let the morphisms η_X and δ_X be defined by the commutativity of the following diagrams*

$$\begin{tikzcd}
X \arrow[r, phantom, "\cong"] \arrow[rr, dashed, bend left, "=:\eta_X=\langle !_X e, 1_X\rangle"] & T^0 \times X \arrow[r, "e\times X"'] & T \times X
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & T \times (T \times X) \arrow[dl, "a_1"'] \arrow[dd, "\exists ! a_7"] \arrow[dd, phantom, shift right=6ex, "\langle a_1, a_2 a_3\rangle = T\times a_3"] \arrow[drrr, "a_2"] \arrow[dddd, dashed, bend left, "=:\delta_X=\langle a_7+, a_2 a_4\rangle"] & & \\
T & & & T \times X \arrow[ddl, "a_3"'] \arrow[ddd, "a_4"] \\
 & T \times T \arrow[ul, "a_5"] \arrow[dr, "a_6"] \arrow[ddl, "+"'] & & \\
 & & T & \\
T & T \times X \arrow[l, "a_3"'] \arrow[rr, "a_4"] & & X
\end{tikzcd}$$

(where the morphisms a_i, $i \neq 7$, are projection morphisms of the respective products). Then

(a) $1_{\mathcal{C}} \xrightarrow{\eta} T \times -$ *and* $T \times (T \times -) \xrightarrow{\delta} T \times -$ *are natural transformations.*

(b) For an object $X \in \mathcal{C}$ and the projection morphism $T \times X \xrightarrow{a_3} T$, the conditions $a_3 \langle !_T e, 1_T\rangle + = a_3 \cong (e \times T)+ = a_3$ and $\eta_{T\times X}\delta_X = 1_{T\times X}$ are equivalent.

(c) If $(T, +, e)$ satisfies condition (3.1a), then the diagrams

$$\begin{tikzcd}
T \times X \arrow[r, "\eta_{T\times X}"] \arrow[dr, "1_{T\times X}"'] & T \times (T \times X) \arrow[d, "\delta_X"] \\
 & T \times X
\end{tikzcd} \tag{4.1}$$

commute for all $X \in \mathcal{C}$. If the projection morphism $T \times X \xrightarrow{a_3} T$ is epi for one $X \in \mathcal{C}$, then the converse implication is true, as well.

(d) For an object $X \in \mathcal{C}$ and the projection morphism $T \times X \xrightarrow{a_3} T$, the conditions $a_3 \langle 1_T, !_T e\rangle + = a_3 \cong (T \times e)+ = a_3$ and $(T \times \eta_X)\delta_X = 1_{T\times X}$ are equivalent.

(e) *If $(T, +, e)$ satisfies condition (3.1b), then the diagrams*

$$\begin{tikzcd} T \times X \arrow[r, "T\times\eta_X"] \arrow[dr, "1_{T\times X}"'] & T \times (T \times X) \arrow[d, "\delta_X"] \\ & T \times X \end{tikzcd} \tag{4.2}$$

commute for all $X \in \mathcal{C}$. If the projection morphism $T \times X \xrightarrow{a_3} T$ is epi for one $X \in \mathcal{C}$, then the converse implication is true, as well.

(f) *For an object $X \in \mathcal{C}$ and the isomorphism $T \times (T \times T) \xrightarrow{a_{12}} (T \times T) \times T$ the following equalities hold*

$$\delta_{T\times X} a_7 + = (T \times a_7) a_{12} (+ \times T) + \tag{4.3}$$

$$(T \times \delta_X) a_7 + = (T \times a_7)(T \times +) + . \tag{4.4}$$

Furthermore, the equality $\delta_{T\times X}\delta_X = (T \times \delta_X)\delta_X$ is equivalent to

$$\delta_{T\times X} a_7 + = (T \times \delta_X) a_7 +$$

and consequently to $(T \times a_7)a_{12}(+ \times T)+ = (T \times a_7)(T \times +)+$.

(g) *If $(T, +, e)$ satisfies condition (3.1c), then the diagrams*

$$\begin{tikzcd} T \times (T \times (T \times X)) \arrow[r, "\delta_{T\times X}"] \arrow[d, "T\times\delta_X"'] & T \times (T \times X) \arrow[d, "\delta_X"] \\ T \times (T \times X) \arrow[r, "\delta_X"'] & T \times X \end{tikzcd} \tag{4.5}$$

commute for all $X \in \mathcal{C}$. If $T \times (T \times (T \times X)) \xrightarrow{T\times a_7} T \times (T \times T)$ is an epimorphism for one $X \in \mathcal{C}$, then the converse implication is true, as well.

(h) *If $(T, +, e)$ is a $\mathcal{C}$-monoid, then $(T \times -, \delta, \eta)$ is a monad. If the projection morphisms $T \times X \xrightarrow{a_3} T$ and $T \times (T \times (T \times Y)) \xrightarrow{T\times a_7} T \times (T \times T)$ are epi for some objects $X, Y \in \mathcal{C}$, then the converse also holds.*

(i) *Let an object $X \in \mathcal{C}$ and a morphism $T \times X \xrightarrow{\varphi} X$ be given. For the isomorphism $T \times (T \times X) \xrightarrow{a_{15}} (T \times T) \times X$ from condition (3.3b) the following equality holds*

$$a_{15}(+ \times X) = \delta_X,$$

whence diagram (3.3b) commutes if and only if

$$\begin{tikzcd} T \times (T \times X) \arrow[r, "T\times\varphi"] \arrow[d, "\delta_X"'] & T \times X \arrow[d, "\varphi"] \\ T \times X \arrow[r, "\varphi"'] & X \end{tikzcd} \tag{4.6a}$$

commutes, and diagram (3.3a) *commutes if and only if*

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "1_X"'] & T \times X \arrow[d, "\varphi"] \\ & X \end{tikzcd} \tag{4.6b}$$

commutes.

Proof: We start the proof with reminding the reader how the functor $T \times -$ operates on morphisms $X \xrightarrow{f} Y$, where X, Y are arbitrary objects of $\mathcal{C}$. The morphism $T \times X \xrightarrow{T \times f} T \times Y$ is uniquely determined by the commutativity of the following diagram (see also Example 2.2.12(e))

$$\begin{tikzcd} & T \\ T \times X \arrow[ur, "a_3"] \arrow[r, dashed, "T \times f"] \arrow[d, "a_4"'] & T \times Y \arrow[u, "b_3"'] \arrow[d, "b_4"] \\ X \arrow[r, "f"'] & Y, \end{tikzcd} \tag{4.7}$$

where the morphisms a_3, a_4, b_3, b_4 are projection morphisms belonging to the products $T \times X$ and $T \times Y$.

(a) To show that η and δ are natural transformations, we fix objects $X, Y \in \mathcal{C}$ and a morphism $X \xrightarrow{f} Y$ between them. It has to be shown that

$$f\eta_Y = \eta_x T \times f \qquad \text{and} \qquad T \times (T \times f)\delta_Y = \delta_X T \times f.$$

For the first equality let us mention that the commutativity of the following diagram is equivalent to the definition of η_X

$$\begin{tikzcd} & X \\ X \arrow[ur, "1_X"] \arrow[r, dashed, "\eta_X"] \arrow[d, "!_X"'] & T \times X \arrow[u, "a_4"'] \arrow[d, "a_3"] \\ T^0 \arrow[r, "e"'] & T, \end{tikzcd} \tag{4.8}$$

since $T \times X$ is a product with projections a_3 and a_4. The proof of the desired equality is contained in the commutativity of the following diagram:

$$
\begin{tikzcd}
 & & & & X \arrow[dddd, "f"] \\
X \arrow[rrr, "\eta_X"] \arrow[urrrr, "1_X"] \arrow[dr, "!_X"] \arrow[dd, "f"'] & & & T \times X \arrow[ur, "a_4"'] \arrow[dl, "a_3"] \arrow[dd, "T\times f"] & \\
 & T^0 \arrow[r, "e"] & T & & \\
Y \arrow[rrr, "\eta_Y"] \arrow[drrrr, "1_Y"] \arrow[ur, "!_Y"] & & & T \times Y \arrow[ul, "b_3"'] \arrow[dr, "b_4"] & \\
 & & & & Y,
\end{tikzcd}
$$

where the triangle on the left commutes by the definition of the terminal object T^0, the central quadrangles and the triangles on top and bottom commute by the definition of η (cf. (4.8)), and the triangle and the quadrangle on the right commute by the definition of $T \times f$ (cf. (4.7)).

From this it follows that

$$
\begin{aligned}
f\eta_Y b_3 &= f!_Y e = !_X e = \eta_X a_3 = \eta_X (T \times f) b_3 \\
f\eta_Y b_4 &= f 1_Y = 1_X f = \eta_X a_4 f = \eta_X (T \times f) b_4,
\end{aligned}
$$

and the conjunction of these two equalities is equivalent to $f\eta_Y = \eta_X T \times f$ since $T \times Y$ together with b_3, b_4 is a product.

The proof of the remaining equality, $T \times (T \times f)\delta_Y = \delta_X T \times f$, is a bit more technical but uses the same ideas as just presented. First we link the defining diagrams for δ_X and δ_Y in the following scheme:

$$
\begin{tikzcd}
 & T \times (T \times X) \arrow[rrrr, "T\times(T\times f)"] \arrow[dl, "a_1"'] \arrow[dd, "a_7"] \arrow[ddrr, "a_2"] \arrow[dddd, bend left, "\delta_X"] & & & & T \times (T \times Y) \arrow[dl, "b_1"'] \arrow[dd, "b_7"] \arrow[ddrr, "b_2"] \arrow[dddd, bend left, "\delta_Y"] & & \\
T \arrow[rrrr, "1_T"] & & & & T & & & \\
 & T \times T \arrow[ul, "a_5"] \arrow[dr, "a_6"] \arrow[dddl, "+"] \arrow[rrrr, bend right=10, "1_{T\times T}"'] & & T \times X \arrow[rr, "T\times f"] \arrow[dl, "a_3"'] \arrow[ddd, "a_4"] & & T \times T \arrow[ul, "a_5"] \arrow[rr] \arrow[dr, "a_6"] \arrow[dddl, "+"] & & T \times Y \arrow[dl, "b_3"'] \arrow[ddd, "b_4"] \\
 & & T \arrow[rrrr, "1_T"] & & & & T & \\
 & T \times X \arrow[rrrr, "T\times f"] \arrow[dl, "a_3"'] \arrow[drr, "a_4"] & & & & T \times Y \arrow[dl, "b_3"'] \arrow[drr, "b_4"] & & \\
T \arrow[rrrr, bend right=10, "1_T"] & & & X \arrow[rrrr, bend right=10, "f"] & T & & & Y.
\end{tikzcd}
$$

Again, as $T \times Y$ is a product with projections b_3, b_4, the desired equality is equivalent to the conjunction of

$$T \times (T \times f)\delta_Y b_3 = \delta_X (T \times f) b_3 \quad \text{and} \quad T \times (T \times f)\delta_Y b_4 = \delta_X (T \times f) b_4.$$

First, note that

$$a_7 1_{T\times T} a_5 = a_7 a_5 1_T \overset{\text{def } a_7}{=} a_1 1_T \overset{\text{def } T\times(T\times f)}{=} T \times (T \times f) b_1 \overset{\text{def } b_7}{=} T \times (T \times f) b_7 a_5$$

and

$$\begin{aligned} a_7 1_{T\times T} a_6 = a_7 a_6 1_T \overset{\text{def } a_7}{=} a_2 a_3 1_T \overset{\text{def } T\times f}{=} a_2 (T \times f) b_3 \overset{\text{def } T\times(T\times f)}{=} T \times (T \times f) b_2 b_3 \\ \overset{\text{def } b_7}{=} T \times (T \times f) b_7 a_6, \end{aligned}$$

whence $a_7 1_{T\times T} = T \times (T \times f) b_7$ follows due to $T \times T$ being a product with projections a_5, a_6. Using this one obtains

$$\begin{aligned} \delta_X (T \times f) b_3 \overset{\text{def } T\times f}{=} \delta_X a_3 1_T \overset{\text{def } \delta_X}{=} a_7 + 1_T = a_7 1_{T\times T} + \overset{\text{v.s.}}{=} T \times (T \times f) b_7 + \\ \overset{\text{def } \delta_Y}{=} T \times (T \times f) \delta_Y b_3. \end{aligned}$$

Likewise, one can show

$$\begin{aligned} \delta_X (T \times f) b_4 \overset{\text{def } T\times f}{=} \delta_X a_4 f \overset{\text{def } \delta_X}{=} a_2 a_4 f \overset{\text{def } T\times f}{=} a_2 (T \times f) b_4 \\ \overset{\text{def } T\times(T\times f)}{=} T \times (T \times f) b_2 b_4 \overset{\text{def } \delta_Y}{=} T \times (T \times f) \delta_Y b_4, \end{aligned}$$

finishing the proof of this item.

