Structure of Random 312-Avoiding Permutations

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September 5, 2018

Abstract

We evaluate the probabilities of various events under the uniform distribution on the set of 312-avoiding permutations of $1, \ldots, N$. We derive exact formulas for the probability that the i^{th} element of a random permutation is a specific value less than i, and for joint probabilities of two such events. In addition, we obtain asymptotic approximations to these probabilities for large N when the elements are not close to the boundaries or to each other. We also evaluate the probability that the graph of a random 312-avoiding permutation has k specified decreasing points, and we show that for large N the points below the diagonal look like trajectories of a random walk.

1 Introduction

Let S_N denote the set of permutations of numbers $1, \ldots, N$ for each positive integer N. Given $\tau \in S_k$ (with $k \leq N$), we say that a permutation $\sigma = \sigma_1 \ldots \sigma_N$ avoids the pattern τ (or "σ is τ-avoiding") if there is no subsequence of σ with length k having the same relative order as τ. The set of τ-avoiding permutations in S_N is denoted by $S_N(\tau)$. For example the permutation 435621 avoids the 312 pattern and hence $435621 \in S_6(312)$ but it is not an element of $S_6(321)$

since 432, 431, 421, 321, 521, and 621 are subsequences in 435621 having the 321 pattern. A permutation $\sigma = \sigma_1 \dots \sigma_N$ can be represented as a function σ that maps i to $\sigma(i) = \sigma_i$. The graph of this function is the set of N points $\{(i, \sigma_i) : i = 1 \dots N\}$ (see Figure 1). Points of the form $(i, \sigma_i) = (i, i)$ are said to be on the diagonal of the graph of σ (these correspond to fixed points of the permutation).

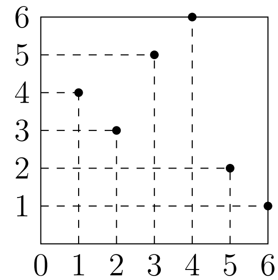

Figure 1: $\sigma = 435621 \in S_6$ viewed as a function $i \mapsto \sigma_i$. This is a 312-avoiding permutation.

The study of pattern-avoiding permutations often reveals connections to other combinatorial objects. For example in [3] it is shown that there is a bijection between 1342-avoiding permutations and plane forests of $\beta(0,1)$-trees, as well as with ordered collections of rooted bicubic planar maps.

Among other well studied connections of permutations excluding or including certain patterns in the literature are Kazhdan-Lusztig polynomials, singularities of Schubert varieties, Chebyshev polynomials, rook polynomials for Ferrers boards. In a recent book [7] Kitaev goes through a vast amount of literature to point out the connections of permutations with other mathematical objects. Also, Bouvel and Rossin [5] show how permutation patterns are related to problems in computational biology.

One of the initial motivations to study pattern avoiding permutations came from computer science. A basic problem in computer science is sorting n distinct elements in increasing order. Stack sorting is an algorithm that does the sorting operation efficiently, although it only works on some permutations. It was observed that a permutation is stack sortable if and only if it avoids the pattern 231. A detailed explanation of the connection can be found in Bóna's book [4]. An important reference on stack sorting is Knuth's book [8].

Two recent papers of Madras and Liu [11] and Atapour and Madras [1] present numerical and probabilistic approaches to investigate the shapes of random pattern avoiding permutations, mainly of length three, four and five. Both papers include Monte Carlo simulations suggesting the limiting distributions of the positions of points of the permutations. For example Atapour and Madras [1] present the result of Monte Carlo simulation (similar to Figure 2 here) which suggests that typical 312-avoiding permutations are accumulated near the diagonal as well as below the diagonal. To generate random 312-avoiding permutations, they run a Markov chain on $S_N(312)$ which is irreducible, symmetric and aperiodic [11], and hence has the uniform distribution on $S_N(312)$ as its limiting distribution. They further prove the following results that support the findings of the simulations.

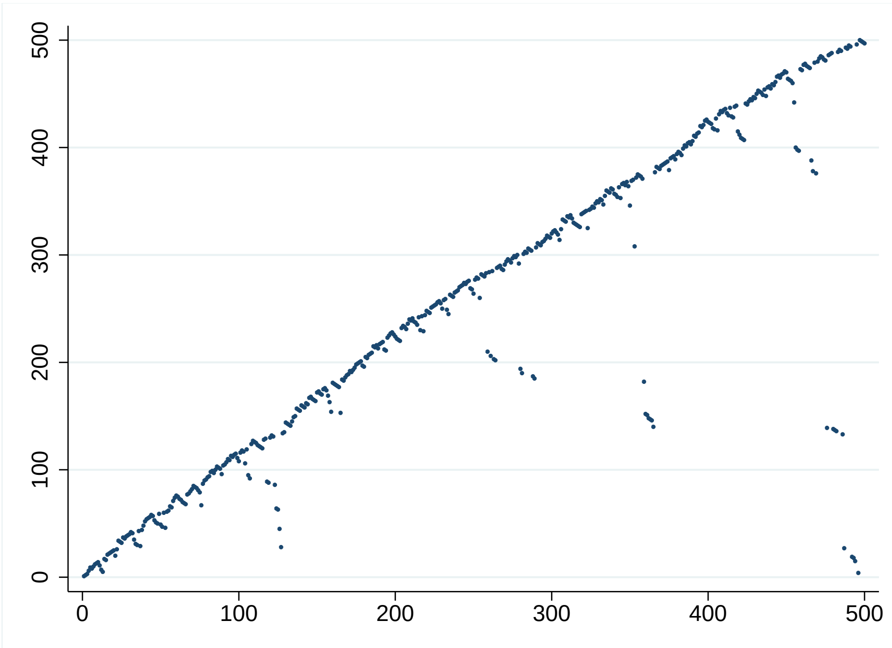

Figure 2: Graph of a randomly generated 312-avoiding permutation with N=500

Definition 1.1. Consider a pattern τ. For each $N \geq 1$, let P_N^τ be the uniform distribution on the set $S_N(\tau)$; that is $P_N^\tau(A) = |A| \,/\, |S_N(\tau)|$. For simplicity we shall write P_N to denote P_N^{312} throughout this paper.

Theorem 1.2. *[1] Define the function K^* on the unit square $[0,1]^2$ by*

$$K^*(s,t) = \begin{cases} 1 & \text{if } 0 \leq t \leq s \leq 1, \\ \frac{1}{4}\frac{(2-s-t)^{2-s-t}(s+t)^{s+t}}{(1-s)^{1-s}(1-t)^{1-t}t^t s^s} & \text{if } 0 \leq s \leq t \leq 1. \end{cases}$$

Then for any relatively open subset D of $[0,1]^2$, we have

$$\lim_{N\to\infty} \left[P_N \left\{ \left(\frac{i}{N}, \frac{\sigma_i}{N} \right) \in D \text{ for some } i \in \{1,\ldots,N\} \right\} \right]^{1/N} = \sup \{ K^*(s,t) : (s,t) \in D \}.$$

Theorem 1.2 concludes that it is rare to have points of the graph well above the diagonal (since $K^*(s,t) < 1$ for $0 < s < t < 1$), but it is not rare to have points well below the diagonal. A related result is the following, which says that the number of points well below diagonal is $o(N)$ with high probability.

Proposition 1.3. *[1] Let $\delta > 0$, $0 < t < 1$ and $K_N(\sigma, \delta N) = |\{i : \sigma_i < i - \delta N\}|$. Then*

$$\lim_{N\to\infty} [P_N(K_N(\sigma, \delta N) > tN)]^{1/N} < 1.$$

The present paper is mainly motivated by these results in [1]. In this paper we investigate the probabilities of having a 312-avoiding permutation that has one or two specified points below the diagonal (i.e., satisfying $\sigma_i = j$ for specified i and j with $j < i$). We also extend our results to k decreasing points below the diagonal. Exact evaluations of the probabilities and the approximation results for these probabilities for large N are stated in the next section. Our main theorems imply that the probability of obtaining 312-avoiding permutations with one specified point below the diagonal is of order $N^{-3/2}$, and the probability of obtaining 312-avoiding permutations with two (well separated) specified points below the diagonal is of order N^{-3}. However, the two-point probability is not approximated by the product of the corresponding one-point probabilities. In particular, Corollary 2.13 describes situations in which two one-point events are positively or negatively correlated. Exact combinatorial results for 312-avoiding permutations with specified points above the diagonal could also be calculated in similar manner; however in this paper we concentrate our attention below the diagonal, as motivated by [1].

While this paper was being written, S. Miner and I. Pak completed a preprint (now [12]) that investigates probabilities of random 123- and 132-avoiding permutations with one specified point and their asymptotics. In particular, [12] independently proves Theorems 2.3 and 2.7, and extends these results considerably in directions that we have not pursued.

This paper is organized as follows. Section 2 collects the main results of this paper. Section 3 states the definitions, the basic terminology and results needed for the remaining parts of the paper. Section 4 consists of the proofs of Theorems 2.3, 2.8 and 2.10 which give exact formulas for probabilities of obtaining a 312-avoiding permutation that has one or two specified points below the diagonal. Theorem 2.3 treats the one point case. Theorem 2.8 considers the case that $\sigma_{i_1} = j_1 > \ldots > \sigma_{i_k} = j_k$ with $j_k < \ldots < j_1 < i_1 < \ldots < i_k$. Finally, Theorem 2.10 is the case that $\sigma_{i_1} = j_1 < \sigma_{i_2} = j_2$ with $j_1 < i_1 < i_2$ and $j_2 < i_2$. Section 5 gives the proofs of Theorems 2.7, 2.9 and 2.11, which are asymptotic approximations of the probabilities calculated in Theorems 2.3, 2.8 and 2.10 respectively. Section 6 proves results related to the limiting distribution of σ near the lower right corner of the square $[1, N]^2$, as well as the limiting conditional distribution of σ northwest of a given point (i, j) below the diagonal given that $\sigma_i = j$.

2 Main Results

Let C_N denote the Catalan number, $C_N = \frac{1}{N+1}\binom{2N}{N}$ for $N \geq 0$. The proof of the well known result that $|S_N(\tau)| = C_N$ for $\tau \in S_3$ and $N \geq 1$ can be found in [4] or [13].

Definition 2.1. For $N \in \mathbb{N}$ and $i, j \in [1, N]$, let

$$S^\bullet(N, i, j) = \{\sigma \in S_N(312) : \sigma_i = j\} \qquad \text{and}$$

$$S^{\square}(N,i,j) \;=\; \{\sigma \in S_N(312) : \sigma_i = j \text{ and } \sigma_k < j \text{ for all } k \in [1,i)\}.$$

Remark 2.2. Observe that $|S^{\square}(N,i,j)| = 0$ if $j < i$. We also have

$$\left|S^{\square}(N,i,j)\right| \;=\; \frac{(j-i+1)^2}{j(N-i+1)}\binom{2N-i-j}{N-i}\binom{i+j-2}{j-1} \quad \text{whenever } 1 \le i \le j \le N.$$

For $i < j$, this was proven in [1]. For the case $i = j$, we note that for every $\sigma \in S^{\square}(N,j,j)$, we have $\sigma_k > j$ for every $k > j$. Therefore $\left|S^{\square}(N,j,j)\right| = |S_{j-1}(312)| \times |S_{N-j}(312)| = C_{j-1}C_{N-j} = \frac{1}{j(N-j+1)}\binom{2N-2j}{N-j}\binom{2j-2}{j-1}$.

Our first result gives the cardinality of $S^{\bullet}(N, N-t, j)$.

Theorem 2.3. *Let $j < N - t$. Then*

$$|S^{\bullet}(N,N-t,j)| = \sum_{i_0=\max\{1,j-t\}}^{j} C_{N-t-i_0}\frac{(j-i_0+1)^2}{j(t+1)}\binom{i_0+2t-j}{t}\binom{i_0+j-2}{j-1}.$$

We shall prove Theorem 2.3 by constructing a natural bijection between $S^{\bullet}(N,N-t,j)$ and $\bigcup_{i_0=\max\{1,j-t\}}^{j} S_{N-t-i_0}(312) \times S^{\square}(t+i_0,i_0,j)$ for fixed N, t and j (see Definition 4.5). We shall use this bijection repeatedly for the proof of Theorem 2.8, and a closely related bijection for the proof of Theorem 2.10.

Remark 2.4. It is not hard to check that $|S^{\bullet}(N,N-t,j)| = |S^{\bullet}(N,N-j,t)|$.

Remark 2.5. Theorem 2.3 also gives the probability $P_N(S^{\bullet}(N,i,j))$, since by definition it is equal to $\frac{|S^{\bullet}(N,N-t,j)|}{|S_N(312)|}$. This applies to Theorems 2.8 and 2.10 as well.

Notation 2.6. In preparation for upcoming results, we state our conventions on asymptotics. We write $f(N) \sim g(N)$ to mean that $\lim_{N\to\infty} f(N)/g(N) = 1$. We write $f(N) \asymp g(N)$ to mean that there exists a constant $C > 0$ such that $C > g(N)/f(N) > C^{-1}$ for all sufficiently large N. We write $f(N) \prec\!\prec g(N)$ to mean that there is an $\epsilon > 0$ such that $g(N) - f(N) > \epsilon N$ for all sufficiently large N.

For example, the statement "$f(i,j,N) = O(N)$ for $i \prec\!\prec j$" would mean that for any $\epsilon > 0$, there exists a C and N_1 such that $|f(i,j,N)| \le CN$ for all i, j, and N such that $j - i > \epsilon N$ and $N \ge N_1$ (where C and N_1 can depend on ϵ).

Theorem 2.7. *Fix $0 < \theta < \frac{1}{6}$. Then*

$$P_N(S^{\bullet}(N,N-t,j)) = \frac{N^{-3/2}}{2\sqrt{\pi}\left(1-\frac{N-t-j}{N}\right)^{3/2}\left(\frac{N-t-j}{N}\right)^{3/2}}(1+O(N^{3\theta-\frac{1}{2}}))$$

for $0 \prec\!\prec j \prec\!\prec i \prec\!\prec N$.

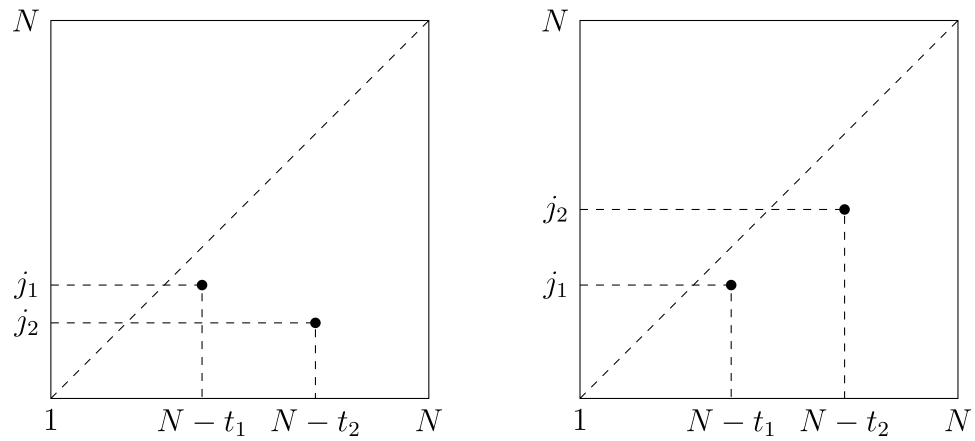

Figure 3: These diagrams represents the requirement that a permutation has specified points $\sigma_{N-t_1} = j_1$ and $\sigma_{N-t_2} = j_2$ where $j_1 < N - t_1$, $j_2 < N - t_2$, and $N - t_1 < N - t_2$. The left diagram corresponds to Theorem 2.8 for $k = 2$ and the right one corresponds to Theorem 2.10.

