Quadratic second-order backward stochastic differential equation and numeric analysis for Sannikov's optimal contracting problem

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Chapter 1

L'introduction

Cette thèse se concentre principalement sur deux sujets: les équations differentielles stochastiques rétrogrades de second ordre à croissance quadratique et le problème de contrat optimal de Sannikov.

Précisément, la première partie présente le problème bien-défini de solution des équations différentielles stochastiques de second ordre (2EDSR en abrégé) à croissance quadratique. Après un résumé concernant les résultats des équations différentielles stochastiques rétrogrades réflechies à croissance quadratique, c'est-à-dire, le problème bien-défini de la solution (l'existence et l'unicité), la comparaison, la stabilité, etc., on centralise essentiellement des études sur le problème bien-défini du 2EDSRs correspondantes. Hormis la croissance quadratique, on suppose aussi que le generateur f soit concave en z avec un gradient linéaire. Ceci nous offre une inégalité nouvelle de déscentées de la fonction valeur (aussi une f-surmartingale) V. Après d'avoir obtenu la régularité de sa limite en t, on s'appuye sur les études de Soner, Touzi et Zhang [STZ12] pour établir une expression de solution et vérifier l'existence et l'unicité.

La deuxième partie traite à fond le problème de contrat optimal de Sannikov sous contrainte de faille du bien de production. Un modèle de mandantement fait partie du problème de contrat, où l'Agent s'occupe du processus de la production et le Principal définit les régles du contrat d'après la performance du processus de la production, en espérant inciter l'Agent à s'engager plus efficacement vers son objectif. Le contrat a besoin de l'accord de l'Agent au début et progresse en temps réel selon des efforts de deux cotés. Et puis il stipule le temps d'arrêt (deterministe ou aléatoire) du contrat et un versement continu du Principal à l'Agent jusqu'au terminal ainsi qu'un versement en une fois à la fin. Notre objectif pincipal est d'étudier l'effet d'imposer des restrictions au temps d'arrêt du contrat avant qu'une faille du bien de production ait lieu. Dans ce but, on propose un nouveau rapprochement numérique, se fondant sur les réseaux neurons de Galerkin, pour examiner le comportement de la fonction valeur issue de problème correspondant du Principal.

Cette thèse comporte trois chapitres et se présente comme suit. Notez que, pour la définition précise de certains symboles apparaissait dans cette partie d'introduction, nous recommandons au lecteur de lire les chapitres correspondants pour une explication complete.

1.1 2EDSR avec des sujets relatifs

1.1.1 2EDSR à croissance quadratique

Soner, Touzi and Zhang [STZ12] généralisent la théorie des équations différentielles stochastiques rétrogrades de second ordre, où ils construisent la théorie complete de l'existence et l'unicité des équations différentielles stochastiques rétrogrades de second order avec un generateur Lipschitzien. Dans la suite, beaucoup d'extensions débordent avec des perspectives différentes, par exemple, la structure de générateur, l'intégrabilité satisfaite par la condition terminale, la régularité pour le principe de programmation dynamique, le cas discontinu, etc.. Jusqu'à présent, il reste encore beaucoup de sujets relatifs et questions ouvertes à traiter. Si on raconte le progrès sur la régularité pour le principe de programmation dynamique dans l'étude de 2EDSR: [STZ12, STZ13] et Possamaï et Zhou [PZ13] en obtiennent la régularité en chemin ω sous des hypothèses assez fortes satisfaites par la condition terminale et le générateur de EDSR, et vérifient la décomposition sémimartingale qui correspond à une fonction valeur du problème contrôle stochastique. Et ceci guarantie l'existence et l'unicité de 2EDSR correspondante. L'hypothèse de régularité a été faite pour prouver la continuité de la fonction valeur au lieu d'utiliser la théorème de selection mesurable.

Possamaï, Tan et Zhou [PTZ18] progressent aussi sur le sujet de la régularité. Grosso modo, ils prouvent d'abord le principe de programmation dynamique en utilisant la théorème de selection mesurable avec la comparaison et la stabilité d'ESDR correspondante, et puis étudient la régularité en chemin de la fonction valeur d'EDSR équipée d'une intégrabilité appropriée et un générateur Lipschitzien, ensuite ils appliquent ces deux résultats à vérifier le problème bien-défini de solution de 2EDSR. Il convient de souligner que dans leur cas, ils ont besoin aussi d'un générateur Lipschitzien.

Dans ce papier, notre but est d'obtenir l'existence et l'unicité des 2EDSRs équipées d'une condition terminale ξ qui est d'exponentielle intégrable et un générateur f à croissance quadratique. On note que, au lieu d'une hypothèse que la filtration est engendrée par le mouvement brownien, on introduit une autre martingale orthogonale M dans la définition d'EDSR et aussi 2EDSR.

Une étape cruciale pour démontrer notre résultat d'existence fait appel au problèlme d'existence et d'unicité pour les équations différentielles stochastiques rétrogrades réfléchies. Générallement, on considère les EDSRs réfléchies comme une extension des EDSRs, pour lesquelles la partie de Y doit rester au-dessus d'un certain processus (on l'appelle barrière) donné au début. D'ailleurs, il existe un processus à variation bornée sur $[0, T]$ dans $\mathbb{R}$ qui rend la partie Y toujours au-dessus de la barrière. El Karoui et al. [KKP+97] introduisent des EDSR réfléchies dans le cas où considèrent une filtration Brownienne et une barrière continue. Lepeltier et Xu [LX07] obtiennent l'existence de solution des EDSRs réfléchies ayant une condition terminale non bornée, mais avec une barrière continue et bornée. Bayraktar et Yao [BY12] généralisent ces résultats pour les EDSRs réfléchies équipées d'une barrière continue et non bornée, précisément, obtiennent l'existence et l'unicité de la solution corréspondante. Tous les deux travaillent sur une filtration engendrée par un mouvement brownien standard.

Plus tard, plusieurs extensions sur une barrière discontinue sont justifiés avec une base plus large que seulement celle de Brownienne (par exemple, Crépey et Matoussi [CM08], Lin, Ren, Touzi et Yang [LRTY20], Essaky, Hassani et Ouknine [EHO15], Essaky, Hassani et Rhazlane [EHR20], Hamadène et Ouknine [HO15]). Tous les extensions ont besoin d'une hypothèse de continuité à droite de la barrière. Particulièrement, on note que l'existence de la solution maximale et minimal des EDSRs réfléchies à croissance quadratique étudiées par [EHO15] and [EHR20] sont justifiées sans aucune hypothèse de $\mathbb{P}$-intégrabilité de la condition terminale, et le résultat de [EHR20] suit mais aussi se distingue de celui de [EHO15] par son hypothèse supplémentaire que la limite à gauche Y_- de la partie Y de la solution soit bornée par une barrière au-dessus et au-dessous qui sont prévisibles. Un point essentiel à remarquer est que leur besoin de la continuité à droite de la barrière, qui est différent des extensions obtenues par Grigorova et al. [GIO+17] où ils prouvent que la continuité à droite n'est pas nécessaire dans l'extension des EDSRs réfléchies équipées d'une filtration Brownienne standard. L'existence et l'unicité y sont établies en utilisant la décomposition de Mertens d'une surmartingale spéciale (pas forcément continue à droite) et certaine généralization appropriée de formule Itô dû à Galchouk et Lenglart.

L'objet du premier chapitre ici est l'étude de l'existence et l'unicité de solution des équations différentielles stochastiques rétrogrades de second ordre à croissance quadratique. A la section §2.3, on donne un résumé contenant des recherches anciennes sur les EDSRs réfléchies équipées d'une barrière non bornée. Ensuite, une estimation nouvelle a priori est justifiée à section §2.3 en utilisant le résultat des EDOs aléatoires obtenu par [LX07]. Cette estimation nouvelle joue un rôle essentiel pour l'étude de l'unicité (inclut la comparaison et la stabilité) pour les EDRS réfléchies mentionnées. Si la barrière dans la définition d'EDSR réfléchie est vers $-\infty$, on obtiendra une EDSR qui hérite l'existence et l'unicité de la solution. Par conséquent, cela explique notre enquête sur les résultats des EDSRs réfléchies ou la solution supérieur des EDSRs à §2.3. Bien noté que la solution des 2EDSRs doit être, dans un certain sens, le supremum des solutions des EDSRs standards, on justifie alors l'existence à §2.4. Précisément, on utilise deux fois le principe de programmaition dynamique pour étudier la mesurabilité de deux fonctions valeurs, V et sa limite continue à droite V^+, et puis on prouve l'existence. Un point à remarquer est l'importance de la deduction d'un deuxième principe de programmation dynamique sur V^+. C'est difficile d'obtenir diretement un résultat de la régulatiré en temps par la définition de V, à cet égard certains problèmes émergent, comme l'inégalité déscentées, etc.. Vu les circonstances, on ajoute une hypothèse naturelle sur le gradient du générateur en variable z et offre une perspective possible pour étudier ce problème bien-défini de la solution de notre 2ESDR. A la section §2.5, on justifie l'unicité.

1.1.2 Commentaire du principe de programmation dynamique

Après avoir défini la fonction de valeur pour des EDSRs correspondantes à croissance quadratique dans l'espace décalé, l'outil principal pour étudier des questions de mesurabilité dans la théorie du contrôle est le principe de programmation dynamique (PPD en abrégé), qui stipule qu'un problème d'optimisation globale peut être divisé en une série de problèmes d'optimisation locale.

A partir des années 1970, la vérification de ce principe intuitif est cependant difficile à réaliser même si simplement énoncé. De nombreux travaux ne considèrent que le cas sublinéaire, mais généralement les problèmes de contrôle d'ici consistent la maximisation d'une famille d'espérances sur l'ensemble des contrôles.

Plusieurs exemples pour le cas sublinéaire:

1. Pour le problème de contrôle stochastique du temps discret, comme dans Bertsekas et Shreve [BS78] ou Dellacherie [Del85], ils ont prouvé le PPD en se basant sur la stabilité des contrôles de conditionnement et de concaténation ainsi que sur un argument de sélection mesurable. Et cela conduit à la mesurabilité de la fonction de valeur associée, et à la construction de contrôles presque optimaux.

2. Pour le problème de contrôle stochastique de temps continu. Du fait qu'un suprêmem essentiel sur les temps d'arrêt peut être approché par un suprêmem sur une famille de variables aléatoires, El Karoui [Kar81] a utilisé deux outils, les fortes propriétés de stabilité des temps d'arrêt et le fait que l'argument de sélection mesurable peut être évité, pour établir le PPD pour le problème d'arrêt optimal dans un temps continu. En outre, pour les problèmes généreux du processus Markov contrôlés (en temps continu), El Karoui, Huu Nguyen et Jeanblanc [NEKJP87] ont interprété les contrôles comme des mesures de probabilité sur l'espace de trajectoire canonique pour fournir un cadre à dériver le PPD en utilisant le théorème de sélection mesurable. El Karoui et Tan [KT13a, KT13b] ont aussi étudié le problème du temps continu dans un contexte plus général, mais toujours basé sur les mêmes arguments que dans [NEKJP87] et l'article de Nutz et al. [NN13] qui sera mentionné plus tard.

3. Certains articles ont porté sur la recherche de contourner l'argument de sélection mesurable. Par exemple, Fleming et Soner [FS06], le faible PPD de Bouchard et Touzi [BT11]. Ce dernier résultat a ensuite été étendu par Bouchard et Nutz [BN12, BN16] et Bouchard, Moreau et Nutz [BBN14] pour optimiser les problèmes de contrôle avec les contraintes de l'état ainsi qu'aux jeux différentiels (voir aussi Dumitrescu, Quenez et Sulem [RDS16] pour un problème combiné d'arrêt et de contrôle des EDSRs). L'outil alternatif qu'ils ont utilisé à la place de l'argument de sélection mesurable est d'ajouter une hypothèse supplémentaire, a priori la régularité de la fonction valeur.

4. Bayraktar et Sîrbu [BS12, BS13] ont développé la méthode stochastique de Perron qui permet aux problèmes de Markov d'obtenir la caractérisation de la solution de viscosité de la fonction de valeur sans utiliser le PPD, puis d'en prouver a posteriori.

5. Nutz et al. [NN13, NvH13] ont montré une nouvelle idée motivée par la théorie de la finance robuste à prouver le PPD pour les espérances sublinéaires (ou un problème de contrôle stochastique non-Markovian), qui partagent certaines similitudes des arguments essentiels dans [NEKJP87].

1.1.3 Commentaire de l'espérance non-linéaire

Le concept d'espérances nonlinéaires sur un espace de probabilité donné (c'est-à-dire des opérateurs agissant sur des variables aléatoires qui préservent toutes les propriétés des espérances sauf de la linéarité) a une longue histoire qui commence avec le but de former la théorie des capacités et est utilisé en économie pour axiomatiser les préférences des agents économiques qui ne satisfont pas les axiomes habituels de von Neumann et Morgenstern, ou de la g-espérance (ou EDSRs) introduit par Peng [Pen97].

Cette théorie est construite pour être cohérente en filtration (ou en temps), c'est-à-dire que sa version conditionnelle satisfait la formule de l'espérance totale, similaire à celle des espérances linéaires, qui est elle-même une sorte de PPD. Ceci explique l'intérêt particulier des recherches sur ce sujet du point de vue du contrôle stochastique.

En outre, Coquet et al. [FCP02] ont prouvé que toutes les espérances nonlinéaires cohérentes de filtration satisfaisant aux propriétés de domination appropriées pouvaient être représentées par des EDSRs.

Pour distinguer l'approche EDP de l'approche probabiliste:

Peng [Pen13] ont introduit une théorie connexe et a développé autour de la notion de G–espérance, qui mène à la G-EDSR. Ils ont renoncé à travailler sur un espace de probabilité fixé portant différentes mesures de probabilité correspondant aux contrôles. Au lieu d'un paramètre probabiliste, ils ont travaillé directement sur un espace d'espérance sublinéaire dans lequel le processus canonique déjà inclut les différentes mesures.

Ils utilisent principalement des arguments EDPs pour construire une solution dans le cas Markovian puis un argument de clôture, où se situe la différence de ces méthodes; mais les résultats finaux sont extrêmement proches de 2EDSRs, avec des restrictions similaires en termes de régularité. En outre, du fait que les EDPs qu'ils estimaient ont besoin d'avoir au moins une solution continue, alors il semble que cette approche exige toujours certaine régularité.

L'approche probabiliste des 2EDSRs présente plus de possibilités, comme [NvH13] dans le cas des espérances linéaires (c'est-à-dire lorsque le générateur des EDSRs est à 0), tout peut être bien défini en supposant seulement que la condition terminale est (Borel) mesurable.

1.1.4 Commentaire de solution de viscosité d'EDP entièrement non-linéaire et dépendante du chemin (EDPP en abrégé)

Cette notion a été introduite par Ekren, Keller, Touzi et Zhang [IEZ14b, IEZ16a, IEZ16b] et partage des liens étroits avec 2EDSRs. Précisément, ils ont montré que la solution de 2EDSR, avec un générateur et une condition terminale uniformément continue (en ω), est exactement la solution de viscosité d'une

EDPP particulier, faisant la théorie précédente de 2EDSRs un cas particulier de la théorie de EDPPs.

Dans [PTZ18], ils ont montré un cas spécial qui ne peut pas être inclus dans la théorie des EDPPs, que (une version appropriée de) la fonction valeur pour laquelle ils ont obtenu le PPD fournit une solution à des 2EDSRs sans exiger aucune hypothèse de régularité. A noter que dans le cadre général qu'ils ont considéré, la méthode classique de preuve de l'existence et de l'unicité échoue (parce que la filtration avec laquelle ils travaillent n'est pas quasi-gauche continue en général), et les estimations découlent d'un résultat général de [BBZ18]. D'ailleurs, ils ont considéré un cas particulier de bien-défini des EDSRs et une classe plus générale de mesures de probabilité. En particulier, les résultats de [PTZ18] contiennent un cas particulier de la théorie des EDSRs, ni relatif aux 2EDSRs de [STZ12], ni relatif à G-EDSR.

1.1.5 Commentaire des équations différentielles stochastiques rétrogrades du second ordre

À la suite des résultats de [STZ12] sur 2EDSRs et motivés par la tarification des créances contingentes américaines sur les marchés incertains en matière de volatilité, Matoussi, Possamaï et Zhou [AMZ13] ont utilisé la méthodologie de [STZ12] pour introduire une notion des équations différentielles stochastiques rétrogrades du second ordre et refléchies, et prouvé l'existence et l'unicité dans le cas d'un obstacle inférieur.

C'était essentiel qu'ils ne considèrent que des obstacles inférieurs. Dans ce cas, les effets dus à la réflexion et au second ordre agissent dans le même sens, où tous les deux ils forcent la solution à rester au-dessus de certains processus. Il suffit donc d'ajouter un processus non-décroissant à la solution de l'équation. Cependant, dès qu'on essaie de considérer les obstacles supérieurs, les deux effets commencent à se contrebalancer et la situation change radicalement. Cette affaire a donc été laissée ouverte en [AMZ13].

D'ailleurs, [IEZ14a] ont donné des résultats spécifiques pour le problème d'arrêt optimal dans le cadre d'une espérance nonlinéaire (qui correspond à peu près aux 2EDSRs réfléchies avec un générateur égale à 0). Toutefois, comme il s'agit d'un problème de 'sup-sup', il n'est lié qu'aux 2EDSRs réfléchies plus faibles. Après, Nutz [Nut12] a réussi à traiter le même problème de l'arrêt optimal en fonction des espérances nonlinéaires, mais avec une formulation 'inf–sup' liée aux 2EDSRs réfléchies supérieures.

1.2 EDSRs réfléchies, EDSRs à horizon aléatoire et à sauts (EDSRS en abrégé)

Dans ce chapitre de revue, nous traitons trois sujets: des équations différentielles stochastiques rétrogrades réfléchies (EDSRs réfléchies en abrégé), des équations différentielles stochastiques rétrogrades à horizon aléatoire et des équations différentielles stochastiques rétrogrades à sauts (EDSRS en abrégé). Il s'agit d'un sondage qui résume les progrès réalisés au cours des dernières années en fonction de nos propres besoins et préférences en matière de recherche. Après l'introduction, la première partie est une nouvelle preuve de l'existence et le principe de comparaison des solutions pour un type des EDOs réfléchie est fourni, qui est utilisé pour obtenir une estimation a priori sur Y-partie de solutions pour les EDSRs réfléchies à croissance quadratique avec une condition terminale ξ exponentielle intégrable et un générateur$f$ à croissance quadratique, équipées des barrières bornés ou non bornés. Ensuite, la deuxième partie étend la solidité de la théorie des EDSRs/EDSRs réfléchies à croissance quadratique (par exemple, [BH08, BH06, BY12]) à une filtration plus générale engendrée par le mouvement Brownien standard et une martingale orthogonale. De plus, la discussion dans les troisième et quatrième parties comprend l'explication des difficultés que nous avons rencontrées dans notre pratique et les sujets de recherche futurs possibles.

Ici on précise surtout les introductions des dernières 3 sections: EDSRs réfléchies, EDSRs à horizon aléatoire et à sauts.

1.2.1 Equations différentielles stochastiques rétrogrades réfléchies (EDSRs réfléchies en abrégé) dans une filtration plus générale

La théorie des EDSRs et EDSRs réfléchies à croissance quadratique joue un rôle important dans notre compréhension de 2EDSRs. Par conséquent, nous présentons ici un aperçu des principaux arguments relatifs à l'existence et à l'unicité des EDSRs/EDSRs réfléchies à croissance quadratique à horizon déterministe.

Equations différentielles stochastiques rétrogrades (EDSRs en abrégé) réfléchies: La notion d'EDSR réfléchie est introduite par El Karoui et al. [KKP+97]. Ici dans ce chapitre, nous limitons la définition d'une solution pour une telle équation, associée à un générateur f, une condition terminale ξ et une barrière continu $L \in \mathbb{C}^0_{\mathbb{F}^{\mathbb{P}}}$, être un quadruple de processus $(Y, Z, M, K) \in \mathbb{C}^0_{\mathbb{F}^{\mathbb{P}}} \times \mathbb{H}^{2,\text{loc}}(\mathbb{P}, \mathbb{R}^d) \times \mathbb{M}(\mathbb{P}) \times \mathbb{I}(\mathbb{P})$, satisfaisant

$$\begin{aligned}
&Y_t = \xi + \int_t^T f_s(Y_s, Z_s)ds - \int_t^T Z_s \cdot dX_s - \int_t^T dM_s + K_T - K_t, 0 \leqslant t \leqslant T, \\
&Y_t \geqslant L_t, 0 \leqslant t \leqslant T, \mathbb{P}\text{-p.s.}, \\
&\int_0^T (Y_s - L_s)dK_s = 0, \mathbb{P}\text{-p.s. (Skorokhod condition)},
\end{aligned} \tag{1.1}$$

où $dX_s := \widehat{\sigma}_s dW_s$, $\widehat{\sigma}_s \in \mathbb{S}_d^{\geqslant 0}$ (c'est-à-dire, l'ensemble des matrices symétriques non négatives-définies $d \times d$), et $f_\cdot(\cdot, \cdot, \cdot)$ est un générateur progressivement mesurable répondant à certaines conditions à préciser ultérieurement. Notez ici que la filtration sous-jacente n'est pas seulement engendrée par un mouvement brownien, donc nous introduisons un autre composant dans la définition d'une solution superieur d'un EDSR, à savoir une martingale M qui est orthogonale au mouvement Brownien standard W. K est un processus continu non décroissant qui pousse à la hausse le processus Y afin de le garder au-dessus de la barrière L. La dernière équation signifie que le processus K agit seulement lorsque le processus Y atteint l'obstacle L. Notez que nous autorisons L à prendre la valeur $-\infty$, de sorte que la notion d'EDSR réfléchie couvre EDSR de Pardoux et Peng [PP90]. Par la paire (f, ξ), nous entendons l'EDSR correspondante, sans les deux dernières conditions concernant la barrière L dans ce qui précède (1.1).

1.2.2 Equations différentielles stochastiques rétrogrades (EDSRs en abrégé) à horizon aléatoire

La théorie des équations différentielles stochastiques rétrogrades (EDSR en abrégé) à horizon fini est introduite par Pardoux et Peng [PP90] avec une condition terminale ξ intégrable au carré et un générateur f uniformément Lipschitzien en y- et z-variable ainsi que l'intégrabilité $\mathbb{E}[\int_0^T |f_t(0,0)|^2 dt] < \infty$ à certain horizon déterministe $T > 0$. Barles, Buckdahn et Pardoux [BBP97] donnent aussi une preuve de l'existence et l'unicité en utilisant la méthode de point fixe dans un contexte où les EDSRs sont définies par rapport à un mouvement Brownien et une mesure aléatoire de Poisson. Des autres structures sont étudiées à part d'une continuité Lipschitzienne. Par exemple, Darling [Dar95] justifie l'existence et l'unicité pour le cas de la continuité Lipschitzienne locale et la convexité de générateur.

Peng [Pen91] étudie l'interprétation probabilistique pour le système des équations différentielles stochastiques paraboliques quasi-linéaires de second ordre en introduisant certain genre des EDSRs avec des Itô EDSs forward classique, où il explique aussi la relation entre la Y-partie de solution d'EDSR à horizon aléatoire non borné et les équations aux dérivées partielles elliptiques semi-linéaires, et puis obtient l'existence et l'unicité d'EDSR sous une hypothèse très particulière. Darling and Pardoux [DP97] établissent d'abord l'existence et l'unicité pour l'EDSR classique (dans le sens où la condition terminale est donnée au temps fixé au lieu d'aléatoire) avec un générateur pas forcément Lipschtzien en variable y et z; et aussi obtiennent l'existence et l'unicité des EDSRs au temps terminal aléatoire en utilisant une hypothèse plus faible que celle de [Pen91]. Précisément, en dépendant sur la monotonie en y avec le paramètre $a \in \mathbb{R}$ et la continuité Lipschitzienne en z avec le paramètre $b \in \mathbb{R}_+$ pour le générateur $f_t(y, z)$, aussi avec certaine condition de l'intégrabilité à propos de la condition terminale ξ et $f_\cdot(0,0)$ ainsi que l'horizon aléatoire τ, ils donnent l'existence et l'unicité (Proposition 3.2-3.3, Théorème 3.4) de ce genre des EDSRs à horizon aléatoire dans certain espace de Hilbert qui dépend au temps d'arrêt τ presque sûrement fini et le nombre particulier $b^2 - 2a$. D'ailleurs, leur condition de l'integrabilité exige que certain paramètre soit plus large que le nombre $b^2 - 2a$.

L'une des techniques utilisées la plus souvent par les recherches suivantes est leur traitement indépendant de la définition de Y et Z sur deux sous-intervalles complémentaires, c'est-à-dire, $[t\wedge\tau, n\wedge\tau]$ et $[n\wedge\tau, \tau]$.

De plus, Fuhrman et Tessitore [FT04] élargissent ce résultat pour la situation équipée d'un temps d'arrêt infini avec une condition nouvellement ajoutée concernant a et b que $b^2 - 2a < 0$ soit satisfaite, et aussi la borne de $f_t(0,0)$.

Enlever la dépendance de ce nombre $b^2 - 2a$ attire l'attention de plusieurs chercheurs à un moment donné. Dans l'intention d'expliquer l'idée principale du progrès obtenu vers cette direction, on modifie d'abord la définition d'EDSR réfléchie dans le contexte de l'horizon aléatoire comme ci-dessous,

$$\begin{aligned} Y_{t\wedge\tau} =& Y_{T\wedge\tau} + \int_{t\wedge\tau}^{T\wedge\tau} [f_s(Y_s,0) + b_s \cdot Z_s] ds - \int_{t\wedge\tau}^{T\wedge\tau} Z_s \cdot dW_s + \int_{t\wedge\tau}^{T\wedge\tau} dK_s \\ =& Y_{T\wedge\tau} + \int_{t\wedge\tau}^{T\wedge\tau} f_s(Y_s,0) ds - \int_{t\wedge\tau}^{T\wedge\tau} Z_s \cdot d\widetilde{W}_s + \int_{t\wedge\tau}^{T\wedge\tau} dK_s, \end{aligned} \tag{1.2}$$

où on définit $\widetilde{W}_s := W_s - \int_0^s b_r dr$ et le processus b comme $b_s := \frac{f_s(Y_s,Z_s)-f_s(Y_s,0)}{|Z_s|^2} Z_s \mathbf{1}_{\{|Z_s|>0\}}$. Si on peut enlever la dépendance du générateur f en z, cela est utile à éviter le nombre particulier $b^2 - 2a$ dessus. En effet, si f est Lipschitzien en z, le processus b est borné et $\mathcal{E}_t := \exp(\int_0^t b_s dW_s - \frac{1}{2}\int_0^t |b_s|^2 ds)$ est une martingale uniformément intégrable dans l'intervalle $t \in [0,T]$. On précise la probabilité originale $\mathbb{P}$, sous laquelle W est un processus de Wiener standard, et puis on prend $\mathcal{E}_T$ comme la densité d'une nouvelle mesure probabilité $\mathbb{Q}_T$ à propos de la restriction $\mathbb{P}_T$ (la mesure probabilité $\mathbb{P}$ restreinte à la $\sigma-$algèbre $\mathcal{F}_T$). Cette nouvelle probabilité $\mathbb{Q}_T$ est équivalent de $\mathbb{P}_T$ et $\{\widetilde{W}_t\}_{0\leqslant t\leqslant T}$ est un processus de Wiener sous $\mathbb{Q}_T$. Ensuite on peut étudier l'équation (1.2) sous cette nouvelle mesure de probabilité $\mathbb{Q}_T$, pour laquelle il n'existe plus dépendance en z du générateur. Cela explique l'idée principale de certaines recherches suivantes comment éviter le nombre particulier $b^2 - 2a$, comme Briand et Hu [BH98], Royer [Roy14], Hu et Tessitore [HT07], Lin Ren Touzi et Yang [LRTY20], etc..

Précisément, dans le cas où le processus Y prend ses valeurs réelles $\mathbb{R}$, on peut faire une référence à [BH98] et [Roy14] pour des extensions corréspondantes du générateur sans croissance quadratique en z. Tous les deux n'ont plus besoin de la dépendance en $b^2 - 2a$ du générateur, néanmoins ils éxigent que $f_t(0,0)$ soit borné (p.468 Hypothèse (A6.3) et p.478 Hypothèse (A7) à [BH98], p.283 Hypothèse (H4) et p.289 Hypothèse (H4') à [Roy14]) et le coefficient $a \in \mathbb{R}_+$ (non plus valide pour $a \in \mathbb{R}_-$) soit stricte supérieur à 0 ainsi que le théorème de Girsanov, pour justifier l'existence et l'unicité où Y est un processus continu borné. La différence entre ces deux résultats est que [BH98] demande la continuité uniformément Lipschitzien en y et z tandis que [Roy14] exige seulement la continuité Lipschitzienne en z et ajoute une autre croissance en partie y (p.283 Hypothèse (H2) à [Roy14]). En même temps, [Roy14] étudie aussi la situation où $a \equiv 0$.

Particulièrement, on aimerait ici mentionner un résultat le plus récent [LRTY20]. Ils exigent une hypothèse similaire comme celle de [DP97], avec une modification sur le coefficient de remise et considèrent une classe d'intégrabilité sous l'espérance non-linéaire dominée $\mathcal{E}^{\mathbb{P}}$ au lieu de $\mathbb{E}^{\mathbb{P}}$. Leur exemple 3.5 (page 6) clarifie cette corréspondance de leurs hypothèses en utilisant une situation simple où un générateur linéaire est considéré. Pour la majorité de recherches qui généralisent [DP97], comme [BH98] et [Roy14] etc., il existe toujours une hypothèse $a > 0$, c'est-à-dire, le générateur est monotone stricte, et ξ, f_0 sont bornés. Et tout ce genre des discussions sont inclut comme un cas spécial de l'hypothèse 3.2 (page 5) à [LRTY20]. Pour $a = 0$, c'est-à-dire le générateur est monotone, [Roy14] étudie l'existence et l'unicité sous l'hypothèse que le générateur f dépendant seulement en z soit borné et ξ soit aussi borné comme on a mentionné dessus. Ce résultat est ensuite généralisé par [HT07], [BC08] et Papapantoleon et al. [PPS18] pour des situations plus généraux. De plus, théorème 3.4 (page 5) de [LRTY20] étendent ces résultats précédents en permettant $a \leqslant 0$, dû aux norms nouveaux sous lesquels ils établissent l'existence et l'unicité.

Précisément, [HT07] justifie l'argument pareil pour le cas d'un processus cylindrique de Wiener avec valeurs dans un espace de Hilbert, où ils considèrent la solution douce de certain type des équations différentielles partielles elliptiques dans l'espace de Hilbert et la technique principale est la discussion sur la différentiabilité de ce genre de solution des équations rétrogrades dans certain système Markovian forward-backward des équations par rapport à la donnée initial x de l'équation forward.

Pour le cas quadratique, Kobylanski [Kob00] donne des résultat des EDSRs à croissance quadratique équipées d'temps d'arrêt borné ou p.s. fini en utilisant la transformation Hopf-Cole et le rapprochement du générateur f.

Plus tard, Briand et Confortola [BC08] traitent le cas équipé d'un temps d'arrêt infini avec une condition terminale bornée sous deux hypothèses supplémentaires ajoutées au générateur $f_t(y,z)$: Lipschitzien locale en z et monotone stricte en y. Précisément, ils supposent que pour $\mathbb{P}$-p.s. et pour tout $t \geqslant 0$, il existe constants $C \geqslant 0$ et $\lambda > 0$ t.q.

- (BC08-A.1.(i)) pour tout $y \in \mathbb{R}$ et

$$z, z' \in \mathbb{R}^d, |f_t(y,z) - f_t(y,z')| \leqslant C(1 + |z| + |z'|)|z - z'|;$$

- (BC08-A.1.(ii)) f est monotone stricte par rapport à y: pour tout $z \in \mathbb{R}^d$,

$$\forall y, y' \in \mathbb{R}, (y - y')(f_t(y,z) - f_t(y',z)) \leqslant -\lambda |y - y'|^2.$$

On précise que $L^2_{\text{loc}}(\mathbb{R}^d)$ définit l'espace des classes équivalentes des processus mesurables progressivement $\psi : [0,\infty) \times \Omega \to \mathbb{R}^d$ t.q. pour tout $t > 0, \int_0^t |\psi_r|^2 dr < \infty, \mathbb{P}$-p.s. et $M^{2,\epsilon}(\mathbb{R}^d)$ définit l'ensemble de processus $\{\mathcal{F}_t\}_{t\geqslant 0}$ (filtration engendrée par le mouvement Brownien standard)-mesurables progressivement $\{\psi_t\}_{t\geqslant 0}$ avec valeurs dans $\mathbb{R}^d$ t.q. $\mathbb{E}[\int_0^{+\infty} e^{-2\epsilon s}|\psi_s|^2 ds] < \infty$, ensuite avec une condition terminale borné ξ et un générateur $f_t(y,z)$ à croissance quadratique en z, ils donnent l'existence et l'unicité de leurs solution, pour laquelle Y est un processus borné et Z fait partie de $L^2_{\text{loc}}(\mathbb{R}^d)$ voire $M^{2,\epsilon}(\mathbb{R}^d)$ pour tout $\epsilon > 0$. L'idée principale de leur résultat: pour tout $n \in \mathbb{N}$, on définit (Y^n, Z^n) la solution unique d'EDSR

$$Y^n_t = \xi \mathbf{1}_{\{\tau \leqslant n\}} + \int_t^n \mathbf{1}_{\{s \leqslant \tau\}} f_s(Y^n_s, Z^n_s) ds - \int_t^n Z^n_s dW_s, 0 \leqslant t \leqslant n. \tag{1.3}$$

Sous l'hypothèse (BC08-A.1.(i)) et (BC08-A.1.(ii)), l'EDSR a une solution unique d'après [Kob00], et puis $Y^n_t = Y^n_{t\wedge\tau}, Z^n_t \mathbf{1}_{t>\tau} = 0$ par [Roy14]. D'ailleurs, c'est nécessaire d'étendre cette solution à l'axe entier de temps en écrivant pour tout $t > n, Y^n_t = Y^n_n = \xi \mathbf{1}_{\tau\leqslant n}, Z^n_t = 0$. Après, l'épreuve suit une procédure rituelle avec une modification nécessaire: d'abord prouver que Y^n soit borné par une constante indépendante de n; et puis vérifier que la convergence de la série $(Y^n)_{n\geqslant 0}$ et sa limite égale à Y au sens de $M^{2,\epsilon}(\mathbb{R})$ pour tout $\epsilon > 0$; ensuite étudier que la série $(Z^n)_{n\geqslant 0}$ soit Cauchy au sens de $M^{2,\epsilon}(\mathbb{R}^d)$, pour tout $\epsilon > 0$; et finalement justifier que les processus (Y, Z) satisfassent l'EDSR

$$-dY_t = \mathbf{1}_{\{t \leqslant \tau\}} f_t(Y_t, Z_t)dt - Z_t dW_t, Y_\tau = \xi \text{ sur } \{\tau < \infty\}. \tag{1.4}$$

[BC08] cherche une représentation probabilistique pour la solution de certain type des EDPs elliptiques, pour lesquelles la formule Feynman-Kac concerne des EDSRs Markovians avec l'horizon infini. Comme [HT07], leur technique principale est de prouver la différentiabilité de la solution bornée d'EDSR correspondante à horizon infini par rapport à la donnée initiale x d'une certaine équation forward. Généralement, cette preuve se fonde sur une borne a priori qui est utilisée pour des rapprochements appropriés des éauqtions pour le gradient de Y en x. Dans ce but, ils exigent aussi certaine structure dissipative et le paramètre de flux (dans l'équation forward) valué sur $\mathbb{R}$. La différence entre [BC08] et [HT07] est que: [HT07] utilise la stratégie pour des EDPs elliptiques avec un génératuer à croissance sous-linéaire par rapport au gradient tandis que [BC08] exige seulement que la constante de monotonicité soit positive.

1.2.3 Equations différentielles stochastiques rétrogrades avec sauts (EDSRS en abrégé)

On se concentre sur les équations différentielles stochastiques rétrogrades avec sauts (EDSRS en abrégé). Ici se trouve un résumé concis des recherches anciennes: sous l'hypothèse d'une condition terminale ξ exponentielle intégrable et un générateur g à croissance quadratique, El Karoui, Matoussi et Ngoupeyou [KMN18] prouvent l'existence de solution des EDSRSs déterminées par une mesure aléatoire (la première partie de p.23 Théorème 5.6); ensuite, Jeanblanc, Matoussi et Ngoupeyou [JMN16] en vérifient l'unicité pour certain générateur spécial ayant une forme explicite, et leur EDSRS est déterminée par une martingale locale continue (par exemple, le mouvement Brownien) et le processus Poisson (p.7 Proposition 2). On remarque que leur unicité profite du générateur explicit, qui ne convient pas d'autre cas plus général. Sur la base de ces considérations, une possibilité de la recherche suivante est l'étude de l'inégalité de déscentées pour certaine g-surmartingale d'après les EDSRSs spéciaux bien-définis dû à [JMN16]. Il est important d'étudier la régularité en chemin de la g-surmartingale, qui jouera un rôle essetiel dans l'existence et l'unicité des EDSRSs.

