Supersaturation for subgraph counts

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Abstract

The classical extremal problem is that of computing the maximum number of edges in an F-free graph. In particular, Turán's theorem entirely resolves the case where $F = K_{r+1}$. Later results, known as supersaturation theorems, proved that in a graph containing more edges than the extremal number, there must also be many copies of K_{r+1}.

Alon and Shikhelman introduced a broader class of extremal problems, asking for the maximum number of copies of a graph T in an F-free graph (so that $T = K_2$ is the classical extremal number). In this paper we determine some of these generalized extremal numbers when T and F are stars or cliques and prove some supersaturation results for them.

1 Introduction

The classic theorem of Turán [30] gives the maximum number of edges in a K_{r+1}-free graph, a number which is asymptotically $(1-\frac{1}{r})\binom{n}{2}$. As is standard, we let $\mathrm{ex}(n, F)$ be the maximum number of edges in an F-free graph on n vertices, so Turán's theorem determines $\mathrm{ex}(n, K_{r+1})$. Of course, if the number of edges in a graph G on n vertices exceeds $\mathrm{ex}(n, K_{r+1})$, we know that G must contain at least one K_{r+1}. One can ask about the minimum number of copies of K_{r+1} that are contained in G. Results of this type are referred to as *supersaturation*

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theorems. To be precise, letting $k_{r+1}(G)$ be the number of copies of K_{r+1} in a graph G, supersaturation questions ask one to determine

$$\min\{k_{r+1}(G) : G \text{ a graph with } n \text{ vertices and } \mathrm{ex}(n, K_{r+1}) + q \text{ edges}\},$$

for some $q \geq 1$. When $q = o(n^2)$, the problem was studied by Rademacher [27], Erdős [11, 7, 8], and then resolved by Lovász and Simonovits [19, 20]. For the case when $q = \Omega(n^2)$, asymptotic solutions have been found by Razborov [28] for $r = 2$, Nikiforov [24] for $r = 3$, and Reiher [29] for general r. See Pikhurko and Yilma [26] for a very informative introduction to supersaturation.

One could also ask if other structures are guaranteed to exist in graphs with more edges than the Turán number. The following theorem of Erdős and Stone [10] shows that, in a graph in which the edge count exceeds this extremal number by a constant multiple of n^2, must not only contain a K_{r+1}, but indeed a blowup of K_{r+1} with large part sizes. For a graph G, we let the blowup $G(b)$ be the graph where each vertex of G is replaced by an independent set of size b and each edge is replaced by a complete bipartite graph.

Theorem 1.1 (Erdős-Stone). *Let $r \geq 1$ be an integer and let $\varepsilon > 0$. Then there exists $n_0 = n_0(r, \varepsilon)$ such that if G is a graph on $n \geq n_0$ vertices and*

$$e(G) \geq \left(1 - \frac{1}{r} + \varepsilon\right)\binom{n}{2},$$

then G contains $K_{r+1}(b)$ for some $b \geq \varepsilon \log n/(2^{r+1}(r-1)!)$.

We will refer to theorems analogous to Theorem 1.1 as *structural supersaturation* results.

In more recent work, Alon and Shikhelman [1] considered generalized extremal problems involving counting copies of some fixed subgraph rather than edges. To be precise, they were interested in determining values of

$$\mathrm{ex}_T(n, F) = \max\{n_T(G) : G \text{ is an } F\text{-free graph on } n \text{ vertices}\},$$

where $n_T(G)$ is the number of copies of T in G. In particular, $\mathrm{ex}_{K_2}(n, F) = \mathrm{ex}(n, F)$. Their paper lead to many investigations by various authors for different choices of T and F; see [14, 15, 17, 18, 22] for a sample of authors and results. In this paper, we consider Alon-Shikhelman-type problems where T and F are either cliques or stars. We also consider supersaturation and structural supersaturation results in this vein. The following subsections will outline the history of these problems and the new results of this paper.

1.1 Cliques without cliques

The most fundamental Alon-Shikhelman-type problems involve cliques. As above, we write $k_t(G)$ for $n_{K_t}(G)$. Zykov [32], along with many others, showed that $\mathrm{ex}_{K_t}(n, K_{r+1}) = k_t(T(n, r))$, where $T(n, r)$ is the r-partite Turán graph on n vertices. Bollobás [2] discussed the general problem of minimizing the number of copies of K_s in a graph with a given number, say N, of copies of K_t, i.e., a supersaturation result for $\mathrm{ex}_{K_t}(n, K_s)$. (Of course, if $N \le \mathrm{ex}_{K_t}(n, K_s)$, then this minimum number is 0.) His result gives a bound of the form

$$k_s(G) \ge \psi(N),$$

where ψ is a function defined implicitly.

Theorem 1.2 (Bollobás)**.** *For a given n, let $\psi(x) = \psi_t^s(x)$ be the maximal convex function defined for $0 \le x \le \binom{n}{t}$ such that*

$$\psi\left(\left(\frac{n}{i}\right)^t \binom{i}{t}\right) \le \left(\frac{n}{i}\right)^s \binom{i}{s}, \qquad \text{for } i = 1, 2, \dots, n.$$

Then, if G is a graph on n vertices with $k_t(G) \ge x$, then $k_s(G) \ge \psi(x)$.

A weaker, but slightly more transparent version is the following.

Theorem 1.3. *Let θ be a real number and s and t be integers with $2 \le t \le s \le \theta + 1$. If G is a graph on n vertices such that $k_t(G) \ge \binom{\theta}{t}(n/\theta)^t$, then $k_s(G) \ge \binom{\theta}{s}(n/\theta)^s$.*

We cannot find this result in the literature, but it can be proved by iteratively applying the following beautiful theorem of Moon and Moser [23] and the method outlined in Lovász's Combinatorial Problems and Exercises [21, Section 10, Question 40].

Theorem 1.4 (Moon-Moser)**.** *For any graph G on n vertices and any $s \ge 2$,*

$$\frac{k_{s+1}(G)}{k_s(G)} \ge \frac{1}{s^2 - 1}\left(s^2 \frac{k_s(G)}{k_{s-1}(G)} - n\right).$$

Both Theorem 1.2 and Theorem 1.3 are supersaturation results. Now we turn our attention to the corresponding structural supersaturation problem. Nikiforov [25] showed that the conclusion of the Erdős-Stone theorem follows even from the weak hypothesis that G contains $\Omega(n^{r+1})$ copies of K_{r+1}.

Theorem 1.5 (Nikiforov)**.** *Let $s \ge 2$ and c and n be such that*

$$0 < c < 1/s! \quad \text{and} \quad n \ge \exp(c^{-s}).$$

If G is a graph with n vertices and $k_s(G) \ge cn^s$, then G contains a $K_s(b)$ with $b = \lfloor c^s \log n \rfloor$.