Theorem 2.8. *Let $j_k < j_{k-1} < \ldots < j_1 < N - t_1 < \ldots < N - t_k$ and define*

$$S^{\searrow_k}(N) \equiv S^{\searrow_k}(N, N-t_1, \ldots, N-t_k, j_1, \ldots, j_k) = \{\sigma \in S_N(312) : \sigma_{N-t_1} = j_1, \ldots \sigma_{N-t_k} = j_k\}.$$

Then,

$$\left|S^{\searrow_k}(N)\right| = \sum_{i=\max\{1, j_k - t_k\}}^{j_k} \frac{(j_k - i + 1)^2}{j_k(t_k + 1)} \binom{i + 2t_k - j_k}{t_k} \binom{i + j_k - 2}{j_k - 1} \times$$
$$\left|S^{\searrow_{k-1}}(N - t_k - i, N - t_1 - i + 1 \ldots, N - t_{k-1} - i + 1, j_1 - j_k, j_2 - j_k, \ldots, j_{k-1} - j_k)\right|.$$

In particular for $k = 2$ we have

$$\left|S^{\searrow_2}(N)\right| \equiv \left|S^{\searrow}(N)\right| = \sum_{i_1=\max\{1, j_2 - t_2\}}^{j_2} \frac{(j_2 - i_1 + 1)^2}{j_2(t_2 + 1)} \binom{i_1 + 2t_2 - j_2}{t_2} \binom{i_1 + j_2 - 2}{j_2 - 1} \times$$
$$\sum_{i_0=\max\{1, (j_1 - j_2) - (t_1 - t_2 - 1)\}}^{j_1 - j_2} C_{N - t_1 - i_1 - i_0 + 1} \frac{(j_1 - j_2 - i_0 + 1)^2}{(j_1 - j_2)(t_1 - t_2)} \times$$
$$\binom{i_0 + 2(t_1 - t_2 - 1) - (j_1 - j_2)}{t_1 - t_2 - 1} \binom{i_0 + j_1 - j_2 - 2}{j_1 - j_2 - 1}.$$

Theorem 2.9. *Fix $0 < \theta < \frac{1}{6}$. Then*

$$P_N\left(S^{\searrow}(N)\right) = \frac{1}{4\pi} \frac{N^{-3}}{\left(\frac{(N - t_2 - j_2) - (N - t_1 - j_1)}{N}\right)^{3/2} \left(\frac{N - t_1 - j_1}{N}\right)^{3/2} \left(1 - \frac{N - t_2 - j_2}{N}\right)^{3/2}} (1 + O(N^{3\theta - 1/2}))$$

for $0 \prec\prec j_2 \prec\prec j_1 \prec\prec N - t_1 \prec\prec N - t_2 \prec\prec N$.

Theorem 2.10. *For $N, j_1, j_2, t_1, t_2 \in \mathbb{N}$ such that $j_1 < N - t_1 < N - t_2$, $j_1 < j_2 < N - t_2$, define $S^{\nearrow}(N) \equiv S^{\nearrow}(N, N-t_1, N-t_2, j_1, j_2) = \{\sigma \in S_N(312) : \sigma_{N-t_1} = j_1, \sigma_{N-t_2} = j_2\}$.*
(a) If $j_2 < N - t_1 + 1$, then $\left|S^{\nearrow}(N)\right| = 0$.
(b) If $j_2 \geq N - t_1 + 1$, then

$$\begin{aligned}\left|S^{\nearrow}(N)\right| \; = & \sum_{i_1=\max\{1, j_1-j_2+N-t_1+1\}}^{j_1} \; \sum_{i_2=\max\{N-t_1+1, j_2-t_2\}}^{j_2} C_{N-t_1-i_1} C_{N-t_2-i_2} \times \\ & \frac{(j_1-i_1+1)(j_2-i_2+1)}{j_1(t_2+1)} \binom{j_1+i_1-2}{j_1-1} \binom{i_2+2t_2-j_2}{t_2} \times \\ & \left(\binom{i_1+i_2+j_2-j_1-2(N-t_1+1)}{j_2+(i_1-j_1-1)-(N-t_1)} - \binom{i_1+i_2+j_2-j_1-2(N-t_1+1)}{j_2-(N-t_1)} \right).\end{aligned}$$

Theorem 2.11. *Fix $0 < \theta < \frac{1}{6}$. Then,*

$$P_N(S^{\nearrow}(N)) = \\ \frac{N^{-3}}{4\pi \left(1 - \frac{(N-t_1-j_1)+(N-t_2-j_2)}{N}\right)^{3/2} \left(\frac{N-t_1-j_1}{N}\right)^{3/2} \left(\frac{N-t_2-j_2}{N}\right)^{3/2}} (1 + O(N^{3\theta - \frac{1}{2}}))$$

for $0 \prec\prec j_1 \prec\prec N - t_1 \prec\prec j_2 \prec\prec N - t_2 \prec\prec N$.

The next corollary restates Theorem 2.7 for the special case that $j = \lfloor \alpha N \rfloor$, $i = N - t = \lfloor \beta N \rfloor$. Corollary 2.13 contains the analogous statements for Theorems 2.9 and 2.11. Corollary 2.13 also frames these results in terms of a random field corresponding to points in the graph of a random σ.

Corollary 2.12. *Assume that $0 < \alpha < \beta < 1$. Then*

$$P_N(S^{\bullet}(N, \lfloor \beta N \rfloor, \lfloor \alpha N \rfloor)) \sim \frac{N^{-3/2}}{2\sqrt{\pi}(1-(\beta-\alpha))^{3/2}(\beta-\alpha)^{3/2}}.$$

Corollary 2.13. *Let $i_1 = \lfloor \beta_1 N \rfloor, i_2 = \lfloor \beta_2 N \rfloor, j_1 = \lfloor \alpha_1 N \rfloor, j_2 = \lfloor \alpha_2 N \rfloor$. Also let $\Delta_1 = \beta_1 - \alpha_1$ and $\Delta_2 = \beta_2 - \alpha_2$. For a random σ having distribution P_N, let $Z(i, j)$ be the indicator of the event that $\sigma_i = j$, and let Cov_N denote covariance with respect to P_N.*
(a) Assume that $0 < \alpha_2 < \alpha_1 < \beta_1 < \beta_2 < 1$. Then

$$P_N\left(S^{\searrow}(N)\right) \sim \frac{1}{4\pi} \frac{N^{-3}}{[(\Delta_2-\Delta_1)\Delta_1(1-\Delta_2)]^{3/2}} \quad \text{and}$$

$$\lim_{N\to\infty} N^3 \, \mathrm{Cov}_N(Z(i_1, j_1), Z(i_2, j_2)) = \frac{\left(\left[\frac{\Delta_2(1-\Delta_1)}{\Delta_2-\Delta_1}\right]^{3/2} - 1\right)}{4\pi \left[\Delta_1 \Delta_2 (1-\Delta_1)(1-\Delta_2)\right]^{3/2}} > 0 \, .$$

(b) Assume that $0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < 1$. *Then*

$$P_N\left(S^{\nearrow}(N)\right) \;\sim\; \frac{1}{4\pi}\frac{N^{-3}}{[(1-\Delta_1-\Delta_2)\Delta_1\Delta_2]^{3/2}} \qquad \textit{and}$$

$$\lim_{N\to\infty} N^3\,\mathrm{Cov}_N(Z(i_1,j_1),Z(i_2,j_2)) \;=\; \frac{\left(\left[\frac{(1-\Delta_1)(1-\Delta_2)}{1-\Delta_1-\Delta_2}\right]^{3/2}-1\right)}{4\pi\,[\Delta_1\Delta_2(1-\Delta_1)(1-\Delta_2)]^{3/2}} \;>\; 0\,.$$

(c) Assume that $0 < \alpha_1 < \alpha_2 < \beta_1 < \beta_2 < 1$. *Then* $P_N\left(S^{\nearrow}(N)\right) \;=\; 0$ *and*

$$\lim_{N\to\infty} N^3\,\mathrm{Cov}_N(Z(i_1,j_1),Z(i_2,j_2)) \;=\; \frac{-1}{4\pi\,[\Delta_1\Delta_2(1-\Delta_1)(1-\Delta_2)]^{3/2}} \;<\; 0\,. \tag{1}$$

The above asymptotic results hold for points well below the diagonal and well away from the sides of the square $[1,N]^2$.

The next results concern the lower right corner of the square. Our starting point is Proposition 2.15, of which part (a) has also been observed by Miner and Pak [12].

Definition 2.14. For all $a,b\in\mathbb{N}$ define

$$\rho(a,b) \;=\; \sum_{i_0=\max\{1,b-a+1\}}^{b} \frac{(b-i_0+1)^2}{ba4^{i_0+a-1}}\binom{i_0+2(a-1)-b}{a-1}\binom{i_0+b-2}{b-1}.$$

We note that we can also write $\rho(a,b) \;=\; \sum_{i=\max\{1,b-a+1\}}^{b}|S^{\square}(i+a-1,i,b)|/4^{i+a-1}$ (recall Definition 2.1 and Remark 2.2).

Proposition 2.15. *(a) Fix* $a,b\in\mathbb{N}$. *Then*

$$\lim_{N\to\infty} P_N(S^{\bullet}(N,N{-}a{+}1,b)) \;=\; \rho(a,b).$$

(b) More generally, let $k\in\mathbb{N}$ *and let* $a_1,\ldots,a_k,b_1,\ldots b_k\in\mathbb{N}$. *Define* $A_m = \sum_{l=1}^m a_l$, $B_m = \sum_{l=1}^m b_l$ *for* $m=1,\ldots,k$. *Then*

$$P_N(S^{\searrow k}(N,N{-}A_k{+}1,\ldots,N{-}A_1{+}1,B_k,\ldots,B_1)) \;=\; \left[\prod_{v=1}^{k}\rho(a_v,b_v)\right](1+O(N^{-1})).$$

By Remark 2.4, we also observe that ρ is symmetric in a and b.

Our next task is to expand part (b) above into a more complete limiting description of σ near the lower right corner of the square $[1,N]^2$. Since we consider $N\to\infty$, we shall translate the lower right corner so that the square expands to fill the second quadrant.

Definition 2.16. Let $Q := \{(-i,j) : i,j\in\mathbb{N}\}$ be the integer points inside the second quadrant. For $N\in\mathbb{N}$, let $W_N := [-N,-1]\times[1,N]$ be the $N\times N$ square in the lower right corner of Q.

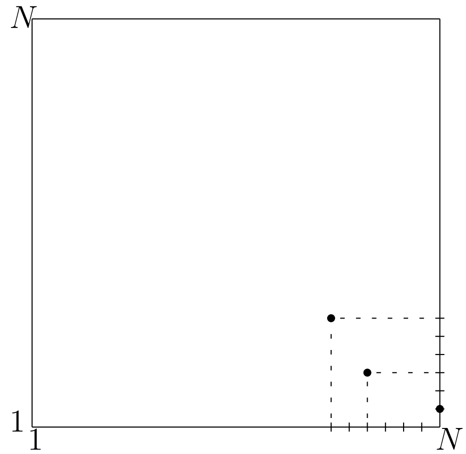

Figure 4: The solid circles show three points of the graph of a permutation in $S^{\searrow 3}(N, N-4, N-6, 2, 4, 7)$. Such a permutation gives the values $X^N_{(-1,2)} = 1 = X^N_{(-5,4)} = X^N_{(-7,7)}$ (and hence $X^N_{(-7,j)} = 0$ for all $j \neq 7$, etc.).

For each N, define the collection of (dependent) binary random variables $\{X^N_{(-i,j)} : (-i, j) \in Q\}$ by

$$X^N_{(-i,j)} = \begin{cases} 1 & \text{if } (-i, j) \in W_N \text{ and } \sigma_{N-i+1} = j \\ 0 & \text{otherwise} \end{cases}$$

where σ has the distribution P_N. We will often write "q" (or sometimes "r" or "s") to represent a generic element $(-i, j)$ of Q; e.g. we can refer to the above collection as $\{X^N_q : q \in Q\}$.

See Figure 4. Note that the random set $\{q \in Q : X^N_q = 1\}$ is essentially the graph of the random permutation σ.

With the above definition, Proposition 2.15(b) concludes that the probability that $X^N_{(-A_m,B_m)} = 1$ for every $m = 1, \ldots, k$ converges to $\rho(a_1, \ldots, b_k)$ as $N \to \infty$. Our next result builds on this to show that there is a limiting collection of random variables indexed by points of Q.

Theorem 2.17. *There exists a collection of $\{0, 1\}$-valued random variables $\{X_q : q \in Q\}$ (with joint distribution P_∞) such that for every finite subset C of Q, the collection $\{X^N_q : q \in C\}$ converges in distribution to $\{X_q : q \in C\}$ as $N \to \infty$.*

The product form of the limit in Proposition 2.15(b) implies that the limiting collection of random variables $\{X_q : q \in Q\}$ has a kind of two-dimensional regenerative property, analogous to the more standard regenerative property on $\mathbb{N}$ possessed by discrete renewal processes (see [6]). Our two-dimensional renewal structure is fully described in Theorem 2.19, using the following notation.

Definition 2.18. For $a, b \in \mathbb{N}$, let

$$\pi(-a,b) \;=\; \frac{C_{a-1}C_{b-1}}{4^{a+b-1}}.$$

It is not hard to see that π is a probability distribution on Q. (Indeed, using the well-known Catalan generating function $G(z) = \sum_{i=0}^{\infty} C_i z^i = (1-\sqrt{1-4z})/2z$ (e.g. [4]), we have $\sum_{a,b=1}^{\infty} \pi(-a,b) = \frac{1}{4}\, G\left(\frac{1}{4}\right)^2 = 1$.) This distribution plays a key role in the following theorem.

Theorem 2.19. *The set $W^* = \{q \in Q : X_q = 1\}$ is an infinite random set of the form $\{\vec{V}_m : m \in \mathbb{N}\}$ where $\{(\vec{V}_m - \vec{V}_{m-1}) : m \in \mathbb{N}\}$ are i.i.d. Q-valued random vectors with distribution $\pi(-a,b)$ [writing $\vec{V}_0 = (0,0)$]. Moreover, the components of $\vec{V}_1$ have infinite means.*

In particular, Theorem 2.19 tells us that, with probability one, the random set $W^* = \{q \in Q : X_q = 1\}$ is an infinite sequence of points $\{(-A_m, B_m)\}$ such that the sequences $\{A_m\}$ and $\{B_m\}$ are both strictly increasing. Moreover, the distribution of W^* is exactly that of the set of points visited by a random walk with jump distribution π. Observe that such a random walk only jumps to the north and west. (Here, "random walk" denotes a process which is the sequence of partial sums of an i.i.d. sequence of vectors.)

To illustrate our results, note that Proposition 2.15(b) tells us that the probability of observing the three points shown in Figure 4 is approximately $\rho(1,2)\rho(4,2)\rho(2,3)$ for large N. In contrast, Theorem 2.19 says that the probability of observing these three points *and having no other points in* $[N-6, N] \times [1,7]$ is approximately $\pi(-1,2)\pi(-4,2)\pi(-2,3)$.

Our final theorem says that if we condition on the event $\{\sigma_{N-t+1} = j\}$ (i.e. $\{X^N_{(-t,j)} = 1\}$) and let N get large while $(N-t, j)$ remains well below the diagonal, then the conditional distribution of points above and to the left of $(N-t, j)$ (and near $(N-t, j)$) approaches the (unconditional) distribution of points in the lower right corner of the square $[1, N]^2$.

Theorem 2.20. *Let D and F be disjoint finite subsets of Q. Then*

$$\lim_{N-t-j\to\infty} P_N(X^N_{q+(-t,j)} = 1 \;\; \forall q \in D \textit{ and } X^N_{r+(-t,j)} = 0 \;\; \forall r \in F \,|\, X^N_{(-t,j)} = 1)$$
$$= P_\infty(X_q = 1 \;\; \forall q \in D \textit{ and } X_r = 0 \;\; \forall r \in F)\,.$$

3 Terminology and Useful Results

3.1 Pattern Avoiding Permutations and Dyck Paths

Although we focus on the pattern 312, we first give a general definition of pattern avoidance.

Definition 3.1. Let k be a positive integer $k \geq 2$ and $\tau = \tau_1 \ldots \tau_k \in S_k$.

(a) We say that a string of k distinct integers $\alpha_1 \ldots \alpha_k$ forms the pattern τ if for each $i = 1, \ldots, k$, α_i is the τ_ith smallest element of $\{\alpha_1, \ldots, \alpha_k\}$. In this case we also write $\tau = \mathbf{Patt}(\alpha_1, \ldots, \alpha_k)$.

(b) We say that $\sigma \in S_N$ contains pattern τ if some k-element subsequence $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_k}$ of σ occurs with the same relative order as $\tau = \tau_1 \ldots \tau_k$, i.e. $\tau = \mathbf{Patt}(\sigma_{i_1}, \ldots, \sigma_{i_k})$. If σ does not contain the pattern τ, then we say σ avoids τ. Let $S_N(\tau)$ be the set of permutations of $\{1, \ldots N\}$ that avoid τ.

Explicitly, a permutation σ avoids the pattern 312 if σ has no subsequence of three elements that has same relative order as 312, i.e., if there does not exist $i_1 < i_2 < i_3$ such that $\sigma_{i_1} > \sigma_{i_3} > \sigma_{i_2}$. See Figure 1 for an example.

Definition 3.2. A Dyck segment from (X_0, Y_0) to (X_K, Y_K) is a sequence $(X_0, Y_0), (X_1, Y_1), \ldots,$ (X_K, Y_K) in $\mathbb{Z}^2$ such that and X_0 and Y_0 are nonnegative integers and

$$X_i - X_{i-1} = 1,\;\; Y_i - Y_{i-1} \in \{-1, +1\}\,, \text{ and }\;\; Y_i \geq 0 \;\text{ for every } i = 1, \ldots, K. \tag{2}$$

A Dyck path of length $2L$ is a Dyck segment from $(0, 0)$ to $(2L, 0)$. See Figure 6 below for an example.