1.3 Le problème de contrat optimal de Sannikov sous contrainte de faille du bien de production

Après un premier aperçu du problème de l'Agent et du problème de Sannikov Principal originalement étudiés par Possamaï et Touzi [PT20] et initialement introduits par Sannikov [San08], nous introduisons un nouveau schéma de délégation dans un contrat, c'est-à-dire le problème de Principal dans un processus de sortie par défaut pour lequel il existe une restriction supplémentaire sur le délai de résiliation. Des équations de programmation dynamiques sont données pour les deux cas ainsi que leurs formes équivalentes, et un résultat de comparaison est prouvé sous le réglage d'un processus de sortie par défaut. En outre, nous proposons une approximation numérique de réseau neural de Galerkin pour comprendre la fonction de valeur dans Sannikov Principal problème avec/sans processus par défaut.

On précise ci-dessous les recherches correspondantes.

Holmström et Milgrom [HM87] révélent la forme linéaire d'un contrat optimal dans un problème à horizon fini avec la fonction d'utilité CARA pour tous les deux parties (le Principal et l'Agent), où seulement la dérive du processus de production est influencée par l'effort de l'Agent. Généralement, on le considère comme le premier et avant-garde papier, soulignant que c'est plus facile de traiter le problème de contrat optimal dans une circonstance au temps continu.

Cette perspective est suivie et vérifiée par une multiplicité des articles avec l'hypothèse au temps continu. Précisément, les extensions suivantes du résultat de Holmström et Milgrom sont Schättler et Sung [SS93], Sung [Sun95b, Sun95a], Müller [M9̈8, M0̈0], Hellwig et Schmidt [Hel07, HS02]. En comparant avec les papiers ci-dessus qui partagent la même circonstance où ils ont tous considéré l'extension au temps continu issue d'une approche du premier ordre émergées de la théorie contrat dans le cas statique (par exemple Rogerson [Rog85]), Williams [Wil08, Wil11, Wil15] et Cvitanić, Wan, et Zhang [CWZ06, CWZ08, CWZ09] prennent un autre chemin en utilisant le principle maximum stochastique et des équations différentielles stochastiques forward–backward à décrire la récompense optimale pour des utilités plus généraux. On peut référer aussi à la monographie excellente de Cvitanić et Zhang [CZ12].

Les papiers de Sannikov [San08, San13] présentent des nouveautés avec des perspectives différentes: d'abord c'est son idée originale de considérer la valeur de continuation dynamique de l'Agent comme une variable d'état pour le problème de Principal; et puis plusieurs interprétations économiques révélées par la situation à horizon infini considérée dans sa recherche. Pour le premier cas, il existe une approche élégante, s'appuyant sur le résultat de réprésentation de la fonction valeur dynamique, qui est présentée par l'implémentation systematique au temps continu. En même temps, au temps discret, Spear et Srivastava [SS87] prouvent que cette idée est bien impliquée dans la présentation du problème. Pour le point second, l'argument principale de Sannikov est que le Principal démissionne optimalement l'Agent en l'offert un parachute d'or, ça veut dire qu'un flux de consomption, constant continu tout au long de la vie, lorsque l'utilité de continuation arrive au niveau assez haut, et l'Agent ayant une petite utilité de reservation possède une location informationnel, au sens qu'il est offert un contrat avec de valeur extrêment haute.

Notre discussion ici se limite au cas où le Principal et l'Agent partagent le même coefficient d'actualisation.

Au delà de la nouveauté méthodologique introduite dans l'article de Sannikov, son modèle met en évidence plusieurs résultats économiques remarquables.

Tout d'abord, son modèle justifie l'optimalité d'une rente fixe sous forme d'un golden parachute quand l'agent atteint une valeur de continuation de son utilité assez haute. Autrement dit, si l'agent atteint un coup assez élevé, le principal préfère terminer son contrat et s'en séparer moyennant le paiement d'une compensation au moment d'arrêter le contrat. Une première motivation du chapitre est d'examiner si ce résultat continue à être vrai si le risque de faillite du bien de production est pris en compte en imposant une clause de fin de contrat en cas de faillite du bien de production.

Le deuxième résultat économique remarquable dans le travail de Sannikov concerne l'existence d'un phénomène de rente informationnelle. Il s'agit d'un gain d'utilité que l'agent pourrait réaliser du fait de l'asymétrie d'information inhérente dans le problème de délégation: le principal ne peut observer l'effort de l'agent et n'a accès qu'à la réalisation de la valeur du bien de production. Cette asymétrie d'information est à l'origine du risque de moralité (moral hazard) modélisé dans le problème de délégation étudié par Sannikov.

Dans le problème de contrat optimal de Sannikov, le principal cherche le meilleur contrat en maximisant son utilité, étant donnée la réponse optimale de l'agent au contrat proposé, et sous la contrainte d'acceptabilité de l'agent qui réclame une valeur d'utilité supérieure à un niveau de réserve. En dessous de ce niveau, l'agent refuse de rentrer dans le jeu de délégation. La rente informationnelle est justement une situation où le contrat optimal offert par le principal confère une utilité pour l'agent strictement supérieure à son niveau de réserve. Autrement dit, grâce à son avantage informationnel, l'agent force le principal à lui offrir un contrat optimal qui lui procure une utilité strictement supérieure à son seuil d'acceptabilité.

Dans le modèle de Sannikov, le phénomène de rente informationnelle est mois en évidence pour des valeurs du seuil d'acceptabilité assez petites. Si le seuil d'acceptabilité de l'agent est en dessous d'une valeur donnée (y^*, celle qui réalise le maximum de l'utilité du principal), il est optimal pour le principal d'offrir un contrat qui induit une valeur d'utilité pour l'agent égale à y^*.

Dans le modèle étudié dans le chapitre 3 où la fin du contrat est conditionnée par la non faillite du bien de production, nos résultats numériques révèlent que cette rente informationnelle est largement diminuée. Plus précisément, si la valeur du bien de production est largement éloignées du risque de faillite, nous constatons que la rente informationnelle existe à l'image des résultats de Sannikov. Cependant plus la valeur du bien de production diminue, plus cette rente informationnelle diminue, jusqu'à disparaitre quand le bien de production est très proche du niveau de faillite.

A propos de la partie numérique, on fait ici une introduction concise sur des méthodes classiques des EDPs (semi-linéaire et entièrement non-linéaire). Pour les EDPs semi-linéaires, le défi principal est issu du fléau de la dimension qui rend la discrétisation inatteignable lorque la dimension est plus que 3. Pour résoudre ce problème, il existe des méthodes probabilistes sans maillage basées sur des EDS rétrogrades, une méthode de branchement, méthodes Picard à plusieurs niveaux. Pour le deuxième cas, les EDPs entièrement non-linéaires, il existe normalement une partie de non-linéarité contenant le deuxième terme dérivé. Plusieurs méthodes classiques, par exemple, les méthodes déterministes comme la méthode de différence finie ou la méthode des éléments finis, des méthodes max-plus pour les équations HJB, ne sont développés que pour les dimensions inférieures. Nous renvoyons les lecteurs à Germain, Pham et Warin [GPW20] pour une description détaillée des références connexes.

Les méthodes d'apprentissage automatique se sont développées rapidement au cours des dernières années. Parmi eux, [BEJ19] a proposé un réseau neuronal global profond visant à résoudre les EDPs nonlinéaires de grande dimension avec la minimisation d'une fonction de perte objective basée sur la représentation des EDS backward de second ordre, ils n'ont pas donné de test concret sur un exemple entièrement nonlinéaire. Plus loin, [GPW20] se concentrent sur la fourniture d'une approximation efficace de la Hessienne, c'est-à-dire, le terme Γ de l'EDSR, pour donner 3 nouveaux algorithmes, à savoir, le DBDP explicite de second ordre en plusieurs étapes, le DBDP multi-étapes de second ordre et le DBDP multi-étapes de second ordre Malliavin.

Etant différent de ces essais d'apprentissage automatique sur les EDP semi-linéaires et entièrement non-linéaires, nous allons discuter ici d'une approximation numérique de réseau Galerkin totalement nouvelle à résoudre notre EDO / EDP dans l'intérêt.

Chapter 2

Quadratic 2BSDE with unbounded terminal condition

Bowen SHENG

After a summary of previous research results on the solutions of quadratic reflected backward SDE, i.e., wellposedness (existence and uniqueness), comparison principle, stability result, etc., we in this paper mainly consider the wellposedness of solutions for the corresponding second order backward SDE. Besides the quadratic growth, we also assume that the generator f is concave in z with linearly growing gradient. This leads to a new downcrossing inequality of the value function (also a f-supermartingale) V. Gaining the regularity of its right limit w.r.t. t, we follow Soner, Touzi and Zhang [STZ12] to get the representation of solutions and implement the proof of wellposedness.

2.1 Introduction

Soner, Touzi and Zhang [STZ12] extend the theory of backward stochastic differential equation(BSDE, hereafter) to the second order case, where they give a complete theory of existence and uniqueness for certain type of second order backward SDE(2BSDE, hereafter). After this, many generalisations emerged in different perspectives, e.g., the structure of generator, the integrability satisfied by the terminal condition, regularity requirement for dynamic programming principle, discontinuous setting, etc.. Until now, there still exists lots of interesting problems in this topic. For example, if we take a look at the progress made in the regularity assumption for dynamic programming principle concerned in 2BSDE:

[STZ12, STZ13] and Possamaï and Zhou [PZ13] obtained the dynamic programming principle under very strong continuity assumptions w.r.t. ω on the terminal condition and the generator of the BSDEs, and obtained a semimartingale decomposition of the value function of the corresponding stochastic control problem, which ensured well-posedness of the associated 2BSDE. These regularity assumptions are made to obtain the continuity of the value function a priori, which allows to avoid completely the use of the measurable selection theorem.

Possamaï, Tan and Zhou [PTZ18] made progress concerning the regularity assumptions. Generally speaking, they firstly proved the dynamic programming principle following directly from the measurable selection theorem together with the comparison and stability of the corresponding BSDEs, then they studied the path regularization of the value function for the BSDEs equipped with appropriate integrability and a Lipschitz-continuous generator, and next they applied the above two results to obtain the well-posedness of 2BSDE, which is also equipped with a terminal condition and a Lipschitz-continuous generator.

In this paper, our final goal is to obtain the wellposedness of the solution of 2BSDE equipped with an exponential integrable terminal condition ξ and a quadratic generator f. Note that we do not assume that the filtration is generated by a Brownian motion, we introduced another orthogonal martingale M in the definition of a solution of BSDE.

Another tool which will be used in our main result is the wellposedness of quadratic reflected backward SDE(RBSDE, hereafter). Generally, we consider RBSDE as an extension of BSDEs for which the Y-part of the solution is required to stay above(precisely, greater than or equal to) some given process, which we name it as the barrier. Besides, there exists a nondecreasing process in addition which keeps the Y-part of the solution stay above the barrier. El Karoui et al. [KKP+97] introduce RBSDEs in the situation where a standard Brownian filtration and a continuous barrier are considered. Lepeltier and Xu [LX07] obtained the existence of solutions for quadratic RBSDEs with an unbounded terminal condition, but it's equipped with a bounded continuous barrier. Bayraktar and Yao [BY12] extended to the unbounded continuous barriers, precisely, getting the wellposedness of the corresponding solutions.

Both of them work also on the filtration generated by a standard Brownian motion.

Later on, several extensions based on these results had been made to the case of a discontinuous barrier with/without a larger stochastic basis than the standard Brownian one(for example, Crépey and Matoussi [CM08], Lin, Ren, Touzi and Yang [LRTY20], Essaky, Hassani and Ouknine [EHO15], Essaky, Hassani and Rhazlane [EHR20], Hamadène and Ouknine [HO15]). All these extensions require an assumption of right continuity for the barrier. In particular, we mention that the existence of maximal and minimal solution for (stochastically) quadratic RBSDE studied in [EHO15] and [EHR20] is validated without assuming any $\mathbb{P}$-integrability conditions on the terminal condition, and [EHR20] inherits also differs from [EHO15] by the additional requirement that the left limit Y_- of the Y-part for the solution is bounded by an upper and lower barriers which are both predictable. One essential point which calls for attention is that the requirement of right continuity for the barrier differs these extensions from the result obtained by Grigorova et al. [GIO+17] where they give a further extension of RBSDEs to the case for which in the standard Brownian filtration the barrier is not necessary to be right continuous. The corresponding wellposedness has been established with the help of Mertens decomposition of optional strong (but not necessarily right continuous) supermartingales and some appropriate generalization of Itô's formula due to Galchouk and Lenglart.

The outline of this paper is as follows. In §2.3, we give a summary of previous research results on RBSDEs w.r.t. unbounded barriers. Based on the discussion of random ODE in [LX07], a new a priori estimate given in §2.3 plays an essential role for the uniqueness results (including comparison principle and monotone stability) for the former mentioned RBSDE. While we set the barrier in the definition of RBSDE to be $-\infty$, it results in a BSDE which inherits the wellposedness of solutions, so this explains our previous interest for RBSDE or the supersolution of BSDE in §2.3. Then, having in mind that the solution of the 2BSDE should be, to some extent, a supremum of solutions of standard BSDEs, we formulate our existence proof in §2.4. Precisely, we use twice the dynamic programming principle to study the measurability issues of two value functions, V and the corresponding right continuous limit V^+, and then we show the existence. One point to mark is the necessity of the second dynamic programming on V^+. Because it's difficult to obtain directly a result on regularity in time by the definition of V, we naturally turn to study V^+, during which certain problems arise, like the downcrossing inequality, etc.. For all of these, we add a natural condition on the gradient of the generator with respect to the z-variable and provide a possible perspective to look into and work on. In §2.5, we update the uniqueness result.

2.1.1 Canonical space

Let $d \in \mathbb{N}^*$. We denote by $\Omega := \mathbb{C}([0,T], \mathbb{R}^d)$ the canonical space of all $\mathbb{R}^d$-valued continuous paths ω on $[0,T]$ s.t. $\omega_0 = 0$, equipped with the canonical process X, that is, $X_t(\omega) := \omega_t$, for all $\omega \in \Omega$. Denote by $\mathbb{F} = (\mathcal{F}_t)_{0 \leqslant t \leqslant T}$ the canonical filtration generated by X, and by $\mathbb{F}_+ = (\mathcal{F}_t^+)_{0 \leqslant t \leqslant T}$ the right limit of $\mathbb{F}$ with $\mathcal{F}_t^+ := \cap_{s>t} \mathcal{F}_s$ for all $t \in [0,T)$ and $\mathcal{F}_T^+ := \mathcal{F}_T$. We equip ω with the uniform convergence norm $\|\omega\|_\infty := \sup_{0 \leqslant t \leqslant T} \|\omega_t\|$, so that the Borel σ-field of Ω coincides with $\mathcal{F}_T$.

Let $\mathscr{M}_1$ denote the collection of all probability measures on $(\Omega, \mathcal{F}_T)$. Note that $\mathscr{M}_1$ is a Polish space equipped with the weak convergence topology. We denote by $\mathfrak{B}$ its Borel σ-field. Then for any $\mathbb{P} \in \mathscr{M}_1$, denote by $\mathcal{F}_t^{\mathbb{P}}$ the completed σ-field of $\mathcal{F}_t$ under $\mathbb{P}$. Denote also the completed filtration by $\mathbb{F}^{\mathbb{P}} = (\mathcal{F}_t^{\mathbb{P}})_{0 \leqslant t \leqslant T}$ and $\mathbb{F}_+^{\mathbb{P}}$ the right limit of $\mathbb{F}^{\mathbb{P}}$, so that $\mathbb{F}_+^{\mathbb{P}}$ satisfies the usual conditions. For $\mathcal{P} \subseteq \mathscr{M}_1$, we say that a property holds $\mathcal{P}$ quasi-surely, abbreviated as $\mathcal{P}$-q.s., if it holds $\mathbb{P}$ a.s. for all $\mathbb{P} \in \mathcal{P}$. Moreover, we introduce the universally completed filtration $\mathbb{F}^U := (\mathcal{F}_t^U)_{0 \leqslant t \leqslant T}, \mathbb{F}^{\mathcal{P}} := (\mathcal{F}_t^{\mathcal{P}})_{0 \leqslant t \leqslant T}$, and $\mathbb{F}^{\mathcal{P}+} := (\mathcal{F}_t^{\mathcal{P}+})_{0 \leqslant t \leqslant T}$, defined as follows:

$$\mathcal{F}_t^U := \cap_{\mathbb{P} \in \mathscr{M}_1} \mathcal{F}_t^{\mathbb{P}}, \quad \mathcal{F}_t^{\mathcal{P}} := \cap_{\mathbb{P} \in \mathcal{P}} \mathcal{F}_t^{\mathbb{P}}, \quad t \in [0,T],$$
$$\mathcal{F}_t^{\mathcal{P}+} := \mathcal{F}_{t+}^{\mathcal{P}}, \quad t \in [0,T), \quad \text{and} \quad \mathcal{F}_T^{\mathcal{P}+} := \mathcal{F}_T^{\mathcal{P}}.$$

We also introduce an enlarged canonical space $\overline{\Omega} := \Omega \times \Omega$. By abuse of notation, we denote by (X, W), its canonical process, that is, $X_t(\bar{\omega}) := \omega_t, W_t(\bar{\omega}) := \omega'_t$ for all $\bar{\omega} := (\omega, \omega') \in \overline{\Omega}$, by $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{0 \leqslant t \leqslant T}$ the canonical filtration generated by (X, W), and by $\overline{\mathbb{F}}^X = (\overline{\mathcal{F}}_t^X)_{0 \leqslant t \leqslant T}$ the filtration generated by X. Similarly, we denote the corresponding right-continuous filtration by $\overline{\mathbb{F}}_+^X$ and $\overline{\mathbb{F}}_+$, and the augmented filtration by $\overline{\mathbb{F}}_+^{X, \overline{\mathbb{P}}}$ and $\overline{\mathbb{F}}_+^{\overline{\mathbb{P}}}$, given a probability measure $\overline{\mathbb{P}}$ on $\overline{\Omega}$.

Define $\mathcal{P}_{\mathsf{loc}}$ the subset of $\mathscr{M}_1$ s.t., for each $\mathbb{P} \in \mathcal{P}_{\mathsf{loc}}, X$ is $\mathbb{P}$-local martingale whose quadratic variation $\langle X\rangle$ is absolutely continuous in t w.r.t. the Lebesgue measure. Note that the $d\times d$-matrix-valued processes $\langle X\rangle$ can be defined pathwisely, and we may introduce the corresponding $\mathbb{F}$-progressively measurable density processes

$$\widehat{a}_t := \limsup_{n\to\infty} \frac{\langle X\rangle_t - \langle X\rangle_{t-\frac{1}{n}}}{\frac{1}{n}},$$

so that $\langle X\rangle_t = \int_0^t \widehat{a}_s ds, t\geqslant 0, \mathbb{P}$-a.s., for all $\mathbb{P}\in\mathcal{P}_{\mathsf{loc}}$. For later use, we observe that, as $\widehat{a}_t\in\mathbb{S}_d^{\geqslant 0}$, the set of $d\times d$ nonnegative-definite symmetric matrices, we may define a measurable generalized inverse $\widehat{a}_t^{-1}$, and a symmetric measurable square root $\widehat{\sigma}_t := \widehat{a}_t^{\frac{1}{2}}$.

Throughout this paper, we shall work with the following subset of $\mathcal{P}_{\mathsf{loc}}$:

$$\mathcal{P}_b := \{\mathbb{P}\in\mathcal{P}_{\mathsf{loc}} : \widehat{\sigma} \text{ and } \widehat{\sigma}^{-1} \text{ are bounded }, dt\otimes\mathbb{P}(d\omega)\text{-a.e.}\}.$$

2.1.2 Spaces and norms

From now on, we will simply keep using symbols like $\mathcal{F}$ and $\mathbb{F}$ etc. to represent the enlarged filtrations. For $p\geqslant 1$,

$(\mathbf{i})$ *One-measure integrability classes:* given a finite time horizon $T>0$, let $\mathcal{T}_{0,T}$ denote the set of all $\mathbb{F}$-stopping times ν s.t. $0\leqslant\nu\leqslant T, \mathbb{P}$-a.s. For any probability measure $\mathbb{P}\in\mathscr{M}_1$, let τ be an $\mathbb{F}_+^{\mathbb{P}}$-stopping time, similarly we can define $\mathcal{T}_{0,\tau}$, etc. Besides, we define,

- Let $\mathbb{L}^0(\mathcal{F}_t^{+,\mathbb{P}},\mathbb{R})$ be the set of $\mathbb{R}$-valued and $\mathcal{F}_t^{+,\mathbb{P}}$-measurable r.v., then,

  $$\mathbb{L}^p(\mathbb{P},\mathbb{R}) := \{\xi\in\mathbb{L}^0(\mathcal{F}_T^{+,\mathbb{P}},\mathbb{R}) : \|\xi\|^p_{\mathbb{L}^p(\mathbb{P},\mathbb{R})} := \mathbb{E}^{\mathbb{P}}[|\xi|^p] < \infty\}.$$

  Besides, $\mathbb{L}^\infty(\mathbb{P},\mathbb{R}) := \{\xi\in\mathbb{L}^0(\mathcal{F}_T^{+,\mathbb{P}},\mathbb{R}), \|\xi\|_{\mathbb{L}^\infty(\mathbb{P},\mathbb{R})} := \operatorname{ess\,sup}^{\mathbb{P}}_{\omega\in\Omega}|\xi(\omega)| < \infty\}$.
  $\mathbb{L}^{\exp}(\mathbb{P},\mathbb{R}) := \{\xi\in\mathbb{L}^0(\mathcal{F}_T^{+,\mathbb{P}},\mathbb{R}), \mathbb{E}[e^{p|\xi|}] < \infty, \forall p\in(1,\infty)\}$.

- Let $\mathbb{B}$ be a generic Banach space with norm $|\cdot|_{\mathbb{B}}$. For any $p,q\in[1,\infty)$, let $\mathbb{H}^{p,q}(\mathbb{P},\mathbb{B})$ be the space of all $\mathbb{B}$-valued $\mathbb{F}_+^{\mathbb{P}}$ -progressive measurable processes X with $\|X\|_{\mathbb{H}^{p,q}(\mathbb{P},\mathbb{B})} := \Big\{\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T |X_t|^p_{\mathbb{B}}dt\Big)^{\frac{q}{p}}\Big]\Big\}^{\frac{1}{q}} <$

∞. We simply write $\mathbb{H}^p$ for $\mathbb{H}^{p,p}$ if $p = q$ and add superscript loc if the integrability only satisfies $\int_0^T |X_t|_{\mathbb{B}}^p dt < \infty, \mathbb{P}$-a.s.. Besides, $\mathbb{M}^p(\mathbb{P})$ denotes the space of $\mathbb{R}$-valued, $\mathbb{F}_+^{\mathbb{P}}$-adapted martingales M, with $\mathbb{P}$-a.s. càdlàg path, s.t. M is orthogonal to W and $\|M\|_{\mathbb{M}^p(\mathbb{P})}^p := \mathbb{E}^{\mathbb{P}}[\langle M\rangle_T^p] < \infty$.

- Let $\mathbb{C}^0_{\mathbb{F}_+^{\mathbb{P}}}$ be the set of all $\mathbb{R}$-valued, $\mathbb{F}_+^{\mathbb{P}}$-adapted continuous processes on $[0, T]$, then,

$$\mathbb{C}^p(\mathbb{P}) := \{X \in \mathbb{C}^0_{\mathbb{F}_+^{\mathbb{P}}} : \|X\|_{\mathbb{C}^p}^p := \mathbb{E}^{\mathbb{P}}[\sup_{0\leqslant t\leqslant T} |X_t|^p] < \infty\}.$$

$$\mathbb{I}(\mathbb{P}) := \{X \in \mathbb{C}^0_{\mathbb{F}_+^{\mathbb{P}}} : X. \text{ is an increasing process with } X_0 = 0\}$$

$$\mathbb{I}^p(\mathbb{P}) := \{X \in \mathbb{I}(\mathbb{P}) : X_T \in \mathbb{L}^p(\mathbb{P}, \mathbb{R})\} \text{ for all } p \in [1, \infty).$$

$$\mathbb{C}^{\exp(\lambda,\lambda')}(\mathbb{P}) := \{X \in \mathbb{C}^0_{\mathbb{F}_+^{\mathbb{P}}} : \mathbb{E}^{\mathbb{P}}[e^{(\lambda X_*^- + \lambda' X_*^+)}] < \infty\} \subseteq \cap_{p\in[1,\infty)}\mathbb{C}^p(\mathbb{P}) \text{ for all } \lambda, \lambda' \in (0, \infty),$$

where $X_*^{\pm} := \sup_{t\in[0,T]}(X_t)^{\pm}$, and $\mathbb{C}^{\exp(\mathrm{p})}(\mathbb{P}) := \{X \in \mathbb{C}^0_{\mathbb{F}_+^{\mathbb{P}}} : \mathbb{E}^{\mathbb{P}}[e^{pX_*}] < \infty\}$ for all $p \in (0, \infty)$, where $X_* := \sup_{t\in[0,T]} X_t$.

Moreover, for any $p \in [1, \infty)$, we set $\mathbb{S}^p(\mathbb{P}) := \mathbb{C}^{\exp(\mathrm{p})}(\mathbb{P}) \times \mathbb{H}^{2,2p}(\mathbb{P}, \mathbb{R}^d) \times \mathbb{M}^p(\mathbb{P}) \times \mathbb{I}^p(\mathbb{P})$. By using $\mathbb{D}$ to replace $\mathbb{C}$ in the above definition, we mean càdlàg instead of continuous process. And correspondingly we denote respectively $\mathbb{I}_{\mathsf{rcll}}(\mathbb{P}), \mathbb{I}^p_{\mathsf{rcll}}(\mathbb{P})$ for the càdlàg version of $\mathbb{I}(\mathbb{P}), \mathbb{I}^p(\mathbb{P})$.

$(\mathbf{ii})$. *Integrability classes under non-dominated nonlinear expectation:* take $\mathcal{P} \subseteq \mathcal{P}_b$ as a subset of probability measures, and define

$$\mathcal{E}^{\mathcal{P}}[\cdot] := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[\cdot].$$

Let $\mathbb{G} := \{\mathcal{G}_t\}_{t\geqslant 0}$ be a filtration with $\mathcal{F}_t \subseteq \mathcal{G}_t$ for any $t \geqslant 0$. We define the subspace $\mathcal{L}^p(\mathcal{P}, \mathbb{R}, \mathbb{G})$ as the set of all $\mathcal{G}_T$-measurable $\mathbb{R}$-valued r.v. ξ, s.t.

$$\|\xi\|_{\mathcal{L}^p(\mathcal{P},\mathbb{R},\mathbb{G})}^p := \mathcal{E}^{\mathcal{P}}[|\xi|^p] < \infty.$$

Similarly we can give the definition of the subspaces $\mathcal{D}^p(\mathcal{P}, \mathbb{G}), \mathcal{D}^{\exp(\mathrm{p})}(\mathcal{P}, \mathbb{G}), \mathcal{H}^p(\mathcal{P}, \mathbb{R}^d, \mathbb{G})$ by replacing $\mathbb{F}_+^{\mathbb{P}}$ by $\mathbb{G}$. Moreover, for any $p \in [1, \infty)$, we set $\mathcal{S}^p(\mathcal{P}, \mathbb{G}) := \mathcal{D}^{\exp(\mathrm{p})}(\mathcal{P}, \mathbb{G}) \times \mathcal{H}^{2,2p}(\mathcal{P}, \mathbb{R}^d, \mathbb{G}) \times \Big(\mathbb{M}^p(\mathbb{P})\Big)_{\mathbb{P}\in\mathcal{P}} \times \Big(\mathbb{I}^p_{\mathsf{rcll}}(\mathbb{P})\Big)_{\mathbb{P}\in\mathcal{P}}$, where $\mathcal{D}^{\exp(\mathrm{p})}(\mathcal{P}, \mathbb{G})$ denotes the set of all $\mathbb{R}$-valued, $\mathbb{G}$-adapted càdlàg processes X on $[0, T]$, satisfying $\|X\|_{\mathcal{D}^{\exp(\mathrm{p})}(\mathcal{P},\mathbb{G})} := \mathcal{E}^{\mathcal{P}}[e^{pX_*}] = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[e^{pX_*}] < \infty$.

2.1.3 Definition of equations

Reflected Backward SDEs(RBSDEs): The notion of reflected backward SDE has been introduced by El Karoui et al. [KKP+97]. Here, a solution of such an equation(f,ξ,L), i.e., associated with a generator f, a terminal condition ξ and a continuous(or càdlàg) barrier L, is a quadruple of processes $(Y,Z,M,K)\in\mathbb{C}^0_{\mathbb{F}^{\mathbb{P}}_+}$(or $\mathbb{D}^0_{\mathbb{F}^{\mathbb{P}}_+}$) $\times\mathbb{H}^{2,\mathrm{loc}}(\mathbb{P},\mathbb{R}^d)\times\mathbb{M}(\mathbb{P})\times\mathbb{I}(\mathbb{P})$(or $\mathbb{I}_{\mathrm{rcll}}(\mathbb{P})$), satisfying

$$\begin{aligned}
&Y_t=\xi+\int_t^T f_s(Y_s,Z_s,\widehat{\sigma}_s)ds-\int_t^T Z_s\cdot dX_s-\int_t^T dM_s+K_T-K_t, 0\leqslant t\leqslant T,\\
&Y_t\geqslant L_t, 0\leqslant t\leqslant T, \mathbb{P}\text{-a.s.},\\
&\int_0^T (Y_s-L_s)dK_s=0, \mathbb{P}\text{-a.s. (Skorokhod condition)},
\end{aligned}\tag{2.1}$$

where $f.(\cdot,\cdot,\cdot)$ is a progressively measurable map satisfying some conditions to be specified later. Note that here the underlying filtration is not only generated by a Brownian motion, so we introduce another component in the definition of a supersolution a backward SDE, namely a martingale M which is orthogonal to standard Brownian motion W. K is a continuous(or càdlàg) nondecreasing process which pushes upwards the process Y in order to keep it above the barrier L. The last equation means that the process K acts only when the process Y reaches the barrier L. Note that we allow L to take the value $-\infty$, so the notion of reflected backward SDE covers backward SDE of Pardoux and Peng [PP90]. By the pair (f,ξ), we mean the corresponding backward SDE, without the last two conditions concerning the barrier L in the above (2.1).

Second order Backward SDEs(2BSDEs): Following [STZ12], we introduce 2BSDE as a family of backward SDEs defined on the supports of a convenient family of singular probability measures. For this reason, we introduce the subset of $\mathcal{P}_b$:

$$\mathcal{P}_0:=\{\mathbb{P}\in\mathcal{P}_b\colon\ \widehat{f}^0_t(\omega)<\infty,\ \text{for Leb}\otimes\mathbb{P}\text{-a.e. } (t,\omega)\in\mathbb{R}_+\times\Omega\},\tag{2.2}$$

where $\widehat{f}^0_t(\omega) = f_t(\omega, 0, 0, \widehat{\sigma}_t(\omega))$. We also define for all stopping times τ_0:

$$\mathcal{P}_{\mathbb{P}}(\tau_0) := \{\mathbb{P}' \in \mathcal{P}_0 \colon \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_{\tau_0}\}, \text{ and } \mathcal{P}^+_{\mathbb{P}}(\tau_0) := \cup_{h>0}\mathcal{P}_{\mathbb{P}}(\tau_0 + h). \tag{2.3}$$

The 2BSDE is defined by

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s, \widehat{\sigma}_s)ds - Z_s \cdot dX_s - dM_s + dK_s, \quad \mathcal{P}_0\text{-q.s.} \tag{2.4}$$

for some nondecreasing process K together with a convenient minimality condition.

Definition 2.1.1. *Let $p, \lambda, \lambda' \in \mathbb{R}_+$. A process triple $(Y, Z, (M^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}) \in$ $\mathcal{D}^{\exp(\lambda,\lambda')}(\mathcal{P}_0, \mathbb{F}^{\mathcal{P}_0+})$(or, $\mathcal{D}^{\exp(\mathrm{p})}(\mathcal{P}_0, \mathbb{F}^{\mathcal{P}_0+})) \times \mathcal{H}^{2,2p}(\mathcal{P}_0, \mathbb{R}^d, \mathbb{F}^{\mathcal{P}_0+}) \times \left(\mathbb{M}^p(\mathbb{P})\right)_{\mathbb{P}\in\mathcal{P}_0}$ is said to be a solution of the 2BSDE(2.4), if*

- $Y_T = \xi, \quad \mathcal{P}_0$*-q.s..*
- *for all $\mathbb{P} \in \mathcal{P}_0$, the process*

$$K^{\mathbb{P}}_t := Y_0 - Y_t + \int_0^t -f_s(Y_s, Z_s, \widehat{\sigma}_s)ds + Z_s \cdot dX_s + dM^{\mathbb{P}}_s, \quad t \geqslant 0, \mathbb{P}\text{-a.s.},$$

  is a non-decreasing process starting from $K^{\mathbb{P}}_0 = 0$.
- *the family $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_0\}$ satisfies the minimality condition: for $0 \leqslant s \leqslant t$, there exists a sequence of stopping time $\{\tau^{\mathbb{P}'}_n\}_{n,\mathbb{P}'}$ with $n \in \mathbb{N}$ and $\mathbb{P}' \in \mathcal{P}^+_{\mathbb{P}}(s)$ (which possibly tends to $+\infty$) s.t.*

$$K^{\mathbb{P}}_s = \underset{\mathbb{P}' \in \mathcal{P}^+_{\mathbb{P}}(s)}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}'}[K^{\mathbb{P}'}_{t\wedge\tau^{\mathbb{P}'}_n} | \mathcal{F}^{+,\mathbb{P}'}_s], \mathbb{P}\text{-a.s.}. \tag{2.5}$$

Remark 2.1.2. *Some words about the relaxation from (2.4) to the very above definition, i.e., here for example we allow for a dependence of $M^{\mathbb{P}}$ on the underlying probability measure $\mathbb{P}$. This dependence is due to the fact that the stochastic integral $Z \cdot X := \int Z_s \cdot dX_s$ is defined $\mathbb{P}$-a.s. under all $\mathbb{P} \in \mathcal{P}_0$ and should rather be denoted by $(Z \cdot X)^{\mathbb{P}}$ in order to emphasize the $\mathbb{P}$-dependence. By Nutz [Nut12](p.3 Theorem 2.2, which works under the Zermelo–Fraenkel set theory with axiom of choice (ZFC) together with the Continuum Hypothesis), the family $\{(Z \cdot X)^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_0\}$ in our framework can be aggregated as a medial limit $Z \cdot X$. In this case, $Z \cdot X$ can be chosen as an $\mathbb{F}^{\mathcal{P}_0+}$ -adapted process, and the family*

$\{M^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_0\}$ *can be aggregated into the resulting medial limit* M*, i.e.,* $M = M^{\mathbb{P}}, \mathbb{P}$*-a.s. for all* $\mathbb{P} \in \mathcal{P}_0$.

2.1.4 Assumptions

Let us consider a 2BSDE (2.4) with terminal condition ξ and generator $f : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d^{\geqslant 0} \to \mathbb{R}$.

Assumption 2.1.3.

1. *The random variable* ξ *is* $\mathcal{F}_T$*-measurable and for every fixed* (y, z, σ)*, the map* $(t, \omega) \mapsto f_t(\omega, y, z, \sigma)$ *is* $\mathbb{F}$*-progressively measurable.*

2. *Lipshitz in* y*: there is a constant* $\mathfrak{L} \geqslant 0,$ *s.t. for each* $(t, z, \sigma, \omega) \in [0,T] \times \mathbb{R}^d \times \mathbb{S}_d^{\geqslant 0} \times \Omega,$ *and any* $y, y' \in \mathbb{R},$
$$|f_t(\omega, y, z, \sigma) - f_t(\omega, y', z, \sigma)| \leqslant \mathfrak{L}|y - y'|.$$

3. *Growth condition: for every* $(t, \omega, y, z, \sigma) \in [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d^{\geqslant 0}$*, there exists* $\alpha \geqslant 0, \beta \geqslant 0$ *and* $\gamma > 0$ *s.t.*
$$|f_t(\omega, y, z, \sigma)| \leqslant \alpha + \beta|y| + \frac{\gamma}{2}|\sigma z|^2.$$

4. *Concave in* z *: for all* $(\zeta, y) \in (0,1) \times \mathbb{R},$ *all* $z_1, z_2 \in \mathbb{R}^d,$ $\sigma \in \mathbb{S}_d^{\geqslant 0}$ *and* $\mathbb{P}$ *is a fixed probability measure,*
$$f_t(\omega, y, \zeta z_1 + (1-\zeta)z_2, \sigma) \geqslant \zeta f_t(\omega, y, z_1, \sigma) + (1-\zeta) f_t(\omega, y, z_2, \sigma), dt \otimes d\mathbb{P}\text{-a.e.}$$

5. *There exists some constant* $\gamma > 0$ *s.t. for* $(t, \omega) \in [0,T] \times \Omega,$
$$\left|\frac{\partial f_t(\omega, y, z, \sigma)}{\partial z}\right| \leqslant \frac{\gamma}{2}(1 + |\sigma z|).$$

2.2 Wellposedness of solutions for quadratic 2BSDE

We state here the main result of this paper.

Theorem 2.2.1. *Consider 2BSDE*(f,ξ)(2.4).