(b) We fix an object X of $\mathcal{C}$ and start to demonstrate a number of auxiliary equalities that are needed for our equivalence. The following diagram derived from diagram (4.8) will be useful

$$\begin{tikzcd}
 & T \times X \\
T \times X \arrow[ur, "1_{T\times X}"] \arrow[r, "\eta_{T\times X}"] \arrow[d, "!_{T\times X}"'] & T \times (T \times X) \arrow[u, "a_2"'] \arrow[d, "a_1"] \\
T^0 \arrow[r, "e"'] & T,
\end{tikzcd}$$

since it expresses the definition of $\eta_{T\times X}$. First, it is

$$\eta_{T\times X} \delta_X a_4 \overset{\text{def } \delta_X}{=} \eta_{T\times X} a_2 a_4 \overset{\text{def } \eta_{T\times X}}{=} 1_{T\times X} a_4 = a_4.$$

Second, we have

$$\eta_{T\times X} a_7 a_5 \overset{\text{def } a_7}{=} \eta_{T\times X} a_1 \overset{\text{def } \eta_{T\times X}}{=} !_{T\times X} e = a_3 !_T e = a_3 \langle !_T e, 1_T \rangle a_5$$

and

$$\eta_{T\times X} a_7 a_6 \overset{\text{def } a_7}{=} \eta_{T\times X} a_2 a_3 \overset{\text{def } \eta_{T\times X}}{=} 1_{T\times X} a_3 = a_3 1_T = a_3 \langle !_T e, 1_T \rangle a_6,$$

whence we obtain $\eta_{T\times X} a_7 = a_3 \langle !_T e, 1_T \rangle$ as $T \times T$ is a product with projections a_5 and a_6. Consequently, we get $\eta_{T\times X} \delta_X a_3 \overset{\text{def } \delta_X}{=} \eta_{T\times X} a_7 + \overset{\text{v.s.}}{=} a_3 \langle !_T e, 1_T \rangle +$. Since $T \times X$ with a_3 and a_4 is a product, the equality $\eta_{T\times X} \delta_X = 1_{T\times X}$ is equivalent to the conjunction of $\eta_{T\times X} \delta_X a_3 = a_3$ and $\eta_{T\times X} \delta_X a_4 = a_4$, the latter of which is generally true by what has been shown above. Hence the equality $\eta_{T\times X} \delta_X = 1_{T\times X}$ holds if and only if $a_3 \langle !_T e, 1_T \rangle + \overset{\text{v.s.}}{=} \eta_{T\times X} \delta_X a_3 = a_3$.

(c) For any X in $\mathcal{C}$, condition (3.1a) implies, by composition from the left with the respective projection morphism $T \times X \xrightarrow{a_3} T$, that $a_3 \langle !_T e, 1_T \rangle + = a_3$. This is, by item (b), equivalent to the commutativity of diagram (4.1).

If, conversely, diagram (4.1) commutes for all X in $\mathcal{C}$, this means that the equality $a_3 \langle !_T e, 1_T \rangle + = a_3$ holds for every object X of $\mathcal{C}$. If a_3 can be cancelled from the left in this equality for at least one object X of $\mathcal{C}$ (e.g. if a_3 is epi), then obviously condition (3.1a), i.e. $\langle !_T e, 1_T \rangle + = 1_T$, follows.

(d) This proof is similar to that of item (b). We fix an object $X \in \mathcal{C}$ and start to show some equalities that are needed for the statement. The following diagram expressing the definition of $T \times \eta_X$ can be obtained from (4.7):

$$\begin{tikzcd}
 & T \\
T \times X \arrow[ur, "a_3"] \arrow[r, "T\times\eta_X"] \arrow[d, "a_4"'] & T \times (T \times X) \arrow[u, "a_1"'] \arrow[d, "a_2"] \\
X \arrow[r, "\eta_X"'] & T \times X.
\end{tikzcd}$$

First, it is

$$(T \times \eta_X)\delta_X a_4 \overset{\text{def } \delta_X}{=} (T \times \eta_X) a_2 a_4 \overset{\text{def } T\times\eta_X}{=} a_4 \eta_X a_4 \overset{\text{def } \eta_X}{=} a_4 1_X = a_4.$$

Second, we have

$$(T \times \eta_X) a_7 a_5 \overset{\text{def } a_7}{=} (T \times \eta_X) a_1 \overset{\text{def } T\times\eta_X}{=} a_3 = a_3 1_T = a_3 \langle 1_T, !_T e \rangle a_5$$

and

$$(T \times \eta_X)a_7 a_6 \stackrel{\text{def } a_7}{=} (T \times \eta_X)a_2 a_3 \stackrel{\text{def } T\times\eta_X}{=} a_4 \eta_X a_3 \stackrel{\text{def } \eta_X}{=} a_4 !_X e = !_{T\times X} e$$
$$= a_3 !_T e = a_3 \langle 1_T, !_T e\rangle a_6,$$

whence we obtain $(T \times \eta_X)a_7 = a_3 \langle 1_T, !_T e\rangle$ as $T \times T$ is a product with projections a_5 and a_6. Consequently, we get

$$(T \times \eta_X)\delta_X a_3 \stackrel{\text{def } \delta_X}{=} (T \times \eta_X)a_7 + \stackrel{\text{v.s.}}{=} a_3 \langle 1_T, !_T e\rangle + .$$

Since $T \times X$ with a_3, a_4 is a product, the equality $(T \times \eta_X)\delta_X = 1_{T\times X}$ is equivalent to the conjunction of $(T \times \eta_X)\delta_X a_3 = a_3$ and $(T \times \eta_X)\delta_X a_4 = a_4$, the latter of which is generally true by what has been shown above. Therefore, the condition $(T \times \eta_X)\delta_X = 1_{T\times X}$ is satisfied if and only if the equality $a_3 \langle 1_T, !_T e\rangle + \stackrel{\text{v.s.}}{=} (T \times \eta_X)\delta_X a_3 = a_3$ holds.

(e) This proof is similar to that of item (c). For any X in $\mathcal{C}$, condition (3.1b) implies, by composition from the left with the respective projection morphism $T \times X \xrightarrow{a_3} T$, the equality $a_3 \langle 1_T, !_T e\rangle + = a_3$. The latter is, by item (d), equivalent to the commutativity of diagram (4.2).

If, conversely, diagram (4.2) commutes for all X in $\mathcal{C}$, this means that the equality $a_3 \langle 1_T, !_T e\rangle + = a_3$ holds for every object X of $\mathcal{C}$. If a_3 can be cancelled from the left in this equality for at least one object X of $\mathcal{C}$ (e.g. if a_3 is epi), then obviously condition (3.1b), i.e. $\langle 1_T, !_T e\rangle + = 1_T$, follows.

(f) Again we consider a fixed object X from $\mathcal{C}$. For this part we will need the defining diagrams for $T \times \delta_X$, $T \times a_7$, $+ \times T$, $\delta_{T\times X}$ and the not yet specified canonical isomorphism a_{12} from diagram (3.1c):

$$\begin{tikzcd}
 & & T \\
T \times (T \times (T \times X)) \arrow[r, "T\times\delta_X"] \arrow[urr, "a_1'"] \arrow[d, "a_2'"'] & T \times (T \times X) \arrow[ur, "a_1"'] \arrow[d, "a_2"] & \\
T \times (T \times X) \arrow[r, "\delta_X"'] & T \times X, &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & T \\
T \times (T \times (T \times X)) \arrow[r, "T\times a_7"] \arrow[urr, "a_1'"] \arrow[d, "a_2'"'] & T \times (T \times T) \arrow[ur, "a_8"'] \arrow[d, "a_9"] & \\
T \times (T \times X) \arrow[r, "a_7"'] & T \times T, &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & T \times (T \times (T \times X)) \arrow[dll, "a'_1"'] \arrow[dd, "T\times a_1 = a'_7"] \arrow[dddd, bend left, "\delta_{T\times X}"] \arrow[drr, "a'_2"] & & \\
T & & & & T \times (T \times X) \arrow[ddl, "a_1"'] \arrow[ddd, "a_2"] \\
 & & T \times T \arrow[ull, "a_5"] \arrow[ddll, "+"'] \arrow[dr, "a_6"] & & \\
 & & & T & \\
T & & T \times (T \times X) \arrow[ll, "a_1"'] \arrow[rr, "a_2"] & & T \times X,
\end{tikzcd}$$

$$\begin{tikzcd}
 & T \times (T \times T) \arrow[r, "a_9"] \arrow[dl, "T\times a_5"'] \arrow[d, dashed, "a_{12} = \langle T\times a_5, a_9 a_6\rangle"] & T \times T \arrow[d, "a_6"] \\
T \times T & (T \times T) \times T \arrow[l, "a_{10}"] \arrow[r, "a_{11}"'] & T,
\end{tikzcd}$$

$$\begin{tikzcd}
T \times T \arrow[d, "+"'] & (T \times T) \times T \arrow[l, "a_{10}"'] \arrow[r, "a_{11}"] \arrow[d, "+\times T"] & T \\
T & T \times T. \arrow[l, "a_5"] \arrow[ur, "a_6"'] &
\end{tikzcd}$$

Now we show equalities (4.3) and (4.4). To this end we note that

$$\begin{aligned}
\delta_{T\times X} a_7 a_5 &\overset{\text{def } a_7}{=} \delta_{T\times X} a_1 \overset{\text{def } \delta_{T\times X}}{=} a'_7 + \overset{\text{def } a'_7}{=} (T \times a_1) + \overset{\text{def } a_7}{=} T \times (a_7 a_5) + \\
&= (T \times a_7)(T \times a_5) + \overset{\text{def } a_{12}}{=} (T \times a_7) a_{12} a_{10} + \overset{\text{def } +\times T}{=} (T \times a_7) a_{12} (+ \times T) a_5
\end{aligned}$$

and

$$\begin{aligned}
\delta_{T\times X} a_7 a_6 &\overset{\text{def } a_7}{=} \delta_{T\times X} a_2 a_3 \overset{\text{def } \delta_{T\times X}}{=} a'_2 a_2 a_3 \overset{\text{def } a_7}{=} a'_2 a_7 a_6 \overset{\text{def } T\times a_7}{=} (T \times a_7) a_9 a_6 \\
&\overset{\text{def } a_{12}}{=} (T \times a_7) a_{12} a_{11} \overset{\text{def } +\times T}{=} (T \times a_7) a_{12} (+ \times T) a_6,
\end{aligned}$$

whence we obtain that $\delta_{T\times X} a_7 = (T \times a_7) a_{12} (+ \times T)$ as $T \times T$ is a product with projections a_5 and a_6. Composition with $+$ on the right-hand side then yields equality (4.3).

Equality (4.4) follows from

$$(T \times \delta_X) a_7 \overset{\text{def } a_7}{=} (T \times \delta_X)(T \times a_3) = T \times (\delta_X a_3) \overset{\text{def } \delta_X}{=} T \times (a_7 +) = (T \times a_7)(T \times +)$$

by composition with $+$ on the right-hand side.

Note that

$$\delta_{T\times X}\delta_X a_4 \overset{\text{def}\,\delta_X}{=} \delta_{T\times X} a_2 a_4 \overset{\text{def}\,\delta_{T\times X}}{=} a_2' a_2 a_4 \overset{\text{def}\,\delta_X}{=} a_2' \delta_X a_4 \overset{\text{def}\,T\times\delta_X}{=} (T\times\delta_X) a_2 a_4$$
$$\overset{\text{def}\,\delta_X}{=} (T\times\delta_X)\delta_X a_4.$$

This implies, as $T \times X$ with a_3, a_4 is a product, that $\delta_{T\times X}\delta_X = (T\times\delta_X)\delta_X$ is equivalent to $\delta_{T\times X}\delta_X a_3 = (T\times\delta_X)\delta_X a_3$. Since $\delta_X a_3 = a_7+$ holds by definition of δ_X, the previous equality is equivalent to $\delta_{T\times X} a_7 + = (T\times\delta_X) a_7 +$. Combining this with equalities (4.3) and (4.4) finishes the proof of this item.

(g) If diagram (3.1c) commutes, then for every X in $\mathcal{C}$, one obtains, by composition with $T \times a_7$ from the left-hand side, the equality

$$(T\times a_7)a_{12}(+\times T)+ = (T\times a_7)(T\times +)+,$$

which, by the previous item, is equivalent to commutativity of diagram (4.5).

If, conversely, diagram (4.5) commutes for all X in $\mathcal{C}$ and for some object X of $\mathcal{C}$ the morphism $T \times a_7$ is cancellable in the equation

$$(T\times a_7)a_{12}(+\times T)+ = (T\times a_7)(T\times +)+,$$

then also the converse implication is true. This is, for instance, the case if $T \times a_7$ is an epimorphism.

(h) If $(T, +, e)$ is a $\mathcal{C}$-monoid, then the three diagrams (3.1) commute. Using the items (c), (e) and (g) above, one obtains from this that for any object X in $\mathcal{C}$ the diagrams (4.1), (4.2) and (4.5) commute, equivalently that $(T\times -, \delta, \eta)$ is a monad.

The additional assumptions on the morphisms in this item ensure that the implications stated in items (c), (e) and (g) are actually logical equivalences. Hence, the shown implication can be reversed and one obtains that $(T, +, e)$ is a $\mathcal{C}$-monoid.

