

# Odd edge-colorings of subdivisions of odd graphs

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## Abstract

An odd graph is a finite graph all of whose vertices have odd degrees. A graph  $G$  is decomposable into  $k$  odd subgraphs if its edge set can be partitioned into  $k$  subsets each of which induces an odd subgraph of  $G$ . The minimum value of  $k$  for which such a decomposition of  $G$  exists is the odd chromatic index,  $\chi'_o(G)$ , introduced by Pyber (1991). For every  $k \geq \chi'_o(G)$ , the graph  $G$  is said to be odd  $k$ -edge-colorable. Apart from two particular exceptions, which are respectively odd 5- and odd 6-edge-colorable, the rest of connected loopless graphs are odd 4-edge-colorable, and moreover one of the color classes can be reduced to size  $\leq 2$ . In addition, it has been conjectured that an odd 4-edge-coloring with a color class of size at most 1 is always achievable. Atanasov et al. (2016) characterized the class of loopless subcubic graphs in terms of the value  $\chi'_o(G) \leq 4$ . In this paper, we extend their result to a characterization of all loopless subdivisions of odd graphs in terms of the value of the odd chromatic index. This larger class  $\mathcal{S}$  is of a particular interest as it collects all ‘least instances’ of non-odd graphs. As a prelude to our main result, we show that every connected graph  $G \in \mathcal{S}$  requiring the maximum number of four colors, becomes odd 3-edge-colorable after removing a certain edge. Thus, we provide support for the mentioned conjecture by proving it for all subdivisions of odd graphs. The paper concludes with few problems for possible further work.

**Keywords:** odd graph, odd edge-coloring, odd chromatic index, subdivision.

## 1 Introduction

### 1.1 Basic terminology

All considered graphs  $G = (V(G), E(G))$  are undirected and finite, loops and parallel edges are allowed. We follow [2] for any terminology and notation not defined here. The parameters  $n(G) = |V(G)|$  and  $m(G) = |E(G)|$  are called the *order* and the *size* of  $G$ , respectively. A graph of order 1 is *trivial*, and a graph of size 0 is *empty*. A path or cycle is either *odd* or *even* depending on the parity of its size. A path (resp. an edge) with endvertices  $x$  and  $y$

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is referred to as an  $x$ - $y$  path (resp. an  $x$ - $y$  edge). Given a path  $P$  and vertices  $x, y \in V(P)$ , the  $x$ - $y$  subpath of  $P$  is denoted  $xPy$ . For every vertex  $v \in V(G)$ ,  $E_G(v)$  denotes the set of edges incident with  $v$ , and the size of  $E_G(v)$  (every loop being counted twice) is the *degree*,  $d_G(v)$ , of  $v$  in  $G$ . The maximum and minimum vertex degree in  $G$  are denoted by  $\Delta(G)$  and  $\delta(G)$ , respectively. A graph  $G$  is *subcubic* if  $\Delta(G) \leq 3$ . Each vertex  $v$  of even (resp. odd) degree  $d_G(v)$  is an *even* (resp. *odd*) vertex. In particular, if  $d_G(v)$  equals 0 (resp. 1), we say that  $v$  is an *isolated* (resp. *pendant*) vertex of  $G$ . Any vertex of degree  $d$  is called a  $d$ -*vertex*. A graph is *even* (resp. *odd*) whenever all its vertices are even (resp. odd). The set of neighboring vertices of  $v \in V(G)$  is denoted by  $N_G(v)$ . For every  $u \in N_G(v)$ , the edge set  $E_G(u) \cap E_G(v)$  is the  $u$ - $v$  *bouquet* in  $G$ , with notation  $\mathcal{B}_{uv}$ . The maximum size of a bouquet in  $G$  is its *multiplicity*,  $\mu(G)$ . A graph  $G$  is *simple* if it is loopless and of multiplicity at most 1.

For  $X \subseteq V(G) \cup E(G)$ ,  $G - X$  is the subgraph of  $G$  obtained by removing  $X$ ; we abbreviate  $G - \{x\}$  to  $G - x$ . Similarly, given a subgraph  $H \subseteq G$ ,  $H + X$  is the subgraph of  $G$  obtained by adding to  $H$  all the vertices and edges from  $X$ . A spanning subgraph of  $G$  is also called a *factor* of  $G$ .

To *split* a vertex  $v$  is to replace it by two (not necessarily adjacent) vertices  $v'$  and  $v''$ , and to replace each edge incident to  $v$  by an edge incident to either  $v'$  or  $v''$  (but not both, unless the edge is a loop at  $v$ ), the other end of the edge remaining unchanged. A vertex of positive degree can be split in several ways, so the resulting graph is not unique in general. Another local operation on graph  $G$  is to *suppress* a 2-vertex  $v$ . The modified graph  $G \% v$  is obtained from  $G - v$  by adding an edge between the neighbors of  $v$  (the new edge is a link unless  $N_G(v)$  is a singleton).

The *connectivity*,  $\kappa(G)$ , of a graph  $G$  is the minimum size of a subset  $S \subseteq V(G)$  such that  $G - S$  is disconnected or of order 1. A graph is said to be  $k$ -*connected* if its connectivity is at least  $k$ . A vertex  $v \in V(G)$  is a *cutvertex* of  $G$  if  $G - v$  has more (connected) components than  $G$ . If  $V_1, \dots, V_k$  are the vertex sets of all components of  $G - v$ , then for  $i = 1, \dots, k$ , the induced subgraph  $G[V_i \cup \{v\}]$  is called a  $v$ -*lobe* of  $G$ . A *block graph* is a connected graph without any cutvertices. Given a nontrivial connected graph  $G$ , a maximal block subgraph is a *block* of  $G$ . Thus each block is either 2-connected or a bouquet, and each cycle is entirely within a single block. For a block  $B$  of  $G$ , each vertex  $v \in V(B)$  which is not a cutvertex of  $G$  is an *internal vertex* of  $B$  (and of  $G$ ). The collection of internal vertices of  $B$  is denoted by  $\text{Int}_G(B)$ . If  $V(B)$  contains at most one cutvertex of  $G$  then  $B$  is an *end-block*. Any connected graph  $G$  is associated with a bipartite graph  $B(G)$  having bipartition  $(\mathcal{B}, \mathcal{V})$ , where  $\mathcal{B}$  is the set of blocks of  $G$  and  $\mathcal{V}$  the set of cutvertices of  $G$ , a block  $B \in \mathcal{B}$  and a cutvertex  $v \in \mathcal{V}$  being adjacent in  $B(G)$  if and only if  $B$  contains  $v$ . The graph  $B(G)$  is connected and acyclic, the former because  $G$  is connected and the latter because a cycle in  $B(G)$  would correspond to a cycle in  $G$  passing through two or more blocks. The graph  $B(G)$  is therefore a tree, called the *block-tree* of  $G$ . If  $\mathcal{V} \neq \emptyset$ , the end-blocks of  $G$  correspond to the leaves of its block-tree. Every vertex  $v$  of a block graph  $G$  has a neighbor among the internal vertices of each end-block of  $G - v$ .

For a nonempty subset  $X \subset V(G)$ , the *edge cut*  $\partial(X)$  is the set of edges with one endvertex in  $X$  and the other endvertex in  $V(G) \setminus X$ ; in case  $X$  is a singleton, we speak of a *trivial* edge cut  $\partial(X)$ . The *edge-connectivity*,  $\kappa'(G)$ , of a nontrivial graph  $G$  is the minimum size of a subset  $S \subseteq E(G)$  such that  $G - S$  is disconnected; equivalently,  $\kappa'(G)$  is the minimum size of an edge cut in  $G$ . A  $k$ -*edge cut* is an edge cut of size  $k$ ; a 1-edge cut is also called a

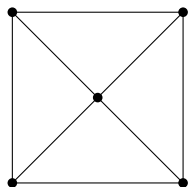
*bridge*. If  $vw$  is a bridge and the vertex  $w$  is not the only neighbor of the vertex  $v$ , then  $v$  is a cutvertex of  $G$ . A graph is said to be  $k$ -edge-connected if its edge-connectivity is at least  $k$ . A  $k$ -edge-connected graph is termed *essentially  $(k + 1)$ -edge-connected* if all of its  $k$ -edge cuts are trivial.

## 1.2 Odd edge-colorings and odd chromatic index

An assignment  $\varphi : E(G) \rightarrow S$  is an *edge-coloring* of  $G$  with *color set*  $S$ . If  $|S| \leq k$ , we speak of a  $k$ -edge-coloring  $\varphi$ . The nature of the colors is irrelevant, and it is conventional to use  $S = [k] := \{1, 2, \dots, k\}$  for a color set of size  $k$ . For each color  $c \in S$ ,  $E_c(G, \varphi)$  denotes the *color class* of  $c$ , that is, the set  $\varphi^{-1}(c)$  of edges colored by  $c$ . Whenever  $G$  and  $\varphi$  are clear from the context, we denote the color class of  $c$  simply by  $E_c$ . Given an edge-coloring  $\varphi$  and a vertex  $v$  of  $G$ , we say that a color  $c$  *appears at*  $v$  if  $E_c \cap E_G(v) \neq \emptyset$ . Any decomposition  $\{H_1, \dots, H_k\}$  of  $G$  can alterably be interpreted as a  $k$ -edge-coloring of  $G$  for which the color classes are  $E(H_1), \dots, E(H_k)$ .

An *odd edge-coloring* of a graph  $G$  is an edge-coloring such that each nonempty color class  $E_c$  induces an odd subgraph of  $G$ . In other words, at each vertex  $v$ , for any appearing color  $c$  the degree  $d_{G[E_c]}(v)$  is odd. Equivalently, an odd edge-coloring can be seen as a decomposition of  $G$  into (edge-disjoint) odd subgraphs. As usual, we are most interested in the least number of colors necessary to create such a coloring. An odd edge-coloring of  $G$  using at most  $k$  colors is referred to as an *odd  $k$ -edge-coloring*, and if such a coloring exists we say that  $G$  is *odd  $k$ -edge-colorable*. Whenever  $G$  admits an odd edge-coloring, the *odd chromatic index*,  $\chi'_o(G)$ , is defined to be the minimum integer  $k$  for which  $G$  is odd  $k$ -edge-colorable.

It is obvious that a necessary and sufficient condition for odd edge-colorability of  $G$  is the absence of vertices incident only to loops. Apart from this, the presence of loops does not influence the existence nor changes the value of the index  $\chi'_o(G)$ . Therefore, the class of loopless graphs comprises a natural framework for the study of the odd chromatic index.

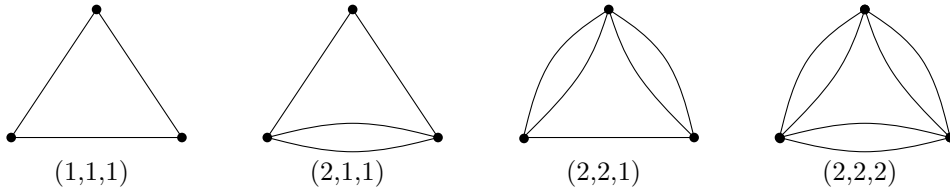


**Figure 1:** The wheel  $W_4$  is a simple graph with  $\chi'_o(W_4) = 4$ .

As a notion, odd edge-coloring was introduced by Pyber in his survey on graph coverings [8]. The mentioned work concerns simple graphs and (among other results) contains a proof of the following.

**Theorem 1.1** (Pyber, 1991). *For every simple graph  $G$ , it holds that  $\chi'_o(G) \leq 4$ .*

Pyber observed that the established upper bound is realized by the wheel on four spokes  $W_4$  (see Figure 1). However, this upper bound of four colors does not apply to the class of all loopless graphs  $G$ . For instance, Figure 2 depicts four graphs with the following characteristic property: each of their odd subgraphs is of order 2 and size 1, that is, a copy of  $K_2$ . Consequently, for each of these graphs the odd chromatic index equals the size.



**Figure 2:** Four Shannon triangles (the smallest one of each type).

As defined in [4], a *Shannon triangle* is a loopless graph on three pairwise adjacent vertices. And if  $p, q, r$  are parities of the sizes of its bouquets in non-increasing order, with 2 (resp. 1) denoting an even-sized (resp. odd-sized) bouquet, then  $G$  is a Shannon triangle of *type*  $(p, q, r)$ . Figure 2 depicts (from left to right) the smallest, in terms of size, Shannon triangle of type  $(1, 1, 1)$ ,  $(2, 1, 1)$ ,  $(2, 2, 1)$ , and  $(2, 2, 2)$ , respectively. It is straightforward that if  $G$  is a Shannon triangle of type  $(p, q, r)$ , then

$$\chi'_o(G) = p + q + r. \quad (1.1)$$

The main result of [4] tells that six colors suffice for an odd edge-coloring of any loopless graph. Furthermore, it characterizes when six colors are necessary.