(i) (Existence) Under Assumption 2.1.3, if for any $\mathbb{P} \in \mathcal{P}_0$, $\mathbb{E}^{\mathbb{P}}[e^{\lambda\gamma\xi^-} + e^{\lambda'\gamma e^{\beta T}\xi^+}] < \infty$ *for some* $\lambda, \lambda' > 0$ *with* $\frac{1}{\lambda} + \frac{1}{\lambda'} < \frac{1}{6\vee 2\gamma}$, *then the 2BSDE (2.4) has a solution* $(Y, Z, (M^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})$ *which belongs to the space* $\cap_{p\in(1,\frac{\lambda\lambda'}{\lambda+\lambda'})}\mathcal{D}^{\exp(\lambda\gamma,\lambda'\gamma)}(\mathcal{P}_0, \mathbb{F}^{\mathcal{P}_0+}) \times \mathcal{H}^{2,2p}(\mathcal{P}_0, \mathbb{R}^d, \mathbb{F}^{\mathcal{P}_0+}) \times \Big(\mathbb{M}^p(\mathbb{P})\Big)_{\mathbb{P}\in\mathcal{P}_0}$.

(ii) (Uniqueness) Under Assumption 2.1.3, assume that $(Y, Z, (M^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})$, *belonging to the space* $\mathcal{D}^{\exp(\lambda,\lambda')}(\mathcal{P}_0, \mathbb{F}^{\mathcal{P}_0+}) \times \mathcal{H}^{2,loc}(\mathcal{P}_0, \mathbb{R}^d, \mathbb{F}^{\mathcal{P}_0+}) \times \Big(\mathbb{M}^p(\mathbb{P})\Big)_{\mathbb{P}\in\mathcal{P}_0}$, *is a solution to the 2BSDE(2.4), then for* $0 \leqslant t \leqslant T$, *the 2BSDE(2.4) has at most one solution in* $\mathcal{D}^{\exp(\lambda,\lambda')}(\mathcal{P}_0, \mathbb{F}^{\mathcal{P}_0+}) \times \mathcal{H}^{2,loc}(\mathcal{P}_0, \mathbb{R}^d, \mathbb{F}^{\mathcal{P}_0+}) \times \Big(\mathbb{M}^p(\mathbb{P})\Big)_{\mathbb{P}\in\mathcal{P}_0}$ *with* $\lambda \in (\gamma, \infty)$ *and* $\lambda' \in (0, \infty)$.

Remark 2.2.2. *Under our main result of 2BSDE, we here try to explain ahead the relation between solutions of 2BSDE and corresponding BSDE. In fact, the solution of 2BSDE(a family of measures) relies on BSDE(single measure) or RBSDE which will be shown in our final existence and uniqueness proof. One natural question is the necessity of reducing 2BSDE to BSDE and how we perform this process. To answer this issue, we can think reversely, i.e., how estimates in [S.22a], w.r.t. quadratic BSDE in a more general filtration, can go from BSDE to 2BSDE? To handle this problem, we will take 2 steps: one is sample paths catenation for which the new path after catenation is still continuous, so this guarantees the feasibility of using [S.22a]; the other or next step is taking supremum over a family of measures, which doesn't bother the extension of estimates in [S.22a] to apply on 2BSDE because now it's only the supremum matters in all estimates concerned. Although the new value function which we will construct in the end is càdlàg no longer continuous, this fact plays NO role in our estimates and only matters in the final existence and uniqueness where then we will turn to [EHO15], no longer [S.22a].*

1. *We give here two a-priori estimates for the solution above, based on the estimates of solution for RBSDEs in [BY12], i.e.,*

$$Y_t \leqslant c_0 + \frac{1}{\gamma}\ln \mathbb{E}^{\mathbb{P}}_t[e^{\gamma e^{\beta T}\xi^+}], \forall t \in [0,T], \forall \mathbb{P} \in \mathcal{P}_0, \text{ where } \mathbb{E}^{\mathbb{P}}_t[\cdot] := \mathbb{E}^{\mathbb{P}}[\cdot|\mathcal{F}^{+,\mathbb{P}}_t];$$

Besides, if $\xi^+ \vee \xi^- \in \mathbb{L}^{\exp}(\mathcal{F}^{+,\mathbb{P}}_T, \mathbb{P}, \mathbb{R})$, *then* (Y, Z, M, K) *lies in* $\mathbb{S}^p(\mathbb{P})$ *for all* $p \in [1 \vee 2\gamma, \infty)$. *To* *be precise, for any* $p \in (1 \vee 2\gamma, \infty)$,

$$\mathbb{E}^{\mathbb{P}}[e^{p\gamma Y_*}] \leqslant \mathbb{E}^{\mathbb{P}}[e^{p\gamma \xi^-}] + c_p \mathbb{E}^{\mathbb{P}}[e^{p\gamma e^{\beta T}\xi^+}] < \infty,$$
$$\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T |Z_s|^2 \hat{\sigma}_s^2 ds + d\langle M\rangle_s\right)^p + K_T^p\right] \leqslant c_p \mathbb{E}^{\mathbb{P}}[e^{3p\gamma Y_*}] < \infty,$$

where c_0 *and* c_p *are all positive constants.*

2. *The condition* $\frac{1}{\lambda} + \frac{1}{\lambda'} < \frac{1}{6\vee 2\gamma}$ *for* $\lambda, \lambda' > 0$ *guarantees the existence of* $a_0 > 1 \vee 2\gamma$ *s.t.* $a_0 = 1 + a_2 + a_3, (\frac{a_2}{a_2-1})^2 = \frac{a_2+a_3-1}{a_2-1}$ *for some* $a_2, a_3 \in \mathbb{R}_+$.

In order to prove the above theorem, we need firstly to recall some previous research results on RBSDEs or BSDEs. Then, we set the barrier in the definition of RBSDE to be $-\infty$, it results in a BSDE which inherits the wellposedness of solutions. Next, having in mind that the solution of 2BSDEs should be, to some extent, a supremum of solution of standard BSDEs, we give the existence and uniqueness proof.

2.3 A review of quadratic BSDE

We list here several results on quadratic BSDE with unbounded terminal condition and generator with quadratic growth in z, which will be needed for the proof of our main result.

2.3.1 Existence

Theorem 2.3.1 ([S.22a] Proposition 3.9)**.** *Let* (f,ξ) *be a pair s.t.* f *satisfies the quadratic growth (i.e., (3) in Assumption 2.1.3) and that*

$$\text{for any } \sigma \in \mathbb{S}_d^{\geqslant 0}, dt \otimes d\mathbb{P}\text{-a.e.}(t,\omega) \in [0,T]\times\Omega, \text{ the mapping } f(t,\omega,\cdot,\cdot,\sigma) \text{ is continuous.} \tag{2.6}$$

If $\mathbb{E}^{\mathbb{P}}[e^{\lambda\gamma L_*^-} + e^{\lambda'\gamma e^{\beta T}(\xi^+\vee L_*^+)}] < \infty$ *for some* $\lambda, \lambda' > 6$ *with* $\frac{1}{\lambda} + \frac{1}{\lambda'} < \frac{1}{6}$, *then BSDE*$(f,\xi)$ *admits a solution* $(Y,Z,M,K) \in \cap_{p\in\left(1,\frac{\lambda\lambda'}{\lambda+\lambda'}\right)} \mathbb{C}^{\exp(\lambda\gamma,\lambda'\gamma)}(\mathbb{P}) \times \mathbb{H}^{2,2p}(\mathbb{P};\mathbb{R}^d) \times \mathbb{M}^p(\mathbb{P}) \times \mathbb{I}^p(\mathbb{P})$ *that satisfies* $Y_t \leqslant c_0 + \frac{1}{\gamma}\ln \mathbb{E}^{\mathbb{P}}[e^{\gamma e^{\beta T}\xi^+}|\mathcal{F}_t^{+,\mathbb{P}}], \forall t \in [0,T]$; *Besides, if* $\xi^+ \in \mathbb{L}^{\exp}(\mathbb{P},\mathbb{R})$, *then this solution* (Y,Z,M,K) *belongs to* $\mathbb{S}^p(\mathbb{P}) = \mathbb{C}^{\exp(p)}(\mathbb{P}) \times \mathbb{H}^{2,2p}(\mathbb{P},\mathbb{R}^d) \times \mathbb{M}^p(\mathbb{P}) \times \mathbb{I}^p(\mathbb{P})$ *for all* $p \in [1,\infty)$. *To be precise, for any*

$p \in (1, \infty)$, *we have the following a priori estimates:*

$$\mathbb{E}^{\mathbb{P}}[e^{p\gamma Y_*}] \leqslant c_p + c_p \mathbb{E}^{\mathbb{P}}[e^{p\gamma e^{\beta T}\xi^+}] < \infty;$$
$$\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \widehat{\sigma}_s^2 |Z_s|^2 ds + d\langle M\rangle_s\right)^p + K_T^p\right] \leqslant c_p \mathbb{E}^{\mathbb{P}}[e^{3p\gamma Y_*}] < \infty.$$

2.3.2 Comparison and uniqueness

Theorem 2.3.2 ([S.22a] Theorem 3.10)**.** *Let* $(f, \xi), (f', \xi')$ *be two pairs and let* (Y, Z, M, K) *(resp.* (Y', Z', M', K')*) be a solution of BSDE* (f, ξ) *(resp. BSDE* (f', ξ')*) s.t.*

(C1) $\xi \leqslant \xi', \mathbb{P}$*-a.s.;*

(C2) $\mathbb{E}^{\mathbb{P}}[e^{\lambda Y_*^+} + e^{\lambda Y_*'^-}] < \infty$ *for all* $\lambda \in (1, \infty)$ *and* $K \in \mathbb{I}^p(\mathbb{P})$ *for some* $p \in (1, \infty)$*;*

(C3) For $\alpha, \beta, \kappa \geqslant 0, \gamma > 0$, f *(resp.,* f'*) satisfies (2), (3) in Assumption 2.1.3,* f *(resp.,* f'*) is concave in* z*, and* $\Delta f(t) := f(t, Y_t', Z_t') - f'(t, Y_t', Z_t') \leqslant 0, dt \otimes d\mathbb{P}$*-a.e. (resp.,* $\Delta f(t) := f(t, Y_t, Z_t) - f'(t, Y_t, Z_t) \leqslant 0, dt \otimes d\mathbb{P}$*-a.e.);*

then $Y_t \leqslant Y_t'$ *for any* $t \in [0, T], \mathbb{P}$*-a.s..*

Based on the above comparison principle, we could follow a routine procedure(e.g., Briand and Hu [BH08] Corollary 6) to naturally verify the uniqueness of corresponding BSDE. Besides, a new uniqueness result on corresponding RBSDE could also be shown by using Legendre-Fenchel transform, for which we only need a given exponential moment on the Y-part of the solution and some special inequality, without using the above comparison result. And this is more general than the former one induced by comparison.

2.3.3 Stability

We finally report the following result which is needed in the mesurability discussion of component concerned in the solution of 2BSDE.

Theorem 2.3.3 ([S.22a] Theorem 3.13)**.** *Let* $\{(f^m, \xi_m)\}_{m\in\mathbb{N}\cup\{0\}}$ *be a sequence of pairs s.t.*

(S1) For the same constants $\alpha, \beta, \kappa \geqslant 0$ *and* $\gamma > 0$, $\{f^n\}_{n\in\mathbb{N}\cup\{0\}}$ *satisfies the Lipschitz condition in* y*(i.e., (2) in Assumption 2.1.3); besides,* $\{f^n\}_{n\in\mathbb{N}}$ *also satisfies quadratic and concave in* z*(i.e., (3), (4) of Assumption 2.1.3);*

(S2) $\mathbb{P}$*-a.s.* ξ_n *converges to* ξ_0*;*

(S3) $\Xi(p) := \sup_{m\in\mathbb{N}\cup\{0\}} \mathbb{E}^{\mathbb{P}}[e^{p\xi_m^+}] < \infty$ *for all* $p \in (1,\infty)$.

For all $n \geqslant 0$, *let* (Y^n, Z^n, M^n, K^n) *be the unique solution of the quadratic BSDE* $(f^n, \xi_n) \in \cap_{p\in[1,\infty)}\mathbb{S}^p(\mathbb{P})$. *If* $f^n(t, Y_t^0, Z_t^0)$ *converges* $dt \otimes d\mathbb{P}$*-a.e. to* $f^0(t, Y_t^0, Z_t^0)$, *then for any* $p \in [1,\infty)$,

$$\lim_{n\to\infty} \mathbb{E}^{\mathbb{P}}[\sup_{t\in[0,T]} |Y_t^n - Y_t^0|^p] = \lim_{n\to\infty} \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T |Z_s^n - Z_s|^2 \widehat{\sigma}_s^2 ds + d\langle M_\cdot^n - M_\cdot\rangle_s\right)^p\right] = 0.$$

Moreover, if it holds $dt \otimes d\mathbb{P}$*-a.e. that* $f^n(t,\omega,y,z)$ *converges to* $f^0(t,\omega,y,z)$ *locally uniformly in* (y,z), *then up to a subsequence, we have further* $\lim_{n\to\infty} \mathbb{E}^{\mathbb{P}}[\sup_{t\in[0,T]} |K_t^n - K_t^0|^p] = 0, \forall p \in [1,\infty)$.

2.4 Existence of solutions for the 2BSDE

2.4.1 Shifted space

We define here the concatenation of two paths ω, ω' at time t:

$$(\omega \otimes_t \omega')_s := \omega_s \mathbf{1}_{[0,t)}(s) + (\omega_t + \omega'_{s-t})\mathbf{1}_{[t,\infty)}(s), s \geqslant 0,$$

and the (t,ω)-shifted r.v.:

$$\xi^{t,\omega}(\omega') := \xi(\omega \otimes_t \omega'), \text{ for all } \omega' \in \Omega.$$

If ξ is $\mathcal{F}_{t+s}$-measurable, then we have that $\xi^{t,\omega}$ is $\mathcal{F}_s$-measurable by the monotone class argument. Besides, $\overrightarrow{\tau}^{t,\omega} := \tau^{t,\omega} - t$ remains to be an $\mathbb{F}$-stopping time for any $\mathbb{F}$-stopping time $\tau, t \leqslant \tau$, and the shifted process

$$Y_s^{t,\omega}(\omega') := Y_{t+s}(\omega \otimes_t \omega'), s \geqslant 0,$$

remains to be $\mathbb{F}$-progressively measurable for any $\mathbb{F}$-progressively measurable process Y. All notations in above can be extended to (τ,ω)-shifted for any finite $\mathbb{F}$-stopping time τ. Next, we define $\widehat{f}_s^{0,t,\omega}(\omega') := f_{t+s}(\omega \otimes_t \omega', 0, 0, \widehat{\sigma}_s(\omega'))$, and

$$\mathcal{P}(t,\omega) := \{\mathbb{P} \in \mathcal{P}_b : \widehat{f}_s^{0,t,\omega}(\omega') < \infty, \text{ for Leb} \otimes \mathbb{P}\text{-a.e. } (s,\omega') \in \mathbb{R}_+ \times \Omega\},$$

and $\mathcal{P}_0 = \mathcal{P}(0, \mathbf{0})$.

We recall from [LRTY20](Lemma 6.1) that the mapping $(\omega, t, \omega') \in \Omega \times \mathbb{R}_+ \times \Omega \mapsto \omega \otimes_t \omega' \in \Omega$ is continuous. And if ξ is $\mathcal{F}_T$-measurable, then $\xi^{\cdot,\cdot}(\cdot)$ is $\mathcal{F}_T \otimes \mathcal{B}([0,\infty)) \otimes \mathcal{F}_\infty$-measurable.

According to Stroock and Varadhan [SV97](Theorem 1.3.4), there exists a family of regular conditional probability distribution (for notation ease, r.c.p.d.) $(\mathbb{P}^\tau_\omega)_{\omega\in\Omega}$ for any probability measure $\mathbb{P}$ on Ω and $\mathbb{F}$-stopping time τ. On $(\Omega, \mathcal{F})$, a probability measure $\mathbb{P}^{\tau,\omega}$ is induced by the r.c.p.d. $\mathbb{P}^\tau_\omega$, precisely,

$$\mathbb{P}^{\tau,\omega}(A) := \mathbb{P}^\tau_\omega(\omega \otimes_\tau A) := \mathbb{P}^\tau_\omega(\{\omega \otimes_\tau \omega' : \omega' \in A\}), A \in \mathcal{F}. \tag{2.7}$$

Immediately, for every $\mathcal{F}$-measurable r.v. ξ, we have $\mathbb{E}^{\mathbb{P}^\tau_\omega}[\xi] = \mathbb{E}^{\mathbb{P}^{\tau,\omega}}[\xi^{\tau,\omega}]$.

2.4.2 Measurability of BSDEs on the shifted spaces

Given $\mathbb{P} \in \mathcal{P}_b$, we introduce here the following shifted deterministic horizon BSDEs

$$\mathcal{Y}^{t,\omega,\mathbb{P}}_s = \xi^{t,\omega} + \int_s^{T-t} f^{t,\omega}_r(\mathcal{Y}^{t,\omega,\mathbb{P}}_r, \mathcal{Z}^{t,\omega,\mathbb{P}}_r, \widehat{\sigma}_r)dr - \mathcal{Z}^{t,\omega,\mathbb{P}}_r \cdot dX_r - d\mathcal{M}^{t,\omega,\mathbb{P}}_r, \quad s \in [0, T-t], \mathbb{P}\text{-a.s.} \tag{2.8}$$

In order to emphasize the dependence on the terminal condition, we shall also denote the Y-component of the solution of this BSDE by $\mathcal{Y}^{t,\omega,\mathbb{P}}_s[\xi, T]$. We repeat once again the above mentioned continuity of the mapping $(\omega, t, \omega') \in \Omega \times \mathbb{R}_+ \times \Omega \mapsto \omega \otimes_t \omega' \in \Omega$ which allows us to set $L = -\infty$, and the wellposedness of solutions for quadratic BSDE follows from Theorem 2.3.1 and the corresponding uniqueness result.

Following Briand and Hu [BH06], we introduce the exponential transform,

$$\begin{aligned}
&\widetilde{\mathcal{Y}}^{t,\omega,\mathbb{P}} := e^{\gamma \mathcal{Y}^{t,\omega,\mathbb{P}}}, \widetilde{\xi}^{t,\omega} := e^{\gamma \xi^{t,\omega}}, \widetilde{\mathcal{Z}}^{t,\omega,\mathbb{P}} := \gamma e^{\gamma \mathcal{Y}^{t,\omega,\mathbb{P}}} \mathcal{Z}^{t,\omega,\mathbb{P}} = \gamma \widetilde{\mathcal{Y}}^{t,\omega,\mathbb{P}} \mathcal{Z}^{t,\omega,\mathbb{P}},\\
&\widetilde{\mathcal{K}}^{t,\omega,\mathbb{P}} := \int_0^\cdot \gamma e^{\gamma \mathcal{Y}^{t,\omega,\mathbb{P}}_s} d\mathcal{K}^{t,\omega,\mathbb{P}}_s = \gamma \int_0^\cdot \widetilde{\mathcal{Y}}^{t,\omega,\mathbb{P}}_s d\mathcal{K}^{t,\omega,\mathbb{P}}_s,\\
&\widetilde{\mathcal{M}}^{t,\omega,\mathbb{P}} := \gamma \int_0^\cdot \widetilde{\mathcal{Y}}^{t,\omega,\mathbb{P}}_s d\mathcal{M}^{t,\omega,\mathbb{P}}_s + \frac{\gamma^2}{2} \int_0^t \widetilde{\mathcal{Y}}^{t,\omega,\mathbb{P}}_s d\langle \mathcal{M}^{t,\omega,\mathbb{P}}_\cdot \rangle_s,
\end{aligned}$$

and

$$F_s^{t,\omega}(p,q,\sigma) := \mathbf{1}_{\{p>0\}}\left(\gamma p f_s^{t,\omega}\left(\frac{\ln p}{\gamma}, \frac{q}{\gamma p}, \sigma\right) - \frac{\sigma^2}{2}\frac{|q|^2}{p}\right),$$

we have

$$\widetilde{\mathcal{Y}}_s^{t,\omega,\mathbb{P}} = \widetilde{\xi}^{t,\omega} + \int_s^{T-t} F_r^{t,\omega}(\widetilde{\mathcal{Y}}_r^{t,\omega,\mathbb{P}}, \widetilde{\mathcal{Z}}_r^{t,\omega,\mathbb{P}}, \widehat{\sigma}_r)dr - \widetilde{\mathcal{Z}}_r^{t,\omega,\mathbb{P}}dX_r - d\widetilde{\mathcal{M}}_r^{t,\omega,\mathbb{P}}, s \in [0, T-t], \mathbb{P}\text{-a.s.} \tag{2.9}$$

Define the value functions

$$V_t(\omega) := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{Y}^{t,\omega,\mathbb{P}}[\xi, T], \quad \text{with } \mathbb{Y}^{t,\omega,\mathbb{P}}[\xi,T] := \mathbb{E}^{\mathbb{P}}[\mathcal{Y}_0^{t,\omega,\mathbb{P}}[\xi,T]]. \tag{2.10}$$

$$\widetilde{V}_t(\omega) := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \widetilde{\mathbb{Y}}^{t,\omega,\mathbb{P}}[\widetilde{\xi}, T], \quad \text{with } \widetilde{\mathbb{Y}}^{t,\omega,\mathbb{P}}[\widetilde{\xi},T] := \mathbb{E}^{\mathbb{P}}[\widetilde{\mathcal{Y}}_0^{t,\omega,\mathbb{P}}[\widetilde{\xi},T]] = \mathbb{E}^{\mathbb{P}}[e^{\gamma\mathcal{Y}_0^{t,\omega,\mathbb{P}}[\xi,T]}]. \tag{2.11}$$

The final objective of this section is to prove a measurability result as below, which is necessary for the dynamic programming. We omit their proofs which are similar to the results in [LRTY20].

Proposition 2.4.1. *Let Assumption 2.1.3 (1)-(4) hold true, the mapping $(t,\omega,\mathbb{P}) \mapsto \mathbb{Y}^{t,\omega,\mathbb{P}}[\xi,T]$ (resp., $\widetilde{\mathbb{Y}}^{t,\omega,\mathbb{P}}[\widetilde{\xi},T]$) is $\mathcal{B}([0,\infty)) \otimes \mathcal{F}_T \otimes \mathcal{B}(\mathscr{M}_1)$-measurable.*

Lemma 2.4.2 ([LRTY20] Lemma 6.3)**.** *There exists a version of $\mathcal{Y}^{t,\omega,\mathbb{P}}$ s.t. the mapping $(t,\omega,s,\omega',\mathbb{P}) \in [0,\infty)\times\Omega\times[0,\infty)\times\Omega\times\mathcal{P}_b \mapsto \mathcal{Y}^{t,\omega,\mathbb{P}}(\omega') \in \mathbb{R}$ is $\mathcal{B}([0,\infty))\times\mathcal{F}_T\times\mathcal{B}([0,\infty))\times\mathcal{F}_\infty\times\mathcal{B}(\mathscr{M}_1)$-measurable.*

2.4.3 Dynamic programming principle

In this section, we prove that the dynamic value process $\widetilde{V}$ satisfies the dynamic programming principle.

Lemma 2.4.3. *Let Assumption 2.1.3 (1)-(4) hold true, and ξ satisfy the corresponding exponential integrability in Theorem 2.3.1 and 2.3.2. Then, for τ being an $\mathbb{F}$-stopping time and taking values in $[t,T](\neq\emptyset)$ and $\mathbb{P}\in\mathcal{P}_b$,*

1. $\mathbb{E}^{\mathbb{P}}[\widetilde{\mathcal{Y}}_t^{\mathbb{P}}|\mathcal{F}_t](\omega) = \widetilde{\mathbb{Y}}^{t,\omega,\mathbb{P}^{t,\omega}}[\widetilde{\xi},T]$, *for $\mathbb{P}$-a.e. $\omega\in\Omega$,*
2. $\widetilde{\mathcal{Y}}_t^{\mathbb{P}}[\widetilde{\xi},T] = \widetilde{\mathcal{Y}}_t^{\mathbb{P}}\left[\widetilde{\mathcal{Y}}_\tau^{\mathbb{P}}[\widetilde{\xi},T],\tau\right] = \widetilde{\mathcal{Y}}_t^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\widetilde{\mathcal{Y}}_\tau^{\mathbb{P}}[\widetilde{\xi},T]|\mathcal{F}_\tau\right],\tau\right]$, *for all $t\geqslant 0$,*

which also works for the couple of $\mathcal{Y}^{\mathbb{P}}$ with $\mathbb{Y}$.

We omit the proof which is similar to [LRTY20](Lemma 6.5) or [PTZ18](Lemma 2.7). Next we claim some properties of the probability family $\{\mathcal{P}(t,\omega)\}_{(t,\omega)\in[0,T]\times\Omega}$, originated from Nutz and van Handel [NvH13](Theorem 4.3), and this is needed for applying the classic measurable selection results.

Lemma 2.4.4 ([NvH13], Theorem 4.3). *Define $[\![\mathcal{P}]\!] := \{(t,\omega,\mathbb{P}) : \mathbb{P} \in \mathcal{P}(t,\omega)\}$, then $[\![\mathcal{P}]\!]$ is Borel-measurable in $\mathbb{R}_+ \times \Omega \times \mathscr{M}_1$. For any $(t,\omega) \in [0,T] \times \Omega$ and all stopping time τ satisfying $t \leqslant \tau \leqslant T($ and $[t, T-t] \neq \emptyset)$, defining $\overrightarrow{\tau}^{t,\omega} := \tau^{t,\omega} - t$, we have*

1. *$\mathcal{P}(t,\omega) = \mathcal{P}(t,\omega_{\cdot\wedge t})$, and for all $\mathbb{P} \in \mathcal{P}(t,\omega)$, the r.c.p.d. $\mathbb{P}^{\overrightarrow{\tau}^{t,\omega},\omega'} \in \mathcal{P}(\tau, \omega \otimes_t \omega')$, for $\mathbb{P}$-a.e. $\omega' \in \Omega$.*

2. *For any $\mathcal{F}_{\overrightarrow{\tau}^{t,\omega}}$-measurable kernel $\nu : \Omega \to \mathscr{M}_1$ with $\nu(\omega') \in \mathcal{P}(\tau, \omega \otimes_t \omega')$ for $\mathbb{P}$-a.e. $\omega' \in \Omega$, the mapping $\mathbb{P}' := \mathbb{P} \otimes_{\overrightarrow{\tau}^{t,\omega}} \nu$ defined by*

$$\mathbb{P}'(A) = \int\int (\mathbf{1}_A)^{\overrightarrow{\tau}^{t,\omega},\omega'}(\omega'')\nu(d\omega''; \omega')\mathbb{P}(d\omega''), A \in \mathcal{F},$$

   is a probability measure in $\mathcal{P}(t,\omega)$.

Consider 2BSDE(f,ξ)(2.4). Due to Remark 2.2.2, we know that under different exponential integrability conditions, we have two cases of exponential integrability for the Y-part of the solution. In the following, we only consider the second case and the first one is similar, so we omit here.

Before the statement of the theorem, we recall that $\mathcal{P}_{\mathbb{P}}(t)$ and $\mathcal{P}(t,\omega)$ are defined as

$$\mathcal{P}_{\mathbb{P}}(t) := \{\mathbb{P} \in \mathcal{P}_0 : \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_t\},$$
$$\mathcal{P}(t,\omega) := \{\mathbb{P} \in \mathcal{P}_b : f_s^{0,t,\omega}(\omega') < \infty, \text{ for Leb} \otimes \mathbb{P}\text{-a.e.}(s,\omega') \in \mathbb{R}_+ \times \Omega\},$$

where $f_s^{0,t,\omega}(\omega') := f_{t+s}(\omega \otimes_t \omega', 0, 0, \widehat{\sigma}(\omega'))$, so that $\mathcal{P}_0 = \mathcal{P}(0, \mathbf{0})$.

Theorem 2.4.5 (Dynamic programming for V). *Let Assumption 2.1.3 (1)-(4) hold true, and ξ satisfy the corresponding exponential integrability in Theorem 2.3.1 and 2.3.2, the mapping $\omega \mapsto V_\tau(\omega)$ is $\mathcal{F}_\tau^U (:= \cap_{\mathbb{P}\in\mathscr{M}_1}\mathcal{F}_\tau^{\mathbb{P}})$-measurable. For any $(t,\omega) \in [0,T] \otimes \Omega$, and an $\mathbb{F}$-stopping time τ with $t \wedge T \leqslant \tau \leqslant$*

$T($ *and* $[t,T] \neq \emptyset)$, *defining* $\overrightarrow{\tau} := \tau^{t,\omega} - t$,

$$\sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}[|(\widetilde{V}_\tau)^{t,\omega}|^p] < +\infty, \quad \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}[|\widetilde{V}_\tau|^p] < +\infty, \text{ for all } p \geqslant 1 \vee 2\gamma, \tag{2.12}$$

$$\widetilde{V}_t(\omega) = \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \widetilde{\mathbb{Y}}^{t,\omega,\mathbb{P}}[\widetilde{V}_\tau, \tau], \tag{2.13}$$

$$\widetilde{V}_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_{\mathbb{P}}(t)} \mathbb{E}^{\mathbb{P}'}\Big[\widetilde{\mathcal{Y}}_t^{\mathbb{P}'}[\widetilde{V}_\tau, \tau]\Big|\mathcal{F}_t\Big], \mathbb{P}\text{-a.s., for all } \mathbb{P}\in\mathcal{P}_0. \tag{2.14}$$

Proof. W.L.O.G., we assume from now on that $(t,\omega) \equiv (0,\mathbf{0})$.

1. By Proposition 2.4.1, $(t,\omega,\mathbb{P}) \mapsto \widetilde{\mathbb{Y}}^{t,\omega,\mathbb{P}}[\widetilde{\xi}, T]$ is $\mathcal{B}([0,\infty))\otimes\mathcal{F}_T\otimes\mathcal{B}(\mathscr{M}_1)$-measurable, and Lemma 2.4.4 shows that $[\![\mathcal{P}]\!]$ is analytic. By Bersekas and Shreve [BS96] (Proposition 7.47), it follows that $(t,\omega) \mapsto \widetilde{V}_t(\omega) := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \widetilde{\mathbb{Y}}^{t,\omega,\mathbb{P}}[\widetilde{\xi}, T]$ is upper semi-analytic and thus universally measurable, i.e., $\mathcal{B}([0,\infty))\otimes\mathcal{F}_T^U$-measurable. Note that $\widetilde{V}_t(\omega) = \widetilde{V}_t(\omega_{t\wedge\cdot})$, we get from Galmarino's test that $\widetilde{V}_\tau$ is $\mathcal{F}^U_{\tau\wedge T}$-measurable.

2. We here prove (2.12). Due to measurable selection theorem (e.g., Proposition 7.50 in [BS96]), for each $\epsilon > 0$, we may find an $\mathcal{F}^U_\tau$-measurable kernel $\nu^\epsilon : \omega \mapsto \nu^\epsilon(\omega) \in \mathcal{P}(\tau(\omega),\omega)$, s.t. for all $\omega\in\Omega$,

$$\widetilde{V}_\tau \leqslant \widetilde{\mathbb{Y}}^{\tau,\omega,\nu^\epsilon(\omega)}[\xi, T] + \epsilon, \text{ i.e., } \exp(\gamma\mathcal{Y}_0^{\tau,\omega,\mathbb{P}'}) \leqslant \exp(\gamma\mathbb{Y}^{\tau,\omega,\nu^\epsilon(\omega)}[\xi,T]) + \epsilon, \tag{2.15}$$

then we have by Lemma 2.4.3(1), for all $\mathbb{P}\in\mathcal{P}_0\subseteq\mathcal{P}_b$,

$$\exp(\gamma\mathbb{Y}^{\tau,\omega,\nu^\epsilon(\omega)}[\xi,T]) = \mathbb{E}^{\mathbb{P}\otimes_\tau\nu^\epsilon}\Big[\exp(\gamma\mathcal{Y}_\tau^{\mathbb{P}\otimes_\tau\nu^\epsilon})\Big|\mathcal{F}_\tau\Big](\omega), \mathbb{P}\text{-a.s.},$$

for all $\mathbb{P}\in\mathcal{P}_0$. Then we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[|\exp(\gamma\mathcal{Y}_0^{\tau,\omega,\mathbb{P}'})|^p] &\leqslant \mathbb{E}^{\mathbb{P}}\Big[\Big|\mathbb{E}^{\mathbb{P}\otimes_\tau\nu^\epsilon}\Big[\exp(\gamma\mathcal{Y}_\tau^{\mathbb{P}\otimes_\tau\nu^\epsilon})\Big|\mathcal{F}_\tau\Big](\omega)+\epsilon\Big|^p\Big] \\
&= \mathbb{E}^{\mathbb{P}}\Big[\Big|\mathbb{E}^{\mathbb{P}\otimes_\tau\nu^\epsilon}\Big[\exp(\gamma\mathcal{Y}_\tau^{\mathbb{P}\otimes_\tau\nu^\epsilon})+\epsilon\Big|\mathcal{F}_\tau\Big](\omega)\Big|^p\Big] \\
&\leqslant C_p\Big(\mathbb{E}^{\mathbb{P}\otimes_\tau\nu^\epsilon}[\exp(p\gamma\mathcal{Y}_\tau^{\mathbb{P}\otimes_\tau\nu^\epsilon})] + \epsilon^p\Big).
\end{aligned}$$

By the a priori estimate for the solution of the corresponding BSDE (Theorem 2.3.1), we have that

$$\mathcal{Y}_\tau^{\mathbb{P}\otimes_\tau\nu^\epsilon} \leqslant c_0 + \frac{1}{\gamma}\ln\mathbb{E}^{\mathbb{P}\otimes_\tau\nu^\epsilon}[e^{\gamma e^{\beta T}(\xi^+\vee L_*^+)}|\mathcal{F}_\tau^{+,\mathbb{P}}], \quad \forall t\in[0,T],$$

i.e. for $p \geqslant 1 \vee 2\gamma$,

$$e^{p\gamma(\mathcal{Y}_\tau^{\mathbb{P}\otimes_\tau\nu^\epsilon}(\omega')-c_0)} \leqslant |\mathbb{E}^{\mathbb{P}\otimes_\tau\nu^\epsilon}[e^{\gamma e^{\beta T}(\xi^+\vee L_*^+)}|\mathcal{F}_\tau^{+,\mathbb{P}}]|^p \leqslant \mathbb{E}^{\mathbb{P}\otimes_\tau\nu^\epsilon}[e^{p\gamma e^{\beta T}(\xi^+\vee L_*^+)}|\mathcal{F}_\tau^{+,\mathbb{P}}],$$

finally we obtain

$$\sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}[|\widetilde{V}_\tau(\omega)|^p] \leqslant C_p\left(\sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}\otimes_\tau\nu^\epsilon}[e^{p\gamma e^{\beta T}(\xi^+\vee L_*^+)}]+c_p\right)<\infty,$$

which gives the required estimate by letting $\epsilon \to 0$.

3. We here prove (2.13). Using Lemma 2.4.3(2), we get that

$$\begin{aligned}\widetilde{V}_0 &= \sup_{\mathbb{P}\in\mathcal{P}_0}\widetilde{\mathbb{Y}}^{0,\mathbf{0},\mathbb{P}}[\widetilde{\xi},T] = \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}[\widetilde{\mathcal{Y}}_0^{0,\mathbf{0},\mathbb{P}}]\\ &= \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}[\widetilde{\mathcal{Y}}_0^{\mathbb{P}}[\widetilde{\xi},T]] = \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[\widetilde{\mathcal{Y}}_0^{\mathbb{P}}[\widetilde{\mathcal{Y}}_\tau^{\mathbb{P}}[\widetilde{\xi},T],\tau]\Big] = \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[\widetilde{\mathcal{Y}}_0^{\mathbb{P}}\Big[\mathbb{E}^{\mathbb{P}}[\widetilde{\mathcal{Y}}_\tau^{\mathbb{P}}[\widetilde{\xi},T]|\mathcal{F}_\tau],\tau\Big]\Big].\end{aligned}$$

For all $\mathbb{P}\in\mathcal{P}_0$, $\mathbb{P}$-a.s. ω, we have,

$$\widetilde{V}_\tau(\omega) = \sup_{\mathbb{P}\in\mathcal{P}(\tau,\omega)}\widetilde{\mathbb{Y}}^{\tau,\omega,\mathbb{P}}[\widetilde{\xi},T] = \sup_{\mathbb{P}\in\mathcal{P}(\tau,\omega)}\mathbb{E}^{\mathbb{P}}\Big[\widetilde{\mathcal{Y}}_0^{\tau,\omega,\mathbb{P}}[\widetilde{\xi},T]\Big] \geqslant \mathbb{E}^{\mathbb{P}}\Big[\widetilde{\mathcal{Y}}_\tau^{\mathbb{P}}[\widetilde{\xi},T]\Big|\mathcal{F}_\tau\Big](\omega),$$

then this leads to

$$\widetilde{V}_0 \leqslant \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[\widetilde{\mathcal{Y}}_0^{\mathbb{P}}[\widetilde{V}_\tau,\tau]\Big] = \sup_{\mathbb{P}\in\mathcal{P}_0}\widetilde{\mathbb{Y}}^{0,\mathbf{0},\mathbb{P}}[\widetilde{V}_\tau,\tau], \tag{2.16}$$

where the last equality is due to the definition (2.11) and the first inequality is by the comparison principle Theorem 2.3.2(for BSDEs' case by letting the barrier tend to $-\infty$). As for the reverse inequality, it follows, by the measurable selection theorem, there exists an $\mathcal{F}_\tau^U$-measurable kernel $\nu^\epsilon : \omega \mapsto \nu^\epsilon(\omega) \in \mathcal{P}(\tau,\omega)$ s.t. (2.15) holds true for any $p \geqslant 1 \vee 2\gamma$. Define $\overline{\mathbb{P}} := \mathbb{P}\otimes_\tau \nu^\epsilon$, clearly we have $\overline{\mathbb{P}}|_{\mathcal{F}_\tau} = \mathbb{P}|_{\mathcal{F}_\tau}$, and then by Lemma 2.4.3 and the stability Theorem 2.3.3 (for BSDEs' case by letting the barrier tend to $-\infty$), we have

$$\widetilde{V}_0 \geqslant \mathbb{E}^{\overline{\mathbb{P}}}\Big[\widetilde{\mathcal{Y}}_0^{\overline{\mathbb{P}}}[\widetilde{\xi},T]\Big] = \mathbb{E}^{\mathbb{P}}\Big[\widetilde{\mathcal{Y}}_0^{\mathbb{P}}\Big[\mathbb{E}^{\overline{\mathbb{P}}^{\tau,\cdot}}[\widetilde{\mathcal{Y}}_0^{\tau,\cdot,\overline{\mathbb{P}}^{\tau,\cdot}}[\widetilde{\xi},T]|\mathcal{F}_\tau],\tau\Big]\Big] = \mathbb{E}^{\mathbb{P}}\Big[\widetilde{\mathcal{Y}}_0^{\mathbb{P}}\Big[\mathbb{E}^{\nu^\epsilon(\cdot)}[\widetilde{\mathcal{Y}}_0^{\tau,\cdot,\nu^\epsilon(\cdot)}[\widetilde{\xi},T]|\mathcal{F}_\tau],\tau\Big]\Big].$$

By (2.15), the RHS is bigger than $\mathbb{E}^{\mathbb{P}}\Big[\widetilde{\mathcal{Y}}_0^{\mathbb{P}}[\widetilde{V}_\tau,\tau]\Big] - C\epsilon$ for some $C>0$ which is independent of ϵ. Therefore, $\widetilde{V}_0 \geqslant \widetilde{\mathbb{Y}}^{0,\mathbf{0},\mathbb{P}}[\widetilde{V}_\tau,\tau]-C\epsilon$, so we obtain the required equality by letting $\epsilon\to 0$.