(i) We fix an object X of $\mathcal{C}$ and a morphism $T \times X \xrightarrow{\varphi} X$. For this part we need the defining diagrams for the morphism $+ \times X$ and the isomorphism a_{15} from diagram (3.3b):

$$\begin{tikzcd}
 & T \times (T \times X) \arrow[r, "a_2"] \arrow[dl, "a_7"'] \arrow[d, dashed, "a_{15} = \langle a_7, a_2 a_4 \rangle"] & T \times X \arrow[d, "a_4"] \\
T \times T & (T \times T) \times X \arrow[l, "a_{13}"] \arrow[r, "a_{14}"'] & X,
\end{tikzcd}$$

$$\begin{tikzcd}
T \times T \arrow[d, "+"'] & (T \times T) \times X \arrow[l, "a_{13}"'] \arrow[r, "a_{14}"] \arrow[d, "+ \times X"] & X \\
T & T \times X. \arrow[l, "a_3"] \arrow[ur, "a_4"'] &
\end{tikzcd}$$

First we infer from the equalities

$$a_{15}(+ \times X)a_3 \overset{\text{def} + \times X}{=} a_{15}a_{13}+ \overset{\text{def}\, a_{15}}{=} a_7+ \overset{\text{def}\, \delta_X}{=} \delta_X a_3$$

and

$$a_{15}(+ \times X)a_4 \overset{\text{def} + \times X}{=} a_{15}a_{14} \overset{\text{def}\, a_{15}}{=} a_2 a_4 \overset{\text{def}\, \delta_X}{=} \delta_X a_4$$

that $a_{15}(+ \times X) = \delta_X$. With this condition diagram (3.3b) becomes

$$\begin{tikzcd}
(T \times T) \times X \arrow[r, "+ \times X"] & T \times X \arrow[dr, "\varphi"] & \\
 & & X, \\
T \times (T \times X) \arrow[uu, "\cong (a_{15})"] \arrow[uur, "\delta_X"] \arrow[r, "T \times \varphi"] & T \times X \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

and since the upper triangle commutes, (3.3b) commutes if and only if (4.6a) commutes. Furthermore, by definition of η, the diagrams (3.3a) and (4.6b) are identical. □

The previous lemma enables us to characterise abstract dynamical systems in terms of monadic algebras for the endo-functor $T \times -$ on $\mathcal{C}$.

4.2 From abstract dynamical systems to monadic algebras

Here we finally relate our definition of abstract dynamical system on finite product categories to the well-known algebraic concept of monadic algebra.

4.2.1 Proposition. *Let $\mathcal{C}$ be a finite product category and T one of its objects. Suppose $(T, +, e)$ is a $\mathcal{C}$-monoid setting and $1_{\mathcal{C}} \xrightarrow{\eta} T \times -$ and $T \times (T \times -) \xrightarrow{\delta} T \times -$ are the associated natural transformations as in Lemma 4.1.1(a). Furthermore, let X be an object of $\mathcal{C}$ with a morphism $T \times X \xrightarrow{\varphi} X$. Provided that*

$$\left((T, +, e), X, T \times X \xrightarrow{\varphi} X\right)$$

is a dynamical system on $\mathcal{C}$ then $(T \times -, \delta, \eta)$ is a monad and $\left(X, T \times X \xrightarrow{\varphi} X\right)$ is a monadic $(T \times -)$-algebra for this monad.

If, for certain objects Y, Z of $\mathcal{C}$, the morphism $T \times Z \xrightarrow{a_3} T$ and the morphism $T \times (T \times (T \times Y)) \xrightarrow{T \times a_7} T \times (T \times T)$ mentioned in Lemma 4.1.1(h) are epi, then also the converse implication holds.

Proof: If $((T, +, e), X, T \times X \xrightarrow{\varphi} X)$ is a dynamical system on $\mathcal{C}$, then $(T, +, e)$ is a $\mathcal{C}$-monoid, so by Lemma 4.1.1(h), $(T \times -, \delta, \eta)$ is a monad. Furthermore, (X, φ) is a $\mathcal{C}$-monoid action, so diagrams (3.3a) and (3.3b) commute, which, by Lemma 4.1.1(i), is equivalent to the commutativity of diagrams (4.6b) and (4.6a). This, however, means that (X, φ) is a monadic $(T \times -)$-algebra w.r.t. the monad $(T \times -, \delta, \eta)$.

Under the additional assumptions, the implication in Lemma 4.1.1(h) can be reversed, which shows the second part of the proposition. $\square$

A much more concise formulation of this result is achieved if one starts with a monoid instead of a monoid setting as in the following corollary:

4.2.2 Corollary. *Let $\mathcal{C}$ be a finite product category and T, X be objects of $\mathcal{C}$ with a morphism $T \times X \xrightarrow{\varphi} X$. Furthermore, let $(T, +, e)$ be a $\mathcal{C}$-monoid and $(T \times -, \delta, \eta)$ the associated monad as in Lemma 4.1.1(h). Then*

$\left((T, +, e), X, T \times X \xrightarrow{\varphi} X\right)$ *is a dynamical system on $\mathcal{C}$*	*if and only if*	$\left(X, T \times X \xrightarrow{\varphi} X\right)$ *is a monadic $(T \times -)$-algebra for $(T \times -, \delta, \eta)$.*

Proof: Note that the additional assumptions in Proposition 4.2.1 have only been needed to show that $(T, +, e)$ is a $\mathcal{C}$-monoid provided that $(T \times -, \delta, \eta)$ is a monad. As the conclusion of this implication is already contained in the assumptions of the corollary, the same proof as for the proposition works, just using the part involving Lemma 4.1.1(i). $\square$

It is now easy to see that the connection exhibited in the previous corollary can be formalised as an isomorphism between categories.

4.2.3 Remark. For a finite product category $\mathcal{C}$, any two objects T, X, a morphism $T \times X \xrightarrow{\varphi} X$ and a $\mathcal{C}$-monoid $(T, +, e)$ with associated monad $(T \times -, \delta, \eta)$ as in Lemma 4.1.1(h), mapping

$$\left((T, +, e), X, T \times X \xrightarrow{\varphi} X\right) \mapsto \left(X, T \times X \xrightarrow{\varphi} X\right)$$

induces a categorical equivalence (even an isomorphism) between the category of abstract dynamical systems on $\mathcal{C}$ w.r.t. the $\mathcal{C}$-monoid $(T, +, e)$ and that of monadic $(T \times -)$-algebras for the associated monad $(T \times -, \delta, \eta)$. This is so because the condition for a morphism $X \xrightarrow{h} Y$ of $\mathcal{C}$ to be a morphism of dynamical systems $\left((T, +, e), X, T \times X \xrightarrow{\varphi} X\right)$ and $\left((T, +, e), Y, T \times Y \xrightarrow{\psi} Y\right)$ is precisely the same as for being a morphism of $(T \times -)$-algebras, namely that the diagram

$$\begin{tikzcd} T \times X \arrow[r, "T\times h"] \arrow[d, "\varphi"'] & T \times Y \arrow[d, "\psi"] \\ X \arrow[r, "h"] & Y \end{tikzcd}$$

commutes. Therefore, the assignment above extends to a functor that maps morphisms identically and has the obvious inverse functor.

In particular, if we combine the latter observation with Corollary 3.2.2, we obtain that the category of topological dynamical systems over a fixed topological monoid $\mathbf{T} = \langle T; +, 0\rangle$, which are the dynamical systems on $\mathcal{T}\!op$ for this particular $\mathcal{T}\!op$-monoid, is isomorphic to the category of $(T \times -)$-algebras for the associated monad $(T \times -, \delta, \eta)$ as given in Lemma 4.1.1(h).

In this context the canonically given notion of isomorphism in the category of $(T \times -)$-algebras translates to the well-known concept of *topological conjugacy* from the world of dynamical systems.

For example, in Section 2.3.2 of [Ber01] a prototypical example of a chaotic dynamical system is studied. It is a discrete time system as introduced at the beginning of Subsection 2.3, induced by iterating the logistic map on a certain Cantor set Λ within the real unit interval, viewed as a topological subspace X of the real numbers with the topology being given by the absolute value metric. In Theorem 2.20, Berger examines the topological dynamical system over the discrete topological monoid $\langle \mathbb{N}; +, 0\rangle$ given by $\varphi\colon \mathbb{N} \times X \to X$, where $\varphi(n, x) := f^n(x)$, and $f\colon X \to X$ is defined by $f(x) := \mu x(1 - x)$ for $x \in X$ and the special choice $\mu := 3.839$. The space X is partitioned into two disjoint parts and each state $x \in X$ is mapped to an ω-sequence $h(x) \in 2^\omega$ of indices zero and one, indicating which of the two parts the respective n-th iterate $\varphi(n, x) = f^n(x)$ belongs to. Thereby, Berger establishes that the particular discrete time dynamical system is isomorphic to a so-called *subshift of finite type*. The latter one is readily seen to fulfil the criteria of a chaotic system.

The condition that needs to be shown in the proof of the mentioned theorem is precisely that the two associated monadic algebras are isomorphic: the mapping h has to be a homeomorphism (an isomorphism in the category $\mathcal{T}\!op$) satisfying the condition that for every $x \in X$ shifting the sequence assigned to x to the left yields the sequence assigned to $f(x)$.

In a similar way, other category theoretic concepts and constructions, e.g. existing limits for monadic algebras, can be shifted both ways between the algebraic world of $(T \times -)$-algebras and the analytic world of dynamical systems.

4.3 Connections to coalgebras

The aim of this part is to establish a link between abstract dynamical systems that have now been understood as monadic algebras for the endo-functor $T \times -$ and coalgebras for another signature functor. It will turn out that these coalgebras will also carry a comonadic structure in a natural way.

The motivation for the rest of this section comes from regarding Corollary 4.2.2 in the special case of $\mathcal{C} = \mathcal{Set}$ and at first forgetting about monadicity conditions. What remains is an algebra $(X, \varphi\colon T \times X \to X)$ of signature[10] T on the state space X. There is an easy construction (recall Example 2.2.20), well-known from computer science as *currying*, that transforms every mapping $\varphi\colon T \times X \to X$ into a mapping $\psi\colon X \to X^T$, where X^T denotes the set of all mappings from T to X. The morphism ψ sends every state $x \in X$ to the mapping $\psi(x)\colon T \to X$ assigning to all time points $t \in T$ the evolved state $\varphi(t, x)$ derived from x. Evidently, the mapping ψ suffices to encode all the information about state transitions that is contained in φ, i.e. the currying operation can be reversed by assigning to every pair $(t, x) \in T \times X$ the state $\psi(x)(t)$, thus re-obtaining φ from ψ.

Consequently, in $\mathcal{Set}$ there is a one-to-one correspondence between mappings of the form $\varphi\colon T \times X \to X$ and $\psi\colon X \to X^T$ or, in other words, between algebras $(X, \varphi\colon T \times X \to X)$ and coalgebras $\left(X, \psi\colon X \to X^T\right)$ for the hom-functor $-^T = \mathrm{Hom}(T, -)$.

This encourages the question, how the latter phenomenon can be generalised to arbitrary abstract categories. To this end the first problem that has to be solved is that in the case of $\mathcal{Set}$, the hom-functor $-^T$ turns out to be an endo-functor, and that in fact the category $\mathcal{Set}$ and its subcategories are basically the only cases when this happens (as hom-sets always have to be sets). Our search for an appropriate replacement (or definition) of the object X^T leads us back to the original idea of currying. In fact the one-to-one correspondence between mappings as described above in the case of dynamical systems on $\mathcal{Set}$ is a bit more general: every mapping $\varphi\colon T \times X \to Y$ in $\mathcal{Set}$ can be translated into a mapping $\psi\colon X \to Y^T$ and vice versa. However, this is the defining property of an adjunction between the endo-functors $T \times -$ and $-^T$. It turns out that this is the right point of view for a generalisation to arbitrary categories, which as a side effect ensures that algebras for $T \times -$ and coalgebras for the other functor are uniquely related.

[10]Readers familiar with the modelling of classical universal algebras as functorial algebras are invited to view this as a unary universal algebra with one unary operation for each point $t \in T$ in time, assigning to each state $x \in X$ its evolved state $\varphi(t, x)$ at the point t.

In every finite product category $\mathcal{C}$ any object T gives rise to an endo-functor $T \times -$ on $\mathcal{C}$. We say that *$\mathcal{C}$ has exponential objects w.r.t. T* if the endo-functor $\mathcal{C} \xrightarrow{T\times -} \mathcal{C}$ has a right adjoint, called $\mathcal{C} \xrightarrow{-^T} \mathcal{C}$. Moreover, *$\mathcal{C}$ has exponential objects*, if it has exponential objects w.r.t. to any object T of $\mathcal{C}$. Such categories having all finite products and exponentials are also called *Cartesian closed*.

These notions enable us to study the connections between dynamical systems as monadic algebras and a possible formalisation as coalgebras on the more general level of adjoint functors. In fact, this discussion can be done independently of the particular functor $T \times -$ and a possible adjoint $-^T$. We will continue with this approach in Subsection 4.5.

Since adjoint functors (if they exist at all) are unique up to isomorphism this method also yields a reasonable definition of the object X^T for our algebras: X^T is *whatever the adjoint functor returns*, not necessarily the set $\mathrm{Hom}(T, X)$ equipped with some structure. However, if $\mathcal{C}$ is a construct (having a faithful forgetful functor U to $\mathcal{Set}$), then it is usually a good idea to start with $\mathrm{Hom}(T, X)$ and to try to find some object X^T satisfying $U(X^T) = \mathrm{Hom}(T, X)$ (cf. [AHS06, Chapter 27]). For example in the category $\mathcal{Top}$ the set $\mathrm{Hom}(T, X)$ equipped with the compact open topology serves as an exponential object provided that the time space T is locally compact Hausdorff. Since topological spaces are a central example of this paper, we give detailed account of this in the following subsection.