Lemma 3.3. *[14] The set D of all Dyck segments from (X_0, Y_0) to (X_K, Y_K) has $|D| = 0$ iff at least one of the following conditions holds:*
(a) $X_0 > X_K$,
(b) $|Y_K - Y_0| > X_K - X_0$,
(c) $Y_K - Y_0 \neq X_K - X_0 \,(mod\, 2)$ (i.e., one of $Y_K - Y_0$ or $X_K - X_0$ is even and the other is odd).
Otherwise,

$$|D| \;=\; \binom{X_K - X_0}{\frac{X_K - X_0 + |Y_K - Y_0|}{2}} - \binom{X_K - X_0}{\frac{X_K - X_0 + Y_K + Y_0 + 2}{2}} \;=\; \binom{X_K - X_0}{\frac{X_K - X_0 + Y_K - Y_0}{2}} - \binom{X_K - X_0}{\frac{X_K - X_0 + Y_K + Y_0 + 2}{2}}.$$

Remark 3.4. For Dyck segments from $(0, 0)$ to (X_K, Y_K), the result of Lemma 3.3 can be rewritten as

$$|D| \;=\; \frac{2Y_k + 2}{X_k + Y_k + 2}\binom{X_k}{\frac{X_k + Y_k}{2}}.$$

Definition 3.5. (X_i, Y_i) is a peak of a given Dyck path if $Y_{i-1} = Y_{i+1} = Y_i - 1$.

Krattenthaler [9] proves that there is a bijection between $S_N(132)$ and the set D_N of all Dyck paths of length $2N$. We restate his result by replacing 132-avoiding permutations by its complement 312-avoiding permutations. A different bijective proof between Dyck paths and $S_N(312)$ can also be found in [2].

Let $\pi \in S_N(312)$ and $\pi = \pi_1\pi_2 \ldots \pi_N$. Following the steps of Krattenthaler's proof we first determine the left-to-right maxima in π. A left-to-right maximum is an element π_i which is greater than all the elements to its left, i.e., larger than all π_j with $j < i$. For example left-to-right maxima in the permutation 25647318 are $2, 5, 6, 7$ and 8. Let the left-to-right maxima in π be $M_1, M_2, \ldots, M_s$ so that $\pi = M_1W_1M_2W_2 \ldots M_sW_s$ where W_i is the (possibly empty) subword of π in between M_i and M_{i+1}. Then any left-to-right maximum is translated into $M_i - M_{i-1}$ up-steps (with the convention that $M_0 = 0$). Any subword W_i is translated into $|W_i| + 1$ down-steps (where $|W_i|$ denotes the number of elements of W_i). Hence, if $\pi_i = M_t$ for some $t \leq i$, then the corresponding point on the Dyck path has its horizontal component equal to $\sum_{k=1}^{t}(M_k - M_{k-1}) + \sum_{k=1}^{t-1} |W_k| + (t-1)$. Observe that $\sum_{k=1}^{t-1} |W_k|$ counts the number of elements that are not a maximum up until the t-th maximum M_t, and $t-1$ counts the previous maxima $M_1, \ldots, M_{t-1}$. Together they count the number of all positions to the left of i, which is $i-1$, i.e., $\sum_{k=1}^{t-1} |W_k| + (t-1) = i-1$. Hence, the horizontal component of the point on the Dyck path corresponding to $\pi_i = M_t$ is $M_t + (i-1)$. Similarly, the vertical component of the point on the Dyck path corresponding to $\pi_i = M_t$ is $M_t - (i-1)$. Therefore, the left-to-right maximum M_t at position i corresponds to a peak $(M_t+(i-1), M_t-(i-1))$ in the corresponding Dyck path. For example, Figures 5 and 6 show the correspondence between the permutation $\pi = 25647318$ and its Dyck path. The fourth left-to-right maximum in Figure 5, namely 7 (circled), corresponds to the peak $(11, 3)$ in Figure 6 (dashed lines). Observe that a clockwise 45° rotation of the dashed lines in Figure 5 produces the diagonal lines and horizontal axis of Figure 6. Explicitly, this rotation maps a point (x, y) in Figure 5 to the point $(x+y-1, y-x+1)$ in Figure 6.

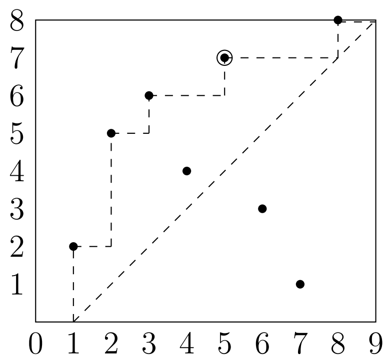

Figure 5: $\pi = 25647318$ with diagonal segment corresponding to x-axis in Figure 6. Here, $M_1 = 2$, $M_4 = 7$, W_1 is empty, and $W_4 = 31$.

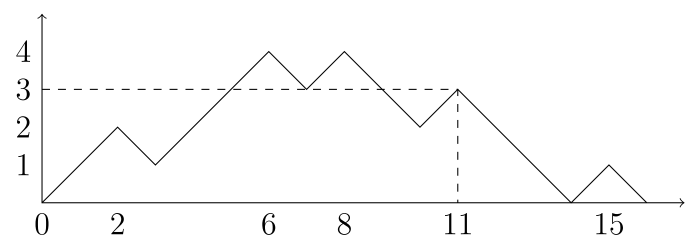

Figure 6: Dyck path of length 16 corresponding to $\pi = 25647318$.

3.2 Approximations of Integrals

We first record two results that will be useful in Section 5 for obtaining the asymptotic behaviors of sums. For a function g, let $\|g\|_\infty$ be the supremum of $|g|$ over its domain.

Proposition 3.6. *Let $f : [0, \infty) \to \mathbb{R}$ be continuous and differentiable. Let $\Delta > 0$, $R \in \mathbb{N}$, and define $X_i = i\Delta$ for $i = 1, 2, \ldots, R$. Also let $J = \sum_{i=1}^{R} f(X_i)\Delta$ and $I = \int_0^{R\Delta} f(x)\,dx$. Then*

$$|J - I| \ \leq\ \|f'\|_\infty \frac{\Delta^2 R}{2}.$$

Proposition 3.7. *Let $\tilde{f} : [0,\infty)^2 \to \mathbb{R}$ be continuous and differentiable. Let $\Delta > 0$, $R \in \mathbb{N}$, and $Y_i = X_i = i\Delta$ for $i = 1, 2, \ldots, R$. Let $\tilde{J} = \sum_{i=1}^{R} \sum_{j=1}^{R} \tilde{f}(X_i, Y_j)\Delta^2$, $\tilde{I} = \int_0^{R\Delta} \int_0^{R\Delta} \tilde{f}(x,y)\, dx\, dy$, and $C_{\tilde{f}} := \max\left\{ \left\| \frac{\partial \tilde{f}}{\partial x} \right\|_\infty, \left\| \frac{\partial \tilde{f}}{\partial y} \right\|_\infty \right\}$. Then*

$$\left| \tilde{J} - \tilde{I} \right| \ \le\ C_{\tilde{f}} \Delta^3 R^2.$$

In particular, suppose $\theta > 0$, $R = R_N = \lceil N^{1/2+\theta} \rceil$, and $\Delta = \Delta_N = N^{-1/2}$. Then in Proposition 3.6, the sums $I = I(N)$ and $J = J(N)$ satisfy

$$|J(N) - I(N)| \ =\ O(N^{\theta - 1/2}),$$

and in Proposition 3.7 the sums $\tilde{I} = \tilde{I}(N)$ and $\tilde{J} = \tilde{J}(N)$ satisfy

$$\left| \tilde{J}(N) - \tilde{I}(N) \right| \ =\ O(N^{2\theta - 1/2}).$$

Such results are well known. For example, Proposition 3.7 holds because

$$\left| \int_{b-\Delta}^{b} \int_{a-\Delta}^{a} (\tilde{f}(x,y) - \tilde{f}(a,b))\, dx\, dy \right| \ \le\ \int_{b-\Delta}^{b} \int_{a-\Delta}^{a} (|a-x| + |b-y|)\, C_{\tilde{f}}\, dx\, dy \ =\ \Delta^3\, C_{\tilde{f}}\,.$$

The following lemma is useful in the proof of Theorem 2.7.

Lemma 3.8. *Let $T > 0$ and $K > 0$. Then $\int_0^T z^2 e^{-Kz^2} dz = \frac{\sqrt{\pi}}{4K^{3/2}}(1 + O(e^{-T\sqrt{K}}))$, where the term $O(e^{-T\sqrt{K}})$ is uniform over T and K such that $T\sqrt{K}$ is sufficiently large.*

Proof. We know that $\int_0^\infty z^2 e^{-Kz^2} dz \ =\ \sqrt{\pi}/4K^{3/2}$. Also, $\int_T^\infty z^2 e^{-Kz^2} dz \ =\ K^{-3/2} \int_{T\sqrt{K}}^\infty u^2 e^{-u^2} du$ $\le\ K^{-3/2} \int_{T\sqrt{K}}^\infty e^{2u-u^2} du$. Since $2u - u^2 \le -u$ for $u \ge 3$, we have

$$\int_{T\sqrt{K}}^\infty e^{2u-u^2} du \ \le\ \int_{T\sqrt{K}}^\infty e^{-u} du \ =\ e^{-T\sqrt{K}} \qquad \text{for } T\sqrt{K} \ge 3.$$

Therefore, $\int_T^\infty z^2 e^{-Kz^2} dz = K^{-3/2} O(e^{-T\sqrt{K}})$. We conclude that

$$\int_0^T z^2 e^{-Kz^2} dz = \int_0^\infty z^2 e^{-Kz^2} dz - \int_T^\infty z^2 e^{-Kz^2} dz = \frac{\sqrt{\pi}}{4K^{3/2}}(1 + O(e^{-T\sqrt{K}})).$$

□

The next result is used in the proof of Theorem 2.11.

Lemma 3.9. *For positive K_1, K_2, K_3, w_1, and w_2, we have*

$$\int_0^{w_2} \int_0^{w_1} xy\, e^{-K_1 x^2} e^{-K_2 y^2} \left(e^{-K_3 (y-x)^2} - e^{-K_3 (x+y)^2} \right) dx\, dy$$

$$=\ \frac{\pi K_3}{4(K_1K_2 + K_1K_3 + K_2K_3)^{3/2}} + O\left(\frac{e^{-K_1 w_1^2} + e^{-K_2 w_2^2}}{K_1 K_2} \right).$$

Proof. Using standard properties of bivariate Gaussian integrals, we know that for positive A and B and real C such that $4AB - C^2 > 0$,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xye^{-(Ax^2+By^2+Cxy)}\,dx\,dy \;=\; \frac{-2\pi C}{(4AB-C^2)^{3/2}}.$$

Letting $A = K_1 + K_3$, $B = K_2 + K_3$ and $C = \pm 2K_3$, we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xye^{-K_1x^2}e^{-K_2y^2}e^{-K_3(x\pm y)^2}\,dx\,dy \;=\; \frac{\mp\pi K_3}{2(K_1K_2+K_1K_3+K_2K_3)^{3/2}}.$$

Let $h(x,y) = xy\,e^{-K_1x^2}e^{-K_2y^2}(e^{-K_3(y-x)^2} - e^{-K_3(x+y)^2})$. Since $h(x,y)$ is an even function of x and of y, we have

$$\int_0^{\infty}\int_0^{\infty} h(x,y)\,dx\,dy \;=\; \frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\,dx\,dy \;=\; \frac{\pi K_3}{4(K_1K_2+K_1K_3+K_2K_3)^{3/2}}. \tag{3}$$

For $a, b \geq 0$, we also have the simple bounds

$$0 \;<\; \int_b^{\infty}\int_a^{\infty} h(x,y)\,dx\,dy \;<\; \int_b^{\infty}\int_a^{\infty} xy\,e^{-K_1x^2}e^{-K_2y^2}\,dx\,dy \;=\; \frac{e^{-(K_1a^2+K_2b^2)}}{4K_1K_2}. \tag{4}$$

Using Equation (4), we see that

$$\left|\int_0^{w_2}\int_0^{w_1} h \;-\; \int_0^{\infty}\int_0^{\infty} h\right| \;\leq\; \int_0^{\infty}\int_{w_1}^{\infty} h \;+\; \int_{w_2}^{\infty}\int_0^{\infty} h \;=\; O\left(\frac{e^{-K_1w_1^2}+e^{-K_2w_2^2}}{K_1K_2}\right).$$

The lemma follows from this and Equation (3). □

4 Proofs of the Exact Results

Definition 4.1. Let σ and τ be permutations of lengths N and M respectively. For $i \in [1, N]$, we define **Insert**(τ, σ, i) (see Figure 7) to be the permutation θ in S_{N+M} given by

$$\theta_k = \begin{cases} \sigma_k & \text{if } k < i \text{ and } \sigma_k < \sigma_i, \\ \sigma_k + M & \text{if } k < i \text{ and } \sigma_k > \sigma_i, \\ \tau_{k-i+1} + \sigma_i & \text{if } i \leq k < i + M, \\ \sigma_{k-M} & \text{if } k \geq i + M \text{ and } \sigma_{k-M} \leq \sigma_i, \\ \sigma_{k-M} + M & \text{if } k \geq i + M \text{ and } \sigma_{k-M} > \sigma_i. \end{cases}$$

The next result shows that the Insert operation preserves 312-avoidance in a certain situation.

Proposition 4.2. *Let* $\sigma \in S^{\square}(N, i, j)$ *and* $\tau \in S_M(312)$*. Then* **Insert**$(\tau, \sigma, i) \in S^{\bullet}(N + M, i + M, j)$*.*

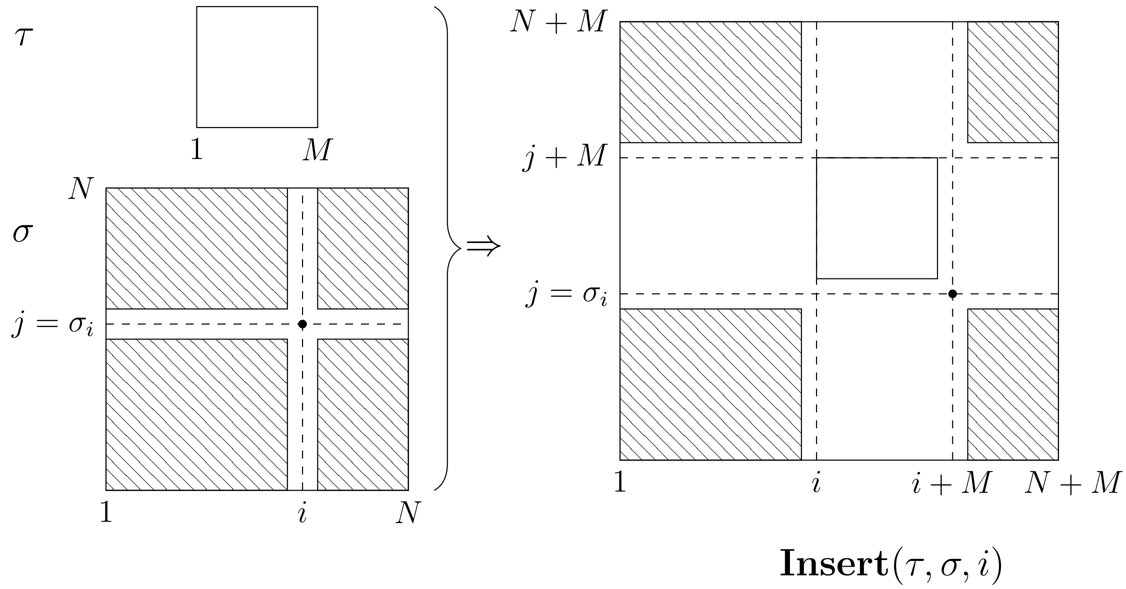

Figure 7: Diagram of how the graphs of τ and σ combine to make the graph of $\textbf{Insert}(\tau, \sigma, i)$. The graph of σ is broken into four rectangles, which are then moved apart to make room for τ.

Proof. Let $\theta = \textbf{Insert}(\tau, \sigma, i)$. Since $\sigma_i = j$, Definition 4.1 implies that $\theta_{i+M} = j$. We must now show that $\textbf{Patt}(\theta_{k_1}\theta_{k_2}\theta_{k_3}) \neq 312$ whenever $1 \leq k_1 < k_2 < k_3 \leq N + M$. It is not hard to do this by checking the cases that k_1 is in $[1, i)$, $[i, i + M)$, or $[i + M, N + M]$. We leave this to the reader, with the aid of Figure 7. □

The next lemma may be viewed as a converse of Proposition 4.2. Roughly speaking, part (f) shows that every element of $S^\bullet(\cdot, \cdot, \cdot)$ may be expressed as the result of an Insert operation. Lemma 4.4 then shows that such an expression is unique. This will permit us to evaluate the cardinality of $S^\bullet(\cdot, \cdot, \cdot)$ by using the Insert operation to construct an explicit bijection.