4. This part goes to prove (2.14). Now the above case $(t,\omega)\equiv(0,\mathbf{0})$ can be generalized so that we have

$$\widetilde{V}_t(\omega)\geqslant\widetilde{\mathbb{Y}}^{t,\omega,\mathbb{P}'}[V_\tau,\tau],\ \text{ for all } \mathbb{P}'\in\mathcal{P}(t,\omega).$$

Fix a probability measure $\mathbb{P}\in\mathcal{P}_0$, it follows from the Lemma 2.4.4(1) that for any $\widetilde{\mathbb{P}}\in\mathcal{P}_{\mathbb{P}}(t)\subseteq\mathcal{P}_0$, we have $\widetilde{\mathbb{P}}^{t,\omega}\in\mathcal{P}(t,\omega)$ and then $\widetilde{V}_t(\omega)\geqslant\widetilde{\mathbb{Y}}^{t,\omega,\widetilde{\mathbb{P}}^{t,\omega}}[\widetilde{V}_\tau,\tau]$. From Lemma 2.4.3(1), we have

$$\widetilde{V}_t\geqslant\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\widetilde{\mathcal{Y}}_t^{\widetilde{\mathbb{P}}}[\widetilde{V}_\tau,\tau]\Big|\mathcal{F}_t\Big],\mathbb{P}\text{-a.s., and then } \widetilde{V}_t\geqslant\underset{\widetilde{P}\in\mathcal{P}_{\mathbb{P}}(t)}{\operatorname{ess\,sup}^{\mathbb{P}}}\,\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\widetilde{\mathcal{Y}}_t^{\widetilde{\mathbb{P}}}[\widetilde{V}_\tau,\tau]\Big|\mathcal{F}_t\Big],$$

The reverse inequality can be validated by using the measurable selection theorem on the optimization problem (2.13). In fact, we can get an $\mathcal{F}_t^U$-measurable kernel $\nu^\epsilon:\omega\mapsto\nu^\epsilon(\omega)\in\mathcal{P}(t,\omega)$ s.t. $\widetilde{V}_t(\omega)\leqslant\widetilde{\mathbb{Y}}^{t,\omega,\nu^\epsilon(\omega)}[\widetilde{V}_\tau,\tau]+\epsilon$. We have $\mathbb{P}^\epsilon:=\mathbb{P}\otimes_t\nu^\epsilon\in\mathcal{P}_0$ by Lemma 2.4.4, and therefore $\mathbb{P}^\epsilon\in\mathcal{P}_{\mathbb{P}}(t)$. This together with Lemma 2.4.3(1) gives

$$\widetilde{V}_t\leqslant\mathbb{E}^{\mathbb{P}^\epsilon}\Big[\widetilde{\mathcal{Y}}_t^{\mathbb{P}^\epsilon}[\widetilde{V}_\tau,\tau]\Big|\mathcal{F}_t\Big]+\epsilon\leqslant\underset{\widetilde{\mathbb{P}}\in\mathcal{P}_{\mathbb{P}}(t)}{\operatorname{ess\,sup}^{\mathbb{P}}}\,\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\widetilde{\mathcal{Y}}_t^{\widetilde{\mathbb{P}}}[\widetilde{V}_\tau,\tau]\Big|\mathcal{F}_t\Big]+\epsilon.$$

By sending $\epsilon\to0$ we get the required inequality.

□

Now that we obtained the measurability issues of V_t and $\widetilde{V}_t$, we need to study their regularities in time. Define for all (t,ω), the $\mathbb{F}_+$-progressively measurable process

$$V_t^+:=\limsup_{r\in\mathbb{Q}\cap(t,T],r\downarrow t}V_r,\quad \widetilde{V}_t^+:=\limsup_{r\in\mathbb{Q}\cap(t,T],r\downarrow t}\widetilde{V}_r. \tag{2.17}$$

Precisely, we expect to prove that

1. these two limits (2.17) exist for any time in $[0,T]$ or equivalently $\liminf_{r\in\mathbb{Q}\cap(t,T],r\downarrow t}V_r$ (resp., $\liminf_{r\in\mathbb{Q}\cap(t,T],r\downarrow t}\widetilde{V}_r$) exists for any $t\in(0,T]$.
2. the new value process V_t^+ (resp., $\widetilde{V}_t^+$) is càdlàg.

Normally, we study the so-called downcrossing inequality for V (resp., $\widetilde{V}$) in order to furthermore get the path regularity. For the up-(resp., down-) crossing inequality for f-sub(resp., super-) martingale while terminal condition ξ satisfies certain type of integrability and generator f is Lipschitz continuous in (y, z) or quadratic in z for the related BSDE, there exists generally two previous results. One is the case of square integrable ξ and Lipschitz f in Chen and Peng [CP00](p.173, theorem 6) where they use the explicit form of y-part of the solution for related BSDE to construct a supermartingale in order to perform classical downcrossing inequality. The other is the case of bounded ξ and quadratic f in Ma and Yao [MY10](p.731, theorem 5.5), which use that $\int_0^\cdot Z_s \cdot dW_s$ is a BMO martingale and f is quadratic combined with Y is in $\mathbb{L}^\infty(\mathbb{P}, \mathbb{R})$ for some probability $\mathbb{P}$.

Compared with the previous literature, the following downcrossing inequality for an f-supermartingale is a new part. On one hand, we don't have Lipschitz continuity jointly w.r.t. (y, z). On the other hand, there is no more BMO martingale here due to the exponential integrability ξ in our setting. So none of the results mentioned work for the problem that we consider.

And below, we only consider the case of V while it's similar for $\widetilde{V}$, so we omit it here.

2.4.4 Existence of a càdlàg version of V

We fix a time horizon $T > 0$ and recall that $\mathcal{T}_{[0,T]}$ denotes the set of stopping times a.s. less than or equal to T. Furthermore, here for all $\iota \in \mathcal{T}_{[0,T]}$, we define $\mathcal{T}_{[\iota,T]}$ to be the set of $\tau \in \mathcal{T}_{[0,T]}$ s.t. $\tau \geqslant \iota, \mathbb{P}$-a.s., where $\mathbb{P}$ is a fixed probability measure. Besides, we say $(\iota, \tau) \in \mathcal{T}_2$ if $\iota \in \mathcal{T}_{[0,T]}$ and $\tau \in \mathcal{T}_{[\iota,T]}$.

Let $p \in (1, +\infty]$ and q denote the conjugate of p (i.e. $\frac{1}{p} + \frac{1}{q} = 1$). Then, we define a nonlinear conditional expectation operator as a family $\mathfrak{E} = \{\mathfrak{E}_{\iota,\tau}, (\iota, \tau) \in \mathcal{T}_2\}$ of maps

$$\mathfrak{E}_{\iota,\tau} : \mathbb{L}^p(\mathcal{F}_\tau, \mathbb{P}) \mapsto \mathbb{L}^p(\mathcal{F}_\iota, \mathbb{P}), \quad \text{for } (\iota, \tau) \in \mathcal{T}_2.$$

For $p \in (0, \infty]$, we denote by $\mathbb{X}^p$ the collection of all optional processes $\mathcal{U}$ s.t. $\mathcal{U}_\tau$ lies in $\mathbb{L}^p(\mathcal{F}_\tau, \mathbb{P})$ for all $\tau \in \mathcal{T}_{[0,T]}$.

We say that $\mathcal{U}$ is a $\mathfrak{E}$-supermartingale if $\mathcal{U} \in \mathbb{X}^p$ and $\mathcal{U}_\iota \geqslant \mathfrak{E}_{\iota,\tau}[\mathcal{U}_\tau]$ a.s. for all $(\iota, \tau) \in \mathcal{T}_2$.

Given $(\iota,\tau)\in\mathcal{T}_2$ and $\xi^{t,\omega}$ satisfies some necessary exponential integrability condition, we set $\mathfrak{E}^f_{\iota,\tau}[\xi]:=\mathcal{Y}^{t,\omega,\mathbb{P}}_\iota$ in which $(\mathcal{Y}^{t,\omega,\mathbb{P}},\mathcal{Z}^{t,\omega,\mathbb{P}},\mathcal{M}^{t,\omega,\mathbb{P}})$ is the unique solution of the following BSDE(2.8)

$$\mathcal{Y}^{t,\omega,\mathbb{P}}_s=\xi^{t,\omega}+\int_s^{T-t}f^{t,\omega}_r(\mathcal{Y}^{t,\omega,\mathbb{P}}_r,\mathcal{Z}^{t,\omega,\mathbb{P}}_r,\widehat{\sigma}_r)dr-\mathcal{Z}^{t,\omega,\mathbb{P}}_r\cdot dX_r-d\mathcal{M}^{t,\omega,\mathbb{P}}_r,\quad s\geqslant 0,\mathbb{P}\text{-a.s.}$$

We define $\mathfrak{E}^f$-supermartingales, also called f-supermartingales, for $\mathfrak{E}=\mathfrak{E}^f$, i.e. $\mathcal{U}$ is a $\mathfrak{E}^f$-supermartingale iff $\mathcal{U}\in\mathbb{X}^p$ and $\mathcal{U}_\iota\geqslant\mathfrak{E}^f_{\iota,\tau}[\mathcal{U}_\tau]$ a.s. for all $(\iota,\tau)\in\mathcal{T}_2$.

For any $m>0$, we denote by $\mathfrak{E}^{\pm m}_{\iota,\tau}$ the nonlinear expectation operator associated to the generator $(t,\omega,y,z)\mapsto\pm m|z|$ and the stopping times $(\iota,\tau)\in\mathcal{T}_2$.

Downcrossing inequality

Theorem 2.4.6. *For any finite set $\mathscr{J}=\{0\leqslant r_0<r_1<\cdots<r_n\leqslant T\}$, let $D^b_a(V,\mathscr{J})$ denote the number of downcrossings of the interval $[a,b]$ by V over $\mathscr{J}$, then $D^b_a(V,\mathscr{J})<\infty,\mathbb{P}$-a.s. for all $\mathbb{P}\in\mathcal{P}_0$.*

Proof. Without loss of generality, we can always suppose that $r_0\equiv 0$ and $r_n\equiv T$, and also that $b>a=0$. Indeed, whenever $b>a\neq 0$, we can consider the barrier constants $(0,b-a)$, and the $\mathfrak{E}^{\overline{f}}$-supermartingale $V-a$, with generator $\overline{f}_r(y,z):=f_r(y+a,z)$, which reduces the problem to the case $b>a=0$.

Step 1. For any $j\in\{1,\cdots,n\}$ and any finite set $\mathscr{J}=\{0\leqslant r_0<r_1<\cdots<r_n\leqslant T\}$, we consider the following quadratic BSDE:

$$\mathcal{Y}^{t,\omega,\mathbb{P},j}_r=V_{r_j}+\int_r^{r_j}f^{t,\omega}_s(\mathcal{Y}^{t,\omega,\mathbb{P},j}_s,\mathcal{Z}^{t,\omega,\mathbb{P},j}_s,\widehat{\sigma}_s)ds-\mathcal{Z}^{t,\omega,\mathbb{P},j}_s\cdot dX_s-d\mathcal{M}^{t,\omega,\mathbb{P},j}_s,\forall r\in[r_{j-1},r_j],\mathbb{P}\text{-a.s.},$$

where the terminal condition V_{r_j}, defined in (2.10), is the f-supermartingale. Note that if we want to use the wellposedness of quadratic BSDE with exponential integrable terminal condition(which corresponds to the results on RBSDE in the former section), we should firstly check the exponential integrability of such ξ that we choose, i.e., we have to verify that $V\in\mathbb{D}^{\exp(p)}(\mathbb{P})$(i.e., $\mathbb{E}^{\mathbb{P}}[e^{pV_*}]<\infty$) for $p\in[1\vee 2\gamma,\infty)$.

Indeed, for any $t \in [0,T]$,

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[e^{pV_t}] = \mathbb{E}^{\mathbb{P}}\left[\exp\left(p \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\left[\mathcal{Y}_0^{t,\omega,\mathbb{P}}[\xi,T]\right]\right)\right] &= \mathbb{E}^{\mathbb{P}}\left[\left(e^{\sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\left[\mathcal{Y}_0^{t,\omega,\mathbb{P}}[\xi,T]\right]}\right)^p\right] \\
&\leqslant \mathbb{E}^{\mathbb{P}}\left[\left(\sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}} e^{\mathcal{Y}_0^{t,\omega,\mathbb{P}}[\xi,T]}\right)^p\right] \quad \text{(Jensen's inequality)} \\
&\leqslant \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\left[\left(\sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}} e^{\mathcal{Y}_0^{t,\omega,\mathbb{P}}[\xi,T]}\right)^p\right] \\
&= \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}[|\widetilde{V}_T|^p] < \infty,
\end{aligned} \tag{2.18}$$

where the last inequality is due to (2.12) and the penultimate equality corresponds to the special case of $\gamma = 1$. So now we have verified the exponential integrability of V and could claim the wellposedness of the above quadratic BSDE.

Next, define a d-dimensional process $\beta_{\mathscr{J}}(r) := (\beta_r^1, \cdots, \beta_r^d), \forall r \in [0, r_n]$ by

$$\beta_r^l := \sum_{j=1}^n \mathbf{1}_{r\in(r_{j-1},r_j]} \frac{\partial f_r^{t,\omega}}{\partial z_l}(\mathcal{Y}_r^{t,\omega,\mathbb{P},j}, (\mathcal{Z}_r^{t,\omega,\mathbb{P},j,1}, \cdots, \zeta^l \mathcal{Z}_r^{t,\omega,\mathbb{P},j,l}, 0, \cdots, 0)), \quad \zeta^l \in [0,1], \quad l \in \{1, \cdots, d\},$$

where z_l denote the l-th component of the d-dimensional r.v. Z. By mean value theorem that for any $r \in (r_{j-1}, r_j]$,

$$\begin{aligned}
& f_r^{t,\omega}(\mathcal{Y}_r^{t,\omega,\mathbb{P},j}, \mathcal{Z}_r^{t,\omega,\mathbb{P},j}) - f_r^{t,\omega}(\mathcal{Y}_r^{t,\omega,\mathbb{P},j}, 0) \\
=& \sum_{l=1}^d \left\{ f_r^{t,\omega}(\mathcal{Y}_r^{t,\omega,\mathbb{P},j}, (\mathcal{Z}_r^{t,\omega,\mathbb{P},j,1}, \cdots, \mathcal{Z}_r^{t,\omega,\mathbb{P},j,l}, 0, \cdots, 0)) - f_r^{t,\omega}(\mathcal{Y}_r^{t,\omega,\mathbb{P},j}, (\mathcal{Z}_r^{t,\omega,\mathbb{P},j,1}, \cdots, \mathcal{Z}_r^{t,\omega,\mathbb{P},j,l-1}, 0, \cdots, 0)) \right\} \\
=& \sum_{l=1}^d \mathcal{Z}_r^{t,\omega,\mathbb{P},j,l} \beta_r^l = \widehat{\sigma}_r \mathcal{Z}_r^{t,\omega,\mathbb{P},j} \cdot \widehat{\sigma}_r^{-1} \beta_{\mathscr{J}}(r).
\end{aligned} \tag{2.19}$$

Moreover, (5) in Assumption 2.1.3 gives that

$$|\beta_r^l| \leqslant \frac{\gamma}{2} \sum_{j=1}^n \mathbf{1}_{r\in(r_{j-1},r_j]}(1 + |\widehat{\sigma}_r \mathcal{Z}_r^{t,\omega,\mathbb{P},j}|), \quad r \in [0, r_n], \quad l \in \{1, \cdots, d\}. \tag{2.20}$$

Another essential point is the Legendre-Fenchel transformation of $f_s(\mathcal{Y}_s^{t,\omega,\mathbb{P},j}, \cdot)$: under (2),(3) and (4) of Assumption 2.1.3, for any $\mathfrak{q} \in \mathbb{R}^d$, we define $\breve{f}_s(\mathcal{Y}_s^{t,\omega,\mathbb{P},j}, \mathfrak{q}) := \sup_{z\in\mathbb{R}^d} \left(f_s(\mathcal{Y}_s^{t,\omega,\mathbb{P},j}, z) + z \cdot \mathfrak{q}\right)$. If we take $\mathfrak{N}^{\mathbb{P}}$ as the $dt \otimes d\mathbb{P}$-null set except on which the above mentioned assumptions hold, then for

$(s,\omega) \in \left(\mathfrak{N}^{\mathbb{P}}\right)^c$, $\breve{f}$ exhibits two properties of interest. One is

$$\breve{f}_s(\mathcal{Y}_s^{t,\omega,\mathbb{P},j}, \mathfrak{q}) \geqslant -\alpha - \beta\left|\mathcal{Y}_s^{t,\omega,\mathbb{P},j}\right| + \frac{1}{2\gamma}|\widehat{\sigma}_s^{-1}\mathfrak{q}|^2, \text{ for any } \mathcal{Y}_s^{t,\omega,\mathbb{P},j} \in \mathbb{R}, \tag{2.21}$$

which is obvious due to the quadratic growth of f in z and $z \cdot \mathfrak{q} \leqslant \frac{1}{2\gamma}|\widehat{\sigma}_s^{-1}\mathfrak{q}|^2 + \frac{\gamma}{2}|\widehat{\sigma}_s z|^2$. The other is the concept of subdifferential. Given $(\mathcal{Y}_s^{t,\omega,\mathbb{P},j}, \mathcal{Z}_s^{t,\omega,\mathbb{P},j}) \in \mathbb{R} \times \mathbb{R}^d$, we define $\partial(-f_s)(\mathcal{Y}_s^{t,\omega,\mathbb{P},j}, \mathcal{Z}_s^{t,\omega,\mathbb{P},j})$ as the subdifferential of $-f_s(\mathcal{Y}_s^{t,\omega,\mathbb{P},j}, \cdot)$ at $\mathcal{Z}_s^{t,\omega,\mathbb{P},j}$, and $\partial(-f_s)(\mathcal{Y}_s^{t,\omega,\mathbb{P},j}, \mathcal{Z}_s^{t,\omega,\mathbb{P},j})$ is a non-empty convex compact subset of $\mathfrak{q} \in \mathbb{R}^d$ s.t. $-f_s(\mathcal{Y}_s^{t,\omega,\mathbb{P},j}, \mathcal{Z}_s'^{t,\omega,\mathbb{P},j}) + f_s(\mathcal{Y}_s^{t,\omega,\mathbb{P},j}, \mathcal{Z}_s^{t,\omega,\mathbb{P},j}) \geqslant (\mathcal{Z}_s'^{t,\omega,\mathbb{P},j} - \mathcal{Z}_s^{t,\omega,\mathbb{P},j}) \cdot \mathfrak{q}$ for any $\mathcal{Z}_s'^{t,\omega,\mathbb{P},j} \in \mathbb{R}^d$. By $\mathcal{Z}_s^{t,\omega,\mathbb{P},j} \cdot \mathfrak{q} + f_s(\mathcal{Y}_s^{t,\omega,\mathbb{P},j}, \mathcal{Z}_s^{t,\omega,\mathbb{P},j}) \geqslant \mathcal{Z}_s^{t,\omega,\mathbb{P},j} \cdot \mathfrak{q} + (\mathcal{Z}_s'^{t,\omega,\mathbb{P},j} - \mathcal{Z}_s^{t,\omega,\mathbb{P},j}) \cdot \mathfrak{q} + f_s(\mathcal{Y}_s^{t,\omega,\mathbb{P},j}, \mathcal{Z}_s'^{t,\omega,\mathbb{P},j})$, and $\breve{f}_s(\mathcal{Y}_s^{t,\omega,\mathbb{P},j}, \mathfrak{q}) := \sup_{\mathcal{Z}_s'^{t,\omega,\mathbb{P},j} \in \mathbb{R}^d}\left(f_s(\mathcal{Y}_s^{t,\omega,\mathbb{P},j}, \mathcal{Z}_s'^{t,\omega,\mathbb{P},j}) + \mathcal{Z}_s'^{t,\omega,\mathbb{P},j} \cdot \mathfrak{q}\right)$, we could obtain

$$\breve{f}_s(\mathcal{Y}_s^{t,\omega,\mathbb{P},j}, \mathfrak{q}) = \mathcal{Z}_s^{t,\omega,\mathbb{P},j} \cdot \mathfrak{q} + f_s(\mathcal{Y}_s^{t,\omega,\mathbb{P},j}, \mathcal{Z}_s^{t,\omega,\mathbb{P},j}). \tag{2.22}$$

Immediately by this definition and (2.19) in the above, we know that $-\beta_{\mathscr{J}}(s)$ belongs to the subdifferential $\partial(-f_s^{t,\omega}(\mathcal{Y}_s^{t,\omega,\mathbb{P},j}, \mathcal{Z}_s^{t,\omega,\mathbb{P},j}))$.

Step 2. Define for $j \in \{1, \cdots, n\}$,

$$\mathscr{E}(\beta_{\mathscr{J}} \cdot X)_{r_j} := e^{-\frac{1}{2}\int_0^{r_j}|\widehat{\sigma}_s^{-1}\beta_{\mathscr{J}}|^2 ds + \int_0^{r_j}\widehat{\sigma}_s^{-2}\beta_{\mathscr{J}} \cdot dX_s}, \text{ with } \beta_{\mathscr{J}}(\cdot) := (\beta_\cdot^1, \cdots, \beta_\cdot^d).$$

Note that here we only require the space $\mathbb{H}^{2,\mathrm{loc}}(\mathbb{P}; \mathbb{R}^d)$ for $\mathcal{Z}^{t,\omega,\mathbb{P}}$-part of the corresponding solution, as in the above uniqueness result(which is directly implied by Theorem 2.3.2), and this is far weaker than the space $\mathbb{H}^{2,2p}(\mathbb{P}, \mathbb{R}^d), p > 1$ in the existence result(Theorem 2.3.1). So we have,

$$\int_r^{r_j}|\widehat{\sigma}_s^{-1}\beta_{\mathscr{J}}(s)|^2 ds \leqslant \int_r^{r_j}\left(\frac{\gamma}{2}\sum_{j=1}^n \mathbf{1}_{s\in(r_{j-1},r_j]}(1 + |\widehat{\sigma}_s\mathcal{Z}_s^{t,\omega,\mathbb{P},j}|)\right)^2|\widehat{\sigma}_s^{-2}|ds < \infty, \mathbb{P}\text{-a.s.}, \tag{2.23}$$

and then we may consider for all $m \in \mathbb{N}$ the $\mathbb{P}$-a.s. finite $\mathbb{F}$-stopping time $\tau_m^j := \inf\Big\{t \in [r_{j-1}, r_j] : \int_{r_{j-1}}^t \left[|\widehat{\sigma}_s^{-1}\beta_{\mathscr{J}}|^2 + |\widehat{\sigma}_s Z_s^{t,\omega,\mathbb{P},j}|^2\right] ds > m\Big\} \wedge r_j$. Define $\ell_t^j := \int_0^t \mathbf{1}_{\{s\geqslant r_{j-1}\}}\widehat{\sigma}_s^{-2}\beta_{\mathscr{J}} \cdot dX_s$, and $L_t^j := \exp\left(\ell_t^j - \frac{1}{2}\langle \ell_\cdot^j\rangle_t\right) = \exp\left(\int_0^t \mathbf{1}_{\{s\geqslant r_{j-1}\}}\widehat{\sigma}_s^{-2}\beta_{\mathscr{J}} \cdot dX_s - \frac{1}{2} \cdot \mathbf{1}_{\{s\geqslant r_{j-1}\}}|\widehat{\sigma}_s^{-1}\beta_{\mathscr{J}}|^2 ds\right)$. Due to the definition of τ_m^j, we have $\mathbb{E}^{\mathbb{P}}\Big[\exp\left(\frac{1}{2}\langle \ell_\cdot^j\rangle_{\tau_m^j\wedge\infty}\right)\Big] = \mathbb{E}^{\mathbb{P}}\Big[\exp\left(\frac{1}{2}\int_0^{\tau_m^j}\mathbf{1}_{\{s\geqslant r_{j-1}\}}|\widehat{\sigma}_s^{-1}\beta_{\mathscr{J}}|^2 ds\right)\Big] < \infty$ and we claim that $\{L_{\tau_m^j\wedge t}^j\}_{t\in[0,r_j]}$ is a uniformly integrable martingale by Novikov's Criterion. By $\frac{d\mathbb{Q}_m^j}{d\mathbb{P}} := L_{\tau_m^j}^j$ and

$X_t^{j,m} := X_t - \int_0^t \widehat{\sigma}_s \mathbf{1}_{\{r_{j-1}\leqslant s\leqslant \tau_m^j\}}\beta_{\mathscr{J}}\widehat{\sigma}_s^{-1}ds = X_t - \int_0^t \mathbf{1}_{\{r_{j-1}\leqslant s\leqslant \tau_m^j\}}\beta_{\mathscr{J}}ds$, we know that $\mathbb{Q}_m^j$ is a probability measure equivalent to $\mathbb{P}$ and $\{X_t^{j,m}\}_{t\in[0,r_j]}$ is a local martingale under $\mathbb{Q}_m^j$,

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\Big[L^j_{\tau^j_m}\ln L^j_{\tau^j_m}\Big] = \mathbb{E}^{\mathbb{Q}^j_m}\Big[\ln L^j_{\tau^j_m}\Big] =&\mathbb{E}^{\mathbb{Q}^j_m}\Big[\int_{r_{j-1}}^{\tau^j_m}\widehat{\sigma}_s^{-2}\beta_{\mathscr{J}}\cdot dX_s^{j,m}+\frac{1}{2}|\beta_{\mathscr{J}}\widehat{\sigma}_s^{-1}|^2ds\Big]\\ =&\frac{1}{2}\mathbb{E}^{\mathbb{Q}^j_m}\Big[\int_{r_{j-1}}^{\tau^j_m}|\widehat{\sigma}_s^{-1}\beta_{\mathscr{J}}|^2ds\Big].\end{aligned} \tag{2.24}$$

Note that in the sub-intersect $[r_{j-1}, r_j]$, the BSDEs w.r.t. different time share the same terminal condition, then $\mathbb{P}$-a.s.,

$$\begin{aligned}\mathcal{Y}^{t,\omega,\mathbb{P},j}_{r_{j-1}}-\mathcal{Y}^{t,\omega,\mathbb{P},j}_{\tau^j_m} &= \int_{r_{j-1}}^{\tau^j_m} f_s^{t,\omega}(\mathcal{Y}_s^{t,\omega,\mathbb{P},j},\mathcal{Z}_s^{t,\omega,\mathbb{P},j},\widehat{\sigma}_s)ds - \int_{r_{j-1}}^{\tau^j_m}\Big(\mathcal{Z}_s^{t,\omega,\mathbb{P},j}\cdot dX_s + d\mathcal{M}_s^{t,\omega,\mathbb{P},j}\Big)\\ &= \int_{r_{j-1}}^{\tau^j_m}\Big(\breve{f}_s^{t,\omega}(\mathcal{Y}_s^{t,\omega,\mathbb{P},j},-\beta_{\mathscr{J}}(s)) - \mathcal{Z}_s^{t,\omega,\mathbb{P},j}\cdot(-\beta_{\mathscr{J}}(s))\Big)ds - \mathcal{Z}_s^{t,\omega,\mathbb{P},j}\cdot dX_s - d\mathcal{M}_s^{t,\omega,\mathbb{P},j}\\ &\geqslant \int_{r_{j-1}}^{\tau^j_m}\Big(-\alpha-\beta|\mathcal{Y}_s^{t,\omega,\mathbb{P},j}|+\frac{1}{2\gamma}|\widehat{\sigma}_s^{-1}(-\beta_{\mathscr{J}}(s))|^2\Big)ds - \mathcal{Z}_s^{t,\omega,\mathbb{P},j}\cdot dX_s^{j,m} - d\mathcal{M}_s^{t,\omega,\mathbb{P},j},\end{aligned} \tag{2.25}$$

where the choice of $-\beta_{\mathscr{J}}$ in the penultimate equality is important under the definition of $\ell_t^j := \int_0^t \mathbf{1}_{\{s\geqslant r_{j-1}\}}\widehat{\sigma}_s^{-2}\beta_{\mathscr{J}}\cdot dX_s$ and $\{L^j_{\tau^j_m\wedge t}\}_{t\in[0,r_j]}$ as well as its corresponding local martingale $\{X_t^{j,m}\}_{t\in[0,r_j]}$. Besides, the penultimate equality is due to the property of subdifferential element $-\beta_{\mathscr{J}}$, as we claimed exactly before the theorem and the last inequality is (2.21). By taking expectation $\mathbb{E}^{\mathbb{Q}^j_m}$, we have

$$\begin{aligned}\frac{1}{2\gamma}\mathbb{E}^{\mathbb{Q}^j_m}\Big[\int_{r_{j-1}}^{\tau^j_m}|\widehat{\sigma}_s^{-1}\beta_{\mathscr{J}}(s)|^2ds\Big] \leqslant&\mathbb{E}^{\mathbb{Q}^j_m}\Big[\mathcal{Y}^{t,\omega,\mathbb{P},j}_{r_{j-1}}-\mathcal{Y}^{t,\omega,\mathbb{P},j}_{\tau^j_m}+\int_{r_{j-1}}^{\tau^j_m}\Big(\alpha+\beta|\mathcal{Y}_s^{t,\omega,\mathbb{P},j}|\Big)ds\Big]\\ \leqslant&\mathbb{E}^{\mathbb{Q}^j_m}\Big[(2+\beta\Delta_j)\cdot\sup_{s\in[0,r_j]}\Big|\mathcal{Y}_s^{t,\omega,\mathbb{P},j}\Big|\Big]+\alpha\Delta_j,\end{aligned} \tag{2.26}$$

where $\Delta_j := r_j - r_{j-1}$. Inside of the above expectation $\mathbb{E}^{\mathbb{Q}^j_m}$,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^j_m}\Big[\sup_{s\in[0,r_j]}\Big|\mathcal{Y}_s^{t,\omega,\mathbb{P},j}\Big|\Big] =&\mathbb{E}^{\mathbb{P}}\Big[\Big(\sup_{s\in[0,r_j]}\Big|\mathcal{Y}_s^{t,\omega,\mathbb{P},j}\Big|\Big)\cdot L^j_{\tau^j_m}\Big]\\ \leqslant&\mathbb{E}^{\mathbb{P}}\Big[e^{p\big(\mathcal{Y}_*^{t,\omega,\mathbb{P},j,+}+\mathcal{Y}_*^{t,\omega,\mathbb{P},j,-}\big)}\Big]+\mathbb{E}^{\mathbb{P}}\Big[L^j_{\tau^j_m}\frac{1}{p}\Big(\ln L^j_{\tau^j_m}-\ln p\Big)\Big]\\ =&\mathbb{E}^{\mathbb{P}}\Big[e^{p\big(\mathcal{Y}_*^{t,\omega,\mathbb{P},j,+}+\mathcal{Y}_*^{t,\omega,\mathbb{P},j,-}\big)}\Big]+\frac{1}{p}\mathbb{E}^{\mathbb{Q}^j_m}\Big[\Big(\ln L^j_{\tau^j_m}-\ln p\Big)\Big]\\ =&\mathbb{E}^{\mathbb{P}}\Big[e^{p\big(\mathcal{Y}_*^{t,\omega,\mathbb{P},j,+}+\mathcal{Y}_*^{t,\omega,\mathbb{P},j,-}\big)}\Big]-\frac{1}{p}\ln p+\frac{1}{2p}\mathbb{E}^{\mathbb{Q}^j_m}\Big[\int_{r_{j-1}}^{\tau^j_m}|\widehat{\sigma}_s^{-1}\beta_{\mathscr{J}}|^2ds\Big],\end{aligned} \tag{2.27}$$

where in the ante-penultimate inequality we use the fact that for all $x \in \mathbb{R}, y \in (0,\infty), xy = px \cdot \frac{y}{p} \leqslant e^{px} + \frac{y}{p}(\ln y - \ln p) \leqslant e^{px} + \frac{y}{p}\ln y$. So we have

$$\left(\frac{1}{2\gamma} - \frac{1}{2p}(2+\beta\Delta_j)\right)\mathbb{E}^{\mathbb{Q}^j_m}\left[\int_{r_{j-1}}^{\tau^j_m} |\widehat{\sigma}_s^{-1}\beta_{\mathscr{J}}(s)|^2 ds\right] \leqslant \mathbb{E}^{\mathbb{P}}\left[e^{p\left(\mathcal{Y}_*^{t,\omega,\mathbb{P},j,+} + \mathcal{Y}_*^{t,\omega,\mathbb{P},j,-}\right)}\right] \cdot (2+\beta\Delta_j) + \alpha\Delta_j, \tag{2.28}$$

and if we consider the above estimate in all intersects which satisfies $\Delta_j < \left(\frac{p}{\gamma} - 2\right)\frac{1}{\beta}$ for $j \in \{1, \cdots, n\}$, then

$$\mathbb{E}^{\mathbb{Q}^j_m}\left[\int_{r_{j-1}}^{\tau^j_m} |\widehat{\sigma}_s^{-1}\beta_{\mathscr{J}}(s)|^2 ds\right] \leqslant \mathbb{E}^{\mathbb{P}}\left[e^{p\left(\mathcal{Y}_*^{t,\omega,\mathbb{P},j,+} + \mathcal{Y}_*^{t,\omega,\mathbb{P},j,-}\right)}\right] \frac{1}{\frac{1}{2\gamma} - \frac{1}{2p}(2+\beta\Delta_j)}\frac{p}{\gamma} + \frac{\alpha\Delta_j}{\frac{1}{2\gamma} - \frac{1}{2p}(2+\beta\Delta_j)} =: \mathscr{B}, \tag{2.29}$$

and note that the condition $p \in \left(2\gamma, +\infty\right)$ guarantees the positivity of $\left(\frac{p}{\gamma} - 2\right)\frac{1}{\beta}$ above. To conclude, we obtained $\mathbb{E}^{\mathbb{P}}\left[L^j_{\tau^j_m} \ln L^j_{\tau^j_m}\right] = \frac{1}{2}\mathbb{E}^{\mathbb{Q}^j_m}\left[\int_{r_{j-1}}^{\tau^j_m} |\widehat{\sigma}_s^{-1}\beta_{\mathscr{J}}(s)|^2 ds\right] \leqslant \frac{1}{2}\mathscr{B}$. By de la Vallée-Poussin's lemma(here the corresponding non-decreasing function for this lemma is $\phi(x) := x\ln x, x \in (0,\infty)$), $\{L^j_{\tau^j_m}\}_{m\in\mathbb{N}}$ is uniformly integrable w.r.t. m(note that this is different to the fact proved above that $\{L^j_{\tau^j_m \wedge t}\}_{t\in[0,r_j]}$ is a uniformly integrable martingale for a given $m \in \mathbb{N}$ and this claim will be used in the 2nd equality below), so $\mathbb{E}^{\mathbb{P}}[L^j_{r_j}] = \lim_{m\to\infty}\mathbb{E}^{\mathbb{P}}[L^j_{\tau^j_m}] = \lim_{m\to\infty}\mathbb{E}^{\mathbb{P}}[L^j_{r_{j-1}}] = \lim_{m\to\infty}\mathbb{E}^{\mathbb{P}}[1] = 1$, which show that L^j is a martingale. Once again by $\frac{d\mathbb{Q}^j}{d\mathbb{P}} := L^j_{t_j}$ and $X^j_t := X_t - \int_0^t \mathbf{1}_{\{s \geqslant r_{j-1}\}}\beta_{\mathscr{J}}(s)ds$, we could claim that $\mathbb{Q}^j$ is a probability measure equivalent to $\mathbb{P}$ and $\{X^j_t\}_{t\in[0,r_j]}$ is a local martingale under $\mathbb{Q}^j$.

