

Point partition numbers: perfect graphs

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Abstract

Graphs considered in this paper are finite, undirected and without loops, but with multiple edges. For an integer $t \geq 1$, denote by \mathcal{MG}_t the class of graphs whose maximum multiplicity is at most t . A graph G is called strictly t -degenerate if every non-empty subgraph H of G contains a vertex v whose degree in H is at most $t - 1$. The point partition number $\chi_t(G)$ of G is smallest number of colors needed to color the vertices of G so that each vertex receives a color and vertices with the same color induce a strictly t -degenerate subgraph of G . So χ_1 is the chromatic number, and χ_2 is known as the point arboricity. The point partition number χ_t with $t \geq 1$ was introduced by Lick and White. If H is a simple graph, then tH denotes the graph obtained from H by replacing each edge of H by t parallel edges. Then $\omega_t(G)$ is the largest integer n such that G contains a tK_n as a subgraph. Let G be a graph belonging to \mathcal{MG}_t . Then $\omega_t(G) \leq \chi_t(G)$ and we say that G is χ_t -perfect if every induced subgraph H of G satisfies $\omega_t(H) = \chi_t(H)$. Based on the Strong Perfect Graph Theorem due to Chudnowsky, Robertson, Seymour and Thomas, we give a characterization of χ_t -perfect graphs of \mathcal{MG}_t by a set of forbidden induced subgraphs. We also discuss some complexity problems for the class of χ_t -critical graphs.

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1 Introduction

In this paper we extend the theory of perfect graphs to graphs having multiple edges. For this purpose we replace the chromatic number χ by the point partition number (respectively t -chromatic number) χ_t introduced in the 1970s by Lick and White [15].

2 Notation for graphs

For integers k and ℓ , let $[k, \ell] = \{x \in \mathbb{Z} \mid k \leq x \leq \ell\}$, let \mathbb{N} be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. By a **graph** we mean a finite undirected graph with multiple edges, but without loops. For a graph G , let $V(G)$ and $E(G)$ denote the **vertex set** and the **edge set** of G , respectively. The number of vertices of G is called the **order** of G and is denoted by $|G|$. A graph G is called **empty** if $|G| = 0$, in this case we also write $G = \emptyset$. For a vertex $v \in V(G)$ let $E_G(v)$ denote the set of edges of G **incident** with v . Then $d_G(v) = |E_G(v)|$ is the **degree** of v in G . As usual, $\delta(G) = \min_{v \in V(G)} d_G(v)$ is the **minimum degree** and $\Delta(G) = \max_{v \in V(G)} d_G(v)$ is the **maximum degree** of G . For different vertices u, v of G , let $E_G(u, v) = E_G(u) \cap E_G(v)$ be the set of edges incident with u and v . If $e \in E_G(u, v)$ then we also say that e is an edge of G **joining** u and v . Furthermore, $\mu_G(u, v) = |E_G(u, v)|$ is the **multiplicity** of the vertex pair u, v ; and $\mu(G) = \max_{u \neq v} \mu_G(u, v)$ is the **maximum multiplicity** of G . The graph G is said to be **simple** if $\mu(G) \leq 1$. For $X, Y \subseteq V(G)$, denote by $E_G(X, Y)$ the set of all edges of G joining a vertex of X with a vertex of Y . If G' is a **subgraph** of G , we write $G' \subseteq G$. The **subgraph induced by** the vertex set X with $X \subseteq V(G)$ is denoted by $G[X]$, i.e., $V(G[X]) = X$ and $E(G[X]) = E_G(X, X)$. Furthermore, $G - X = G[V(G) \setminus X]$. For a vertex v , let $G - v = G - \{v\}$. For $F \subseteq E(G)$, let $G - F$ denote the subgraph of G with vertex set $V(G)$ and edge set $E(G) \setminus F$. For an edge $e \in E(G)$, let $G - e = G - \{e\}$. We denote by K_n the **complete graph** of order n with $n \geq 0$, and by C_n the **cycle** of order n with $n \geq 2$. A cycle is called **odd** or **even** depending on whether its order is odd or even. The order of a cycle is also called its **length**.