4.4 Exponential objects in $\mathcal{Top}$ for locally compact Hausdorff spaces

In this subsection it will be proven that the category $\mathcal{Top}$ has exponential objects with respect to locally compact Hausdorff spaces. In the first instance, we address some notational issues. The main result of this subsection is revealed in the third statement of the subsequent proposition.

4.4.1 Definition. For topological spaces $\boldsymbol{X} = (X, \rho)$ and $\boldsymbol{Y} = (Y, \sigma)$, a compact set $K \in K(\boldsymbol{X})$ and $U \in \sigma$ we let $[K, U] := \{ f \in C(\boldsymbol{X}, \boldsymbol{Y}) \mid f[K] \subseteq U \}$. Then we define $\kappa(\boldsymbol{X}, \boldsymbol{Y})$ to be the *compact-open topology* on $C(\boldsymbol{X}, \boldsymbol{Y})$, i.e. the topology generated by the subbase $\{ [K, U] \mid K \in K(\boldsymbol{X}), U \in \sigma \}$. Moreover, we put $\boldsymbol{Y}^{\boldsymbol{X}} := (C(\boldsymbol{X}, \boldsymbol{Y}), \kappa(\boldsymbol{X}, \boldsymbol{Y}))$. ◊

Note that the set $C(\boldsymbol{X}, \boldsymbol{Y})$ was called $\mathcal{Top}(\boldsymbol{X}, \boldsymbol{Y})$ in the general category theoretic setting introduced in Subsection 2.2.

As we will see in the next proposition, the category of locally compact Hausdorff spaces has exponential objects.

4.4.2 Proposition. *Let $\boldsymbol{T} = (T, \tau) \in \mathcal{Top}$.*

(a) The assignment $-^{\boldsymbol{T}}\colon \mathcal{Top} \to \mathcal{Top}\colon\ \boldsymbol{X} \mapsto \boldsymbol{X}^{\boldsymbol{T}}$ *defines a functor, operating on morphisms* $\boldsymbol{X} \xrightarrow{f} \boldsymbol{Y}$ *via* $\boldsymbol{X}^{\boldsymbol{T}} \xrightarrow{f^{\boldsymbol{T}}} \boldsymbol{Y}^{\boldsymbol{T}}\colon g \mapsto f \circ g$.

(b) The family of morphisms given by

$$\begin{array}{llll} \Phi^{\boldsymbol{T}}_{\boldsymbol{X},\boldsymbol{Y}}\colon & \mathcal{Top}(\boldsymbol{T} \times \boldsymbol{X}, \boldsymbol{Y}) & \to & \mathcal{Top}(\boldsymbol{X}, \boldsymbol{Y}^{\boldsymbol{T}}) \\ & \boldsymbol{T} \times \boldsymbol{X} \xrightarrow{f} \boldsymbol{Y} & \mapsto & \boldsymbol{X} \xrightarrow{\Phi^{\boldsymbol{T}}_{\boldsymbol{X},\boldsymbol{Y}}(f)} \boldsymbol{Y}^{\boldsymbol{T}}\colon [x \mapsto [t \mapsto f(t,x)]], \end{array}$$

constitutes a natural transformation.

(c) If $\boldsymbol{T}$ *is locally compact Hausdorff, then* $\Phi^{\boldsymbol{T}}\colon \mathcal{Top}\left(\boldsymbol{T} \times -_1, -_2\right) \to \mathcal{Top}\left(-_1, -_2^{\boldsymbol{T}}\right)$ *is a natural equivalence.*

Proof: (a) Functoriality of $-^{\boldsymbol{T}}$ is clear. For a continuous map $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ between spaces $\boldsymbol{X} = (X, \rho)$ and $\boldsymbol{Y} = (Y, \sigma)$, the resulting map $f^{\boldsymbol{T}}\colon \boldsymbol{X}^{\boldsymbol{T}} \to \boldsymbol{Y}^{\boldsymbol{T}}$ is indeed continuous in every point $g \in C(\boldsymbol{T}, \boldsymbol{X})$. Namely, for every basic open neighbourhood $[K, U]$ of $f^{\boldsymbol{T}}(g) = f \circ g$, i.e. $K \in K(\boldsymbol{T})$ and $U \in \sigma$ such that $f \circ g[K] \subseteq U$, the set $[K, f^{-1}[U]]$ is an open neighbourhood of g, and every $h \in [K, f^{-1}[U]]$ satisfies $h[K] \subseteq f^{-1}[U]$, so $f \circ h[K] \subseteq f[f^{-1}[U]] \subseteq U$. This means $f^{\boldsymbol{T}}(h) = f \circ h \in [K, U]$.

(b) Let $\boldsymbol{X} = (X, \rho)$ and $\boldsymbol{Y} = (Y, \sigma)$ be topological spaces, $f \in C(\boldsymbol{T} \times \boldsymbol{X}, \boldsymbol{Y})$. Obviously, for each $x \in X$ it is $[t \mapsto f(t,x)] \in C(\boldsymbol{T}, \boldsymbol{Y})$. In order to prove that the mapping $[x \mapsto [t \mapsto f(t,x)]]$ belongs to $C\left(\boldsymbol{X}, \boldsymbol{Y}^{\boldsymbol{T}}\right)$, consider $x \in X$, $K \subseteq T$ compact w.r.t. τ, $U \in \sigma$ such that $[t \mapsto f(t,x)] \in [K, U]$. By continuity of f, for

$$\mathcal{V} := \bigcup_{t \in K} \{V \in \mathcal{U}_t(\boldsymbol{T}) \mid \exists W \in \mathcal{U}_x(\boldsymbol{X}) : f[V \times W] \subseteq U\}$$

we have $K \subseteq \bigcup_{V \in \mathcal{V}} \operatorname{int}_{\boldsymbol{T}}(V)$. Since K is compact, there exist $V_1, \ldots, V_n \in \mathcal{V}$ such that $K \subseteq \bigcup_{i=1}^n \operatorname{int}_{\boldsymbol{T}}(V_i)$. For each $i \in \{1, \ldots, n\}$, we can find some neighbourhood $W_i \in \mathcal{U}_x(\boldsymbol{Y})$ with the property $f[V_i \times W_i] \subseteq U$. Define $V := \bigcup_{i=1}^n V_i$, $W := \bigcap_{i=1}^n W_i$. Then it follows $W \in \mathcal{U}_x(\boldsymbol{X})$ and $f[K \times W] \subseteq f[V \times W] \subseteq U$. So we have $[t \mapsto f(t,x)] \in [K, U]$ for all $x \in W$. Moreover, it is easy to see that the naturality of the transformation $\Phi^{\boldsymbol{T}}\colon \mathcal{Top}(\boldsymbol{T} \times -_1, -_2) \to \mathcal{Top}(-_1, -_2{}^{\boldsymbol{T}})$ is satisfied.

(c) Let $\boldsymbol{X} = (X, \rho), \boldsymbol{Y} = (Y, \sigma) \in \mathcal{Top}$. It is easy to see that $\Phi^{\boldsymbol{T}}_{\boldsymbol{X},\boldsymbol{Y}}$ is injective. Hence it is left to prove that it is surjective. Let $g \in \mathcal{Top}\left(\boldsymbol{X}, \boldsymbol{Y}^{\boldsymbol{T}}\right)$. Define the mapping

$$\begin{array}{llll} f\colon & T \times X & \longrightarrow & Y \\ & (t,x) & \longmapsto & g(x)(t). \end{array}$$

Let us show that f is continuous. To this end, let $(t, x) \in T \times X$ and $W \in \sigma$ such that $f(t, x) \in W$. Since T is locally compact Hausdorff, there exists a compact neighbourhood K of t such that $f[K \times \{x\}] = g(x)[K] \subseteq W$. Yet now, due to the continuity of g, there exists a neighbourhood U of x such that $g[U] \subseteq [K, W]$. Thus, $f[K \times U] \subseteq W$, that is, f is continuous. Evidently, $\Phi^{\boldsymbol{T}}_{\boldsymbol{X},\boldsymbol{Y}}(f) = g$, so we are done. $\square$

In the theory of dynamical systems, state spaces are often chosen as metric spaces. This motivates the search for those topological spaces $\boldsymbol{T}$ for which the space $\boldsymbol{X}^{\boldsymbol{T}}$ is metrisable whenever $\boldsymbol{X}$ is metrisable.

We recall that a topological space is σ-compact if it has a countable exhaustion by compact subsets (cf. Definition 2.1.1). Note, furthermore, that σ-compact Hausdorff spaces are necessarily locally compact (see Lemma 2.1.2). The following proposition now answers the previously stated question.

4.4.3 Proposition. *If $\boldsymbol{T}$ is a σ-compact topological space and $\boldsymbol{X}$ a metrisable topological space, then $\boldsymbol{X}^{\boldsymbol{T}}$ is metrisable, too.*

Proof: Let $(K_n)_{n\in\mathbb{N}}$ be a countable exhaustion of $\boldsymbol{T}$ by compact subsets, and let d be a metric generating the topology of $\boldsymbol{X}$. Then it is not difficult to see that

$$\begin{array}{llll} d^*\colon & C(\boldsymbol{T},\boldsymbol{X})^2 & \to & \mathbb{R}, \\ & (f,g) & \mapsto & \sum_{n=0}^{\infty} \frac{1}{2^n} \min\left\{\sup_{x\in K_n} d(f(x),g(x)), 1\right\}. \end{array}$$

is a metric on $C(\boldsymbol{T},\boldsymbol{X})$ that generates $\kappa(\boldsymbol{T},\boldsymbol{X})$ (cp. also [Wil04, 43G.1., p. 289]). $\square$

Many well-known topological spaces are σ-compact, such as all finite powers of $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$. However, as it turns out, a slight generalization of metrisable spaces allows us to use a notably larger class of time spaces. Namely, if $\boldsymbol{X}$ is uniformisable, we shall see that $\boldsymbol{T}$ may indeed be an arbitrary topological space.

Obviously, the notion of uniform space generalises that of metric space. Namely, with every metric space (X, d), we associate a uniformity Θ on X generated by the entourages $U_\varepsilon := \{(x, y) \in X^2 \mid d(x, y) \leq \varepsilon\}$, $\varepsilon \in \mathbb{R}_{>0}$.

To give an example of uniform spaces that properly generalise metric spaces, let $\boldsymbol{X}$ be a topological space and consider the space $C(\boldsymbol{X}) := C(\boldsymbol{X},\mathbb{R})$ of continuous real-valued functions on $\boldsymbol{X}$, equipped with the topology of compact convergence. That is, convergence in $C(\boldsymbol{X})$ means uniform convergence on every compact subset of $\boldsymbol{X}$. The topology underlying this notion of convergence is given by the base $\{[f, K, \varepsilon] \mid f \in C(\boldsymbol{X}), K \in K(\boldsymbol{X}), \varepsilon \in \mathbb{R}_{>0}\}$, where

$$[f, K, \varepsilon] := \left\{ g \in C(\boldsymbol{X}) \;\middle|\; \sup_{x\in K} |f(x) - g(x)| < \varepsilon \right\}.$$

For completeness we mention that one can show that this topology coincides with the compact-open topology on $C(\boldsymbol{X})$. This is in fact an instance of a general non-trivial observation, depending only on uniformisability of the image space $\mathbb{R}$ (cf. [Wil04, Theorem 43.7, p. 284]).

It is, furthermore, easy to see that the topology of compact convergence on $C(\boldsymbol{X})$ is induced by[11] the uniform structure generated by the uniformity base $\{\Theta_{K,\varepsilon} \mid K \in K(\boldsymbol{X}), \varepsilon \in \mathbb{R}_{>0}\}$ where

$$\Theta_{K,\varepsilon} := \left\{ (f,g) \in (C(\boldsymbol{X}))^2 \;\middle|\; \sup_{x \in K} |f(x) - g(x)| < \varepsilon \right\}.$$

However, it does not follow in general that the induced topology or the uniform structure, respectively, is metrisable. Namely, if $\boldsymbol{X}$ is a locally compact Hausdorff space, it is well-known that metrisability of $C(\boldsymbol{X})$ is equivalent to σ-compactness of $\boldsymbol{X}$ (cf. [Are46, Theorem 8] for more details). Thus, choosing for $\boldsymbol{X}$ any locally compact Hausdorff space which is not σ-compact, we obtain that $C(\boldsymbol{X})$ is a space with a uniform structure that cannot be given by a metric. For instance, we may take for $\boldsymbol{X}$ the subspace of a Tychonoff cube $[0,1]^I$ with an uncountable index set I that results from deleting an arbitrary single point from $[0,1]^I$.

Such function spaces $C(\boldsymbol{X})$ as state spaces promise a wide variety of dynamic behaviour, much more than just $\mathbb{R}^n$, which corresponds to the case, when $\boldsymbol{X}$ is discrete and finite (in particular compact). It goes beyond the scope of this article to study them in more detail, but in [SKBS13], we examine topological dynamical systems on function spaces over topological groups more closely. In particular, we study and characterise faithful strongly chaotic continuous actions of locally compact Hausdorff topological groups on such spaces.