**Theorem 1.2.** *For every connected loopless graph  $G$ , it holds that  $\chi'_o(G) \leq 6$ . Moreover, equality is attained if and only if  $G$  is a Shannon triangle of type  $(2, 2, 2)$ .*

Recently, the following improvement of Theorems 1.1 and 1.2 has been shown in [6].

**Theorem 1.3.** *Let  $G$  be a connected loopless graph that is not a Shannon triangle of type  $(2, 2, 1)$  or  $(2, 2, 2)$ . Then  $G$  admits an odd edge-coloring with color set  $\{1, 2, 3, 4\}$  such that the color class  $E_4$  satisfies two additional conditions:*

- (i)  $|E_4| \in \{0, 1, 2\}$ , and if  $|E_4| = 2$  then the pair of edges colored by 4 are at distance 2 (i.e., are second-neighbors in the line graph);
- (ii) if  $\mathcal{B}_{xy} \cap E_4 \neq \emptyset$  then another common color (besides 4) appears at  $x$  and  $y$ .



**Figure 3:** Two odd edge-colorings of  $W_4$  that satisfy conditions (i) and (ii) from Theorem 1.3. In the coloring depicted on the left,  $|E_4| = 2$  and at both endvertices of any edge colored by 4 either the color 2 or the color 3 appears. In the coloring depicted on the right,  $|E_4| = 1$  and at both endvertices of the only edge colored by 4 each of the colors 2 and 3 occurs.

It is not known whether there exists a connected graph  $G$  with  $\chi'_o(G) = 4$  that does not admit an odd 4-edge-coloring with a color class of size 1. In this regard, the following has been conjectured in [7].

**Conjecture 1.4.** *Every connected graph  $G$  with  $\chi'_o(G) = 4$  becomes odd 3-edge-colorable by removing a particular edge.*

Regarding odd 2-edge-colorability of graphs, Kano et al. [3] have shown the following.

**Theorem 1.5.** *The decision problem whether a given graph  $G$  is odd 2-edge-colorable is solvable in polynomial time. Moreover, in the affirmative case, such a coloring can be found in polynomial time.*

In view of Theorem 1.3, the decision problem whether a given graph  $G$  is odd 4-edge-colorable is also solvable in polynomial (in fact, linear) time. Moreover, its proof can be used as an efficient algorithm for exhibiting such a coloring. The analogous complexity questions regarding odd 3-edge-colorability of general graphs are still open. Nevertheless, these questions have been answered (in the affirmative) for subcubic graphs. Namely, a complete characterization of the class of loopless subcubic graphs in terms of their odd chromatic index was obtained in [1] through the following:

**Theorem 1.6.** *Let  $G$  be a connected loopless subcubic graph. Then*

$$\chi'_o(G) = \begin{cases} 0 & \text{if } G \text{ is empty;} \\ 1 & \text{if } G \text{ is odd;} \\ 2 & \text{if } G \text{ has 2-vertices, with an even number of them on each cycle;} \\ 4 & \text{if } G \text{ is obtainable from a cubic bipartite graph by a single edge subdivision;} \\ 3 & \text{otherwise.} \end{cases}$$

In this paper we focus on loopless subdivisions of odd graphs, which are in some sense the least non-odd graphs among all. Let us denote this collection by  $\mathcal{S}$ , and similarly, let  $\mathcal{O}$  be the class of loopless odd graphs, with the understanding that  $\mathcal{O} \subset \mathcal{S}$ . Our main result, Theorem 4.3 at the very end of Section 4, is a characterization of the members of  $\mathcal{S}$  in terms of the value of their odd chromatic index. Thus, we achieve a generalization of Theorem 1.6, and at the same time answer a question raised by the end of [6]. Our findings here also provide support for Conjecture 1.4 over the class  $\mathcal{S}$ .

The rest of the article is divided into four sections. In the next, preliminary one, we collect several ‘easy’ results (most of them previously known). Sections 3 and 4 are devoted to a derivation of our main result - a characterization of  $\mathcal{S}$  in terms of the value of the odd chromatic index. The final section briefly conveys some possible directions for further related study.

## 2 Preliminaries

The *edge-complement*,  $\widehat{H}$ , of a subgraph  $H \subseteq G$  is the spanning subgraph  $\widehat{H} = G - E(H)$ . A *co-forest* in  $G$  is a subgraph whose edge-complement is a forest. For a graph  $G$ , let  $T$  be an even-sized subset of  $V(G)$ . Following [2], a spanning subgraph  $H$  of  $G$  is said to be a  *$T$ -join* of  $G$  if  $d_H(v)$  is odd for all  $v \in T$  and even for all  $v \in V(G) \setminus T$ . For instance, if  $P \subseteq G$  is a nontrivial path with endvertices  $x$  and  $y$ , the spanning subgraph of  $G$  with edge set  $E(P)$  is an  $\{x, y\}$ -join of  $G$ . As another example, every even spanning subgraph is an  $\emptyset$ -join of  $G$ . Observe that the symmetric difference of an  $S$ -join and a  $T$ -join is an  $S \oplus T$ -join. (We shall

use  $\oplus$  to denote both the symmetric difference operation on spanning subgraphs and on sets.) Hence, the symmetric difference,  $H \oplus K$ , of a  $T$ -join  $H$  and a spanning even subgraph  $K$  of  $G$  is again a  $T$ -join. In particular, the removal (resp. addition) of all edges of an edge-disjoint cycle from (resp. to) a  $T$ -join, produces another  $T$ -join. Therefore, if a  $T$ -join of  $G$  exists, there also exists such a forest (resp. co-forest). By the handshake lemma, necessary for the existence of a  $T$ -join is that the intersection of  $T$  with the vertex set of every component of  $G$  is even-sized, and a straightforward implementation of the above mentioned facts (see [9]) is that this condition also suffices. Consequently, given a connected graph  $G$  and an even-sized subset  $T$  of  $V(G)$ ,

- (1) there exists a  $T$ -join of  $G$  that is a forest;
- (2) there exists a  $T$ -join of  $G$  that is a co-forest;
- (3) additionally, if  $G$  is of even order, then it contains a spanning odd co-forest.

An edge-coloring  $\varphi$  is said to be *odd* (resp. *even*) *at* a vertex  $v$  if each color appearing at  $v$  is odd (resp. even). Similarly, we say that  $\varphi$  is *odd* (resp. *even*) *away from*  $v$  if  $\varphi$  is odd (resp. even) at every vertex  $w \in V(G) \setminus \{v\}$ , without any assumptions about the behavior of  $\varphi$  at  $v$  being made. The following useful result appears in [5, 6].

**Proposition 2.1.** *Let  $v$  be a vertex of a forest  $F$ . Any local coloring of  $E_F(v)$  which uses at most two colors extends to a 2-edge-coloring of  $F$  that is odd away from  $v$ . In particular,  $F$  is odd 2-edge-colorable.*

An immediate consequence of Proposition 2.1 is the result below, which concerns a graph all of whose cycles (if any) share a vertex.

**Proposition 2.2.** *If  $v$  is a vertex of a graph  $G$  such that  $G - v$  is a forest, then  $G$  admits a 2-edge-coloring that is odd away from  $v$ . Additionally, if  $d_G(v)$  is odd, then  $G$  admits an edge-coloring with color set  $\{1, 2\}$  that is odd away from  $v$  and the color 1 (resp. 2) is odd (resp. even) at  $v$ .*

*Proof.* We may assume that  $G$  is loopless. It suffices to prove the first part. Split  $v$  into  $k = d_G(v)$  pendant vertices  $v_1, \dots, v_k$  in order to obtain a forest  $F$ . By Proposition 2.1,  $F$  admits an odd 2-edge-coloring. Re-identify  $v_1, \dots, v_k$  into  $v$  while keeping the colors on all edges. We thus regain  $G$  along with a required edge-coloring.  $\square$

As observed in [8], the odd 2-edge-colorability of forests implies odd 3-edge-colorability for all connected graphs of even order, which in turn yields odd 3-edge-colorability for all graphs with edge-connectivity 1. The following proof comes from [6].

**Proposition 2.3.** *If  $G$  is a connected graph such that  $n(G)$  is even or  $\kappa'(G) = 1$  then  $\chi'_o(G) \leq 3$ .*

*Proof.* Let  $n(G)$  be even and let  $H$  be a spanning odd co-forest of  $G$ . Take an odd edge-coloring of the forest  $\widehat{H}$  with color set  $\{1, 2\}$  and extend to  $E(G)$  by coloring  $E(H)$  with 3. This gives an odd 3-edge-coloring of  $G$ .

Assume now that  $n(G)$  is odd. First we consider the case when the minimum degree  $\delta(G) = 1$ . Select a pendant vertex  $u$  and take a spanning odd co-forest  $H$  of  $G - u$ . As

$F = G - E(H)$  is a forest, combine an odd 2-edge-coloring of  $F$  with a monochromatic coloring of  $E(H)$  that uses a third color.

So suppose that there are no pendant vertices in  $G$ , but nevertheless  $\kappa'(G) = 1$ . Let  $vw$  be a bridge in  $G$ . Denote by  $G_v$  and  $G_w$ , respectively, the components of  $G - vw$  containing  $v$  and  $w$ . By the previous case, the subgraphs  $G' = G[V(G_v) \cup \{w\}]$  and  $G'' = G[V(G_w) \cup \{v\}]$  admit respective odd 3-edge-colorings  $\varphi'$  and  $\varphi''$  with the same color set. Moreover, by permuting colors if necessary, we can achieve that  $\varphi'(vw) = \varphi''(vw)$ . Then  $\varphi' \cup \varphi''$  is an odd 3-edge-coloring of  $G$ .  $\square$

The next result may be used to characterize odd 2-edge-colorability of unicyclic graphs.

**Proposition 2.4.** *Let  $G$  be a unicyclic loopless graph, and let  $C \subseteq G$  be the (unique) cycle. Then  $\chi'_o(G) \leq 3$ . Moreover, the upper bound is attained if and only if the following two conditions hold simultaneously:*

- (i)  $\{v \in V(C) : d_G(v) = 2\}$  is odd-sized;
- (ii)  $\{v \in V(C) : d_G(v) \neq 2 \text{ and } d_G(v) \text{ is even}\} = \emptyset$ .

*Proof.* Let us first show that  $G$  is odd 3-edge-colorable. Since  $G$  is unicyclic, Proposition 2.1 allows for the assumption that  $G$  is connected. Moreover, in view of Proposition 2.3, we may further assume that  $G$  is bridgeless. However, from all assumed it readily follows that  $G = C$ . Hence  $\chi'_o(G) = \chi'(C) \leq 3$ .

Now we show that fulfilment of the conditions (i) and (ii) is both necessary and sufficient for the equality  $\chi'_o(G) = 3$  to hold. If  $G = C$  then condition (ii) is clearly met, and the characterization is trivially true (as by then the notions ‘odd edge-coloring’ and ‘proper edge-coloring’ become equivalent). Assuming  $G \neq C$ , let  $S = \{v \in V(C) : d_G(v) \neq 2\}$  and  $\widehat{S} = V(C) \setminus S$ . Denote by  $S'$  and  $S''$ , respectively, the subsets of  $S$  comprised of those vertices  $v$  for which the degree  $d_G(v)$  is odd or even. Observe that for every  $v \in S'$ , a coloring of  $E_C(v)$  extends to an odd 2-edge-coloring of  $E_G(v)$  if and only if it is monochromatic. Otherwise, for every  $v \in S''$  each coloring of  $E_C(v)$  extends to an odd 2-edge-coloring of  $E_G(v)$ . Consequently, in view of Proposition 2.1, a given 2-edge-coloring of  $C$  extends to an odd 2-edge-coloring of  $G$  if and only if the coloring is dichromatic at each  $v \in \widehat{S}$  and monochromatic at each  $v \in S'$ .

So, if condition (ii) fails to hold, then  $\chi'_o(G) \leq 2$ . Indeed, simply select a vertex  $v \in S''$ , take a 2-edge-coloring of  $C$  that is monochromatic at each vertex from  $S'$  and dichromatic at each vertex from  $V(C) \setminus (S' \cup \{v\})$ ; by the above observation, such a coloring of  $E(C)$  extends to an odd 2-edge-coloring of  $G$ .

