Join-Reachability Problems in Directed Graphs*

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November 11, 2018

Abstract

For a given collection $\mathcal{G}$ of directed graphs we define the *join-reachability graph* of $\mathcal{G}$, denoted by $\mathcal{J}(\mathcal{G})$, as the directed graph that, for any pair of vertices a and b, contains a path from a to b if and only if such a path exists in all graphs of $\mathcal{G}$. Our goal is to compute an efficient representation of $\mathcal{J}(\mathcal{G})$. In particular, we consider two versions of this problem. In the *explicit* version we wish to construct the smallest join-reachability graph for $\mathcal{G}$. In the *implicit* version we wish to build an efficient data structure (in terms of space and query time) such that we can report fast the set of vertices that reach a query vertex in all graphs of $\mathcal{G}$. This problem is related to the well-studied *reachability problem* and is motivated by emerging applications of graph-structured databases and graph algorithms. We consider the construction of join-reachability structures for two graphs and develop techniques that can be applied to both the explicit and the implicit problem. First we present optimal and near-optimal structures for paths and trees. Then, based on these results, we provide efficient structures for planar graphs and general directed graphs.

1 Introduction

In the *reachability problem* our goal is to preprocess a (directed or undirected) graph G into a data structure that can quickly answer queries that ask if a vertex b is reachable from a vertex a. This problem has numerous and diverse applications, including internet routing, geographical navigation, and knowledge-representation systems [20]. Recently, the interest in graph reachability problems has been rekindled by emerging applications of graph data structures in areas such as the semantic web, bio-informatics and social networks. These developments together with recent applications in graph algorithms [7, 8, 9] have motivated us to introduce the study of the *join-reachability problem* that we define as follows: We are given a collection $\mathcal{G}$ of λ directed graphs $G_i = (V_i, A_i)$, $1 \le i \le \lambda$, where each graph G_i represents a binary relation R_i over a set of elements

*This research project has been funded by the John S. Latsis Public Benefit Foundation. The sole responsibility for the content of this paper lies with its authors.

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$V \subseteq V_i$ in the following sense: For any $a, b \in V$, we have aR_ib if and only if b is reachable from a in G_i. Let $\mathcal{R} \equiv \mathcal{R}(\mathcal{G})$ be the binary relation over V defined by: $a\mathcal{R}b$ if and only if aR_ib for all $i \in \{1, \ldots, \lambda\}$ (i.e., b is reachable from a in all graphs in $\mathcal{G}$). We can view $\mathcal{R}$ as a type of JOIN operation on graph-structured databases. Our objective is to find an efficient representation of this relation. To the best of our knowledge, this problem has not been previously studied. We will restrict our attention to the case of two input graphs ($\lambda = 2$).

Contribution. In this paper we explore two versions of the join-reachability problem. In the *explicit* version we wish to represent $\mathcal{R}$ with a directed graph $\mathcal{J} \equiv \mathcal{J}(\mathcal{G})$, which we call the *join-reachability graph of* $\mathcal{G}$, i.e., for any $a, b \in V$, we have $a\mathcal{R}b$ if and only if b is reachable from a in $\mathcal{J}$. Our goal is to minimize the size (i.e., the number of vertices plus arcs) of $\mathcal{J}$. We consider this problem in Sections 2 and 3, and present results on the computational and combinatorial complexity of $\mathcal{J}$. In the *implicit* version we wish to represent $\mathcal{R}$ with an efficient data structure (in terms of space and query time) that can report fast all elements $a \in V$ satisfying $a\mathcal{R}b$ for any query element $b \in V$. We deal with the implicit problem in Section 4. First we describe efficient join-reachability structures for simple graph classes. Then, based on these results, we consider planar graphs and general directed graphs. Also, in Appendix B and Appendix C we consider join-reachability structures for planar st-graphs and lattices. Although we focus on the case of two directed graphs ($\lambda = 2$), we note that some of our results are easily extended for $\lambda \geq 3$ with the use of appropriate multidimensional geometric structures.

Applications. Instances of the join-reachability problem appear in various applications. For example, in the rank aggregation problem [5] we are given a collection of rankings of some elements and we may wish to report which (or how many) elements have the same ranking relative to a given element. This is a special version of join-reachability since the given collection of rankings can be represented by a collection of directed paths with the elements being the vertices of the paths. Similarly, in a graph-structured database with an associated ranking of its vertices we may wish to find the vertices that are related to a query vertex and have higher or lower ranking than this vertex. Instances of join-reachability also appear in graph algorithms arising from program optimization. Specifically, in [7] we need a data structure capable of reporting which vertices satisfy certain ancestor-descendant relations in a collection of rooted trees. Moreover, in [9] it is shown that any directed graph G with a distinguished source vertex s has two spanning trees rooted at s such that a vertex a is a dominator of a vertex b (meaning that all paths in G from s to b pass through a) if and only if a is an ancestor of b in both spanning trees. This generalizes the graph-theoretical concept of *independent spanning trees*. Two spanning trees of a graph G are independent if they are both rooted at the same vertex r and for each vertex v the paths from r to v in the two trees are internally vertex disjoint. Similarly, λ spanning trees of G are independent if they are pairwise independent. In this setting, we can apply a join-reachability structure to decide if λ given spanning trees are independent. Finally we note that a variant of the join-reachability problem we defined here appears in the context of a recent algorithm for computing two internally vertex-disjoint paths for any pair of query vertices in a 2-vertex connected directed graph [8].

Preliminaries and Related Work. The reachability problem is easy in the undirected case since it suffices to compute the connected components of the input graph. Similarly, the undirected version of the join-reachability problem is also easy, as given the connected components of two undirected graphs G_1 and G_2 with n vertices, we can compute the connected components of $\mathcal{J}(\{G_1, G_2\})$ in $O(n)$ time. On the other hand, no reachability data structure is currently known to simultaneously achieve $o(n^2)$ space and $o(n)$ query time for a general directed graph with n vertices [20]. Nevertheless, efficient reachability structures do exist for several important cases. First, asymptotically optimal structures exist for rooted trees [1] and planar directed graphs with one source and one sink [12, 17]. For general planar graphs Thorup [18] gives an $O(n \log n)$-space structure with constant query time. Talamo and Vocca [16] achieve constant query time for lattice partial orders with an $O(n\sqrt{n})$-space structure.

Notation. In the description of our results we use the following notation and terminology. We denote the vertex set and the arc set of a directed graph (digraph) G by $V(G)$ and $A(G)$, respectively. Without loss of generality we assume that $V(G) = V$ for all $G \in \mathcal{G}$. The size of G, denoted by $|G|$, is equal to the number of arcs plus vertices, i.e., $|G| = |V| + |E|$. We use the notation $a \rightsquigarrow_G b$ to denote that b is reachable from a in G. (By definition $a \rightsquigarrow_G a$ for any $a \in V$.) The *predecessors* of a vertex b are the vertices that reach b, and the *successors* of a vertex b are the vertices that are reached from b. Let P be a directed path (dipath); the *rank* of $a \in P$, $r_P(a)$, is equal to the number of predecessors of a in P minus one, and the *height* of $a \in P$, $h_P(a)$, is equal to the number of successors of a in P minus one. For a rooted tree T, we let $T(a)$ denote the subtree rooted at a and let $nca_T(a, b)$ denote the nearest common ancestor of a and b. We will deal with two special types of directed rooted trees: In an *in-tree*, each vertex has exactly one outgoing arc except for the root which has none; in an *out-tree*, each vertex has exactly one incoming arc except for the root which has none. We use the term *unoriented tree* for a directed tree with no restriction on the orientation of its arcs. Similarly, we use the term *unoriented dipath* to refer to a path in the undirected sense, where the arcs can have any orientation. In our constructions we map the vertices of V to objects in a d-dimensional space and use the notation $x_i(a)$ to refer to the ith coordinate that vertex a receives. Finally, for any two vectors $\xi = (\xi_1, \ldots, \xi_d)$ and $\zeta = (\zeta_1, \ldots, \zeta_d)$, the notation $\xi \leq \zeta$ means that $\xi_i \leq \zeta_i$ for $i = 1, \ldots, d$.

1.1 Preprocessing: Computing Layers and Removing Cycles

Thorup's Layer Decomposition. In [18] Thorup shows how to reduce the reachability problem for any digraph G to reachability in some digraphs with special properties, called *2-layered digraphs*. A *t-layered spanning tree* T of G is a rooted directed tree such that any path in T from the root (ignoring arc directions) is the concatenation of at most t dipaths in G. A digraph G is *t-layered* if it has such a spanning tree. Now we provide an overview of Thorup's reduction. The vertices of G are partitioned into layers $L_0, L_1, \ldots, L_{\mu-1}$ that define a sequence of digraphs $G^0, G^1, \ldots, G^{\mu-1}$ as follows. An arbitrary vertex $v_0 \in V(G)$ is chosen as a root. Then, layer L_0 contains v_0 and the vertices that are reachable from v_0. For odd i, layer L_i contains the vertices that reach the previous layers L_j, $j < i$. For even i, layer L_i contains the vertices that are reachable from the previous layers L_j, $j < i$. To form G^i for $i > 0$ we contract the vertices in layers L_j for $j \leq i - 1$ to a single root vertex r_0; for $i = 0$ we set $r_0 = v_0$. Then G^i is induced by L_i, L_{i+1} and r_0. It follows that each G^i is a 2-layered digraph. Let $\iota(v)$ denote the index of the layer containing v, that is, $\iota(v) = i$ if and only if $v \in L_i$. The key properties of the decomposition are: (i) all the predecessors of v in G are contained in $G^{\iota(v)-1}$ and $G^{\iota(v)}$, and (ii) $\sum_i |G^i| = O(|G|)$.

Removing Cycles. In the standard reachability problem, a useful preprocessing step that can reduce the size of the input digraph is to contract its strongly connected components (strong components) and consider the resulting acyclic graph. When we apply the same idea to join-reachability we have to deal with the complication that the strong components in the two digraphs may differ. Still, we can construct two acyclic digraphs $\hat{G}_1$ and $\hat{G}_2$ such that, for any $a, b \in V$, $a \leadsto_{\mathcal{J}(\{G_1,G_2\})} b$ if and only if $a \leadsto_{\mathcal{J}(\{\hat{G}_1,\hat{G}_2\})} b$, and $|\hat{G}_i| \le |G_i|$, $i = 1, 2$. This is accomplished as follows. First, we compute the strong components of G_1 and G_2 and order them topologically. Let G'_i, $i = 1, 2$, denote the digraph produced after contracting the strong components of G_i. (We remove loops and duplicate arcs so that each G'_i is a simple digraph.) Also, let C_i^j denote the jth strong component of G_i. We partition each component C_i^j into subcomponents such that two vertices are in the same subcomponent if and only if they are in the same strong component in both G_1 and G_2. The subcomponents are the vertices of $\hat{G}_1$ and $\hat{G}_2$. Next we describe how to add the appropriate arcs. The process is similar for the two digraphs so we consider only $\hat{G}_1$.

Let $C_1^{j,1}, C_1^{j,2}, \ldots, C_1^{j,l_j}$ be the subcomponents of C_1^j, which are ordered with respect to the topological order of G'_2. That is, if $x \in C_1^{j,i}$ and $y \in C_1^{j,i'}$, where $i < i'$, then in the topological order of G'_2 the component of x precedes the component of y. We connect the subcomponents by adding the arcs $(C_1^{j,i}, C_1^{j,i+1})$ for $1 \le i < l_j$. Moreover, for each arc (C_1^i, C_1^j) in $A(G'_1)$ we add the arc $(C_1^{i,l_i}, C_1^{j,1})$ to $A(\hat{G}_1)$, where C_1^{i,l_i} is the last subcomponent of C_1^i. See Figure 1. It is straightforward to verify that $a \leadsto_{\mathcal{J}} b$ if and only if a and b are in the same subcomponent or the subcomponent of a is a predecessor of the subcomponent of b in both $\hat{G}_1$ and $\hat{G}_2$.