3 Point partition numbers

In what follows let $t \in \mathbb{N}$. Denote by \mathcal{MG}_t the class of graphs G with $\mu(G) \leq t$. So \mathcal{MG}_1 is the class of simple graphs. If G is a graph, then $H = tG$ denotes the graph obtained from G by replacing each edge of G by t parallel edges, that is, $V(H) = V(G)$ and for any two different vertices $u, v \in V(G)$ we have $\mu_H(u, v) = t\mu_G(u, v)$. The graph $H = tG$ is called a **t -uniform inflation** of G .

Let G be an arbitrary graph. We call G **strictly t -degenerate** if every non-empty subgraph H of G has a vertex v such that $d_H(v) \leq t - 1$. Let \mathcal{SD}_t denote the class of strictly t -degenerate graphs. Clearly, \mathcal{SD}_1 is the class of edgeless graphs, and \mathcal{SD}_2 is the class of forests.

A **coloring** of G with **color set** Γ is a mapping $\varphi : V(G) \rightarrow \Gamma$ that assigns to each vertex $v \in V(G)$ a **color** $\varphi(v) \in \Gamma$. For a color $c \in \Gamma$, the preimage $\varphi^{-1}(c) = \{v \in V(G) \mid \varphi(v) = c\}$ is called a **color class** of G with respect to φ . A subgraph H of G is called **monochromatic** with respect to φ if $V(H)$ is a subset of a color class of G with respect to φ .

A coloring φ of G with color set Γ is called an **\mathcal{SD}_t -coloring** of G if for each

color $c \in \Gamma$ the subgraph of G induced by the color class $\varphi^{-1}(c)$ belongs to \mathcal{SD}_t . We denote by $\mathcal{CO}_t(G, k)$ the set of \mathcal{SD}_t -colorings of G with color set $\Gamma = [1, k]$. The **point partition number** $\chi_t(G)$ of the graph G is defined as the least integer k such that $\mathcal{CO}_t(G, k) \neq \emptyset$. Note that χ_1 equals the **chromatic number** χ , and χ_2 is known as the **point aboricity**.

The graph classes \mathcal{SD}_t and the corresponding coloring parameters χ_t with $t \geq 1$ were introduced in 1970 by Lick and White [15]. Bollobás and Manvel [6] used the term **t -chromatic number** for the parameter χ_t . The point aboricity χ_2 was already introduced in 1968 by Hedetniemi [14].

Clearly, an \mathcal{SD}_t -coloring of a graph G induces an \mathcal{SD}_t -coloring with the same color set for each of its subgraphs, and so

$$H \subseteq G \text{ implies } \chi_t(H) \leq \chi_t(G). \quad (3.1)$$

Furthermore, it is easy to check that if we delete a vertex or an edge from a graph, then its t -chromatic number decreases by at most one. Clearly, $sK_2 \notin \mathcal{SD}_t$ if and only if $s \geq t$. Consequently, if a graph G has a vertex pair (u, v) such that $\mu_G(u, v) \geq t + 1$ and $e \in E_G(u, v)$, then $\chi_t(G - e) = \chi_t(G)$. So it suffices to investigate the t -chromatic number for graphs belonging to \mathcal{MG}_t . Furthermore, if G is a simple graph, then $tG \in \mathcal{MG}_t$ and a coloring φ of G with color set $\Gamma = [1, k]$ satisfies

$$\varphi \in \mathcal{CO}_1(G, k) \text{ if and only if } \varphi \in \mathcal{CO}_t(tG, k), \quad (3.2)$$

which implies that

$$\chi(G) = \chi_t(tG). \quad (3.3)$$

Let G be a graph. A vertex set $X \subseteq V(G)$ is called an \mathcal{SD}_t -set of G if $G[X]$ belongs to \mathcal{SD}_t . In particular, an \mathcal{SD}_1 -set is also called an **independent set** of G . Let $\alpha_t(G)$ be the maximum cardinality of an \mathcal{SD}_t -set of G . Note that $\chi_t(G)$ is the least integer k such that $V(G)$ has a partition into k sets each of which is an \mathcal{SD}_t -set of G . Consequently, G satisfies

$$|G| \leq \chi_t(G)\alpha_t(G). \quad (3.4)$$

We call $X \subseteq V(G)$ a **t -fold clique** of G if $\mu_G(u, v) \geq t$ for any pair (u, v) of distinct vertices of X . A 1-fold clique of G is also called a **clique** of G . Let $\omega_t(G)$ the maximum cardinality of a t -fold clique of G . If $G \in \mathcal{MG}_t$, then a vertex set $X \subseteq V(G)$ of cardinality n is a t -fold clique of G if and only if $G[X]$ is a tK_n . Furthermore, (3.1) and (3.3) implies that

$$\omega_t(G) \leq \chi_t(G). \quad (3.5)$$

A graph $G \in \mathcal{MG}_t$ is called χ_t -**perfect** if every induced subgraph H of G satisfies $\chi_t(H) = \omega_t(H)$.