On the other hand, assuming (ii), odd 2-edge-colorability of  $G$  is equivalent to the existence of a 2-edge-coloring of  $C$  that is dichromatic precisely at each vertex of  $\widehat{S}$ . The latter is clearly equivalent to the requirement that the set  $\{v \in V(C) : d_G(v) = 2\}$  is even-sized.  $\square$

**Corollary 2.5.** *Let  $G$  be a connected unicyclic loopless graph,  $C \subseteq G$  be the (unique) cycle and let  $\{v \in V(C) : d_G(v) \text{ is odd}\} = \emptyset$ . Then  $\chi'_o(G) \leq 2$  unless  $G = C$  is an odd cycle.*

We end the preliminaries with two more already known results (proofs can be found in [6]).

**Proposition 2.6.** *In a connected loopless graph  $G$ , let  $v$  be an internal vertex and  $e \in E_G(v)$ . If  $T \subseteq V(G)$  is even-sized, then there exists a  $T$ -join  $H$  of  $G$  which is a co-forest such that  $E_{\widehat{H}}(v) \subseteq \{e\}$ .*

If additionally the graph  $G$  from Proposition 2.6 is of even order, we derive the following by setting  $T = V(G)$ .

**Corollary 2.7.** *In a connected loopless graph  $G$  of even order, let  $v$  be an internal vertex and  $e \in E_G(v)$ . Then there exists a spanning odd co-forest  $H$  of  $G$  such that  $E_{\widehat{H}}(v) \subseteq \{e\}$ .*

### 3 Subdivisions of odd graphs

Recall that  $\mathcal{S}$  denotes the class of all loopless subdivisions of odd graphs. The following proposition is an overture to our subsequent study of  $\mathcal{S}$  in terms of the value of the odd chromatic index. The final product of the study, our main result, shall be formulated by the end of Section 4. As a warm-up, we commence by showing that four colors always suffice for an odd edge-coloring of any member of  $\mathcal{S}$ . Moreover, the fourth color can be reduced to at most one appearance per component.

**Proposition 3.1.** *Let  $v$  be a 2-vertex of a connected graph  $G \in \mathcal{S}$ , and let  $e \in E_G(v)$ . Then  $G$  admits an odd edge-coloring with color set  $\{1, 2, 3, 4\}$  such that the color class  $E_4 \subseteq \{e\}$ . Moreover, if  $\chi'_o(G) = 4$  then it holds that:*

- (i) *Every 2-vertex is internal;*
- (ii) *No 2-vertices are adjacent.*

*Proof.* If there exists a cutvertex  $u$  in  $G$  such that  $d_G(u) = 2$ , then  $u$  must be incident with two bridges. Consequently, Proposition 2.3 yields odd 3-edge-colorability of  $G$ . Assuming (i), the vertex  $v$  is internal. Therefore, the graph  $G - e$  is connected and of minimum degree  $\delta(G - e) = 1$ , so it admits an odd edge-coloring with color set  $\{1, 2, 3\}$ . By assigning the color 4 to the edge  $e$  we obtain the promised coloring of  $E(G)$ . This proves the first part and, in addition, confirms that (i) is necessary for  $\chi'_o(G) = 4$ .

As for (ii), still assuming  $\chi'_o(G) = 4$ , suppose there is an edge  $f$  whose endvertices  $u$  and  $w$  are 2-vertices. Let  $g_u$  and  $g_w$  be the other edges (besides  $f$ ) incident with  $u$  and  $w$ , respectively. Since  $g_u$  and  $g_w$  are pendant edges in the (connected) graph  $G - f$ , Proposition 2.3 guarantees that there is an odd edge-coloring  $\varphi$  of  $G - f$  with color set  $\{1, 2, 3\}$ . Extend  $\varphi$  to  $E(G)$  by assigning  $f$  with a color from  $\{1, 2, 3\} \setminus \{\varphi(g_u), \varphi(g_w)\}$ . This completes an odd 3-edge-coloring of  $G$ , a contradiction.  $\square$

Note that the first part of Proposition 3.1 supports Conjecture 1.4. For our intended characterization of all members of the class  $\mathcal{S}$  in terms of their odd chromatic index, let us denote by  $\mathcal{S}_i$  ( $i = 1, 2, 3, 4$ ) the subclass consisting of those  $G \in \mathcal{S}$  having  $\chi'_o(G) = i$ . Clearly,  $\mathcal{S}_1$  comprises the class of loopless odd graphs,  $\mathcal{O}$ . At the other end of the spectrum, the second part of Proposition 3.1 gives a pair of necessary conditions for membership in  $\mathcal{S}_4$ ; equivalently, it describes two sufficient conditions for odd 3-edge-colorability of a loopless subdivision of an odd graph. The following result provides another such condition (which shall be useful on more than one occasion later on in this section).

**Proposition 3.2.** *Let  $G \in \mathcal{S}$  be a connected graph, and let  $C \subseteq G$  be a cycle passing through a 2-vertex  $v$  of  $G$ . If the cycle  $C$  is even or it passes through another 2-vertex of  $G$ , then  $\chi'_o(G) \leq 3$ .*



*Proof.* We argue by contradiction, that is, suppose  $G$  is not odd 3-edge-colorable. Then, by Proposition 2.3, the order  $n(G)$  is odd. So, in view of Proposition 3.1, we have that the graph  $G - v$  is connected and of even order. Let  $\mathcal{H}$  be the collection of all spanning odd co-forests of  $G - v$ . Thus  $\mathcal{H} \neq \emptyset$ . As  $d_G(v) = 2$ , the neighborhood  $N_G(v)$  is either a 1-set or a 2-set. We show it is the latter.

**Claim 1.**  $|N_G(v)| = 2$ .

Otherwise, the cycle  $C$  is of length 2 (namely,  $E(C) = E_G(v)$ ). Consider a member  $H \in \mathcal{H}$ , along with its edge-complement in regard to  $G$ : the former subgraph (the odd co-forest  $H$ ) is odd 1-edge-colorable, whereas the latter is a unicyclic graph (the unique cycle is  $C$ ) and its component containing the cycle satisfies all assumptions of Corollary 2.5. Hence,  $G$  admits an odd edge-coloring that uses at most  $1 + 2 = 3$  colors, a contradiction.  $\diamond$

Let  $N_G(v) = \{u, w\}$ . By Proposition 3.1 (ii), the degrees  $d_G(u), d_G(w)$  are odd. Let  $P = C - v$  and observe that (by the initial assumptions)  $P$  is a  $u$ - $w$  path in  $G - v$  which is even or it passes through a 2-vertex of  $G$ . For any  $H \in \mathcal{H}$  denote  $\widehat{H} = G - v - E(H)$ . Since  $H$  is a (spanning odd) co-forest of  $G - v$ , the graph  $\widehat{H}$  is a forest. We show next that there is a particular component in  $\widehat{H}$ .

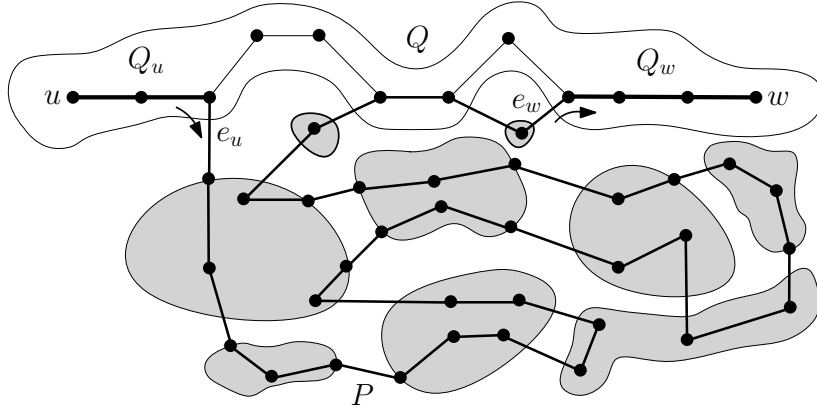
**Claim 2.** For every  $H \in \mathcal{H}$ , a component of  $\widehat{H}$  is an odd  $u$ - $w$  path  $Q = Q(\widehat{H})$ . Moreover,  $Q \neq P$ .

Note that the vertices of odd degree in  $\widehat{H}$  are precisely  $u, w$  and all 2-vertices of  $G$  that are within  $G - v$ . Thus, looking at  $G - E(H) = \widehat{H} + \{uv, vw\}$ , the odd vertices of this graph are precisely the 2-vertices of  $G$  that are  $\neq v$ ; moreover, each such vertex is pendant in regard to  $G - E(H)$ . There are two possibilities for  $G - E(H)$ : either it is a forest (if  $u, w$  do not share a component of  $\widehat{H}$ ), or it is a unicyclic graph such that no vertex of the cycle has an odd degree. Therefore, in view of Corollary 2.5, the graph  $G - E(H)$  is odd 2-edge-colorable (and consequently  $G$  is odd 3-edge-colorable), unless it is always the case that a component of  $\widehat{H}$  is an odd  $u$ - $w$  path, say  $Q$ . As we are supposing  $\chi'_o(G) > 3$ , we have thus established the existence of the path-component  $Q$ . Let us show that  $Q \neq P$ . If the cycle  $C$  is even, then  $P$  and  $Q$  are of different parities, and hence cannot be the same. And if  $C$  passes through a 2-vertex of  $G$  contained within  $G - v$ , then so does  $P$  but not  $Q$  (because every such 2-vertex of  $G$  is a pendant vertex of  $\widehat{H}$ ).  $\diamond$

Let  $Q_u$  and  $Q_w$  be, respectively, the components of  $u$  and  $w$  in  $P \cap Q$ . (Here  $P$  and  $Q$  are seen as spanning subgraphs of  $G - v$  with respective edge sets  $E(P)$  and  $E(Q)$ .) Since  $Q \neq P$ , the paths  $Q_u$  and  $Q_w$  are disjoint. The former has  $u$ , whereas the latter has  $w$  as an endvertex. Notice that  $Q_u \cup Q_w \subset P$ . Say  $e_u$  and  $e_w$  are, respectively, the first and the last edge lying outside  $Q_u \cup Q_w$  on a traversal of  $P$  from  $u$  to  $w$  (it is not excluded that  $e_u = e_w$ ).

Take the symmetric difference  $H \oplus P \oplus Q$ . The obtained graph is clearly another odd factor of  $G - v$ , though not necessarily a co-forest. Let  $H'$  be a maximal odd factor of  $G - v$  subjected to the condition  $H \oplus P \oplus Q \subseteq H'$ . Then obviously  $H' \in \mathcal{H}$ . According to Claim 2, an odd  $u$ - $w$  path  $Q'$  constitutes a component of the forest  $\widehat{H}' = G - v - E(H')$ . Let  $Q'_u, Q'_w$  be defined analogously as before, that is, let  $Q'_u$  and  $Q'_w$  be the respective components of  $u$  and  $w$  in  $P \cap Q'$ .

**Claim 3.**  $Q_u \subset Q'_u$  and  $Q_w \subset Q'_w$ .



**Figure 4:** The path  $P$  and the components of the forest  $\widehat{H}$ . Apart from the path-component  $Q$ , the rest of the components of  $\widehat{H}$  are shaded. The edges of  $P$  are depicted as heavier, and those of  $Q_u \cup Q_w$  are fat. On a traversal of  $P$  from  $u$  to  $w$ , the arrows notify the first embarkment and the last disembarkment of  $P - E(Q)$ . Since this happens precisely along the edges  $e_u$  and  $e_w$ , respectively, these two cannot be cycle edges of  $\widehat{H} \oplus P \oplus Q$ . The same is obviously true for any edge of  $Q_u \cup Q_w$ .

Begin by observing that  $H' - E(H \oplus P \oplus Q)$  is an even subgraph of  $\widehat{H} \oplus P \oplus Q$ , the edge-complement of  $H \oplus P \oplus Q$  with respect to  $G - v$ . Also note that  $Q_u \cup \{e_u\} \cup \{e_w\} \cup Q_w$  is fully contained in  $\widehat{H} \oplus P \oplus Q$ . Moreover, for any vertex  $x \in V(Q_u \cup Q_w)$  it holds that  $d_{\widehat{H} \oplus P \oplus Q}(x) = d_P(x) \leq 2$ . In particular,  $d_{\widehat{H} \oplus P \oplus Q}(u) = d_{\widehat{H} \oplus P \oplus Q}(w) = 1$ . Therefore, no edge from  $Q_u \cup \{e_u\} \cup \{e_w\} \cup Q_w$  belongs to a cycle contained entirely in  $\widehat{H} \oplus P \oplus Q$ . Consequently,  $H'$  is edge-disjoint from  $Q_u \cup \{e_u\} \cup \{e_w\} \cup Q_w$ . Equivalently,  $Q_u \cup \{e_u\} \cup \{e_w\} \cup Q_w \subseteq P \cap Q'$ . It follows that  $Q_u \cup \{e_u\} \subseteq Q'_u$  and  $Q_w \cup \{e_w\} \subseteq Q'_w$ .  $\diamond$

So for any  $H \in \mathcal{H}$ , there exists another  $H' \in \mathcal{H}$  such that  $Q_u \cup Q_w \subset Q'_u \cup Q'_w \subset P$ . This is the desired contradiction.  $\square$

Our next result concerns odd 2-edge-colorability of subdivisions of odd graphs, and thus yields a structural characterization of  $\mathcal{S}_2$ .