2 Computational Complexity of Computing the Smallest $\mathcal{J}(\{G_1, G_2\})$

We explore the computational complexity of computing the smallest $\mathcal{J}(\{G_1, G_2\})$: Given two digraphs $G_1 = (V, A_1)$ and $G_2 = (V, A_2)$ we wish to compute a digraph $\mathcal{J} \equiv \mathcal{J}(\{G_1, G_2\})$ of minimum size such that for any $a, b \in V$, $a \leadsto_{\mathcal{J}} b$ if and only if $a \leadsto_{G_1} b$ and $a \leadsto_{G_2} b$. We consider two versions of this problem, depending on whether $\mathcal{J}$ is allowed to have Steiner vertices (i.e., vertices not in V) or not: In the *unrestricted* version $V(\mathcal{J}) \supseteq V$, while in the *restricted* version $V(\mathcal{J}) = V$. Computing $\mathcal{J}$ is NP-hard in the unrestricted case. This is implied by a straightforward reduction to the *reachability substitute problem*, which was shown to be NP-hard by Katriel et al. [13]. In this problem we are given a digraph H and a subset $U \subseteq V(H)$, and ask for the smallest digraph H^* such that for any $a, b \in U$, $a \leadsto_{H^*} b$ and only if $a \leadsto_H b$. For the reduction, we let $G_1 = H$ and let G_2 contain all the arcs connecting vertices in U only, that is, $A(G_2) = U \times U$. Clearly, for any $a, b \in U$ we have $a \leadsto_{\mathcal{J}} b$ if and only if $a \leadsto_H b$. Therefore computing the smallest join-reachability graph is equivalent to computing H^*. In the restricted case, on the other hand, we can compute $\mathcal{J}$ using transitive closure and transitive reduction computations, which can be

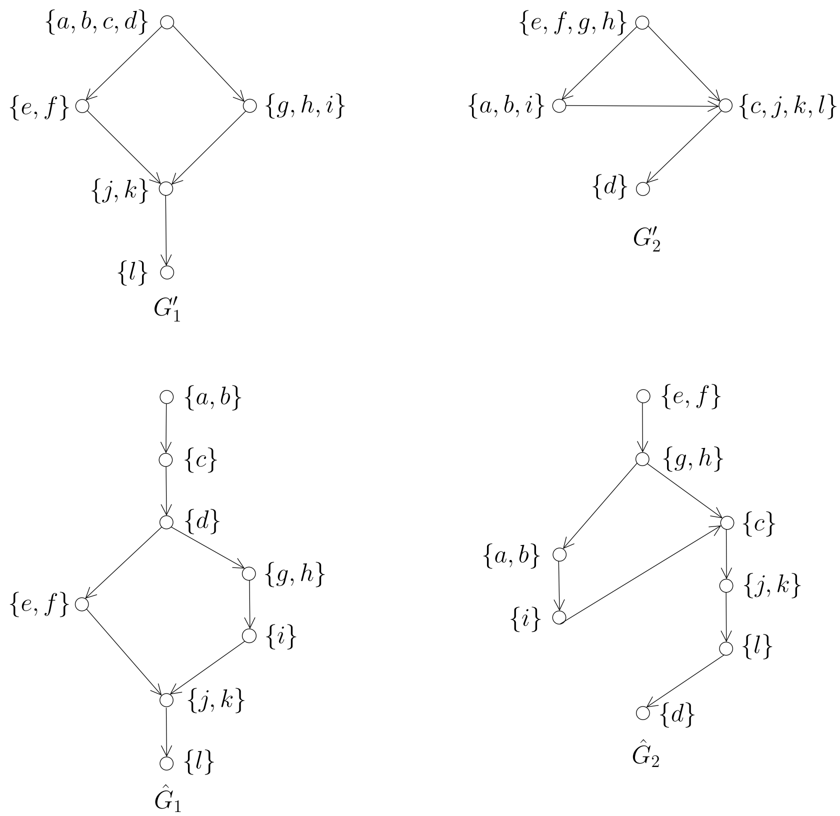


Figure 1: The contracted digraphs G'_1 and G'_2 and their corresponding acyclic digraphs $\hat{G}_1$ and $\hat{G}_2$.

done in polynomial time [2]. (This is done as follows: First we compute the transitive closure matrices M_1 and M_2 of G_1 and G_2 respectively. Then we form the transitive closure matrix M of $\mathcal{J}$ by taking the AND operation of corresponding entries in M_1 and M_2. Finally we compute the transitive reduction of the resulting transitive closure matrix M.) This implies the next theorem.

Theorem 2.1. *Let $\mathcal{J}$ be the smallest join-reachability graph of a collection of digraphs. The computation of $\mathcal{J}$ is feasible in polynomial time if Steiner vertices are not allowed, and NP-hard otherwise.*

The existence of Steiner vertices can reduce the size of $\mathcal{J}$ significantly. Consider for example a complete bipartite digraph G with $V(G) = X \cup Y$ and $A(G) = X \times Y$. This digraph has the same transitive closure as the digraph G' with $V(G') = V(G) \cup \{z\}$ and $A(G') = \{(x, z), (z, y) \mid x \in X, y \in Y\}$. In Section 3 we explore the combinatorial complexity of the unrestricted join-reachability graph and provide bounds for $|\mathcal{J}|$ in several cases.

3 Combinatorial Complexity of $\mathcal{J}(\{G_1, G_2\})$

In this section we provide bounds on the size of $\mathcal{J}(\{G_1, G_2\})$ in several cases. These results are summarized in the next theorem.

Theorem 3.1. *Given two digraphs G_1 and G_2 with n vertices, the following bounds on the size of the join-reachability graph $\mathcal{J}(\{G_1, G_2\})$ hold:*

(a) $\Theta(n \log n)$ in the worst case when G_1 is an unoriented tree and G_2 is an unoriented dipath.

(b) $O(n \log^2 n)$ when both G_1 and G_2 are unoriented trees.

(c) $O(n \log^2 n)$ when G_1 is a planar digraph and G_2 is an unoriented dipath.

(d) $O(n \log^3 n)$ when both G_1 and G_2 are planar digraphs.

(e) $O(\kappa_1 n \log n)$ when G_1 is a digraph that can be covered with κ_1 vertex-disjoint dipaths and G_2 is an unoriented dipath.

(f) $O(\kappa_1 n \log^2 n)$ when G_1 is a digraph that can be covered with κ_1 vertex-disjoint dipaths and G_2 is a planar graph.

(g) $O(\kappa_1 \kappa_2 n \log n)$ when each G_i, $i = 1, 2$, is a digraph that can be covered with κ_i vertex-disjoint dipaths.

In the following sections we prove Theorem 3.1. In each case we provide a construction of the corresponding join-reachability graph that achieves the claimed bound. In Section 4 we provide improved space bounds for the implicit representation of $\mathcal{J}(\{G_1, G_2\})$, i.e., data structures that answer join-reachability reporting queries fast. Still, a process that computes an explicit representation of $\mathcal{J}(\{G_1, G_2\})$ can be useful, as it provides a natural way to handle collections of more than two digraphs (i.e., it allows us to combine the digraphs one pair at a time).

3.1 Two Paths

We start with the simplest case where G_1 and G_2 are dipaths with n vertices. First we show that we can construct a join-reachability graph of size $O(n \log n)$. Given this result we can provide bounds for trees, planar digraphs, and general digraphs. Then we show this bound is tight, i.e., there are instances for which $\Omega(n \log n)$ size is needed. We begin by mapping the vertices of V to a two-dimensional rank space: Each vertex a receives coordinates $(x_1(a), x_2(a))$ where $x_1(a) = r_{G_1}(a)$ and $x_2(a) = r_{G_2}(a)$. Note that these ranks are integers in the range $[0, n-1]$. Now we can view these vertices as lying on an $n \times n$ grid, such that each row and each column of the grid contains exactly one vertex. Clearly, $a\mathcal{R}b$ if and only if $(x_1(a), x_2(a)) \leq (x_1(b), x_2(b))$.

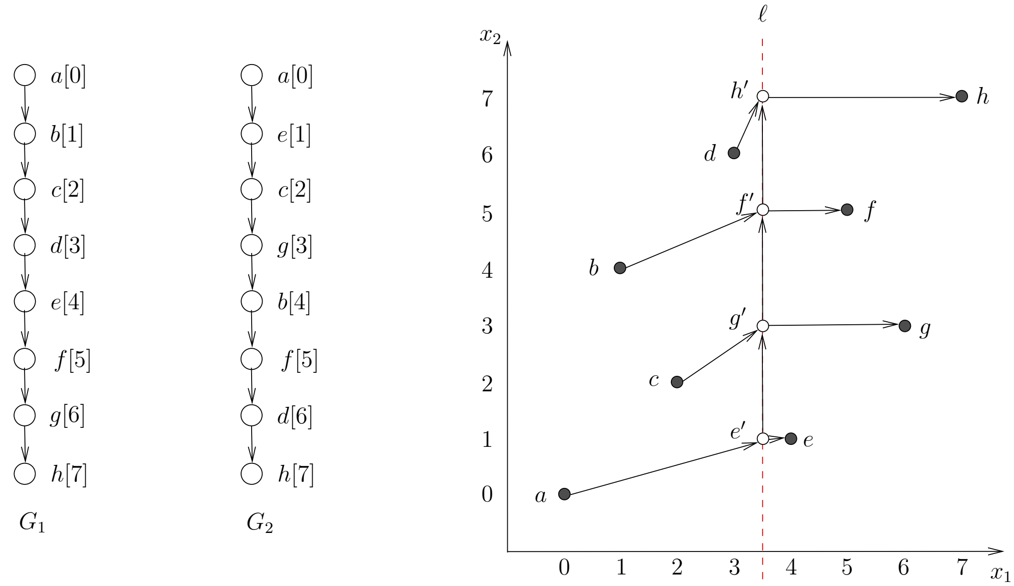


Figure 2: The mapping of the vertices of two dipaths to 2d rank space and the construction of $\mathcal{J}_\ell$; Steiner vertices in $\mathcal{J}_\ell$ are white.

Upper bound. We use a simple divide-and-conquer method. Let ℓ be the vertical line with x_1-coordinate equal to $n/2$. A vertex z is *to the right of* ℓ if $x_1(z) \geq n/2$ and *to the left of* ℓ otherwise. The first step is to construct a subgraph $\mathcal{J}_\ell$ of $\mathcal{J}$ that connects the vertices to the left of ℓ to the vertices to the right of ℓ. For each vertex b to the right of ℓ we create a Steiner vertex b' and add the arc (b', b). Also, we assign to b' the coordinates $(n/2, x_2(b))$. We connect these Steiner vertices in a dipath starting from the vertex with the lowest x_2-coordinate. Next, for each vertex a to the left of ℓ we locate the Steiner vertex b' with the smallest x_2-coordinate such that $x_2(a) \leq x_2(b')$. If b' exists we add the arc (a, b'). See Figure 2. Finally we recurse for the vertices to the left of ℓ and for the vertices to the right of ℓ. It is easy to see that $\mathcal{J}$ contains a path from a to b if and only if $(x_1(a), x_2(a)) \leq (x_1(b), x_2(b))$. To bound $|\mathcal{J}|$ note that we have $O(\log n)$ levels of recursion, and at each level the number of added Steiner vertices and arcs is $O(n)$. Hence, the $O(n \log n)$ bound for two dipaths follows.