If a graph G is χ_t -perfect, then not only G , but all its induced subgraphs fulfill a min-max equation. This is one of the reasons why χ_t -perfect graphs are interesting objects for graph theorists. The study of χ_1 -perfect graphs has attracted a

lot of attention over the last six decades and was mainly motivated by a conjecture proposed in the 1960s by Berge [5]. However, it took more than forty years until Chudnovsky, Robertson, Seymour, and Thomas [9] succeeded in proving this conjecture; the result is now commonly known as the Strong Perfect Graph Theorem (SPGT). The SPGT provides a characterization of χ_1 -perfect graphs by forbidden induced subgraphs.

Strong Perfect Graph Theorem. *A simple graph is χ_1 -perfect if and only if it contains no odd cycle of length at least five, or its complement, as an induced subgraph.*

4 Characterizing χ_t -perfect graphs

We need some more notation. Let G be a graph belonging to \mathcal{MG}_t . We call a graph H the t -**complement** of G , written $H = \overline{G}^t$, if $V(H) = V(G)$, $E(G) \cap E(H) = \emptyset$, and $\mu_H(u, v) + \mu_G(u, v) = t$ for every pair (u, v) of distinct vertices of G . In particular, for $t = 1$, the 1-complement is the ordinary **complement** \overline{G} of the simple graph G . Clearly, $H = \overline{G}^t$ if and only if $G = \overline{H}^t$, and H is an induced subgraph of G if and only if \overline{H}^t is an induced subgraph of \overline{G}^t . If G has order n , then $G \cup \overline{G}^t = tK_n$. Furthermore, let $S_t(G)$ denote the simple graph with vertex set $V(S_t(G)) = V(G)$ and edge set $E(S_t(G)) = \{uv \mid \mu_G(u, v) = t\}$. Note that G satisfies

$$\omega_t(G) = \omega_1(S_t(G)). \quad (4.1)$$

Theorem 4.1. *For any graph $G \in \mathcal{MG}_2$ the following statements are equivalent:*

- (a) *The graph G is χ_2 -perfect.*
- (b) *The graph $S_2(G)$ is χ_1 -perfect and the graph G contains no cycle of length at least three as an induced subgraph.*
- (c) *The graph $S_2(G)$ contains no odd cycle of length at least five, or its complement, as an induced subgraph, and the graph G contains no cycle of length at least three as an induced subgraph.*
- (d) *G contains no induced subgraph H such that $S_2(H)$ is an odd cycle of length at least five, or its complement, or H is a cycle of length at least three.*

Proof. To show that (a) implies (b), suppose that G is χ_2 -perfect. If $S_2(G)$ is not χ_1 -perfect, then there exists an induced subgraph H of $S_2(G)$ such that $\omega_1(H) < \chi_1(H)$. For $X = V(H)$, we have $S_2(G[X]) = H$. Then it follows from (4.1) that

$$\omega_2(G[X]) = \omega_1(S_2(G[X])) = \omega_1(H) < \chi_1(H) = \chi_1(S_2(G[X])) \leq \chi_2(G[X]),$$

which implies that G is not χ_2 -perfect, a contradiction. If G contains a cycle C_n with $n \geq 3$ as an induced subgraph, then $\omega_2(C_n) = 1 < 2 = \chi_2(C_n)$, and so G is