Even though function spaces $C(\boldsymbol{X})$ sometimes lack metrisability and thus Proposition 4.4.3 fails to be applicable, these spaces are certainly uniformisable as said before. Hence, one may instead rely on the following well-known variant of Proposition 4.4.3, which, as a side-effect, allows us to drop the assumption of σ-compactness w.r.t. the time space $\boldsymbol{T}$. For a proof of this fact we refer to [Wil04, Theorem 43.7].

4.4.4 Proposition. *If $\boldsymbol{T}$ is a topological space and $\boldsymbol{X}$ a uniformisable space, then $\boldsymbol{X}^{\boldsymbol{T}}$ is uniformisable.*

According to Proposition 4.4.3, if $\boldsymbol{X}$ is a metrisable space and the time space $\boldsymbol{T}$ is σ-compact, then also $\boldsymbol{X}^{\boldsymbol{T}}$ is metrisable, i.e. the compact-open topology on $\boldsymbol{X}^{\boldsymbol{T}}$ is induced by some metric. Such a metric in a natural way defines a uniform structure Θ on $\boldsymbol{X}^{\boldsymbol{T}}$ (v.s.), which is indeed the same as the uniform structure constructed by Proposition 4.4.4 applied to $\boldsymbol{T}$ and the uniform space on X derived from the metric on X.

[11] see also the definition on page 7

With the previous result we have established the existence of exponential objects w.r.t. any time space in the full subcategory $\mathcal{Unif}$ of uniformisable spaces. With Proposition 4.4.3 we have done the same for the subcategory $\mathcal{Met}$ of metrisable spaces and σ-compact times spaces. Furthermore, Proposition 4.4.2 solves this question for the category of locally compact Hausdorff spaces in general and for the category of topological spaces w.r.t. locally compact Hausdorff time spaces.

Thus, in many familiar situations, one can ensure that the functor taking products with the time space $\boldsymbol{T}$ has a right adjoint endo-functor $-^{\boldsymbol{T}}$. It is on the level of adjoint endo-functors that we will now explore, how to understand dynamical systems in abstract categories in a different manner than as monadic algebras.

4.5 (Co)Monadic (co)algebras and adjoint functors

In this part we will show that monadic algebras correspond closely to comonadic coalgebras if the respective signature functors are adjoint. Since under fairly weak assumptions on the considered category $\mathcal{C}$, general dynamical systems have been modelled as monadic algebras for the signature functor $T \times -$ (cf. Corollary 4.2.2), this result will in particular apply to an adjoint functor $-^T$ on $\mathcal{C}$ provided it exists. However, our treatment of this topic allows the functors $T \times -$ and $-^T$ to be replaced by any other adjoint pair of endo-functors $F \dashv G$.

Our first aim is to show how a monad (F, δ, η) for an endo-functor $F \in \operatorname{End} \mathcal{C}$ can be transformed into a comonad $(G, \tilde{\delta}, \tilde{\eta})$ for a right adjoint endo-functor $G \in \operatorname{End} \mathcal{C}$. We will put the technical part of the construction into the following lemma:

4.5.1 Lemma. *Let $\mathcal{C}$ be a category and $F, G \in \operatorname{End} \mathcal{C}$ be two adjoint endo-functors ($F \dashv G$). We denote the corresponding natural equivalence between the hom-sets by $\operatorname{Hom}(F,-) \xrightarrow{\nu} \operatorname{Hom}(-,G)$, the unit by $1_{\mathcal{C}} \xrightarrow{\vartheta} GF$ and the co-unit by $FG \xrightarrow{\varepsilon} 1_{\mathcal{C}}$.*

Furthermore, let two natural transformations $FF \xrightarrow{\delta} F$ and $1_{\mathcal{C}} \xrightarrow{\eta} F$, and an arbitrary object $X \in \mathcal{C}$ be given.

For every object $Y \in \mathcal{C}$ we define the following morphisms:

$$FFGY \xrightarrow{\zeta_Y := \delta_{GY}\varepsilon_Y} Y \qquad GY \xrightarrow{\tilde{\delta}_Y := \nu_{GY,GY}(\mu_Y)} GGY$$

$$FGY \xrightarrow{\mu_Y := \nu_{FGY,Y}(\zeta_Y)} GY \qquad GY \xrightarrow{\tilde{\eta}_Y := \eta_{GY}\varepsilon_Y} Y.$$

Then the following assertions are true:

(a) $FFG \xrightarrow{\zeta} 1_{\mathcal{C}}$, $FG \xrightarrow{\mu} G$, $G \xrightarrow{\tilde{\delta}} GG$ and $G \xrightarrow{\tilde{\eta}} 1_{\mathcal{C}}$ are natural transformations.

(b) $\tilde{\delta}_X G\tilde{\eta}_X = \nu_{GX,X}(\eta_{FGX}\delta_{GX}\varepsilon_X) = \nu_{GX,X}(\eta_{FGX}\zeta_X)$ and
$\eta_{FX}\delta_X = F\vartheta_X\eta_{FGFX}\zeta_{FX}$.

(c) $\tilde{\delta}_X\tilde{\eta}_{GX} = \nu_{GX,X}(F\eta_{GX}\delta_{GX}\varepsilon_X) = \nu_{GX,X}(F\eta_{GX}\zeta_X)$ and
$F\eta_X\delta_X = F\vartheta_X F\eta_{GFX}\zeta_{FX}$.

(d) $\tilde{\delta}_X G\tilde{\delta}_X = \nu_{GX,G^2X}(\nu_{FGX,GX}(\nu_{F^2GX,X}(\delta_{FGX}\zeta_X)))$ $\quad F\delta_X\delta_X = F^3\vartheta_X F\delta_{GFX}\zeta_{FX}$
$\tilde{\delta}_X\tilde{\delta}_{GX} = \nu_{GX,G^2X}(\nu_{FGX,GX}(\nu_{F^2GX,X}(F\delta_{GX}\zeta_X)))$ $\quad \delta_{FX}\delta_X = F^3\vartheta_X\delta_{FGFX}\zeta_{FX}$.

Proof: Before we start with the actual proof we remind the reader about some basic facts regarding adjunctions $F \dashv G$: for all objects $X, Y \in \mathcal{C}$ and every morphism $FX \xrightarrow{g} Y$ the following equations hold:

$$1_{FX} = F\vartheta_X\varepsilon_{FX} \tag{4.9}$$
$$1_{GY} = \vartheta_{GY}G\varepsilon_Y \tag{4.10}$$
$$\vartheta_X Gg = \nu_{X,Y}(g) \tag{4.11}$$
$$F\nu_{X,Y}(g)\varepsilon_Y = g \tag{4.12}$$

Equations 4.9 and 4.10 characterise adjunctions and are known as co-unit-unit equations (cf. Definition 2.2.16). The other two relate the natural equivalences ν and ν^{-1} to the unit and co-unit, respectively (see also Proposition 2.2.17).

In the course of the proof we are going to need the characterising commutative diagrams for each of the involved natural transformations. We will refer to them using the names of the respective transformations if we apply the commutativity condition for some morphism in a calculation. Such an application is indicated by underlining the corresponding part of the formula which has to be replaced.

For objects $X, Y \in \mathcal{C}$ and any morphism $X \xrightarrow{h} Y$ the following diagrams commute:

$$\begin{tikzcd} FX \arrow[r, "Fh"] & FY \\ X \arrow[u, "\eta_X"] \arrow[r, "h"] & Y \arrow[u, "\eta_Y"'] \end{tikzcd} \tag{4.13η}$$

$$\begin{tikzcd} FFX \arrow[r, "FFh"] \arrow[d, "\delta_X"'] & FFY \arrow[d, "\delta_Y"] \\ FX \arrow[r, "Fh"] & FY \end{tikzcd} \tag{4.13δ}$$

$$\begin{tikzcd} FFGX \arrow[r, "FFGh"] \arrow[d, "\zeta_X"'] & FFGY \arrow[d, "\zeta_Y"] \\ X \arrow[r, "h"] & Y \end{tikzcd} \tag{4.13ζ}$$

$$\begin{tikzcd} FGX \arrow[r, "FGh"] \arrow[d, "\varepsilon_X"'] & FGY \arrow[d, "\varepsilon_Y"] \\ X \arrow[r, "h"] & Y \end{tikzcd} \tag{4.13ε}$$

$$\begin{tikzcd} GFX \arrow[r, "GFh"] & GFY \\ X \arrow[u, "\vartheta_X"] \arrow[r, "h"] & Y \arrow[u, "\vartheta_Y"'] \end{tikzcd} \tag{4.13ϑ}$$

$$\begin{tikzcd} FGX \arrow[r, "FGh"] \arrow[d, "\mu_X"'] & FGY \arrow[d, "\mu_Y"] \\ GX \arrow[r, "Gh"] & GY \end{tikzcd} \tag{4.13μ}$$

(a) Taking into account the co-unit-unit equations it is easy to see that the defined families of morphisms are each obtained using compositions of natural transformations with functors and with each other (cf. Remark 2.2.14). Indeed, we have

$$\zeta = \delta_G\varepsilon, \qquad \tilde{\delta} = \vartheta_G G\mu,$$
$$\mu = \vartheta_{FG}G\zeta, \qquad \tilde{\eta} = \eta_G\varepsilon.$$

As such compositions yield again natural transformations this item is proven.

(b) First we note that

$$\begin{aligned}
\nu_{FGX,X}(\zeta_X)\tilde{\eta}_X &\overset{(4.11)}{=} \vartheta_{FGX}G\zeta_X\tilde{\eta}_X \overset{\text{def}\,\tilde{\eta}_X}{=} \vartheta_{FGX}\underline{G\zeta_X\eta_{GX}}\varepsilon_X \\
&\overset{(4.13\eta)}{=} \vartheta_{FGX}\eta_{GFFGX}\underline{FG\zeta_X\varepsilon_X} \overset{(4.13\varepsilon)}{=} \underline{\vartheta_{FGX}\eta_{GFFGX}}\varepsilon_{FFGX}\zeta_X \\
&\overset{(4.13\eta)}{=} \eta_{FGX}\underline{F\vartheta_{FGX}\varepsilon_{FFGX}}\zeta_X \overset{(4.9)}{=} \eta_{FGX}1_{FFGX}\zeta_X = \eta_{FGX}\zeta_X.
\end{aligned}$$

Then, by definition,

$$\begin{aligned}
\tilde{\delta}_X G\tilde{\eta}_X = \nu_{GX,GX}(\nu(FGX,X)(\zeta_X))G\tilde{\eta}_X &\overset{(4.11)}{=} \vartheta_{GX}G(\nu_{FGX,X}(\zeta_X)\tilde{\eta}_X) \\
&\overset{\text{v.s.}}{=} \vartheta_{GX}G(\eta_{FGX}\zeta_X) \overset{(4.11)}{=} \nu_{GX,X}(\eta_{FGX}\zeta_X) \overset{\text{def}\,\zeta_X}{=} \nu_{GX,X}(\eta_{FGX}\delta_{GX}\varepsilon_X).
\end{aligned}$$

The other equality holds because

$$\begin{aligned}
\eta_{FX}\delta_X = \eta_{FX}\delta_X\underline{1_{FX}} \overset{(4.9)}{=} \eta_{FX}\underline{\delta_X F\vartheta_X}\varepsilon_{FX} &\overset{(4.13\delta)}{=} \eta_{FX}FF\vartheta_X\delta_{GFX}\varepsilon_{FX} \\
&\overset{\text{def}\,\zeta_{FX}}{=} \eta_{FX}FF\vartheta_X\zeta_{FX}.
\end{aligned}$$

(c) A long calculation

$$\begin{aligned}
&\tilde{\delta}_X\tilde{\eta}_{GX} \overset{\text{def}\,\tilde{\delta}_X}{=} \nu_{GX,GX}(\mu_X)\tilde{\eta}_{GX} \overset{(4.11)}{=} \vartheta_{GX}G\mu_X\tilde{\eta}_{GX} \overset{\text{def}\,\tilde{\eta}_{GX}}{=} \vartheta_{GX}\underline{G\mu_X\eta_{GGX}}\varepsilon_{GX} \\
&\overset{(4.13\eta)}{=} \vartheta_{GX}\eta_{GFGX}\underline{FG\mu_X\varepsilon_{GX}} \overset{(4.13\varepsilon)}{=} \underline{\vartheta_{GX}\eta_{GFGX}}\varepsilon_{FGX}\mu_X \overset{(4.13\eta)}{=} \eta_{GX}\underline{F\vartheta_{GX}\varepsilon_{FGX}}\mu_X \\
&\overset{(4.9)}{=} \eta_{GX}1_{FGX}\mu_X \overset{\text{def}\,\mu_X}{=} \eta_{GX}\nu_{FGX,X}(\zeta_X) \overset{(4.11)}{=} \underline{\eta_{GX}\vartheta_{FGX}}G\zeta_X \\
&\overset{(4.13\vartheta)}{=} \vartheta_{GX}GF\eta_{GX}G\zeta_X = \vartheta_{GX}G(F\eta_{GX}\zeta_X) \overset{(4.11)}{=} \nu_{GX,X}(F\eta_{GX}\zeta_X) \\
&\overset{\text{def}\,\zeta_X}{=} \nu_{GX,X}(F\eta_{GX}\delta_{GX}\varepsilon_X)
\end{aligned}$$

shows the first equality. The second one can be seen from

$$\begin{aligned}
F\eta_X\delta_X = F\eta_X\delta_X\underline{1_{FX}} \overset{(4.9)}{=} F\eta_X\underline{\delta_X F\vartheta_X}\varepsilon_{FX} &\overset{(4.13\delta)}{=} F\eta_X FF\vartheta_X\delta_{GFX}\varepsilon_{FX} \\
\overset{\text{def}\,\zeta_{FX}}{=} F(\underline{\eta_X F\vartheta_X})\zeta_{FX} &\overset{(4.13\eta)}{=} F(\vartheta_X\eta_{GFX})\zeta_{FX} = F\vartheta_X F\eta_{GFX}\zeta_{FX}.
\end{aligned}$$