The case of two unoriented dipaths G_1 and G_2 can be reduced to that of dipaths, yielding the same $O(n \log n)$ bound. This is accomplished by splitting G_1 and G_2 to maximal subpaths that consist of arcs with the same orientation. Then $\mathcal{J}$ is formed from the union of separate join-reachability graphs for each pair of subpaths of G_1 and G_2. The $O(n \log n)$ bound follows from the fact that each vertex appears in at most two subpaths of each unoriented dipath, so in at most four subgraphs. We remark that our construction can be generalized to handle more dipaths, with an $O(\log n)$ factor blowup per additional dipath.

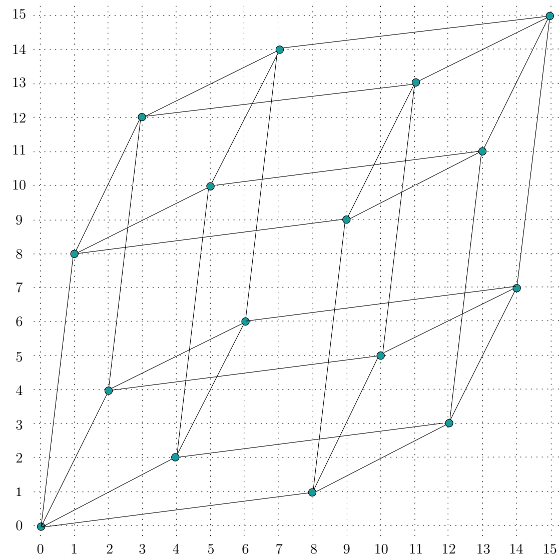


Figure 3: The digraph used in the lower bound proof of Section 2 for $n = 16$. The arcs are directed towards northeast. The x_2-coordinate of each vertex is produced by reversing the bits of its x_1-coordinate.

Lower bound. Let G_1 be any dipath, and let $x_1(a) = r_{G_1}(a)$. Also let $x_1^i(a)$ denote the ith bit in the binary representation of $x_1(a)$ and let $\beta = \lceil \log_2 n \rceil$ be the number of bits in this representation. We use similar notation for $x_2(a)$. We define G_2 such that the rank of a in G_2 is $x_2(a) = x_1(a)^R$, where $x_1(a)^R$ is the integer formed by the bit-reversal in the binary representation of $x_1(a)$, i.e., $x_2^i(a) = x_1^{\beta-1-i}(a)$ for $0 \le i \le \beta - 1$. Let $\mathcal{P}$ be the set that contains all pairs of vertices (a, b) that satisfy $x_1^i(a) = 0$, $x_1^i(b) = 1$ and $x_1^j(a) = x_1^j(b), j \neq i$, for $0 \le i \le \beta - 1$. Notice that for a pair $(a, b) \in \mathcal{P}$, $x_1(a) < x_1(b)$ and $x_1(a)^R < x_1(b)^R$. Hence $(x_1(a), x_2(a)) < (x_1(b), x_2(b))$, which implies $a \rightsquigarrow_{\mathcal{J}} b$. Now let G be the digraph that is formed by the arcs $(a, b) \in \mathcal{P}$. See Figure 3. Then $a \rightsquigarrow_G b$ only if $a \rightsquigarrow_{\mathcal{J}} b$. Moreover, the transitive reduction of G is itself and has size $\Omega(n \log n)$. We also observe that any two vertices in G share at most one immediate successor. Therefore the size of G cannot be reduced by introducing Steiner vertices. This implies that size of $\mathcal{J}$ is also $\Omega(n \log n)$.

3.2 Tree and Path

Let G_1 be a rooted (in- or out-)tree and G_2 a dipath. First we note that the ancestor-descendant relations in a rooted tree can be described by two linear orders (corresponding to a preorder and a postorder traversal of the tree) and therefore we can get an $O(n \log^2 n)$ bound on the size of $\mathcal{J}$ using the result of Section 3.1. Here we provide an $O(n \log n)$ bound, which also holds when G_1 is unoriented. This upper bound together with the $\Omega(n \log n)$ lower bound of Section 3.1 implies Theorem 3.1(a).

Let T be the rooted tree that results from G_1 after removing arc directions. We associate each vertex $x \in T$ with a label $h(x) = h_{G_2}(x)$, the height of x in G_2. If G_1 is an out-tree then any vertex b must be reachable from all its ancestors a in T with $h(a) > h(b)$. Similarly, if G_1 is an in-tree then any vertex b must be reachable from all its descendants a in T with $h(a) > h(b)$. We begin by assigning a depth-first search interval to each vertex in T. Let $I(a) = [s(a), t(a)]$ be the interval of a vertex $a \in T$; $s(a)$ is the time of the first visit to a (during the depth-first search) and $t(a)$ is the time of the last visit to a. These times are computed by incrementing a counter after visiting or leaving a vertex during the search. This way all the $s()$ and $t()$ values that are assigned are distinct and for any vertex a we have $1 \leq s(a) < t(a) \leq 2n$. Moreover, by well-known properties of depth-first search, we have that a is an ancestor of b in T if and only if $I(b) \subseteq I(a)$; if a and b are unrelated in T then $I(a)$ and $I(b)$ do not intersect. Now we map each vertex a to the x_1-axis-parallel segment $S(a) = I(a) \times h(a)$.

As in Section 3.1 we use a divide-and-conquer method to build $\mathcal{J}$. We will consider G_1 to be an out-tree; the in-tree case is handled similarly and yields the same asymptotic bound. Let ℓ be the horizontal line with x_2-coordinate equal to $n/2$. A vertex x is *above* ℓ if $h(x) \geq n/2$; otherwise ($h(x) < n/2$), x is *below* ℓ. We create a subgraph $\mathcal{J}_\ell$ of $\mathcal{J}$ that connects the vertices above ℓ to the vertices below ℓ. To that end, for each vertex u above ℓ we create a Steiner vertex u' together with the arc (u, u'). Let z be the nearest ancestor of u in T that is above ℓ. If z exists then we add the arc (z', u'). Then, for each vertex y below ℓ we locate the nearest ancestor u of y in T that is above ℓ. If u exists then we add the arc (u', y). See Figure 4. Finally, we recurse for the vertices above ℓ and for the vertices below ℓ.

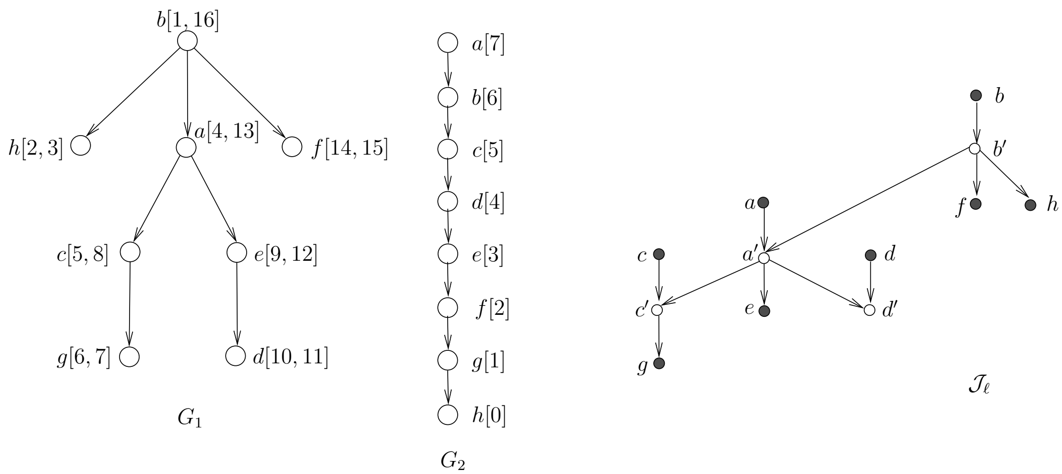
It is not hard to verify the correctness of the above construction. The size of the resulting graph can be bounded by $O(n \log n)$ as in Section 3.1. Furthermore, we can generalize this construction for an unoriented tree and an unoriented path, and accomplish the same $O(n \log n)$ bound as required by Theorem 3.1(a). (We omit the details which are similar to the more complicated construction of Section 3.4.)

3.3 Two Trees

The construction of Section 3.2 can be extended to handle more than one dipath. We show how to apply this extension in order to get an $O(n \log^2 n)$ bound for the join-reachability graph of two rooted trees. We consider the case where G_1 is an out-tree and G_2 is an in-tree; the other two cases (two out-trees and two in-trees) are handled similarly.

Let T_1 and T_2 be the corresponding undirected trees. We assign to each vertex a two depth-first search intervals $I_1(a) = [s_1(a), t_1(a)]$ and $I_2(a) = [s_2(a), t_2(a)]$, where $I_j(a)$ corresponds to T_j, $j = 1, 2$. We create two linear orders (i.e., dipaths), P_1 and P_2, from the I_2-intervals as follows: In P_1 the vertices are ordered by decreasing s_2-value and in P_2 by increasing t_2-value. Each vertex a is mapped to an x_1-axis-parallel segment $I_1(a) \times x_2(a) \times x_3(a)$ (in three dimensions), where $x_2(a) = h_{P_1}(a)$ and $x_3(a) = h_{P_2}(a)$. Then $a \rightsquigarrow_{\mathcal{J}} b$ if and only if $I_1(b) \subseteq I_1(a)$ and $(x_2(b), x_3(b)) \leq (x_2(a), x_3(a))$. See Figure 5.

Again we employ a divide-and-conquer approach and use the method of Section 3.2 as a subroutine. The details are as follows. Let p be the plane with x_3-coordinate equal to $n/2$. We construct a subgraph $\mathcal{J}_p$ of $\mathcal{J}$ that connects the vertices above p (i.e., vertices z with $x_3(z) \geq n/2$) to the



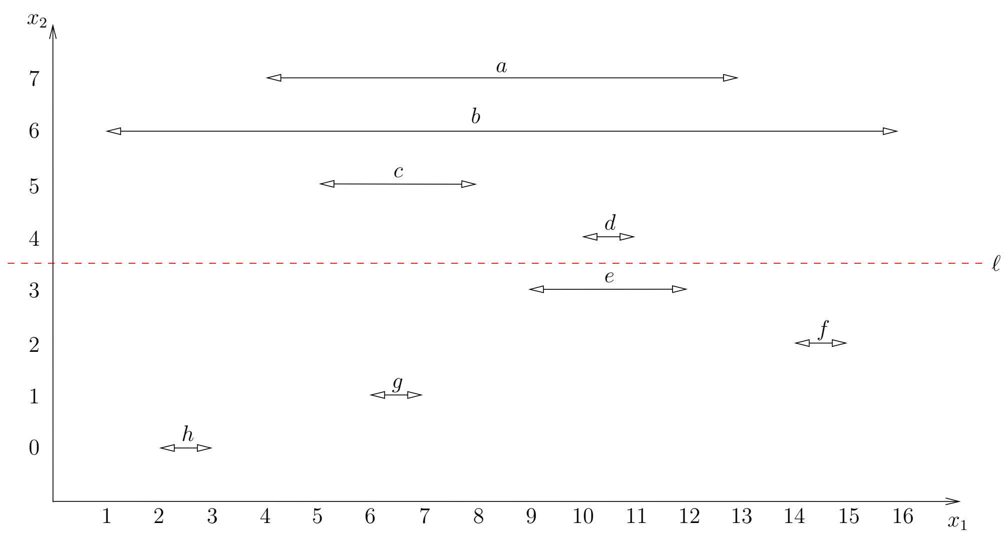
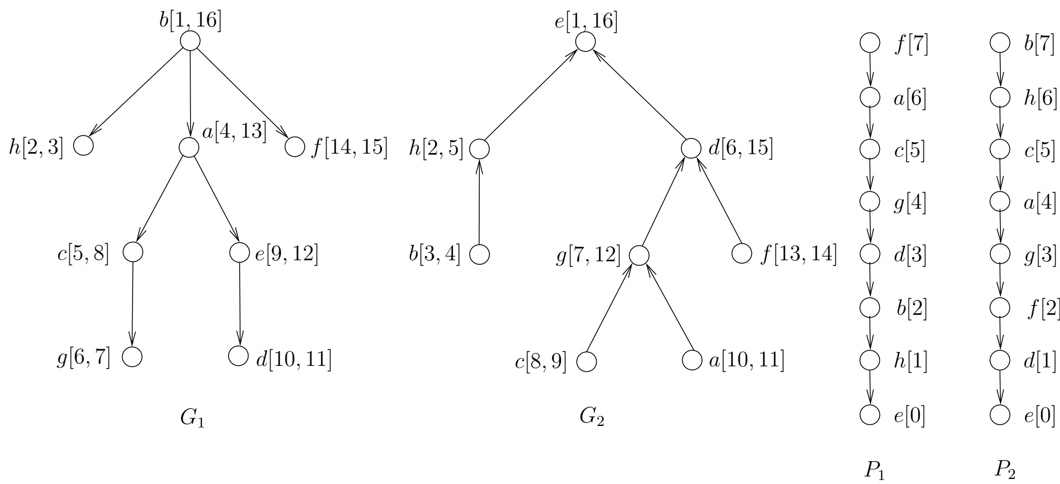


Figure 4: The mapping of the vertices of a rooted tree and a dipath to horizontal segments in a 2d rank space and the construction of $\mathcal{J}_\ell$.