This, together with Theorem 1.3, proves a structural supersaturation extension of Zykov's result.

Theorem 1.6. *For all $\varepsilon > 0$, there is a $\delta > 0$ and an $n_0 \in \mathbb{N}$ such that if G is a graph on $n \ge n_0$ vertices and $k_t(G) \ge (1+\varepsilon)k_t(T(n, r))$, then G contains a $K_{r+1}(C \log n)$ for some $C = C(\varepsilon, r) > 0$.*

Proof. The hypothesis on $k_t(G)$ implies that, for some $\theta > r$, we have $k_t(G) \ge \binom{\theta}{t}(n/\theta)^t$. Thus, by Corollary 1.3, $k_{r+1}(G) \ge \binom{\theta}{r+1}(n/\theta)^{r+1}$, a constant multiple of n^{r+1}. Now, by Theorem 1.5, G contains a large blowup of K_{r+1}. □

1.2 Cliques without stars

If we write S_r for $K_{1,r}$, a recent result of Chase [5], building on work of Gan, Loh, and Sudakov [13], completely determines $\mathrm{ex}_{K_t}(n, S_{r+1})$.

Theorem 1.7 (Chase)**.** *Fix $t \geq 3$. For any positive integers $n, r \geq 1$, if $n = a(r+1) + b$ where $0 \leq b \leq r$, then*

$$\mathrm{ex}_{K_t}(n, S_{r+1}) = a\binom{r+1}{t} + \binom{b}{t}.$$

It will be useful for us later to state and prove here a "signpost" version of Theorem 1.7 due to Wood [31], and Engbers and Galvin [6]. For $v \in V(G)$, we write $k_t(v)$ for the number of copies of K_t in G that contain vertex v.

Theorem 1.8 (Wood, Engbers-Galvin)**.** *For any $1 \leq r \leq n$, we have*

$$\mathrm{ex}_{K_t}(n, S_{r+1}) \leq \frac{n}{t}\binom{r}{t-1} = \frac{n}{r+1}\binom{r+1}{t}.$$

Proof. Note that being S_{r+1}-free is equivalent to having maximum degree at most r. Let G be such a graph on n vertices. If we count pairs (v, S) where v is a vertex of G, S is a t-clique in G and $v \in S$ then

$$tk_t(G) = \sum_{v \in V(G)} k_t(v) = \sum_{v \in V(G)} k_{t-1}(G[N(v)]) \leq n\binom{r}{t-1}.$$

□

Note that though Theorem 1.8 does not give the exact value of $\mathrm{ex}_{K_t}(n, S_{r+1})$, it is asymptotically sharp since the graph aK_{r+1} achieves the bound whenever n is divisible by $r+1$.

In Section 2, we prove the following supersaturation result showing that if G contains many copies of K_t, then there must be many copies of S_r in G. We write $s_r(G)$ for $n_{S_r}(G)$ and note that

$$s_r(G) = \sum_{v \in V(G)} \binom{d(v)}{r}.$$

Theorem 1.9. *Given $2 \leq t \leq r$, for all $\varepsilon > 0$ there exists $\delta > 0$ such that if G is a graph on n vertices having*

$$k_t(G) \geq (1+\varepsilon)\frac{n}{r+1}\binom{r+1}{t},$$

then $s_{r+1}(G) \geq \delta n$.

Note that the bound in Theorem 1.9 is asymptotically sharp. To see this, let $s > r$ and consider the graph G on $n = k(s+1)$ vertices that is the disjoint union of k copies of K_{s+1}, i.e., $G = kK_{s+1}$. Then,

$$k_t(G) = k\binom{s+1}{t} = \frac{n}{t}\binom{s}{t-1}.$$

A straightforward calculation shows that, provided $s \geq r+1$,

$$\frac{\frac{n}{t}\binom{s}{t-1}}{\frac{n}{r+1}\binom{r+1}{t}} > 1,$$

and so the conditions of Theorem 1.9 are met. Further, note that

$$s_{r+1}(G) = n\binom{s}{r+1}.$$

Thus, equality is achieved in the conclusion of Theorem 1.9 with $\delta = \binom{s}{r+1}$.

For structural supersaturation, since the star is not vertex transitive, there are different notions of a blowup of S_{r+1}; they are all of the form $K_{a,b}$. The discussion above implies that having a surplus of K_ts does not imply even the existence of a $K_{1,r+2}$. In addition, the classic construction of Füredi [12] demonstrates that it is also not possible to guarantee the existence of a $K_{2,r+1}$ (at least in the case when $t = 3$). The following theorem can be read out of his paper.

Theorem 1.10 (Füredi)**.** *For any $r \geq 1$ there exist infinitely many n so that there is a graph on n vertices which is $K_{2,r+1}$-free and contains $\Omega(n^{3/2})$ triangles. In particular, knowing that $k_3(G)$ is at least $(1+\varepsilon)\,\mathrm{ex}_{K_3}(n, S_{r+1})$ does not imply the existence of a $K_{2,r+1}$ in G.*

1.3 Stars without stars

Although this case is rather uninteresting, we include it for completeness.

Proposition 1.11. *If $t > 1$, then for $n \geq r+1$,*

$$\mathrm{ex}_{S_t}(n, S_{r+1}) = \begin{cases} n\binom{r}{t} & \text{if } nr \text{ is even,} \\ (n-1)\binom{r}{t} + \binom{r-1}{t} & \text{otherwise.} \end{cases}$$

Proof. Since each degree is at most r, we have that $s_t(G) = \sum \binom{d(v)}{t}$ is maximized when G is as close to r-regular as possible. If nr is even, then there is an r-regular graph and otherwise there is a graph where one vertex has degree $r-1$ and all others have degree r. □

One can also prove a rather uninteresting supersaturation result in this case. Since both the number of S_ts and the number of S_{r+1}s are a function of the degree sequence, it is easy to check that an excess of $\varepsilon n\binom{r}{t}$ copies of S_t yields at least $\varepsilon n(r-t+1)/t$ copies of S_{r+1}. The extremal graph is as regular as possible.

No structural supersaturation theorem for this case is true. Any $(r+1)$-regular graph has a fixed fraction more S_ts than $\mathrm{ex}_{S_t}(n, S_{r+1})$, without containing any S_{r+2}. The same Füredi example from the previous section is almost regular and hence contains at least $(1+\varepsilon)\,\mathrm{ex}_{S_t}(n, S_{r+1})$ copies of S_t without having a $K_{2,r+1}$.