not χ_2 -perfect, a contradiction, too. This shows that (a) implies (b). To show the converse implication, suppose that $S_2(G)$ is χ_1 -perfect, but G is not χ_2 -perfect. Our aim is to show that G contains a cycle C_n with $n \geq 3$ as an induced subgraph. Since G is not χ_2 -perfect, there is an induced subgraph H of G such that $\omega_2(H) < \chi_2(H)$. Let $k = \omega_2(H) = \omega_1(S_2(H))$ (see (4.1)). Clearly, $S_2(H)$ is an induced subgraph of $S_2(G)$, and so $S_2(H)$ is χ_1 -perfect as $S_2(G)$ is χ_1 -perfect. Hence there is a coloring $\varphi \in \mathcal{CO}_1(S_2(H), k)$. Then (3.2) implies that $\varphi \in \mathcal{CO}_2(2S_2(H), k)$. As $k < \chi_2(H)$, $\varphi \notin \mathcal{CO}_2(H, k)$ and so there is a color $c \in [1, k]$ such that $H[\varphi^{-1}(c)] \notin \mathcal{SD}_2$. As $\varphi \in \mathcal{CO}_1(S_2(H), k)$, $H[\varphi^{-1}(c)]$ contains a cycle, but no C_2 . Consequently, $H[\varphi^{-1}(c)]$ contains an induced cycle of length at least three, which is also an induced cycle of G . This completes the proof that (b) implies (a). The equivalence of (b) and (c) follows from the SPGT, the equivalence of (c) and (d) is evident. \square

Statement (d) of the above theorem provides a characterization of the class of χ_t -perfect graphs by an infinite family of forbidden induced subgraphs. As an immediate consequence of the proof of Theorem 4.1 we obtain the following two corollaries.

Corollary 4.2. *Let $G \in \mathcal{MG}_2$ be a χ_2 -perfect graph. Then every \mathcal{SD}_1 -coloring of $S_2(G)$ is a \mathcal{SD}_2 -coloring of G with the same color set.*

Corollary 4.3. *Let $G \in \mathcal{MG}_2$ be graph. Then G is χ_2 -perfect, or there are three distinct vertices u, v and w of G such that $\mu_G(u, v) \leq 1$, $\mu_G(v, w) \leq 1$, and $\mu_G(u, w) \geq 1$.*

Corollary 4.2 implies that there is a polynomial time algorithm that computes for a given χ_2 -perfect graph G an optimal \mathcal{SD}_2 -coloring of G . Clearly, we can compute the simple graph $G' = S_2(G)$ in polynomial time and Theorem 4.1 implies that G' is χ_1 -perfect. Then it follows from a result by Grötschel, Lovász, and Schrijver [11] that an optimal \mathcal{SD}_1 -coloring φ of G' can be computed in polynomial time. Then φ is an optimal \mathcal{SD}_2 -coloring of G (by Corollary 4.2 and (4.1)).

Corollary 4.3 is interesting as we wanted to use it to prove a Hajós-type result for the point aboricity χ_2 . In 1961 Hajós [13] proved that, for any fixed integer $k \geq 3$, a simple graph G has chromatic number at least k if and only if G contains a **k -constructible** subgraph, that is, a graph that can be obtained from disjoint copies of K_k by repeated application of the Hajós join and the identification of non-adjacent vertices. Recall that the **Hajós join** of two disjoint graphs G_1 and G_2 with edges $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$ is the graph G obtained from the union $G_1 \cup G_2$ by deleting both edges u_1v_1 and u_2v_2 , identifying v_1 with v_2 , and adding the new edge u_1u_2 ; we then write $G = G_1 \nabla G_2$. The "if" implication of Hajós' theorem follows from the facts that $\chi_1(K_k) = k$, $\chi_1(G_1 \nabla G_2) \geq \max\{\chi_1(G_1), \chi_2(G_2)\}$ (provided that $E(G_i) \neq \emptyset$), and $\chi_1(G/I) \geq \chi_1(G)$, where G/I denotes the (simple) graph obtained from G by identifying an independent set I of G to a single vertex. The proof of the "only if" implication is by reductio ad absurdum. So we consider a simple graph G with $\chi_1(G) \geq k$ and without a k -constructible subgraph. The

graph G may be assumed to be maximal in the sense that the addition of any edge $e \in E(\overline{G})$ to G gives rise to a k -constructible subgraph G_e of $G + e$ with $e \in E(G_e)$. If non-adjacency is an equivalence relation on $V(G)$, then the number of equivalence classes is at least k (as $\chi_1(G) \geq k$), which implies that G contains a K_k , a contradiction. Therefore, there are three vertices u, v and w such that $uv, vw \in E(\overline{G})$ and $uw \in E(G)$. Then there are two k -constructible graphs G_{uv} and G_{vw} . Now let $G' = (G_{uv} - uv) \cup (G_{vw} - vw) + uw$. Then G' is a subgraph of G which can be obtained from disjoint copies of G_{uv} and G_{vw} by removing the copies of the edges uv and vw , identifying the two copies of v and adding the copy of the edge uw . Then, for each vertex x belonging to both G_{uv} and G_{vw} we identify the two copies of x , thereby obtaining the k -constructible subgraph G' of G , a contradiction.