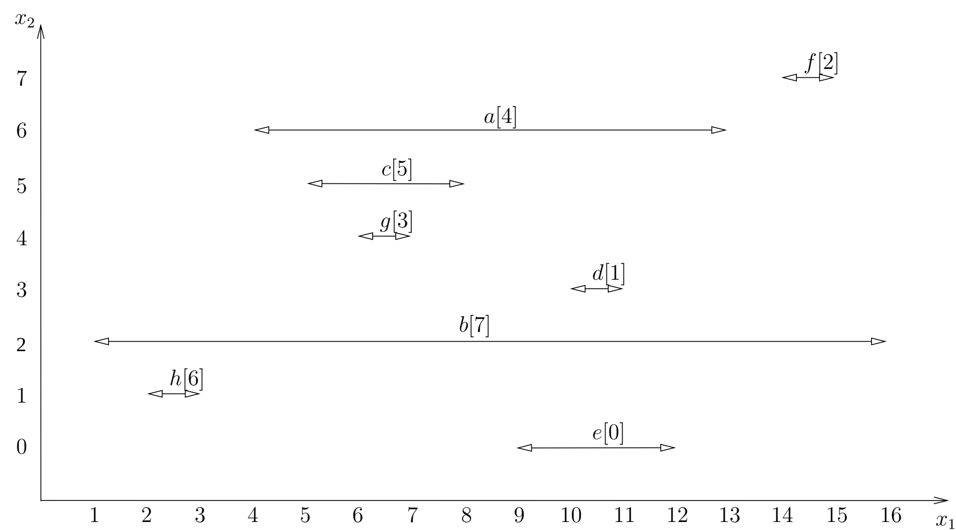


Figure 5: The mapping of the vertices of two rooted trees to horizontal segments in a 3d rank space. The value in brackets above the segments correspond to the x_3-coordinates.

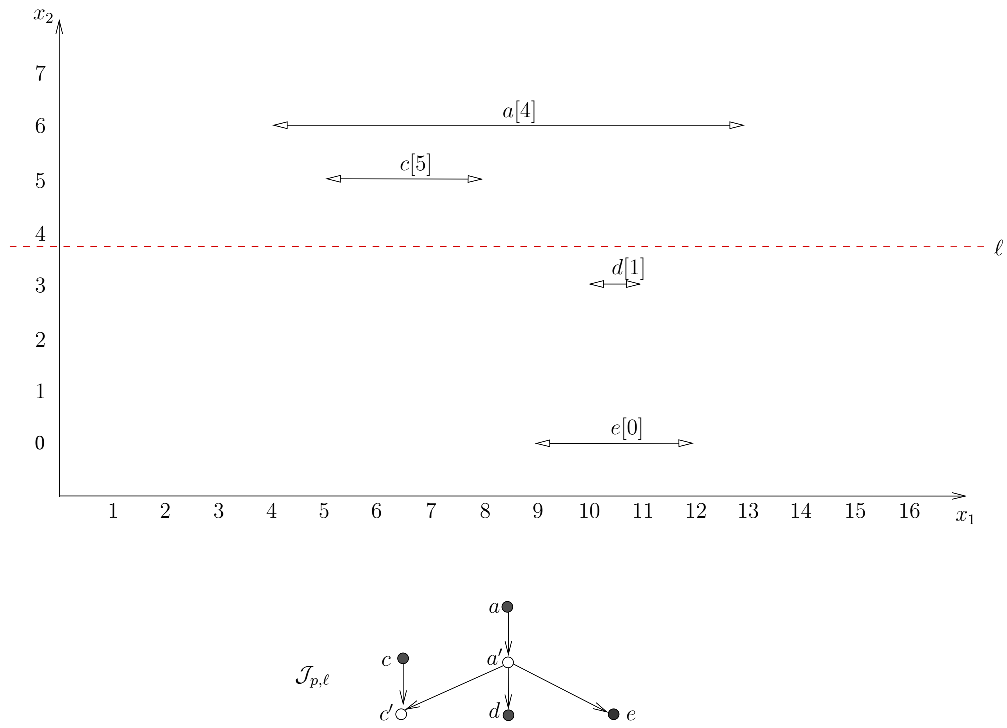


Figure 6: The construction of $\mathcal{J}_{p,\ell}$.

vertices below p (i.e., vertices z with $x_3(z) < n/2$). Then we use recursion for the vertices above p and the vertices below p.

We construct $\mathcal{J}_p$ using the method of Section 3.2 with some modifications. Let ℓ be the horizontal line with x_2-coordinate equal to $n/2$. We create a subgraph $\mathcal{J}_{p,\ell}$ of $\mathcal{J}_p$ that connects the vertices above p and ℓ to the vertices below p and ℓ. To that end, for each vertex z with $(x_2(z), x_3(z)) \geq (n/2, n/2)$ we create a Steiner vertex z' together with the arc (z, z'). Let u be the nearest ancestor of z in T_1 such that $(x_2(u), x_3(u)) \geq (n/2, n/2)$. If u exists then we add the arc (u', z'). Finally, for each vertex y with $(x_2(y), x_3(y)) < (n/2, n/2)$ we locate the nearest ancestor z of y in T_1 such that $(x_2(z), x_3(z)) \geq (n/2, n/2)$. If z exists then we add the arc (z', y). See Figure 6. Finally, we recurse for the vertices above ℓ and for the vertices below ℓ.

Now we bound the size of our construction. From Section 3.2 we have that the size of each substructure $\mathcal{J}_p$ is $O(n \log n)$. Since each vertex participates in $O(\log n)$ such substructures, the total size is bounded by $O(n \log^2 n)$.

3.4 Unoriented Trees

We can reduce the case of unoriented trees to that of rooted trees by applying Thorup's layer decomposition (see Section 1.1). We apply this decomposition to both G_1 and G_2. Let $G_i^0, G_i^2, \ldots, G_i^{\mu_i - 1}$

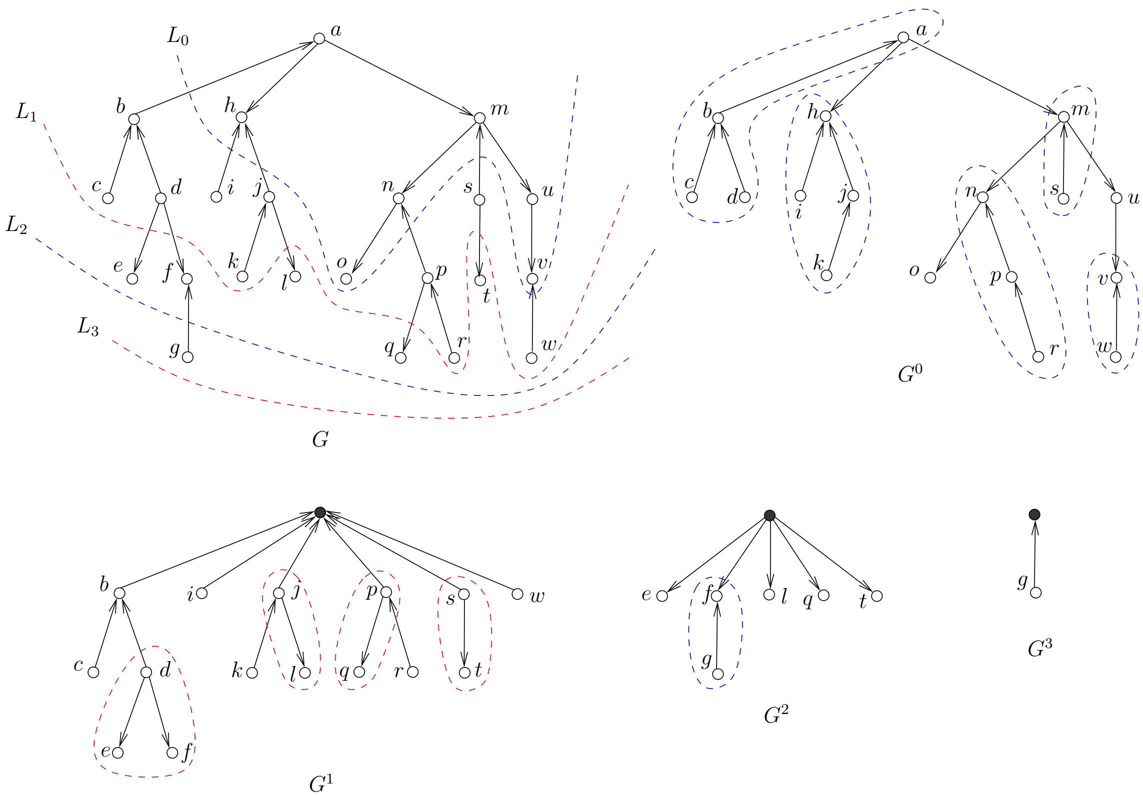


Figure 7: An unoriented tree and its sequence of 2-layered tree. Fringe trees are encircled.

be the sequence of rooted trees produced from G_i, $i = 1, 2$, where each G_i^j is a 2-layered tree. See Figure 7. For even j, G_i^j consists of a *core* out-tree, formed by the arcs directed away from the root, and a collection of *fringe* in-trees. The situation is reversed for odd j, where the core tree is an in-tree and the fringe trees are out-trees. We call a vertex of the core tree a *core vertex*; we call a vertex of a fringe tree (excluding its root) a *fringe vertex*.

We build $\mathcal{J}$ as the union of join-reachability graphs $\mathcal{J}_{i,j}$ for each pair (G_1^i, G_2^j). Each graph $\mathcal{J}_{i,j}$ is constructed similarly to Section 3.3, with the exception that we have to take special care for the fringe vertices. (We also remark that in general $\mathcal{J}_{i,j} \neq \mathcal{J}(\{G_1^i, G_2^j\})$.) A vertex $z \in V(G_1^i) \cap V(G_2^j)$ is included in $\mathcal{J}_{i,j}$ if one of the following cases hold: (i) z is a core vertex in at least one of G_1^i and G_2^j, or (ii) z is a fringe vertex in both G_1^i and G_2^j and the corresponding fringe trees containing z are either both in-trees or both out-trees. Let $V_{i,j}$ be the vertices in $V(G_1^i) \cap V(G_2^j)$ that satisfy the above condition.