1.4 Stars without cliques

This case is substantially more difficult than the others we've encountered up to this point. Caro and Yuster [4] considered the related problem of determining, for a graph H and $t \geq 1$,

$$\mathrm{ex}_t(n, H) = \max\Big\{ f_t(G) \,:\, G \text{ is } H\text{-free on } n \text{ vertices} \Big\},$$

where

$$f_t(G) = \sum_{v \in V(G)} d_G(v)^t.$$

Note that the values of $\mathrm{ex}_t(n, K_{r+1})$ and $\mathrm{ex}_{S_t}(n, K_{r+1})$ are asymptotically equal as $n \to \infty$. Caro and Yuster showed that the extremal graph for $H = K_{r+1}$ and $t = 1, 2, 3$ is the Turán graph, and asked if this was true for larger t. Bollobás and Nikiforov [3] gave a nearly complete answer. We sum up their results in the following theorem:

Theorem 1.12 (Bollobás, Nikiforov)**.** *For all $r \geq 2$ and $t > 0$ there exists $c = c(t, r)$ such that if some K_{r+1}-free graph G of order n satisfies $f_t(G) = \mathrm{ex}_t(n, K_{r+1})$, then G is a complete r-partite graph having $r-1$ vertex classes of size $cn + o(n)$. Furthermore, if $t < r$ and n is sufficiently large, then the Turán graph $T_r(n)$ realizes $\mathrm{ex}_t(n, K_{r+1})$, but for $t \geq r + \sqrt{2r}$, $\mathrm{ex}_t(n, K_{r+1}) > f(t, T_r(n))$.*

Considering the graphon version $\mathrm{ex}_{S_t}(W, K_{r+1})$ of the stars without cliques problem, we extend the Bollobás-Nikiforov result in two ways. First, we specify more precisely the sizes of the vertex classes of non-Turán solutions. Second, we prove the existence of a value $t^*(r)$ such that when $t < t^*$, the Turán graphon uniquely realizes $\mathrm{ex}_{S_t}(W, K_{r+1})$ and when $t \geq t^*$, the non-Turán solution is the unique maximum. We have not been able to make any progress on either the supersaturation or structural supersaturation versions of the problem. All the details can be found in Section 3.

2 Supersaturation for cliques without stars

In order to prove Theorem 1.9, we start with a lemma concerning the function $\binom{x}{s}$ where x is a postive real number. We define, for $x \in [0, \infty)$ and $s \in \mathbb{N}_{\geq 1}$,

$$f_s(x) = \begin{cases} \dbinom{x}{s} = \dfrac{1}{s!}\, x(x-1)\cdots(x-s+1) & \text{if } x \geq s-1 \\ 0 & \text{if } 0 \leq x < s-1. \end{cases}$$

Note that, for $x > s+1$,

$$f_s'(x) = \frac{1}{s!}\sum_{i=0}^{s-1} x(x-1)\cdots\cancel{(x-i)}\cdots(x-s+1)$$

and

$$f_s''(x) = \frac{2}{s!}\sum_{0\le i<j\le s-1} x(x-1)\cdots\cancel{(x-i)}\cdots\cancel{(x-j)}\cdots(x-s+1).$$

Also, note that f_s is strictly increasing on $[s-1,\infty)$. We denote the inverse of $f_s|_{[s-1,\infty)}$ by f_s^{-1}.

Lemma 2.1. *For all $1 \le t < s$ the function $f_s \circ f_t^{-1}$ is convex on $(0,\infty)$ and strictly convex on $(\binom{s-1}{t},\infty)$.*

Proof. Note that $f_s \circ f_t^{-1}(x) = 0$ if $x \le \binom{s-1}{t}$. Further, the derivative is positive if $x > \binom{s-1}{t}$ and thus it's enough to show strict convexity on $(\binom{s-1}{t},\infty)$. For convenience we'll denote $f_t^{-1}(x)$ by u, and we may assume that $u > s-1$. Note that

$$(f_s \circ f_t^{-1})'(x) = f_s'(u)\cdot u', \qquad \text{and} \qquad u' = \frac{1}{f_t'(u)}.$$

Thus

$$\begin{aligned}(f_s \circ f_t^{-1})'' &= f_s''(u)\cdot(u')^2 + f_s'(u)\cdot u'' = f_s''(u)\cdot\frac{1}{(f_t'(u))^2} - \frac{f_s'(u)}{(f_t'(u))^2}\cdot f_t''(u)\cdot u' \\ &= \frac{f_s''(u)f_t'(u) - f_s'(u)f_t''(u)}{(f_t'(u))^3}.\end{aligned}$$

Since $u > t-1$, we have $f_t'(u) > 0$ so we need only that the numerator of the above is positive. To this end, since $s > t$, note that

$$\begin{aligned}&f_s''(u)f_t'(u) - f_s'(u)f_t''(u) \\ &\quad= \frac{2}{s!t!}\Bigg[\sum_{\substack{0\le i<j\le s-1\\ 0\le k\le t-1}} u(u-1)\cdots\cancel{(u-i)}\cdots\cancel{(u-j)}\cdots(u-s+1) \\ &\qquad\qquad\qquad \cdot u(u-1)\cdots\cancel{(u-k)}\cdots(u-t+1) \\ &\qquad - \sum_{\substack{0\le i\le s-1\\ 0\le j<k\le t-1}} u(u-1)\cdots\cancel{(u-i)}\cdots(u-s+1) \\ &\qquad\qquad\qquad \cdot u(u-1)\cdots\cancel{(u-j)}\cdots\cancel{(u-k)}\cdots(u-t+1)\Bigg].\end{aligned}$$

We'll show that this is non-negative by proving that all the negative terms are canceled by positive ones. If we write $T_{ij|k}$ for a typical term in the first sum and $T_{i|jk}$ for one in the second, then we see that all the terms with $i, j, k < t$ cancel since $T_{i|jk}$ cancels with $T_{jk|i}$. The remaining negative terms are of the form $T_{i|jk}$ with $i \geq t$. We have that each such term $T_{i|jk}$ cancels with $T_{ji|k}$. Strictness of convexity is guaranteed since some strictly positive terms remain, e.g., the $T_{ij|k}$ with $i = k$ and $j \geq t$. □

We are now ready for the proof of the main theorem of this section, which we recall here.

Theorem 1.9. *Given $2 \leq t \leq r$, for all $\varepsilon > 0$ there exists $\delta > 0$ such that if G is a graph on n vertices having*

$$k_t(G) \geq (1+\varepsilon)\frac{n}{r+1}\binom{r+1}{t},$$

then $s_{r+1}(G) \geq \delta n$.

Proof. As in Theorem 1.8,

$$n\binom{r}{t-1}(1+\varepsilon) \leq tk_t(G) = \sum_v k_t(v) \leq \sum_v \binom{d(v)}{t-1}.$$

Define $\ell(v) = f_{t-1}(d(v)) = \binom{d(v)}{t-1}$. We have

$$\sum_v \ell(v) \geq n\binom{r}{t-1}(1+\varepsilon) \qquad \text{and} \qquad s_{r+1} = \sum_v \binom{d(v)}{r+1} = \sum_v \binom{f_{t-1}^{-1}(\ell(v))}{r+1}.$$

The last equality is true term-by-term noting that if $d(v) < t-1$, and hence $d(v) \neq f_{t-1}^{-1}(f_{t-1}(d(v)))$, the v term in both these sums is zero.