(d) We start by showing $\tilde{\delta}_X G\tilde{\delta}_X = \nu_{GX,G^2X}(\nu_{FGX,GX}(\nu_{F^2GX,X}(\delta_{FGX}\zeta_X)))$. This equality follows from

$$\begin{aligned}
\tilde{\delta}_X G\tilde{\delta}_X \overset{\text{def}\,\tilde{\delta}_X}{=} \nu_{GX,GX}(\mu_X)G\tilde{\delta}_X &\overset{(4.11)}{=} \vartheta_{GX}G\mu_X G\tilde{\delta}_X = \vartheta_{GX}G(\mu_X\tilde{\delta}_X) \\
&\overset{(4.11)}{=} \nu_{GX,G^2X}(\mu_X\tilde{\delta}_X),
\end{aligned}$$

together with

$$\begin{aligned}\mu_X\tilde{\delta}_X &\overset{\text{def }\mu_X}{=} \nu_{FGX,X}(\zeta_X)\tilde{\delta}_X \overset{(4.11)}{=} \vartheta_{FGX}G\zeta_X\tilde{\delta}_X \overset{\text{def }\tilde{\delta}_X}{=} \vartheta_{FGX}G\zeta_X\nu_{GX,GX}(\mu_X)\\ &\overset{(4.11)}{=} \vartheta_{FGX}\underline{G\zeta_X\vartheta_{GX}}G\mu_X \overset{(4.13\vartheta)}{=} \underline{\vartheta_{FGX}\vartheta_{GFFGX}}GFG\zeta_XG\mu_X\\ &\overset{(4.13\vartheta)}{=} \vartheta_{FGX}\underline{GF\vartheta_{FGX}GFG\zeta_XG\mu_X} = \vartheta_{FGX}G(F\vartheta_{FGX}FG\zeta_X\mu_X)\\ &\overset{(4.11)}{=} \nu_{FGX,GX}(F\vartheta_{FGX}FG\zeta_X\mu_X),\end{aligned}$$

$$\begin{aligned}&F\vartheta_{FGX}\underline{FG\zeta_X\mu_X} \overset{(4.13\mu)}{=} F\vartheta_{FGX}\mu_{FFGX}G\zeta_X\\ &\overset{\text{def }\mu_{FFGX}}{=} F\vartheta_{FGX}\nu_{FGFFGX,FFGX}(\zeta_{FFGX})G\zeta_X\\ &\overset{(4.11)}{=} \underline{F\vartheta_{FGX}\vartheta_{FGFFGX}}G\zeta_{FFGX}G\zeta_X \overset{(4.13\vartheta)}{=} \vartheta_{FFGX}\underline{GFF\vartheta_{FGX}G\zeta_{FFGX}G\zeta_X}\\ &= \vartheta_{FFGX}G(FF\vartheta_{FGX}\zeta_{FFGX}\zeta_X) \overset{(4.11)}{=} \nu_{FFGX,X}(FF\vartheta_{FGX}\zeta_{FFGX}\zeta_X)\end{aligned}$$

and

$$\begin{aligned}F^2\vartheta_{FGX}\underline{\zeta_{F^2GX}\zeta_X} &\overset{(4.13\zeta)}{=} \underline{F^2\vartheta_{FGX}F^2G\zeta_X}\zeta_X = F^2(\vartheta_{FGX}G\zeta_X)\zeta_X\\ &\overset{\text{def }\zeta_X}{=} \underline{F^2(\vartheta_{FGX}G\zeta_X)\delta_{GX}}\varepsilon_X\\ &\overset{(4.13\delta)}{=} \delta_{FGX}F(\vartheta_{FGX}G\zeta_X)\varepsilon_X = \delta_{FGX}F\vartheta_{FGX}\underline{FG\zeta_X\varepsilon_X}\\ &\overset{(4.13\varepsilon)}{=} \delta_{FGX}\underline{F\vartheta_{FGX}\varepsilon_{FFGX}}\zeta_X \overset{(4.9)}{=} \delta_{FGX}1_{FFGX}\zeta_X = \delta_{FGX}\zeta_X.\end{aligned}$$

We continue with the equality $\tilde{\delta}_X\tilde{\delta}_{GX} = \nu_{GX,G^2X}(\nu_{FGX,GX}(\nu_{F^2GX,X}(F\delta_{GX}\zeta_X)))$, following from

$$\begin{aligned}\tilde{\delta}_X\tilde{\delta}_{GX} &\overset{\text{def }\tilde{\delta}}{=} \nu_{GX,GX}(\mu_X)\nu_{G^2X,G^2X}(\mu_{GX}) \overset{(4.11)}{=} \underline{(\vartheta_{GX}G\mu_X)\,\vartheta_{G^2X}}G\mu_{GX}\\ &\overset{(4.13\vartheta)}{=} \vartheta_{GX}\underline{GF\,(\vartheta_{GX}G\mu_X)G\mu_{GX}} = \vartheta_{GX}G\,(F\vartheta_{GX}FG\mu_X\mu_{GX})\\ &\overset{(4.11)}{=} \nu_{GX,G^2X}\,(F\vartheta_{GX}FG\mu_X\mu_{GX})\,,\end{aligned}$$

$$\begin{aligned}F\vartheta_{GX}\underline{FG\mu_X\mu_{GX}} &\overset{(4.13\mu)}{=} F\vartheta_{GX}\mu_{FGX}G\mu_X \overset{\text{def }\mu_{FGX}}{=} F\vartheta_{GX}\nu_{FGFGX,FGX}(\zeta_{FGX})G\mu_X\\ &\overset{(4.11)}{=} \underline{F\vartheta_{GX}\vartheta_{FGFGX}}G\zeta_{FGX}G\mu_X \overset{(4.13\vartheta)}{=} \vartheta_{FGX}\underline{GFF\vartheta_{GX}G\zeta_{FGX}G\mu_X}\\ &= \vartheta_{FGX}G\left(F^2\vartheta_{GX}\zeta_{FGX}\mu_X\right) \overset{(4.11)}{=} \nu_{FGX,GX}\left(F^2\vartheta_{GX}\zeta_{FGX}\mu_X\right)\end{aligned}$$

and

$$\begin{aligned}
F^2\vartheta_{GX}\underline{\zeta_{FGX}\mu_X} &\overset{(4.13\zeta)}{=} \underline{F^2\vartheta_{GX}F^2G\mu_X}\zeta_{GX} = F^2\left(\vartheta_{GX}G\mu_X\right)\zeta_{GX} \\
&\overset{\mathrm{def}\,\zeta_{GX}}{=} \underline{F^2\left(\vartheta_{GX}G\mu_X\right)\delta_{G^2X}}\varepsilon_{GX} \overset{(4.13\delta)}{=} \delta_{GX}F\left(\vartheta_{GX}G\mu_X\right)\varepsilon_{GX} \\
&= \delta_{GX}F\vartheta_{GX}\underline{FG\mu_X\varepsilon_{GX}} \overset{(4.13\varepsilon)}{=} \delta_{GX}\underline{F\vartheta_{GX}\varepsilon_{FGX}}\mu_X \overset{(4.9)}{=} \delta_{GX}1_{FGX}\mu_X \\
&\overset{\mathrm{def}\,\mu_X}{=} \delta_{GX}\nu_{FGX,X}\left(\zeta_X\right) \overset{(4.11)}{=} \underline{\delta_{GX}\vartheta_{FGX}}G\zeta_X \overset{(4.13\vartheta)}{=} \vartheta_{FFGX}\underline{GF\delta_{GX}G\zeta_X} \\
&= \vartheta_{FFGX}G\left(F\delta_{GX}\zeta_X\right) \overset{(4.11)}{=} \nu_{F^2GX,X}\left(F\delta_{GX}\zeta_X\right).
\end{aligned}$$

Furthermore, the remaining equalities,

$$\begin{aligned}
F\delta_X\delta_X = F\delta_X\delta_X\underline{1_{FX}} &\overset{(4.9)}{=} F\delta_X\underline{\delta_XF\vartheta_X}\varepsilon_{FX} \overset{(4.13\delta)}{=} F\delta_XF^2\vartheta_X\underline{\delta_{GFX}\varepsilon_{FX}} \\
&\overset{\mathrm{def}\,\zeta_{FX}}{=} F\left(\underline{\delta_XF\vartheta_X}\right)\zeta_{FX} \overset{(4.13\delta)}{=} F\left(F^2\vartheta_X\delta_{GFX}\right)\zeta_{FX} = F^3\vartheta_XF\delta_{GFX}\zeta_{FX}
\end{aligned}$$

and

$$\begin{aligned}
\delta_{FX}\delta_X = \delta_{FX}\delta_X\underline{1_{FX}} &\overset{(4.9)}{=} \delta_{FX}\underline{\delta_XF\vartheta_X}\varepsilon_{FX} \overset{(4.13\delta)}{=} \delta_{FX}F^2\vartheta_X\underline{\delta_{GFX}\varepsilon_{FX}} \\
&\overset{\mathrm{def}\,\zeta_{FX}}{=} \underline{\delta_{FX}F^2\vartheta_X}\zeta_{FX} \overset{(4.13\delta)}{=} F^2F\vartheta_X\delta_{FGFX}\zeta_{FX} = F^3\vartheta_X\delta_{FGFX}\zeta_{FX}
\end{aligned}$$

can be verified. □

4.5.2 Proposition. *Let $\mathcal{C}$ be a category and $F, G \in \operatorname{End}\mathcal{C}$ be two adjoint endofunctors ($F \dashv G$). We denote the corresponding natural equivalence between the hom-sets by $\operatorname{Hom}(F,-) \xrightarrow{\nu} \operatorname{Hom}(-,G)$, the unit by $1_{\mathcal{C}} \xrightarrow{\vartheta} GF$ and the co-unit by $FG \xrightarrow{\varepsilon} 1_{\mathcal{C}}$.*

Furthermore, let two natural transformations $FF \xrightarrow{\delta} F$ and $1_{\mathcal{C}} \xrightarrow{\eta} F$ be given and the corresponding natural transformations $FFG \xrightarrow{\zeta} 1_{\mathcal{C}}$, $FG \xrightarrow{\mu} G$, $G \xrightarrow{\tilde{\delta}} GG$ and $G \xrightarrow{\tilde{\eta}} 1_{\mathcal{C}}$ be defined as in Lemma 4.5.1.

Then the following equivalences hold:

(a) $\forall X \in \mathcal{C}\colon\ \eta_{FX}\delta_X = 1_{FX} \iff \forall X \in \mathcal{C}\colon\ \tilde{\delta}_XG\tilde{\eta}_X = 1_{GX}$
$\iff \forall X \in \mathcal{C}\colon\ \eta_{FGX}\zeta_X = \varepsilon_X.$

(b) $\forall X \in \mathcal{C}\colon\ F\eta_X\delta_X = 1_{FX} \iff \forall X \in \mathcal{C}\colon\ F\eta_{GX}\zeta_X = \varepsilon_X$
$\iff \forall X \in \mathcal{C}\colon\ \tilde{\delta}_X\tilde{\eta}_{GX} = 1_{GX}.$

(c) $\forall X \in \mathcal{C}\colon\ F\delta_X\delta_X = \delta_{FX}\delta_X \iff \forall X \in \mathcal{C}\colon\ \tilde{\delta}_X\tilde{\delta}_{GX} = \tilde{\delta}_XG\tilde{\delta}_X.$

(d) (F,δ,η) *is a monad if and only if* $\left(G,\tilde{\delta},\tilde{\eta}\right)$ *is a comonad.*

Proof: We rely upon Lemma 4.5.1 to prove the stated equivalences. Furthermore, note that for every $X \in \mathcal{C}$ the following holds:

$$\nu_{GX,X}(\varepsilon_X) \overset{(4.11)}{=} \vartheta_{GX} G\varepsilon_X \overset{(4.10)}{=} 1_{GX}. \tag{4.14}$$

(a) First, for a fixed object $X \in \mathcal{C}$ it is easy to see, using Equation (4.14) and Lemma 4.5.1(b) that the equalities

$$\tilde{\delta}_X G\tilde{\eta}_X = 1_{GX} \qquad \text{and} \qquad \nu_{GX,X}\left(\eta_{FGX}\delta_{GX}\varepsilon_X\right) = \nu_{GX,X}(\varepsilon_X)$$

are equivalent. As $\nu_{GX,X}$ is a bijection, the latter equality is equivalent to $\eta_{FGX}\delta_{GX}\varepsilon_X = \varepsilon_X$. Taking into account that $\zeta_X = \delta_{GX}\varepsilon_X$, the equivalence of the two statements on the right-hand side is proven.