**Proposition 3.3.** *The following statements are equivalent for every graph  $G \in \mathcal{S}$ :*

(i)  $\chi'_o(G) \leq 2$ ;

(ii) *For every cycle  $C$  of  $G$  the set  $\{v : v \in V(C) \text{ and } d_G(v) = 2\}$  is even-sized.*

*Proof.* We may assume that  $G$  is connected. Notice that a 2-edge-coloring of  $G$  is odd if and only if every edge set  $E_G(v)$  is monochromatic or dichromatic depending on whether  $v$  is an odd vertex or a 2-vertex of  $G$ .

Now (i)  $\Rightarrow$  (ii) follows easily as moving around any given cycle  $C \subseteq G$ , there must occur an even number of color changes; in other words,  $C$  must contain an even number (possibly 0) of 2-vertices of  $G$ .

To show (ii)  $\Rightarrow$  (i), select a spanning tree  $T$  rooted at an odd vertex  $v_0$ . First we color  $E(T)$  as follows. Assign  $E_T(v_0)$  with the color 1, and repeatedly apply the following procedure until  $E(T)$  becomes fully colored: choose a vertex  $v \neq v_0$  that has just one incident edge already

colored, say by a color  $c \in \{1, 2\}$ ; color the rest of  $E_T(v)$  by the color  $c$  (resp.  $3 - c$ ) if  $d_G(v)$  is odd (resp. equal to 2). This gives a 2-edge-coloring  $\varphi$  of  $T$  that is dichromatic precisely at the 2-vertices of  $G$  which are not pendant in regard to  $T$ .

Let us extend  $\varphi$  to  $E(G)$ . Consider an edge  $e \in E(G) \setminus E(T)$ , say  $x$  and  $y$  are its endvertices. Denote by  $e_x$  and  $e_y$  the (not necessarily distinct) edges of the  $x$ - $y$  path  $P$  in  $T$  that are incident with  $x$  and  $y$ , respectively. Note that the equality  $\varphi(e_x) = \varphi(e_y)$  holds if and only if an even number (possibly 0) of internal vertices of  $P$  are 2-vertices in  $G$ . Therefore, since  $P + e$  is a cycle,  $\varphi(e_x) = \varphi(e_y)$  if and only if an even number (both or neither) of the vertices  $x, y$  are 2-vertices in  $G$ . So, we assign one of the colors 1, 2 to  $e$  as follows: (1) if both  $x, y$  are 2-vertices in  $G$ , then set  $\varphi(e) \neq \varphi(e_x)$ ; (2) if neither  $x, y$  are 2-vertices in  $G$ , then set  $\varphi(e) = \varphi(e_x)$ ; if just one of the vertices  $x, y$  is a 2-vertex in  $G$ , say such is  $x$ , then set  $\varphi(e) = \varphi(e_y)$ . The resulting  $\varphi$  is an odd 2-edge-coloring of  $G$  since on every edge set  $E_G(v)$  it is monochromatic or dichromatic depending on whether  $v$  is an odd vertex or a 2-vertex.  $\square$

The above proof shows that the given characterization of odd 2-edge-colorability within  $\mathcal{S}$  is good and, in the affirmative, such a coloring can be found in polynomial time.

**Corollary 3.4.** *Let  $G \in \mathcal{S}$ . Then  $\chi'_o(G) = 2$  if and only if  $G \notin \mathcal{O}$  and for every cycle  $C$  of  $G$  the set  $\{v : v \in V(C) \text{ and } d_G(v) = 2\}$  is even-sized.*

In the remainder of the paper we provide a structural characterization of the class  $\mathcal{S}_4$ . The next result shall allow us to confine to 2-connected graphs.

**Proposition 3.5.** *If  $G \in \mathcal{S}$  is a connected graph, then  $\chi'_o(G) = 4$  if and only if every block of  $G$  belongs to  $\mathcal{S}_4$  and for every cutvertex  $v$  there is a unique block  $B$  such that  $d_B(v)$  is odd.*

*Proof.* The essential part of our proof is to establish property (P) below, which sheds some light on the structure of graphs  $G \in \mathcal{S}$  of connectivity  $\kappa(G) = 1$  that require four colors for an odd edge-coloring.

(P) *Let  $v$  be a cutvertex of a connected graph  $G \in \mathcal{S}$ . If  $G_1, \dots, G_k$  are the  $v$ -lobes of  $G$ , then the following statements are equivalent:*

(i)  $\{G_1, G_2, \dots, G_k\} \subseteq \mathcal{S}_4$  and there is a unique  $j$  such that  $d_{G_j}(v)$  is odd; in particular,  $d_{G_i}(v) = 2$  for every  $i \neq j$ .

(ii)  $G \in \mathcal{S}_4$ .

Notice that, once the equivalence stated in (P) is verified, the proposition may be derived by inducting on the number  $t$  of cutvertices in  $G$ . Namely, the case  $t = 0$  is trivial, and the case  $t = 1$  follows immediately from (P). For  $t > 1$ , consider a cutvertex  $v$  which is an internal leaf of the block-tree  $B(G)$ ; in other words,  $v$  is such that all but one of the blocks containing it are end-blocks of  $G$ . Let  $G_1, \dots, G_k$  be an enumeration of the  $v$ -lobes of  $G$ , so that  $G_2, \dots, G_k$  are end-blocks of  $G$ . Notice that the blocks of  $G_1$  are precisely the blocks of  $G$  that are  $\neq G_2, \dots, G_k$ , whereas the cutvertices of  $G_1$  are the cutvertices of  $G$  distinct from  $v$ . In particular, the number of cutvertices of  $G_1$  is  $t - 1$ . Within the graph  $G_1$ , the vertex  $v$  is an internal vertex of some block  $B_1$ , and thus  $d_{G_1}(v) = d_{B_1}(v)$ . By applying (P) to the pair  $G, v$  we deduce that  $G \in \mathcal{S}_4$  if and only if  $G_1, G_2, \dots, G_k \in \mathcal{S}_4$  and there is a unique  $j$  such that  $d_{G_j}(v)$  is odd, where it may happen that  $j \neq 1$ . Combine this equivalence with the inductive hypothesis applied to  $G_1$ , and we are done.

In what follows, we verify the property (P) by proving the implications (i)  $\Rightarrow$  (ii) and (ii)  $\Rightarrow$  (i). First we show the easy part, which is the direction (i)  $\Rightarrow$  (ii). Arguing by contradiction, suppose that both (i) and  $\neg$ (ii) hold. Consider an odd 3-edge-coloring  $\varphi$  of  $G$ . For every  $i = 1, \dots, k$ , let  $\varphi_i$  be the restriction of  $\varphi$  to  $E(G_i)$ , that is,  $\varphi_i = \varphi|_{E(G_i)}$ . Clearly, each  $\varphi_i$  is odd away from  $v$ . Moreover, since  $\chi'_o(G_i) = 4$ , the edge-coloring  $\varphi_i$  is not odd at  $v$ . Consequently, whenever  $d_{G_i}(v) = 2$  the edge set  $E_{G_i}(v)$  is monochromatic under  $\varphi_i$ . However, then  $\varphi_j$  must be an odd 3-edge-coloring of  $G_j$ , a contradiction.

Now let us prove direction (ii)  $\Rightarrow$  (i). Assuming (ii), note that degree  $d_G(v)$  is odd by Proposition 3.1. We break the argument into several claims which eventually lead to (i).

**Claim 1.** *If  $G' \cup G'' = G$  and  $G' \cap G'' = \{v\}$ , then both orders  $n(G'), n(G'')$  are odd. Moreover, if  $d_{G'}(v)$  is even then any 3-edge-coloring of  $G'$  which is odd away from  $v$  must be even at  $v$ ; in particular,  $G'$  is not odd 3-edge-colorable.*

By Proposition 2.3,  $n(G)$  is odd. Hence,  $n(G'), n(G'')$  are of the same parity. Suppose  $n(G'), n(G'')$  are even. Then there exist spanning odd co-forests  $H'$  and  $H''$  of  $G'$  and  $G''$ , respectively. Denote  $F = G - E(H' \cup H'')$ . Note that  $F$  is a forest and  $d_F(v)$  is odd. This enables construction of an odd 3-edge-coloring of  $G$  as follows: color  $E(H')$  by 1 and  $E(H'')$  by 2; color  $E_F(v)$  by 3; extend the coloring of  $E_{F \cap G'}(v)$  to an edge-coloring of the forest  $F \cap G'$  with color set  $\{2, 3\}$  that is odd away from  $v$ ; similarly, extend the coloring of  $E_{F \cap G''}(v)$  to an edge-coloring of the forest  $F \cap G''$  with color set  $\{1, 3\}$  that is odd away from  $v$ . The obtained contradiction shows that  $n(G'), n(G'')$  are both odd.

Suppose that  $d_{G'}(v)$  is even and  $G'$  admits a 3-edge-coloring  $\varphi'$  which is odd away from  $v$  so that at least one, and hence two, of the colors are odd at  $v$ . Assume the color set of  $\varphi'$  is  $\{1, 2, 3\}$  and the colors 1 and 2 are odd at  $v$ . Since  $d_{G'}(v)$  is even, it follows that the color 3 is even at  $v$ . We construct an accompanying edge-coloring  $\varphi''$  of  $G''$ . In order to do so, consider an auxiliary graph  $G^* = G'' + vv^*$ , where  $v^*$  is a new vertex. Since  $G^*$  is a connected graph of even order and the degree  $d_{G^*}(v)$  is even, Proposition 2.3 yields an odd edge-coloring of  $G^*$  with color set  $\{1, 2, 3\}$  such that  $E_{G^*}(v)$  is colored by 2 and 3 with the edge  $vv^*$  colored by 2. Let  $\varphi''$  be the restriction to  $E(G'')$  of this coloring of  $E(G^*)$ . However, then  $\varphi' \cup \varphi''$  is an odd 3-edge-coloring of  $G$ . The obtained contradiction proves our point.  $\diamond$

From the first part of Claim 1 it follows that every  $v$ -lobe of  $G$  has an odd order. Next we use the last part of Claim 1 to show that the degree of  $v$  is odd in regard to precisely one  $v$ -lobe.

**Claim 2.** *There is a unique  $j \in \{1, 2, \dots, k\}$  such that  $d_{G_j}(v)$  is odd.*

Since  $d_G(v)$  is odd, so is  $d_{G_j}(v)$  for some  $j$ . Suppose there are at least two such indices, say  $j = 1$  and  $j = 2$ . It follows that  $k \geq 3$ . Let  $G' = G_1 \cup G_2$  and  $G'' = \bigcup\{G_i : i = 3, \dots, k\}$ . As both  $G_1 - v, G_2 - v$  are connected graphs of even order, there exist spanning odd co-forests  $H_1$  and  $H_2$  of  $G_1 - v$  and  $G_2 - v$ , respectively. By Proposition 2.2, take an edge-coloring of  $\widehat{H}_j = G_j - E(H_j)$  with color set  $\{1, 2\}$  which is odd away from  $v$  and so that the color  $j$  is odd at  $v$  in  $\widehat{H}_j$ ,  $j = 1, 2$ . Extend to  $E(G')$  by coloring  $E(H_1 \cup H_2)$  with 3. This furnishes an odd 3-edge-coloring of  $G'$ . However, as  $d_{G'}(v)$  is even, the obtained coloring contradicts the last part of Claim 1.  $\diamond$

Proceed by showing that each  $v$ -lobe is a subdivision of an odd graph.