We define

$$\tilde{f}_{r+1,t-1}(\ell) = f_{r+1}(f_{t-1}^{-1}(\ell)).$$

We will determine the minimum of $\sum_{i=1}^n \tilde{f}_{r+1,t-1}(\ell_i)$ subject to $\sum_{i=1}^n \ell_i \geq n\binom{r}{t-1}(1+\varepsilon)$. To be precise, we solve the relaxation where $\ell_i \in \mathbb{R}_{\geq 0}$. Since $\tilde{f}_{r+1,t-1}$ is convex by Lemma 2.1, we have

$$\sum_{i=1}^n \tilde{f}_{r+1,t-1}(\ell_i) \geq n\tilde{f}_{r+1,t-1}\Big(\sum_{i=1}^n \ell_i\Big) \geq n\tilde{f}_{r+1,t-1}\left(\binom{r}{t-1}(1+\varepsilon)\right).$$

Thus we are done, setting $\delta = \tilde{f}_{r+1,t-1}\left(\binom{r}{t-1}(1+\varepsilon)\right)$. □

3 Many stars, no K_{r+1}

We first note that any extremal graph is complete r-partite. This simplifies the problem of finding the largest number of S_ts. We then address the graphon version of the problem, determining $\mathrm{ex}_{S_t}(W, K_{r+1}) = \mathrm{ex}_t(W, K_{r+1})$. Erdős [9] proved that, given any K_{r+1}-free graph G, there is an r-partite graph H on the same vertex set satisfying $d_G(v) \le d_H(v)$ for all vertices v. Hence, there is an r-partite optimizer for $\mathrm{ex}_{S_t}(n, K_{r+1})$. Indeed, Győri, Pach, and Simonovits [16] proved the following substantially stronger result.

Theorem 3.1 (Győri-Pach-Simonovits)**.** *Let T be a complete k-partite graph with $t+1$ vertices, let $r \ge k$, and let $n \ge \max(t+2, r+1)$. Then all K_{r+1}-free graphs G on n vertices satisfying $n_T(G) = \mathrm{ex}_T(n, K_{r+1})$ are complete r-partite.*

Sketch of proof. For a graph G and non-adjacent vertices $u, v \in V(G)$, the *Zykov symmetrization* of v by u, denoted $Z_{u \to v}(G)$, turns v into a clone of u by setting $N(v) = N(u)$. This technique was introduced by Zykov in [32]. Note that Zykov symmetrizations cannot increase the clique number of a graph, and indeed, one can be chosen that (weakly) increases the number of copies of T until G is complete multipartite. (That G becomes complete multipartite is only guaranteed because T is complete multipartite itself.) At any cloning step where the graph becomes complete multipartite, one can show that the graph is transformed from a strict subgraph of a complete multipartite graph to a complete multipartite graph, strictly increasing the number of copies of T in the process. □

Knowing that the optimal graph is complete multipartite leaves only the question of what part sizes are optimal. We solve the problem asymptotically, i.e., we show that there are optimal proportions $\alpha_1, \alpha_2, \ldots, \alpha_r$ for the part sizes. The optimization problem we are trying to solve then is (asymptotically, and ignoring a factor of $1/t!$)

$$\begin{array}{llcl} \text{Maximize} & F(\rho_1, \rho_2, \ldots, \rho_r) & = & \displaystyle\sum_{i=1}^{r} \rho_i (1-\rho_i)^t \\ \text{subject to} & \rho_i & \ge & 0 \\ & \displaystyle\sum_{i=1}^{r} \rho_i & = & 1. \end{array} \tag{1}$$

We will naturally start by finding the interior critical points, which must satisfy

$$\nabla F(\rho) = \lambda(1, 1, \ldots, 1)$$

for some λ. Writing $f(\rho) = (1-\rho)^t \rho$ we require that the vector $(f'(\rho_1), f'(\rho_2), \ldots, f'(\rho_r))$ is constant. We start with a basic lemma concerning the derivatives of f.

Lemma 3.2. *With $f(\rho) = (1-\rho)^t \rho$ and $k \ge 1$ we have*

$$f^{(k)}(\rho) = (-1)^k t_{(k-1)} (1-\rho)^{t-k} ((t+1)\rho - k).$$

In particular the first and second derivatives of f are

$$g(\rho) = f'(\rho) = (1-\rho)^{t-1}(1-(t+1)\rho)$$
$$h(\rho) = f''(\rho) = t(1-\rho)^{t-2}((t+1)\rho - 2).$$

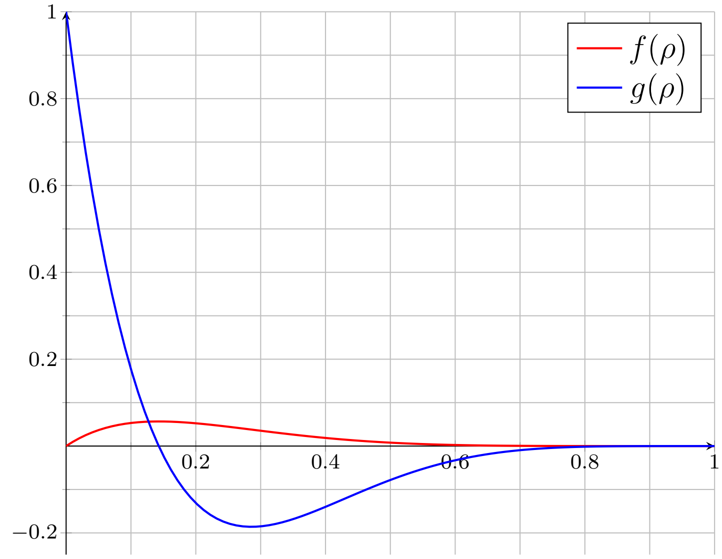

Figure 1: Graphs of $f(\rho)$ and $g(\rho)$ with $t = 6$

Proof. Straightforward. ☐

We denote values of $g(\rho)$ by ϕ. If ϕ is a value of g with $\phi > 0$ there is exactly one solution of $g(\rho) = \phi$, whereas if $\phi \in (\phi_{\min}, 0]$ (where $\phi_{\min} = g(2/(t+1))$ is the minimum value of $g(\rho)$ on $[0, 1]$) then there are exactly two solutions. One of these solutions satisfies $1/(t+1) \leq \rho < 2/(t+1)$, and the other satisfies $2/(t+1) < \rho \leq 1$.