If $V_{i,j} = \emptyset$ then $\mathcal{J}_{i,j}$ is empty. Now suppose $V_{i,j} \neq \emptyset$. First consider the case where the core of G_1^i is an out-tree. We contract each fringe in-tree to its root and let the new core supervertex correspond to the vertices of the contracted fringe tree. Let $\hat{G}_1^i$ be the out-tree produced from this process. Equivalently, if the core of G_1^i is an in-tree then the contraction of the fringe out-trees produces an in-tree $\hat{G}_1^i$. We repeat the same process for G_2^j. Next, we assign a depth-first search interval $I_1(z)$ to each vertex z in $\hat{G}_1^i$ and a depth-first search interval $I_2(z)$ to each vertex z in $\hat{G}_2^j$, as in Section 3.3. The vertices in $V_{i,j}$ are assigned a depth-first search interval in both trees, and therefore can be mapped to horizontal segments in a 3d space, as in Section 3.3. Hence, we can employ the method of Section 3.3 with some necessary changes that involve the fringe vertices. Let $z \in V_{i,j}$ be a fringe vertex in at least one of G_1^i and G_2^j. If the fringe tree containing z is an in-tree then we only include in $\mathcal{J}_{i,j}$ arcs leaving z; otherwise we only include arcs entering z.

Finally we need to show that the size of the resulting graph is $O(n \log^2 n)$. This follows from the fact that each subgraph $\mathcal{J}_{i,j}$ has size $O(n \log^2 n)$ and that each vertex can appear in at most four such subgraphs. Theorem 3.1(b) follows.

3.5 Planar Digraphs

Now we turn to planar digraphs and combine our previous constructions with Thorup's reachability oracle [18]. From this combination we derive the bounds stated in Theorem 3.1(c) and (d). First we need to provide some details for the reachability oracle of [18].

Let G be a planar digraph, and let $G^0, G^1, \ldots, G^{\mu-1}$ be the sequence of 2-layered digraphs produced from G as described in Section 1.1. Consider one of these digraphs G^i. The next step is to obtain a separator decomposition of G^i. To that end, we treat G^i as an undirected graph and compute a separator S whose removal separates G^i into components, each with at most half the vertices. The separator S consists of three root paths of a spanning tree of G^i rooted at r_0. Because G^i is 2-layered, each root path in S corresponds to at most two dipaths in G^i. The key idea now is to process each separator dipath Q and find the connections between $V(G^i)$ and Q. For each $v \in V(G^i)$ two quantities are computed: (i) $\mathrm{from}_v[Q]$ which is equal to $r_Q(u)$, where $u \in Q$ is the vertex with the highest rank in Q such that $u \rightsquigarrow_{G^i} v$, and (ii) $\mathrm{to}_v[Q]$ which is equal to $r_Q(u)$, where $u \in Q$ is the vertex with the lowest rank in Q such that $v \rightsquigarrow_{G^i} u$. Clearly there is a path from a to b that passes though Q if and only if $\mathrm{to}_a[Q] \leq \mathrm{from}_b[Q]$. The same process is carried out recursively for each component of $G^i \setminus V(S)$. The depth of this recursion is $O(\log n)$, so each vertex is connected to $O(\log n)$ separator dipaths. The space and construction time for this structure is $O(n \log n)$.

Now we consider how to construct a join-reachability graph when G_1 is a planar digraph. We begin with the case where G_2 is a dipath. First we perform the layer decomposition of G_1 and construct the corresponding graph sequence $G_1^0, G_1^1, \ldots, G_1^{\mu-1}$. Then we form pairs of digraphs $P_i = \{G_1^i, G_2^i\}$ where G_2^i is a dipath containing only the vertices in $V(G_1^i)$ in the order they appear in G_2. Clearly $a \rightsquigarrow_{\mathcal{J}} b$ if and only if $a \rightsquigarrow_{\mathcal{J}_{\iota(b)-1}} b$ or $a \rightsquigarrow_{\mathcal{J}_{\iota(b)}} b$, where $\mathcal{J}_i$ is the join-reachability graph of P_i. Then $\mathcal{J}$ is formed from the union of $\mathcal{J}_0, \ldots, \mathcal{J}_{\mu-1}$.

To construct $\mathcal{J}_i$ we perform the separator decomposition of G_1^i, so that each vertex is associated with $O(\log n)$ separator dipaths. Let Q be such a separator dipath. Also, let V_Q be the set of vertices that have a successor or a predecessor in Q. We build a subgraph $\mathcal{J}_{i,Q}$ of $\mathcal{J}_i$ for the vertices in V_Q; $\mathcal{J}_i$ is formed from the union of the subgraphs $\mathcal{J}_{i,Q}$ for all the separator dipaths of G_1^i. The construction of $\mathcal{J}_{i,Q}$ is carried out as follows. Let $z \in V_Q$. If z has a predecessor in Q then we create a vertex z^- which is assigned coordinates $x_1(z^-) = \mathrm{from}_z[Q]$ and $x_2(z^-) = r_{G_2}(z)$, and add the arc (z, z^-). Similarly, if z has a successor in Q then we create a vertex z^+ which is assigned coordinates $x_1(z^+) = \text{to}_z[Q]$ and $x_2(z^+) = r_{G_2}(z)$, and add the arc (z^+, z).

Now we can use the method of Section 3.1 to build the rest of $\mathcal{J}_{i,Q}$, so that $a \rightsquigarrow_{\mathcal{J}_{i,Q}} b$ if and only if $(x_1(a^+), x_2(a^+)) \leq (x_1(b^-), x_2(b^-))$. Let ℓ be the vertical line with x_1-coordinate equal to $n/2$. The first step is to construct the subgraph of $\mathcal{J}_{i,Q}$ that connects the vertices a^+ with $x_1(a^+) \leq n/2$ to the vertices b^- with $x_1(b^-) \geq n/2$. For each such b^- we create a Steiner vertex b' and add the arc (b', b^-). Also, we assign to b' the coordinates $(n/2, x_2(b^-))$. We connect these Steiner vertices in a dipath starting from the vertex with the lowest x_2-coordinate. Next, for each vertex a^+ with $x_1(a^+) \leq n/2$ we locate the Steiner vertex b' with the smallest x_2-coordinate such that $x_2(a^+) \leq x_2(b')$. If b' exists we add the arc (a^+, b'). Finally we recurse for the vertices with x_1-coordinate in $[1, n/2)$ and for the vertices with x_1-coordinate in $(n/2, n]$.

It remains to bound the size of $\mathcal{J}$. From Section 3.1, we have $|\mathcal{J}_{i,Q}| = O(|V_Q| \log |V_Q|)$. Moreover, the bound $\sum_Q |V_Q| = O(|V(G_1^i)| \log |V(G_1^i)|)$, where the sum is taken over all separator paths of G_1^i, implies $|\mathcal{J}_i| \leq \sum_Q |\mathcal{J}_{i,Q}| = O(|V(G_1^i)| \log^2 |V(G_1^i)|)$. Finally, since $\sum_i |V(G_1^i)| = O(n)$ we obtain $|\mathcal{J}| \leq \sum_i |\mathcal{J}_i| = O(n \log^2 n)$.

We handle the case where G_2 is an unordered dipath as noted in Section 3.1, which implies Theorem 3.1(c). The methods we developed here in combination with the structures of Section 3.4 result to a join-reachability graph of size $O(n \log^3 n)$ for a planar digraph and an unoriented tree. The same bound of $O(n \log^3 n)$ is achieved for two planar digraphs, as stated in Theorem 3.1(d).

3.6 General Graphs

A technique that is used to speed up transitive closure and reachability computations is to cover a digraph with simple structures such as dipaths, chains, or trees (e.g., see [1]). Such techniques are well-suited to our framework as they can be combined with the structures we developed earlier. We also remark that the use of the preprocessing steps of Section 1.1 reduces the problem from general digraphs to acyclic and 2-layered digraphs. In this section we describe how to obtain join-reachability graphs with the use of dipath covers. This gives the bounds stated in Theorem 3.1(e)-(g); similar results can be derived with the use of tree covers. Again for simplicity, we first consider the case where G_1 is a general digraph and G_2 is a dipath.

A *dipath cover* is a decomposition of a digraph into vertex-disjoint dipaths. Let $P_1^1, P_1^2, \ldots P_1^{\kappa_1}$ be a dipath cover of G_1. For each vertex v and each path P_1^i we compute $\text{from}_v[P_1^i]$, i.e., $r_{P_1^i}(z)$ where $z \in P_1^i$ is the vertex with the highest rank in P_1^i such that $z \rightsquigarrow_{G_1} v$. Let P_2^i be the dipath that consists of the vertices in P_1^i ordered by increasing rank in G_2. Also, set $\text{from}_v[P_2^i] = r_{P_2^i}(z)$ where $z \in P_2^i$ is the vertex with the largest rank such that $r_{G_2}(z) \leq r_{G_2}(v)$. Let $V_{P_1^i}$ be set of vertices that have a predecessor in P_1^i. We build a subgraph $\mathcal{J}_i$ of $\mathcal{J}$ that connects the vertices of P_1^i to $V_{P_1^i}$. Then $\mathcal{J}$ is formed from the union of the subgraphs $\mathcal{J}_i$. For each $z \in V_{P_1^i}$ we create a vertex z^- which is assigned coordinates $x_1(z^-) = \text{from}_z[P_1^i]$ and $x_2(z^-) = \text{from}_z[P_2^i]$, and add the arc (z^-, z). Also, for each $z \in P_1^i$ we create a vertex z^+ which is assigned coordinates $x_1(z^+) = r_{P_1^i}(z)$ and $x_2(z^+) = r_{P_2^i}(z)$, and add the arc (z, z^+). Now we can build a join-reachability graph, so that $a \rightsquigarrow_{\mathcal{J}_i} b$ if and only if $(x_1(a^+), x_2(a^+)) \leq (x_1(b^-), x_2(b^-))$, as in Section 3.5.

The size of this graph is bounded by $\sum_i |V_{P_1^i}| \log |V_{P_1^i}| = O(\kappa_1 n \log n)$, which implies the result of Theorem 3.1(e). We can extend this method to handle two general digraphs and obtain the bound of Theorem 3.1(g). The case where G_2 is planar digraph is handled by combining the above method with the techniques of Section 3.5, resulting to Theorem 3.1(f).

4 Data Structures for Join-Reachability

Now we deal with the data structure version of the join-reachability problem. Our goal is to construct an efficient data structure for $\mathcal{J} \equiv \mathcal{J}(\{G_1, G_2\})$ such that given a query vertex b it can report all vertices a satisfying $a \leadsto_{\mathcal{J}} b$. We state the efficiency of a structure using the notation $\langle s(n), q(n,k)\rangle$ which refers to a data structure with $O(s(n))$ space and $O(q(n,k))$ query time for reporting k elements. In order to design efficient join-reachability data structures we apply the techniques we developed in Section 3. The bounds that we achieve this way are summarized in the following theorem.

Theorem 4.1. *Given two digraphs G_1 and G_2 with n vertices we can construct join-reachability data structures with the following efficiency:*

(a) $\langle n, k\rangle$ when G_1 is an unoriented tree and G_2 is an unoriented dipath.

(b) $\langle n, \log n + k\rangle$ when G_1 is an out-tree and G_2 is an unoriented tree.

(c) $\langle n \log^{\varepsilon} n, \log\log n + k\rangle$ (for any constant $\varepsilon > 0$), when G_1 and G_2 are unoriented trees.

(d) $\langle n \log n, k \log n\rangle$ when G_1 is planar digraph and G_2 is an unoriented tree.

(e) $\langle n \log^2 n, k \log^2 n\rangle$ when both G_1 and G_2 are planar digraphs.