Step 3. Based on the last results, we have for any $r \in [r_{j-1}, r_j]$,

$$\begin{aligned}
\mathcal{Y}_r^{t,\omega,\mathbb{P},j} &= V_{r_j} + \int_r^{r_j} [f_s^{t,\omega}(\mathcal{Y}_s^{t,\omega,\mathbb{P},j}, 0) + \mathcal{Z}_s^{t,\omega,\mathbb{P},j} \cdot \beta_{\mathscr{J}}]ds - \int_r^{r_j} \mathcal{Z}_s^{t,\omega,\mathbb{P},j} \cdot dX_s - \int_r^{r_j} d\mathcal{M}_s^{t,\omega,\mathbb{P},j} \\
&= V_{r_j} + \int_r^{r_j} f_s^{t,\omega}(\mathcal{Y}_s^{t,\omega,\mathbb{P},j}, 0)ds - \int_r^{r_j} \mathcal{Z}_s^{t,\omega,\mathbb{P},j} \cdot dX^j_s - \int_r^{r_j} d\mathcal{M}_s^{t,\omega,\mathbb{P},j} \\
&= V_{r_j} + \int_r^{r_j} f_s^{t,\omega}(0,0)ds + \int_r^{r_j} [f_s^{t,\omega}(\mathcal{Y}_s^{t,\omega,\mathbb{P},j}, 0) - f_s^{t,\omega}(0,0)]ds - \int_r^{r_j} \mathcal{Z}_s^{t,\omega,\mathbb{P},j} \cdot dX^j_s - \int_r^{r_j} d\mathcal{M}_s^{t,\omega,\mathbb{P},j} \\
&= V_{r_j} + \int_r^{r_j} f_s^{t,\omega}(0,0)ds + \int_r^{r_j} \lambda^j_s\mathcal{Y}_s^{t,\omega,\mathbb{P},j}ds - \int_r^{r_j} \mathcal{Z}_s^{t,\omega,\mathbb{P},j} \cdot dX^j_s - \int_r^{r_j} d\mathcal{M}_s^{t,\omega,\mathbb{P},j}.
\end{aligned} \tag{2.30}$$

We also introduce another linear BSDE

$$\begin{aligned}\overline{\mathcal{Y}}_r^{t,\omega,\mathbb{P},j} &= V_{r_j} + \int_r^{r_j} (-|f_s^{t,\omega}(0,0)|)ds + \int_r^{r_j} \lambda_s^j \overline{\mathcal{Y}}_s^{t,\omega,\mathbb{P},j} ds + \overline{\mathcal{Z}}_s^{t,\omega,\mathbb{P},j} \cdot \beta_{\mathscr{J}}]ds - \int_r^{r_j} \overline{\mathcal{Z}}_s^{t,\omega,\mathbb{P},j} \cdot dX_s - \int_r^{r_j} d\overline{\mathcal{M}}_s^{t,\omega,\mathbb{P},j} \\ &= V_{r_j} + \int_r^{r_j} (-|f_s^{t,\omega}(0,0)|)ds + \int_r^{r_j} \lambda_s^j \overline{\mathcal{Y}}_s^{t,\omega,\mathbb{P},j} ds - \int_r^{r_j} \overline{\mathcal{Z}}_s^{t,\omega,\mathbb{P},j} \cdot dX_s^j - \int_r^{r_j} d\overline{\mathcal{M}}_s^{t,\omega,\mathbb{P},j}.\end{aligned} \tag{2.31}$$

By the comparison principle for BSDEs (see e.g., Kruse and Popier [KP16]) and the fact that V is an $\overline{f}$-supermartingale, it's clear that $\overline{\mathcal{Y}}_{r_{j-1}}^{t,\omega,\mathbb{P},j} \leqslant \mathcal{Y}_{r_{j-1}}^{t,\omega,\mathbb{P},j} \leqslant V_{r_{j-1}}$. Solving the above linear BSDE (2.31), it follows that

$$\overline{\mathcal{Y}}_{r_{j-1}}^{t,\omega,\mathbb{P},j} = \mathbb{E}^{\mathbb{Q}^j}\left[V_{r_j} e^{\int_{r_{j-1}}^{r_j} \lambda_r^j dr} - \int_{r_{j-1}}^{r_j} e^{\int_{r_{j-1}}^{s} \lambda_r^j dr} |f_s^{t,\omega}(0,0)| ds \middle| \mathcal{F}_{r_{j-1}} \right]. \tag{2.32}$$

Step 4. Our final goal is to give an estimate of the down-crossing number $D_0^b(V, \mathscr{J})$, which corresponds to the f-supermartingale V. To do this, the idea is that we will try to firstly define a $\mathbb{Q}^j$-supermartingale(namely y defined below) and show the relation between this constructed $\mathbb{Q}^j$-supermartingale and the above f-supermartingale V; then refer to the classical down-crossing inequality for this $\mathbb{Q}^j$-supermartingale.

To be precise, let $\lambda_s := \sum_{j=1}^n \lambda_s^j \mathbf{1}_{[r_{j-1}, r_j)}(s)$, it follows that the discrete process $\{y_{r_j}\}_{0 \leqslant j \leqslant n}$ defined by

$$y_{r_j} := V_{r_j} e^{\int_0^{r_j} \lambda_r dr} - \int_0^{r_j} e^{\int_0^s \lambda_r dr} |f_s^{t,\omega}(0,0)| ds$$

is a $\mathbb{Q}^j$-supermartingale. Indeed, the fact that V is an f-supermartingale gives that for $\widetilde{r} \in [r_{j-1}, r_j]$,

$$\overline{\mathcal{Y}}_{\widetilde{r}}^{t,\omega,\mathbb{P},j} = \mathfrak{E}_{\widetilde{r},r_j}^{0}[V_{r_j}] \leqslant V_{\widetilde{r}}. \tag{2.33}$$

If we take $\widetilde{r} \equiv r_{j-1}$ and use the (2.32), we obtain that

$$\mathbb{E}^{\mathbb{Q}^j}\left[V_{r_j} e^{\int_{r_{j-1}}^{r_j} \lambda_r^j dr} - \int_{r_{j-1}}^{r_j} e^{\int_{r_{j-1}}^{s} \lambda_r^j dr} |f_s^{t,\omega}(0,0)| ds \middle| \mathcal{F}_{r_{j-1}} \right] \leqslant V_{r_{j-1}}. \tag{2.34}$$

After multiplying a constant $e^{\int_0^{r_{j-1}} \lambda_r dr}$(not $e^{\int_0^{r_{j-1}} \lambda_r^j dr}$) in both sides,

$$\mathbb{E}^{\mathbb{Q}^j}\left[V_{r_j} e^{\int_0^{r_j} \lambda_r dr} - \int_{r_{j-1}}^{r_j} e^{\int_0^{s} \lambda_r dr} |f_s^{t,\omega}(0,0)| ds \middle| \mathcal{F}_{r_{j-1}} \right] \leqslant V_{r_{j-1}} e^{\int_0^{r_{j-1}} \lambda_r dr}. \tag{2.35}$$

This shows that $\left\{V_{r_j}e^{\int_0^{r_j}\lambda_r dr} - \int_0^{r_j} e^{\int_0^s \lambda_r dr}|f_s^{t,\omega}(0,0)|ds\right\}_{0\leqslant j\leqslant n}$ (i.e., $\{y_{r_j}\}_{0\leqslant j\leqslant n}$) is a $\mathbb{Q}^j$-supermartingale. While the upper and lower boundary for V are respectively b and 0, here we have to define the upper and lower boundary for the above constructed $\mathbb{Q}^j$-supermartingale y, i.e., for $q>0$, let

$$u_q := be^{\int_0^q \lambda_r dr} - \int_0^q e^{\int_0^s \lambda_r dr}|f_s^{t,\omega}(0,0)|ds,$$

and

$$l_q := -\int_0^q e^{\int_0^s \lambda_r dr}|f_s^{t,\omega}(0,0)|ds.$$

Denote then by $D_l^u(y,\mathscr{J})$ the number of down-crossing of the process y from the upper boundary u to the lower boundary l. Notice that

$$\left.\begin{array}{l}\{V_q\leqslant 0\}\Rightarrow\{y_q\leqslant l_q\}\\ \{V_q\geqslant b\}\Rightarrow\{y_q\geqslant u_q\}\end{array}\right\}\begin{array}{l}\Rightarrow\{V \text{ downcrosses from } b \text{ to } 0\Rightarrow y \text{ downcrosses from } u \text{ to } l\}\\ \qquad\qquad\Rightarrow D_0^b(V,\mathscr{J})\leqslant D_l^u(y,\mathscr{J}),\end{array}$$

so we can apply the classical down-crossing theorem for supermartingales to y, and obtain that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^j}[D_0^b(V,\mathscr{J})]\leqslant\mathbb{E}^{\mathbb{Q}^j}[D_l^u(y,\mathscr{J})]&\leqslant\frac{\mathbb{E}^{\mathbb{Q}^j}\left[y_{r_{j-1}}\wedge u_{r_{j-1}}-y_{r_j}\wedge u_{r_j}\right]}{\min_{q\in[r_{j-1},r_j]}(u_q-l_q)}\\
\leqslant&\frac{e^{LT}}{b}\mathbb{E}^{\mathbb{Q}^j}\left[(y_{r_{j-1}}\wedge be^{LT}-y_{r_j})\mathbf{1}_{\{u_{r_j}\geqslant y_{r_j}\}}+(y_{r_{j-1}}\wedge be^{LT}-u_{r_j})\mathbf{1}_{\{u_{r_j}<y_{r_j}\}}\right]\\
=&\frac{e^{LT}}{b}\mathbb{E}^{\mathbb{Q}^j}\left[(y_{r_{j-1}}\wedge be^{LT}-y_{r_j})-(u_{r_j}-y_{r_j})\wedge 0\right]\\
\leqslant&\frac{e^{LT}}{b}\mathbb{E}^{\mathbb{Q}^j}\left[(V_{r_{j-1}}\wedge b)e^{LT}-e^{\int_0^T\lambda_s ds}(V_{r_j}\wedge b)+e^{LT}\int_0^T|f_s^{t,\omega}(0,0)|ds\right]<\infty,\end{aligned}$$

which holds true for all $j\in\{1,2,\cdots,n\}$, and now this estimate implies the required claim thanks to the equivalence of the measures $\mathbb{Q}^j$ and $\mathbb{P}$. □

A càdlàg version of the value function

Using the estimate of Theorem 2.4.6 and following the same argument as in Karatzas and Shreve [KS12] (Theorem 1.3.8(v), p.14), we could then claim that for any $p \geqslant 1 \vee 2\gamma$, the right limit

$$V_t^+(\omega) := \lim_{r\in\mathbb{Q}\cap[t,T], r\downarrow t} V_r(\omega) = \lim_{r\in\mathbb{Q}, r\downarrow t} \sup_{\mathbb{P}\in\mathcal{P}(r,\omega)} \mathbb{Y}^{r,\omega,\mathbb{P}}[\xi, T] = \lim_{r\in\mathbb{Q}\cap[t,T], r\downarrow t} \sup_{\mathbb{P}\in\mathcal{P}(r,\omega)} \mathbb{E}^{\mathbb{P}}\Big[\mathcal{Y}_0^{r,\omega,\mathbb{P}}\Big],$$
$$(\widetilde{V}_t^+)^p(\omega) := \lim_{r\in\mathbb{Q}\cap[t,T], r\downarrow t} \widetilde{V}_r^p(\omega) = \lim_{r\in\mathbb{Q}, r\downarrow t} \sup_{\mathbb{P}\in\mathcal{P}(r,\omega)} (\widetilde{\mathbb{Y}}^{r,\omega,\mathbb{P}}[\widetilde{\xi}, T])^p \leqslant \lim_{r\in\mathbb{Q}\cap[t,T], r\downarrow t} \sup_{\mathbb{P}\in\mathcal{P}(r,\omega)} \mathbb{E}^{\mathbb{P}}\Big[e^{p\gamma\mathcal{Y}_0^{r,\omega,\mathbb{P}}}\Big],$$

exists $\mathcal{P}_0$-q.s. and the process V^+ (resp., $\widetilde{V}^+$) is càdlàg $\mathcal{P}_0$-q.s.

Proposition 2.4.7 (Dynamic programming for V^+)**.** *Let Assumption 2.1.3 (1)-(4) hold true, and ξ satisfy the corresponding exponential integrability in Theorem 2.3.1 and 2.3.2 , then: if $\mathbb{E}[e^{2\lambda\gamma\xi^-} + e^{2\lambda'\gamma e^{\beta T}\xi^+}] < \infty$, we have $V^+ \in \mathcal{D}^{\exp(\lambda\gamma,\lambda'\gamma)}(\mathcal{P}_0, \mathbb{F}^{\mathcal{P}_0+})$, for $\lambda, \lambda' > 0$ with $\frac{1}{\lambda} + \frac{1}{\lambda'} < \frac{1}{6\vee 2\gamma}$; if $\xi^+ \vee \xi^-$ has exponential moments of orders $p \in [1\vee 2\gamma, \infty)$, we have $V^+ \in \mathcal{D}^{\exp(p)}(\mathcal{P}_0, \mathbb{F}^{\mathcal{P}_0+})$, for $p \geqslant 1\vee 2\gamma$. And for $0 \leqslant t_1 \leqslant t_2 \leqslant T$, and $\mathbb{P} \in \mathcal{P}_0$, we have*

$$V_{t_1}^+ = \operatorname*{ess\,sup^{\mathbb{P}}}_{\mathbb{P}'\in\mathcal{P}_{\mathbb{P}}^+(t_1)} \mathcal{Y}_{t_1}^{\mathbb{P}'}[V_{t_2}^+, t_2], \mathbb{P}\text{-a.s.}$$

Proof. We consider here only the second case, i.e., $V^+ \in \mathcal{D}^{\exp(p)}(\mathcal{P}_0, \mathbb{F}^{\mathcal{P}_0+})$, for $p \geqslant 1 \vee 2\gamma$.

1. To verify that $V^+ \in \mathcal{D}^{\exp(p)}(\mathcal{P}_0, \mathbb{F}^{\mathcal{P}_0+})$ and $\lim_{n\to\infty} \mathcal{E}^{\mathbb{P}}[|\widetilde{V}_{t^n}^+ - \widetilde{V}_t^+|^p] = 0, \forall p \geqslant 1 \vee 2\gamma, t \leqslant T$. Indeed, letting $\{t_n\}_{n\geqslant 1}$ be the sequence decreasingly approximating t, satisfying $t_n - t \leqslant 2^{-n}$ and $t \leqslant T$ as well as for all $\mathbb{P} \in \mathcal{P}_0$, we deduce that

$$\mathbb{E}^{\mathbb{P}}[(\widetilde{V}_t^+)^p] = \lim_{n\to\infty} \mathbb{E}^{\mathbb{P}}[(\widetilde{V}_{t_n})^p] \leqslant \sup_{t'\leqslant T} \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}[(\widetilde{V}_{t'})^p] =: \nu < \infty,$$

by (2.12) in Theorem 2.4.5, and this implies that $\sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}[(\widetilde{V}_{t_n}^+)^p] \leqslant \nu$.

Define $\delta_n := |\widetilde{V}_{t^n}^+ - \widetilde{V}_t^+|$. Then for any $m \geqslant 1$, we have

$$\mathcal{E}^{\mathbb{P}}[\delta_n^p] \leqslant \mathcal{E}^{\mathbb{P}}[\delta_n^p \mathbf{1}_{\{t\geqslant m\}}] + \mathcal{E}^{\mathbb{P}}[\delta_n^p \mathbf{1}_{\{t<m\}}] \leqslant \ 2\nu^{\frac{p}{p'}} \mathcal{E}^{\mathbb{P}}[\mathbf{1}_{\{t\geqslant m\}}]^{1-\frac{p}{p'}} + C_m(\mathcal{E}^{\mathbb{P}}[\delta_n^{p'}])^{\frac{p}{p'}},$$

letting $n, m \to \infty$, we get the required convergence.

2. Prove that $V_{t_1}^+ \geqslant \mathcal{Y}_{t_1}^{\mathbb{P}}[V_{t_2}^+, t_2], \mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}_0$. Clearly we had verified the wellposedness of

the RHS in step 1, precisely, by the integrability of V^+. By (2.15) in Theorem 2.4.5, we have

$$V_{t_1^m} \geqslant \mathbb{E}^{\mathbb{P}}[\mathcal{Y}^{\mathbb{P}}_{t_1^m}[V_{t_2^n}, t_2^n]|\mathcal{F}_{t_1^m}], \mathbb{P}\text{-a.s.}, \tag{2.36}$$

where t_1^m and t_2^n are defined from t_1 and t_2 as before. Due to the stability Theorem 2.3.3 (for BSDEs' case by letting the barrier tend to $-\infty$) together with Step 1, we have

$$\lim_{n\to\infty} \|\mathcal{Y}^{\mathbb{P}}_{t_1^m}[V_{t_2^n}, t_2^n] - \mathcal{Y}^{\mathbb{P}}_{t_1^m}[V^+_{t_2}, t_2]\|_{\mathbb{L}^1(\mathbb{P})} \leqslant \lim_{n\to\infty} \|\mathcal{Y}^{\mathbb{P}}_{t_1^m}[V_{t_2^n}, t_2^n] - \mathcal{Y}^{\mathbb{P}}_{t_1^m}[V^+_{t_2}, t_2]\|_{\mathcal{L}^1(\mathbb{P})} = 0, \tag{2.37}$$

Combining (2.36) and (2.37), we have

$$V^+_{t_1} = \lim_{m\to\infty} V_{t_1^m} \geqslant \lim_{m\to\infty}\lim_{n\to\infty} \mathbb{E}^{\mathbb{P}}[\mathcal{Y}^{\mathbb{P}}_{t_1^m}[V_{t_2^n}, t_2^n]|\mathcal{F}_{t_1^m}] = \lim_{m\to\infty} \mathbb{E}^{\mathbb{P}}[\mathcal{Y}^{\mathbb{P}}_{t_1^m}[V^+_{t_2}, t_2]|\mathcal{F}_{t_1^m}] = \mathbb{E}^{\mathbb{P}}[\mathcal{Y}^{\mathbb{P}}_{t_1}[V^+_{t_2}, t_2]|\mathcal{F}_{t_1}],$$

where we used the fact that $\mathcal{Y}^{\mathbb{P}}[V^+_{t_2}, t_2] \in \mathcal{D}^{\exp(\mathrm{p})}(\mathbb{P}, \mathbb{F}^{\mathbb{P}})$ in the last equality.

3. Here we prove the reverse inequality. The comparison principle Theorem 2.3.2(for BSDEs' case by letting the barrier tend to $-\infty$), together with step 2, gives that

$$\operatorname*{ess\,sup^{\mathbb{P}}}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(t_1)} \mathcal{Y}^{\mathbb{P}'}_{t_1}[V^+_{t_2}, t_2] \geqslant \operatorname*{ess\,sup^{\mathbb{P}}}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(t_1)} \mathcal{Y}^{\mathbb{P}'}_{t_1}[\mathcal{Y}^{\mathbb{P}'}_{t_2}[\xi, T], t_2] = \operatorname*{ess\,sup^{\mathbb{P}}}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(t_1)} \mathcal{Y}^{\mathbb{P}'}_{t_1}[\xi, T],$$

where the last equality is similar to the 1st equality in (2) of Lemma 2.4.3. Then we only need to prove that $V^+_{t_1} \leqslant \operatorname{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(t_1)} \mathcal{Y}^{\mathbb{P}'}_{t_1}[\xi, T]$.

$$\begin{aligned}
V^+_{t_1} &= \lim_{n\to\infty} \mathbb{E}[V_{t_1^n}|\mathcal{F}^+_{t_1}] = \lim_{n\to\infty} \mathbb{E}[\operatorname*{ess\,sup^{\mathbb{P}}}_{\mathbb{P}'\in\mathcal{P}_{\mathbb{P}}(t_1^n)} \mathbb{E}^{\mathbb{P}'}[\mathcal{Y}^{\mathbb{P}'}_{t_1^n}|\mathcal{F}_{t_1^n}]|\mathcal{F}^+_{t_1}] \\
&\leqslant \lim_{n\to\infty} \mathbb{E}[\operatorname*{ess\,sup^{\mathbb{P}}}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(t_1)} \mathbb{E}^{\mathbb{P}'}[\mathcal{Y}^{\mathbb{P}'}_{t_1^n}|\mathcal{F}_{t_1^n}]|\mathcal{F}^+_{t_1}] \\
&= \lim_{n\to\infty} \operatorname*{ess\,sup^{\mathbb{P}}}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(t_1)} \mathbb{E}^{\mathbb{P}'}[\mathcal{Y}^{\mathbb{P}'}_{t_1^n}|\mathcal{F}^+_{t_1}] \\
&= \lim_{n\to\infty} \operatorname*{ess\,sup^{\mathbb{P}}}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(t_1)} \left\{\mathcal{Y}^{\mathbb{P}'}_{t_1} + \mathbb{E}^{\mathbb{P}'}\left[\int_{t_1^n}^{t_1} f_s(\mathcal{Y}^{\mathbb{P}'}_s, \mathcal{Z}^{\mathbb{P}'}_s, \widehat{\sigma}_s)ds - \mathcal{Z}^{\mathbb{P}'}_s \cdot dX_s - d\mathcal{M}^{\mathbb{P}'}_s \middle| \mathcal{F}^+_{t_1}\right]\right\} < \infty,
\end{aligned}$$

where we used the fact that $t_1^n - t_1 \leqslant 2^{-n}$ and the estimate of the last inequality is just similar to Theorem 2.3.1.

4. So far, we had proved the equality required in this proposition, so we have that $V_t^+ = \operatorname{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}^+_{\mathbb{P}}(t)} \mathcal{Y}_t^{\mathbb{P}'}[\xi, T]$. Here we prove that V^+ also has the integrability property of V, i.e. $V^+ \in \mathcal{D}^{\exp(\mathrm{p})}(\mathcal{P}_0, \mathbb{F}^{\mathcal{P}_0+})$. For any $p \geqslant 1 \vee 2\gamma$,

$$|\widetilde{V}_t^+|^p = |\operatorname*{ess\,sup^{\mathbb{P}}}_{\mathbb{P}' \in \mathcal{P}^+_{\mathbb{P}}(t)} \widetilde{\mathcal{Y}}_t^{\mathbb{P}'}[\widetilde{\xi}, T]|^p = \operatorname*{ess\,sup^{\mathbb{P}}}_{\mathbb{P}' \in \mathcal{P}^+_{\mathbb{P}}(t)} |\widetilde{\mathcal{Y}}_t^{\mathbb{P}'}[\widetilde{\xi}, T]|^p < \infty,$$

where the last inequality is obtained from the a-priori estimate in Theorem 2.3.1.

□

2.4.5 Proof of Theorem 2.2.1: existence

Proof. As before, we consider here only the second case, i.e., $V^+ \in \mathcal{D}^{\exp(\mathrm{p})}(\mathcal{P}_0, \mathbb{F}^{\mathcal{P}_0+})$, for $p \geqslant 1 \vee 2\gamma$, and we adapt the argument in [STZ12] and [PTZ18].

1. We first prove the existence of a process $Z, (M^{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}_0}, (K^{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}_0}$ s.t. for all $p \geqslant 1 \vee 2\gamma$,

$$\begin{aligned} &(Z, M^{\mathbb{P}}, K^{\mathbb{P}}) \in \mathcal{H}^{2,2p}(\mathbb{P}, \mathbb{R}^d, \mathbb{F}^{\mathbb{P}}_+) \times \mathbb{M}^p(\mathbb{P}) \times \mathbb{I}^p_{\text{rcll}}(\mathbb{P}), \\ &\text{and } V_t^+ = \xi + \int_t^T f_s(V_s^+, Z_s, \widehat{\sigma}_s) ds - \int_t^T Z_s \cdot dX_s + dM_s^{\mathbb{P}} - dK_s^{\mathbb{P}}, t \geqslant 0, \mathbb{P}\text{-a.s.} \end{aligned} \tag{2.38}$$

Fix $\mathbb{P} \in \mathcal{P}_0$. By Proposition 2.4.7, $V^+ \in \mathcal{D}^{\exp(\mathrm{p})}(\mathcal{P}_0, \mathbb{F}^{\mathcal{P}_0+})$ for $p \geqslant 1 \vee 2\gamma$. It follows from the existence of RBSDE equipped with lower càdlàg barrier [EHO15](Theorem 2.1, the corresponding claim on minimality solution, for which we may verify line by line that their proof applies well to the case here of an infinite upper bound and an additional orthogonal martingale) that there exists a minimal solution $(Y^{\mathbb{P}}, Z^{\mathbb{P}}, M^{\mathbb{P}}, K^{\mathbb{P}})$ for all $\mathbb{P} \in \mathcal{P}_0$(with spaces to identify in a while), to the RBSDE

$$\begin{aligned} Y_t^{\mathbb{P}} = \xi + \int_t^T f_s(Y_s^{\mathbb{P}}, Z_s^{\mathbb{P}}, \widehat{\sigma}_s) ds - (Z_s^{\mathbb{P}} \cdot dX_s + dM_s^{\mathbb{P}} - dK_s^{\mathbb{P}}), Y^{\mathbb{P}} \geqslant V^+, \mathbb{P}\text{-a.s.} \\ \text{and } \int_0^t \Big(1 \wedge (Y_{r-}^{\mathbb{P}} - V_{r-}^+)\Big) dK_r^{\mathbb{P}} = 0, \text{ for all } t \geqslant 0. \end{aligned} \tag{2.39}$$

We now claim that $Y^{\mathbb{P}} = V^+, \mathbb{P}$-a.s.. Indeed, by contradiction we assume that $2\epsilon := Y_0^{\mathbb{P}} - V_0^+ > 0$, then $\tau_\epsilon := \inf\{t > 0 : Y_t^{\mathbb{P}} \leqslant V_t^+ + \epsilon\} > 0, \mathbb{P}$-a.s. Clearly $\tau_\epsilon \leqslant T$, because the two processes are equal to ξ at time T. By Skorokhod condition, $K^{\mathbb{P}} \equiv 0$ on $[0, \tau_\epsilon]$, thus we can reduce the RBSDE to a BSDE on this time interval. Denoting the corresponding solution by $\mathcal{Y}^{\mathbb{P}}[V^+_{\tau_\epsilon}, \tau_\epsilon]$, we obtain that,

$$Y_0^{\mathbb{P}} \leqslant \mathcal{Y}_0^{\mathbb{P}}[V^+_{\tau_\epsilon}, \tau_\epsilon] + \mathbb{E}^{\mathbb{P}}[Y_{\tau_\epsilon} - V^+_{\tau_\epsilon}] \leqslant \mathcal{Y}_0^{\mathbb{P}}[V^+_{\tau_\epsilon}, \tau_\epsilon] + \epsilon \leqslant V_0^+ + \epsilon,$$

where we use the dynamic programming principle of Proposition 2.4.7 in the last inequality. Then this is contradictory to the definition of ϵ. So we proved $Y^{\mathbb{P}} = V^+$. And now by the exponential integrability proved in Proposition 2.4.7, we could claim that the lying space of $(Y^{\mathbb{P}}, Z^{\mathbb{P}}, M^{\mathbb{P}}, K^{\mathbb{P}})$ as $\mathcal{D}^{\exp(p)}(\mathbb{P}) \times \mathcal{H}^{2,2p}(\mathbb{P}, \mathbb{R}^d, \mathbb{F}^{\mathbb{P}}_+) \times \mathbb{M}^p(\mathbb{P}) \times \mathbb{I}^p_{\text{rcll}}(\mathbb{P})$, for all $\mathbb{P} \in \mathcal{P}_0$, and $p \in [1 \vee 2\gamma, \infty)$. The next step is to prove that the family $\{Z^{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}_0}$ can be aggregated to some general process in the universal sense, then V^+ is a càdlàg semimartingale which satisfies (2.38). In fact, we know that the quadratic covariation process $\langle V^+, X\rangle$ can be defined on $\mathbb{R}_+ \times \Omega$ by [Kar95]. Indeed, $\langle V^+, X\rangle$ is $\mathcal{P}_0$-q.s. continuous and therefore is $\mathbb{F}^{\mathcal{P}_0+}$-predictable, or equivalently, $\mathbb{F}^{\mathcal{P}_0+}$-predictable. We can define a universal $\mathbb{F}^{\mathcal{P}_0+}$-predictable process Z by $Z_t dt := \widehat{\sigma}_t^{-2} d\langle V^+, X\rangle_t$, and compare with the corresponding variation for each $\mathbb{P} \in \mathcal{P}_0$, it follows $Z = Z^{\mathbb{P}}, \mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}_0$. Then we finished the proof of (2.38).

2. In this part we prove that the family of the increasing processes $\{K^{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}_0}$ satisfies the minimality condition. Let $0 \leqslant s \leqslant T, \mathbb{P} \in \mathcal{P}_0, \mathbb{P}' \in \mathcal{P}^+_{\mathbb{P}}(0)$, and denote by $(\mathcal{Y}^{\mathbb{P}'}, \mathcal{Z}^{\mathbb{P}'}, \mathcal{M}^{\mathbb{P}'})$ the solution of the BSDE with parameters (f, ξ). In the following, we will claim an equivalent probability measure $\mathbb{Q}^j(\sim \mathbb{P}')$ and its underlying local martingale X^j, all defined later. Define $\delta Y := V^+ - \mathcal{Y}^{\mathbb{P}'}, \delta Z := Z - \mathcal{Z}^{\mathbb{P}'}, \delta M := M - \mathcal{M}^{\mathbb{P}'}$. Once again, we define $\mathfrak{N}^{\mathbb{P}'}$ as the $ds \otimes d\mathbb{P}'$-null set except on which Assumption 2.1.3 holds. For any $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, let $\partial(-f_s)(y, z)$ denote the subdifferential of the function $-f_s(y, \cdot)$ at z. For any $(s, \omega) \in \left(\mathfrak{N}^{\mathbb{P}'}\right)^c$, by measurable selection theorem(see e.g., Beněs [Ben70](Lemma 1) or Elliott [Ell82](Lemma 16.34)), there exists respectively a progressively measurable process $\widetilde{\mathfrak{q}}^1_s \in \partial(-f_s)(V^+_s(\omega), Z_s(\omega))$ and $\widetilde{\mathfrak{q}}^2_s \in \partial(-f_s)(\mathcal{Y}^{\mathbb{P}'}_s(\omega), \mathcal{Z}^{\mathbb{P}'}_s(\omega))$ s.t.,

$$f_s(V^+_s, Z_s) = \breve{f}_s(V^+_s, \widetilde{\mathfrak{q}}^1_s) - Z_s \cdot \widetilde{\mathfrak{q}}^1_s, f_s(\mathcal{Y}^{\mathbb{P}'}_s, \mathcal{Z}^{\mathbb{P}'}_s) = \quad \breve{f}_s(\mathcal{Y}^{\mathbb{P}'}_s, \widetilde{\mathfrak{q}}^2_s) - \mathcal{Z}^{\mathbb{P}'}_s \cdot \widetilde{\mathfrak{q}}^2_s. \tag{2.40}$$

In particular, we note that

$$\begin{aligned} f_s(V^+_s, Z_s) - f_s(\mathcal{Y}^{\mathbb{P}'}_s, \mathcal{Z}^{\mathbb{P}'}_s) &\geqslant \breve{f}_s(V^+_s, \widetilde{\mathfrak{q}}^1_s) - Z_s \cdot \widetilde{\mathfrak{q}}^1_s - \breve{f}_s(\mathcal{Y}^{\mathbb{P}'}_s, \widetilde{\mathfrak{q}}^1_s) + \mathcal{Z}^{\mathbb{P}'}_s \cdot \widetilde{\mathfrak{q}}^1_s, \\ f_s(V^+_s, Z_s) - f_s(\mathcal{Y}^{\mathbb{P}'}_s, \mathcal{Z}^{\mathbb{P}'}_s) &\leqslant \breve{f}_s(V^+_s, \widetilde{\mathfrak{q}}^2_s) - Z_s \cdot \widetilde{\mathfrak{q}}^2_s - \breve{f}_s(\mathcal{Y}^{\mathbb{P}'}_s, \widetilde{\mathfrak{q}}^2_s) + \mathcal{Z}^{\mathbb{P}'}_s \cdot \widetilde{\mathfrak{q}}^2_s. \end{aligned} \tag{2.41}$$

Since $(V^+, Z, M^{\mathbb{P}'}, K^{\mathbb{P}'})$, which is the solution to the RBSDE(f, ξ, V^+), belongs to the space $\cap_{p\in(1\vee2\gamma,\infty)}\mathcal{D}^{\exp(p)}(\mathbb{P}') \times \mathcal{H}^{2,2p}(\mathbb{P}', \mathbb{R}^d, \mathbb{F}^{\mathbb{P}'}_+) \times \mathbb{M}^p(\mathbb{P}') \times \mathbb{I}^p_{\text{rcll}}(\mathbb{P}')$, it holds $\mathbb{P}'$-a.s., $\int_0^T |f_s(V^+_s, Z_s)|ds +$

$|V_*^+| + \int_0^T |\widehat{\sigma}_s^{-1} Z_s|^2 ds < \infty$. Based on the fact that

$$\begin{aligned} f_s(V_s^+, Z_s) &= \breve{f}_s(V_s^+, \widetilde{\mathfrak{q}}_s^1) - Z_s \cdot \widetilde{\mathfrak{q}}_s^1 \\ &\geqslant -\alpha - \beta|V_s^+| + \frac{1}{2\gamma}|\widehat{\sigma}_s^{-1}\widetilde{\mathfrak{q}}_s^1|^2 - \frac{1}{2}\Big(2\gamma|\widehat{\sigma}_s Z_s|^2 + \frac{1}{2\gamma}|\widehat{\sigma}_s^{-1}\widetilde{\mathfrak{q}}_s^1|^2\Big), \end{aligned} \tag{2.42}$$

we then have

$$\frac{1}{4\gamma}\int_0^T |\widehat{\sigma}_s^{-1}\widetilde{\mathfrak{q}}_s^1|^2 ds \leqslant \int_0^T |f_s(V_s^+, Z_s)|ds + (\alpha + \beta V_*^+)T + \gamma\int_0^T |\widehat{\sigma}_s Z_s|^2 ds < \infty, \mathbb{P}'\text{-a.s.}. \tag{2.43}$$

Taking $N \in \mathbb{N}$ s.t. $\frac{T}{N} \leqslant \left(\frac{p}{\gamma} - 1\right)\frac{1}{\beta}$. Let $t_0 \equiv 0$ and $t_j := \frac{jT}{N}$ for $j \in \{1, \cdots, N\}$. We define the process $M_t^j := \exp\Big(-\int_0^t \mathbf{1}_{\{s \geqslant t_{j-1}\}} \widehat{\sigma}_s^{-2}\widetilde{\mathfrak{q}}_s^1 \cdot dX_s - \frac{1}{2}\int_0^t \mathbf{1}_{\{s\geqslant t_{j-1}\}}|\widehat{\sigma}_s^{-1}\widetilde{\mathfrak{q}}_s^1|^2 ds\Big), t \in [0, t_j]$. For $n \in \mathbb{N}$, we define the stopping time

$$\tau_n^j := \inf\left\{ t \in [t_{j-1}, t_j] : \int_{t_{j-1}}^t [|\widehat{\sigma}_s(Z_s + \mathcal{Z}_s^{\mathbb{P}'})|^2 + |\widehat{\sigma}_s^{-1}\widetilde{\mathfrak{q}}_s^1|^2]ds > n \text{ or } \frac{M_t^j}{M_{t_{j-1}}^j} < \frac{1}{n}\right\} \wedge t_j. \tag{2.44}$$

Then, $\lim_{n\to\infty} \uparrow \tau_n^j = t_j, \mathbb{P}'$-a.s. by (2.43), and $\left\{M^j_{\tau_n^j \wedge t}\right\}_{t\in[0,t_j]}$ is a uniformly integrable martingale by Novikov's criterion. Hence, $\frac{d\mathbb{Q}_n^j}{d\mathbb{P}'} := M^j_{\tau_n^j\wedge t}$ induces a probability $\mathbb{Q}_n^j$ which is equivalent to $\mathbb{P}'$. Besides, $\{X_t^{j,n} := X_t + \int_0^t \mathbf{1}_{\{t_{j-1}\leqslant s \leqslant \tau_n^j\}}\widetilde{\mathfrak{q}}_s^1 ds\}_{t\in[0,t_j]}$ is a local martingale under $\mathbb{Q}_n^j$.