Corollary 3.3. *Interior critical points for* (1) *are either of the form* $(1/r, 1/r, \ldots, 1/r)$*, the Turán solution, or* $(\alpha, \alpha, \ldots, \alpha, \beta, \beta, \ldots, \beta)$*, where*

$$\frac{1}{t+1} \leq \alpha < \frac{2}{t+1} \qquad \textit{and} \qquad \frac{2}{t+1} < \beta \leq 1. \tag{2}$$

and for some $\phi \in (\phi_{min}, 0]$ *we have* $g(\alpha) = \phi = g(\beta)$*, which we will refer to as a skew solution. In the skew solution case we also require that* $a\alpha + b\beta = 1$*, where* a *is the number of* α*s and* b *is the number of* β*s.*

In this section we will prove the following theorem describing the optimal solution to (1).

Theorem 3.4. *Let* $r, t \geq 2$*. The objective function* F *is maximized at an interior critical point. There are at most two possibilities for this maximizing critical point. One is the Turán solution. The only other possibility is the skew solution* $(\alpha, \alpha, \ldots, \alpha, \beta)$ *associated to* $a = r - 1$ *and* $b = 1$ *having* $g(\alpha) = g(\beta)$ *largest. If any skew solution exists, then this skew solution exists.*

Our approach will be to fix t, a, and b, and consider α, β as functions of ϕ. We are then looking for solutions to

$$L_{a,b}(\phi) = a\alpha + b\beta = 1,$$

which maximize

$$F_{a,b} = af(\alpha) + bf(\beta).$$

If the context makes it clear, we will omit the subscripts. We will then consider a critical point $(\alpha, \alpha, \ldots, \alpha, \beta, \beta, \ldots, \beta)$ with a copies of α and b copies of β and $\phi = g(\alpha) = g(\beta)$. If $a < r-1$, we will show that there is a critical point associated to some $\phi' = g(\alpha') = g(\beta') > \phi$ with $a+1$ copies of α', $b-1$ copies of β', and a larger value for the objective function. Thus, we need only consider which critical point associated with the case $a = r-1$ and $b = 1$ is best. We show it is the one with ϕ largest.

We begin with some preliminary lemmas.

Lemma 3.5. *For any real numbers a, b summing to r, we have*

$$\frac{dL_{a,b}}{d\phi} = \frac{a}{h(\alpha)} + \frac{b}{h(\beta)},$$
$$\frac{dF_{a,b}}{d\phi} = \phi\frac{dL_{a,b}}{d\phi}, \text{ and}$$
$$\frac{d^2L_{a,b}}{d\phi^2} = \frac{ah'(\alpha)(h(\beta))^3 + bh'(\beta)(h(\alpha))^3}{-(h(\alpha)h(\beta))^3}.$$

Proof. Since $\phi = g(\alpha)$, we have that $d\alpha/d\phi = 1/h(\alpha)$. Similarly, $d\beta/d\phi = 1/h(\beta)$ from which the first equation follows. For the second,

$$\frac{dF}{d\phi} = ag(\alpha)\frac{d\alpha}{d\phi} + bg(\beta)\frac{d\beta}{d\phi} = \frac{a\phi}{h(\alpha)} + \frac{b\phi}{h(\beta)} = \phi\Big(\frac{a}{h(\alpha)} + \frac{b}{h(\beta)}\Big) = \phi\frac{dL}{d\phi}.$$

The third is a straightforward calculation. □

As a consequence, for $\phi_2 < \phi_1$, we have

$$F(\phi_1) - F(\phi_2) = \int_{\phi_2}^{\phi_1} \frac{dF}{d\phi}\,d\phi = \int_{\phi_2}^{\phi_1} \phi\frac{dL}{d\phi}\,d\phi = \phi L\Big|_{\phi_2}^{\phi_1} - \int_{\phi_2}^{\phi_1} L\,d\phi. \tag{3}$$

Note that, in the expression for $d^2L/d\phi^2$ in Lemma 3.5, the denominator and the first term on the numerator are always positive and the second term on the numerator is positive provided $\beta > 3/(t+1)$. Hence, for $\phi > \phi_{\text{key}} := g(3/(t+1))$, we see that L is a convex function of ϕ. Our proof will depend on the fact that if L is concave at ϕ, then $\phi \leq \phi_{\text{key}}$.

Now we are ready to begin the proof in earnest. The following sequence of technical lemmas builds our understanding of the relationship between the values of the objective function at the possible internal critical points.

Lemma 3.6. *If there is a critical point with parameters ϕ, a, and b, and $a < r-1$, then there is a critical point associated to ϕ', $a+1$, and $b-1$, with $\phi' > \phi$ and $F_{a,b}(\phi) < F_{a+1,b-1}(\phi')$.*

Proof. We have $L_{a,b}(\phi) = 1$ and $\alpha(\phi) < \beta(\phi)$, hence $L_{a+1,b-1} < 1$. Also, note that $F_{a+1,b-1}(\phi) = F_{a,b}(\phi) + f(\alpha) - f(\beta)$. By the Intermediate Value Theorem, there is a root of $L_{a+1,b-1} = 1$ between ϕ and 0. (Note that $L_{a+1,b-1}(0) \geq 2(a+1)/(t+1) + b - 1 > 1$.) Let ϕ' be the smallest such root. By (3), we have

$$\begin{aligned} F_{a+1,b-1}(\phi') - F_{a+1,b-1}(\phi) &= \phi' L_{a+1,b-1}(\phi') - \phi L_{a+1,b-1}(\phi) - \int_{\phi}^{\phi'} L_{a+1,b-1}(\rho)\, d\rho \\ &= \phi' - \phi(1 + \alpha - \beta) - \int_{\phi}^{\phi'} L_{a+1,b-1}(\rho)\, d\rho \\ &= \phi(\beta - \alpha) + (\phi' - \phi) - \int_{\phi}^{\phi'} L_{a+1,b-1}(\rho)\, d\rho \\ &> \phi(\beta - \alpha), \end{aligned}$$

where the inequality is a consequence of the fact that $L_{a+1,b-1}(\rho) < 1$ for $\rho \in (\phi, \phi')$. Thus,

$$\begin{aligned} F_{a+1,b-1}(\phi') - F_{a,b}(\phi) &= (F_{a+1,b-1}(\phi') - F_{a+1,b-1}(\phi)) + (F_{a+1,b-1}(\phi) - F_{a,b}(\phi)) \\ &> \phi(\beta - \alpha) + f(\alpha) - f(\beta). \end{aligned}$$

So, it suffices to show

$$\frac{f(\beta) - f(\alpha)}{\beta - \alpha} \leq \phi.$$

But by the Mean Value Theorem for some $\rho \in (\alpha, \beta)$, we have

$$\frac{f(\beta) - f(\alpha)}{\beta - \alpha} = g(\rho).$$

For all $\rho \in (\alpha, \beta)$, we have $g(\rho) < g(\alpha) = g(\beta) = \phi$ and so we are done. □