(f) $\langle n\kappa_1, k\rangle$ when G_1 is a general digraph that can be covered with κ_1 vertex-disjoint dipaths and G_2 is an unoriented tree.

(g) $\langle n(\kappa_1+\log n), k\kappa_1 \log n\rangle$ or $\langle n\kappa_1 \log n, k \log n\rangle$ when G_1 is a general digraph that can be covered with κ_1 vertex-disjoint dipaths and G_2 is planar digraph.

(h) $\langle n(\kappa_1 + \kappa_2), \kappa_1\kappa_2 + k\rangle$ or $\langle n\kappa_1\kappa_2, k\rangle$ when each G_i, $i = 1, 2$, is a digraph that can be covered with κ_i vertex-disjoint dipaths.

Next we provide the constructions that prove the bounds stated in Theorem 4.1. Throughout this section k denotes the size of the output of a join-reachability reporting query.

4.1 Two Paths

Let G_1 and G_2 be two dipaths. We use the mapping of Section 2. Recall that each vertex a is mapped to a point $(x_1(a), x_2(a))$ on an $n \times n$ grid so that $a \leadsto_{\mathcal{J}} b$ if and only if $(x_1(a), x_2(a)) \leq (x_1(b), x_2(b))$. This is a *two-dimensional point dominance problem* that can be solved optimally with a Cartesian tree [6]. Thus, we immediately get an $\langle n, k\rangle$ join-reachability structure for two dipaths. We provide the details of this structure as we will need them in later constructions. A Cartesian tree T is a binary tree defined recursively as follows. The root of T is the point a with minimum x_2-coordinate. The left subtree of the root is a Cartesian tree for the points b with $x_1(b) < x_1(a)$ and the right subtree of the root is a Cartesian tree for the points b with $x_1(b) > x_1(a)$. Clearly this structure uses linear space, and moreover it can be constructed in linear time [6]. The reporting algorithm uses the following property of Cartesian trees. Consider two points a and b, and let c be the point with minimum x_2-coordinate such that $x_1(a) \leq x_1(c) \leq x_1(b)$. Then, $c = nca_T(a, b)$. Now let ζ be the point with the smallest x_1-coordinate. In order to find all points a such that $(x_1(a), x_2(a)) \leq (x_1(b), x_2(b))$ we first locate $y = nca_T(\zeta, b)$. The returned point y has the smallest x_2-coordinate in the x_1-range $[0, x_1(b)]$. If $x_2(y) > x_2(b)$ then the answer is null and we stop our search. Otherwise we return y and search recursively in the x_1-ranges $[0, x_1(y)-1]$ and $[x_1(y)+1, x_1(b)]$. Using the fact that nearest common ancestor queries in a tree can be answered in constant time after linear time preprocessing [10], it follows that the time to report k vertices is $O(k)$.

As in Section 3.1, we can achieve the same bounds when G_1 and G_2 are unoriented dipaths by splitting them into maximal subpaths consisting of arcs with the same orientation.

4.2 Tree and Path

Next we consider the case where G_1 is a rooted tree and G_2 is a dipath. As in Section 3.2, we note that a rooted tree can be described by two linear orders, and therefore we can get an $\langle n, \log n + k \rangle$ solution using a three-dimensional dominance reporting structure [11]. Here we develop an alternative method that reduces the dimension of our problem and as a result it achieves an $\langle n, k \rangle$ bound. Furthermore, this method can be extended to give more efficient structures for two trees (compared to four-dimensional dominance reporting [11]). We will distinguish two cases depending on whether G_1 is an out-tree or an in-tree. In any case, let T be the rooted tree that results from G_1 after removing arc directions. We associate each vertex $x \in T$ with a label $h(x) = h_{G_2}(x)$, the height of x in G_2. For an in-tree we wish to support the following query: Given a vertex b and a label j find all vertices $a \in T(b)$ with $h(a) > j$. Equivalently, for an out-tree the query algorithm needs to find all ancestors a of b in T with $h(a) > j$. We present a geometry-based method, which achieves $O(\log n + k)$ reporting time for an in-tree and $O(k)$ for an out-tree. An alternative method, based on a heavy-path decomposition of T [15], is given in Appendix A.

We use the mapping of Section 3.2. Each vertex a is assigned a depth-first search interval $I(a) = [s(a), t(a)]$ in T and is mapped to the x_1-axis-parallel segment $S(a) = I(a) \times h(a)$. Now the choice of the structure we use depends on the arc directions in G_1. For an out-tree we have that $a \rightsquigarrow_{\mathcal{J}} b$ if and only if $S(a)$ is above $S(b)$ and the x_1-projection of $S(a)$ covers the x_1-projection of $S(b)$. The fact that interval endpoints are distinct implies that $a \rightsquigarrow_{\mathcal{J}} b$ if and only if the vertical ray v_b emanating from $(s(b), h(b))$ towards the $(+x_2)$-direction intersects $S(a)$. Indeed, if $a \rightsquigarrow_{\mathcal{J}} b$ then $h(b) \leq h(a)$ and $b \in T(a)$, so $I(b) \subseteq I(a)$. Similarly, if $S(a)$ is above $S(b)$ and $I(b) \subseteq I(a)$ then v_b intersects $S(a)$. Therefore, we have reduced our problem to a *planar segment intersection* problem. We can get an $\langle n, k \rangle$ structure by adapting either the hive graph of Chazelle [4] or the persistence-based planar point location structure of Sarnak and Tarjan [14]. Both these data structures require $O(n \log n)$ preprocessing time as they need to sort the endpoint coordinates. In our case sorting is not necessary, since the x_1-coordinates are produced in sorted order by the depth-first search, and the x_2-coordinates correspond to the height of the vertices in G_2. Hence our preprocessing time is $O(n)$. Furthermore, the reporting time using either the hive graph or the persistence-based structure is $O(\log n + k)$, where the $\log n$ term is due to a point location query. In our case this term can be reduced to constant; point location is not necessary since the segment endpoints are the only possible query locations. Hence our reporting time is $O(k)$.

We turn to the case where G_1 is an in-tree. Here we have that $a \leadsto_{\mathcal{J}} b$ if and only if $S(a)$ is below $S(b)$ and the x_1-projection of $S(b)$ covers the x_1-projection of $S(a)$. Since the interval endpoints are distinct we have $a \leadsto_{\mathcal{J}} b$ if and only if the endpoints of $S(a)$ are contained inside the rectangle $[s(b), t(b)] \times [0, h(b)]$. This is a *two-dimensional grounded range search* problem (one side of the query rectangle always lies on the x_1-axis). Since we have integer coordinates in $[1, 2n] \times [0, n-1]$ we can get an $\langle n, k \rangle$ structure again with the use of a Cartesian tree [6].

The $\langle n, k \rangle$ bound is also achieved when G_1 is an unoriented tree, as stated in Theorem 4.1(a), by applying the method of Section 3.4. Let $G_1^0, G_1^2, \ldots, G_1^{\mu_i - 1}$ be the sequence of 2-layered rooted trees produced from G_1. We construct a join-reachability structure for each pair $P_i = \{G_1^i, G_2^i\}$, where G_2^i is a dipath containing only the vertices in $V(G_1^i)$ in the order they appear in G_2. A query for a vertex b needs to search the structures for the pairs $P_{\iota(b)-1}$ and $P_{\iota(b)}$. The structure for P_i is constructed as follows. We contract each fringe tree to its root and let the new core supervertex correspond to the vertices of the contracted fringe tree. Let $\hat{G}_1^i$ be the tree produced from this process. Next, we assign a depth-first search interval $I_1(z)$ to each vertex z in $\hat{G}_1^i$, and map z to the x_1-axis-parallel segment $I_1(z) \times x_2(z)$, where $x_2(z) = h_{G_2^i}(z)$. Using this mapping we can construct the data structures developed above depending on whether $\hat{G}_1^i$ is an out-tree or an in-tree. One important detail is that if $\hat{G}_1^i$ is an in-tree then the data structure for P_i does not store the segments that correspond to fringe vertices; the segment of such a fringe vertex z is needed however in order to answer a join-reachability query for z. Equivalently, if $\hat{G}_1^i$ is an out-tree and the query vertex b is an fringe in-tree of G_1^i then we do not search the structure for P_i.

4.3 Two Trees

We extend the method of Section 4.2 in order to deal with two rooted trees G_1 and G_2. We distinguish three cases depending on the type, in-tree or out-tree, of each tree. Then, by applying the layer decomposition method of Section 3.4, we can extend our structures to handle unoriented trees. This way we achieve the bounds stated in Theorem 4.1(b) and (c).

Let T_1 and T_2 be the corresponding undirected trees. We assign each vertex a two depth-first search intervals $I_1(a) = [s_1(a), t_1(a)]$ and $I_2(a) = [s_2(a), t_2(a)]$, where $I_j(a)$ corresponds to T_j, for $j = 1, 2$. We use the two intervals $I_1(a) = [s_1(a), t_1(a)]$ and $I_2(a) = [s_2(a), t_2(a)]$ to map each vertex a to an axis-parallel rectangle $R(a) = I_1(a) \times I_2(a)$. See Figure 8. Again we exploit the fact that for any two vertices a and b, the intervals $I_j(a)$ and $I_j(b)$ are either disjoint or one contains the other. If $I_1(a) \cap I_1(b) = \emptyset$ or $I_2(a) \cap I_2(b) = \emptyset$, then $R(a)$ and $R(b)$ do not intersect. Now suppose that both $I_1(a) \cap I_1(b) \neq \emptyset$ and $I_2(a) \cap I_2(b) \neq \emptyset$. Without loss of generality, consider that $I_1(b) \subseteq I_1(a)$. If $I_2(b) \subseteq I_2(a)$ then $R(b)$ is contained in $R(a)$. Otherwise, if $I_2(a) \subseteq I_2(b)$ then both horizonal edges of $R(a)$ intersect both vertical edges of $R(b)$. Next, we distinguish three cases

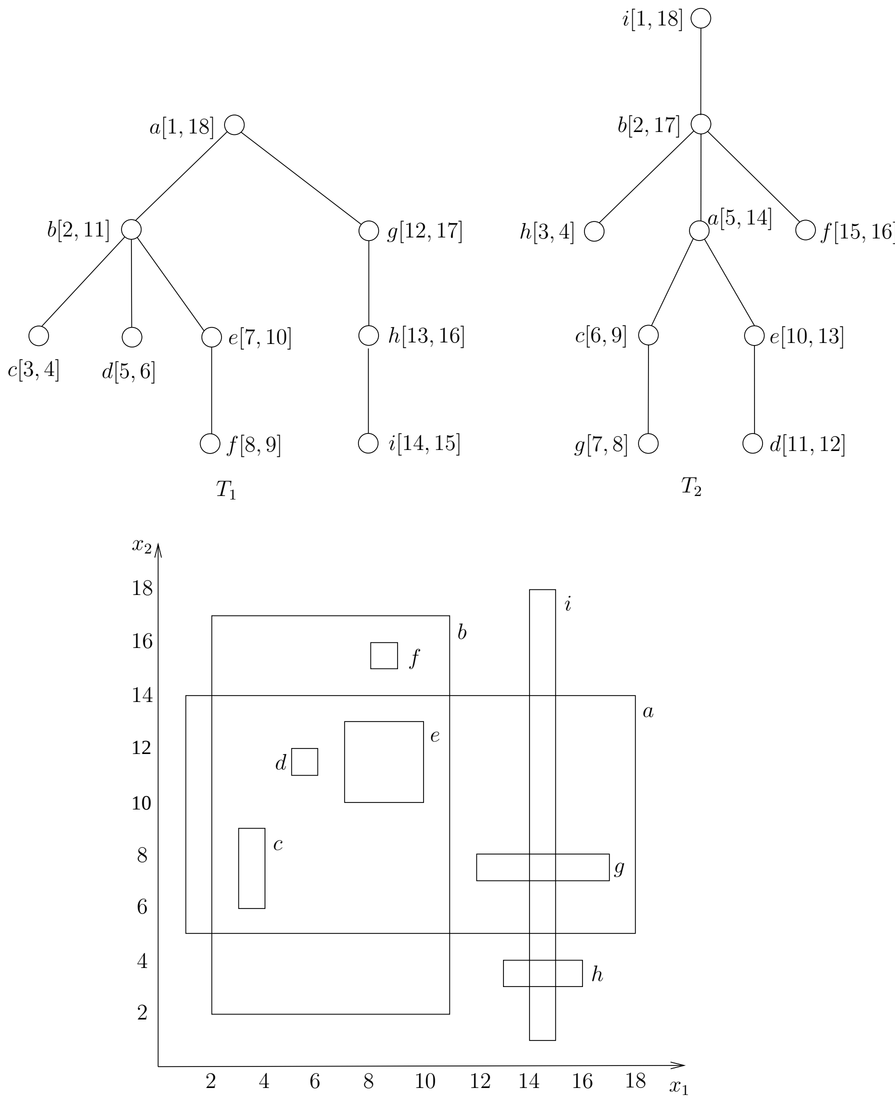


Figure 8: Example of the mapping of Section 4.3

depending on the type of the two trees.