Lemma 4.3. *Let $\sigma \in S^\bullet(M, M - t, j)$ with $j < M - t$. Let $i_0 = \min\{i \in [1, M] : \sigma_i > j\}$. Then*
(a) $i_0 \in [1, M - t)$;
(b) $\sigma_i > j$ for every $i \in [i_0, M - t)$;
(c) $\sigma_i \in [1, j)$ for every $i \in [1, i_0)$;
(d) $j \geq i_0 \geq \max\{1, j - t\}$;
(e) The image of the domain $[i_0, M - t)$ under σ equals $(j, j + M - t - i_0]$;
(f) Let $\hat{\sigma} = \textbf{Patt}(\sigma_1, \ldots, \sigma_{i_0-1}, \sigma_{M-t}, \ldots, \sigma_M)$ and $\tilde{\sigma} = \textbf{Patt}(\sigma_{i_0}, \ldots, \sigma_{M-t-1})$. Then $\hat{\sigma} \in S^\square(t + i_0, i_0, j)$, $\tilde{\sigma} \in S_{M-t-i_0}(312)$ and $\sigma = \textbf{Insert}(\tilde{\sigma}, \hat{\sigma}, i_0)$.

Figure 8 depicts the properties of $\sigma \in S^\bullet(M, M - t, j)$ that are stated in Lemma 4.3.

Proof. Let $\sigma \in S_M(312)$ such that $\sigma_{M-t} = j$.
(a) $j < M - t$ implies that there is at least one element $i \in [1, M - t)$ such that $\sigma_i > j$. Hence, $i_0 \in [1, M - t)$.

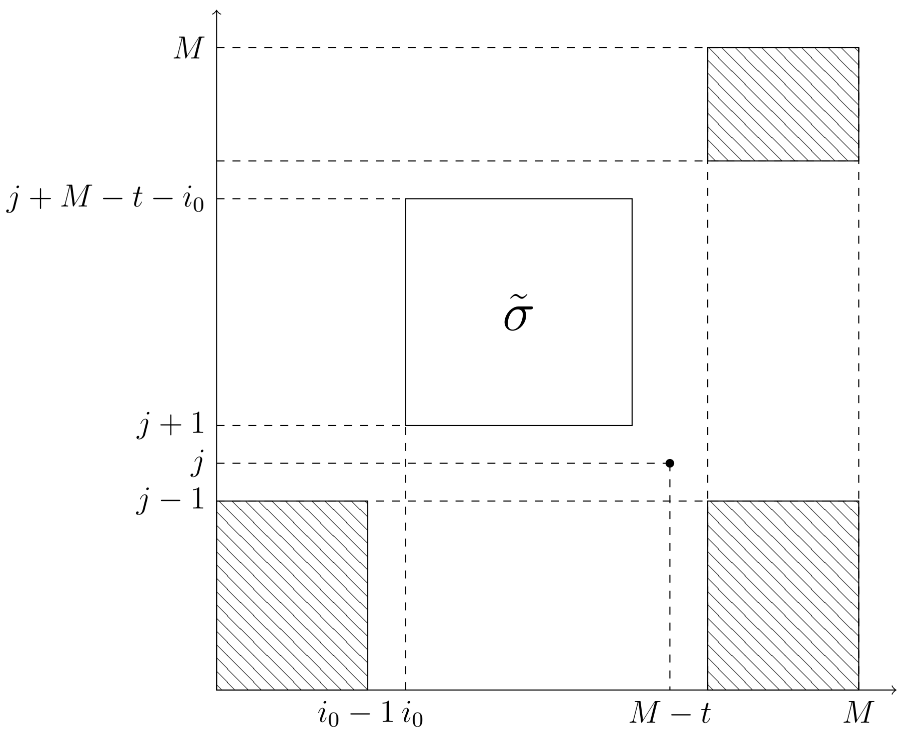

Figure 8: Illustration of properties described in Lemma 4.3 for a permutation $\sigma \in S_M(312)$ with $\sigma_{M-t} = j$. Except for the point $(M-t, j)$, all points of the graph of σ are inside one of the four rectangles bounded by solid lines.

(b) If this were false, then there would exist $i \in (i_0, M-t)$ such that $\sigma_i = j_i < j < \sigma_{i_0}$. But then we would have $\mathbf{Patt}(\sigma_{i_0}, \sigma_i, \sigma_{M-t}) = 312$.

(c) The definition of i_0 and the fact that $M-t \notin [1, i_0)$ imply that $\forall i \in [1, i_0)$, $\sigma_i \in [1, j)$.

(d) Part (c) and the Pigeonhole Principle imply that $i_0 \leq j$. Once $i_0 - 1$ elements are mapped to the range $[1, j)$, the remaining $j - i_0$ elements in $[1, j)$ should be the images of elements in the domain $(M-t, M]$ (using part (b) and $\sigma_{M-t} = j$). Hence, $j - i_0 \leq |(M-t, M]| = t$ and therefore $i_0 \geq \max\{1, j-t\}$.

(e) Assume $\exists\ I \in [i_0, M-t)$ such that $\sigma_I = J > j + M - t - i_0$. This implies that there is an element N in $(j, j+M-t-i_0]$ that is not the image of any element in $[i_0, M-t)$. Hence, there is an element K in $(M-t, M]$ such that $\sigma_K = N$. Therefore, $\sigma_I \sigma_{M-t} \sigma_K = JjN$ is a 312-pattern. This contradiction, together with part (b), proves part (e).

(f) Since σ avoids 312, clearly $\hat{\sigma} \in S_{t+i_0}(312)$ and $\tilde{\sigma} \in S_{M-t-i_0}(312)$. By part (b), $\hat{\sigma}_{i_0} = \sigma_{M-t} = j$. Also, for all $i < i_0$, $\hat{\sigma}_i = \sigma_i < j$ by part (c). Hence, $\hat{\sigma} \in S^{\square}(t+i_0, i_0, j)$. Finally, from part (e), we deduce that $\sigma_i = \tilde{\sigma}_{i-i_0+1} + j$ for all $i \in [i_0, M-t)$. Using this and part (c) we conclude that $\sigma = \mathbf{Insert}(\tilde{\sigma}, \hat{\sigma}, i_0)$. □

Lemma 4.4. *Fix N, t, and j. Assume that $\mathbf{Insert}(\tau^0, \rho^0, i^0) = \mathbf{Insert}(\tau^1, \rho^1, i^1)$, where $\tau^k \in S_{N-t-i^k}(312)$ and $\rho^k \in S^{\square}(t+i^k, i^k, j)$ for $k = 0, 1$. Then $i^0 = i^1$, $\tau^0 = \tau^1$ and $\rho^0 = \rho^1$.*

Proof. If $i^0 = i^1$ it is easy to see that $(\tau^0, \rho^0) = (\tau^1, \rho^1)$. Assume that $i^0 < i^1$ (or similarly, we could consider $i^0 > i^1$). By its definition, $(\mathbf{Insert}(\tau^0, \rho^0, i^0))_{i^0} = \tau^0_1 + \rho^0_{i^0} = \tau^0_1 + j > j$. On the other hand, $(\mathbf{Insert}(\tau^1, \rho^1, i^1))_{i^0} = \rho^1_{i^0} < j$ since $\rho^1 \in S^{\square}(t+i^1, i^1, j)$. This gives a contradiction and hence we conclude the result. □

Proposition 4.2 allows us to make the following definition.

Definition 4.5. Fix N, t, and j. Let

$$\mathbf{Dom}_{\bullet}(N, t, j) \ = \bigcup_{i_0=\max\{1, j-t\}}^{j} S_{N-t-i_0}(312) \times S^{\square}(t+i_0, i_0, j).$$

We define the map $\phi_{\bullet;N,t,j} : \mathbf{Dom}_{\bullet}(N, t, j) \to S^{\bullet}(N, N-t, j)$ by

$$\phi_{\bullet}((\tilde{\sigma}, \hat{\sigma})) = \mathbf{Insert}(\tilde{\sigma}, \hat{\sigma}, i_0) \qquad \text{for } (\tilde{\sigma}, \hat{\sigma}) \in S_{N-t-i_0}(312) \times S^{\square}(t+i_0, i_0, j).$$

Lemma 4.6. *Fix N, t, and j with $j < N-t$. Let $\phi_{\bullet;N,t,j} : \mathbf{Dom}_{\bullet}(N, t, j) \to S^{\bullet}(N, N-t, j)$ be the map in Definition 4.5. Then $\phi_{\bullet;N,t,j}$ is a bijective map.*

Proof. Assume that $(\tilde{\sigma}^0, \hat{\sigma}^0) \in S_{N-t-i_0}(312) \times S^{\square}(t+i_0, i_0, j)$ and $(\tilde{\sigma}^1, \hat{\sigma}^1) \in S_{N-t-i_1}(312) \times S^{\square}(t+i_1, i_1, j)$ and that $\phi_{\bullet}((\tilde{\sigma}^0, \hat{\sigma}^0)) = \phi_{\bullet}((\tilde{\sigma}^1, \hat{\sigma}^1))$. We first apply Lemma 4.4 with $i^k = i_k$,$\tau^k = \tilde{\sigma}^k$, $\rho^k = \hat{\sigma}^k$ for $k = 0, 1$ and conclude that $\phi_{\bullet}$ is one-to-one. Next we apply Lemma 4.3(f), substituting $M = N$, and conclude that for each $\sigma \in S^{\bullet}(N, N-t, j)$ there exists an i_0 such that $\max\{1, j-t\} \leq i_0 \leq j$ and that $\sigma = \mathbf{Insert}(\tilde{\sigma}, \hat{\sigma}, i_0)$ where $\hat{\sigma} = \mathbf{Patt}(\sigma_1, \ldots, \sigma_{i_0-1}, \sigma_{N-t}, \ldots, \sigma_N) \in S^{\square}(t+i_0, i_0, j)$ and $\tilde{\sigma} = \mathbf{Patt}(\sigma_{i_0}, \ldots, \sigma_{N-t-1}) \in S_{N-t-i_0}(312)$. Therefore, $\phi_{\bullet;N,t,j}$ is a surjective map and hence a bijection. □

It is now straightforward to prove Theorem 2.3.

Proof of Theorem 2.3. By Lemma 4.6, we have

$$|S^{\bullet}(N,N-t,j)| \;=\; |\mathbf{Dom}_{\bullet}(N,t,j)| \;=\; \sum_{i_0=\max\{1,j-t\}}^{j} C_{N-t-i_0} \left|S^{\square}(t+i_0,i_0,j)\right|.$$

Hence, Remark 2.2 completes the proof of Theorem 2.3. □

Next we look at cardinality of the set of 312-avoiding permutations that has k decreasing points below the diagonal. That is, for $j_k < j_{k-1} < \ldots < j_1 < N-t_1 < \ldots < N-t_k$, we shall prove the recursive formula in Theorem 2.8 for the cardinality of

$$S^{\searrow_k}(N) \;\equiv\; S^{\searrow_k}(N,N-t_1,\ldots,N-t_k,j_1,\ldots,j_k) = \{\sigma\in S_N(312) : \sigma_{N-t_1}=j_1,\,\ldots\,\sigma_{N-t_k}=j_k\}\,.$$

Proof of Theorem 2.8. Let $\sigma\in S^{\searrow_k}(N)$. Since $S^{\searrow_k}(N)\subseteq S^{\bullet}(N,N-t_k,j_k)$, we apply Lemma 4.3 with $M=N$, $t=t_k$, $j=j_k$ and writing i instead of i_0. See Figure 9. By part (f) of this Lemma we conclude that $\sigma=\mathbf{Insert}(\tilde\sigma,\hat\sigma,i)$, where $\hat\sigma=\mathbf{Patt}(\sigma_1,\ldots,\sigma_{i-1},\sigma_{N-t_k},\ldots,\sigma_N)\in S^{\square}(t_k+i,i,j_k)$ and $\tilde\sigma=\mathbf{Patt}(\sigma_i,\ldots,\sigma_{N-t_k-1})\in S_{N-t_k-i}(312)$. By Lemma 4.3(d) we also know that $\max\{1,j_k-t_k\}\le i\le j_k$. Since $i\le j_k<\ldots<j_1<N-t_1<\ldots<N-t_k$ and using Lemma 4.3(e), we see that $\tilde\sigma_{N-t_r-i+1}=\sigma_{N-t_r}-\sigma_{N-t_k}=j_r-j_k$ for $r=1,\ldots,k-1$. Hence, $\tilde\sigma\in S^{\searrow_{k-1}}(N-t_k-i,N-t_1-(i-1),N-t_2-(i-1),\ldots,N-t_{k-1}-(i-1),j_1-j_k,\ldots,j_{k-1}-j_k)$.

Let

$$\mathbf{Dom}_{\searrow_k} \;\equiv\; \mathbf{Dom}_{\searrow_k}(N,j_1,j_2,\ldots,j_k,t_1,t_2,\ldots t_k) \;=\; \bigcup_{i=\max\{1,j_k-t_k\}}^{j_k} S^{\searrow_{k-1}}(N-t_k-i,N-t_1-i+1,\ldots,N-t_{k-1}-i+1,j_1-j_k,\ldots,j_{k-1}-j_k)\times S^{\square}(t_k+i,i,j_k).$$

Observe that $\mathbf{Dom}_{\searrow_k}\subset\mathbf{Dom}_{\bullet}(N,t_k,j_k)$. We define the map $\phi_{\searrow_k}$ such that $\phi_{\searrow_k}:\mathbf{Dom}_{\searrow_k}\to S^{\searrow_k}(N)$ is the restriction of $\phi_{\bullet;N,t_k,j_k}$ to $\mathbf{Dom}_{\searrow_k}$.

In the discussion of the first paragraph, we see that $(\tilde\sigma,\hat\sigma)\in\mathbf{Dom}_{\searrow_k}$, and that $\sigma=\mathbf{Insert}(\tilde\sigma,\hat\sigma,i)=\phi_{\searrow_k}(\tilde\sigma,\hat\sigma)$. Hence $\phi_{\searrow_k}$ is surjective, and $\phi_{\searrow_k}$ is injective (since by Lemma 4.6 $\phi_{\bullet;N,t_k,j_k}$ is), so $\phi_{\searrow_k}$ is a bijection. This implies that

$$\left|S^{\searrow_k}(N)\right| \;=\; \sum_{i=\max\{1,j_k-t_k\}}^{j_k} \left|S^{\square}(t_k+i,i,j_k)\right| \times \left|S^{\searrow_{k-1}}(N-t_k-i,N-t_1-i+1,\ldots,N-t_{k-1}-i+1,j_1-j_k,\ldots,j_{k-1}-j_k)\right|.$$

This proves the recursion. The formula for the case $k=2$ follows, using Theorem 2.3. □

Now we turn to the proof of Theorem 2.10, which establishes the cardinality of

$$S^{\nearrow}(N)\equiv S^{\nearrow}(N,N-t_1,N-t_2,j_1,j_2)=\{\sigma\in S_N(312):\sigma_{N-t_1}=j_1,\sigma_{N-t_2}=j_2\}$$

for the situation $j_1<N-t_1<N-t_2$, $j_1<j_2<N-t_2$.

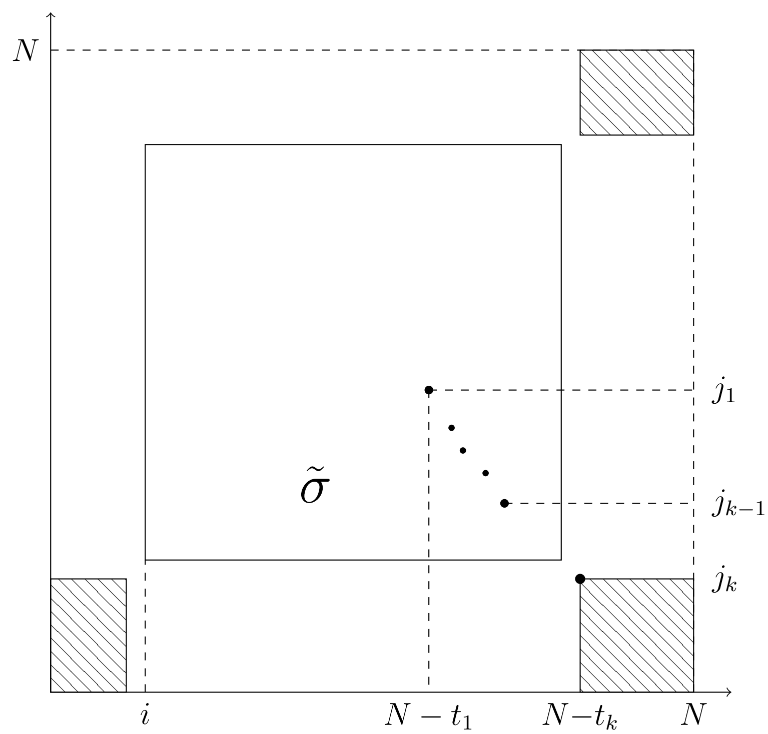

Figure 9: Proof of Theorem 2.8. The shaded regions correspond to the permutation $\hat{\sigma}$.