Lemma 3.7. *If $\alpha < \beta \leq 3/(t+1)$ satisfy $g(\alpha) = \phi = g(\beta)$, then*

$$\frac{2 - (t+1)\alpha}{(t+1)\beta - 2} \leq 1.$$

Proof. First note that if $2/(t+1) \leq \rho \leq 3/(t+1)$ then we have

$$-h\Big(\frac{4}{t+1} - \rho\Big) = t\Big(1 + \rho - \frac{4}{t+1}\Big)^{t-2}((t+1)\rho - 2) \geq t(1-\rho)^{t-2}((t+1)\rho - 2) = h(\rho),$$

since by hypothesis $2\rho \geq \frac{4}{t+1}$. As a consequence if $2/(t+1) < \beta \leq 3/(t+1)$ then

$$g\Big(\frac{4}{t+1} - \beta\Big) \geq g(\beta), \text{ since } g(\beta) - g\Big(\frac{4}{t+1} - \beta\Big) = \int_{\frac{2}{t+1}}^{\beta} h(\rho) + h\Big(\frac{4}{t+1} - \rho\Big)\, d\rho \leq 0.$$

Now to prove the result we note that since $g\big(\frac{4}{t+1} - \beta\big) \geq \phi = g(\beta)$ while $g(\alpha) = \phi$, and g is decreasing on the interval $\big(\frac{1}{t+1}, \frac{2}{t+1}\big)$ we must have $\alpha \geq \frac{4}{t+1} - \beta$, which implies the claim. □

Lemma 3.8. *If* $r-1 \geq 2/((t+1)\alpha(\phi_{key})-1)$*, that is,* $(t+1)\alpha(\phi_{key}) \geq (r+1)/(r-1)$*, and* $\phi \leq \phi_{key}$ *then* $\frac{dL}{d\phi} \leq 0$*.*

Proof. We have, with $L = L_{r-1,1}$,

$$\begin{aligned}\frac{dL}{d\phi} &= \frac{r-1}{h(\alpha)} + \frac{1}{h(\beta)} \\ &= \frac{(r-1)t(1-\beta)^{t-2}((t+1)\beta-2)+t(1-\alpha)^{t-2}((t+1)\alpha-2)}{h(\alpha)h(\beta)}.\end{aligned}$$

The first term in the numerator is positive and the second is negative. The denominator is negative. Thus $\frac{dL}{d\phi} \leq 0$ precisely if

$$(r-1)(1-\beta)^{t-2}((t+1)\beta-2) \geq (1-\alpha)^{t-2}(2-(t+1)\alpha),$$

i.e.,

$$(\frac{1-\alpha}{1-\beta})^{t-2} \cdot \frac{2-(t+1)\alpha}{(t+1)\beta-2} = \frac{(t+1)\beta-1}{(t+1)\alpha-1} \cdot \frac{1-\beta}{1-\alpha} \cdot \frac{2-(t+1)\alpha}{(t+1)\beta-2} \leq r-1,$$

where we used the fact that

$$\Big(\frac{1-\alpha}{1-\beta}\Big)^{t-1} = \frac{(t+1)\beta-1}{(t+1)\alpha-1},$$

which is a simple consequence of the fact that $g(\alpha) = g(\beta)$. Both of the ratios $\frac{1-\beta}{1-\alpha}$ and $\frac{2-(t+1)\alpha}{(t+1)\beta-2}$ are at most one; the first because $\beta \geq \alpha$, the second because of Lemma 3.7, so it is sufficient to prove that $\frac{(t+1)\beta-1}{(t+1)\alpha-1} \leq r-1$. This fraction is monotonically increasing in ϕ and we have, by hypothesis,

$$\frac{(t+1)\beta-1}{(t+1)\alpha-1} \leq \frac{(t+1)\beta(\phi_{\text{key}})-1}{(t+1)\alpha(\phi_{\text{key}})-1} = \frac{2}{(t+1)\alpha(\phi_{\text{key}})-1} \leq r-1.$$

□

We require separate arguements for different pairs (r,t). Call a pair (r,t)

- *type A* if $t \leq r$.
- *type B* if $r \geq 7$ and $t \geq r+1$.
- *type C* if $2 \leq r \leq 6$ and $t \geq 3r-1$.

Lemma 3.9. *If* (r,t) *is a type A pair, then no skew solution exists.*

Proof. Recall from (2) that $\alpha > 1/(t+1)$. By repeated application of Lemma 3.6 we have $a = (r-1)$ and thus

$$\beta = 1 - (r-1)\alpha > \frac{2}{t+1} \implies \alpha < \frac{t-1}{(r-1)(t+1)}.$$

We conclude

$$\alpha \in \left(\frac{1}{t+1}, \frac{t-1}{(r-1)(t+1)}\right).$$

Of course, if this interval is empty then we have no choices for α and, due to Lemma 3.6, no skew solution exists. Thus in order for a skew solution to exist we must have

$$\frac{t-1}{(r-1)(t+1)} > \frac{1}{t+1},$$

which implies $t > r$, so for $t \leq r$ no skew solution exists. □

Lemma 3.10. *The hypothesis of Lemma 3.8 holds, that is,*

$$(t+1)\alpha(\phi_{key}) \geq \frac{r+1}{r-1},$$

for all (r,t) that are type B.

Proof. It is sufficient to prove that, for type B pairs (r,t), we have

$$g\Big(\frac{r+1}{(r-1)(t-1)}\Big) \geq g\Big(\frac{3}{t+1}\Big).$$

Noting that both of these expressions are negative, this is equivalent to

$$\frac{2}{r-1}\Big(1 - \frac{r+1}{(r-1)(t+1)}\Big)^{t-1} \leq 2\Big(1 - \frac{3}{t+1}\Big)^{t-1},$$

i.e.,

$$\left(\frac{t - \frac{2}{r-1}}{t-2}\right)^{t-1} = \left(1 + \frac{1 + \frac{r-3}{r-1}}{t-2}\right)^{t-1} \leq r-1.$$

The left-hand side converges, as t tends to infinity, to $\exp(1 + (r-3)/(r-1))$. The smallest r for which $\exp(1 + (r-3)/(r-1)) \leq r-1$ is $r = 6$ so the lemma holds for all type B pairs. □

Corollary 3.11. *For any type B pair (r,t), there is no root of $L = L_{r-1,1} = 1$ with $\phi \leq \phi_{key}$ and $\frac{dL}{d\phi} > 0$.*

Lemma 3.12. *For type B pairs (r,t), there are at most two roots of $L_{r-1,1} = 1$.*