First suppose that both G_1 and G_2 are out-trees. Then $a \rightsquigarrow_{\mathcal{J}} b$ implies $b \in T_1(a)$ and $b \in T_2(a)$. So here we have $I_1(b) \subseteq I_1(a)$ and $I_2(b) \subseteq I_2(a)$, thus $R(b)$ is contained in $R(a)$. In particular, the rectangle arrangement has the property that $a \rightsquigarrow_{\mathcal{J}} b$ if and only if $R(a)$ encloses a corner of $R(b)$. This property implies that we have a *two-dimensional point enclosure*: In order to report all vertices a such that $a \rightsquigarrow_{\mathcal{J}} b$ we need to find all rectangles $R(a)$ that enclose a corner of $R(b)$. To that end, we can use the point enclosure structure of Chazelle [4] to get an $\langle n, \log n + k \rangle$ join-reachability structure.

Next, suppose that G_1 is an out-tree and G_2 is an in-tree. In this case $a \rightsquigarrow_{\mathcal{J}} b$ if and only if $b \in T_1(a)$ and $a \in T_2(b)$, which implies $I_1(b) \subseteq I_1(a)$ and $I_2(a) \subseteq I_2(a)$. Thus, $R(a)$ intersects $R(b)$. Furthermore, the properties of the depth-first search intervals imply that $a \rightsquigarrow_{\mathcal{J}} b$ if and only if the segment $s_1(b) \times I_2(b)$ intersects $I_1(a) \times s_2(a)$. This is an orthogonal segment intersection problem, for which we can get an $\langle n, k \rangle$ join-reachability structure as in Section 4.2.

The last case is when G_1 and G_2 are in-trees. Now $a \rightsquigarrow_{\mathcal{J}} b$ if and only if $a \in T_1(b)$ and $a \in T_2(b)$. Then we have $I_1(a) \subseteq I_1(b)$ and $I_2(a) \subseteq I_2(b)$, which implies that $a \rightsquigarrow_{\mathcal{J}} b$ if and only if $R(b)$ encloses a corner of $R(a)$. Thus, our reporting query reduces to *orthogonal range searching*. Here the results of Alstrup et al. [3] imply an $\langle n \log^{\varepsilon} n, \log\log n + k \rangle$ join-reachability structure (for any constant $\varepsilon > 0$).

4.4 Planar Digraphs

With the help of Thorup's reachability oracle [18] we can develop efficient structures for join-reachability in planar digraphs. Suppose first that G_2 is a dipath. We perform the layer decomposition of G_1 and construct the corresponding graph sequence $G_1^0, G_1^1, \ldots, G_1^{\mu-1}$. Then we form pairs of digraphs $P_i = \{G_1^i, G_2^i\}$ where G_2^i is a dipath containing only the vertices in $V(G_1^i)$ in the order they appear in G_2. Clearly $a \rightsquigarrow_{\mathcal{J}} b$ if and only if $a \rightsquigarrow_{\mathcal{J}_{\iota(b)-1}} b$ or $a \rightsquigarrow_{\mathcal{J}_{\iota(b)}} b$, where $\mathcal{J}_i$ is the join-reachability graph of P_i. For each pair P_i we build a join-reachability structure. In order to answer a reporting query for b we query the structures for $P_{\iota(b)-1}$ and $P_{\iota(b)}$ independently and return the union of the results. It remains to describe the structure for a pair $P_i = \{G_1^i, G_2^i\}$. We perform the separator decomposition of G_1^i, so that each vertex is associated with $O(\log n)$ separator dipaths. For each vertex $v \in V(G_1^i)$ we record a set $S(v)$ containing the separator dipaths Q that reach v together with the number $\mathrm{from}_v[Q]$ (see Section 3.5). For each separator dipath Q we record the vertices v that reach Q together with the numbers $\mathrm{to}_v[Q]$. Next, for each separator dipath Q we build the data structure of Section 4.1 for the vertices that reach Q. Each such vertex a receives coordinates $(x_1(a), x_2(a))$ where $x_1(a) = \mathrm{to}_a[Q]$ and $x_2(a)$ is the rank of a in G_2 among the vertices that reach Q. Now we can report the vertices that reach b through Q by finding the vertices a that satisfy $(x_1(a), x_2(a)) \leq (\mathrm{from}_b[Q], x_2(b))$. To that end, we use a Cartesian tree T as in Section 4.1. Here we need to modify this structure in order to allow points with identical x_1-coordinates. Since the x_1-coordinates are integers in the range $[0, |Q|-1]$ we find for each integer i in that range the point a_i with $x_1(a_i) = i$ and minimum x_2-coordinate. Then we build a Cartesian tree for the points a_i, $0 \leq i \leq |Q|-1$. Also, we associate with a_i a list of the remaining points with x_1-coordinate equal to i in increasing x_2-coordinate. Next, in order to initiate the search we also need to locate the vertex c with $x_1(c) = \mathrm{from}_b[Q]$. We can do that easily in $O(1)$ time by using an array of size

$|Q|$ to map the x_1-coordinates to the corresponding locations in T. Recall that the basic step of the reporting algorithm is to locate the point with smallest x_2-coordinates in an x_1-range $[\alpha, \beta]$. If y is the corresponding point, then we check if $x_2(y) \le \text{from}_b[Q]$. If this is the case, then we report y and search the list associated with y and report all points with $x_2(z) \le \text{from}_b[Q]$. Clearly the reporting time for k points is still $O(k)$. Also the required space and preprocessing is $O(|V(G_1^i)|)$. Therefore, the asymptotic preprocessing time and space are the same as in Thorup's structure, i.e., $O(n \log n)$. Finally we need to specify how to report all vertices a such that $a \leadsto_{\mathcal{J}} b$. We query the structures for $P_{\iota(b)-1}$ and $P_{\iota(b)}$. To perform a query for P_i we use the list of separator dipaths that reach b, and for each such dipath Q we use the corresponding Cartesian tree to report the vertices a that satisfy $(x_1(a), x_2(a)) \le (\text{from}_b[Q], x_2(b))$. Let k_Q be the number of reported vertices. The total reporting time is bounded by $\sum_{Q \in S(b)} k_Q = O(k \log n)$.

Using the results of Section 4.2 we can get join-reachability structures when G_2 is a rooted or an unoriented tree. Let $I_2(a)$ be the depth-first search interval assigned to each vertex a in T_2, where T_2 is the undirected version of G_2. If G_2 is an out-tree then we report the vertices a that satisfy $\text{to}_a[Q] \le \text{from}_b[Q]$ and $I_2(b) \subseteq I_2(a)$, which by Section 4.2 can be done in $O(k_Q)$ time. So, the total reporting time is $O(k \log n)$. Similarly, if G_2 is an in-tree then we report the vertices a that satisfy $\text{to}_a[Q] \le \text{from}_b[Q]$ and $I_2(a) \subseteq I_2(b)$, which again takes $O(k_Q)$ time with the structure of Section 4.2. So, the total reporting time in both cases is bounded by $O(k \log n)$. Theorem 4.1(d) follows. With similar ideas we can obtain an $\langle n \log^2 n, k \log^2 n \rangle$ structure when G_2 is also a planar digraph, as stated by Theorem 4.1(e).

4.5 General Digraphs

Here we examine how to obtain join-reachability strucures for general digraphs with the use of dipath covers. We begin with the case where G_2 is a dipath.

Let $P_1^1, P_1^2, \ldots P_1^{\kappa_1}$ be a dipath cover of G_1, and let P_2^i be the dipath that consists of the vertices in P_1^i ordered by increasing rank in G_2. Also, let $V_{P_1^i}$ be set of vertices that have a predecessor in P_1^i. We build a join-reachability structure for each pair $\{P_1^i, P_2^i\}$ which we use in order to report the vertices in P_1^i that reach a query vertex in both G_1 and G_2. To that end, each vertex a in P_1^i is assigned coordinates $x_1(a) = r_{P_1^i}(a)$ and $x_2(a) = r_{P_2^i}(a)$, and we build a join-reachability structure for these vertices as in Section 4.1. With this structure we can answer a reporting query for vertex b by finding the vertices a that satisfy $(x_1(a), x_2(a)) \le (\text{from}_b[P_1^i], \text{from}_b[P_2^i])$ for each $i \in \{1, \ldots, \kappa_1\}$. The reporting time is $O(k + \kappa_1)$ using $O(\kappa_1 n)$ space. The reporting time can be reduced to $O(k)$ if we store for each vertex v a list $I(v)$ of the indices $i \in \{1, \ldots, \kappa_1\}$ such that the reporting query for v in the join-reachability structure for the pair $\{P_1^i, P_2^i\}$ is non-empty. Then we only need to query the structures for $i \in I(v)$. The asymptotic space bound remains $O(\kappa_1 n)$.

We can extend the above method in order to handle two general graphs. The resulting bounds, however, are interesting only when the product $\kappa_1 \kappa_2$ is small compared to n, where κ_2 is the number of disjoint dipaths in a dipath cover of G_2. Specifically, we can get either $O((\kappa_1 + \kappa_2)n)$ space and $O(\kappa_1 \kappa_2 + k)$ reporting time, or $O((\kappa_1 \kappa_2)n)$ space and $O(k)$ reporting time. (In the latter structure we improve the reporting time by storing for each vertex v the pairs of dipaths in the cover of G_1 and G_2 that contain a common predecessor of v.) This implies Theorem 4.1(h). By combining the dipath cover method with the techniques of Section 4.2 we obtain the bound of Theorem 4.1(f). Similarly, the techniques of Section 4.4 imply Theorem 4.1(g).

5 Conclusions and Open Problems

We explored the computational and combinatorial complexity of the join-reachability graph, and the design of efficient join-reachability data structures for a variety of graph classes. We believe that several open problems deserve further investigation. For instance, from the aspect of combinatorial complexity, it would be interesting to prove or disprove that an $O(m \cdot \mathrm{polylog}(n))$ bound on the size of the join-reachability graph $\mathcal{J}(\{G_1, G_2\})$ is attainable when G_1 is a general digraph with n vertices and m arcs and G_2 is a dipath. Another direction is to consider the problem of approximating the smallest join-reachability graph for specific graph classes. From the aspect of data structures, we can consider the following type of join-reachability query: Given vertices b and c, report (or count) all vertices a such that $a \rightsquigarrow_{G_1} b$ and $a \rightsquigarrow_{G_2} c$.

Acknowledgement. We would like to thank Li Zhang for several useful discussions.