That the Hajós join well behaves with respect to the point aboricity χ_2 was proved by the authors in [18]; the Hajós join not only preserves the point aboricity, but also criticality. So we were hopeful to establish a counterpart of Hajós' theorem for the point aboricity, with $2K_k$ as the basic graphs. For the proof of the "only if" implication we could use Corollary 4.3. However, we were not able to control the identification operation for graphs in \mathcal{MG}_2 to handle the "if" implication. So we did not succeed in finding a constructive characterization for the class of graphs $G \in \mathcal{MG}_2$ with $\chi_2(G) \geq k$.

The proof of Theorem 4.1 can easily be extended to obtain a characterization for the class of χ_t -perfect graphs with $t \geq 3$ by a family of forbidden induced subgraphs. For $t \in \mathbb{N}$ with $t \geq 2$, let \mathcal{GD}_t denote the class of connected graphs $G \in \mathcal{MG}_{t-1}$ with $\delta(G) \geq t$. Note that $\mathcal{GD}_t \cap \mathcal{SD}_t = \emptyset$.

Theorem 4.4. *Let $t \in \mathbb{N}$ with $t \geq 2$. For any graph $G \in \mathcal{MG}_t$ the following statements are equivalent:*

- (a) *The graph G is χ_t -perfect.*
- (b) *The graph $S_t(G)$ is χ_1 -perfect and no induced subgraph of G belongs to \mathcal{GD}_t .*
- (c) *G contains no induced subgraph H such that $S_2(H)$ is an odd cycle of length at least five, or its complement, or $H \in \mathcal{GD}_t$.*

Proof. To show that (a) implies (b), suppose that G is χ_t -perfect. If $S_t(G)$ is not χ_1 -perfect, then there exists an induced subgraph H of $S_t(G)$ such that $\omega_1(H) < \chi_1(H)$. For $X = V(H)$, we have $S_t(G[X]) = H$. Then it follows from (4.1) that

$$\omega_t(G[X]) = \omega_1(S_t(G[X])) = \omega_1(H) < \chi_1(H) = \chi_1(S_t(G[X])) \leq \chi_t(G[X]),$$

which implies that G is not χ_t -perfect, a contradiction. If G has an induced subgraph $H \in \mathcal{GD}_t$, then $\omega_t(H) = 1$ (as $H \in \mathcal{MG}_{t-1}$) and $\chi_t(H) \geq 2$ (as $H \notin \mathcal{SD}_t$), which implies that G is not χ_2 -perfect, a contradiction, too. This shows that (a) implies (b). To show the converse implication, suppose that $S_t(G)$ is χ_1 -perfect, but G is not χ_t -perfect. Our aim is to show that G contains a graph $H \in \mathcal{GD}_t$ as an

induced subgraph. Since G is not χ_t -perfect, there is an induced subgraph G' of G such that $\omega_t(G') < \chi_t(G')$. Let $k = \omega_t(G') = \omega_1(S_t(G'))$ (see (4.1)). Clearly, $S_t(G')$ is an induced subgraph of $S_t(G)$, and so $S_t(G')$ is χ_1 -perfect as $S_t(G)$ is χ_1 -perfect. Hence there is a coloring $\varphi \in \mathcal{CO}_1(S_t(G'), k)$. Then (3.2) implies that $\varphi \in \mathcal{CO}_t(tS_t(G'), k)$. As $k < \chi_t(G')$, $\varphi \notin \mathcal{CO}_t(G', k)$ and so there is a color $c \in [1, k]$ such that $G'[\varphi^{-1}(c)] \notin \mathcal{SD}_t$. As $\varphi \in \mathcal{CO}_1(S_t(G'), k)$, $G'[\varphi^{-1}(c)]$ contains no tK_2 and so $G'[\varphi^{-1}(c)] \in \mathcal{MG}_{t-1}$. Then we conclude that $G'[\varphi^{-1}(c)]$ contains an induced subgraph H such that H is connected and $\delta(H) \geq t$. Then H is an induced subgraph of G belonging to \mathcal{GD}_t , as required. This completes the proof that (b) implies (a). The equivalence of (b) and (c) follows from the SPGT. \square