**Claim 3.**  $G_i \in \mathcal{S}$  for every  $i = 1, 2, \dots, k$ .

Supposing the opposite, there is a  $v$ -lobe  $G_r$  such that  $d_{G_r}(v)$  is even and  $d_{G_r}(v) \geq 4$ . First we show that there exists of an even cycle  $C \subseteq G_r$  with  $v \in V(C)$ . For this we may assume that every bouquet of  $G_r$  incident with  $v$  is a singleton (otherwise, there is a 2-cycle through  $v$ ). Since  $d_{G_r}(v) \geq 4$ , consider a triplet  $x, y, z \in N_{G_r}(v)$ . If there is an even  $x$ - $y$  path in  $G_r - v$ , we are obviously done. So, as  $G_r - v$  is connected, let  $Q$  be an odd  $x$ - $y$  path. We exhibit an even path in  $G_r - v$  going from  $z$  to the set  $\{x, y\}$ . In view of the connectedness of  $G_r - v$ , let  $P$  be an odd  $z$ - $x$  path. On a traversal of  $P$  from  $z$  to  $x$ , say  $w$  is the first vertex that belongs to  $Q$ . Then  $zPw \cup wQx$  and  $zPw \cup wQy$  are paths (by the choice of  $w$ ). Because  $wQx$  and  $wQy$  are of opposite parities, we have found an even  $z$ - $\{x, y\}$  path in  $G_r - v$ , which in turn yields an even cycle  $C \subseteq G_r$  passing through  $v$ .

Next, we use the presence of the even cycle  $C$  to show that there exists a 3-edge-coloring  $\varphi_r$  of  $G_r$  which is odd away from  $v$ , and at  $v$  two of the colors are odd. Consider the following local modification of  $G_r$  that consists of splitting out  $v$  entirely into 2-vertices and then suppressing all of them except one: create a 2-vertex  $v'$  incident to the two edges forming  $E_C(v)$ ; then arbitrarily split out the rest of  $v$  into 2-vertices and suppress them all (except  $v'$ ). Every even vertex of the resulting connected graph  $G'_r$  is a 2-vertex, and  $C$  is an even cycle (properly contained) in  $G'_r$  that passes through its 2-vertex  $v'$ . By Proposition 3.2,  $G'_r$  admits an odd 3-edge-coloring. Returning to  $G_r$ , we obtain the desired  $\varphi_r$ . However, since  $d_{G_r}(v)$  is even, the edge-coloring  $\varphi_r$  contradicts with the second part of Claim 1.  $\diamond$

The final piece of our argument is showing that no  $v$ -lobe is odd 3-edge-colorable.

**Claim 4.**  $\chi'_o(G_i) = 4$  for every  $i = 1, 2, \dots, k$ .

By Claim 1, no  $v$ -lobe  $G_i$  with  $d_{G_i}(v) = 2$  is odd 3-edge-colorable. Suppose that the (unique)  $v$ -lobe  $G_j$  having odd  $d_{G_j}(v)$  is odd 3-edge-colorable. Take an odd edge-coloring  $\varphi_j$  of  $G_j$  with color set  $\{1, 2, 3\}$  such that the color 1 appears on  $E_{G_j}(v)$ . Consider now an arbitrary  $G_i$  with  $i \neq j$ . Letting  $v^*$  be a new vertex, the graph  $G_i + vv^*$  is connected and of even order, hence it admits an odd 3-edge-coloring with color set  $\{1, 2, 3\}$  under which the pendant edge  $vv^*$  receives the color 1. Denote by  $\varphi_i$  the restriction to  $E(G_i)$  of the constructed edge-coloring of  $G_i + vv^*$ . Note that  $\varphi_i$  is odd away from  $v$  and colors  $E_{G_i}(v)$  with 1, for otherwise  $G_i$  would be odd 3-edge-colorable. However then the union  $\varphi_1 \cup \dots \cup \varphi_k$  is an odd 3-edge-coloring of  $G$ , a contradiction.  $\diamond$

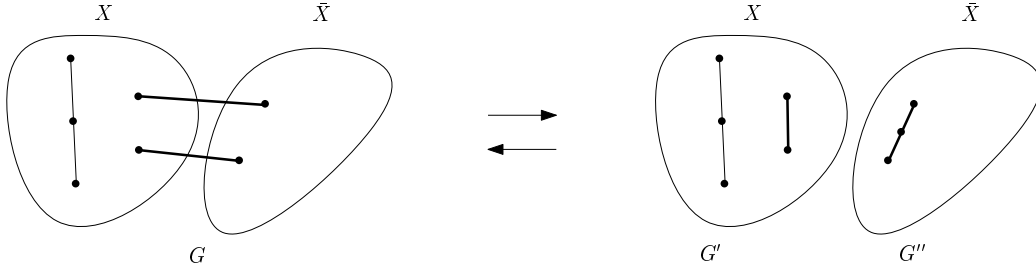
This completes the verification of  $(ii) \Rightarrow (i)$ , and thus settles the property  $(P)$  and the proposition.  $\square$

A straightforward implication of Propositions 3.2 and 3.5 is that each connected member of  $\mathcal{S}_4$  must have precisely one vertex of degree 2.

**Corollary 3.6.** *If  $G \in \mathcal{S}_4$  is connected, then the set of its 2-vertices is a singleton.*

*Proof.* We prove the corollary by induction on  $n(G)$ . Assume first that  $G$  has a cutvertex  $v$ . Then by the induction hypothesis, every  $G_i$  with  $d_{G_i}(v) = 2$  in  $(P)$  has the unique 2-vertex  $v$ . So the unique 2-vertex of  $G_j$  with  $d_{G_j}(v)$  odd is the unique 2-vertex of  $G$ . Hence we may assume that  $G$  is a block graph. Since  $G \in \mathcal{S}_4$ , its order is odd (by Proposition 2.3), and there is a 2-vertex  $v \in V(G)$ . Hence,  $G$  is 2-connected, that is, every pair of its vertices lie on a cycle. Therefore, by Proposition 3.2,  $v$  is the only 2-vertex in  $G$ .  $\square$

In view of Proposition 4.1 and Corollary 3.6, we are left with the task of determining which 2-connected loopless graphs with a single 2-vertex belong to  $\mathcal{S}_4$ . We proceed to describe a construction which shall enable us to narrow down the search to essentially 3-edge-connected graphs.



**Figure 5:** A 2-connected graph  $G$  (left), and graphs  $G', G''$  (right). The fat edges form the symmetric difference  $E(G) \oplus (E(G') \cup E(G''))$ .

Consider a 2-connected loopless graph  $G$  that is obtainable from an odd graph by a single edge subdivision. Thus  $\kappa(G) = \kappa'(G) = \delta(G) = 2$ . Assume  $G$  is not essentially 3-edge-connected, that is, let there be a nontrivial 2-edge cut  $\partial(X)$ . Say the unique 2-vertex of  $G$  falls in  $X$ , and consider the graphs  $G'$  and  $G''$  constructed as follows (see also Figure 5):

- (i)  $G'$  is derived from  $G[X]$  by adding an edge between the two endvertices of  $\partial(X)$  in  $X$ ;
- (ii)  $G''$  is obtained from  $G[\bar{X}]$  by introducing a new vertex and joining it with the two endvertices of  $\partial(X)$  in  $\bar{X} = V(G) \setminus X$ ; (equivalently,  $G''$  is derived from  $G$  by shrinking  $X$  to a single new vertex, i.e.  $G'' = G/X$ ).

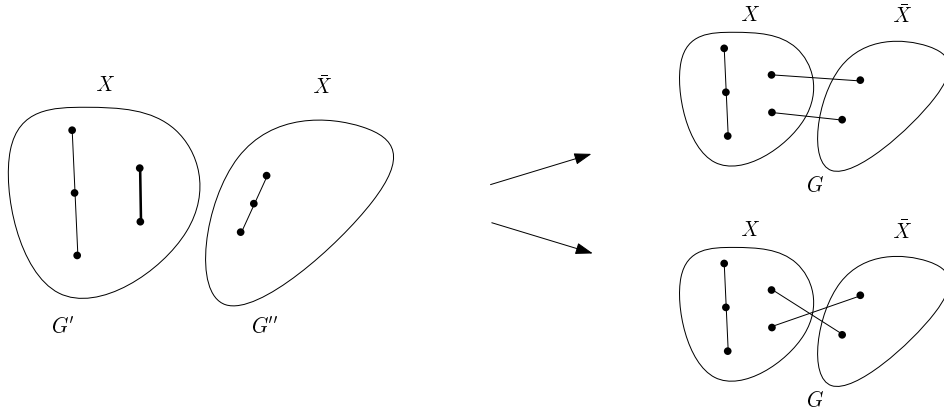
Notice that both  $G', G''$  are 2-connected and obtainable from odd graphs by single edge subdivisions. Reversing the process, let  $G', G''$  be disjoint 2-connected loopless graphs that are obtained from odd graphs by single edge subdivisions. Break a selected edge of  $G'$  into two half-edges, then remove the 2-vertex of  $G''$  along with its incident half-edges, and finally pair up and glue the half-edges emanating from  $G'$  with the corresponding half-edges emanating from  $G''$  so that two new (whole) edges are created. Call this process *gluing* of  $G'$  and  $G''$  (with respect to a selected edge of  $G'$ ). The resulting graph  $G$  is also 2-connected and obtainable from an odd graph by a single edge subdivision, moreover it has a nontrivial 2-edge cut.

We point out here that the result of gluing such a disjoint pair  $G', G''$  is not unique, because of an apparent 2-fold freedom involved in the process: first, there is a freedom of choice due to the arbitrariness of the selected edge from  $G'$ ; and second, there is freedom concerning the pairing the half-edges emanating from  $G'$  with the half-edges emanating from  $G''$  (cf. Figure 6).

The importance of transforming  $G$  into the pair  $G', G''$  and vice versa comes from the following.

**Proposition 3.7.** *Let  $G$  be a 2-connected loopless graph that is obtained from an odd graph by a single edge subdivision, and let  $\partial(X)$  be a nontrivial 2-edge cut in  $G$  such that the unique 2-vertex is in  $X$ . With  $G', G''$  as described above, the following equivalence holds:*

$$\chi'_o(G) = 4 \quad \text{if and only if} \quad \chi'_o(G') = \chi'_o(G'') = 4.$$



**Figure 6:** Graphs  $G', G''$  (left), and the two ways to obtain a graph  $G$  (right) by gluing  $G', G''$  in respect of a selected edge of  $G'$  (depicted as fat).

*Proof.* Assuming  $\chi'_o(G) = 4$ , we argue by contradiction that  $\chi'_o(G') = \chi'_o(G'') = 4$ . In view of Proposition 3.1, more than four colors are never required. Let  $e \in E(G') \setminus E(G)$ . Suppose  $\chi'_o(G') \leq 3$  and consider an odd 3-edge-coloring  $\varphi$  of  $G'$ . Extend the restriction  $\varphi|_{E(G[X])}$  to  $E(G)$  by using the color  $\varphi(e)$  for  $E(G) \setminus E(G[X])$ . This gives an odd 3-edge-coloring of  $G$ , a contradiction.

Suppose now that  $\chi'_o(G'') \leq 3$ . We already know from Corollary 3.6 (or from Proposition 2.3) that the graph  $G' * e$ , obtained from  $G'$  by introducing a 2-vertex on the edge  $e$ , is odd 3-edge-colorable. However then an odd 3-edge-coloring of  $G$  arises by combining an odd 3-edge-coloring of  $G' * e$  and an odd 3-edge-coloring of  $G''$  with the same color set (after possibly permuting colors in the latter). This contradiction settles the issue that  $\chi'_o(G) = 4$  implies  $\chi'_o(G') = \chi'_o(G'') = 4$ .

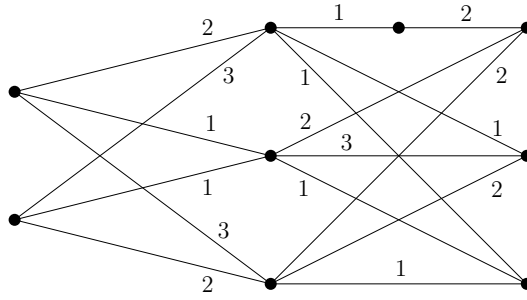
Let us show the reversed implication by contrapositive. Assume an odd 3-edge-coloring of  $G$  exists and consider its restrictions over the 2-edge cut  $\partial(X)$ . If the two edges forming this cut are colored the same, then we have an odd 3-edge-coloring of  $G'$ . Otherwise, if the two edges are colored differently, then an odd 3-edge-coloring of  $G''$  readily appears.  $\square$

We end this section with a property shared by all 2-connected members of  $\mathcal{S}_4$ .