We consider the estimate on $\widetilde{\mathfrak{q}}_s^1$ by formulating the calculus below,

$$\begin{aligned} V_{t_{j-1}}^+ - V_{\tau_n^j}^+ &= \int_{t_{j-1}}^{\tau_n^j} f_s(V_s^+, Z_s, \widehat{\sigma}_s)ds - \int_{t_{j-1}}^{\tau_n^j} Z_s \cdot dX_s + dM_s^{\mathbb{P}'} + K_{\tau_n^j}^{\mathbb{P}'} - K_{t_{j-1}}^{\mathbb{P}'} \\ &\geqslant \int_{t_{j-1}}^{\tau_n^j} \breve{f}_s(V_s^+, \widetilde{\mathfrak{q}}_s^1) - \int_{t_{j-1}}^{\tau_n^j} Z_s \cdot dX_s^{j,n} + dM_s^{\mathbb{P}'} \\ &\geqslant \int_{t_{j-1}}^{\tau_n^j}\Big(-\alpha - \beta|V_s^+| + \frac{1}{2\gamma}|\widehat{\sigma}_s^{-1}\widetilde{\mathfrak{q}}_s^1|^2\Big)ds - \int_{t_{j-1}}^{\tau_n^j} Z_s \cdot dX_s^{j,n} + dM_s^{\mathbb{P}'}, \end{aligned} \tag{2.45}$$

then by taking expectation $\mathbb{E}^{\mathbb{Q}_n^j}[\cdot]$ on both sides, we get

$$\begin{aligned} \frac{1}{2\gamma}\mathbb{E}^{\mathbb{Q}_n^j}\left[\int_{t_{j-1}}^{\tau_n^j} |\widehat{\sigma}^{-1}\widetilde{\mathfrak{q}}_s^1|^2 ds\right] &\leqslant \mathbb{E}^{\mathbb{Q}_n^j}\left[V_{t_{j-1}}^+ - V_{\tau_n^j}^+ + \int_{t_{j-1}}^{\tau_n^j}(\alpha + \beta|V_s^+|)ds\right] \\ &\leqslant \mathbb{E}^{\mathbb{Q}_n^j}\left[(2 + \beta\Delta_j)\sup_{s\in[0,t_j]}|V_s^+|\right] + \alpha\Delta_j. \quad (\Delta_j := t_j - t_{j-1}) \end{aligned} \tag{2.46}$$

And from now on, we could step by step follow the deduction in Theorem 2.4.6 to get $\mathbb{Q}^j$ is a probability measure equivalent to $\mathbb{P}'$ and $\{X_t^j\}_{t\in[0,t_j]}$ is a local martingale under $\mathbb{Q}^j$, where $\frac{d\mathbb{Q}^j}{d\mathbb{P}'} = M_{t_j}^j$ and $X_t^j := X_t + \int_0^t \mathbf{1}_{\{s\geqslant t_{j-1}\}}\widetilde{\mathfrak{q}}_s^1 ds$.

Now we could verify the minimality condition on $t\in[t_{j-1},t_j], j\in\{1,2,\cdots,N\}$. In fact,

$$\begin{aligned}\delta Y_{t_{j-1}} - \delta Y_{\tau_n^j} &= \int_{t_{j-1}}^{\tau_n^j} \Big[f_s(V_s^+, Z_s, \widehat{\sigma}_s) - f_s(\mathcal{Y}_s^{\mathbb{P}'}, \mathcal{Z}_s^{\mathbb{P}'}, \widehat{\sigma}_s)\Big]ds - \delta Z_s\cdot dX_s - d\delta M_s^{\mathbb{P}'} + dK_s^{\mathbb{P}'}\\ &\geqslant \int_{t_{j-1}}^{\tau_n^j} \Big[\breve{f}_s(V_s^+, \widetilde{\mathfrak{q}}_s^1) - Z_s\widehat{\sigma}_s\cdot\widetilde{\mathfrak{q}}_s^1 - \breve{f}_s(\mathcal{Y}_s^{\mathbb{P}'}, \widetilde{\mathfrak{q}}_s^1) + \mathcal{Z}_s^{\mathbb{P}'}\widehat{\sigma}_s\cdot\widetilde{\mathfrak{q}}_s^1\Big]ds - \delta Z_s\cdot dX_s - d\delta M_s^{\mathbb{P}'} + dK_s^{\mathbb{P}'}\\ &= \int_{t_{j-1}}^{\tau_n^j} \Big[\breve{f}_s(V_s^+, \widetilde{\mathfrak{q}}_s^1) - \breve{f}_s(\mathcal{Y}_s^{\mathbb{P}'}, \widetilde{\mathfrak{q}}_s^1)\Big]ds - \delta Z_s\cdot dX_s^j - d\delta M_s^{\mathbb{P}'} + dK_s^{\mathbb{P}'}\\ &= \int_{t_{j-1}}^{\tau_n^j} \widetilde{\kappa}_s\delta Y_s ds - \delta Z_s\cdot dX_s^j - d\delta M_s^{\mathbb{P}'} + dK_s^{\mathbb{P}'},\end{aligned} \tag{2.47}$$

where $\widetilde{\kappa}$ is a bounded process due to the Lipschitz continuity satisfied by the Legendre-Fenchel transformation $\breve{f}$ of f. Set $\Gamma_{t_{j-1},t} := \exp\Big(\int_{t_{j-1}}^t \widetilde{\kappa}_s ds\Big)$, or say, $\Gamma_{t_{j-1},t} = 1 + \int_{t_{j-1}}^t \widetilde{\kappa}_s\Gamma_{t_{j-1},s}ds$, then

$$\begin{aligned}d(\Gamma_{t_{j-1},t}\delta Y_t) =& d\Gamma_{t_{j-1},t}\delta Y_t + \Gamma_{t_{j-1},t}d\delta Y_t + d\langle\Gamma_{t_{j-1},\cdot}, \delta Y_\cdot\rangle_t\\ =&\widetilde{\kappa}_t\Gamma_{t_{j-1},t}\delta Y_t dt + \Gamma_{t_{j-1},t}(-\widetilde{\kappa}_t\cdot\delta Y_t dt + \delta Z_t\cdot dX_t^j + d\delta M_t^{\mathbb{P}'} - dK_t^{\mathbb{P}'})\\ =&\Gamma_{t_{j-1},t}(\delta Z_t\cdot dX_t^j + d\delta M_t^{\mathbb{P}'} - dK_t^{\mathbb{P}'}),\end{aligned} \tag{2.48}$$

i.e.,

$$\Gamma_{t_{j-1},t_{j-1}}\delta Y_{t_{j-1}} = \Gamma_{t_{j-1},\tau_n^j}\delta Y_{\tau_n^j} - \int_{t_{j-1}}^{\tau_n^j} \Gamma_{t_{j-1},t}(\delta Z_t\cdot dX_t^j + d\delta M_t^{\mathbb{P}'} - dK_t^{\mathbb{P}'}). \tag{2.49}$$

Note that $\delta Y\geqslant 0$, then we take conditional expectation $\mathbb{E}_{t_{j-1}}^{\mathbb{Q}^j}[\cdot] := \mathbb{E}^{\mathbb{Q}^j}[\cdot|\mathcal{F}_{t_{j-1}}^+]$ in (2.49), we get

$$\begin{aligned}\delta Y_{t_{j-1}} \geqslant \mathbb{E}_{t_{j-1}}^{\mathbb{Q}^j}\left[\int_{t_{j-1}}^{\tau_n^j}\Gamma_{t_{j-1},t}dK_t^{\mathbb{P}'}\right] \geqslant \gamma_{t_{j-1},t_j}\mathbb{E}_{t_{j-1}}^{\mathbb{Q}^j}\left[\int_{t_{j-1}}^{\tau_n^j} dK_t^{\mathbb{P}'}\right] \geqslant& \gamma_{t_{j-1},t_j}\mathbb{E}_{t_{j-1}}^{\mathbb{P}'}\left[\frac{M_{\tau_n^j}^j}{M_{t_{j-1}}^j}\int_{t_{j-1}}^{\tau_n^j} dK_t^{\mathbb{P}'}\right]\\ \geqslant&\frac{\gamma_{t_{j-1},t_j}}{n}\mathbb{E}_{t_{j-1}}^{\mathbb{P}'}\left[\int_{t_{j-1}}^{\tau_n^j} dK_t^{\mathbb{P}'}\right],\end{aligned} \tag{2.50}$$

with $\gamma_{t_{j-1},t_j} := e^{-|\widetilde{\kappa}|_\infty(t_j-t_{j-1})} \leqslant \inf_{t_{j-1}\leqslant t\leqslant t_j}\Gamma_{t_{j-1},t}$. By removing parameters to the LHS together with dynamic programming principle in Proposition 2.4.7, we get that

$$\underset{\mathbb{P}'\in\mathcal{P}_{\mathbb{P}}^+(t_{j-1})}{\overset{\mathbb{P}}{\operatorname{ess\,inf}}}\ \mathbb{E}_{t_{j-1}}^{\mathbb{P}'}\Big[\int_{t_{j-1}}^{\tau_n^j} dK_t^{\mathbb{P}'}\Big] = 0.$$

Note that $j \in \{1, \cdots, N\}$, then we obtain the minimality condition on the whole interval $[0, T]$.

□

2.5 Uniqueness

In this section, we prove the uniqueness result which is stated in detail including its representation.

Theorem 2.5.1 (Uniqueness). *Under Assumption 2.1.3 and assuming that* $(Y, Z, (M^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})$, *belonging to the space* $\mathcal{D}^{\exp(\lambda,\lambda')}(\mathcal{P}_0, \mathbb{F}^{\mathcal{P}_0+}) \times \mathcal{H}^{2,loc}(\mathcal{P}_0, \mathbb{R}^d, \mathbb{F}^{\mathcal{P}_0+}) \times \Big(\mathbb{M}(\mathbb{P})\Big)_{\mathbb{P}\in\mathcal{P}_0}$, *is a solution to the 2BSDE(2.4), then for all* $\mathbb{P} \in \mathcal{P}_0$ *and* $0 \leqslant t_1 < t_2 \leqslant T$,

$$Y_{t_1\wedge T} = \underset{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(t_1\wedge T)}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathcal{Y}^{\mathbb{P}'}_{t_1\wedge T}(Y_{t_2\wedge T}, t_2\wedge T) \tag{2.51}$$

$$= \underset{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(t_1\wedge T)}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathcal{Y}^{\mathbb{P}'}_{t_1\wedge T}(\xi, T), \mathbb{P}\text{-a.s.}, \tag{2.52}$$

the 2BSDE(2.4) has at most one solution in $\mathcal{D}^{\exp(\lambda,\lambda')}(\mathcal{P}_0, \mathbb{F}^{\mathcal{P}_0+}) \times \mathcal{H}^{2,loc}(\mathcal{P}_0, \mathbb{R}^d, \mathbb{F}^{\mathcal{P}_0+}) \times \Big(\mathbb{M}(\mathbb{P})\Big)_{\mathbb{P}\in\mathcal{P}_0}$ *with* $\lambda \in (\gamma, \infty)$ *and* $\lambda' \in (0, \infty)$.

Proof. The proof follows the idea of [STZ12] (Theorem 4.4), and the new part is that we have to deal with quadratic difference term. First, if (2.51) holds, this implies that

$$Y_{r_1} = \underset{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(r_1)}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathcal{Y}^{\mathbb{P}'}_{r_1}(T, \xi), \quad \mathbb{P}\text{-a.s. for all } \mathbb{P}\in\mathcal{P}_0, \tag{2.53}$$

and thus is unique. Then, since we have $d\langle Y, X\rangle_t = Z_t d\langle X\rangle_t, \mathcal{P}_0$-q.s., Z is also unique. We now prove (2.51) in two steps. The outline is that, we will use comparison theorem to get one inequality, and use the the minimal condition (2.5) to get the second(i.e., the reverse) one.

1. Fix $0 \leqslant r_1 < r_2 \leqslant T$ and $\mathbb{P} \in \mathcal{P}_0$. For any $\mathbb{P}' \in \mathcal{P}^+_{\mathbb{P}}(r_1)$, we have

$$Y_t = Y_{r_2} + \int_t^{r_2} f_s(Y_s, Z_s, \widehat{\sigma}_s)ds - Z_s \cdot dX_s - dM^{\mathbb{P}'}_s + K^{\mathbb{P}'}_{r_2} - K^{\mathbb{P}'}_{r_1}, \quad r_1 \leqslant t \leqslant r_2 \quad \mathbb{P}'\text{-a.s.}$$

   and that $K^{\mathbb{P}'}$ is non-decreasing, $\mathbb{P}'$-a.s.. The existence of the solution to the above BSDEs has been validated in the previous section. Then, we can always apply the comparison theorem of quadratic BSDE under $\mathbb{P}'$ to obtain $Y_{r_1} \geqslant \mathcal{Y}^{\mathbb{P}'}_{r_1}(r_2, Y_{r_2}), \mathbb{P}'$-a.s. Since $\mathbb{P}' = \mathbb{P}$ on $\mathcal{F}^+_{r_1}$, we get $Y_{r_1} \geqslant \mathcal{Y}^{\mathbb{P}'}_{r_1}(r_2, Y_{r_2}), \mathbb{P}$-a.s. and thus

$$Y_{r_1} \geqslant \underset{\mathbb{P}' \in \mathcal{P}^+_{\mathbb{P}}(r_1)}{\operatorname{ess\,sup}^{\mathbb{P}}} \; \mathcal{Y}^{\mathbb{P}'}_{r_1}(r_2, Y_{r_2}), \quad \mathbb{P}\text{-a.s.}$$

2. We now prove the reverse inequality. Fix $\mathbb{P} \in \mathcal{P}_0$. In the existence proof of last section, we have proved that there exists a sequence of stopping time $\{\tau_n^{\mathbb{P}'}\}$ with $n \in \mathbb{N}$ and $\mathbb{P}' \in \mathcal{P}^+_0(\mathbb{P})$, for which we use the superscript $\mathbb{P}'$ to emphasize its dependence on this probability measure, s.t.

$$\underset{\mathbb{P}' \in \mathcal{P}^+_{\mathbb{P}}(0)}{\operatorname{ess\,inf}^{\mathbb{P}}} \; \mathbb{E}^{\mathbb{P}'}_{t_{j-1}}[K^{\mathbb{P}'}_{\tau_n^{\mathbb{P}'}} - K^{\mathbb{P}'}_{t_{j-1}}] = 0, \mathbb{P}\text{-a.s.}, j \in \{1, \cdots, N\}. \tag{2.54}$$

For every $\mathbb{P}' \in \mathcal{P}^{\mathbb{P}}_+(r_1)$, denote $\delta Y := Y - \mathcal{Y}^{\mathbb{P}'}(r_2, Y_{r_2})$, $\delta Z := Z - \mathcal{Z}^{\mathbb{P}'}(r_2, Y_{r_2})$, and $\delta M := M - \mathcal{M}^{\mathbb{P}'}$. Using the same notation in the above existence proof (notice that we here take the division $\{t_j\}_{j \in \{1, \cdots, N\}}$ on the interval $[r_1, r_2]$), we have the following formulation similar to (2.47) with

$$\vartheta_n^{\mathbb{P}'} := \tau_n^{\mathbb{P}'} \wedge \inf\left\{ t \in [t_{j-1}, t_j] : \int_{t_{j-1}}^t [|\widehat{\sigma}_s Z_s|^2 + |\widehat{\sigma}_s \mathcal{Z}^{\mathbb{P}'}_s|^2 + |\widehat{\sigma}_s^{-1} \widetilde{\mathfrak{q}}_s^2|^2] ds > n \text{ or } \frac{M^j_t}{M^j_{t_{j-1}}} > n \right\} \wedge t_j, \tag{2.55}$$

where we emphasize that $\{\tau_n^{\mathbb{P}'}\}$ is the localizing sequence in the above minimality condition.

$$\begin{aligned}
\delta Y_{t_{j-1}} &= \int_{t_{j-1}}^{\vartheta_n^{\mathbb{P}'}} \Big[f_s(Y_s, Z_s, \widehat{\sigma}_s) - f_s(\mathcal{Y}^{\mathbb{P}'}_s, \mathcal{Z}^{\mathbb{P}'}_s, \widehat{\sigma}_s) \Big] ds - \delta Z_s \cdot dX_s - d\delta M^{\mathbb{P}'}_s + dK^{\mathbb{P}'}_s \\
&\leqslant \int_{t_{j-1}}^{\vartheta_n^{\mathbb{P}'}} \Big[\breve{f}_s(Y_s, \widetilde{\mathfrak{q}}_s^2) - Z_s \cdot \widetilde{\mathfrak{q}}_s^2 - \breve{f}_s(\mathcal{Y}^{\mathbb{P}'}_s, \widetilde{\mathfrak{q}}_s^2) + \mathcal{Z}^{\mathbb{P}'}_s \cdot \widetilde{\mathfrak{q}}_s^2 \Big] ds - \delta Z_s \cdot dX_s - d\delta M^{\mathbb{P}'}_s + dK^{\mathbb{P}'}_s \\
&= \int_{t_{j-1}}^{\vartheta_n^{\mathbb{P}'}} \Big[\breve{f}_s(Y_s, \widetilde{\mathfrak{q}}_s^2) - \breve{f}_s(\mathcal{Y}^{\mathbb{P}'}_s, \widetilde{\mathfrak{q}}_s^2) \Big] ds - \delta Z_s \cdot dX^{j,n}_s - d\delta M^{\mathbb{P}'}_s + dK^{\mathbb{P}'}_s \\
&= \int_{t_{j-1}}^{\vartheta_n^{\mathbb{P}'}} \widetilde{\kappa}_s \delta Y_s ds - \delta Z_s \cdot dX^{j,n}_s - d\delta M^{\mathbb{P}'}_s + dK^{\mathbb{P}'}_s,
\end{aligned} \tag{2.56}$$

where the ante-penultimate inequality uses the second inequality in (2.41) and $X^{j,n}_t := X_t + \int_0^t \mathbf{1}_{\{t_{j-1} \leqslant s \leqslant \vartheta_n^{\mathbb{P}'}\}} \widetilde{\mathfrak{q}}_s^2 ds$ for $t \in [0, t_j]$. Recall that for $t_{j-1} \leqslant t \leqslant t_j, \Gamma_{t_{j-1}, t} = e^{\int_{t_{j-1}}^t \widetilde{\kappa}_s ds}$, then we obtain that

$$\begin{aligned}\delta Y_{t_{j-1}} \leqslant \mathbb{E}^{\mathbb{Q}^j}_{t_{j-1}}\left[\int_{t_{j-1}}^{\vartheta_n^{\mathbb{P}'}} \Gamma_{t_{j-1},t} dK_t^{\mathbb{P}'}\right] &\leqslant \mathbb{E}^{\mathbb{Q}^j}_{t_{j-1}}\left[\sup_{t_{j-1}\leqslant t\leqslant \tau_n^{\mathbb{P}'}} \Gamma_{t_{j-1},t}(K^{\mathbb{P}'}_{\tau_n^{\mathbb{P}'}} - K^{\mathbb{P}'}_{t_{j-1}})\right] \\ &\leqslant e^{|\widetilde{\kappa}|_\infty (t_j - t_{j-1})}\mathbb{E}^{\mathbb{Q}^j}_{t_{j-1}}\left[K^{\mathbb{P}'}_{\tau_n^{\mathbb{P}'}} - K^{\mathbb{P}'}_{t_{j-1}}\right] \\ &\leqslant n e^{|\widetilde{\kappa}|_\infty (t_j - t_{j-1})}\mathbb{E}^{\mathbb{P}'}_{t_{j-1}}\left[K^{\mathbb{P}'}_{\tau_n^{\mathbb{P}'}} - K^{\mathbb{P}'}_{t_{j-1}}\right],\end{aligned}$$

where we use $\vartheta_n^{\mathbb{P}'} \leqslant \tau_n^{\mathbb{P}'}$ in the second inequality. Similar to the above existence proof, we here use the minimality condition (2.54), the fact that $\mathbb{P}' \in \mathcal{P}^+_{\mathbb{P}}(r_1)$ are arbitrary as well as that $j \in \{1, \cdots, N\}$, we end the proof.

□

Chapter 3

RBSDE, BSDE with random horizon and BSDEJ

Bowen SHENG

In this review paper, we handle with three subjects: quadratic reflected backward SDEs, random horizon quadratic backward SDEs and quadratic backward SDE with jumps. It's a survey to summarize the corresponding progress made in the latest years according to our own research need and preference. After the introduction, the first part is a new proof of existence and comparison principle of solutions for one type of reflected ODEs is provided, which is used for obtaining an a priori estimate on Y-part of solutions for the quadratic reflected backward SDE with an exponential-integrable terminal condition ξ and a quadratic generator f, for both bounded and unbounded obstacles. Then, the second part extends the wellposedness of quadratic BSDE/RBSDE(e.g., [BH08, BH06, BY12]) to a more general filtration generated by the standard Brownian motion and an orthogonal martingale. Next, the discussion in the third and fourth parts includes the explanation of difficulties which we encountered in our practice and possible future research topics.

3.1 Introduction

3.1.1 Canonical space

Let $d \in \mathbb{N}^*$. We denote by $\Omega := \mathbb{C}([0,T],\mathbb{R}^d)$ the canonical space of all $\mathbb{R}^d$-valued continuous paths ω on $[0,T]$ s.t. $\omega_0 = 0$, equipped with the canonical process X, that is, $X_t(\omega) := \omega_t$, for all $\omega \in \Omega$. Denote by $\mathbb{F} = (\mathcal{F}_t)_{0 \leqslant t \leqslant T}$ the canonical filtration generated by X. We equip ω with the uniform convergence norm $\|\omega\|_\infty := \sup_{0 \leqslant t \leqslant T} \|\omega_t\|$, so that the Borel σ-field of Ω coincides with $\mathcal{F}_T$.

Let $\mathscr{M}_1$ denote the collection of all probability measures on $(\Omega, \mathcal{F}_T)$. Note that $\mathscr{M}_1$ is a Polish space equipped with the weak convergence topology. We denote by $\mathfrak{B}$ its Borel σ-field. Then for any $\mathbb{P} \in \mathscr{M}_1$, denote by $\mathcal{F}_t^{\mathbb{P}}$ the completed σ-field of $\mathcal{F}_t$ under $\mathbb{P}$. Denote also the completed filtration by $\mathbb{F}^{\mathbb{P}} = (\mathcal{F}_t^{\mathbb{P}})_{0 \leqslant t \leqslant T}$.

We also introduce an enlarged canonical space $\overline{\Omega} := \Omega \times \Omega$, and denote by (X, W) as its canonical process, that is, $X_t(\bar{\omega}) := \omega_t, W_t(\bar{\omega}) := \omega'_t$ for all $\bar{\omega} := (\omega, \omega') \in \overline{\Omega}$, by $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{0 \leqslant t \leqslant T}$ the canonical filtration generated by (X, W), and by $\overline{\mathbb{F}}^X = (\overline{\mathcal{F}}_t^X)_{0 \leqslant t \leqslant T}$ the filtration generated by X. By abuse of notation, we will keep using $\mathbb{P}$ to represent $\overline{\mathbb{P}}$ on $\overline{\Omega}$.

3.1.2 Spaces and norms

For $p \geqslant 1$. $(\mathbf{i})$ *One-measure integrability classes:* given a finite time horizon $T > 0$, let $\mathcal{T}_{0,T}$ denote the set of all $\overline{\mathbb{F}}^{\mathbb{P}}$-stopping times ν s.t. $0 \leqslant \nu \leqslant T, \mathbb{P}$-a.s. For any probability measure $\mathbb{P} \in \mathscr{M}_1$, let τ be an $\overline{\mathbb{F}}^{\mathbb{P}}$-stopping time, similarly we can define $\mathcal{T}_{0,\tau}$, etc. Besides, we define,

- Let $\mathbb{L}^0(\overline{\mathcal{F}}_t^{\mathbb{P}}, \mathbb{R})$ be the set of $\mathbb{R}$-valued and $\overline{\mathcal{F}}_t^{\mathbb{P}}$ -measurable r.v., then,

$$\mathbb{L}^p(\mathbb{P}, \mathbb{R}) := \{\xi \in \mathbb{L}^0(\overline{\mathcal{F}}_T^{\mathbb{P}}, \mathbb{R}) : \|\xi\|_{\mathbb{L}^p(\mathbb{P},\mathbb{R})}^p := \mathbb{E}^{\mathbb{P}}[|\xi|^p] < \infty\}.$$

  Besides, $\mathbb{L}^\infty(\mathbb{P}, \mathbb{R}) := \{\xi \in \mathbb{L}^0(\overline{\mathcal{F}}_T^{\mathbb{P}}, \mathbb{R}), \|\xi\|_{\mathbb{L}^\infty(\mathbb{P},\mathbb{R})} := \operatorname{ess\,sup}_{\omega \in \Omega}^{\mathbb{P}} |\xi(\omega)| < \infty\}.$

$$\mathbb{L}^{\exp}(\mathbb{P},\mathbb{R}) := \{\xi \in \mathbb{L}^0(\overline{\mathcal{F}}^{\mathbb{P}}_T,\mathbb{R}), \mathbb{E}[e^{p|\xi|}] < \infty, \forall p \in (1,\infty)\}.$$

- Let $\mathbb{B}$ be a generic Banach space with norm $|\cdot|_{\mathbb{B}}$. For any $p, q \in [1,\infty)$, let $\mathbb{H}^{p,q}(\mathbb{P},\mathbb{B})$ be the space of all $\mathbb{B}$-valued $\overline{\mathbb{F}}^{\mathbb{P}}$ -progressive measurable processes X with $\|X\|_{\mathbb{H}^{p,q}(\mathbb{P},\mathbb{B})} := \Big\{\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T |X_t|^p_{\mathbb{B}} dt\Big)^{\frac{q}{p}}\Big]\Big\}^{\frac{1}{q}} < \infty$. We simply write $\mathbb{H}^p$ for $\mathbb{H}^{p,p}$ if $p = q$ and add superscript `loc` if the integrability only satisfies $\Big(\int_0^T |X_t|^p_{\mathbb{B}} dt\Big)^{\frac{q}{p}} < \infty, \mathbb{P}$-a.s.. Besides, $\mathbb{M}^p(\mathbb{P})$ denotes the space of $\mathbb{R}$-valued, $\overline{\mathbb{F}}^{\mathbb{P}}$ -adapted martingales M, with $\mathbb{P}$-a.s. càdlàg path, s.t. M is orthogonal to W and $\|M\|_{\mathbb{M}^p(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}[\langle M\rangle^{\frac{p}{2}}_T] < \infty$.

- Let $\mathbb{C}^0_{\overline{\mathbb{F}}^{\mathbb{P}}}$ be the set of all $\mathbb{R}$-valued, $\overline{\mathbb{F}}^{\mathbb{P}}$ -adapted continuous processes on $[0,T]$, then,

$$\mathbb{C}^{\infty}(\mathbb{P}) := \{X \in \mathbb{C}^0_{\overline{\mathbb{F}}^{\mathbb{P}}} : \|X\|_{\mathbb{C}^\infty} := \operatorname*{ess\,sup}^{\mathbb{P}}_{\omega\in\Omega}(\sup_{t\in[0,T]} |X_t(\omega)|) < \infty\}.$$

$$\mathbb{C}^{p}(\mathbb{P}) := \{X \in \mathbb{C}^0_{\overline{\mathbb{F}}^{\mathbb{P}}} : \|X\|^p_{\mathbb{C}^p} := \mathbb{E}^{\mathbb{P}}[\sup_{0\leqslant t\leqslant T} |X_t|^p] < \infty\}.$$

$$\mathbb{I}(\mathbb{P}) := \{X \in \mathbb{C}^0_{\overline{\mathbb{F}}^{\mathbb{P}}} : X_{\cdot} \text{ is an increasing process with } X_0 = 0\}$$

$$\mathbb{I}^{p}(\mathbb{P}) := \{X \in \mathbb{I}(\mathbb{P}) : X_T \in \mathbb{L}^p(\mathbb{P},\mathbb{R})\} \text{ for all } p \in [1,\infty).$$

$$\mathbb{C}^{\exp(\lambda,\lambda')}(\mathbb{P}) := \{X \in \mathbb{C}^0_{\overline{\mathbb{F}}^{\mathbb{P}}} : \mathbb{E}^{\mathbb{P}}[e^{(\lambda X^-_* + \lambda' X^+_*)}] < \infty\} \subseteq \cap_{p\in[1,\infty)}\mathbb{C}^p(\mathbb{P}) \text{ for all } \lambda, \lambda' \in (0,\infty),$$

where $X^{\pm}_* := \sup_{t\in[0,T]}(X_t)^{\pm}$, and $\mathbb{C}^{\exp(p)}(\mathbb{P}) := \{X \in \mathbb{C}^0_{\overline{\mathbb{F}}^{\mathbb{P}}} : \mathbb{E}^{\mathbb{P}}[e^{pX_*}] < \infty\}$ for all $p \in (0,\infty)$, where $X_* := \sup_{t\in[0,T]} X_t$.

Moreover, for any $p \in [1,\infty)$, we set $\mathbb{S}^p(\mathbb{P}) := \mathbb{C}^{\exp(p)}(\mathbb{P}) \times \mathbb{H}^{2,2p}(\mathbb{P},\mathbb{R}^d) \times \mathbb{M}^p(\mathbb{P}) \times \mathbb{I}^p(\mathbb{P})$.

3.2 Existence of a solution for reflected ODEs

Inspired by Lepeltier and Xu [LX07], here we give a new proof of their result (p.12 Theorem 6.2). Firstly, we introduce here some background(two necessary technical tools). One is an ODE as Briand and Hu [BH06] points out: define

$$\begin{aligned} H(p) :=& p(\alpha\gamma + \beta \log p)\mathbf{1}_{[1,+\infty)}(p) + \gamma\alpha\mathbf{1}_{(-\infty,1)}(p) \\ =& (1 \vee p) \cdot (\alpha\gamma + \beta \log(1 \vee p)), \forall p \in \mathbb{R}, \end{aligned}$$

the solution of the integral equation for any $x \in \mathbb{R}$,

$$\phi_t = x + \int_t^T H(\phi_s)ds, 0 \leqslant t \leqslant T, \tag{3.1}$$

is as follows,

$$\phi_t(x) = \begin{cases} \exp\left(\gamma\alpha \dfrac{e^{\beta(T-t)}-1}{\beta}\right) x^{e^{\beta(T-t)}}, & x \geqslant 1, \quad \beta > 0, \\ e^{\gamma\alpha(T-t)}x, & x \geqslant 1, \quad \beta = 0, \\ x + \gamma\alpha(T-t), & x < 1, \quad x + T\gamma\alpha \leqslant 1, \\ [x+\gamma\alpha(T-t)]\mathbf{1}_{(S,T]}(t) + \exp\left(\gamma\alpha \dfrac{e^{\beta(S-t)}-1}{\beta}\right)\mathbf{1}_{[0,S)}(t), & x < 1, \quad x + T\gamma\alpha > 1, \end{cases}$$

where $S \in [0,T]$ satisfies that $x + \gamma\alpha(T-S) = 1$. Particularly, we would like to claim explicitly certain necessary properties of (3.1) to be used later on, with the special setting of using $e^{\gamma x}$ to replace its x: for any $\widetilde{T} \in [0,T]$,

$$\phi(t) = e^{\gamma x} + \int_t^{\widetilde{T}} H(\phi(s))ds, t \in [0,\widetilde{T}]. \tag{3.2}$$

(i)for $x \geqslant 0 : \phi_t^{\widetilde{T}} = \exp\{\mu\varphi(\widetilde{T}-t) + \gamma x e^{\beta(\widetilde{T}-t)}\}$, where $\varphi(s) := \frac{e^{\beta s}-1}{\beta}\mathbf{1}_{\beta>0} + s\mathbf{1}_{\beta=0}, \forall s \in [0,T]$;
(ii)for $x < 0$:

$$\phi_t^{\widetilde{T}}(x) = \begin{cases} e^{\gamma x} + \mu(\widetilde{T}-t) < 1 + \mu(\widetilde{T}-t) \leqslant e^{\mu(\widetilde{T}-t)} \leqslant e^{\mu\varphi(\widetilde{T}-t)}, & \text{if } e^{\gamma x} + \mu(\widetilde{T}-t) < 1, \\ \exp\left\{\mu\varphi\left(\widetilde{T} - t + \dfrac{e^{\gamma x}-1}{\mu}\right)\right\} \leqslant e^{\mu\varphi(\widetilde{T}-t)}, & \text{if } e^{\gamma x} + \mu(\widetilde{T}-t) \geqslant 1. \end{cases} \tag{3.3}$$

And we have the following properties:

(ϕ1) for any $x \in \mathbb{R}$ and $\widetilde{T} \in [0,T], t \mapsto \phi_t^{\widetilde{T}}(x)$ is a decreasing and continuous function on $[0,\widetilde{T}]$.

(ϕ2) for any $x \in \mathbb{R}$ and $\widetilde{T} \in [0,T], \widetilde{T} \mapsto \phi_t^{\widetilde{T}}(x)$ is an increasing and continuous function on $[0,\widetilde{T}]$.

(ϕ3) for any $0 \leqslant t \leqslant \widetilde{T} \leqslant T, x \mapsto \phi_t^{\widetilde{T}}(x)$ is an increasing and continuous function on $\mathbb{R}$.

(ϕ4) for any $x \in \mathbb{R}$ and $0 \leqslant t \leqslant \widetilde{T} \leqslant T, \phi_t^{\widetilde{T}}(x) \leqslant \exp\{\mu\varphi(T) + \gamma x^+ e^{\beta T}\}$.

The other tool is the following reflected ordinary differential equation. Now, consider the reflected ordinary differential equation(RODE in short) reflected to one continuous obstacle l on $[0,T]$, with terminal value $x \in \mathbb{R}$, whose solution is a couple $(y_t, k_t)_{0\leqslant t\leqslant T}$, and k is a continuous increasing process, $k_0 = 0$, and the following holds

$$\begin{aligned} y_t &= x + \int_t^T H(y_s)ds + k_T - k_t, \\ y_t &\geqslant l_t, \int_0^T (y_s - l_s)dk_s = 0. \end{aligned} \tag{3.4}$$

Here we suppose

Assumption 3.2.1. *The function $H : \mathbb{R} \to \mathbb{R}$ is continuous, strictly positive (or strictly negative) and there exists a strictly positive function i_0, s.t. $|H(y)| \leqslant i_0(y)$, with $\int_0^\infty \frac{dy}{i_0(y)} = \infty$.*

Then we have

Theorem 3.2.2. *Under Assumption 3.2.1 there exists a unique solution $(y_t, k_t)_{0\leqslant t\leqslant T}$ for the RODE (3.4). Moreover,*

$$y_t = \sup_{t\leqslant s\leqslant T} u_t^s, \tag{3.5}$$

where $(u_t^s)_{0\leqslant t\leqslant s}$ is the unique solution of the following ODE defined on $[0, s]$,

$$u_t^s = (x\mathbf{1}_{\{s=T\}} + l_s\mathbf{1}_{\{s<T\}}) + \int_t^s H(u_r^s)dr. \tag{3.6}$$

Existence

Proof. By Assumption 3.2.1, $\int_0^\infty \frac{dy}{|H(y)|} \geqslant \int_0^\infty \frac{dy}{i_0(y)} = \infty$. For convenience, here we only consider the strictly positive $H > 0$. Due to lemma 1 of Lepeltier and San Martín[LM98] (a sufficient and necessary condition), for $-\infty < x < \infty$, the following ODE

$$y_t = x + \int_t^T H(y_s)ds \tag{3.7}$$

has a unique global bounded solution on $[0, T]$. It's explicit: for $z \geqslant x$, we define $J(z) := \int_x^z \frac{1}{H(y)}dy$, then $y_t = J^{-1}(T - t)$. Then, we go back to the RODE (3.4), it's easy to get the following expression satisfied by the solution (for this type Skorohod equation, see e.g., Karatzas and Shreve [KS12] p.210 Lemma 6.14)

$$(y_t, k_T - k_t) = \left(\max\{J^{-1}(T-t), l_t\}, \left[l_t - \left(x + \int_t^T H(l_s)ds \right) \right] \mathbb{1}_{\{J^{-1}(T-t)\leqslant l_t\}} \right)$$

so we confirmed the existence. □

Moreover, we have a comparison theorem.

Theorem 3.2.3. *We consider the equations associated to* $(H^i, x^i, l), i = 1, 2$, *and assume that* H^1 *and* H^2 *satisfy Assumption 3.2.1. Let* (y^i, k^i) *be the respective solutions of their equations. Moreover, we assume for* $t \in [0, T]$,

$$x^1 \geqslant x^2, H^1(y_t^1) \geqslant H^2(y_t^1), l_t^1 \geqslant l_t^2.$$

Then, $y_t^1 \geqslant y_t^2$.