Corollary 4.5. *Let $G \in \mathcal{MG}_t$ be a χ_t -perfect graph with $t \geq 2$. Then every \mathcal{SD}_1 -coloring of $S_t(G)$ is a \mathcal{SD}_t -coloring of G with the same color set.*

Corollary 4.6. *Let $t \geq 2$. Then there exists a polynomial time algorithm that computes for any χ_t -perfect graph $G \in \mathcal{MG}_t$ an optimal \mathcal{SD}_t -coloring.*

It is well known that many decision problems that are NP-complete for the class \mathcal{MG}_1 are polynomial solvable for the class of χ_1 -perfect graphs, e.g. the clique problem, the independent set problem, the coloring problem, and the clique covering problem (see [12, Corollaries 9.3.32, 9.3.33, 9.4.8, Theorem 9.4.3]).

Let $G \in \mathcal{MG}_t$ be a χ_t -perfect graph with $t \geq 2$. Then $G' = S_t(G)$ belongs to \mathcal{MG}_1 , and G' is a χ_1 -perfect graph and no induced subgraph of G belongs to \mathcal{GD}_t (by Theorem 4.4(b)). Now let $I \subseteq V(G)$. We claim that $G[I] \in \mathcal{SD}_t$ if and only if $G'[I] \in \mathcal{SD}_1$ (i.e., I is an independent set of G'). If $G'[I] \in \mathcal{SD}_1$, then $G[I]$ does not contain a tK_2 as a subgraph, and so I is an independent set of $G' = S_t(G)$. Now assume that I is an independent set of G' . Then $G[I] \in \mathcal{MG}_{t-1}$ and, since no induced subgraph of G belongs to \mathcal{GD}_t , it follows that $G[I] \in \mathcal{SD}_t$. By a result due to Grötschel, Lovász, and Schrijver [11] it is possible to find an independent set I of G with $|I| = \alpha_1(G')$ in polynomial time. Hence, we have the following result.

Corollary 4.7. *Let $t \geq 2$. Then there exists a polynomial time algorithm that computes for any χ_t -perfect graph $G \in \mathcal{MG}_t$ an induced subgraph H of G such that $H \in \mathcal{SD}_t$ and $|H| = \alpha_t(G)$.*

5 A weak perfect graph theorem

In 1972 Lovász [16, 17] proved the following result, which was proposed by A. Hajnal.

Theorem 5.1 (Lovász 1972). *A simple graph G is χ_1 -perfect if and only if $|H| \leq \omega_1(H)\alpha_1(H)$ for every induced subgraph H of G .*

On the one hand, this result is an immediate consequence of the SPGT. On the other hand, Lovász gave a proof avoiding the use of the SPGT, in fact he proved it before the SPGT was established. In 1996 Gasparian [10] applied an argument

from linear algebra in order to give a very short proof of Lovász' result. We shall use Theorem 4.4 to extend Lovász' theorem to χ_t -perfect graphs.

Theorem 5.2. *Let $t \geq 2$. A graph $G \in \mathcal{MG}_t$ is χ_t -perfect if and only if $|H| \leq \omega_t(H)\alpha_t(H)$ for every induced subgraph H of G .*

Proof. First assume that G is χ_t -perfect. If H is an induced subgraph of G , then H is χ_t -perfect, too. Based on (3.4), we obtain that $|H| \leq \chi_t(H)\alpha_t(H) = \omega_t(H)\alpha_t(H)$. Thus the forward implication is proved.