Proof. Suppose that there are at least three roots of $L_{r-1,1} = 1$, and let $0 > \phi_1 > \phi_2 > \phi_3$ be the three largest. We must have $\frac{dL}{d\phi} > 0$ at ϕ_1, so by Corollary 3.11, $\phi_1 > \phi_{\text{key}}$. As we observed after Lemma 3.5, for L to be concave requires $\phi \leq \phi_{\text{key}}$. Between ϕ_1 and ϕ_3, L must be concave at some point, so $\phi_3 < \phi_{\text{key}}$. Also, we must have $\frac{dL}{d\phi} > 0$ at ϕ_3 and this combination is ruled out by Corollary 3.11. □

Corollary 3.13. *For type B pairs (r, t), if $L = L_{r-1,1} = 1$ has multiple solutions, then the one at which F is maximized is the one with ϕ largest.*

Proof. By the previous Lemma, there cannot be three roots of $L_{r-1,1} = 1$. If there are two, say $0 > \phi_1 > \phi_2$, then by (3), we have

$$F(\phi_1) - F(\phi_2) = \phi L\Big|_{\phi_2}^{\phi_1} - \int_{\phi_2}^{\phi_1} L\, d\phi = (\phi_1 - \phi_2) - \int_{\phi_2}^{\phi_1} L\, d\phi > 0,$$

since $L < 1$ for $\phi \in (\phi_2, \phi_1)$. □

Lemma 3.14. *If (r, t) is a type C pair, there is exactly one solution to $L = L_{r-1,1} = 1$.*

Proof. We use a weaker condition than that of (2): if a skew solution exists, $\alpha \in [0, \frac{2}{t+1})$. We claim that for $r \geq 2$ and $t \geq 3r - 1$, F has exactly one critical point in this range. Consider the derivative

$$G(\alpha) = F'(\alpha) = (r-1)g(\alpha) - (r-1)g(1-(r-1)\alpha).$$

We start by proving that $G(0) > 0$ and $G(\frac{2}{t+1}) < 0$. We have

$$G(0) = (r-1)f'(0) - (r-1)f'(1-(r-1)\cdot 0) = (r-1)[f'(0) - f'(1)] > 0,$$

and

$$\begin{aligned} G\left(\frac{2}{t+1}\right) &= (r-1)\left[-\left(1-\frac{2}{t+1}\right)^{t-1} + \left(\frac{2(r-1)}{t+1}\right)^{t-1}(t-2(r-1))\right] \\ &\leq (r-1)\left[-\left(\frac{r-1}{r}+\frac{1}{3r}\right)^{t-1} + \left(\frac{2}{3}\cdot\frac{(r-1)}{r}\right)^{t-1}(t-2(r-1))\right] \\ &< (r-1)\left[-\left(\frac{r-1}{r}\right)^{t-1} + \left(\frac{2}{3}\cdot\frac{(r-1)}{r}\right)^{t-1} t\right] \\ &= \frac{3}{2}(r-1)\left(\frac{r-1}{r}\right)^{t-1}\left[t\cdot\left(\frac{2}{3}\right)^{t} - \frac{2}{3}\right], \end{aligned}$$

where we note that

$$t \geq 3r-1 \implies -\left(1-\frac{2}{t+1}\right)^{t-1} \leq -\left(\frac{r-1}{r}+\frac{1}{3r}\right)^{t-1}.$$

Each term but the last is positive. As $r \geq 2$ and $t \geq 3r - 1$, we may take $t \geq 5$ and thus the last term is negative. Therefore $G(\frac{2}{t+1}) < 0$ as claimed.

As G is continuous, by the Intermediate Value Theorem it has at least one root in $[0, \frac{2}{t+1})$. As $G = F'$, a root of G indicates a critical point of F. To prove F has at most one critical point, we start by show that G is concave up on $[0, \frac{2}{t+1})$. We have

$$G''(\alpha) = F^{(3)}(\alpha) = (r-1)f^{(3)}(\alpha) - (r-1)^3 f^{(3)}(1-(r-1)\alpha).$$

Now

$$(r-1)f^{(3)}(\alpha) = (r-1)t(t-1)(1-\alpha)^{t-3}(3-(t+1)\alpha) > (r-1)t(t-1)\left(\frac{t-1}{t+1}\right)^{t-3},$$

as $(1-\alpha) > (t-1)/(t+1)$ and $(t+1)\alpha < 2$ for $\alpha \in [0, \frac{2}{t+1})$. Also, note that

$$\begin{aligned}
(r-1)^3 f^{(3)}(1-(r-1)\alpha) &= (r-1)^3 t(t-1)((r-1)\alpha)^{t-3}(t-2-(r-1)(t+1)\alpha) \\
&< (r-1)^3 t(t-1)\left(\frac{2(r-1)}{t+1}\right)^{t-3}(t-2) \\
&\leq (r-1)\left(\frac{t-2}{3}\right)^2 t(t-1)\left(\frac{2(t-2)}{3(t+1)}\right)^{t-3}(t-2) \\
&< (r-1)t(t-1)\left(\frac{t-1}{t+1}\right)^{t-3}\left(\frac{2}{3}\right)^{t-3}\frac{(t-2)^3}{9},
\end{aligned}$$

as $t \geq 3r-1$ implies $r - 1 \leq \frac{t-2}{3}$. Thus

$$G''(\alpha) > (r-1)t(t-1)\left(\frac{t-1}{t+1}\right)^{t-3}\left[1 - \left(\frac{2}{3}\right)^{t-3}\frac{(t-2)^3}{9}\right].$$

The last term is positive for $t \geq 19$. One can check that in the remaining cases, when $5 \leq t \leq 18$ and $2 \leq r \leq \frac{t+1}{3}$, $G''(\alpha)$ is still positive.

Thus G is concave up on $[0, \frac{2}{t+1})$ and is positive at one endpoint but negative at the other, so it has at most one zero on that interval. We conclude F has at most one critical point on $[0, \frac{2}{t+1})$ and thus at most one skew solution of type $(\alpha, \alpha, \ldots, \alpha, \beta)$ exists. If that solution exists it trivially has $f'(\alpha)$ largest. □

Now we're ready to complete the proof of our main result.

Proof of Theorem 3.4. First we show that F is not maximized on the boundary of the domain. Suppose, without loss of generality, that $\rho_1 = 0$ and $\rho_r \neq 0$. Let $\rho_1' = \rho_r' = \frac{\rho_r}{2}$. Each term of the sum defining F, other than the first and last, remains unchanged. Originally, the first term was 0 and the last was $\rho_r(1-\rho_r)^t$. Now each term is $\frac{\rho_r}{2}(1-\frac{\rho_r}{2})^t$, giving a sum of $\rho_r(1-\frac{\rho_r}{2})^t > \rho_r(1-\rho_r)^t$. We conclude points on the boundary cannot be maximizers.