Proof. We consider $[(y_t^2 - y_t^1)^+]^2$. Notice that on the set $y_t^2 \geqslant y_t^1, y_t^2 \geqslant y_t^1 \geqslant l_t^1 \geqslant l_t^2$, so we have

$$\begin{aligned}\int_t^T (y_s^2 - y_s^1)^+ d(k_s^2 - k_s^1) &= \int_t^T [(y_s^2 - l_s^2) + (l_s^2 - y_s^1)]^+ dk_s^2 - \int_t^T (y_s^2 - y_s^1)^+ dk_s^1 \\ &= \int_t^T [(y_s^2 - l_s^2) - (y_s^1 - l_s^2)]\mathbf{1}_{\{y_s^2 \geqslant y_s^1\}} dk_s^2 - \int_t^T (y_s^2 - y_s^1)^+ dk_s^1 \\ &= \int_t^T (y_s^2 - l_s^2)\mathbf{1}_{\{y_s^2 \geqslant y_s^1\}} dk_s^2 - \int_t^T (y_s^1 - l_s^2)\mathbf{1}_{\{y_s^2 \geqslant y_s^1\}} dk_s^2 - \int_t^T (y_s^2 - y_s^1)^+ dk_s^1,\end{aligned} \tag{3.8}$$

where in the final inequality the first term is not bigger than 0 by (3.4) and the last two terms are not negative, then we could claim that $\int_t^T (y_s^2 - y_s^1)^+ d(k_s^2 - k_s^1) \leqslant 0$. Taking the differential of $[(y_t^2 - y_t^1)^+]^2$ on $s \in [t, T]$,

$$[(y_T^2 - y_T^1)^+]^2 = [(y_t^2 - y_t^1)^+]^2 + 2\int_t^T (y_s^2 - y_s^1)^+ d(y_s^2 - y_s^1)^+.$$

Note that

$$\begin{aligned} y_t^2 - y_t^1 &= x^2 - x^1 + \int_t^T [H^2(y_s^2) - H^1(y_s^1)]ds + k_T^2 - k_T^1 - k_t^2 + k_t^1, \\ \Rightarrow d(y_t^2 - y_t^1) &= -[H^2(y_t^2) - H^1(y_t^1)]dt - d(k_t^2 - k_t^1).\end{aligned}$$

and

$$d(y_s^2 - y_s^1)^+ = \mathbf{1}_{\{y_s^2 > y_s^1\}} d(y_s^2 - y_s^1),$$

so we have,

$$0 = [(y_T^2 - y_T^1)^+]^2 = [(y_t^2 - y_t^1)^+]^2 + 2\int_t^T (y_s^2 - y_s^1)^+ \mathbf{1}_{\{y_s^2 > y_s^1\}} d(y_s^2 - y_s^1),$$

i.e.

$$\begin{aligned}
[(y_t^2 - y_t^1)^+]^2 &= -2\int_t^T (y_s^2 - y_s^1)^+ \mathbf{1}_{\{y_s^2 > y_s^1\}} d(y_s^2 - y_s^1) \\
&= 2\int_t^T (y_s^2 - y_s^1)^+ \mathbf{1}_{\{y_s^2 > y_s^1\}} [H^2(y_s^2) - H^1(y_s^1)]ds + 2\int_t^T (y_s^2 - y_s^1)^+ \mathbf{1}_{\{y_s^2 > y_s^1\}} d(k_s^2 - k_s^1) \\
&\leqslant_{(3.8)} 2\int_t^T (y_s^2 - y_s^1)^+ \mathbf{1}_{\{y_s^2 > y_s^1\}} [H^2(y_s^2) - H^1(y_s^1)]ds.
\end{aligned}$$

Notice the property of the integrand inside that $H^2(y_s^2) - H^1(y_s^1) = [H^2(y_s^2) - H^2(y_s^1)] + H^2(y_s^1) - H^1(y_s^1) \leqslant |[H^2(y_s^2) - H^2(y_s^1)]| + 0 \leqslant \mu_{H^2}|y_s^2 - y_s^1|$, where μ_{H^2} is the Lipschitz coefficient of function $H^2(\cdot)$ over $\{x \in \mathbb{R} : \|x\| \leqslant |y_s^1| \vee |y_s^2|\}$. This gives that

$$\begin{aligned}
[(y_t^2 - y_t^1)^+]^2 &\leqslant 2\mu_{H^2} \int_t^T (y_s^2 - y_s^1)^+ \mathbf{1}_{\{y_s^2 > y_s^1\}} |y_s^2 - y_s^1| ds \\
&\leqslant 2\mu_{H^2} \int_t^T [(y_s^2 - y_s^1)^+]^2 ds.
\end{aligned}$$

Then Gronwall's inequality gives that $(y_t^2 - y_t^1)^+ = 0$, i.e., $y_t^1 \geqslant y_t^2$. □

Remark 3.2.4. *The result is still true under the assumption* $H^1(y_t^2) \geqslant H^2(y_t^2), t \in [0, T]$.

For the uniqueness and characterization of the solution, we suggest to refer to [LX07].

3.3 Reflected backward SDEs in a more general filtration

The theory of quadratic backward SDE and quadratic reflected backward SDE play an important role in our understaning of second order backward SDE. Therefore, we give here an survey of principle arguments concerned in the existence and uniqueness of quadratic backward SDE/reflected backward SDE with deterministic terminal horizon.

3.3.1 Definition of equations

Reflected Backward SDEs(RBSDEs): The notion of reflected backward SDE(RBSDE here after) has been introduced by El Karoui et al. [KKP+97]. Here in this paper, we limit the definition of a solution for such an equation, associated with a generator f, a terminal condition ξ and a continuous obstacle

$L \in \mathbb{C}^0_{\overline{\mathbb{F}}^{\mathbb{P}}}$, to be a quadruple of processes $(Y, Z, M, K) \in \mathbb{C}^0_{\overline{\mathbb{F}}^{\mathbb{P}}} \times \mathbb{H}^{2,\mathrm{loc}}(\mathbb{P}, \mathbb{R}^d) \times \mathbb{M}(\mathbb{P}) \times \mathbb{I}(\mathbb{P})$, satisfying

$$\begin{aligned}
&Y_t = \xi + \int_t^T f_s(Y_s, Z_s)ds - \int_t^T Z_s \cdot dX_s - \int_t^T dM_s + K_T - K_t, 0 \leqslant t \leqslant T,\\
&Y_t \geqslant L_t, 0 \leqslant t \leqslant T, \mathbb{P}\text{-a.s.,}\\
&\int_0^T (Y_s - L_s)dK_s = 0, \mathbb{P}\text{-a.s. (Skorokhod condition),}
\end{aligned} \tag{3.9}$$

where $dX_s := \widehat{\sigma}_s dW_s$, $\widehat{\sigma}_s \in \mathbb{S}_d^{\geqslant 0}$(i.e., the set of $d \times d$ nonnegative-definite symmetric matrices), and $f_\cdot(\cdot, \cdot, \cdot)$ is a progressively measurable map satisfying some conditions to be specified later. Note that here the underlying filtration is not only generated by a Brownian motion, so we introduce another component in the definition of a supersolution a backward SDE, namely a martingale M which is orthogonal to standard Brownian motion W. K is a continuous nondecreasing process which pushes upwards the process Y in order to keep it above the obstacle L. The last equation means that the process K acts only when the process Y reaches the obstacle L. Note that we allow L to take the value $-\infty$, so the notion of reflected backward SDE covers backward SDE of Pardoux and Peng [PP90]. By the pair (f, ξ), we mean the corresponding backward SDE, without the last two conditions concerning the obstacle L in the above (3.9).

Assumption 3.3.1. *Given the terminal condition ξ and generator $f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$. For ease of notation, we denote $f_t^0 := f_t(\omega, 0, 0)$.*

1. *The random variable ξ is $\overline{\mathcal{F}}_T^{\mathbb{P}}$-measurable and for every fixed (y, z), the map $(t, \omega) \mapsto f_t(\omega, y, z)$ is $\overline{\mathbb{F}}^{\mathbb{P}}$-progressively measurable.*

2. *Lipschitz in y: there is a constant $\kappa \geqslant 0$, s.t. for each $(t, z, \omega) \in [0, T] \times \mathbb{R}^d \times \Omega$, and any $y, y' \in \mathbb{R}$,*

$$|f_t(\omega, y, z) - f_t(\omega, y', z)| \leqslant \kappa |y - y'|.$$

3. *Growth condition: for every $(t, \omega, y, z) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d$, there exists $\alpha \geqslant 0, \beta \geqslant 0$ and $\gamma > 0$*

s.t.

$$|f_t(\omega, y, z)| \leqslant \alpha + \beta|y| + \frac{\gamma}{2}|\sigma_t z|^2.$$

4. *Concave in* z : *for all* $(\zeta, y) \in (0,1) \times \mathbb{R}$, *all* $z_1, z_2 \in \mathbb{R}^d$, *and* $\mathbb{P}$ *is a fixed probability measure,*

$$f_t(\omega, y, \zeta z_1 + (1-\zeta)z_2) \geqslant \zeta f_t(\omega, y, z_1) + (1-\zeta) f_t(\omega, y, z_2), dt \otimes d\mathbb{P}\text{-a.e.}$$

3.3.2 Two a priori estimates

Proposition 3.3.2. *Let* (f, ξ, L) *be a triple s.t.* $\xi^+ \in \mathbb{L}^\infty(\mathbb{P}, \mathbb{R})$, $L^+ \in \mathbb{C}^\infty(\mathbb{P})$, *and* f *satisfies the quadratic growth (i.e., (3) in Assumption 3.3.1). If* (Y, Z, M, K) *is a solution of the quadratic RBSDE* (f, ξ, L), *and* $Y^+ \in \mathbb{C}^\infty(\mathbb{P})$, *then it holds* $\mathbb{P}$*-a.s. that*

$$Y_t \leqslant c_0 + \frac{1}{\gamma} \ln \mathbb{E}[e^{\gamma e^{\beta T}(\xi^+ \vee L_*^+)} | \overline{\mathcal{F}}_t^{\mathbb{P}}], \quad \forall t \in [0,T]. \tag{3.10}$$

Proof. **(1).** This part is for finding a 'bound generator' of the original generator f and solving the corresponding random ODE(3.2) as well as giving some estimate of its solution. Note that we use the exponential transform originally used in Briand and Hu [BH08] to search this 'bound generator'. Indeed, $(Y, Z, M, K) \in \mathbb{C}^0_{\overline{\mathbb{F}}^{\mathbb{P}}} \times \mathbb{H}^2(\mathbb{P}, \mathbb{R}^d) \times \mathbb{M}(\mathbb{P}) \times \mathbb{I}(\mathbb{P})$ with $Y^+ \in \mathbb{C}^\infty(\mathbb{P})$ is a solution of RBSDE(f, ξ, L) if and only if $(\widetilde{Y}, \widetilde{Z}, \widetilde{M}, \widetilde{K}) := (e^{\gamma Y}, \gamma e^{\gamma Y} Z, \gamma \int_0^\cdot e^{\gamma Y_s} dM_s, \gamma \int_0^\cdot e^{\gamma Y_s} dK_s) \in \mathbb{C}^\infty(\mathbb{P}) \times \mathbb{H}^2(\mathbb{P}, \mathbb{R}^d) \times \mathbb{M}(\mathbb{P}) \times \mathbb{I}(\mathbb{P})$ is a solution of RBSDE$(\widetilde{f}, e^{\gamma\xi}, e^{\gamma L})$ with

$$\begin{aligned}\widetilde{f}_s(Y_s, Z_s, M) &:= \widetilde{f}_s(Y_s, Z_s) - \mathbf{1}_{\{Y_s>0\}} \frac{\gamma^2}{2} Y_s \frac{d\langle M\rangle_s}{ds} \\ &:= \mathbf{1}_{\{Y_s>0\}} \gamma Y_s \left[f_s\left(\frac{\ln Y_s}{\gamma}, \frac{Z_s}{\gamma Y_s}\right) - \frac{\gamma}{2}\left(\frac{Z_s}{\gamma Y_s}\right)^2 \widehat{\sigma}_s^2 \right] - \mathbf{1}_{\{Y_s>0\}} \frac{\gamma^2}{2} Y_s \frac{d\langle M\rangle_s}{ds},\end{aligned} \tag{3.11}$$

for any $s \in [0,T]$. We define $\mu := \alpha\gamma \vee \beta \vee 1$, then by the quadratic growth (3) in Assumption 3.3.1 that $ds \otimes d\mathbb{P}$-a.e.,

$$\widetilde{f}_s(y, z) \leqslant H(y) := y(\mu + \beta \ln y) \mathbf{1}_{\{y \geqslant 1\}} + \mu \mathbf{1}_{\{y<1\}}, \tag{3.12}$$

for any $(s, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$. From its definition, we know that $H(\cdot)$ is strictly positive, increasing, continuous and convex function satisfying $\int_0^\infty \frac{1}{H(y)} dy = \infty$. Note that properties of the following ODE w.r.t. $H(\cdot)$

$$\phi(t) = e^{\gamma x} + \int_t^{\widetilde{T}} H(\phi(s))ds, t \in [0, \widetilde{T}].$$

has been precisely discussed as (3.2) in §3.2 and we continue to adopt their corresponding notations here.

(2). For any $\omega \in \Omega$, [LX07](Theorem 6.2) shows that the following reflected ODE

$$\begin{aligned} &e^{\gamma L_t(\omega)} \leqslant \Lambda_t(\omega) = e^{\gamma \xi(\omega)} + \int_t^T H(\Lambda_s(\omega))ds + k_T(\omega) - k_t(\omega), t \in [0, T], \\ &\int_0^T (\Lambda_s(\omega) - e^{\gamma L_s(\omega)})dk_s(\omega) = 0, \end{aligned}$$

has a unique solution $(\Lambda_\cdot(\omega), k_\cdot(\omega)) \in \mathbb{C}^0_{\mathbb{F}^{\mathbb{P}}_+} \times \mathbb{I}(\mathbb{P})$, satisfying

$$\begin{aligned} \Lambda_t(\omega) &= \sup_{s \in [t,T]} \left(\int_t^s H(\Lambda_r(\omega))dr + e^{\gamma \xi} \mathbf{1}_{\{s=T\}} + e^{\gamma L_s} \mathbf{1}_{\{s<T\}} \right) \\ &= \sup_{s \in [t,T]} u_t^s(\omega), t \in [0, T], \end{aligned} \tag{3.13}$$

where $u_r^s(\omega)_{r \in [0,s]}$ denotes the unique solution of the following ODE

$$u_r^s(\omega) = e^{\gamma \xi(\omega)} \mathbf{1}_{\{s=T\}} + e^{\gamma L_s} \mathbf{1}_{\{s<T\}} + \int_r^s H(u_b^s(\omega))db, r \in [0, s], \tag{3.14}$$

and by comparing (3.2) and (3.14), it's direct to get that $u_r^s(\omega) = \phi_r^s\big(\xi(\omega)\mathbf{1}_{\{s=T\}} + L_s(\omega)\mathbf{1}_{\{s<T\}}\big)$. Then, we get from (3.13) and ($\phi 4$) that for $t \in [0, T]$,

$$0 \leqslant e^{\gamma L_t} \leqslant \Lambda_t(\omega) = \sup_{s \in [t,T]} u_t^s(\omega) \leqslant \exp\left\{\mu\varphi(T) + \gamma e^{\beta T}(\xi^+(\omega) \vee L_*^+(\omega))\right\}, t \in [0, T]. \tag{3.15}$$

Besides, for any $0 \leqslant t_1 < t_2 \leqslant T$, we get from (3.13) and the fact ($\phi 1$) that $\phi_r^s(x)$ is decreasing w.r.t. r,

$$\Lambda_{t_1}(\omega) = \sup_{s \in [t_1,T]} u_{t_1}^s(\omega) \geqslant \sup_{s \in [t_2,T]} u_{t_1}^s(\omega) \geqslant \sup_{s \in [t_2,T]} u_{t_2}^s(\omega) = \Lambda_{t_2}(\omega), \tag{3.16}$$

which, together with (3.13), means that $t \mapsto \Lambda_t(\omega)$ is decreasing and continuous. Besides, we get from

(3.13) and the fact ($\phi 2$) that $\phi_t^s(x)$ is increasing w.r.t. s,

$$\Lambda_t(\omega) = \sup_{s\in[t,T]} u_t^s(\omega) = \sup\{u_t^s(\omega) : s \in ([t,T)\cap\mathbb{Q})\cup\{T\}\}. \tag{3.17}$$

For any $0 \leqslant t \leqslant s \leqslant T$, we get from ($\phi 3$) the continuity of $\phi_t^s(\cdot)$ that $u_t^s(\omega) = \phi_t^s(\xi\mathbf{1}_{\{s=T\}} + L_s\mathbf{1}_{\{s<T\}})$ is $\overline{\mathcal{F}}_s^{\mathbb{P}}$-measurable. We get from (3.17) that for any $t \in [0,T]$, Λ_t is $\overline{\mathcal{F}}_T^{\mathbb{P}}$-measurable (but not necessarily $\overline{\mathcal{F}}_t^{\mathbb{P}}$-measurable).

(3). We now introduce an $\overline{\mathbb{F}}^{\mathbb{P}}$-adapted process $\mathfrak{f}_t := \mathbb{E}[H(\Lambda_t)|\overline{\mathcal{F}}_t^{\mathbb{P}}], t \in [0,T]$. By (3.16), we know that Λ_t is decreasing w.r.t. t, and since $H(\cdot)$ is increasing, then for $0 \leqslant t \leqslant s \leqslant T$, it holds $\mathbb{E}[\mathfrak{f}_s|\overline{\mathcal{F}}_t^{\mathbb{P}}] = \mathbb{E}[H(\Lambda_s)|\overline{\mathcal{F}}_t^{\mathbb{P}}] \leqslant \mathbb{E}[H(\Lambda_t)|\overline{\mathcal{F}}_t^{\mathbb{P}}] = \mathfrak{f}_t, \mathbb{P}$-a.s., so $\mathfrak{f}$ is a supermartingale. As we know $Y^+ \in \mathbb{C}^\infty(\mathbb{P})$, we have $(\xi^+, L^+) \in \mathbb{L}^\infty(\mathbb{P},\mathbb{R}) \times \mathbb{C}^\infty(\mathbb{P})$.

By the continuity of $H(\Lambda_\cdot)$, (3.15) and bounded convergence theorem, we have $\mathbb{E}[\mathfrak{f}_t] = \mathbb{E}[H(\Lambda_t)] = \lim_{s\downarrow t}\mathbb{E}[H(\Lambda_s)] = \lim_{s\downarrow t}\mathbb{E}[\mathfrak{f}_s], t \in [0,T]$. By [KS12](Theorem 1.3.13), $\mathfrak{f}$ has a right-continuous modification $\widetilde{\mathfrak{f}}$, so $\mathfrak{f}$ could be seen as a generator which is independent of (y,z). By Fubini's theorem, Jensen's inequality and (3.15), we have

$$\begin{aligned}\mathbb{E}\bigg[\int_0^T |\widetilde{\mathfrak{f}}_s|^2 ds\bigg] &= \int_0^T \mathbb{E}[|\widetilde{\mathfrak{f}}_s|^2]ds = \int_0^T \mathbb{E}[|\mathfrak{f}_s|^2]ds \leqslant_{\text{(Jensen)}} \int_0^\infty \mathbb{E}\Big[\mathbb{E}[|H(\Lambda_s)|^2|\overline{\mathcal{F}}_s^{\mathbb{P}}]\Big]ds \\ &= \int_0^T \mathbb{E}[|H(\Lambda_s)|^2]ds < \infty.\end{aligned} \tag{3.18}$$

Because $e^{\gamma\xi} \in \mathbb{L}^\infty(\mathbb{P},\mathbb{R})$ and $e^{\gamma L} \in \mathbb{C}^\infty(\mathbb{P})$, we could follow [KKP+97](Theorem 5.2 and Proposition 2.3) to obtain that RBSDE($\widetilde{f}, e^{\gamma\xi}, e^{\gamma L}$) admits a unique solution $(Y,Z,M,K) \in \mathbb{C}^2(\mathbb{P}) \times \mathbb{H}^2(\mathbb{P},\mathbb{R}^d) \times \mathbb{M}^2(\mathbb{P}) \times \mathbb{I}^2(\mathbb{P})$ and that for any $t \in [0,T]$,

$$\mathcal{Y}_t = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\bigg[\int_t^\tau \widetilde{\mathfrak{f}}_s ds + e^{\gamma\xi}\mathbf{1}_{\{\tau=T\}} + e^{\gamma L_\tau}\mathbf{1}_{\{\tau<T\}}\bigg|\overline{\mathcal{F}}_t^{\mathbb{P}}\bigg], \mathbb{P}\text{-a.s.}. \tag{3.19}$$

By Fubini's theorem, for any $t \in [0,T]$, $\tau \in \mathcal{T}_{t,T}$,

$$\begin{aligned}\mathbb{E}\Big[\int_t^\tau \widetilde{\mathfrak{f}}_s ds\Big|\overline{\mathcal{F}}_t^{\mathbb{P}}\Big] &= \int_t^T \mathbb{E}[\mathbf{1}_{\{s\leqslant\tau\}}\widetilde{\mathfrak{f}}_s|\overline{\mathcal{F}}_t^{\mathbb{P}}]ds = \int_t^T \mathbb{E}\Big[\mathbf{1}_{\{s\leqslant\tau\}}\mathbb{E}[H(\Lambda_s)|\overline{\mathcal{F}}_s^{\mathbb{P}}]\Big|\overline{\mathcal{F}}_t^{\mathbb{P}}\Big]ds \\ &= \int_t^T \mathbb{E}\Big[\mathbf{1}_{\{s\leqslant\tau\}}H(\Lambda_s)|\overline{\mathcal{F}}_t^{\mathbb{P}}\Big]ds = \mathbb{E}\Big[\int_t^\tau H(\Lambda_s)ds\Big|\overline{\mathcal{F}}_t^{\mathbb{P}}\Big], \mathbb{P}\text{-a.s.}.\end{aligned} \tag{3.20}$$

Then from (3.19), (3.13) and (3.15) (by order), we get that for any $t \in [0,T], \mathbb{P}$-a.s.,

$$\begin{aligned}\mathcal{Y}_t =& \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\Big[\int_t^\tau H(\Lambda_s)ds + e^{\gamma\xi}\mathbf{1}_{\{\tau=T\}} + e^{\gamma L_\tau}\mathbf{1}_{\tau<T}\Big|\overline{\mathcal{F}}_t^{\mathbb{P}}\Big] \\ \leqslant& \mathbb{E}[\Lambda_t|\overline{\mathcal{F}}_t^{\mathbb{P}}] \leqslant e^{\mu\varphi(T)}\mathbb{E}\Big[e^{\gamma e^{\beta T}(\xi^+\vee L_*^+)}\Big] \leqslant C_*, \mathbb{P}\text{-a.s.},\end{aligned} \tag{3.21}$$

where $C_* := \exp\Big\{\mu\varphi(T) + \gamma e^{\beta T}\Big(\|\xi^+\|_{\mathbb{L}^\infty(\mathbb{P},\mathbb{R})} \vee \|L^+\|_{\mathbb{C}^\infty(\mathbb{P})}\Big)\Big\}$. By the continuity of process $\mathcal{Y}$, it holds that $\mathbb{P}$-a.s. for $t \in [0,T]$,

$$0 < e^{\gamma L_t} \leqslant \mathcal{Y}_t \leqslant e^{\mu\varphi(T)}\mathbb{E}\Big[e^{\gamma e^{\beta T}(\xi^+\vee L_*^+)}\Big|\overline{\mathcal{F}}_t^{\mathbb{P}}\Big] \leqslant C_*, \tag{3.22}$$

which shows that $\mathcal{Y} \in \mathbb{C}^\infty(\mathbb{P})$ with $\|\mathcal{Y}\|_{\mathbb{C}^\infty(\mathbb{P})} \leqslant C_*$.

(4). The last step is to show that $\mathbb{P}(\widetilde{Y}_t \leqslant \mathcal{Y}_t, \forall t \in [0,T]) = 1$. In fact, given $n \in \mathbb{N}$, we define the $\overline{\mathbb{F}}^{\mathbb{P}}$-stopping time $\tau_n := \inf\{t \in [0,T] : \int_0^t |\widehat{\sigma}_s\mathcal{Z}_s|^2 ds > n\} \wedge T$. Clearly, $\lim_{n\to\infty}\tau_n = T, \mathbb{P}$-a.s. Using Tanaka's formula in the process $(\widetilde{Y} - \mathcal{Y})^+$, we get that

$$\begin{aligned}(\widetilde{Y}_{\tau_n\wedge t} - \mathcal{Y}_{\tau_n\wedge t})^+ =& (\widetilde{Y}_{\tau_n} - \mathcal{Y}_{\tau_n})^+ + \int_{\tau_n\wedge t}^{\tau_n} \mathbf{1}_{\{\widetilde{Y}_s>\mathcal{Y}_s\}}(\widetilde{f}_s(Y_s,Z_s,M) - \widetilde{\mathfrak{f}}_s)ds \\ &+ \int_{\tau_n\wedge t}^{\tau_n} \mathbf{1}_{\{\widetilde{Y}_s>\mathcal{Y}_s\}}(d\widetilde{K}_s - d\mathcal{K}_s) - \int_{\tau_n\wedge t}^{\tau_n} \mathbf{1}_{\{\widetilde{Y}_s>\mathcal{Y}_s\}}(\widetilde{Z}_s - \mathcal{Z}_s)dX_s \\ &- \int_{\tau_n\wedge t}^{\tau_n} \mathbf{1}_{\{\widetilde{Y}_s>\mathcal{Y}_s\}}(d\widetilde{M}_s - dM_s) - \frac{1}{2}\int_{\tau_n\wedge t}^{\tau_n} d\mathfrak{L}_s, t \in [0,T],\end{aligned} \tag{3.23}$$

where $\mathfrak{L}$ is the local time, a $\mathbb{R}$-valued, $\overline{\mathbb{F}}^{\mathbb{P}}$-adapted, increasing and continuous process. Because $H(\cdot)$ is increasing, continuous and convex, by Jensen's inequality and (3.21) we could show that

$$H(\widetilde{Y}_s) - \mathfrak{f}_s = H(\widetilde{Y}_s) - \mathbb{E}[H(\Lambda_s)|\overline{\mathcal{F}}_s^{\mathbb{P}}] \leqslant_{(H(\cdot)\uparrow,(3.21))} H(\widetilde{Y}_s) - H(\mathcal{Y}_s) \leqslant C_H|\widetilde{Y}_s - \mathcal{Y}_s|, s \in [0,T], \tag{3.24}$$

where C_H is the Lipschitz coefficient of $H(\cdot)$ over $\{x \in \mathbb{R}, |x| \leqslant \|\widetilde{Y}\|_{\mathbb{C}^\infty(\mathbb{P})} \vee \|\mathcal{Y}\|_{\mathbb{C}^\infty(\mathbb{P})}\}$. Besides, by using (3.22) and the flat-off condition of (Y,Z,M,K) we could get that

$$\int_0^T \mathbf{1}_{\{\widetilde{Y}_s>\mathcal{Y}_s\}} d\widetilde{K}_s \leqslant \int_0^T \mathbf{1}_{\{\widetilde{Y}_s>\mathcal{Y}_s\geqslant \widetilde{L}_s}\} d\widetilde{K}_s = 0, \mathbb{P}\text{-a.s.} \tag{3.25}$$

Taking the expectation in (3.23), we can deduce from (3.12), Fubini's theorem, (3.25) and (3.24) that

$$\begin{aligned}
&\mathbb{E}[(\widetilde{Y}_{\tau_n\wedge t}-\mathcal{Y}_{\tau_n\wedge t})^+]-\mathbb{E}[(\widetilde{Y}_{\tau_n}-\mathcal{Y}_{\tau_n})^+] \leqslant_{(3.23,3.12,3.25)} \int_t^T \mathbb{E}\Big[\mathbf{1}_{\{s\leqslant\tau_n\}}\mathbf{1}_{\{\widetilde{Y}_s>\mathcal{Y}_s\}}(H(\widetilde{Y}_s)-\mathfrak{f}_s)\Big]ds \\
\leqslant_{(3.24)}& C_H\int_t^T \mathbb{E}\Big[\mathbf{1}_{\{s\leqslant\tau_n\}}\mathbf{1}_{\{\widetilde{Y}_s>\mathcal{Y}_s\}}|\widetilde{Y}_s-\mathcal{Y}_s|\Big]ds \leqslant C_H\int_t^T \mathbb{E}\Big[\mathbf{1}_{\{s\leqslant\tau_n\}}\mathbf{1}_{\{\widetilde{Y}_s>\mathcal{Y}_s\}}(\widetilde{Y}_s-\mathcal{Y}_s)^+\Big]ds \\
\leqslant & C_H\int_t^T \mathbb{E}\Big[(\widetilde{Y}_{\tau_n\wedge s}-\mathcal{Y}_{\tau_n\wedge s})^+\Big]ds, t\in[0,T].
\end{aligned}$$

By Gronwall's inequality, we have that $\mathbb{E}[(\widetilde{Y}_{\tau_n\wedge t}-\mathcal{Y}_{\tau_n\wedge t})^+]\leqslant e^{C_HT}\mathbb{E}\Big[(\widetilde{Y}_{\tau_n}-\mathcal{Y}_{\tau_n})^+\Big], \forall t\in[0,T]$. Letting $n\to\infty$, by the continuity of processes $\widetilde{Y},\mathcal{Y}$ and bounded convergence theorem, we have that for any $t\in[0,T]$,

$$\mathbb{E}[(\widetilde{Y}_t-\mathcal{Y}_t)^+]=0, \text{ then } \widetilde{Y}_t\leqslant\mathcal{Y}_t, \mathbb{P}\text{-a.s.} \tag{3.26}$$

By once again the continuity of processes $\widetilde{Y}$ and $\mathcal{Y}$, we get that $\mathbb{P}(\widetilde{Y}_t\leqslant\mathcal{Y}_t,\forall t\in[0,T])=1$, which together with (3.22) gives (3.10). □

By letting L go to $-\infty$, we could obtain the following version for corresponding BSDE.

Proposition 3.3.3 (BSDE version)**.** *Let (f,ξ) be s.t. f satisfies the quadratic growth (i.e., (3) in Assumption 3.3.1). If (Y,Z,M,K) is a solution of the quadratic BSDE (f,ξ), and $Y^+\in\mathbb{C}^\infty(\mathbb{P})$, then it holds $\mathbb{P}$-a.s. that*

$$Y_t\leqslant c_0+\frac{1}{\gamma}\ln\mathbb{E}[e^{\gamma e^{\beta T}\xi^+}|\overline{\mathcal{F}}^{\mathbb{P}}_t], \quad \forall t\in[0,T]. \tag{3.27}$$

Proposition 3.3.4. *Let (f,ξ,L) be a triple s.t. f satisfies the quadratic growth (i.e., (3) in Assumption 3.3.1). If (Y,Z,M,K) is a solution of the quadratic RBSDE (f,ξ,L) s.t. $Y\in\mathbb{C}^{\exp(\lambda\gamma,\lambda'\gamma)}(\mathbb{P})$ for some $\lambda,\lambda'>1$ with $\frac{1}{\lambda}+\frac{1}{\lambda'}<1$, then for all $p\in\Big(1,\frac{\lambda\lambda'}{\lambda+\lambda'}\Big)$,*

$$\mathbb{E}\Bigg[\bigg(\int_0^T|\widehat{\sigma}_sZ_s|^2ds+d\langle M\rangle_s\bigg)^p+K_T^p\Bigg]\leqslant c_{\lambda,\lambda',p}\mathbb{E}[e^{\lambda\gamma Y_*^-}+e^{\lambda'\gamma Y_*^+}]<\infty. \tag{3.28}$$

Proof. From the definition of p_0, we know that $p_0=\sqrt{\frac{\lambda\lambda'}{p(\lambda+\lambda')}}\wedge 2>1$, and define the stopping times $\tau_n:=\inf\{t\in[0,T]:\langle\int_0^\cdot e^{\frac{-p_0\gamma Y_s}{2}}(Z_s\cdot dX_s+dM_s)\rangle_t>n\}\wedge T,\forall n\in\mathbb{N}$. Because $\mathbb{E}[e^{\lambda\gamma Y_*^-}]<\infty$ and

$Z \in \mathbb{H}^{2,\text{loc}}(\mathbb{P},\mathbb{R}^d)$, $M \in \mathbb{M}(\mathbb{P})$, we get that $Y_*^- + \langle \int_0^\cdot Z_s \cdot dX_s + dM_s \rangle_T < \infty, \mathbb{P}$-a.s.. Then, we have $\langle \int_0^\cdot e^{\frac{-p_0\gamma Y_s}{2}}(Z_s \cdot dX_s + dM_s)\rangle_T \leqslant e^{p_0\gamma Y_*^-}\langle \int_0^\cdot Z_s \cdot dX_s + dM_s\rangle_T < \infty, \mathbb{P}$-a.s., and then $\lim_{n\to\infty}\tau_n = T, \mathbb{P}$-a.s.

Next, we could apply Itô's formula to $e^{-p_0\gamma Y}$ on $[0,\tau_n]$ for any $n \in \mathbb{N}$. Note that $\alpha + \beta x \leqslant \Big(\alpha \vee \frac{\beta}{(p_0^2-p_0)\gamma}\Big)e^{(p_0^2-p_0)\gamma x}, \forall x \geqslant 0$, we obtain $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\frac{p_0^2\gamma^2}{2}\int_0^{\tau_n} e^{-p_0\gamma Y_s}\big[Z_s^2\hat{\sigma}_s^2 ds + d\langle M.\rangle_s\big] \\
\leqslant & e^{-p_0\gamma Y_{\tau_n}} - p_0\gamma\int_0^{\tau_n} e^{-p_0\gamma Y_s} f_s(Y_s,Z_s)ds + e^{-p_0\gamma Y_s}(Z_s\cdot dX_s + dM_s - dK_s) \\
\leqslant & e^{p_0\gamma Y_*^-} + p_0\gamma\int_0^{\tau_n} e^{-p_0\gamma Y_s}\bigg[\Big(\alpha \vee \frac{\beta}{(p_0^2-p_0)\gamma}\Big)e^{(p_0^2-p_0)\gamma|Y_s|}\bigg]ds + \frac{\gamma}{2}e^{-p_0\gamma Y_s}\big[|Z_s|^2\hat{\sigma}_s^2 ds + d\langle M\rangle_s\big] \\
& + p_0\gamma\bigg|\int_0^{\tau_n} e^{-p_0\gamma Y_s}(Z_s\cdot dX_s + dM_s)\bigg|.
\end{aligned} \tag{3.29}$$

Note that $\int_0^{\tau_n} e^{-p_0\gamma Y_s + (p_0^2-p_0)\gamma|Y_s|}ds \leqslant \int_0^{\tau_n} e^{-p_0^2\gamma \mathbf{1}_{\{Y_s<0\}}Y_s}ds \leqslant Te^{p_0^2\gamma Y_*^-}, \mathbb{P}$-a.s.. Then we could combine this with Burkholder-Davis-Gundy inequality and (3.29) to show that

$$\begin{aligned}
&\mathbb{E}\bigg[\langle\int_0^\cdot e^{\frac{-p_0\gamma Y_s}{2}}(Z_s\cdot dX_s + dM_s)\rangle_{\tau_n}^{\frac{\lambda}{p_0^2}}\bigg] \\
\leqslant & C\bigg\{\mathbb{E}[e^{\lambda\gamma Y_*^-}] + \mathbb{E}\bigg\{\bigg[p_0\gamma\Big(\alpha\vee\frac{\beta}{(p_0^2-p_0)\gamma}\Big)Te^{p_0^2\gamma Y_*^-}\bigg]^{\frac{\lambda}{p_0^2}}\bigg\} + \mathbb{E}\bigg[\Big(\int_0^{\tau_n} e^{-p_0\gamma Y_s}(Z_s\cdot dX_s + dM_s)\Big)^{\frac{\lambda}{p_0^2}}\bigg]\bigg\} \\
\leqslant & C\bigg\{\mathbb{E}[e^{\lambda\gamma Y_*^-}] + \mathbb{E}\bigg[\langle\int_0^\cdot e^{-p_0\gamma Y_s}(Z_s\cdot dX_s + dM_s)\rangle_{\tau_n}^{\frac{\lambda}{2p_0^2}}\bigg]\bigg\};
\end{aligned} \tag{3.30}$$

besides, we can apply Young's inequality $Cab \leqslant \frac{1}{2}C^2a^2 + \frac{1}{2}b^2, \forall a,b \in \mathbb{R}_+$ on the second part of the last term, i.e.,

$$\begin{aligned}
&C\langle\int_0^\cdot e^{-p_0\gamma Y_s}(Z_s\cdot dX_s + dM_s)\rangle_{\tau_n}^{\frac{\lambda}{2p_0^2}} \leqslant Ce^{\frac{\lambda}{2p_0}\gamma Y_*^-}\langle\int_0^\cdot e^{\frac{-p_0\gamma Y_s}{2}}(Z_s\cdot dX_s + dM_s)\rangle_{\tau_n}^{\frac{\lambda}{2p_0^2}} \\
\leqslant_{(\text{Young})} & \frac{1}{2}C^2e^{\frac{\lambda}{p_0}\gamma Y_*^-} + \frac{1}{2}\langle\int_0^\cdot e^{\frac{-p_0\gamma Y_s}{2}}(Z_s\cdot dX_s + dM_s)\rangle_{\tau_n}^{\frac{\lambda}{p_0^2}} \\
\leqslant_{(p_0>1)} & \frac{1}{2}C^2e^{\lambda\gamma Y_*^-} + \frac{1}{2}\langle\int_0^\cdot e^{\frac{-p_0\gamma Y_s}{2}}(Z_s\cdot dX_s + dM_s)\rangle_{\tau_n}^{\frac{\lambda}{p_0^2}}.
\end{aligned} \tag{3.31}$$

To conclude the last two steps, we have

$$\mathbb{E}\bigg[\langle\int_0^\cdot e^{\frac{-p_0\gamma Y_s}{2}}(Z_s\cdot dX_s + dM_s)\rangle_{\tau_n}^{\frac{\lambda}{p_0^2}}\bigg] \leqslant C\mathbb{E}[e^{\lambda\gamma Y_*^-}] + \frac{1}{2}\mathbb{E}\bigg[\langle\int_0^\cdot e^{\frac{-p_0\gamma Y_s}{2}}(Z_s\cdot dX_s + dM_s)\rangle_{\tau_n}^{\frac{\lambda}{p_0^2}}\bigg]. \tag{3.32}$$

Since $\mathbb{E}\Big[\langle\int_0^\cdot e^{\frac{-p_0\gamma Y_s}{2}}(Z_s\cdot dX_s+dM_s)\rangle_{\tau_n}^{\frac{\lambda}{p_0^2}}\Big]\leqslant n^{\frac{\lambda}{p_0^2}}<\infty$ due to the definition of τ_n, it follows that

$$\mathbb{E}\Big[\langle\int_0^\cdot e^{\frac{-p_0\gamma Y_s}{2}}(Z_s\cdot dX_s+dM_s)\rangle_{\tau_n}^{\frac{\lambda}{p_0^2}}\Big]\leqslant C\mathbb{E}[e^{\lambda\gamma Y_*^-}].$$

Let $n\to\infty$, we have by monotone convergence theorem that

$$\mathbb{E}\Big[\langle\int_0^\cdot e^{\frac{-p_0\gamma Y_s}{2}}(Z_s\cdot dX_s+dM_s)\rangle_{T}^{\frac{\lambda}{p_0^2}}\Big]\leqslant C\mathbb{E}[e^{\lambda\gamma Y_*^-}].$$

Note that $\frac{\lambda p_0 p}{\lambda-p_0^2p}<\frac{\lambda p_0^2 p}{\lambda-p_0^2p}\leqslant\lambda'$, we use Young's inequality with $\widetilde{p}=\frac{\lambda}{\lambda-p_0^2p}$ and $\widetilde{q}=\frac{\lambda}{p_0^2p}$ (note that by the definition of $p_0, p_0^2p=\frac{\lambda\lambda'}{\lambda+\lambda'}<1$ and $\lambda>1$, so $\widetilde{q}=\frac{\lambda}{p_0^2p}>1$) to get that

$$\begin{aligned}\mathbb{E}\Big[\langle\int_0^\cdot Z_s\cdot dX_s+dM_s\rangle_T^{\frac{p}{2}}\Big]\leqslant&\mathbb{E}\Big[e^{p_0p\gamma Y_*^+}\langle\int_0^\cdot e^{\frac{-p_0\gamma Y_s}{2}}(Z_\cdot\cdot dX_\cdot+dM_\cdot)\rangle_T^{\frac{p}{2}}\Big]\\ \leqslant&C\Big(\mathbb{E}[e^{\lambda'\gamma Y_*^+}]+\mathbb{E}\Big[e^{\lambda\gamma Y_*^-}\Big]\Big)<\infty,\end{aligned}\tag{3.33}$$

where $e^{p_o p\gamma Y_*^+\cdot\frac{\lambda}{\lambda-p_0^2p}}=e^{\frac{\lambda p_0 p}{\lambda-p_0^2p}\cdot\gamma Y_*^+}\leqslant e^{\lambda'\gamma Y_*^+}$. Besides, we have $\mathbb{P}$-a.s. that

$$\begin{aligned}K_T=&Y_0-\xi-\int_0^T f_s(Y_s,Z_s)ds+\int_0^T Z_s\cdot dX_s+\int_0^T dM_s\\ \leqslant&\alpha T+(2+\beta T)(Y_*^-+Y_*^+)+\frac{\gamma}{2}\int_0^T|Z_s|^2\widehat{\sigma}_s^2ds+\Big|\int_0^T Z_s dX_s+dM_s\Big|.\end{aligned}$$

By Burkholder-Davis-Gundy inequality and (3.33), we have

$$\begin{aligned}&\mathbb{E}[K_T^p]\leqslant C\mathbb{E}\Big[1+(Y_*^-)^p+(Y_*^+)^p+\langle\int_0^\cdot Z_s\cdot dX_s+dM_s\rangle_T^{\frac{p}{2}}\Big]\\ \leqslant&C\Big(\mathbb{E}[e^{\lambda'\gamma Y_*^+}]+\mathbb{E}\Big[e^{\lambda\gamma Y_*^-}\Big]\Big)<\infty.\end{aligned}$$

□

Proposition 3.3.5 (BSDE version)**.** *Let (f,ξ) be s.t. f satisfies the quadratic growth (i.e., (3) in Assumption 3.3.1). If (Y,Z,M,K) is a solution of the quadratic BSDE (f,ξ) s.t. $Y\in\mathbb{C}^{\exp(\lambda\gamma,\lambda'\gamma)}(\mathbb{P})$*

for some $\lambda, \lambda' > 1$ *with* $\frac{1}{\lambda} + \frac{1}{\lambda'} < 1$, *then for all* $p \in \left(1, \frac{\lambda\lambda'}{\lambda+\lambda'}\right)$,

$$\mathbb{E}\left[\left(\int_0^T |\widehat{\sigma}_s Z_s|^2 ds + d\langle M\rangle_s\right)^p + K_T^p\right] \leqslant c_{\lambda,\lambda',p}\mathbb{E}[e^{\lambda\gamma Y_*^-} + e^{\lambda'\gamma Y_*^+}] < \infty. \tag{3.34}$$

3.3.3 A monotone stability result

The existence is based on the following monotone stability result.