To prove the backward implication assume that G is not χ_t -perfect. It then suffices to show that G has an induced subgraph H such that $|H| > \omega_t(H)\alpha_t(H)$. By Theorem 4.4, G has an induced subgraph H such that $S_t(H)$ is an odd cycle of length at least five, or its complement, or $H \in \mathcal{GD}_t$. If $S_t(H)$ is an odd cycle of length $\ell \geq 5$, then $\omega_t(H) = 2$ and $\alpha_t(H) \leq (\ell - 1)/2$, which leads to $|H| = \ell > \omega_t(H)\alpha_t(H)$. If $S_t(H)$ is the complement of an odd cycle of length $\ell \geq 5$, then $\alpha_t(H) = 2$ and $\omega_t(H) \leq (\ell - 1)/2$, which leads to $|H| = \ell > \omega_t(H)\alpha_t(H)$. If $H \in \mathcal{GD}_t$, then $\omega_t(H) = 1$ and $\alpha_t(H) \leq |H| - 1$, which leads to $|H| > \omega_t(H)\alpha_t(H)$. \square

Note that if G is a simple graph, then H is an induced subgraph of G if and only if \overline{H} is an induced subgraph of \overline{G} . Furthermore, if H is an induced subgraph of G , then $\alpha_1(\overline{H}) = \omega_1(H)$ and $\omega_1(\overline{H}) = \alpha_1(H)$. Consequently, Theorem 5.1 implies that a simple graph is χ_1 -perfect if and only if its complement is χ_1 -perfect. This immediate corollary of Lovász' theorem, nowadays known as the Weak Perfect Graph Theorem (WPGT), was also conjectured by Berge in the 1960s. The WPGT has no direct counterpart for χ_t -perfect graphs. However, we shall prove a result for χ_2 -perfect graphs that might be considered as a weak χ_2 -perfect graph theorem (see Theorem 5.4).

First we need some notation. Let G be an arbitrary graph. We denote by $S(G)$ the **underlying simple graph** of G , that is, $V(S(G)) = V(G)$ and $E(S(G)) = \{uv \mid \mu_G(u, v) \geq 1\}$. Note that a vertex set $X \subseteq V(G)$ is a clique of G if and only if $S(G)[X]$ is a complete graph. A subgraph C of G is called a **simple cycle** of G if C is a cycle and any pair (u, v) of vertices that are adjacent in C satisfies $\mu_G(u, v) = 1$. We say that G is a **normal graph** if G contains no simple cycle whose vertex set is a clique of G . If G is a normal graph and $H = S(G)$ is its underlying simple graph, then we also say that G is a **clique-acyclic inflation** of H .

Proposition 5.3. *Let $G \in \mathcal{MG}_2$ be a graph. Then G contains no induced cycle if and only if \overline{G}^2 is a normal graph.*

Proof. First assume that G contains an induced cycle C . Then \overline{C}^2 is a simple cycle of \overline{G}^2 whose vertex set is a clique of \overline{G}^2 , and so \overline{G}^2 is not normal. Now assume that \overline{G}^2 is not normal. Then \overline{G}^2 contains a simple cycle whose vertex set is a clique of \overline{G}^2 . Let C be such a cycle whose length is minimum. Then \overline{C}^2 is an induced cycle of G . \square

Theorem 5.4. *Let $G \in \mathcal{MG}_2$ be a graph. Then G is a χ_2 -perfect graph if and only if $\overline{G^2}$ is a clique-acyclic inflation of a perfect graph.*

Proof. Since $G \in \mathcal{MG}_2$, it follows that $S(\overline{G^2}) = \overline{S_2(G)}$. By combining Theorem 4.1 with Proposition 5.3, G is a χ_2 -perfect graph if and only if $S_2(G)$ is a χ_1 -perfect graph and $\overline{G^2}$ is a normal graph. By the WPGT, this implies that G is a χ_2 -perfect graph if and only if $\overline{S_2(G)}$ is a χ_1 -perfect graph and $\overline{G^2}$ is a normal graph, which is equivalent to $\overline{G^2}$ is a clique-acyclic inflation of a perfect graph. \square

6 Recognizing χ_2 -perfect graphs

In 2005, Chudnovsky, Cornuéjols, Liu, Seymour, Vušković [8] proved that the decision problem whether a simple graph does not contain an odd cycle of length at least five, or its complement, as an induced subgraph belongs to the complexity class P. So the SPGT then implies that the recognition problem for perfect graphs is polynomial time solvable. To test whether a graph $G \in \mathcal{MG}_2$ is χ_2 -perfect, by Theorem 4.1 we have to test

1. whether $S_2(G)$ is χ_1 -perfect, and
2. whether G does not contain an induced cycle of length at least three.