As the domain of F is closed and bounded and F is continuous, it must achieve its maximum and thus that maximum must occur at an interior point. By Corollary 3.3, such points only occur at points of the form $(\alpha, \alpha, \ldots, \alpha, \beta, \beta, \ldots, \beta)$ where $\alpha < 2/(t+1) < \beta$ and $g(\alpha) = \phi = g(\beta)$ or at points of the form $(1/r, 1/r, \ldots, 1/r)$.

If there are no critical points of the first type, and in particular if (r,t) is a type A pair, then the only interior critical point is the Turán solution. In this case, F must attain its maximum here.

Otherwise, there exists at least one skew critical point $(\alpha, \alpha, \ldots, \alpha, \beta, \beta, \ldots, \beta)$, say with a many αs and b many βs. Repeatedly applying Lemma 3.6, we see that the critical point at which F attains its maximum is either the Turán solution or the skew solution associated with $a = r-1$ and $b = 1$. For type C pairs, Lemma 3.14 guarantees there is only one such solution, and for type B pairs Corollary 3.13 assures it is the solution having ϕ largest. There are only finitely many pairs (r,t) with $r, t \geq 2$ that are not of type A, B, or C and in each case manual inspection shows no such pair has a critical point other than the Turán solution or skew solution with $a = r-1$ and $b = 1$ with ϕ largest. □

We know from [3] that the asymptotic solution to the $\mathrm{ex}_{S_t}(W, K_{r+1})$ problem is either the Turán solution or some skew solution. Theorem 3.4 specifies exactly which skew solution is the possible maximum. In the last theorem, we prove the existence of a sharp threshold $t^*(r)$ where the solution to the $\mathrm{ex}_{S_t}(W, K_{r+1})$ problem transitions from the Turán solution to the skew solution.

Theorem 3.15. *For any $r \geq 2$, there is $t^* = t^*(r)$ such that for any integer t, F is maximized by the Turán solution when $t < t^*$ and by a skew solution for $t \geq t^*$.*

Proof. Theorem 1.12, together with Theorem 3.4 establishes that for all $r \geq 2$ there are values of t for which a skew solution is optimal. Therefore for each r there is a smallest integer τ, $r < \tau \leq r + \sqrt{2r}$ for which a skew solution is optimal.

Let $f(\rho, t) = \rho(1-\rho)^t$ and, for fixed constant r, define

$$F(\rho, t) = (r-1)f(\rho, t) + f(1-(r-1)\rho, t).$$

Let $\alpha \in (\frac{1}{\tau+1}, \frac{1}{r})$ be such that $F(\alpha, \tau) > F(\frac{1}{r}, \tau)$. We will prove $F(\alpha, \tau+1) > F(\frac{1}{r}, \tau+1)$ and thus as $\alpha \in (\frac{1}{\tau+2}, \frac{1}{r}) \supseteq (\frac{1}{\tau+1}, \frac{1}{r})$,

$$\max_{\alpha' \in (\frac{1}{\tau+2}, \frac{1}{r})} F(\alpha', \tau+1) \geq F(\alpha, \tau+1) > F(\tfrac{1}{r}, \tau+1).$$

Note that $F(\frac{1}{r}, \tau) = (1 - \frac{1}{r})^\tau$. Thus we have

$$F(\alpha, \tau) \cdot (1 - \tfrac{1}{r}) > \left(1 - \frac{1}{r}\right)^{\tau+1} = F(\tfrac{1}{r}, \tau+1).$$

Next note that

$$\begin{aligned}
F(\alpha,\tau)\cdot(1-\tfrac{1}{r}) &= \left(1-\frac{1}{r}\right)\left((r-1)\alpha(1-\alpha)^\tau + (1-(r-1)\alpha)((r-1)\alpha)^\tau\right)\\
&= \left(\left(1-\frac{1}{r}\right)-(1-\alpha)+(1-\alpha)\right)(r-1)\alpha(1-\alpha)^\tau\\
&\qquad + \left(\frac{r-1}{r}-(r-1)\alpha+(r-1)\alpha\right)(1-(r-1)\alpha)((r-1)\alpha)^\tau\\
&= \left(\alpha-\frac{1}{r}\right)(r-1)\alpha(1-\alpha)^\tau + (r-1)\alpha(1-\alpha)^{\tau+1}\\
&\qquad + (r-1)\left(\frac{1}{r}-\alpha\right)(1-(r-1)\alpha)((r-1)\alpha)^\tau\\
&\qquad + (1-(r-1)\alpha)((r-1)\alpha)^{\tau+1}\\
&= F(\alpha,\tau+1)-(r-1)\left(\frac{1}{r}-\alpha\right)(f(\alpha,\tau)-f(1-(r-1)\alpha,\tau)).
\end{aligned}$$

As $\alpha < \frac{1}{r}$, the sign of this second term depends entirely on $f(\alpha,\tau)-f(1-(r-1)\alpha,\tau)$. Recall that

$$\frac{\partial}{\partial\rho}f(\rho,t) = (1-\rho)^{t-1}(1-(t+1)\rho)$$

and note that for fixed t, $\frac{\partial f}{\partial \rho} < 0$ for ρ such that $\frac{1}{t+1} < \rho < 1$. Thus f is decreasing on this range, and as

$$\frac{1}{\tau+1} < \alpha < 1-(r-1)\alpha < 1$$

because $\alpha < \frac{1}{r}$, we have $f(\alpha,\tau) > f(1-(r-1)\alpha,\tau)$. We conclude that

$$F(\alpha,\tau+1) > F(\alpha,\tau+1)-(r-1)\left(\frac{1}{r}-\alpha\right)(f(\alpha,\tau)-f(1-(r-1)\alpha,\tau)) > F(\tfrac{1}{r},\tau+1)$$

as claimed.

Finally, by induction we may apply the same reasoning to any $t > \tau$ to see that if α is the parameter of a skew solution maximizing $F(\alpha,t)$, then $F(\alpha,t+1) > F(\frac{1}{r},t+1)$ and so the pair $(r,t+1)$ also has a skew solution. Defining $t^*(r)$ to be the τ corresponding to r completes the proof. □

Remark. Though Bollobás and Nikiforov did not prove the existence of a sharp threshold, the best known bounds for t^* come from Theorem 1.12: $r < t^* \leq r+\sqrt{2r}$. By looking at the proofs of their theorem and Lemma 3.9, a nearly complete picture of the optimal graphon for fixed r and varying t emerges. First, while $t \leq r$, no skew solutions exist and thus the Turán solution is optimal. Then skew solutions emerge but do not immediately beat the Turán solution. When t reaches t^*, the optimal skew solution overtakes the Turán solution and will continue to outperform it indefinitely. While obtaining a precise estimate of t^* remains difficult, by $t = r + \lceil\sqrt{2r}\rceil$ there is a skew solution that is not optimal (but whose stars are easy to count) that outperforms the Turán solution. We believe, based on numeric evidence, that the true value of t^* is closer to the upper bound, but improving either bound remains an open question.

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