**Proposition 3.8.** *If  $G \in \mathcal{S}_4$  is a block graph, then  $G$  can be obtained from a bipartite block odd graph by a single edge subdivision.*

*Proof.* Since  $G \in \mathcal{S}_4$ , from Corollary 3.6 it follows that there is a single 2-vertex  $v \in V(G)$ . Our task is to prove that the graph  $G \% v$ , obtained from  $G$  by suppressing  $v$ , is a bipartite block odd graph. The graph  $G$  is 2-connected. Hence, as  $v$  is the only 2-vertex of  $G$ , we have that  $G \% v$  is a block odd graph. Concerning the bipartiteness of  $G \% v$ , by Proposition 3.2, every cycle of  $G$  passing through  $v$  is odd. We are left to show that every cycle of  $G$  that avoids  $v$  is even.

Letting  $N_G(v) = \{u, w\}$ , suppose there is an odd cycle  $C_o$  in  $G - v$ . By the 2-connectedness of  $G$ , there exist two disjoint  $\{u, w\} - V(C_o)$  paths, say a  $u - u'$  path  $P$  and a  $w - w'$  path  $Q$ . Let  $R$  denote the  $u' - w'$  path along  $C_o$  which is even (resp. odd) if  $P$  and  $Q$  have same (resp. opposite) parities. Then  $P \cup R \cup Q$  is an even  $u - w$  path in  $G - v$ . However, we have already established that no cycle through  $v$  is even. This contradiction proves our point, that is, the graph  $G \% v$  is a bipartite block odd graph.  $\square$



**Figure 7:** An odd 3-edge-coloring of the graph obtained from  $K_{3,5}$  by a single edge subdivision.

Not every graph which can be obtained from a bipartite block odd graph by a single edge subdivision belongs in  $\mathcal{S}_4$ . For example, it can be readily seen from Figure 7 that the graph obtained from  $K_{3,5}$  by a single edge subdivision is odd 3-edge-colorable. On the other hand, in view of Theorem 1.6, this is not the case for the analogous graph obtained from  $K_{3,3}$ .

Note in passing that if  $G', G''$  are disjoint 2-connected graphs each obtainable from a bipartite odd graph by a single edge subdivision, then the result of any gluing of  $G', G''$  is another such graph. In the section we resolve the question which 2-connected graphs obtainable from a bipartite odd graph by a single edge subdivision belong to the class  $\mathcal{S}_4$ .

## 4 Characterization of $\mathcal{S}_4$

As a result of Proposition 3.5, we may confine to 2-connected graphs. Let us denote by  $\mathcal{F}$  the family defined inductively as follows:

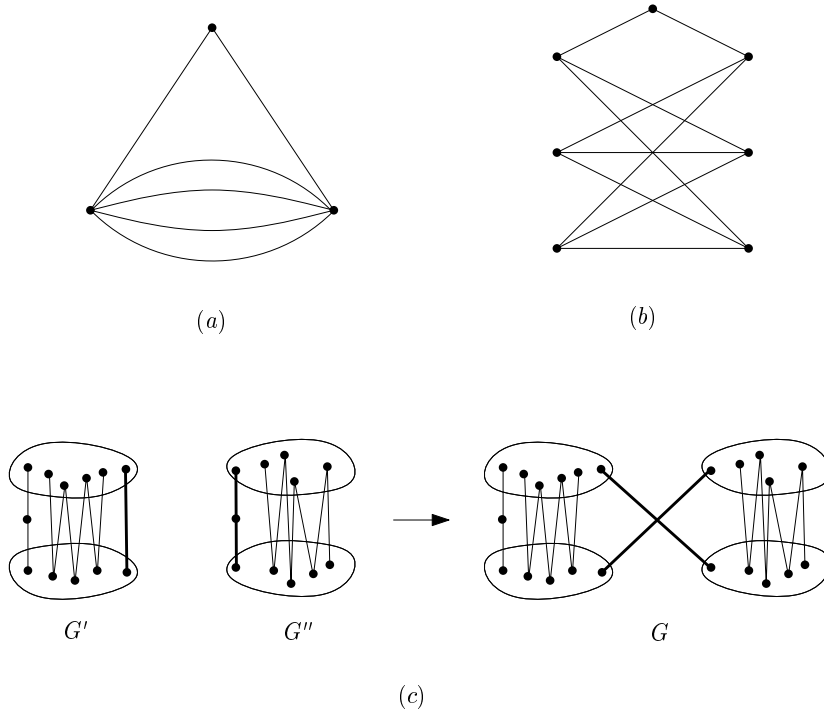
- (a) every Shannon triangle of type  $(2, 1, 1)$  and minimum degree 2 belongs to  $\mathcal{F}$ ;
- (b) every graph that can be obtained by a single edge subdivision from a 3-edge-connected bipartite cubic graph of order at least 4 belongs to  $\mathcal{F}$ ;
- (c) every other graph  $G$  in  $\mathcal{F}$  can be constructed by taking disjoint members  $G', G'' \in \mathcal{F}$ , and gluing them together.

We point out that the condition  $\delta = 2$  is included in (a) in order to stay within  $\mathcal{S}$ . Clearly,  $\mathcal{F} \subseteq \mathcal{S}$  and every member of  $\mathcal{F}$  is a 2-connected graph. Note in passing that parts (a),(b) and (c) of the above constructive definition of  $\mathcal{F}$  are pairwise disjoint: every graph from (a) is of order 3 whereas every graph from (b) or (c) is of odd order at least 5; every graph from (b) is essentially 3-edge-connected whereas every graph from (c) has a nontrivial 2-edge cut (cf. Figure 8). The family  $\mathcal{F}$  happens to be vital for our desired characterization. Namely, it turns out that a block graph  $G$  belongs to  $\mathcal{S}_4$  if and only if it belongs to  $\mathcal{F}$ , which we prove next.

**Theorem 4.1.** *Let  $G \in \mathcal{S}$  be a block graph. Then the following statements are equivalent:*

- (i)  $G \in \mathcal{F}$ ;
- (ii)  $G \in \mathcal{S}_4$ .





**Figure 8:** (a) A Shannon triangle of type  $(2, 1, 1)$  with  $\delta = 2$ ; (b) A single edge subdivision of  $K_{3,3}$ ; (c) Disjoint graphs  $G', G'' \in \mathcal{F}$  (left) and a graph  $G$  (right) obtained by gluing  $G', G''$  (the fat edges form the symmetric difference  $(E(G') \cup E(G'')) \oplus E(G)$ ).

*Proof.* We shall establish both  $(i) \Rightarrow (ii)$  and  $(ii) \Rightarrow (i)$ . Since  $\mathcal{F} \subseteq \mathcal{S}$ , the former implication consists of showing that every graph  $G \in \mathcal{F}$  has odd chromatic index  $\chi'_o(G) = 4$ . So, in view of the equality (1.1) and Proposition 3.7, we only need to use the following fact: *Every graph that can be obtained from a bipartite cubic graph by a single edge subdivision is not odd 3-edge-colorable.* The proof of this is a straightforward double-counting argument (see [1] for the details).

The key ingredient for proving the implication  $(ii) \Rightarrow (i)$  is provided by the next auxiliary result.

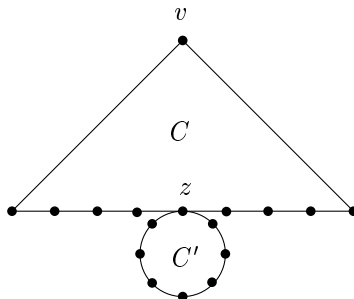
**Lemma 4.2.** *Let  $G \in \mathcal{S}$  be a 2-connected and essentially 3-edge-connected graph such that both the order  $n(G)$  and the maximum degree  $\Delta(G)$  are greater than 3. Then  $\chi'_o(G) \leq 3$ .*

*Proof.* Arguing by contradiction, let  $G$  be a minimal counter-example. By Proposition 3.8 and assuming  $v$  is the unique 2-vertex of  $G$ , the graph  $G \setminus v$  is bipartite; that is, every cycle through  $v$  is odd and every cycle avoiding  $v$  is even. Since, apart from  $v$ , every other vertex in  $G$  is of odd degree we have the following.

**Claim 1.** *No connected even subgraph  $H \subseteq G$  satisfies that  $v \in V(H)$  and  $n(H)$  is even.*

Arguing by contradiction, note that in the edge-complement  $\widehat{H} = G - E(H)$ , the vertex  $v$  is isolated whereas every other vertex has an odd degree. Take an odd factor  $K$  of  $H$ , and color  $E(K)$  by 1,  $E(H) \setminus E(K)$  by 2 and  $E(G) \setminus E(H)$  by 3. This gives an odd 3-edge-coloring of  $G$ , a contradiction.  $\diamond$

In particular, it follows from Claim 1 that there is no pair of cycles  $C, C'$ , one of which passes through  $v$ , such that  $V(C) \cap V(C')$  is a singleton, say  $\{z\}$ ; call this formation a *forbidden cycle pair at  $z$*  (cf. Figure 9). Several structural constraints arise from the absence of forbidden cycle pairs.



**Figure 9:** A forbidden cycle pair at a vertex  $z$ .

Let  $N_G(v) = \{u, w\}$ . Since  $G \in \mathcal{S}_4$ , both  $n(G)$  and  $\Delta(G)$  are odd and  $\geq 5$ . Consider an arbitrary ‘large’ vertex  $z$ , that is, a vertex of degree  $d_G(z) \geq 5$ . By the 2-connectedness of  $G$ , there exists a cycle  $C \subseteq G$  such that  $v, z \in V(C)$  and  $|V(C)| \geq 5$ . Indeed, if  $z \neq u, w$  then any cycle through  $v$  and  $z$  works; otherwise, select a vertex from  $V(G) \setminus \{u, v, w\}$  and use a cycle passing through  $v$  and that vertex. Let  $P = C - v$  be the  $u$ - $w$  path that goes through  $z$  and is contained within  $C$  (it is not excluded that  $z$  is an endvertex of  $P$ ). We consider the collection  $\mathcal{P}_z$  of paths  $Q$  in  $G - v$  such that  $Q$  connects  $z$  and another vertex of  $P$  and  $P \cap Q$  consists of these two vertices. Let us refer to the other endvertex of  $Q \in \mathcal{P}_z$  as its *ending*. We denote by  $\text{In}(Q)$  the set of *internal vertices* of the path  $Q$ , those that are not its endvertices.<sup>1</sup>

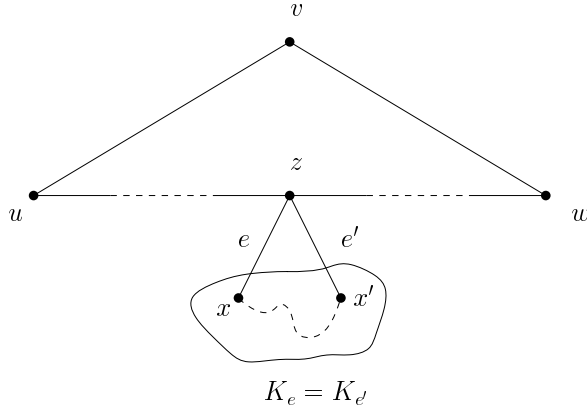
**Claim 2.** *There is a mapping from  $E_G(z) \setminus E_C(z)$  to  $\mathcal{P}_z$ , sending  $e \mapsto Q_e$ , such that  $e \in E(Q_e)$ . Moreover, for every such mapping it holds that*

$$e \neq e' \quad \Rightarrow \quad Q_e \text{ and } Q_{e'} \text{ are internally disjoint, i.e., } \text{In}(Q_e) \cap \text{In}(Q_{e'}) = \emptyset.$$

Let  $x$  be the other endvertex of  $e$  (besides  $z$ ). If  $x \in V(P)$ , all of the claimed is trivially true. Indeed, by then the path  $Q_e$  is uniquely determined and  $\text{In}(Q_e) = \emptyset$  since  $Q_e$  is the 1-path with edge set  $\{e\}$ . Otherwise, if  $x \notin V(P)$ , then  $x$  falls into a component,  $K_e$ , of  $G - V(C)$ . Note that then  $e \neq e'$  implies  $K_e \neq K_{e'}$ , for otherwise a forbidden cycle pair at  $z$  (that includes  $C$ ) is present (cf. Figure 10); in particular,  $e$  and  $e'$  are not parallel edges. From this readily it follows that  $Q_e$  exists in this case as well. Namely, every edge in  $\partial(V(K_e))$  has an endvertex in  $K_e$  and an endvertex on  $P$ ; moreover,  $|\partial(V(K_e))| \geq 3$  since  $G$  is essentially 3-edge-connected. Let us note in passing that neither  $Q_e$  nor its ending are no longer uniquely determined (as we already established that no two edges from the edge cut  $\partial(V(K_e))$  can have the same endvertex on  $P$ ). Observe that  $\text{In}(Q_e) \subseteq V(K_e)$ . Therefore, since  $e \neq e'$  implies  $K_e \neq K_{e'}$ , we have that  $e \neq e' \Rightarrow \text{In}(Q_e) \cap \text{In}(Q_{e'}) = \emptyset$ .  $\diamond$

Any subsequent use of notation  $Q_e$  is to be understood in the context of Claim 2. We study next the following situation:  $e, e' \in E_G(z) \setminus E_C(z)$  are distinct edges and  $Q_e, Q_{e'}$  have endings on the same side of  $P$  in respect of  $z$ .