While the first can indeed be tested efficiently (by the SPGT and [8]), the second is a co-NP-complete problem (see Theorem 6.1). In 2012 Bang-Jensen, Havet, and Trotignon [4, Theorem 11] proved that the decision problem whether a digraph contains an induced directed cycle of length at least three is NP-complete. Here a digraph may have antiparallel arcs, but no parallel arcs. We can use the same reduction as in [4, Theorem 11] (just by ignoring the directions) to prove the following result.

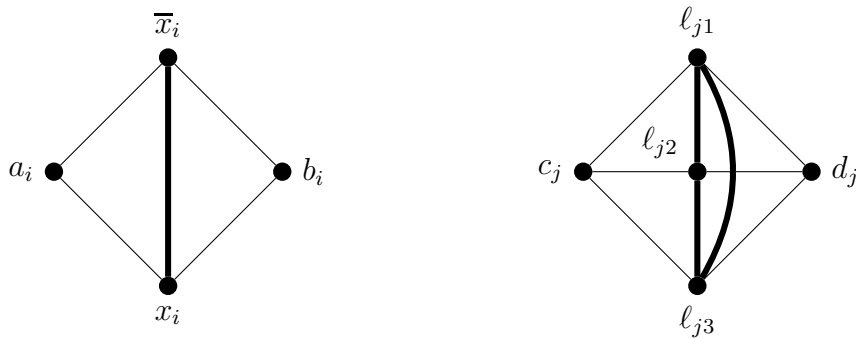


Figure 1: Variable gadget $VG(i)$ and clause gadget $CG(j)$. Bold edges represent parallel edges.

Theorem 6.1. *The decision problem whether a graph of \mathcal{MG}_2 contains an induced cycle of length at least three is NP-complete.*

Proof. The reduction is from 3-SAT. Let I be an instance of 3-SAT with variable x_1, x_2, \dots, x_n and clauses C_1, C_2, \dots, C_m , where $n, m \in \mathbb{N}$ and

$$C_j = (\ell_{j1} \vee \ell_{j2} \vee \ell_{j3}) \text{ with } \ell_{jk} \in \{x_1, x_2, \dots, x_n, \bar{x}_1, \bar{x}_2, \dots, \bar{x}_n\}.$$

For each variable x_i let $VG(i)$ be the variable gadget, and for each clause C_j let $CG(j)$ be the clause gadget, as shown in Figure 1. From the union of all the gadgets we form a graph $G(I)$ by adding edges according to the following two rules:

1. We add the single edges $b_i a_{i+1}$ (with $i \in [1, n-1]$), $b_n c_1$, $d_j c_{j+1}$ (with $j \in [1, m-1]$), and $d_m a_1$.
2. For each literal ℓ_{jk} (which is either x_i or \bar{x}_i) we add two parallel edges joining the vertex ℓ_{jk} of the clause gadget $CG(j)$ with the vertex $\overline{\ell_{jk}}$ in the variable gadget $VG(i)$.

Similar as in the proof of [4, Theorem 11] it is easy to show that $G(I)$ has an induced cycle of length at least three if and only if I is satisfiable. \square

Corollary 6.2. *The decision problem whether a graph from \mathcal{MG}_2 is χ_2 -perfect is co-NP-complete.*

Proof. We use the same reduction as in the proof of Theorem 6.1. So for an instance I of 3-SAT we construct the graph $G(I)$. If necessary, we subdivide the edge $d_m a_1$, so the the graph $S_2(G(I))$ is bipartite and hence χ_1 -perfect. Then Theorem 4.1 implies that $G(I)$ is χ_2 -perfect if and only if $G(I)$ has no induced cycle of length at least three. \square

7 Concluding remarks

The results about χ_2 -perfect graphs, in particular, Theorems 4.1 and 5.4, resemble the results about perfect digraphs (with respect to the directed chromatic number) obtained in 2015 by Andres and Hochstättler [2]. The characterization of perfect digraphs was used by Bang-Jensen, Bellitto, Schweser, and Stiebitz [3] to establish a Hajós-type result for the dichromatic number of digraphs. Andres [1] started to investigate game-perfect graphs, based on maker-breaker games. This concept can also be extended to the point partition number.

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