Proof of Theorem 2.10. Let $\sigma \in S^{\nearrow}(N)$. For every $i \in [1, N-t_1)$, σ_i cannot belong to the interval $(j_2, N]$ (otherwise $\sigma_i\sigma_{N-t_1}\sigma_{N-t_2}$ will form a 312 pattern). This implies that only domain elements in $(N-t_1, N] \setminus \{N-t_2\}$ will be mapped into $(j_2, N]$. Hence $|(j_2, N]| \leq |(N-t_1, N] \setminus \{N-t_2\}|$, which says $j_2 \geq N - t_1 + 1$. This proves part (a).

For the rest of the proof we assume that $j_2 \geq N - t_1 + 1$. Let $\sigma \in S^{\nearrow}(N)$. Observing that $S^{\nearrow}(N) \subseteq S^{\bullet}(N, N-t_2, j_2)$, we apply Lemma 4.3 with $M = N$, $t = t_2$, $j = j_2$, and writing i_2 for i_0. In Lemma 4.3(d,f), we get that $j_2 \geq i_2 \geq j_2 - t_2$, and that $\sigma = \textbf{Insert}(\tilde{\sigma}, \hat{\sigma}, i_2)$ where $\hat{\sigma} = \textbf{Patt}(\sigma_1, \ldots, \sigma_{i_2-1}, \sigma_{N-t_2}, \ldots, \sigma_N) \in S^{\square}(t_2 + i_2, i_2, j_2)$ and $\tilde{\sigma} = \textbf{Patt}(\sigma_{i_2}, \ldots, \sigma_{N-t_2-1}) \in S_{N-t_2-i_2}(312)$.

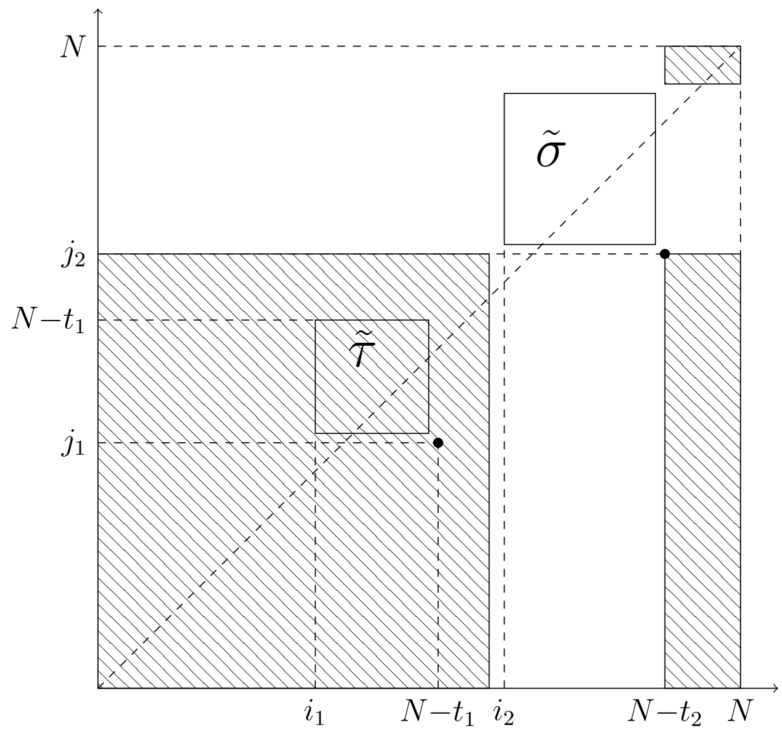

Figure 10: Proof of Theorem 2.10(b). The shaded regions correspond to the permutation $\hat{\sigma}$. The diagonal is drawn for reference.

By Lemma 4.3(b), $\sigma_i > j_2$ for all $i \in [i_2, N-t_2)$, and since $\sigma_{N-t_1} = j_1 < j_2$ it follows that $N - t_1 < i_2$ and hence $\hat{\sigma}_{N-t_1} = \sigma_{N-t_1} = j_1$. We conclude that $\hat{\sigma} \in S^{\bullet}(t_2 + i_2, N - t_1, j_1)$. Also, we have shown $i_2 \in [\max\{N - t_1 + 1, j_2 - t_2\}, j_2]$. Now we will apply Lemma 4.3 one more time to $\hat{\sigma}$ instead of σ. Here we take $M = t_2 + i_2$, $t = (t_2 + i_2) - (N - t_1)$, $j = j_1$ and $i_0 = i_1$. By Lemma 4.3 we get that $i_1 \in [\max\{1, j_1 - t_1 - t_2 - i_2 + N\}, j_1]$ and that $\hat{\sigma} = \textbf{Insert}(\tilde{\tau}, \hat{\tau}, i_1)$ with $\tilde{\tau} \in S_{N-t_1-i_1}(312)$ and $\hat{\tau} \in S^{\square}(t_1+t_2+i_1+i_2-N, i_1, j_1)$. Furthermore, since $\hat{\sigma} \in S^{\square}(t_2 + i_2, i_2, j_2)$, we know that $\hat{\sigma}_i < j_2$ $\forall i \in [i_1, N-t_1) \subset [1, i_2)$. Hence, by Lemma 4.3(e) we conclude that $j_1 + (t_2 + i_2) - (t_2 + i_2 - N + t_1) - i_1 < j_2$, i.e., $j_1 - j_2 + N - t_1 < i_1$.

Let $M^* = N - t_1 - i_1$ (the size of $\tilde{\tau}$). We shall now show that $\hat{\tau} \in S^{\square}(i_1 + i_2 + t_1 + t_2 - N, i_2 -$

$M^*, j_2 - M^*)$. We know $\hat{\sigma}_{i_2} = j_2$ and $\hat{\sigma}_i < j_2$ for all $i < j_2$. We also know $i_2 > N - t_1 = i_1 + M^*$, so by the definition of **Insert** we see that either

(α) $\hat{\sigma}_{i_2} = \hat{\tau}_{i_2 - M^*}$ and $\hat{\tau}_{i_2-M^*} \leq \hat{\tau}_{i_1}$, or
(β) $\hat{\sigma}_{i_2} = \hat{\tau}_{i_2 - M^*} + M^*$ and $\hat{\tau}_{i_2-M^*} > \hat{\tau}_{i_1}$.

Now, $\hat{\tau}_{i_1} = j_1 < j_2 = \hat{\sigma}_{i_2}$, so (α) does not hold. Therefore (β) holds, so $\hat{\tau}_{i_2-M^*} = \hat{\sigma}_{i_2} - M^* = j_2 - M^*$. It remains to show that $\hat{\tau}_u < j_2 - M^*$ for all $u < i_2 - M^*$. On the one hand, if $u < i_1$, then $\hat{\tau}_u \leq \hat{\sigma}_u < j_1 < j_2 - M^*$ (by the last inequality of the preceding paragraph). On the other hand, if $i_1 \leq u < i_2 - M^*$, then $\hat{\tau}_u \leq \max\{j_1, \hat{\sigma}_{u+M^*} - M^*\} < j_2 - M^*$. This completes the proof that $\hat{\tau} \in S^{\square}(i_1 + i_2 + t_1 + t_2 - N, i_2 - M^*, j_2 - M^*)$.

Let

$$\mathbf{Dom}_{\nearrow} \equiv \mathbf{Dom}_{\nearrow}(N, j_1, j_2, t_1, t_2) = \bigcup_{i_1=\max\{1, j_1-j_2+N-t_1+1\}}^{j_1} \bigcup_{i_2=\max\{N-t_1+1, j_2-t_2\}}^{j_2} S_{N-t_2-i_2}(312) \times S_{N-t_1-i_1}(312) \times S_{\cap}(N, i_1, i_2)$$

where

$$S_{\cap}(N, i_1, i_2) = S^{\square}(i_1 + i_2 + t_1 + t_2 - N, i_1, j_1) \cap S^{\square}(i_1 + i_2 + t_1 + t_2 - N, i_2 - M^*, j_2 - M^*)\,.$$

We define the map $\phi_{\nearrow} \equiv \phi_{\nearrow; N, t_1, t_2, j_1, j_2} : \mathbf{Dom}_{\nearrow} \to S^{\nearrow}(N)$ by

$$\phi_{\nearrow}(\alpha, \beta, \gamma) = \mathbf{Insert}(\alpha, \mathbf{Insert}(\beta, \gamma, i_1), i_2) \quad \text{for} \quad (\alpha, \beta, \gamma) \in S_{N-t_2-i_2}(312) \times S_{N-t_1-i_1}(312) \times S_{\cap}(N, i_1, i_2).$$

From the above, we know that for all $\sigma \in S^{\nearrow}(N)$ there exists $(\tilde{\sigma}, \tilde{\tau}, \hat{\tau}) \in \mathbf{Dom}_{\nearrow}$ such that $\sigma = \mathbf{Insert}(\tilde{\sigma}, \mathbf{Insert}(\tilde{\tau}, \hat{\tau}, i_1), i_2)$. Hence, $\phi_{\nearrow}$ is a surjective map. We claim that $\phi_{\nearrow}$ is one-to-one. Assume that $(\alpha^1, \beta^1, \gamma^1) \in S_{N-t_2-i_2^1}(312) \times S_{N-t_1-i_1^1}(312) \times S_{\cap}(N, i_1^1, i_2^1)$ and $(\alpha^2, \beta^2, \gamma^2) \in S_{N-t_2-i_2^2}(312) \times S_{N-t_1-i_1^2}(312) \times S_{\cap}(N, i_1^2, i_2^2)$ and that $\phi_{\nearrow}((\alpha^1, \beta^1, \gamma^1)) = \phi_{\nearrow}((\alpha^2, \beta^2, \gamma^2))$. We have $\alpha^k \in S_{N-t_2-i_2^k}$ and $\mathbf{Insert}(\beta^k, \gamma^k, i_1^k) \in S^{\square}(t_2 + i_2^k, i_2^k, j_2)$ for $k = 1, 2$. We apply Lemma 4.4 with $i^k = i_2^k$, $\tau^k = \alpha^k$ and $\rho^k = \mathbf{Insert}(\beta^k, \gamma^k, i_1^k)$ and conclude that $i_2^1 = i_2^2$, $\alpha^1 = \alpha^2$ and $\mathbf{Insert}(\beta^1, \gamma^1, i_1^1) = \mathbf{Insert}(\beta^2, \gamma^2, i_1^2)$. Noting that $\beta^k \in S_{N-t_1-i_1^k}(312)$ and $\gamma^k \in S^{\square}(t_1 + t_2 + i_1^k + i_2^k - N, i_1^k, j_1)$, we apply Lemma 4.4 one more time. This second time we replace t by $(t_2 + i_2^k) - (N - t_1)$ and hence conclude that $i_1^1 = i_1^2$, $\beta^1 = \beta^2$ and $\gamma^1 = \gamma^2$. This proves that $\phi_{\nearrow}$ is a one-to-one map. Therefore $\phi_{\nearrow}$ is a bijection and

$$\left|S^{\nearrow}(N)\right| = \sum_{i_1=\max\{1, j_1-j_2+N-t_1+1\}}^{j_1} \sum_{i_2=\max\{N-t_1+1, j_2-t_2\}}^{j_2} C_{N-t_1-i_1} C_{N-t_2-i_2} \left|S_{\cap}(N, i_1, i_2)\right|. \tag{5}$$

For given $i_1, i_2, j_1, j_2, t_1, t_2, N$, the set $S_{\cap}(N, i_1, i_2)$ consists of all σ in $S_{i_1+i_2+t_1+t_2-N}(312)$ such that $\sigma_{i_1} = j_1$ is a left-to-right maximum and $\sigma_{i_2-(N-t_1-i_1)} = j_2 - (N - t_1 - i_1)$ is a left-to-right maximum. Hence, using Krattenthaler's bijection from Section 3.1, these two points correspond to peaks at $(j_1+(i_1-1), j_1-(i_1-1))$ and $(j_2+i_2-2(N-t_1-i_1)-1, j_2-(i_2-1))$ on the Dyck path associated with σ. We deduce that $|S_\cap(N,i_1,i_2)|$ equals the product of the cardinalities of the three sets of Dyck segments $D_{(1)}$, $D_{(2)}$, $D_{(3)}$, where

• $D_{(1)}$ is the set of Dyck segments from $(0,0)$ to (j_1+i_1-2, j_1-i_1),

• $D_{(2)}$ is the set of Dyck segments from (j_1+i_1, j_1-i_1) to $(j_2+i_2-2(N-t_1-i_1)-2, j_2-i_2)$, and

• $D_{(3)}$ is the set of Dyck segments from $(j_2+i_2-2(N-t_1-i_1), j_2-i_2)$ to $(2(i_1+i_2+t_1+t_2-N), 0)$, which has the same cardinality as the set of Dyck segments from $(0,0)$ to $(2(i_1+i_2+t_1+t_2-N)-(j_2+i_2-2(N-t_1-i_1)), j_2-i_2) = (i_2+2t_2-j_2, j_2-i_2)$.

Recalling Lemma 3.3 and Remark 3.4, we obtain

$$\begin{aligned}
\left|D_{(1)}\right| &= \frac{j_1-i_1+1}{j_1}\binom{j_1+i_1-2}{j_1-1},\\
\left|D_{(2)}\right| &= \binom{i_1+i_2+j_2-j_1-2(N-t_1+1)}{j_2+(i_1-j_1-1)-(N-t_1)} - \binom{i_1+i_2+j_2-j_1-2(N-t_1+1)}{j_2-(N-t_1)},\\
\left|D_{(3)}\right| &= \frac{j_2-i_2+1}{t_2+1}\binom{i_2+2t_2-j_2}{t_2}.
\end{aligned}$$

Using the above and $|S_\cap(N,i_1,i_2)| = \left|D_{(1)}\right|\left|D_{(2)}\right|\left|D_{(3)}\right|$ in Equation (5), the proof of part (b) is now complete. □

Notice that when $j_2 = N-t_1+1$, each sum in the formula for $|S^{\nearrow}(N)|$ has only one term, namely $i_1 = j_1$ in the outer sum and $i_2 = j_2$ in the inner sum. Hence in this case we obtain $|D_{(1)}| = C_{j_1-1}$, $|D_{(3)}| = C_{t_2}$, and $\left|D_{(2)}\right| = \binom{0}{0} - \binom{0}{1} = 1$, which yields the expression

$$\left|S^{\nearrow}(N)\right| = C_{N-t_1-j_1}\, C_{t_1-t_2-1}\, C_{j_1-1}\, C_{t_2} \quad \text{when } j_2 = N-t_1+1.$$

5 Asymptotics of Probabilities

The main goal of this section is to prove asymptotic formulas for the probabilities of $S^\bullet(N,i,j)$, $S^{\searrow}(N)$, and $S^{\nearrow}(N)$, as described in Section 2. The asymptotics are based on well known asymptotics of binomial probabilities, of which the following is a particularly useful form.

Proposition 5.1. *([10], pp. 61–63) We have the relation*

$$\binom{A}{B}2^{-A} = \sqrt{\frac{2}{\pi A}}\, e^{-\frac{(2B-A)^2}{2A}}\, e^{O\left(\frac{1}{A}+\frac{(2B-A)^4}{A^3}\right)}$$

and hence, if $|2B-A| \le A^{3/4}$,

$$\binom{A}{B}2^{-A} = \sqrt{\frac{2}{\pi A}}\, e^{-\frac{(2B-A)^2}{2A}}\left(1+O\left(\frac{1}{A}+\frac{(2B-A)^4}{A^3}\right)\right).$$

The following notation will be used throughout this section. Let

$$h(u) \;=\; \frac{\binom{2u}{u}}{2^{2u}(u+1)} \;=\; \frac{C_u}{2^{2u}} \qquad \text{and} \qquad \gamma(t,r) \;=\; \frac{\binom{2t-r+1}{t}}{2^{2t-r+1}}. \tag{6}$$

Remark 5.2. The functions $h(u)$ and $\gamma(t,r)$ have the following properties.
(a) $\gamma(t,r)$ is decreasing in r for $r \geq 1$. This is because

$$\frac{\gamma(t,r+1)}{\gamma(t,r)} \;=\; \frac{2(t+1-r)}{2t+1-r} \;=\; \frac{2t+1-r-(r-1)}{2t+1-r} \;\leq\; 1.$$

(b) Applying Proposition 5.1 to $h(u)$, we get $h(u) = \frac{1}{\sqrt{\pi}u(u+1)}(1+O(\frac{1}{u})) = \frac{1}{\sqrt{\pi}u^{3/2}}(1+O(\frac{1}{u}))$.
(c) $\frac{h(u)}{h(N)} = (\frac{N}{u})^{3/2}(1+O(\frac{1}{N}))$ if $u \asymp N$. (This says that the O term is uniform over all u and N such that $cN > u > N/c$ for some fixed $c > 1$.)