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Appendices

In the Appendices we provide additional join-reachability data structures. In Appendix A we apply the *heavy-path decomposition* of trees [15] in order to get alternative join-reachability data structures for trees and paths. In Appendix B we consider the case of planar *st*-graphs [17], and in Appendix C we consider lattices.

A Join-Reachability for Trees based on Heavy-Path Decomposition

Let T be the rooted tree that results from G_1 after removing arc directions. We develop a method based on partitioning T into *heavy paths* [15]. This is done as follows. A child a' of a is *heavy* if $|T(a')| \geq |T(a)|/2$, and *light* otherwise. The *light level* of a vertex a is the number of light vertices on the path from a to the root of T. Each vertex has at most one heavy child and its light level is $O(\log n)$. The heavy paths are formed by the edges connecting a heavy child to its parent and the topmost vertex of a heavy path is light.

First we consider the case where G_2 is a dipath, and then the case where G_2 is a (in- or out-)tree.

A.1 Tree and Path

Based on the heavy-path decomposition of T, we describe a structure with $O(k \log n)$ reporting time for an in-tree and $O(\log n + k)$ reporting time for an out-tree. These bounds are inferior to the ones given in Section 4.2, but are achieved with simpler structures.

Consider the in-tree query first. Here our method is inspired by a routing scheme for trees by Thorup and Zwick [19]. Let $h(T(a))$ be the maximum label in $T(a)$. Obviously, we need to search $T(a)$ only if $h(T(a)) > j$. Let $h'(T(a))$ be the maximum label in $T(a) \setminus T(a')$, where a' is the heavy child of a (if it exists). The search proceeds top-down starting from b. Let a be the current vertex such that $h(T(a)) > j$. If $h(a) > j$, we report a. Then we identify the light children c of a such that $h(T(c)) > j$. Moreover, if a is the topmost vertex of its heavy path P, then we identify the vertices $d \in P$ such that $h'(T(d)) > j$. Then, we repeat this process at each vertex that we have identified. In order to locate these vertices quickly, for each vertex a we order its light children c by $h(T(c))$, and for each heavy path P we order the vertices $d \in P$ by $h'(T(d))$. Note that when we visit a light child c, the light level increases and there is at least one $x \in T(c)$ with $h(x) > j$. The $O(k \log n)$ bound follows.

For the out-tree query we use the same heavy-path decomposition and construct a Cartesian tree for each heavy path P. (See Section 4.1). The Cartesian tree for P stores the vertices in $a \in P$ according to coordinates $(x_1(a), x_2(a)) = (h_P(a), h_{G_2}(a))$. Furthermore, each vertex has a pointer to the topmost vertex of its heavy path, and each topmost vertex of a heavy path has a pointer to its parent in T. Let b be the query vertex and let Q be the tree path from the root of T to b. The goal is to identify the vertices $a \in Q$ with $h(a) > j$. We locate the heavy paths that intersect Q and query them individually. For each such heavy path P we identify the bottommost vertex $p \in P \cap Q$. The query for P has to report the vertices $a \in P$ such that $(x_1(a), x_2(a)) \geq (x_1(p), j)$.

As mentioned in Section 4.1, Cartesian trees can report these vertices in constant time per vertex. Since Q intersects $O(\log n)$ heavy paths the total query time is $O(\log n + k)$.

A.2 Two Trees

With the heavy-path decomposition method we can get an efficient join-reachability structure when one of the two trees is an out-tree. Without loss of generality we assume that G_1 is an out-tree. We perform the heavy-path decomposition of T as earlier and associate with each heavy path P a secondary data structure D_P; the choice of the secondary structure depends on the type of G_2. Also for each vertex $a \in P$ we store $h_P(a)$, the height of a in P. Given a query vertex b we want to report the ancestors a of b in T that reach b in G_2. Let Q be the path in T from the root to b. Our algorithm queries the structure D_P for each heavy path P that intersects Q. For each such heavy path P we identify the bottommost vertex $p \in P \cap Q$. If G_2 is an out-tree then we need to report the vertices $a \in P$ that satisfy $I_2(b) \subseteq I_2(a)$ and $h_P(a) \geq h_P(p)$. In this case, a suitable choice for D_P is a join-reachability structure for an out-tree and a path. Either of the two solutions we developed earlier (Sections 4.2 and A.1) achieves $O(\log |P| + k_P)$ reporting time (because here we need to locate b in D_P), where k_P is the number of reported vertices on P. This results to an overall $\langle n, \log^2 n + k \rangle$ structure. For the case where G_2 is an in-tree we need to report the vertices $a \in P$ that satisfy $I_2(a) \subseteq I_2(b)$ and $h_P(a) \geq h_P(p)$. Here we choose D_P to be a join-reachability structure for an in-tree and a path. Using the geometry-based structure of Section 4.2 results to an overall $\langle n, \log^2 n + k \rangle$ structure.

B Planar st-Graphs

Here we consider the case where G_1 is a *planar st-graph* [17] and G_2 is a dipath. A planar st-graph is planar acyclic digraph with a single source s and a single sink t, such that s and t are on the boundary of the same face. For these graphs Kameda [12] gave an $O(n)$-space structure that answers reachability queries in constant time. His algorithm performs two modified depth-first searches and assigns to each vertex a two integer labels $\ell_1(a)$ and $\ell_2(a)$ both in the range $[1, n]$. Kameda then shows that these labels satisfy the property that $a \rightsquigarrow_{G_1} b$ if and only if $\ell_1(a) \leq \ell_1(b)$ and $\ell_2(a) \leq \ell_2(b)$. Our data structure for the join-reachability problem also assigns each vertex a a third label $\ell_3(a)$ equal to the rank of a in G_3. Now each vertex corresponds to a point in a three-dimensional rank space and $a \rightsquigarrow_{\mathcal{J}} b$ if and only if $(\ell_1(a), \ell_2(a), \ell_3(a)) \leq (\ell_1(b), \ell_2(b), \ell_3(b))$. Using a three-dimensional dominance structure we can get an $\langle n, \log n + k \rangle$ join-reachability structure [11]. With minor adjustments we can get an efficient data structure for the more general class of *spherical st-graphs* [17], which are planar st-graphs without the requirement that s and t appear on the boundary of the same face. Tamassia and Tollis [17] showed how to reduce the reachability problem on these graphs to a reachability problem on planar st-graphs.

C Lattices

Let $(\leq, V)$ be a partial order. An element $z \in V$ is an *upper bound* of $x, y \in V$ if $x \leq z$ and $y \leq z$. If z is an upper bound of x, y and moreover $z \leq w$ for all upper bounds w of x, y then z is a *least upper bound* of x, y. Similarly, if $z \leq x$ and $z \leq y$ then z is a *lower bound* of x, y, and if $w \leq z$ for all lower bounds w of x, y then z is a *greatest lower bound* of x, y. A partial order $(\leq, V)$ is a *lattice* if any two $x, y \in V$ have both a least upper bound and a greatest lower bound. A *partial lattice* $(\leq, V)$ is a partial order that can be extended to lattice by adding elements s and t such that $s \leq x$ and $x \leq t$ for any $x \in V$. Any acyclic digraph $G = (V, A)$ has an associated partial order $P_G = (\leq, V)$ such that for $u, v \in V$, $u \leq v$ if and only if $u \leadsto_G v$. We say that G *satisfies the lattice property* if and only if its associated partial order is a lattice. For this class of digraphs, Talamo and Vocca presented an $O(n\sqrt{n})$-space structure that answers reachability queries in constant time [16]. Their structure is also capable of reporting the predecessors of a query vertex in $O(k)$ time. In this section we show how their structure can be extended in order to support efficient join-reachability. Roughly speaking, the Talamo-Vocca structure represents G as a collection of disjoint clusters with $O(\sqrt{n})$ vertices each. Moreover, we can assume that there are $\Theta(\sqrt{n})$ clusters; refer to [16] for details. Each cluster C has a root vertex c and consists of either a subset of the predecessors of c, in which case it is an *in-cluster*, or of a subset of the successors of c, in which case it is an *out-cluster*. A vertex $x \in C$ is an *internal vertex of* C; a vertex $x \notin C$ that either reaches or is reachable from a vertex in C is an *external vertex of* C. External vertices have the following key property: If x is an external vertex that reaches (resp. is reachable from) a subset $S \subseteq C$ then S contains the greatest lower bound (resp. least upper bound) of S, which is the *representative of* x *in* C. Now each vertex x is associated with a subgraph $G(x)$ consisting of two trees rooted at x; an internal spanning tree $I(x)$ and an external spanning tree $E(x)$. If the cluster C containing x is an in-cluster then the internal tree is an in-tree that contains the predecessors of x in C and the external tree is an out-tree that contains the external vertices of C with x as their representative. Similarly, if the cluster C containing x is an out-cluster then the internal tree is an in-tree that contains the successors of x in C and the external tree is an in-tree that contains the external vertices of C with x as their representative. In order to be able to report all the predecessors of a query vertex b this data structure can explicitly store the predecessors of each vertex x that are located in the same cluster with x. Since each cluster has $O(\sqrt{n})$ vertices the data structure still occupies $O(n\sqrt{n})$ space. The predecessors of b outside its cluster are the predecessors of the vertices that are representatives of b in other clusters for which b is an external vertex.

We can easily enhance the above structure so that it supports efficient join-reachability. We demonstrate this first for the case where G_2 is a dipath. For each vertex x we construct a list $L_1(x)$ of the internal predecessors of x sorted in increasing rank in G_2. Also we keep track of the minimum rank in G_2 of the vertices in $L_1(x)$. Then we construct another list $L_2(x)$ which contains the representatives of x in the clusters where x is an external vertex. Furthermore, $y \in L_2(x)$ only if the minimum rank in $L_1(y)$ is less than the rank of x. Now in order to report the vertices reaching b in the join-reachability graph, we report the vertices in $L_1(a)$ with rank less than b, for all $a \in L_2(b) \cup \{b\}$. Notice that we only visit clusters that contain a least one predecessor of b.

Therefore the reporting time is $O(k)$.

Now we show how the same bounds are achieved when G_2 is a rooted tree. Let $I_2(a) = [s_2(a), t_2(a)]$ be the depth-first search interval assigned to each vertex a in T_2, where T_2 is the undirected version of G_2. For each vertex x we construct a structure $D(x)$ that contains the vertices in $L_1(x)$. In order to report the vertices that reach b in the join-reachability graph, we query the structure $D(a)$ for all $a \in L_2(b) \cup \{b\}$. This structure reports the vertices $\gamma \in L_1(a)$ that satisfy $I_2(b) \subseteq I_2(\gamma)$ if G_2 is an out-tree, or $I_2(\gamma) \subseteq I_2(b)$ if G_2 is an in-tree. Note that during the construction of the join-reachability data structure we can ensure that $a \in L_2(b) \cup \{b\}$ only if the answer of $D(a)$ to query b is nonempty. Finally, we need to specify how $D(a)$ operates. If G_2 is an out-tree then $D(a)$ stores the intervals $I_2(\gamma)$ for all $\gamma \in L_1(a)$; a query asks for those $\gamma \in L_1(a)$ such that $I_2(\gamma)$ contains the point $s_2(b)$. Otherwise, when G_2 is an in-tree, $D(a)$ stores the points $s_2(\gamma)$ for all $\gamma \in L_1(a)$; now a query asks for those $\gamma \in L_1(a)$ such that $s_2(\gamma)$ is contained in $I_2(b)$. Such queries can be answered optimally by Chazelle's *interval overlap* structure [4], which gives us the desired result.

Theorem C.1. *Given a lattice G_1 and an unoriented tree G_2 with n vertices we can construct an $\langle n\sqrt{n}, k\rangle$ join-reachability data structure.*