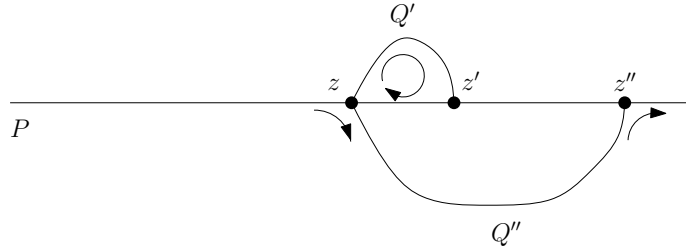
<sup>1</sup>Not to be confused with ‘internal vertices of a block’.



**Figure 10:** A forbidden cycle pair at  $z$  if  $K_e = K_{e'}$ . Letting  $x$  and  $x'$  be the other endvertices (besides  $z$ ) of  $e$  and  $e'$ , respectively, any  $x$ - $x'$  path within the shared component combines with  $e$  and  $e'$  to produce a cycle.

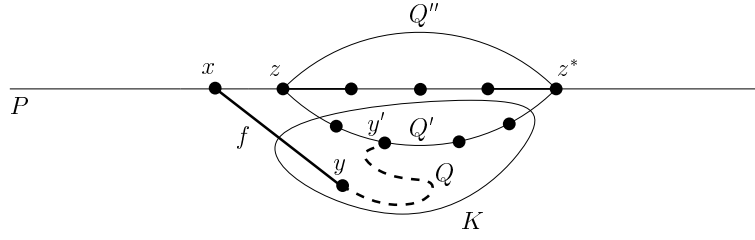
**Claim 3.** *Let  $Q', Q'' \in \mathcal{P}_z$  be internally disjoint, and have their respective endings lying on the same side of  $z$  along  $P$ . Then  $Q', Q''$  are 1-paths and their shared ending is in  $N_P(z)$ .*

First we show that  $Q'$  and  $Q''$  share the same ending. Arguing by contradiction, suppose their respective endings, say  $z'$  and  $z''$ , differ. Without loss of generality, let  $z'$  be an internal vertex of the subpath  $zPz''$ . Denote  $C' = zPz' \cup Q'$  and  $C'' = zPz'' \cup Q''$ . Then  $C' \oplus C''$  and  $C'$  constitute a forbidden cycle pair at  $z$  (cf. Figure 11). The obtained contradiction confirms that  $Q', Q''$  have the same ending, say  $z^*$ . Note in passing that  $z^*$  is another large vertex along  $P$ .



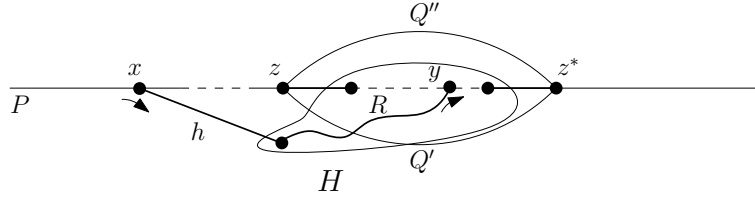
**Figure 11:** A detour from  $P$  that yields a forbidden cycle pair at  $z$ .

Next we prove that  $Q', Q''$  are actually 1-paths. For argument's sake, suppose  $\text{In}(Q') \neq \emptyset$ . Then  $\text{In}(Q')$  is contained within a single component  $K$  of  $G - V(C)$ . Note in passing that  $V(K) \cap \text{In}(Q'') = \emptyset$ , by the proof of Claim 2. Consider the edge cut  $\partial(V(K))$ . The essential 3-edge-connectedness of  $G$  and Claim 2 together guarantee that there is a  $V(P)$ - $V(K)$  edge  $f \notin E(Q')$  such that the endvertex of  $f$  on  $P$  is neither  $z$  nor  $z^*$ . However, such an edge  $f$  would contradict the already established feature of shared path endings. Indeed, let  $x$  and  $y$  be the respective endvertices of  $f$  in  $V(P)$  and  $V(K)$ , and let  $Q$  be a  $y$ - $\text{In}(Q')$  path within  $K$ , say  $y'$  is the other endvertex of  $Q$ . Then each of the paths  $zQ'y' \cup Q + f$  and  $z^*Q'y' \cup Q + f$  is internally disjoint with  $Q''$ ; moreover,  $zQ'y' \cup Q + f \in \mathcal{P}_z$  and  $z^*Q'y' \cup Q + f \in \mathcal{P}_{z^*}$ . Hence, depending on the position of  $x$  along  $P$ , at least one of the pairs  $zQ'y' \cup Q + f, Q''$  and  $z^*Q'y' \cup Q + f, Q''$  yields the mentioned contradiction (either in regard to the large vertex  $z$  or to the large vertex  $z^*$ ; see Figure 13). Consequently, both  $Q'$  and  $Q''$  are  $z$ - $z^*$  edges.



**Figure 12:** A pair of internally disjoint paths  $z^*Q'y' \cup Q + f, Q'' \in \mathcal{P}_{z^*}$  with distinct endings ( $x$  and  $z$ , respectively) on the same side of  $z^*$  along  $P$ .

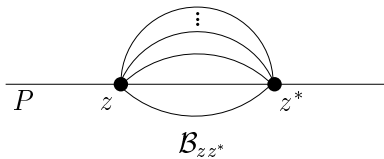
Finally, let us show that  $z^* \in N_P(z)$ , that is,  $z^*$  is a neighbor of  $z$  along  $P$ . Once again we argue by contradiction and evoke essential 3-edge-connectedness. Suppose  $\text{In}(zPz^*) \neq \emptyset$ , and let  $H$  be the component of  $G - (V(C) \setminus \text{In}(zPz^*))$  that includes  $\text{In}(zPz^*)$ . Since  $|\partial(H)| \geq 3$  there exists an edge  $h \in \partial(H) \setminus E(P)$ . It cannot be that  $h$  has an endvertex within  $\{z, z^*\}$ . Indeed, for otherwise,  $h$  would contradict the feature of shared path endings in regard to its endvertex in  $\{z, z^*\}$  (see Figure 13). Thus, without loss of generality, assume  $h$  meets  $P$  on the side of  $z$  not including  $z^*$ . If  $x$  is the endvertex of  $h$  on  $P$ , then there is a path  $R$  that starts at  $x$  along  $h$  and goes through  $H$  until it reaches  $P$  again, say at a vertex  $y \in \text{In}(zPz^*)$ .



**Figure 13:** A detour from  $P$  along  $R$  yielding a forbidden cycle pair at  $z^*$ .

Define  $\overline{C} = xPy \cup R$  and  $\overline{C'} = Q' \cup Q''$ . Then  $C \oplus \overline{C}$  and  $\overline{C}$  form a forbidden cycle pair at  $z^*$ . The obtained contradiction settles the claim.  $\diamond$

Since  $z$  is a large vertex, we have that  $|E_G(z) \setminus E_C(z)| \geq 3$ . Consequently, on at least one side along  $P$  the vertex  $z$  is incident with a  $3^+$ -bouquet  $\mathcal{B}_{zz^*}$  (see Figure 14); call it a *large bouquet*. Thus, every large vertex lying on  $P$  is incident with at least one large bouquet (shared with an adjacent large vertex along  $P$ ). Moreover, every large bouquet incident with a vertex of  $P$  is of this kind, for otherwise a forbidden cycle pair occurs. From Claims 2 and 3 it also follows that for each  $e \in E_G(z) \setminus E_C(z)$ , all the paths  $Q_e$  have endings on the same side of  $P$  with regard to  $z$ .



**Figure 14:** A large bouquet at  $z$  (and at a neighbor  $z^*$ ) along  $P$ .

**Claim 4.** *The multiplicity  $\mu(G) = 3$  whereas the maximum degree  $\Delta(G) = 5$ . Moreover, every 5-vertex on  $P$  is incident with a 3-bouquet, which it shares with a neighboring 5-vertex along  $P$ .*

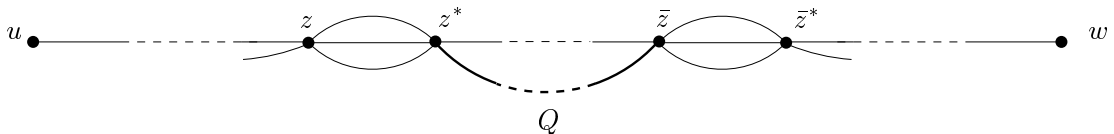
We already noted prior to this claim that every large vertex along  $P$  is incident with a large bouquet (shared with a large neighbor on  $P$ ). Let us first show that  $\mu(G) = 3$ . Consider in  $G$  a bouquet  $\mathcal{B}$  of maximum size. Thus  $|\mathcal{B}| \geq 3$ , implying that its endvertices are large vertices, say  $z'$  and  $z''$ . So (by taking  $z = z'$ ) we may assume that  $\mathcal{B}$  is a large bouquet along  $P$ . Select two edges  $e, f \in \mathcal{B} \setminus E(P)$ . Now it is important to observe that in case  $|\mathcal{B}| \geq 4$  the graph  $G - \{e, f\}$  satisfies all assumptions of the lemma. We proceed with clarifying this.

If  $|\mathcal{B}| \geq 5$  then it is obvious that  $G - \{e, f\} \in \mathcal{S}$  is a 2-connected and essentially 3-edge-connected graph having both order and maximum degree greater than 3. On the other hand, supposing  $|\mathcal{B}| = 4$ , the sets  $E_G(z') \setminus (E(P) \cup \mathcal{B}_{z'z''})$  and  $E_G(z'') \setminus (E(P) \cup \mathcal{B}_{z'z''})$  are even-sized. If neither of the vertices  $z', z''$  is incident with another large bouquet along  $P$ , then (by Claims 2 and 3) the sets  $E_G(z') \setminus (E(P) \cup \mathcal{B}_{z'z''})$  and  $E_G(z'') \setminus (E(P) \cup \mathcal{B}_{z'z''})$  are actually empty. However, that would imply  $|\partial(\{z', z''\})| = 2$ , contradicting the essential 3-edge-connectedness of  $G$ . So, at least one of  $z', z''$  must be incident with a 3-bouquet along  $P$ . This yields the same conclusion that  $G - \{e, f\}$  satisfies all the assumptions of the lemma. Indeed, the 2-connectedness, order and degree assumptions are clearly preserved. As for the essential 3-edge-connectedness of  $G - \{e, f\}$ , suppose there is a nontrivial 2-edge cut. Then  $z', z''$  must be on different sides of this cut, and hence the cycle  $C$  must have at least two edges in common with the cut. We have thus detected at least three edges in a 2-edge cut, a contradiction.

Therefore,  $G - \{e, f\}$  is odd 3-edge-colorable (by the minimality choice of  $G$ ). But such a coloring of  $E(G) \setminus \{e, f\}$  readily extends to an odd 3-edge-coloring of  $G$  by using for both  $e, f$  one color already appearing on  $\mathcal{B} \setminus \{e, f\}$ , a contradiction. Hence, it must be that  $|\mathcal{B}| = 3$ , confirming  $\mu(G) = 3$  and also showing that every large vertex along  $P$  is incident with a 3-bouquet.

Finally, suppose there is a large vertex  $z$  of degree greater than 5. It is incident with a 3-bouquet  $\mathcal{B}_{zz^*}$  along  $P$ , and has  $|E_G(z) \setminus (E(P) \cup \mathcal{B}_{zz^*})| \geq 3$ . In view of Claims 2 and 3, this inequality grants a  $4^+$ -bouquet along  $P$  which is incident with  $z$  and lies on the other side of  $z^*$ . However, such a bouquet contradicts with the already established equality  $\mu(G) = 3$ . Consequently,  $\Delta(G) = 5$ .  $\diamond$

So the 5-vertices along  $P$  come in pairs, each *pair* consisting of two neighbors on  $P$  which are the endvertices of a 3-bouquet. Two such pairs  $(z, z^*)$  and  $(\bar{z}, \bar{z}^*)$  are said to be *successive* if the vertices  $z, z^*, \bar{z}, \bar{z}^*$  are in that relative order on a traversal of  $P$  from  $u$  to  $w$  and  $\mathcal{P}_{z^*} \cap \mathcal{P}_{\bar{z}} \neq \emptyset$  (cf. Figure 15).