Before proceeding, we shall need the following particular form of the asymptotics of $\gamma(t,r)$.

Lemma 5.3. *Fix $\theta \in (0,1/6)$, $\epsilon \in (0,1)$, $C > 0$, and $a \in \mathbb{Z}$. Let $R_N \;=\; \lceil N^{1/2+\theta} \rceil$. Then*

$$\gamma(t+s,r+a) \;=\; \frac{1}{\sqrt{\pi t}}\, e^{-r^2/4t} \left(1 + O\left(N^{3\theta-1/2}\right)\right)\,, \tag{7}$$

where the error term is uniform over N, t, r, and s satisfying $|s| \leq CR_N$, $|r| \leq CR_N$, $\epsilon N \leq t \leq N$, and $N > N_1$ for some N_1. (Note that the error term and N_1 are not uniform over ϵ, a, C, or θ.)

Proof. Using Proposition 5.1 with $A = 2t+2s-r-a+1$ and $B = t+s$, we have

$$\gamma(t+s,r+a) \;=\; \sqrt{\frac{2}{\pi(2t+2s-r-a+1)}}\, e^{-\frac{(r+a-1)^2}{2(2t+2s-r-a+1)}} \left(1 + O\left(\frac{1}{N} + \frac{R_N^4}{N^3}\right)\right). \tag{8}$$

Since $R_N^4/N^3 \asymp N^{4\theta-1}$ and $\theta > 0$, the final term in the above expression is $(1+O(N^{4\theta-1}))$. We also have

$$\left|\frac{r^2}{4t} - \frac{(r+a-1)^2}{2(2t+2s-r-a+1)}\right| \;=\; O\left(\frac{R_N^3}{N^2}\right) \;=\; O(N^{3\theta-1/2}), \qquad \text{and} \tag{9}$$

$$\sqrt{2t+2s-r-a+1} \;=\; \sqrt{2t\left(1+O\left(\frac{R_N}{N}\right)\right)} \;=\; \sqrt{2t}\,(1+O(N^{\theta-1/2})). \tag{10}$$

By Equations (8), (9) and (10),

$$\gamma(t+s,r+a) \;=\; \frac{1}{\sqrt{\pi t}}\, e^{-\frac{r^2}{4t}}\, (1+O(N^{\theta-1/2}))(1+O(N^{3\theta-1/2}))(1+O(N^{4\theta-1}))\,.$$

Since $0 \;>\; 3\theta - \frac{1}{2} \;>\; \max\{4\theta-1, \theta-\frac{1}{2}\}$ for $0 < \theta < \frac{1}{6}$, the lemma follows. ∎

We shall now prove Theorem 2.7, which asserts that, for fixed $\epsilon > 0$ and $0 < \theta < \frac{1}{6}$, we have

$$P_N(S^\bullet(N, N-t, j)) \;=\; \frac{N^{-3/2}}{2\sqrt{\pi}\left(1 - \frac{N-t-j}{N}\right)^{3/2}\left(\frac{N-t-j}{N}\right)^{3/2}}\;\left(1 + O(N^{3\theta-1/2})\right)$$

for $\min\{j, t, N-t-j\} > \epsilon N$.

Proof of Theorem 2.7. For $j < N-t$, we know from Theorem 2.3 that

$$P_N(S^\bullet(N, N-t, j)) = \sum_{i_0=\max\{1,j-t\}}^{j} \frac{C_{N-t-i_0}}{C_N}\frac{(j-i_0+1)^2}{j(t+1)}\binom{i_0+2t-j}{t}\binom{i_0+j-2}{j-1}.$$

Let $r = j - i_0 + 1$, that is, $i_0 = j - r + 1$. Then

$$\begin{aligned}
P_N(S^\bullet(N, N-t, j)) &= \sum_{r=1}^{\min\{j,t+1\}} \frac{r^2}{j(t+1)}\frac{C_{N-t-j+r-1}}{C_N}\binom{2t-r+1}{t}\binom{2j-r-1}{j-1} \\
&= \frac{1}{4}\sum_{r=1}^{\min\{j,t+1\}} \frac{r^2}{j(t+1)}\frac{C_{N-t-j+r-1}}{2^{2(N-t-j+r-1)}}\frac{2^{2N}}{C_N}\frac{\binom{2t-r+1}{t}}{2^{2t-r+1}}\frac{\binom{2j-r-1}{j-1}}{2^{2j-r-1}} \\
&= \frac{1}{4}\sum_{r=1}^{\min\{j,t+1\}} \frac{r^2}{j(t+1)}\,\gamma(t,r)\,\gamma(j-1,r)\,\frac{h(N-t-j+r-1)}{h(N)}.
\end{aligned}$$

To analyze the sum we choose the truncation point $R_N = \lceil N^{1/2+\theta}\rceil$ and consider the sums $S'_N = \sum_{r=1}^{R_N}$ and $T_N = \sum_{r=R_N+1}^{\min\{j,t+1\}}$ separately. Observe that $R_N = o(\min\{j,t\})$.

We shall first prove that T_N is very small. For all $r > R_N$ we have

$$\begin{aligned}
\gamma(t,r) &\le \gamma(t, R_N+1) \qquad \text{(by Remark 5.2(a))} \\
&= \frac{\binom{2t-R_N}{t}}{2^{2t-R_N}} \\
&= O\left(e^{-\frac{R_N^2}{2(2t-R_N)}}\right) \qquad \text{(by Proposition 5.1 with } A = 2t-R_N \text{ and } B = t\text{)}.
\end{aligned}$$

Since $2t - R_N \le 2N$ and since $R_N = \lceil N^{1/2+\theta}\rceil$, we have

$$\gamma(t,r) \;=\; O\left(e^{-(N^{1+2\theta})/4N}\right) \;=\; O\left(e^{-N^\theta}\right). \tag{11}$$

By Remark 5.2(c) and the fact that $\frac{j}{N} < \frac{N-t}{N} - \epsilon$, we obtain for $r > R_N$ that

$$\begin{aligned}
\frac{h(N-t-j+r-1)}{h(N)} &= \frac{N^{3/2}}{(N-t-j+r-1)^{3/2}}\left(1 + O\left(\frac{1}{N}\right)\right) \\
&\le \left(\frac{N}{N-t-j}\right)^{3/2}\left(1 + O\left(\frac{1}{N}\right)\right) \\
&< \epsilon^{-3/2}\left(1 + O\left(\frac{1}{N}\right)\right) \;=\; O(1).
\end{aligned} \tag{12}$$

Moreover, $\gamma(j-1,r) \le 1$ and $\frac{r^2}{j(t+1)} \le 1$ (since $r \le \min\{t+1, j\}$ in the sum). Therefore

$$T_N \;=\; \frac{1}{4}\sum_{r=R_N+1}^{\min\{t+1,j\}} \frac{r^2}{j(t+1)}\,\gamma(t,r)\gamma(j-1,r)\,\frac{h(N-t-j-1+r)}{h(N)} \;=\; O\left(Ne^{-N^{\theta}}\right). \tag{13}$$

Next we approximate S'_N. Using Lemma 5.3 and Remark 5.2(c), we rewrite the truncated sum as

$$\begin{aligned}
S'_N \;&=\; \frac{1}{4}\sum_{r=1}^{R_N} \frac{r^2}{j(t+1)}\,\gamma(t,r)\gamma(j-1,r)\,\frac{h(N-t-j+r-1)}{h(N)} \\
&=\; \frac{1}{4\pi}\sum_{r=1}^{R_N} \frac{r^2}{j(t+1)}\,\frac{e^{-\frac{r^2}{4t}}}{\sqrt{t}}\,\frac{e^{-\frac{r^2}{4j}}}{\sqrt{j}}\left(\frac{N}{N-t-j+r-1}\right)^{3/2}(1+O(N^{3\theta-1/2}))^2\left(1+O\left(\frac{1}{N}\right)\right) \\
&=\; \frac{1}{4\pi}\sum_{r=1}^{R_N} \frac{r^2}{j^{3/2}t^{3/2}(\frac{N-t-j}{N})^{3/2}} e^{-\frac{r^2}{4t}}e^{-\frac{r^2}{4j}}(1+O(N^{3\theta-1/2})),
\end{aligned}$$

where the last step used $t+1 = t\,(1+O(N^{-1}))$ and

$$(N-t-j+r-1)^{3/2} \;=\; (N-t-j)^{3/2}\left(1+\frac{r-1}{N-t-j}\right)^{3/2} \;=\; (N-t-j)^{3/2}\left(1+O\left(\frac{R_N}{N}\right)\right).$$

Define $X_r := \frac{r}{\sqrt{N}}$ for $r = 1, 2\ldots$ Then

$$N^{3/2}S'_N \;=\; \frac{1}{4\pi}\sum_{r=1}^{R_N} \frac{X_r^2\exp\left(\frac{-X_r^2}{4t/N}\right)\exp\left(\frac{-X_r^2}{4j/N}\right)}{\left(\frac{j}{N}\right)^{3/2}\left(\frac{t}{N}\right)^{3/2}\left(\frac{N-t-j}{N}\right)^{3/2}}\,\frac{1}{\sqrt{N}}(1+O(N^{3\theta-1/2})). \tag{14}$$

Let $K = \frac{1}{4}(\frac{N}{t}+\frac{N}{j})$. By our assumptions, $K \asymp 1$. By Proposition 3.6 with $\Delta = \frac{1}{\sqrt{N}}$, we get

$$\sum_{r=1}^{R_N} X_r^2 e^{-KX_r^2}N^{-1/2} \;=\; \int_0^{R_N/\sqrt{N}} z^2e^{-Kz^2}dz \;+\; O(N^{\theta-1/2}).$$

Hence, by Lemma 3.8,

$$\begin{aligned}
\sum_{r=1}^{R_N} X_r^2 e^{-KX_r^2}N^{-1/2} \;&=\; \frac{\sqrt{\pi}}{4K^{3/2}}\left(1+O\left(e^{-R_N\sqrt{K/N}}\right)\right) \;+\; O\left(N^{\theta-1/2}\right) \\
&=\; \frac{\sqrt{\pi}}{4K^{3/2}} \;+\; O(N^{\theta-1/2}).
\end{aligned}$$

As we plug this into Equation (14) we obtain

$$S'_N \;=\; N^{-3/2}\left(\frac{1}{2\sqrt{\pi}\left(1-\frac{N-t-j}{N}\right)^{3/2}\left(\frac{N-t-j}{N}\right)^{3/2}}+O(N^{\theta-1/2})\right)\left(1+O\left(N^{3\theta-1/2}\right)\right).$$

Recalling Equation (13), the theorem follows. □

Remark 5.4. For future reference, we note that Theorem 2.7 and Remark 5.2(b) show that for $0 \prec\prec j \prec\prec N-t \prec\prec N$ we have

$$\sum_{r=1}^{\min\{j,t+1\}} \frac{r^2}{j(t+1)}\,\gamma(t,r)\,\gamma(j-1,r)\,h(N-t-j+r-1) \tag{15}$$

$$= 4\left(\frac{h(N)N^{3/2}}{2\sqrt{\pi}(t+j)^{3/2}(N-t-j)^{3/2}}\right)\left(1+O(N^{3\theta-1/2})\right)$$

$$= \frac{2}{\pi(t+j)^{3/2}(N-t-j)^{3/2}}\left(1+O(N^{3\theta-1/2})\right)\left(1+O\left(\frac{1}{N}\right)\right) \tag{16}$$

$$= \sum_{r=1}^{\min\{j,t+1\}} \frac{r^2}{j(t+1)}\,\gamma(t,r)\,\gamma(j-1,r)\,\frac{(1+O(\frac{1}{N}))}{\sqrt{\pi}(N-t-j+r-1)^{3/2}}. \tag{17}$$

Next we shall prove Theorem 2.9, which says that, for fixed $\epsilon > 0$ and $0 < \theta < \frac{1}{6}$, we have

$$P_N\left(S^{\searrow}(N)\right) = \frac{1}{4\pi}\frac{N^{-3}(1+O(N^{3\theta-1/2}))}{\left(\frac{(N-t_2-j_2)-(N-t_1-j_1)}{N}\right)^{3/2}\left(\frac{N-t_1-j_1}{N}\right)^{3/2}\left(1-\frac{N-t_2-j_2}{N}\right)^{3/2}}$$

for $\min\{j_2, j_1-j_2, N-t_1-j_1, t_1-t_2, t_2\} > \epsilon N$.

Proof of Theorem 2.9. Let $u = j_2 - i_1 + 1$ and $r = j_1 - j_2 - i_0 + 1$ in Theorem 2.8. Then we obtain

$$\begin{aligned}
P_N\left(S^{\searrow}(N)\right) &= \sum_{u=1}^{\min\{j_2,t_2+1\}} \frac{u^2}{j_2(t_2+1)}\binom{2t_2-u+1}{t_2}\binom{2j_2-u-1}{j_2-1} \\
&\quad \sum_{r=1}^{\min\{j_1-j_2,t_1-t_2\}} \frac{r^2}{(j_1-j_2)(t_1-t_2)}\binom{2(t_1-t_2-1)-r+1}{t_1-t_2-1}\times \\
&\quad \binom{2(j_1-j_2-1)-r+1}{j_1-j_2-1}\frac{C_{N-t_1-j_1+u+r-1}}{C_N} \\
&= \frac{1}{2^4}\sum_{u=1}^{\min\{j_2,t_2+1\}} \frac{u^2}{j_2(t_2+1)}\frac{\binom{2t_2-u+1}{t_2}}{2^{2t_2-u+1}}\frac{\binom{2j_2-u-1}{j_2-1}}{2^{2j_2-u-1}} \\
&\quad \sum_{r=1}^{\min\{j_1-j_2,t_1-t_2\}} \frac{r^2}{(j_1-j_2)(t_1-t_2)}\frac{\binom{2(t_1-t_2-1)-r+1}{t_1-t_2-1}}{2^{2(t_1-t_2-1)-r+1}}\frac{\binom{2(j_1-j_2-1)-r+1}{j_1-j_2-1}}{2^{2(j_1-j_2-1)-r+1}}\frac{2^{2N}}{C_N}\times \\
&\quad \frac{C_{N-t_1-j_1+u+r-1}}{2^{2(N-t_1-j_1+u+r-1)}} \\
&= \frac{1}{2^4}\sum_{u=1}^{\min\{j_2,t_2+1\}} \frac{u^2}{j_2(t_2+1)}\gamma(t_2,u)\gamma(j_2-1,u)\times
\end{aligned}$$

$$\sum_{r=1}^{\min\{j_1-j_2,t_1-t_2\}} \frac{r^2\gamma(t_1-t_2-1,r)\gamma(j_1-j_2-1,r)}{(j_1-j_2)(t_1-t_2)} \frac{h(N-t_1-j_1+u+r-1)}{h(N)}.$$

First we consider the inner sum. We substitute t for t_1-t_2-1, j for j_1-j_2, and N_u for $N-t_2-j_2+u-1$, and use Remark 5.4 [Equations (15) and (16)] and Remark 5.2(b) to conclude that

$$\begin{aligned}
&\sum_{r=1}^{\min\{j_1-j_2,t_1-t_2\}} \frac{r^2\,\gamma(t_1-t_2-1,r)\gamma(j_1-j_2-1,r)}{(j_1-j_2)(t_1-t_2)} \frac{h(N-t_1-j_1+u+r-1)}{h(N)} \\
&\quad = \sum_{r=1}^{\min\{j,t+1\}} \frac{r^2}{j(t+1)}\,\gamma(t,r)\gamma(j-1,r)\,\frac{h(N_u-t-j+r-1)}{h(N)} \\
&\quad = \frac{2(1+O(N^{3\theta-1/2}))}{\pi(t+j)^{3/2}(N_u-t-j)^{3/2}}\sqrt{\pi}N^{3/2}\left(1+O\left(\frac{1}{N}\right)\right) \qquad (\text{since } N_u \geq \epsilon N).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
P_N\left(S^{\searrow}(N)\right) \;=\; & \frac{N^{3/2}}{8}\frac{(1+O(N^{3\theta-1/2}))}{(t+j)^{3/2}}\times \\
& \sum_{u=1}^{\min\{j_2,t_2+1\}} \frac{u^2}{j_2(t_2+1)}\,\gamma(t_2,u)\,\gamma(j_2-1,u)\,\frac{(1+O(\frac{1}{N}))}{\sqrt{\pi}(N_u-t-j)^{3/2}}.
\end{aligned}$$

We notice that $N_u-t-j=(N-t-j)-t_2-j_2+u-1$. Finally, we apply Remark 5.4 [Equations (16–17)] one more time, replacing N by $N-t-j$, t by t_2, and j by j_2, obtaining

$$\begin{aligned}
P_N\left(S^{\searrow}(N)\right) \;&=\; \frac{N^{3/2}(1+O(N^{3\theta-1/2}))}{8(t+j)^{3/2}}\frac{2(1+O(N^{3\theta-1/2}))}{\pi(t_2+j_2)^{3/2}((N-t-j)-t_2-j_2)^{3/2}} \\
&=\; \frac{N^{3/2}(1+O(N^{3\theta-1/2}))}{4\pi(t_1-t_2+j_1-j_2-1)^{3/2}(t_2+j_2)^{3/2}(N-t_1-j_1+1)^{3/2}}\,.
\end{aligned}$$

Theorem 2.9 follows. □

Next we prove Theorem 2.11, which says that, for fixed $\epsilon>0$ and $0<\theta<\frac{1}{6}$, we have

$$P_N(S^{\nearrow}(N)) \;=\; \frac{N^{-3}(1+O(N^{3\theta-1/2}))}{4\pi\left(1-\frac{(N-t_1-j_1)+(N-t_2-j_2)}{N}\right)^{3/2}\left(\frac{N-t_1-j_1}{N}\right)^{3/2}\left(\frac{N-t_2-j_2}{N}\right)^{3/2}}$$

for $\min\{j_1, N-t_1-j_1, j_2-(N-t_1), N-t_2-j_2, t_2\} > \epsilon N$.