Now assume that $\eta_{FX}\delta_X = 1_{FX}$ holds for all $X \in \mathcal{C}$. For every $Y \in \mathcal{C}$, substituting $X = GY$ in this equality and composition with ε_Y yields

$$\eta_{FGY}\delta_{GY}\varepsilon_Y = 1_{FGY}\varepsilon_Y = \varepsilon_Y.$$

Conversely, suppose that this holds for all $Y \in \mathcal{C}$. Considering any $X \in \mathcal{C}$ and substituting $Y = FX$ yields the equality $\eta_{FGFX}\delta_{GFX}\varepsilon_{FX} = \varepsilon_{FX}$. From this and Lemma 4.5.1(b) one obtains

$$\eta_{FX}\delta_X \overset{4.5.1(b)}{=} F\vartheta_X \underline{\eta_{FGFX}\delta_{GFX}\varepsilon_{FX}} \overset{\text{v.s.}}{=} F\vartheta_X\varepsilon_{FX} \overset{(4.9)}{=} 1_{FX}.$$

(b) Again, for fixed objects $X \in \mathcal{C}$ the equalities

$$\tilde{\delta}_X\tilde{\eta}_{GX} = 1_{GX} \qquad \text{and} \qquad \nu_{GX,X}\left(F\eta_{GX}\delta_{GX}\varepsilon_X\right) = \nu_{GX,X}(\varepsilon_X)$$

are equivalent, using Equation (4.14) and Lemma 4.5.1(c). As $\nu_{GX,X}$ is bijective, the latter equality is equivalent to $F\eta_{GX}\delta_{GX}\varepsilon_X = \varepsilon_X$, showing that the two assertions on the right-hand side are equivalent.

As above assume now that $F\eta_X\delta_X = 1_{FX}$ holds for all $X \in \mathcal{C}$. Substituting $X = GY$ for an arbitrary $Y \in \mathcal{C}$ and composing with ε_Y, yields

$$F\eta_{GY}\delta_{GY}\varepsilon_Y = 1_{FGY}\varepsilon_Y = \varepsilon_Y.$$

Conversely, if this equation holds for all $Y \in \mathcal{C}$, then substitution of $Y = FX$ and application of Lemma 4.5.1(c) yield

$$F\eta_X\delta_X \overset{4.5.1(c)}{=} F\vartheta_X \underline{F\eta_{GFX}\zeta_{FX}} \overset{\text{v.s.}}{=} F\vartheta_X\varepsilon_{FX} \overset{(4.9)}{=} 1_{FX}.$$

(c) Assume that $F\delta_X\delta_X = \delta_{FX}\delta_X$ holds for all $X \in \mathcal{C}$ and consider any $Y \in \mathcal{C}$. Substituting $X = GY$ in the given equality yields $F\delta_{GY}\delta_{GY} = \delta_{FGY}\delta_{GY}$, whence one obtains $F\delta_{GY}\zeta_Y = \delta_{FGY}\zeta_Y$ by composition with ε_Y. From the latter equality one obtains

$$\begin{aligned}\tilde{\delta}_Y\tilde{\delta}_{GY} &\overset{4.5.1(d)}{=} \nu_{GY,G^2Y}(\nu_{FGY,GY}(\nu_{F^2GY,Y}(F\delta_{GY}\zeta_Y))) \\ &\overset{\text{v.s.}}{=} \nu_{GY,G^2Y}(\nu_{FGY,GY}(\nu_{F^2GY,Y}(\delta_{FGY}\zeta_Y))) \overset{4.5.1(d)}{=} \tilde{\delta}_Y G\tilde{\delta}_Y.\end{aligned}$$

Conversely, assume that this equality holds for all $Y \in \mathcal{C}$. Substituting $Y = GX$ with an arbitrary $X \in \mathcal{C}$ yields

$$\begin{aligned}&\nu_{GFX,G^2FX}(\nu_{FGFX,GFX}(\nu_{F^2GFX,FX}(F\delta_{GFX}\zeta_{FX}))) \overset{4.5.1(d)}{=} \tilde{\delta}_{FX}\tilde{\delta}_{GFX} \\ &= \tilde{\delta}_{FX}G\tilde{\delta}_{FX} \overset{4.5.1(d)}{=} \nu_{GFX,G^2FX}(\nu_{FGFX,GFX}(\nu_{F^2GFX,FX}(\delta_{FGFX}\zeta_{FX}))),\end{aligned}$$

whence $F\delta_{GFX}\zeta_{FX} = \delta_{FGFX}\zeta_{FX}$ as ν is a natural equivalence, so all its mappings are bijective. Composing the result with $F^3\vartheta_X$ yields

$$F\delta_X\delta_X \overset{4.5.1(d)}{=} F^3\vartheta_X \underline{F\delta_{GFX}\zeta_{FX}} \overset{\text{v.s.}}{=} \underline{F^3\vartheta_X\delta_{FGFX}\zeta_{FX}} \overset{4.5.1(d)}{=} \delta_{FX}\delta_X.$$

(d) This statement is a combination of the equivalences just shown. □

In the previous result we have established a relationship between monads for F and comonads for an adjoint endo-functor G. This connection extends to monadic algebras and comonadic coalgebras:

4.5.3 Proposition. *Let $\mathcal{C}$ be a category and $F, G \in \operatorname{End}\mathcal{C}$ two adjoint endo-functors ($F \dashv G$). We denote the corresponding natural equivalence between the hom-sets by $\operatorname{Hom}(F,-) \xrightarrow{\nu} \operatorname{Hom}(-,G)$, the unit if the adjunction by $1_{\mathcal{C}} \xrightarrow{\vartheta} GF$ and the co-unit by $FG \xrightarrow{\varepsilon} 1_{\mathcal{C}}$.*

Furthermore, let two natural transformations $FF \xrightarrow{\delta} F$ and $1_{\mathcal{C}} \xrightarrow{\eta} F$ be given and the corresponding natural transformations $FFG \xrightarrow{\zeta} 1_{\mathcal{C}}$, $FG \xrightarrow{\mu} G$, $G \xrightarrow{\tilde{\delta}} GG$ and $G \xrightarrow{\tilde{\eta}} 1_{\mathcal{C}}$ be defined as in Lemma 4.5.1.

Assume that (F, δ, η) is a monad with its associated comonad $\left(G, \tilde{\delta}, \tilde{\eta}\right)$ as in Proposition 4.5.2. Let $X \in \mathcal{C}$ be an object and $FX \xrightarrow{\varphi} X$ a morphism. Defining the morphism $X \xrightarrow{\psi := \nu_{X,X}(\varphi) = \vartheta_X G\varphi} GX$, we have

(a) $\psi\tilde{\delta}_X = \nu_{X,GX}(\nu_{FX,X}(\delta_X\varphi))$ and $\psi G\psi = \nu_{X,GX}(\nu_{FX,X}(F\varphi\varphi))$.

(b) $\psi\tilde{\eta}_X = \eta_X\varphi$.

(c) (X, φ) is a monadic algebra w.r.t. (F, δ, η) if and only if (X, ψ) is a comonadic coalgebra w.r.t. $\left(G, \tilde{\delta}, \tilde{\eta}\right)$.

(d) Defining $\Phi\left((X,\varphi)\right) := (X,\psi)$ on objects and

$$\Phi\Big((X,\varphi) \xrightarrow{h} (X',\varphi')\Big) := \Phi((X,\varphi)) \xrightarrow{h} \Phi((X',\varphi'))$$

on homomorphisms yields a well-defined functor between the category of monadic F-algebras and comonadic G-coalgebras making both categories isomorphic.

At this point it should be noted that the previous result is not entirely new. It seems to be the case that it has passed into common knowledge, yet we found it hard to give a specific reference, e.g. to one of the common text books on category theory. In [BBW09, 2.6(1)] the authors collect the proof from different references, one of which is the original paper by Eilenberg and Moore, [EM65, Theorem 3.1], showing that every monad arises from a naturally given adjunction. Namely, this adjunction is the one between the free and the forgetful functor of the category of monadic algebras[12] belonging to the given monad, later also known as Eilenberg-Moore algebras of the monad and Eilenberg-Moore category of the monad, respectively.

Our motivation for giving an explicit proof here was in particular to show in detail the concrete constructions that link the monadic algebras and comonadic coalgebras for an adjoint pair of endo-functors, so that they are easily applicable in the concrete case of dynamical systems.

Proof: The results of the proposition are proven using similar manipulations as in the proof of Lemma 4.5.1. Recall that for objects $X, Y \in \mathcal{C}$ and any morphism $X \xrightarrow{h} Y$ the following diagrams commute by the naturality of the transformations $\tilde{\delta}$ and $\tilde{\eta}$:

$$\begin{tikzcd} GGX \arrow[r, "GGh"] & GGY \\ GX \arrow[u, "\tilde{\delta}_X"] \arrow[r, "Gh"] & GY \arrow[u, "\tilde{\delta}_Y"'] \end{tikzcd} \qquad (4.15\tilde{\delta})$$

$$\begin{tikzcd} GX \arrow[r, "Gh"] \arrow[d, "\tilde{\eta}_X"] & GY \arrow[d, "\tilde{\eta}_Y"'] \\ X \arrow[r, "h"] & Y \end{tikzcd} \qquad (4.15\tilde{\eta})$$

(a) We start with the longest calculation:

$$\begin{aligned} \psi\tilde{\delta}_X &\overset{\text{def}\,\psi}{=} \vartheta_X \underline{G\varphi\tilde{\delta}_X} \overset{(4.15\tilde{\delta})}{=} \vartheta_X\tilde{\delta}_{FX}G^2\varphi \overset{\text{def}\,\tilde{\delta}_{FX}}{=} \underline{\vartheta_X\vartheta_{GFX}}G\mu_{FX}G^2\varphi \\ &\overset{(4.13\vartheta)}{=} \vartheta_X \underline{GF\vartheta_X G\mu_{FX}G^2\varphi} = \vartheta_X G\left(F\vartheta_X\mu_{FX}G\varphi\right) \overset{(4.11)}{=} \nu_{X,GX}\left(F\vartheta_X\mu_{FX}G\varphi\right), \end{aligned}$$

$$\begin{aligned} F\vartheta_X\mu_{FX}G\varphi &\overset{\text{def}\,\mu_{FX}}{=} \underline{F\vartheta_X\vartheta_{FGFX}}G\zeta_{FX}G\varphi \overset{(4.13\vartheta)}{=} \vartheta_{FX}\underline{GFF\vartheta_X G\zeta_{FX}G\varphi} \\ &= \vartheta_{FX}G\left(FF\vartheta_X\zeta_{FX}\varphi\right) \overset{(4.11)}{=} \nu_{FX,X}\left(F^2\vartheta_X\zeta_{FX}\varphi\right) \end{aligned}$$

[12] In [BBW09] these algebras are called $\mathbb{F}$-modules of the monad $\mathbb{F} = (F,\delta,\eta)$.

and

$$
\begin{aligned}
F^2\vartheta_X\underline{\zeta_{FX}\varphi} &\overset{(4.13\zeta)}{=} \underline{F^2\vartheta_X F^2 G\varphi}\zeta_X = F^2\left(\vartheta_X G\varphi\right)\zeta_X \overset{\text{def }\zeta_X}{=} \underline{F^2\left(\vartheta_X G\varphi\right)\delta_{GX}}\varepsilon_X \\
&\overset{(4.13\delta)}{=} \delta_X F\left(\vartheta_X G\varphi\right)\varepsilon_X = \delta_X F\vartheta_X \underline{FG\varphi\varepsilon_X} \overset{(4.13\varepsilon)}{=} \delta_X \underline{F\vartheta_X\varepsilon_{FX}}\varphi \\
&\overset{(4.9)}{=} \delta_X 1_{FX}\varphi = \delta_X\varphi.
\end{aligned}
$$

The second part is less complicated:

$$
\psi G\psi \overset{\text{def }\psi}{=} \vartheta_X G\varphi G\psi\vartheta_X = \vartheta_X G(\varphi\psi) \overset{(4.11)}{=} \nu_{X,GX}(\varphi\psi)
$$

and

$$
\varphi\psi \overset{\text{def }\psi}{=} \underline{\varphi\vartheta_X}G\varphi \overset{(4.13\vartheta)}{=} \vartheta_{FX}\underline{GF\varphi G\varphi} = \vartheta_{FX}G\left(F\varphi\varphi\right) \overset{(4.11)}{=} \nu_{FX,X}\left(F\varphi\varphi\right).
$$

(b) Applying similar methods one can verify

$$
\begin{aligned}
\psi\tilde{\eta}_X &\overset{\text{def }\psi}{=} \vartheta_X\underline{G\varphi\tilde{\eta}_X} \overset{(4.15\tilde{\eta})}{=} \vartheta_X\tilde{\eta}_{FX}\varphi \overset{\text{def }\tilde{\eta}_{FX}}{=} \underline{\vartheta_X\eta_{GFX}}\varepsilon_{FX}\varphi \overset{(4.13\eta)}{=} \eta_X\underline{F\vartheta_X\varepsilon_{FX}}\varphi \\
&\overset{(4.9)}{=} \eta_X 1_{FX}\varphi = \eta_X\varphi.
\end{aligned}
$$

(c) By Item (a) and the bijectivity of the morphisms ν the equality $\delta_X\varphi = F\varphi\varphi$ is equivalent to $\psi\tilde{\delta}_X = \psi G\psi$. Likewise, by Item (b) $\eta_X\varphi = 1_X$ holds if and only if $\psi\tilde{\eta}_X = 1_X$. Therefore, (X,φ) is a monadic algebra exactly if (X,ψ) is a comonadic coalgebra.