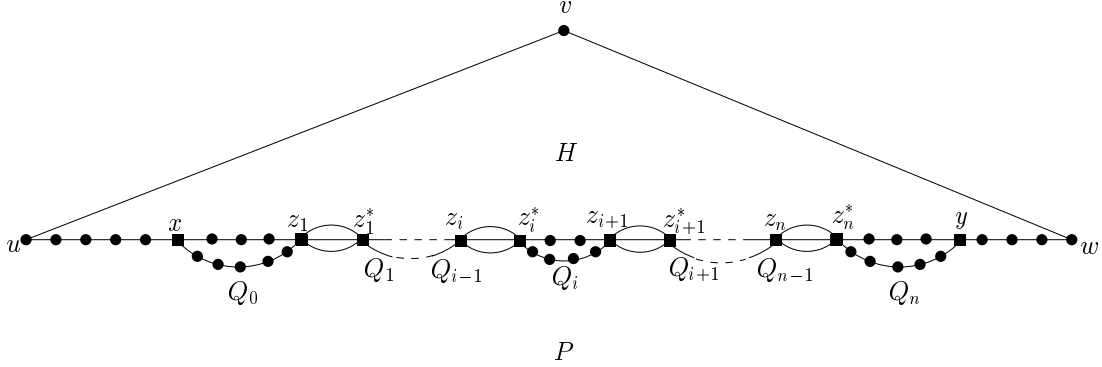


**Figure 15:** Successive pairs  $z, z^*$  and  $\bar{z}, \bar{z}^*$ . A  $z^*$ - $\bar{z}$  path  $Q$  (depicted as fat) is internally disjoint from  $P$ , that is, it holds that  $Q \in \mathcal{P}_{z^*} \cap \mathcal{P}_{\bar{z}}$ .

Consider a maximal sequence  $\mathcal{Z} : (z_1, z_1^*), (z_2, z_2^*), \dots, (z_n, z_n^*)$  of pairs (of 5-vertices along  $P$ ) subjected to the condition that the pairs  $(z_i, z_i^*)$  and  $(z_{i+1}, z_{i+1}^*)$  are successive, for each

$i = 1, 2, \dots, n - 1$ . Select a path  $Q_i \in \mathcal{P}_{z_i^*} \cap \mathcal{P}_{z_{i+1}}$ , for each  $i = 1, 2, \dots, n - 1$ . Also take paths  $Q_0 \in \mathcal{P}_{z_1} \setminus \mathcal{P}_{z_1^*}$  and  $Q_n \in \mathcal{P}_{z_n^*} \setminus \mathcal{P}_{z_n}$ . If  $x$  and  $y$  are the other endvertices of  $Q_0$  and  $Q_n$ , respectively, besides  $z_1$  and  $z_n^*$ . By the maximality choice of  $\mathcal{Z}$ , the vertices  $x$  and  $y$  are 3-vertices of  $G$ . Moreover, the vertices  $x, z_1, z_1^*, z_2, z_2^*, \dots, z_n, z_n^*, y$  are in that relative order on traversal of  $P$  from  $u$  to  $w$ . Consider the subgraph  $H$  of  $G$  defined as follows (cf. Figure 16):

$$H = C \cup \bigcup \{Q_i : i = 0, \dots, n\} \cup \bigcup \{B_{z_i z_i^*} : i = 1, \dots, n\}.$$



**Figure 16:** A sketch of the subgraph  $H \subseteq G$ .

Denote  $Z = \{x, z_1, z_1^*, \dots, z_n, z_n^*, y\}$ . Observe that every vertex from  $Z$  has the same degree in regards to both  $H$  and  $G$ . Thus, every vertex from  $Z$  is isolated in the edge-complement  $\widehat{H} = G - E(H)$ . Every other vertex of  $H$  has degree 2. Moreover, the set  $E(H)$  has the following important features: (i) it contains at least one 3-bouquet (surely  $\mathcal{B}_{z_1 z_1^*}$  is such), and (ii) if all 3-bouquets are to be removed from  $H$ , then all that remains is the path  $xPuvwPy$  and a collection of even pairwise disjoint cycles  $C_0, C_1, \dots, C_{n-1}, C_n$ , where the cycle  $C_i$  consists of the path  $Q_i$  and a suitable portion of  $P$ .

The above mentioned features of  $E(H)$  enable the following construction of a particular edge-coloring  $\varphi_H$  of  $H$  with color set  $\{1, 2, 3\}$ : start by taking a proper edge-coloring of the path  $xPuvwPy$  with color set  $\{1, 2\}$ ; for the remaining two uncolored edges at  $x$  (belonging in  $C_0$ ) use the already appearing color (1 or 2), and then extend to an edge-coloring of  $C_0$  with color set  $\{1, 2\}$  which is proper at each vertex  $\neq x, z_1$ ; similarly, for the remaining two uncolored edges at  $y$  (belonging in  $C_n$ ) use the already appearing color (1 or 2), and extend to an edge-coloring of  $C_n$  with color set  $\{1, 2\}$  which is proper at each vertex  $\neq y, z_n^*$ ; proceed by using all three colors 1, 2, 3 on each of the 3-bouquets  $\mathcal{B}_{z_1 z_1^*}, \dots, \mathcal{B}_{z_n z_n^*}$ ; finally, to each of the remaining uncolored (even and disjoint) cycles  $C_1, C_2, \dots, C_{n-1}$  apply an edge-coloring with color set  $\{1, 2\}$  which is proper at each vertex outside  $Z$  (i.e., alternate here between the colors 1 and 2), whereas the coloring is improper at each vertex from  $Z$  (i.e., repeat here the same color). Notice that the color 3 occurs only on edges having both endvertices in  $Z \setminus \{x, y\}$ . Moreover,  $\varphi_H$  is monochromatic (and thus odd) at each of the vertices  $x, y$ .

Now extend the above constructed  $\varphi_H$  from  $E(H)$  to  $E(G)$  by coloring  $E(G) \setminus E(H)$  with 3. Since every vertex from  $Z$  is isolated in  $\widehat{H}$ , the described extension is an odd 3-edge-coloring of  $G$ . This contradiction settles Lemma 4.2.  $\square$

The freshly proved Lemma 4.2, combined with Propositions 3.7 and 3.8, yields the implication (ii)  $\Rightarrow$  (i). Indeed, arguing by contradiction, let  $G \in \mathcal{S}_4 \setminus \mathcal{F}$  be a block graph of

minimum order  $n(G)$ . Taking into account Proposition 3.7, it is implied by part (c) of the construction of  $\mathcal{F}$  that  $G$  is essentially 3-edge-connected, besides being 2-connected. Consequently, by Lemma 4.2, it holds that  $n(G) = 3$  or  $\Delta(G) = 3$ . However, if  $n(G) = 3$  then  $G$  must be a Shannon triangle of type  $(2, 2, 1)$  with  $\delta(G) = 2$ ; hence  $G \in \mathcal{F}$  (due to part (a) of the construction). Otherwise, if  $\Delta(G) = 3$  then Proposition 3.8 assures that  $G \in \mathcal{F}$  (due to part (b) of the construction). The obtained contradiction settles the implication  $(ii) \Rightarrow (i)$ , which completes the proof of Theorem 4.1.  $\square$

Finally, we arrive at the main result, which succinctly summarizes our findings.

**Theorem 4.3.** *Let  $G \in \mathcal{S}$  be a connected graph. If  $(\mathcal{B}, \mathcal{V})$  is the bipartition of the block-tree  $B(G)$  of  $G$ , where  $\mathcal{B}$  is the set of blocks and  $\mathcal{V}$  the set of cutvertices of  $G$ , the following holds:*

$$\chi'_o(G) = \begin{cases} 1 & \text{if } G \text{ is odd;} \\ 2 & \text{if } G \text{ has 2-vertices, with an even number of them on each cycle;} \\ 4 & \text{if } \mathcal{B} \subseteq \mathcal{F} \text{ and for every } v \in \mathcal{V} \text{ there is a unique } B \in \mathcal{B} \text{ with odd } d_B(v); \\ 3 & \text{otherwise.} \end{cases}$$

*Proof.* Straightforward from Corollary 3.4, Proposition 3.5 and Theorem 4.1.  $\square$

## 5 Further work

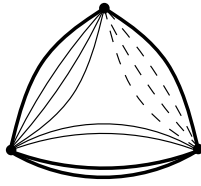
It is implied by Theorem 4.3 that the problem of determining the odd chromatic index of a subdivision of an odd graph is solvable in polynomial time. In view of Theorems 1.3 and 1.5, the complexity questions concerning the value of the odd chromatic index of general graphs amount to deciding on their odd 3-edge-colorability. As already mentioned in Section 1.2, this question is still open. Let us propose the study of a related set of questions arising from the following line of reasoning. The class  $\mathcal{S}$  can be captured by using the notion of *maximum even degree*, defined as follows. Let  $\Delta_{\text{even}}(G)$  denote the maximum even value among the vertex degrees of  $G$ . Thus  $\mathcal{S} = \{G : G \text{ is a loopless graph with } \Delta_{\text{even}}(G) \leq 2\}$ , and Theorem 4.3 tells that the problem of determining  $\chi'_o(G)$  whenever  $\Delta_{\text{even}}(G) \leq 2$  can be efficiently solved. For every  $k = 0, 1, 2, \dots$ , let  $\mathcal{S}^{(2k)} = \{G : G \text{ is a loopless graph with } \Delta_{\text{even}}(G) \leq 2k\}$ . So, by ignoring isolated vertices,  $\mathcal{S}^{(0)} = \mathcal{O}$ ; and obviously  $\mathcal{S}^{(2)} = \mathcal{S}$ . We find the next question interesting.

**Question 5.1.** *Is the decision problem whether a graph  $G \in \mathcal{S}^{(4)}$  has  $\chi'_o(G) \leq 3$  solvable in polynomial time?*

A positive answer to Question 5.1 would open the door for considering the following more general problem.

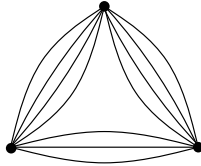
**Question 5.2.** *Given a positive integer  $k$ , is the decision problem whether a graph  $G \in \mathcal{S}^{(2k)}$  has  $\chi'_o(G) \leq 3$  solvable in polynomial time?*

Another possible field of study is to consider  $\mathcal{S}^{(2k)}$ -edge-colorability of graphs for a fixed positive integer  $k$  (instead of odd edge-colorability); that is, define a new type of edge-coloring by requiring that each color class is a member of  $\mathcal{S}^{(2k)}$  (rather than of  $\mathcal{O} = \mathcal{S}^{(0)}$ ). Say the



**Figure 17:** A Shannon triangle of type  $(2, 2, 2)$  that requires three colors for an  $\mathcal{S}$ -edge-coloring. The edges falling in distinct color classes of an optimal coloring are respectively depicted as dashed, normal and heavier.

corresponding index (representing the minimum sufficient number of colors) is  $\chi'_{\mathcal{S}^{(2k)}}(G)$ . For example, it is readily observed that  $\chi'_{\mathcal{S}}(W_4) = 2$  as opposed to  $\chi'_o(W_4) = 4$ . Similarly, if  $G$  is a Shannon triangle of type  $(2, 2, 2)$ , then  $\chi'_{\mathcal{S}}(G) \leq 3$  in contrast to  $\chi'_o(G) = 6$  (cf. Figure 17). Note that there are graphs requiring at least four colors for an  $\mathcal{S}$ -edge-coloring. Namely, every Shannon triangle  $G$  of type  $(2, 2, 1)$  and multiplicity  $\mu(G) \geq 3$  has  $\chi'_{\mathcal{S}}(G) = 4$  (cf. Figure 18).



**Figure 18:** A Shannon triangle of type  $(2, 2, 1)$  that requires four colors for an  $\mathcal{S}$ -edge-coloring. Its bouquets are of size 4, 4 and 3, respectively.

We are tempted to end our discussion here with the following.

**Conjecture 5.3.** *If  $G$  is a connected loopless graph that is not a Shannon triangle of type  $(2, 2, 1)$  and multiplicity  $\mu(G) \geq 3$ , then  $\chi'_{\mathcal{S}}(G) \leq 3$ .*

One wonders whether the bound 3 in Conjecture 5.3 may drop to 2 if  $\mathcal{S}$  is replaced with a certain  $\mathcal{S}^{(2k)}$  of sufficiently large  $k$ . Understandably, the list of excluded graphs might become longer.

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