Proof of Theorem 2.11. By Theorem 2.10,

$$\begin{aligned}P_N(S^{\nearrow}(N)) \;=\; & \sum_{i_1=\max\{1,j_1-j_2+N-t_1+1\}}^{j_1} \; \sum_{i_2=\max\{N-t_1+1,j_2-t_2\}}^{j_2} \frac{C_{N-t_1-i_1}C_{N-t_2-i_2}}{C_N}\times \\ & \frac{(j_1-i_1+1)(j_2-i_2+1)}{j_1(t_2+1)}\binom{j_1+i_1-2}{j_1-1}\binom{i_2+2t_2-j_2}{t_2}\times \\ & \left(\binom{i_1+i_2+j_2-j_1-2(N-t_1+1)}{j_2+(i_1-j_1-1)-(N-t_1)}-\binom{i_1+i_2+j_2-j_1-2(N-t_1+1)}{j_2-(N-t_1)}\right).\end{aligned}$$

Let $r_1=j_1-i_1+1$ and $r_2=j_2-i_2+1$. As in the proof of Theorem 2.9, we use $h(u)$ and $\gamma(t,r)$ to rewrite the probability as

$$\begin{aligned}P_N(S^{\nearrow}(N)) \;=\; & \frac{1}{2^4}\times \\ & \sum_{r_1=1}^{\min\{j_1,j_2-N+t_1\}} \; \sum_{r_2=1}^{\min\{t_2+1,j_2-N+t_1\}} \frac{r_1}{j_1}\frac{r_2}{t_2+1}h(N-t_1-j_1+r_1-1)\gamma(j_1-1,r_1)\gamma(t_2,r_2)\times \\ & \frac{h(N-t_2-j_2+r_2-1)}{h(N)}\left(\gamma(j_2+t_1-N-r_1,r_2-r_1+1)-\gamma(j_2+t_1-N,r_1+r_2+1)\right).\end{aligned}$$

To analyze the sum we choose the truncation point $R_N=\lceil N^{1/2+\theta}\rceil$ and consider the sums $S'_N=\sum_{r_1=1}^{R_N}\sum_{r_2=1}^{R_N}$ and $T_N=P_N(S^{\nearrow}(N))-S'_N$.

We first show that T_N is very small. We view T_N as a sum over pairs (r_1,r_2) in which at least one of r_1 or r_2 is greater than R_N. If $r_1>R_N$, then $\gamma(j_1-1,r_1)=O(e^{-N^\theta})$ (recalling the argument for Equation (11)). Similarly, if $r_2>R_N$, then $\gamma(t_2,r_2)=O(e^{-N^\theta})$. As in Equation (12), we know that $\dfrac{h(N-t_2-j_2+r_2-1)}{h(N)}=O(1)$. Also, $h(N-t_1-j_1+r_1-1)\le 1$, $\gamma(t_2,r_2)\le 1$, and $\gamma(j_1-1,r_1)\le 1$. Moreover, for $r_1\le\min\{j_1,j_2-N+t_1\}$ and $r_2\le\min\{t_2+1,j_2-N+t_1\}$ we get $\dfrac{r_1}{j_1}\dfrac{r_2}{t_2+1}\le 1$. Thus the largest term in T_N is $O(e^{-N^\theta})$, and hence $T_N=O(N^2)\,O(e^{-N^\theta})=O(e^{-N^\theta/2})$.

Next, we approximate S'_N. For the rest of the proof, we will write $-\xi=3\theta-1/2$. By Lemma 5.3 we have that $\gamma(t_2,r_2)=\dfrac{1}{\sqrt{\pi t_2}}e^{-\frac{r_2^2}{4t_2}}(1+O(N^{-\xi}))$, $\gamma(j_1-1,r_1)=\dfrac{1}{\sqrt{\pi j_1}}e^{-\frac{r_1^2}{4j_1}}(1+O(N^{-\xi}))$ and

$$\begin{aligned}&\gamma(j_2+t_1-N-r_1,r_2-r_1+1)-\gamma(j_2+t_1-N,r_1+r_2+1)\\ &\quad=\;\frac{1}{\sqrt{\pi}\sqrt{(j_2+t_1-N)}}\left(e^{-\frac{(r_2-r_1)^2}{4(j_2+t_1-N)}}-e^{-\frac{(r_1+r_2)^2}{4(j_2+t_1-N)}}+O(N^{-\xi})\right).\end{aligned}$$

(For the difference, we need to be careful about relative errors: we use $A_N(1+O(N^{-\xi}))-B_N(1+O(N^{-\xi}))=A_N-B_N+\max\{A_N,B_N\}O(N^{-\xi})$.) We also use Remark 5.2(b,c), and rewrite S_N' as

$$\begin{aligned} S_N' &= \frac{1}{2^4}\sum_{r_1=1}^{R_N}\sum_{r_2=1}^{R_N}\frac{r_1}{j_1}\frac{r_2}{t_2+1}\frac{e^{-r_1^2/4j_1}\,e^{-r_2^2/4t_2}}{\sqrt{\pi j_1}\sqrt{\pi t_2}\sqrt{\pi}(N-t_1-j_1+r_1-1)^{3/2}}\times \\ &\quad \frac{1}{\sqrt{\pi(j_2+t_1-N)}}\left(e^{-\frac{(r_2-r_1)^2}{4(j_2+t_1-N)}}-e^{-\frac{(r_1+r_2)^2}{4(j_2+t_1-N)}}+O(N^{-\xi})\right)\times \\ &\quad \left(\frac{N}{N-t_2-j_2+r_2-1}\right)^{3/2}\left(1+O\left(N^{-1}\right)\right)(1+O(N^{-\xi})) \\ &= \frac{1}{2^4\pi^2}\sum_{r_1=1}^{R_N}\sum_{r_2=1}^{R_N}\frac{r_1}{j_1}\frac{r_2}{t_2+1}\frac{e^{-r_1^2/4j_1}\,e^{-r_2^2/4t_2}\,(1+O(N^{-\xi}))}{\sqrt{j_1}\sqrt{t_2}(N-t_1-j_1)^{3/2}(\frac{N-t_2-j_2}{N})^{3/2}}\times \\ &\quad \frac{1}{\sqrt{(j_2+t_1-N)}}\left(e^{-\frac{(r_2-r_1)^2}{4(j_2+t_1-N)}}-e^{-\frac{(r_1+r_2)^2}{4(j_2+t_1-N)}}+O(N^{-\xi})\right). \end{aligned}$$

Define $X_{r_1}=\frac{r_1}{\sqrt{N}}$ and $X_{r_2}=\frac{r_2}{\sqrt{N}}$. Then

$$\begin{aligned} N^3S_N' &= \frac{1}{2^4\pi^2}\sum_{r_1=1}^{R_N}\sum_{r_2=1}^{R_N}\frac{X_{r_1}X_{r_2}\exp(\frac{-X_{r_1}^2}{4j_1/N})\exp(\frac{-X_{r_2}^2}{4t_2/N})(1+O(N^{-\xi}))}{\left(\frac{j_1}{N}\right)^{3/2}\left(\frac{t_2}{N}\right)^{3/2}\left(\frac{N-t_1-j_1}{N}\right)^{3/2}\left(\frac{N-t_2-j_2}{N}\right)^{3/2}\left(\frac{j_2+t_1-N}{N}\right)^{1/2}}\times \\ &\left(\exp\left(\frac{-(X_{r_2}-X_{r_1})^2}{\frac{4(j_2+t_1-N)}{N}}\right)-\exp\left(\frac{-(X_{r_1}+X_{r_2})^2}{\frac{4(j_2+t_1-N)}{N}}\right)+O(N^{-\xi})\right)\frac{1}{\sqrt{N}}\frac{1}{\sqrt{N}}. \end{aligned}$$

Let $K_1=\frac{N}{4t_2}$, $K_2=\frac{N}{4j_1}$ and $K_3=\frac{N}{4(j_2+t_1-N)}$. Hence, by Proposition 3.7,

$$\begin{aligned} &N^3S_N' = \\ &\frac{\int_0^{R_N/\sqrt{N}}\int_0^{R_N/\sqrt{N}} xye^{-K_1x^2}e^{-K_2y^2}(e^{-K_3(y-x)^2}-e^{-K_3(x+y)^2}+O(N^{-\xi}))dx\,dy+O(N^{-\xi-\theta})}{2^4\,\pi^2\,\left(\frac{j_1}{N}\right)^{3/2}\left(\frac{t_2}{N}\right)^{3/2}\left(\frac{N-t_1-j_1}{N}\right)^{3/2}\left(\frac{N-t_2-j_2}{N}\right)^{3/2}\left(\frac{j_2+t_1-N}{N}\right)^{1/2}} \\ &\qquad\times\left(1+O(N^{-\xi})\right). \end{aligned}$$

Hence, we apply Lemma 3.9 with $w_1=w_2=\frac{R_N}{\sqrt{N}}$ and obtain that

$$P_N(S^{\nearrow}(N)) = \frac{N^{-3}(1+O(N^{-\xi}))}{4\pi\left(1-\frac{(N-t_1-j_1)+(N-t_2-j_2)}{N}\right)^{3/2}\left(\frac{N-t_1-j_1}{N}\right)^{3/2}\left(\frac{N-t_2-j_2}{N}\right)^{3/2}}.$$

□

6 The Lower Right Corner

We begin with the proof of Proposition 2.15. Recall the function ρ from Definition 2.14.

Proof of Proposition 2.15. (a) Fix $a, b \in \mathbb{N}$. We shall prove that $P_N(S^\bullet(N, N-a+1, b)) = \rho(a,b)(1+O(N^{-1}))$. For $N > a+b$, we have from Theorem 2.3 that

$$\frac{|S^\bullet(N, N-a+1, b)|}{C_N} = \sum_{i_0=\max\{1,b-a+1\}}^{b} \frac{C_{N-a-i_0+1}}{C_N} \frac{(b-i_0+1)^2}{ba} \binom{i_0+2(a-1)-b}{a-1} \binom{i_0+b-2}{b-1}.$$

By the fact that $\frac{C_k}{C_{k+1}} = \frac{1}{4}(1+\frac{3}{2k+1})$ and since i_0 is bounded, we get that

$$\begin{aligned} &P_N(S^\bullet(N,N-a+1,b)) \\ &\quad = \sum_{i_0=\max\{1,b-a+1\}}^{b} \frac{(1+O(N^{-1}))}{4^{i_0+a-1}} \frac{(b-i_0+1)^2}{ba} \binom{i_0+2(a-1)-b}{a-1} \binom{i_0+b-2}{b-1} \\ &\quad = \rho(a,b)\,(1+O(N^{-1})). \end{aligned}$$

(b) The proof is by induction on k. The $k=1$ case is part (a). Assume that the result holds for $k-1$. Assume N is large enough that $B_m < N - A_m + 1$ for all $m = 1, \ldots, k$. By Theorem 2.8, for all $k \geq 2$ we have that

$$\begin{aligned} &\frac{\left|S^{\searrow k}(N, N-A_k+1, \ldots, N-A_1+1, B_k, \ldots, B_1)\right|}{C_N} \\ &= \sum_{i=\max\{1,B_1-A_1+1\}}^{B_1} \frac{\left|S^{\square}(i+A_1-1, i, B_1)\right|}{C_N} \left|S^{\searrow k-1}(\tilde{N}, \tilde{N}-\tilde{A}_k+1, \ldots, \tilde{N}-\tilde{A}_2+1, \tilde{B}_k, \ldots, \tilde{B}_2)\right| \end{aligned}$$

where $\tilde{N} \equiv \tilde{N}(i) = N - A_1 - i + 1$, $\tilde{A}_m = A_m - A_1$, $\tilde{B}_m = B_m - B_1$ for all $2 \leq m \leq k$. Then

$$\begin{aligned} &P_N(S^{\searrow k}(N, N-A_k+1, \ldots, N-A_1+1, B_k, \ldots, B_1)) \\ &= \sum_{i=\max\{1,B_1-A_1+1\}}^{B_1} \left|S^{\square}(i+A_1-1, i, B_1)\right| \frac{C_{\tilde{N}}}{C_N} \times \\ &\qquad \frac{\left|S^{\searrow k-1}(\tilde{N}, \tilde{N}-\tilde{A}_k+1, \ldots, \tilde{N}-\tilde{A}_2+1, \tilde{B}_k, \ldots, \tilde{B}_2)\right|}{C_{\tilde{N}}} \\ &= \sum_{i=\max\{1,B_1-A_1+1\}}^{B_1} \frac{\left|S^{\square}(i+A_1-1, i, B_1)\right|}{4^{i+A_1-1}} (1+O(N^{-1})) P_{\tilde{N}}(\sigma_{\tilde{N}-\tilde{A}_2+1} = \tilde{B}_2, \ldots, \sigma_{\tilde{N}-\tilde{A}_k+1} = \tilde{B}_k) \\ &= \sum_{i=\max\{1,B_1-A_1+1\}}^{B_1} \frac{\left|S^{\square}(i+A_1-1, i, B_1)\right|}{4^{i+A_1-1}} (1+O(N^{-1})) \times \prod_{v=2}^{k} \rho(a_v, b_v) \\ &= \left[\prod_{v=1}^{k} \rho(a_v, b_v)\right] (1+O(N^{-1})). \end{aligned}$$

The last two equations follow by the inductive step assumption. □

Our next task is to prove Theorem 2.17, which says that the random variables $\{X_q^N\}$ have a limit $\{X_q\}$ as $N \to \infty$. This is facilitated with some notation.

Let $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{Z}^2$. We write $u \nwarrow v$ if u is "northwest" of v, i.e. if $u_1 < v_1$ and $u_2 > v_2$. Thus $Q = \{v \in \mathbb{Z}^2 : v \nwarrow 0\}$. Let Seq$\nwarrow$ be the set of all finite and infinite subsets $T = \{t^{(1)}, t^{(2)}, t^{(3)}, \ldots\}$ of Q with $t^{(i+1)} \nwarrow t^{(i)}$ for each i.

For finite subsets D and F of Q, let

$$\theta_N(D, F) \;=\; P_N(X_q^N = 1 \;\forall q \in D, \text{ and } X_r^N = 0 \;\forall r \in F) \quad \text{for } N \in \mathbb{N},$$

and let $\theta(D, F) = \lim_{N\to\infty} \theta_N(D, F)$ if this limit exists.

Proof of Theorem 2.17. By Kolmogorov's Extension Theorem, it suffices to prove that the limit $\theta(D, F)$ exists for all disjoint finite subsets D and F of Q. Proposition 2.15 shows that $\theta(D, \emptyset)$ exists whenever D is a finite subset of Seq$\nwarrow$. If $-i_1 < -i_2$ and $j_1 < j_2$, then Proposition 2.10(a) shows that $\theta_N(\{(-i_1, j_1), (-i_2, j_2)\}, \emptyset) = 0$ for sufficiently large N. It follows that $\theta(D, \emptyset)$ exists and equals 0 whenever D is a finite subset of Q that is <u>not</u> in Seq$\nwarrow$.