(d) It remains to be shown that the exhibited correspondence extends nicely to homomorphisms. Functoriality of Φ is trivial once it has been shown that Φ is well-defined. To this end consider arbitrary F-algebras (X,φ) and (X',φ') and a morphism $X \xrightarrow{h} X'$. Name the images of Φ $(X,\psi) := \Phi\left((X,\varphi)\right)$ and $(X',\psi') := \Phi\left((X',\varphi')\right)$, i.e. $\psi := \vartheta_X G\varphi$ and $\psi' := \vartheta_{X'}G\varphi'$. It will be shown that h satisfies the homomorphism property w.r.t. (X,φ) and (X',φ'), i.e. $\varphi h = Fh\varphi'$, if and only if it satisfies it w.r.t. (X,ψ) and (X',ψ'), i.e. $h\psi' = \psi Gh$. So the task is to verify that the left diagram commutes if and only if the one on the right-hand side commutes:

$$
\begin{tikzcd}
FX \arrow[r, "Fh"] \arrow[d, "\varphi"] & FX' \arrow[d, "\varphi'"'] \\
X \arrow[r, "h"] & X'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
GX \arrow[r, "Gh"] & GX' \\
X \arrow[r, "h"] \arrow[u, "\psi"] & X' \arrow[u, "\psi'"']
\end{tikzcd}
$$

This can be seen as follows

$$
\begin{aligned}
h\psi' &\overset{\text{def }\psi'}{=} \underline{h\vartheta_{X'}}G\varphi' \overset{(4.13\vartheta)}{=} \vartheta_X\underline{GFhG\varphi'} = \vartheta_X G\left(Fh\varphi'\right) \overset{(4.11)}{=} \nu_{X,X'}(Fh\varphi') \\
\psi Gh &\overset{\text{def }\psi}{=} \vartheta_X\underline{G\varphi Gh} = \vartheta_X G\left(\varphi h\right) \overset{(4.11)}{=} \nu_{X,X'}(\varphi h).
\end{aligned}
$$

As $\nu_{X,X'}$ is bijective, the desired equivalence holds.

As $\nu_{X,X}$ is bijective, one can define for any G-coalgebra (X,ψ) an F-algebra $(X,\varphi) := \Phi^{-1}((X,\psi)) := \left(X, \nu_{X,X}^{-1}(\psi)\right)$. By Item (c) and since $\nu_{X,X}$ is bijective, this transforms comonadic G-coalgebras into monadic F-algebras. By the equivalence just proven, also

$$\Phi^{-1}\left((X,\psi) \xrightarrow{h} (X',\psi')\right) := \Phi^{-1}((X,\psi)) \xrightarrow{h} \Phi^{-1}((X,\psi'))$$

is well-defined on homomorphisms, yielding an inverse functor for Φ. □

It is evident from the proof, that Φ and its inverse can be seen as inverse functors between arbitrary F-algebras and G-coalgebras that restrict docilely to monadic algebras w.r.t. (F,δ,η) and comonadic coalgebras w.r.t. $\left(G,\tilde{\delta},\tilde{\eta}\right)$, respectively.

Dualising the two previous results yields the converse implication:

4.5.4 Proposition. *Let $\mathcal{C}$ be a category and $F,G \in \operatorname{End}\mathcal{C}$ be two adjoint endofunctors ($F \dashv G$). We denote the corresponding natural equivalence between the hom-sets by $\operatorname{Hom}(F,-) \xrightarrow{\nu} \operatorname{Hom}(-,G)$, the unit by $1_\mathcal{C} \xrightarrow{\vartheta} GF$ and the co-unit by $FG \xrightarrow{\varepsilon} 1_\mathcal{C}$.*

Furthermore, let two natural transformations $G \xrightarrow{\tilde{\delta}} GG$ and $G \xrightarrow{\tilde{\eta}} 1_\mathcal{C}$ be given. Define natural transformations $FF \xrightarrow{\hat{\tilde{\delta}}} F$ and $1_\mathcal{C} \xrightarrow{\hat{\tilde{\eta}}} F$ dually as in Lemma 4.5.1, i.e. $\hat{\tilde{\delta}}_Y := \nu_{FY,FY}^{-1}(\nu_{Y,GFY}^{-1}(\vartheta_Y\tilde{\delta}_{FY}))$ and $\hat{\tilde{\eta}}_Y := \vartheta_Y\tilde{\eta}_{FY}$.

Let $X \in \mathcal{C}$ be an object and $X \xrightarrow{\psi} GX$ a morphism. Defining the morphism $FX \xrightarrow{\varphi:=\nu_{X,X}^{-1}(\psi)} X$, we have

(a) $FF \xrightarrow{\hat{\tilde{\delta}}} F$ and $1_\mathcal{C} \xrightarrow{\hat{\tilde{\eta}}} F$ are indeed natural transformations.

(b) $\left(F,\hat{\tilde{\delta}},\hat{\tilde{\eta}}\right)$ is a monad if and only if $\left(G,\tilde{\delta},\tilde{\eta}\right)$ is a comonad.

(c) (X,φ) is a monadic algebra w.r.t. $\left(F,\hat{\tilde{\delta}},\hat{\tilde{\eta}}\right)$ if and only if (X,ψ) is a comonadic coalgebra w.r.t. $\left(G,\tilde{\delta},\tilde{\eta}\right)$.

Proof: Consider the situation given in the proposition. Then $F,G \in \operatorname{End}\mathcal{C}$ can also be considered as endo-functors $F,G \in \operatorname{End}\mathcal{C}^\partial$ w.r.t. the opposite category of $\mathcal{C}$ (see Remark 2.2.2). They are still adjoint, but $G \dashv F$ (see e.g. [AHS06, 19.6]) and the corresponding natural equivalence is

$$\left(\operatorname{Hom}(-,G) \xrightarrow{\nu^{-1}} \operatorname{Hom}(F,-)\right) = \left(\operatorname{Hom}^\partial(G,-) \xrightarrow{\nu^{-1}} \operatorname{Hom}^\partial(-,F)\right),$$

the unit is $1_{\mathcal{C}^\partial} \xrightarrow{\varepsilon} FG$ and the co-unit is $GF \xrightarrow{\vartheta} 1_{\mathcal{C}^\partial}$. In $\mathcal{C}^\partial$ the natural transformations $G \xrightarrow{\tilde{\delta}} GG$ and $G \xrightarrow{\tilde{\eta}} 1_\mathcal{C}$ become $GG \xrightarrow{\tilde{\delta}} G$ and $1_{\mathcal{C}^\partial} \xrightarrow{\tilde{\eta}} G$, and the morphism

$X \xrightarrow{\psi} GX$ becomes $GX \xrightarrow{\psi} X$. Applying Propositions 4.5.2 and 4.5.3 to this situation (in $\mathcal{C}^{\partial}$) and reinterpreting the results in $\mathcal{C}$ yields exactly the stated claims. □

4.5.5 Proposition. *Let $\mathcal{C}$ be a category and $F, G \in \operatorname{End}\mathcal{C}$ be two adjoint endofunctors ($F \dashv G$). We denote the corresponding natural equivalence between the hom-sets by $\operatorname{Hom}(F,-) \xrightarrow{\nu} \operatorname{Hom}(-,G)$, the unit by $1_{\mathcal{C}} \xrightarrow{\vartheta} GF$ and the co-unit by $FG \xrightarrow{\varepsilon} 1_{\mathcal{C}}$.*

The constructions of comonads out of monads and vice versa, presented in the two previous propositions are mutually inverse, i.e.

(a) $(F, \delta, \eta) \mapsto \left(G, \tilde{\delta}, \tilde{\eta}\right) \mapsto \left(F, \hat{\tilde{\delta}}, \hat{\tilde{\eta}}\right) = (F, \delta, \eta).$

(b) $\left(G, \tilde{\delta}, \tilde{\eta}\right) \mapsto \left(F, \hat{\tilde{\delta}}, \hat{\tilde{\eta}}\right) \mapsto \left(G, \tilde{\hat{\tilde{\delta}}}, \tilde{\hat{\tilde{\eta}}}\right) = \left(G, \tilde{\delta}, \tilde{\eta}\right).$

An analogous statement holds w.r.t. the monadic algebras and comonadic coalgebras.

Proof: The final remark about algebras and coalgebras is trivial once the assertions about the monads and comonads have been shown. It follows directly from the bijectivity of the mapping $\nu_{X,X}$ and its inverse. Thus, we will only prove the results dealing with monads.

(a) By definition one has $\tilde{\delta}_X = \nu_{GX,GX}\left(\nu_{FGX,X}\left(\zeta_X\right)\right)$ and $\tilde{\eta}_X = \eta_{GX}\varepsilon_X$. It first has to be verified that $\hat{\tilde{\delta}}_X \overset{\text{def}\,\hat{\tilde{\delta}}}{=} \nu^{-1}_{FX,FX}\left(\nu^{-1}_{X,GFX}\left(\vartheta_X\tilde{\delta}_{FX}\right)\right) = \delta_X$, or equivalently $\vartheta_X\tilde{\delta}_{FX} = \nu_{FX,FX}\left(\nu_{X,GFX}\left(\delta_X\right)\right)$. Indeed, in detail we have

$$\begin{aligned}
\vartheta_X\tilde{\delta}_{FX} &\overset{\text{def}\,\tilde{\delta}_{FX}}{=} \underline{\vartheta_X\vartheta_{GFX}}G\mu_{FX} \overset{(4.13\vartheta)}{=} \vartheta_X\underline{GF\vartheta_XG\mu_{FX}} = \vartheta_XG\left(F\vartheta_X\mu_{FX}\right)\\
&\overset{(4.11)}{=} \nu_{X,GFX}\left(F\vartheta_X\mu_{FX}\right),\\
F\vartheta_X\mu_{FX} &\overset{\text{def}\,\mu_{FX}}{=} \underline{F\vartheta_X\vartheta_{FGFX}}G\zeta_{FX} \overset{(4.13\vartheta)}{=} \vartheta_{FX}\underline{GF^2\vartheta_XG\zeta_{FX}} = \vartheta_{FX}G\left(F^2\vartheta_X\zeta_{FX}\right)\\
&\overset{(4.11)}{=} \nu_{FX,FX}\left(F^2\vartheta_X\zeta_{FX}\right)
\end{aligned}$$

and

$$F^2\vartheta_X\zeta_{FX} \overset{\text{def}\,\zeta_{FX}}{=} \underline{F^2\vartheta_X\delta_{GFX}}\varepsilon_{FX} \overset{(4.13\delta)}{=} \delta_X\underline{F\vartheta_X\varepsilon_{FX}} \overset{(4.9)}{=} \delta_X1_{FX} = \delta_X.$$

The remaining equality is easier to see:

$$\hat{\tilde{\eta}}_X \overset{\text{def}\,\hat{\tilde{\eta}}_X}{=} \vartheta_X\tilde{\eta}_{FX} \overset{\text{def}\,\tilde{\eta}_{FX}}{=} \underline{\vartheta_X\eta_{GFX}}\varepsilon_{FX} \overset{(4.13\eta)}{=} \eta_X\underline{F\vartheta_X\varepsilon_{FX}} \overset{(4.9)}{=} \eta_X1_{FX} = \eta_X.$$

(b) This fact follows from the previous item by dualisation, similarly as in the proof of Proposition 4.5.4. □

As a consequence of Propositions 4.5.3, 4.5.4 and 4.5.5, it does not matter if we regard (abstract) dynamical systems as monadic algebras or comonadic coalgebras. The coalgebraic perspective however offers us some variability that is not necessarily available on the side of algebras. Slight modifications of the signature $-^T$ may result in a functor that fails to have a left-adjoint, and thus the corresponding coalgebras may lack a counterpart on the algebraic side.

From the applications of coalgebra to transition systems in computer science, a decent choice of related signature functors for the coalgebraic formulation suggests itself. For our convenience and since it is very common for transition systems, we state these functors for the case $\mathcal{C} = \mathcal{S}et$. Using $-^T$ every state x of the coalgebra is mapped to a T-sequence of successor states, the trajectory of $x \in X$. This closely corresponds to a deterministic automaton with state set X and possibly infinite alphabet T. Its behaviour can be extended by observations (or outputs) from a fixed set A. This is possible in at least two ways: one may add one observation per trajectory, resulting in the functor $X \mapsto A \times X^T$, or one per each successor state, yielding the functor $X \mapsto (A \times X)^T$. Instead of using a fixed set A as observables, one may also try the state space itself, giving rise to $X \mapsto X \times X^T$.

Besides, non-determinism can be represented without any difficulties: instead of assigning to each state a sequence of future states, one may assign a T-sequence of subsets of possible successor states. This is expressible using the endo-functor $X \mapsto \mathfrak{P}(X)^T$.

Of course, all these different features may also be combined in one functor, such as $X \mapsto (A \times \mathfrak{P}(X))^T$.

We leave it as a task for future investigations to generalise these functors to categories like $\mathcal{T}op$ or $\mathcal{M}easrbl$, and to explore the increased expressivity for particular examples of dynamical systems.

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