Let D and F be disjoint finite subsets of Q. The following argument is a generalization of the proof in Kingman [6] for one-dimensional regenerative sequences. We have

$$\begin{aligned}
\theta_N(D, F) &= E_N\left(\left(\prod_{q\in D} X_q^N\right) \prod_{r\in F}(1 - X_r^N)\right) \\
&= E_N\left(\left(\prod_{q\in D} X_q^N\right) \sum_{G\subset F}\prod_{s\in G}(-X_s^N)\right) \\
&= \sum_{G\subset F}(-1)^{|G|} E_N\left(\prod_{s\in D\cup G} X_s^N\right) \\
&= \sum_{G\subset F}(-1)^{|G|}\theta_N(D\cup G, \emptyset). \qquad (18)
\end{aligned}$$

From the previous paragraph, we know that the final expression converges as $N \to \infty$. Hence $\theta_N(D, F)$ converges. □

Having proven the above theorem, we know that

$$\theta(D, F) \;=\; P_\infty(X_q = 1 \;\forall q \in D, \text{ and } X_r = 0 \;\forall r \in F) \quad \text{for finite } D, F \subset Q.$$

Some properties of the limiting collection $\{X_q\}$ follow immediately from Proposition 2.15. In particular, $P_\infty(X_{(-i,j)} = 1) = \rho(i, j) = 1 - P_\infty(X_{(-i,j)} = 0)$. More generally, we see from Proposition 2.15(b) that if T is a finite subset of Q of the form

$$\begin{aligned}
T &= \{(-A_m, B_m) \,:\, m = 1, \ldots, k\} \quad \text{where} \\
A_m &= \sum_{l=1}^{m} a_l, \;\; B_m = \sum_{l=1}^{m} b_l \text{ for } m = 1, \ldots, k \text{ and } a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{N} \qquad (19)
\end{aligned}$$

(in particular $T \in \text{Seq}\nwarrow$), then $\theta(T,\emptyset) = \prod_{m=1}^{k} \rho(a_m,b_m)$. As we mentioned in section 2, this is a kind of two-dimensional regenerative property that corresponds to the fact (Theorem 2.19, proven below) that the random set of points q where X_q is 1 follows the law of a trajectory of a random walk that can only move up and left—i.e., a two-dimensional analogue of a renewal process. Another special case we consider is $\theta(T, B(T))$ where $B(T) = [-A_k,-1] \times [1,B_k] \setminus T$; this is the probability that T is exactly the set of locations of all the 1's in the smallest rectangle containing T and the bottom right corner of Q. Proposition 6.3 below shows that $\theta(T, B(T))$ also has a product form.

Notation 6.1. For T of the form (19) let

$$\Lambda_N(T) = \{\sigma \in S_N(312) : \sigma_{N-A_m+1} = B_m \quad \text{for} \quad m = 1,\dots,k \quad \text{and} \quad \sigma_{N-t+1} > B_k \quad \text{for} \\ t \in [1,A_k] \setminus \{A_1,\dots,A_k\}\} \,.$$

Theorem 6.2. *Let T be of the form (19). Then for $N \geq A_k + B_k$,*

$$P_N(\Lambda_N(T)) \;=\; \frac{C_{N-A_k-B_k+k}}{C_N} \prod_{m=1}^{k} C_{a_m-1} C_{b_m-1}.$$

Proof. We will give a proof by induction on k. First we shall prove the $k=1$ case, i.e.

$$|\Lambda_N(\{(-A_1,B_1)\})| \;=\; C_{N-A_1-B_1+1}\, C_{a_1-1}\, C_{b_1-1}\,.$$

Consider the map $\phi_{\bullet;N,t,j}$ in Definition 4.5. We take $t = A_1 - 1$ and $j = B_1$. Let $\mathbb{I}$ be the image of $S_{N-(A_1-1)-B_1}(312) \times S^{\square}(A_1+B_1-1, B_1, B_1)$ under $\phi_{\bullet;N,t,j}$. We will show that $\mathbb{I} = \Lambda_N(\{(-A_1,B_1)\})$.

Let $\tau \in S_{N-A_1-B_1+1}(312)$ and $\rho \in S^{\square}(A_1+B_1-1, B_1, B_1)$, and let $\psi = \phi_{\bullet;N,(A_1-1),B_1}((\tau,\rho)) = \textbf{Insert}(\tau,\rho,B_1)$. Then $\psi_{N-A_1+1} = B_1$ by Proposition 4.2. By the Pigeonhole Principle, $\rho_m > B_1$ for all $m > B_1$. Therefore, by Definition 4.1, for $u \in [1, A_1)$ we have

$$\psi_{N-u+1} \;\geq\; \rho_{N-u+1-(N-A_1-B_1+1)} \;=\; \rho_{B_1+A_1-u} \;>\; B_1$$

(using the above observation, since $B_1 + A_1 - u > B_1$). Hence $\psi \in \Lambda_N(\{(-A_1,B_1)\}$, which proves that $\mathbb{I} \subseteq \Lambda_N(\{(-A_1,B_1)\})$.

Now let $\sigma \in \Lambda_N(\{(-A_1,B_1)\})$ and let $i_0 = \min\{i \in [1,N] : \sigma_i > B_1\}$. By Lemma 4.3(a,b), $i_0 \in [1, N-A_1]$ and $\sigma_i > B_1$ for all $i \in [i_0, N-A_1]$. Since also $\sigma_i > B_1$ for all $i \in (N-A_1+1, N]$, we see that $\sigma_i < B_1$ if and only if $i < i_0$; hence $i_0 = B_1$. By Lemma 4.3(f), we can write $\sigma = \textbf{Insert}(\tilde{\sigma}, \hat{\sigma}, B_1)$ where $\hat{\sigma} \in S^{\square}(A_1+B_1-1, B_1, B_1)$ and $\tilde{\sigma} \in S_{N-A_1-B_1+1}(312)$. Hence $\sigma \in \mathbb{I}$ and we conclude that $\mathbb{I} = \Lambda_N(\{(-A_1,B_1)\})$. By Lemma 4.4 the restricted map is one-to-one and hence a bijection with $\Lambda_N(\{(-A_1,B_1)\})$. We conclude that

$$\begin{aligned} |\Lambda_N(\{(-A_1,B_1)\}| \;&=\; \left|S_{N-(A_1-1)-B_1}(312)\right| \cdot \left|S^{\square}(A_1+B_1-1, B_1, B_1)\right| \\ &=\; C_{N-A_1-B_1+1}\, C_{B_1-1}\, C_{A_1-1} = C_{N-A_1-B_1+1}\, C_{b_1-1}\, C_{a_1-1}, \end{aligned}$$

where we have used Remark 2.2. This proves the result for $k = 1$.

For the induction step, we assume that the statement is true for $k-1$. Given $T = \{(-A_m, B_m) : m = 1, \dots, k\} \in \mathrm{Seq}^{\nwarrow}$, let $T^* = \{(-A_m, B_m) - (-A_1, B_1) : m = 2, \dots, k\} = \{(-A_m^*, B_m^*) : m = 2, \dots, k\} \in \mathrm{Seq}^{\nwarrow}$ where $A_m^* = \sum_{l=2}^m a_l = A_m - A_1$ and $B_m^* = \sum_{l=2}^m b_l = B_m - B_1$. Let $\hat{\phi}$ be the restriction of the map $\phi_{A;N,A_1-1,B_1}$ to the domain $\Lambda_{N-A_1-B_1+1}(T^*) \times S^{\square}(A_1 + B_1 - 1, B_1, B_1)$, and let $\hat{\mathbb{I}}$ be the image of $\hat{\phi}$. We claim that $\hat{\mathbb{I}} = \Lambda_N(T)$ and that $\hat{\phi}$ is a bijection. Let $\tau \in \Lambda_{N-A_1-B_1+1}(T^*)$ and $\rho \in S^{\square}(A_1 + B_1 - 1, B_1, B_1)$, and let $\psi = \hat{\phi}(\tau, \rho) = \mathbf{Insert}(\tau, \rho, B_1)$. As in the k=1 case, we have $\psi_{N-A_1+1} = B_1$. For $m \in [2, k]$, we have $B_1 \le N - A_m + 1 < N - A_1 + 1$, so

$$\psi_{N-A_m+1} \;=\; \tau_{(N-A_m+1)-B_1+1} + \rho_{B_1} \;=\; \tau_{(N-A_1-B_1+1)-A_m^*+1} + B_1 \;=\; B_m^* + B_1 \;=\; B_m \,.$$

Now it is not hard to see that $\psi \in \Lambda_N(T)$. This shows that $\hat{\mathbb{I}} \subset \Lambda_N(T)$.

Next we show that $\Lambda_N(T) \subset \hat{\mathbb{I}}$. Let $\sigma \in \Lambda_N(T)$. Let $i_0 = \min\{i \in [1, N] : \sigma_i > B_1\}$. Then $i_0 = B_1$ (as shown in $k = 1$ case), and, by Lemma 4.3(f), $\sigma = \mathbf{Insert}(\tilde{\sigma}, \hat{\sigma}, B_1)$ where $\hat{\sigma} = \mathbf{Patt}(\sigma_1, \dots, \sigma_{B_1-1}, \sigma_{N-A_1+1}, \dots, \sigma_N) \in S^{\square}(A_1 + B_1 - 1, B_1, B_1)$ and $\tilde{\sigma} = \mathbf{Patt}(\sigma_{B_1}, \dots, \sigma_{N-A_1}) \in S_{N-A_1-B_1+1}(312)$. Moreover, for $2 \le m \le k$,

$$\tilde{\sigma}_{(N-A_1-B_1+1)-A_m^*+1} = \tilde{\sigma}_{(N-A_m+1)-B_1+1} = \sigma_{N-A_m+1} - B_1 = B_m - B_1 = B_m^*.$$

From here it is not hard to show that $\tilde{\sigma} \in \Lambda_{N-A_1-B_1+1}(T^*)$ and hence $\psi \in \hat{\mathbb{I}}$. This proves $\Lambda_N(T) \subset \hat{\mathbb{I}}$. Again, by Lemma 4.4 the map $\hat{\phi}$ is one-to-one. This verifies the claimed bijection, and we conclude that

$$|\Lambda_N(T)| = |\Lambda_{N-A_1-B_1+1}(T^*)| \left|S^{\square}(A_1 + B_1 - 1, B_1, B_1)\right| .$$

By the inductive step and Remark 2.2, the result follows. □

The following result is a direct corollary. Recall that for T of the form (19), we define $B(T) = [-A_k, -1] \times [1, B_k] \setminus T$. Also recall the function $\pi(-a, b)$ from Definition 2.18.

Proposition 6.3. *Let T be of the form (19). Then*

$$\theta(T, B(T)) \;=\; \lim_{N\to\infty} P_N(\Lambda_N(T)) \;=\; \prod_{m=1}^{k} \frac{C_{b_m-1} C_{a_m-1}}{4^{b_m+a_m-1}} \;=\; \prod_{m=1}^{k} \pi(-a_m, b_m)\,.$$

Proof. This follows from Theorem 6.2 and Remark 5.2(c) together with the observation that $\theta_N(T, B(T)) = P_N(\Lambda_N(T))$. □

Next we give the proof of Theorem 2.19 which states that the set $W^* = \{q \in Q : X_q = 1\}$ is an infinite random member of $\mathrm{Seq}^{\nwarrow}$ of the form $\left\{\vec{V}_m : m \in \mathbb{N}\right\}$ where $\left\{(\vec{V}_m - \vec{V}_{m-1}) : m \in \mathbb{N}\right\}$ are i.i.d. Q-valued random vectors with distribution $\pi(-a, b)$. Recall that for finite $D, F \subset Q$, we have $\theta(D, F) = P_\infty(D \subset W^*, F \cap W^* = \emptyset)$.

Proof of Theorem 2.19. As shown in the proof of Theorem 2.17, $\theta(D,\emptyset) = 0$ whenever D is a finite subset of Q that is *not* in Seq$\nwarrow$. Therefore, with probability one, W^* is a (finite or infinite) member of Seq$\nwarrow$. Write the elements of W^* as $\vec{V}_1, \vec{V}_2, \ldots$, where $\vec{V}_m \nwarrow \vec{V}_{m-1}$ for every $m = 1, \ldots, |W^*|$ (where $\vec{V}_0 = (0,0)$).

For $k \in \mathbb{N}$, consider T of the form (19). Then

$$\theta(T, B(T)) \;=\; P_\infty(|W^*| \geq k \text{ and } \vec{V}_m = (-A_m, B_m) \text{ for } m = 1, \ldots, k)\,.$$

Hence, by Proposition 6.3 and the fact that π is a probability distribution on Q, we see that

$$P_\infty\left(|W^*| \geq k\right) \;=\; \sum_{\substack{T \,\in\, Seq\nwarrow \\ |T| \,=\, k}} \theta(T, B(T)) \;=\; \left(\sum_{(-a,b)\in Q} \pi(-a,b) \right)^k \;=\; 1.$$

Since k is arbitrary, W^* must be infinite with probability 1. The above product form of $\theta(T, B(T))$ shows that the jumps $\{\vec{V}_m - \vec{V}_{m-1}\}$ are i.i.d. with common distribution π.

Finally, each component of $\vec{V}_1$ has infinite mean since $\frac{C_{a-1}}{4^a} = \frac{1}{a\,4^a}\binom{2a-2}{a-1} \;\asymp\; a^{-3/2}$. $\square$

We now shift our focus from the points northwest of the origin to the points northwest of a given point $(N-t, j)$ below the diagonal, and show that, conditional on $X^N_{(-t,j)} = 1$, we get the same limiting probabilities of nearby configurations as we do in the bottom right corner.

Proposition 6.4. *For $m = 1, \ldots, u$, let $a_m, b_m \in \mathbb{N}$ and let $A_m = \sum_{l=1}^m a_l$, $B_m = \sum_{l=1}^m b_l$. Then*

$$\lim_{N-t-j\to\infty} P_N(\sigma_{N-t-A_1+1} = j{+}B_1, \ldots, \sigma_{N-t-A_u+1} = j{+}B_u \,|\, \sigma_{N-t+1} = j) = \prod_{v=1}^{u} \rho(a_v, b_v).$$

Proof. Let $k = u+1$. By Theorem 2.8,

$$\begin{aligned} &\frac{\left|S^{\searrow k}(N, N{-}t{-}A_{k-1}{+}1, \ldots, N{-}t{+}1, j{+}B_{k-1}, \ldots, j)\right|}{|S^{\bullet}(N, N{-}t{+}1, j)|} \\ &\quad= \frac{1}{|S^{\bullet}(N, N{-}t{+}1, j)|} \sum_{i=\max\{1,j-t+1\}}^{j} \left|S^{\square}(i+t-1, i, j)\right| C_{\tilde{N}} \times \\ &\qquad\quad \frac{\left|S^{\searrow k-1}(\tilde{N}, \tilde{N}{-}A_{k-1}{+}1, \ldots, \tilde{N}{-}A_1{+}1, B_{k-1}, \ldots, B_1)\right|}{C_{\tilde{N}}} \end{aligned}$$

where $\tilde{N} \equiv \tilde{N}(i) = N{-}t{-}i{+}1$. By Proposition 2.15(b) and since $\tilde{N} \geq N{-}t{-}j{+}1$, we know that

$$\frac{\left|S^{\searrow k-1}(\tilde{N}, \tilde{N}{-}A_{k-1}{+}1, \ldots, \tilde{N}{-}A_1{+}1, B_{k-1}, \ldots, B_1)\right|}{C_{\tilde{N}}} \;=\; \left[\prod_{v=1}^{k-1} \rho(a_v, b_v)\right] \left(1 + O\left(\frac{1}{N{-}t{-}j}\right)\right).$$

Using this and Theorem 2.3 gives

$$\begin{aligned}
&\frac{\left|S^{\searrow k}(N, N{-}t{-}A_{k-1}{+}1, \ldots, N{-}t{+}1, j{+}B_{k-1}, \ldots, j)\right|}{|S^{\bullet}(N, N-t+1, j)|} \\
&\qquad = \sum_{i=\max\{1,j-t+1\}}^{j} \left|S^{\square}(i+t-1, i, j)\right| C_{\tilde{N}} \, \frac{\left[\prod_{v=1}^{k-1} \rho(a_v, b_v)\right] (1 + O(\frac{1}{N-t-j}))}{|S^{\bullet}(N, N-t+1, j)|} \\
&\qquad = \left[\prod_{v=1}^{k-1} \rho(a_v, b_v)\right] \left(1 + O\left(\frac{1}{N{-}t{-}j}\right)\right) .
\end{aligned}$$

The result follows from the last equality. □

Our final result, Theorem 2.20, follows from Proposition 6.4 and an argument very similar to the proof of Theorem 2.17.

7 Acknowledgements

We thank Alexey Kuznetsov and Sam Miner for helpful discussions. The research of NM was supported in part by a Discovery Grant from NSERC of Canada.

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