Theorem 3.3.6. *For any* $n \in \mathbb{N}$, *let* $\{(f^n, \xi_n, L^n)\}_{n\in\mathbb{N}}$ *be a triple and let* $(Y^n, Z^n, M^n, K^n) \in \mathbb{C}^0(\mathbb{P}) \times \mathbb{H}^{2,loc}(\mathbb{P};\mathbb{R}^d) \times \mathbb{M}(\mathbb{P}) \times \mathbb{I}(\mathbb{P})$ *be the corresponding solution of RBSDE* (f^n, ξ_n, L^n) *s.t.*

(M1) All $f^n, n \in \mathbb{N}$ *satisfy the quadratic growth (i.e., (3) in Assumption 3.3.1) where they share the same constants* $\alpha, \beta \geqslant 0$ *and* $\gamma > 0$;

(M2) There exists a function $f : [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}_d^{\geqslant 0} \to \mathbb{R}$ *s.t. for* $dt\otimes d\mathbb{P}-$*a.e.* $(t,\omega) \in [0,T]\times\Omega$, *the mapping* $f_t(\omega,\cdot,\cdot)$ *is continuous and* $f_t^n(\omega,y,z)$ *converges to* $f_t(\omega,y,z)$ *locally uniformly in* (y,z);

(M3) For some $L \in \mathbb{C}^0(\mathbb{P})$ *and some* $\mathbb{R}$*-valued,* $\overline{\mathbb{F}}_+^{\mathbb{P}}-$*adapted process* Y, *it holds* $\mathbb{P}$*-a.s. that for any* $t \in [0,T]$, $\{L_t^n\}_{n\in\mathbb{N}}$ *and* $\{Y_t^n\}_{n\in\mathbb{N}}$ *are both increasing (resp. decreasing) sequences w.r.t.* n *with* $\lim_{n\to\infty}\uparrow L_t^n = L_t$ *(resp.,* $\lim_{n\to\infty}\downarrow L_t^n = L_t$*) and* $\lim_{n\to\infty}\uparrow Y_t^n = Y_t$ *(resp.,* $\lim_{n\to\infty}\downarrow Y_t^n = Y_t$ *).*

Denote $\mathcal{L}_t := (L_t^1)^- \vee L_t^-$ *and* $\mathcal{Y}_t := (Y_t^1)^+ \vee Y_t^+, \forall t \in [0,T]$. *If* $\Xi := \mathbb{E}[e^{\lambda\gamma\mathcal{L}_*} + e^{\lambda'\gamma\mathcal{Y}_*}] < \infty$ *for some* $\lambda, \lambda' > 6$ *with* $\frac{1}{\lambda}+\frac{1}{\lambda'} < \frac{1}{6}$, *then* $Y \in \mathbb{C}^{\exp(\lambda\gamma,\lambda'\gamma)}(\mathbb{P})$ *and there exists* $(Z,M,K) \in \bigcap_{p\in\left(1,\frac{\lambda\lambda'}{\lambda+\lambda'}\right)} \mathbb{H}^{2,2p}(\mathbb{P};\mathbb{R}^d)\times\mathbb{M}^p(\mathbb{P})\times\mathbb{I}^p(\mathbb{P})$ *s.t. the quadruplet* (Y,Z,M,K) *is a solution of the RBSDE* (f,ξ,L) *with* $\xi := Y_T$.

In order to prove this result, we will divide it into several steps to verify. First of all, we know from the above statement that it holds $\mathbb{P}$-a.s. that

$$-\mathscr{L}_t \leqslant L_t^1 \wedge L_t \leqslant L_t^n \leqslant Y_t^n \leqslant Y_t^1 \vee Y_t \leqslant \mathscr{Y}_t, t \in [0,T], \forall n \in \mathbb{N}. \tag{3.35}$$

By setting $\lambda_0 := 5 + \frac{1}{2}\left(\frac{\lambda\lambda'}{\lambda+\lambda'} - 6\right) < \frac{\lambda\lambda'}{\lambda+\lambda'} - 1$, we have $p_0 := \frac{\lambda\lambda'}{\lambda\lambda'-\lambda_0(\lambda+\lambda')} \in \left(1, \frac{\lambda\lambda'}{\lambda+\lambda'}\right)$. Given $n \in \mathbb{N}$, (3.35) gives that $\mathbb{E}[e^{\lambda\gamma(Y^n)_*^-} + e^{\lambda'\gamma(Y^n)_*^+}] \leqslant \mathbb{E}[e^{\lambda\gamma\mathscr{L}_*} + e^{\lambda'\gamma\mathscr{Y}_*}] < \infty$. By setting $p = p_0$, Proposition 3.3.4

gives that

$$\mathbb{E}\Big[\langle\int_0^\cdot Z_s^n\cdot dX_s+M^n\rangle_T^{p_0}+(K_T^n)^{p_0}\Big]\leqslant C\Big(\mathbb{E}[e^{\lambda'\gamma(Y^n)_*^+}]+\mathbb{E}\Big[e^{\lambda\gamma(Y^n)_*^-}\Big]\Big)\leqslant C\Xi<\infty. \tag{3.36}$$

We know from above that $\{Z^n\}_{n\in\mathbb{N}}$ is a bounded subset of the reflexive Banach space $\mathbb{H}^{2,2p_0}(\mathbb{P},\mathbb{R}^d)$, which is also true for $\{M^n\}_{n\in\mathbb{N}}$ in the space $\mathbb{M}^{p_0}(\mathbb{P})$. By Yosida [Yos80](Theorem 5.2.1), $\{Z^n\}_{n\in\mathbb{N}}$ has a weakly convergent subsequence (for notation ease, we still represent it by $\{Z^n\}_{n\in\mathbb{N}}$) and the limit $Z\in\mathbb{H}^{2,2p_0}(\mathbb{P},\mathbb{R}^d)$. And $\{M^n\}_{n\in\mathbb{N}}$ has a weakly convergent subsequence and the limit $M\in\mathbb{M}^{p_0}(\mathbb{P})$.

Lemma 1.A. *Under the setting of Theorem 3.3.6, $\{Z^n\}_{n\in\mathbb{N}}$ strongly converges to Z in $\mathbb{H}^2(\mathbb{P},\mathbb{R}^d)$, and $\{M^n\}_{n\in\mathbb{N}}$ strongly converges to M in $\mathbb{M}(\mathbb{P})$.*

Proof. **Step 1.** Define $\phi(x):=\frac{1}{\lambda_0\gamma}(e^{\lambda_0\gamma x}-\lambda_0\gamma x-1)\geqslant 0,\forall x\in\mathbb{R}$. For any $m\geqslant n$, we define $\xi_{m,n}:=\xi_m-\xi_n$ and $\Theta_\cdot^{m,n}:=\Theta_\cdot^m-\Theta_\cdot^n$ for $\Theta=Y,Z,M,K,L($ and $U:=M-K)$. Using Itô's formula to the process $\phi(Y_\cdot^{m,n})$ on $[t,T]$ yields that

$$\begin{aligned}&\phi(Y_t^{m,n})+\frac{1}{2}\int_t^T\phi''(Y_s^{m,n})d\langle Z_\cdot^{m,n}\cdot X_\cdot+M_\cdot^{m,n}\rangle_s\\=&\phi(\xi_{m,n})+\int_t^T\phi'(Y_s^{m,n})[f^m(s,Y_s^m,Z_s^m)-f^n(s,Y_s^n,Z_s^n)]ds\\&-\int_t^T\phi'(Y_s^{m,n})(Z_s^{m,n}\cdot dX_s+dM_s^{m,n}),t\in[0,T].\end{aligned} \tag{3.37}$$

We can deduce from (3.35) and (3.36) that

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t\phi'(Y_s^{m,n})(Z_s^{m,n}\cdot dX_s+dM_s^{m,n})\Big|\Big]\leqslant_{\text{(BDG)}}C\mathbb{E}\Big[\langle\int_0^\cdot\phi'(Y_s^{m,n})\Big(Z_s^{m,n}\cdot dX_s+dM_s^{m,n}\Big)\rangle_T^{\frac12}\Big]\\=&C\mathbb{E}\Big[\langle\int_0^\cdot(e^{\lambda_0\gamma Y_s^{m,n}}-1)\Big(Z_s^{m,n}\cdot dX_s+dM_s^{m,n}\Big)\rangle_T^{\frac12}\Big]\quad(\phi'(x)=e^{\lambda_0\gamma x}-1,x\in\mathbb{R})\\\leqslant&C\mathbb{E}\Big[\langle\int_0^\cdot e^{\lambda_0\gamma Y_s^{m,n}}\Big(Z_s^{m,n}\cdot dX_s+dM_s^{m,n}\Big)\rangle_T^{\frac12}\Big],\end{aligned} \tag{3.38}$$

where we use $\lambda_0\gamma Y^{m,n}\geqslant 0$ because for $m\geqslant n,\{Y_n\}_n\uparrow$ and therefore $Y_m\geqslant Y_n$; note also that in the last term above, we have $e^{\lambda_0\gamma Y_s^{m,n}} \leqslant e^{\lambda_0\gamma(\mathscr{L}_*+\mathscr{Y}_*)}$ and then

$$\begin{aligned}
&C\mathbb{E}\bigg[e^{\lambda_0\gamma(\mathscr{L}_*+\mathscr{Y}_*)}\bigg(1+\langle\int_0^\cdot Z_s^{m,n}\cdot dX_s+dM_s^{m,n}\rangle_T\bigg)\bigg]\\
\leqslant& C\mathbb{E}\bigg[e^{\lambda_0 p_1\gamma\mathscr{L}_*}+e^{\lambda_0 p_2\gamma\mathscr{Y}_*}+\bigg(1+\langle\int_0^\cdot Z_s^{m,n}\cdot dX_s+dM_s^{m,n}\rangle_T\bigg)^{p_0}\bigg] \quad (\text{ Young's inequality })\\
\leqslant& C(1+\Xi)<\infty,
\end{aligned} \tag{3.39}$$

where we used Young's inequality with the setting of $p_1 := \frac{\lambda}{\lambda_0}, p_2 := \frac{\lambda'}{\lambda_0}$ and $p_0 = \frac{\lambda\lambda'}{\lambda\lambda'-\lambda_0(\lambda+\lambda')} = \Big(1-\frac{1}{p_1}-\frac{1}{p_2}\Big)^{-1}$. To conclude, this shows that $\int_0^\cdot \phi'(Y_s^{m,n})Z_s^{m,n}\cdot dX_s+dM_s^{m,n}$ is a uniformly integrable martingale. By setting $t\equiv 0$, we get from (3.37) that

$$\begin{aligned}
&\mathbb{E}[\phi(Y_0^{m,n})]+\frac{1}{2}\mathbb{E}\bigg[\int_0^T\phi''(Y_s^{m,n})d\langle Z_\cdot^{m,n}\cdot X_\cdot+M_\cdot^{m,n}\rangle_s\bigg]\\
\leqslant&\mathbb{E}[\phi(\xi_{m,n})]+\mathbb{E}\bigg[\int_0^T\phi'(Y_s^{m,n})dK_s^{m,n}\bigg]+\mathbb{E}\int_0^T|\phi'(Y_s^{m,n})|\bigg[2\alpha+\beta\Big(|Y_s^m|+|Y_s^n|\Big)\\
&+\frac{\gamma}{2}\widehat{\sigma}_s^2\bigg(2|Z_s^{m,n}|^2+(\lambda_0-2)|(Z_s-Z_s^n)|^2+\bigg(3+\frac{9}{\lambda_0-5}\bigg)|Z_s|^2\bigg)\bigg]ds.
\end{aligned} \tag{3.40}$$

In above, we used the fact that $|Z_s^m|^2+|Z_s^n|^2 \leqslant 2|Z_s^{m,n}|+3|Z_s^n|^2 = |Z_s^m|^2+|Z_s^n|^2+|Z_s^m|^2-4Z_s^mZ_s^n+4|Z_s^n|^2$ and $|Z_s^n|^2 \leqslant (1+\frac{\lambda_0-5}{3})|Z_s-Z_s^n|^2+(1+\frac{3}{\lambda_0-5})|Z_s|^2$.
Note that $|Y_t^{m,n}| \leqslant |Y_t-Y_t^n| \leqslant |Y_t-Y_t^1|, \forall t\in[0,T], \mathbb{P}$-a.s., we have that $\mathbb{P}$-a.s.

$$\phi(\xi_{m,n}) \leqslant \phi(\xi-\xi_n) \text{ and } |\phi'(Y_t^{m,n})| \leqslant |\phi'(Y_t-Y_t^n)| \leqslant |\phi'(Y_t-Y_t^1)|, \forall t\in[0,T], \tag{3.41}$$

by the monotonicity of ϕ and ϕ'. Similarly, we have $\mathbb{P}$-a.s. that

$$|\phi'(L_t^{m,n})| \leqslant |\phi'(L_t-L_t^n)| \leqslant |\phi'(L_t-L_t^1)|, \forall t\in[0,T]. \tag{3.42}$$

By (3.39), we have that

$$\mathbb{E}\bigg[\int_0^T|\phi'(Y_s^{m,n})|d\langle Z_\cdot^{m,n}\cdot X_\cdot+M_\cdot^{m,n}\rangle_s\bigg] \leqslant \mathbb{E}\bigg[\sup_{s\in[0,T]}|\phi'(Y_s^{m,n})|\langle\int_0^\cdot Z_s^{m,n}\cdot dX_s+dM_s^{m,n}\rangle_T\bigg]<\infty, \tag{3.43}$$

which together with (3.40), (3.41) and (3.35) gives that

$$\mathbb{E}\bigg[\int_0^T(\phi''-2\gamma\phi')(Y_s^{m,n})\langle Z_\cdot^{m,n}\cdot dX_\cdot+dM_\cdot^{m,n}\rangle_s\bigg]\leqslant 2\mathbb{E}[\phi(\xi-\xi_n)]+2\mathbb{E}\bigg[\int_0^T\phi'(Y_s^{m,n})dK_s^{m,n}\bigg]$$
$$+\mathbb{E}\bigg\{\int_0^T|\phi'(Y_s-Y_s^n)|\Big[4\alpha+2\beta\Big(|Y_s^m|+|Y_s^n|\Big)$$
$$+\gamma\widehat{\sigma}_s^2\bigg((\lambda_0-2)|(Z_s-Z_s^n)|^2+\bigg(3+\frac{9}{\lambda_0-5}\bigg)|Z_s|^2\bigg)\Big]ds\bigg\}. \tag{3.44}$$

Step 2. Now we handle with the second part on the RHS of (3.44) by our assumption (M3). Assume firstly the increasing case,

$$\mathbb{E}\bigg[\int_0^T\phi'(Y_s^{m,n})dK_s^{m,n}\bigg]\leqslant\mathbb{E}\bigg[\int_0^T\phi'(Y_s^{m,n})dK_s^{m}\bigg]$$
$$\leqslant\mathbb{E}\bigg[\int_0^T\phi'(Y_s^m-L_s^n)dK_s^m\bigg]\quad(\text{ the flat-off condition and monotonicity of }\phi')$$
$$\leqslant\mathbb{E}\bigg[\int_0^T\mathbf{1}_{\{Y_s^m=L_s^m\}}\phi'(L_s^{m,n})dK_s^m\bigg];$$

besides, we could continue to estimate the last term above that $\mathbb{E}\Big[\int_0^T\mathbf{1}_{\{Y_s^m=L_s^m\}}\phi'(L_s^{m,n})dK_s^m\Big]\leqslant\|K_T^m\|_{\mathbb{L}^{p_0}(\mathbb{P},\mathbb{R})}\|\phi'(L^{m,n})\|_{\mathbb{C}^{\frac{p_0}{p_0-1}}(\mathbb{P})}\leqslant C\Xi^{\frac{1}{p_0}}\|\phi'(L-L^n)\|_{\mathbb{C}^{\frac{p_0}{p_0-1}}(\mathbb{P})}$ by using (3.36) and (3.42) in the last inequality; to conclude,

$$\mathbb{E}\bigg[\int_0^T\phi'(Y_s^{m,n})dK_s^{m,n}\bigg]\leqslant C\Xi^{\frac{1}{p_0}}\|\phi'(L-L^n)\|_{\mathbb{C}^{\frac{p_0}{p_0-1}}(\mathbb{P})}. \tag{3.45}$$

Besides, it holds for the decreasing case of assumption (M3) that

$$\mathbb{E}\bigg[\int_0^T\phi'(Y_s^{m,n})dK_s^{m,n}\bigg]\leqslant-\mathbb{E}\bigg[\int_0^T\mathbf{1}_{\{Y_s^n=L_s^n\}}\phi'(L_s^{m,n})dK_s^n\bigg]\leqslant C\Xi^{\frac{1}{p_0}}\|\phi'(L-L^n)\|_{\mathbb{C}^{\frac{p_0}{p_0-1}}(\mathbb{P})}.$$

Step 3. Since $\{\sqrt{|\phi'(Y_\cdot^{m,n})|}\widehat{\sigma}_\cdot Z_\cdot^{m,n}\}_{m\geqslant n}$ weakly converges to $\sqrt{|\phi'(Y_\cdot-Y_\cdot^n)|}\widehat{\sigma}_\cdot(Z_\cdot-Z_\cdot^n)$ in $\mathbb{H}^2(\mathbb{P},\mathbb{R}^d)$ together with the same fact between $\{\sqrt{|\phi'(Y_\cdot-Y_\cdot^n)d\langle M-M^n\rangle_\cdot/d\cdot|}\}_{m\geqslant n}$ and $\{\sqrt{|\phi'(Y_\cdot^{m,n})d\langle M^{m,n}\rangle_\cdot/d\cdot|}\}_{m\geqslant n}$, which we could verify exactly as in [BY12](p.1201 Appendix corresponding to its (3.12)), then by Theorem 5.1.1(ii) of [Yos80] we have that

$$
\begin{aligned}
&\mathbb{E}\bigg[\int_0^T |\phi'(Y_s - Y_s^n)| d\langle (Z_s - Z_s^n)\cdot X_s + (M_s - M_s^n)\rangle_s\bigg] \\
\leqslant &\liminf_{m\to\infty} \mathbb{E}\bigg[\int_0^T |\phi'(Y_s^{m,n})| d\langle Z_\cdot^{m,n}\cdot X_\cdot + M_\cdot^{m,n}\rangle_s\bigg],
\end{aligned}
\tag{3.46}
$$

Note that $\mathbb{H}^{2,2p_0}(\mathbb{P},\mathbb{R}^d) \subseteq \mathbb{H}^2(\mathbb{P},\mathbb{R}^d)$, the sequence $\{Z^m\}_{m\geqslant n}$ also weakly converges to Z in $\mathbb{H}^2(\mathbb{P},\mathbb{R}^d)$.

Step 4. Applying Theorem 5.1.1(ii) of [Yos80] once again together with the fact that $\lambda_0\gamma = \phi'' - \lambda_0\gamma\phi'$ due to the definition of ϕ, we have

$$
\begin{aligned}
&\lambda_0\gamma\mathbb{E}\bigg[\langle\int_0^\cdot (Z_s - Z_s^n)\cdot dX_s + d(M_s - M_s^n)\rangle_T\bigg] \\
\leqslant &\limsup_{m\to\infty} \mathbb{E}\bigg[\int_0^T (\phi'' - \lambda_0\gamma\phi')(Y_s^{m,n}) d\langle Z_\cdot^{m,n}\cdot X_\cdot + M_\cdot^{m,n}\rangle_s\bigg] \\
= &\limsup_{m\to\infty} \mathbb{E}\bigg[\int_0^T (\phi'' - 2\gamma\phi')(Y_s^{m,n}) d\langle Z_\cdot^{m,n}\cdot X_\cdot + M_\cdot^{m,n}\rangle_s\bigg] \\
&- (\lambda_0 - 2)\gamma \liminf_{m\to\infty} \mathbb{E}\bigg[\int_0^T |\phi'(Y_s^{m,n})| d\langle Z_\cdot^{m,n}\cdot X_\cdot + M_\cdot^{m,n}\rangle_s\bigg];
\end{aligned}
\tag{3.47}
$$

and then we can deduce from (3.44), (3.45) and (3.46) that the last term above has an upper bound

$$
2\mathbb{E}[\phi(\xi - \xi_n)] + C\Xi^{\frac{1}{p_0}} \|\phi'(L - L^n)\|_{\mathbb{C}^{\frac{p_0}{p_0-1}}(\mathbb{P})} + C\mathbb{E}\bigg[\int_0^T |\phi'(Y_s - Y_s^n)|(1 + \mathscr{L}_s + \mathscr{Y}_s + \widehat{\sigma}_s^2|Z_s|^2)ds\bigg];
$$

i.e., we have proved that

$$
\begin{aligned}
&\lambda_0\gamma\mathbb{E}\bigg[\langle\int_0^\cdot (Z_s - Z_s^n)\cdot dX_s + d(M_s - M_s^n)\rangle_T\bigg] \\
\leqslant &2\mathbb{E}[\phi(\xi - \xi_n)] + C\Xi^{\frac{1}{p_0}} \|\phi'(L - L^n)\|_{\mathbb{C}^{\frac{p_0}{p_0-1}}(\mathbb{P})} + C\mathbb{E}\bigg[\int_0^T |\phi'(Y_s - Y_s^n)|(1 + \mathscr{L}_s + \mathscr{Y}_s + \widehat{\sigma}_s^2|Z_s|^2)ds\bigg].
\end{aligned}
\tag{3.48}
$$

Step 5. Note that $\lambda_0 < \frac{\lambda\lambda'}{\lambda+\lambda'}$, then $\lambda' > \frac{\lambda_0\lambda}{\lambda-\lambda_0}$. By Young's inequality with $\widetilde{p} = \frac{\lambda}{\lambda_0}$ and $\widetilde{q} = \frac{\lambda}{\lambda-\lambda_0}$, we get from (3.35) that $\mathbb{P}$-a.s., $0 \leqslant \phi(\xi - \xi_n) \leqslant \frac{1}{\lambda_0\gamma} e^{\lambda_0\gamma(\mathscr{L}_* + \mathscr{Y}_*)} \leqslant C(e^{\lambda\gamma\mathscr{L}_*} + e^{\lambda'\gamma\mathscr{Y}_*}), \forall n \in \mathbb{N}$. By the fact that $\mathbb{E}[e^{\lambda\gamma\mathscr{L}_*}+e^{\lambda'\gamma\mathscr{Y}_*}]<\infty$, ϕ is continuous and dominated convergence theorem, we have that

$$\lim_{n\searrow\infty}\mathbb{E}[\phi(\xi-\xi_n)]=0. \quad (n\searrow\infty \text{ here represents } \xi-\xi_n\searrow 0) \tag{3.49}$$

Dini's theorem and (M3) imply that $\mathbb{P}$-a.s.,

$$\lim_{n\to\infty}\sup_{t\in[0,T]}|L_t^n-L_t|=\lim_{n\to\infty}\sup_{t\in[0,T]}|Y_t^n-Y_t|=0. \tag{3.50}$$

Step 6. The fact that ϕ' is continuous shows that $\mathbb{P}$-a.s.,

$$0=\lim_{n\to\infty}|\phi'(\sup_{t\in[0,T]}|L_t^n-L_t|)|=\lim_{n\to\infty}\sup_{t\in[0,T]}\exp\{\lambda_0\gamma|L_t^n-L_t|\}-1=\lim_{n\to\infty}\sup_{t\in[0,T]}|\phi'(L_t^n-L_t)|.$$

From (3.42), we have that $\mathbb{P}$-a.s. $\sup_{t\in[0,T]}|\phi'(L_t-L_t^n)|^{\frac{p_0}{p_0-1}}\leqslant\sup_{t\in[0,T]}|\phi'(L_t-L_t^1)|^{\frac{p_0}{p_0-1}},\forall n\in\mathbb{N}$. Using Young's inequality with $\widetilde{p}=\frac{\lambda+\lambda'}{\lambda'}$ and $\widetilde{q}=\frac{\lambda+\lambda'}{\lambda}$, we get from (3.35) that

$$\mathbb{E}\bigg[\sup_{t\in[0,T]}|\phi'(L_t-L_t^1)|^{\frac{p_0}{p_0-1}}\bigg]\leqslant\mathbb{E}\bigg[e^{\frac{\lambda\lambda'\gamma}{\lambda+\lambda'}(\mathscr{L}_*+\mathscr{Y}_*)}\bigg]\leqslant C\mathbb{E}[e^{\lambda\gamma\mathscr{L}_*}+e^{\lambda'\gamma\mathscr{Y}_*}]<\infty. \tag{3.51}$$

Then by dominated convergence theorem we have that

$$\lim_{n\searrow\infty}\mathbb{E}\bigg[\sup_{t\in[0,T]}|\phi'(L_t-L_t^n)|^{\frac{p_0}{p_0-1}}\bigg]=0. \tag{3.52}$$

Step 7. It follows from (3.35) and (3.41) that $\mathbb{P}$-a.s.

$$\begin{aligned}&|\phi'(Y_t-Y_t^n)|(1+\mathscr{L}_t+\mathscr{Y}_t+\widehat{\sigma}_t^2Z_t^2)\\ \leqslant&|\phi'(Y_t-Y_t^n)|(1+\mathscr{L}_t+\mathscr{Y}_t)+|\phi'(Y_t-Y_t^n)|\widehat{\sigma}_t^2Z_t^2\\ \leqslant&Ce^{\frac{\lambda\lambda'\gamma}{\lambda+\lambda'}(\mathscr{L}_t+\mathscr{Y}_t)}+|\phi'(Y_t-Y_t^1)|\widehat{\sigma}_t^2Z_t^2,\forall t\in[0,T],\forall n\in\mathbb{N}.\end{aligned} \tag{3.53}$$

Similar to (3.51), one has $\mathbb{E}[\sup_{t\in[0,T]}|\phi'(Y_t-Y_t^1)|^{\frac{p_0}{p_0-1}}]\leqslant C\Xi<\infty$, which together with Young's inequality and (3.51) shows that

$$\begin{aligned} &\mathbb{E}\left[\int_0^T \left(e^{\frac{\lambda\lambda'\gamma}{\lambda+\lambda'}(\mathscr{L}_t+\mathscr{Y}_t)} + |\phi'(Y_t - Y_t^1)|\widehat{\sigma}_t^2 Z_t^2\right) dt\right] \\ \leqslant & C\mathbb{E}\left[e^{\frac{\lambda\lambda'\gamma}{\lambda+\lambda'}(\mathscr{L}_*+\mathscr{Y}_*)} + \sup_{t\in[0,T]} |\phi'(Y_t - Y_t^1)|^{\frac{p_0}{p_0-1}} + \left(\int_0^T \widehat{\sigma}_t^2 Z_t^2 dt\right)^{p_0}\right] < \infty. \end{aligned} \tag{3.54}$$

Then the continuity of ϕ' and dominated convergence theorem imply that $\lim_{n\to\infty}\mathbb{E}[\int_0^T \phi'(Y_t - Y_t^n)(1 + \mathscr{L}_t + \mathscr{Y}_t + \widehat{\sigma}_t^2|\widehat{Z}_t|^2)ds] = 0$.

Step 8. We conclude with the above calculus. Combining (3.49), (3.52), (3.54) and Doob's martingale inequality leads to the following deduction on (3.47):

$$\begin{aligned} &\lim_{n\to\infty}\mathbb{E}\left[\langle\int_0^\cdot (Z_s - Z_s^n)\cdot dX_s + d(M_s - M_s^n)\rangle_T\right] = 0, \\ \text{and } &\lim_{n\to\infty}\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t (Z_\cdot - Z_\cdot^n)\cdot dX_\cdot + d(M_\cdot - M_\cdot^n)\right|^2\right] = 0. \end{aligned} \tag{3.55}$$

□

Lemma 1.B. *Under the setting of Theorem 3.3.6,* $Y \in \mathbb{C}^{exp(\lambda\gamma,\lambda'\gamma)}$.

Proof. **Step 1.** In this part, we prove that by (3.55), we take a subsequence of $\{Z^n\}_{n\in\mathbb{N}}$ (we still denote it by $\{Z^n\}_{n\in\mathbb{N}}$) s.t. $\lim_{n\to\infty} Z_t^n = Z_t, dt\otimes d\mathbb{P}$-a.e. In fact, we can choose this subsequence s.t. $Z^* := \sup_{n\in\mathbb{N}}|Z^n| \in \mathbb{H}^2(\mathbb{P},\mathbb{R}^d)$, see Lepeltier and San Martin [LM97] or Kobylanski [Kob00](Lemma 2.5). By (M2), it follows $ds\otimes d\mathbb{P}$-a.e. that

$$f(s,\omega,y,z) = \lim_{n\to\infty} f_n(s,\omega,y,z), \forall (y,z)\in\mathbb{R}\times\mathbb{R}^d, \tag{3.56}$$

which leads to that f is also $\overline{\mathbb{F}}^{\mathbb{P}}\times\mathcal{B}(\mathbb{R})\times\mathcal{B}(\mathbb{R}^d)\setminus\mathcal{B}(\mathbb{R})$ -measurable by the measurability of $f_n, \forall n\in\mathbb{N}$. By (3.56) and (M1), we also get that f satisfies (3) in Assumption 3.3.1. Besides, by (M3) we have $\lim_{n\nearrow\infty} Y_\cdot^n = Y_\cdot, \mathbb{P}$-a.s.. For $ds\otimes d\mathbb{P}$-a.e. $(s,\omega)\in[0,T]\times\Omega$, the continuity of mapping $f(s,\omega,\cdot,\cdot)$ shows that

$$\lim_{n\to\infty}|[f_s(Y_s^n, Z_s^n) - f_s(Y_s, Z_s)]| = 0. \tag{3.57}$$

We could also see that (M2) gives that for $ds \otimes d\mathbb{P}$-a.e. $(t,\omega) \in [0,T] \times \Omega$,

$$\begin{aligned}
0 \leqslant & \liminf_{n\to\infty} |[f_n(s,Y_s^n,Z_s^n) - f(s,Y_s^n,Z_s^n)]| \\
& \leqslant \lim_{n\to\infty} \sup \Big\{ |[f_n(s,\omega,y,z) - f(s,\omega,y,z)]| : |y| \leqslant |Y_s^1(\omega)| \vee |Y_s(\omega)| < \infty, |z| \leqslant Z_s^*(\omega) < \infty \Big\} \\
& = 0,
\end{aligned}$$

which together with (3.57) shows that $ds \otimes d\mathbb{P}$-a.e.

$$\lim_{n\to\infty} |[f_n(s,Y_s^n,Z_s^n) - f(s,Y_s,Z_s)]| = 0, \tag{3.58}$$

By (3) in Assumption 3.3.1 and (3.35), we have $ds \otimes d\mathbb{P}$-a.e.,

$$|f_n(s,Y_s^n,Z_s^n) - f(s,Y_s,Z_s)| \leqslant 2\alpha + 2\beta(\mathscr{L}_* + \mathscr{Y}_*) + \frac{\gamma}{2}\widehat{\sigma}_s^2(|Z_s^*|^2 + |Z_s|^2), \forall n \in \mathbb{N}, s \in [0,T]. \tag{3.59}$$

On a $\mathbb{P}$-null set $\mathcal{N}$, we assume that (3.58), (3.59) hold for a.e. $s \in [0,T]$ and $\mathscr{L}_* + \mathscr{Y}_* + \int_0^T \widehat{\sigma}_s^2(|Z_s^*|^2 + |Z_s|^2)ds < \infty$. By dominated convergence theorem, we have, for any $\omega \in \mathcal{N}^c$,

$$\lim_{n\to\infty} \int_0^T |f_n(s,\omega,Y_s^n,Z_s^n) - f(s,\omega,Y_s,Z_s)|ds = 0. \tag{3.60}$$

For any $n \in \mathbb{N}$, the integration on s in (3.59) gives that

$$\begin{aligned}
& \int_0^T |f_n(s,Y_s^n,Z_s^n) - f(s,Y_s,Z_s)|ds \\
\leqslant & C e^{\frac{\lambda\lambda'\gamma}{(\lambda+\lambda')p_0}(\mathscr{L}_*(\omega)+\mathscr{Y}_*(\omega))} + \frac{\gamma}{2}\int_0^T \widehat{\sigma}_s^2(|Z_s^n|^2 + |Z_s|^2)ds.
\end{aligned} \tag{3.61}$$

Then together with (3.51) and (3.36), we have

$$\mathbb{E}\bigg[\bigg(\int_0^T \Big|f_n(s,Y_s^n,Z_s^n) - f(s,Y_s,Z_s)\Big|ds\bigg)^{p_0}\bigg] \leqslant C\Xi + C\mathbb{E}\bigg[\langle \int_0^\cdot Z_t dt\rangle_T^{p_0}\bigg] < \infty, \forall n \in \mathbb{N}, \tag{3.62}$$

which implies that

$$\bigg\{\bigg(\int_0^T \Big|f_n(s,Y_s^n,Z_s^n) - f(s,Y_s,Z_s)\Big|ds\bigg)^{\frac{1+p_0}{2}}\bigg\}_{n\in\mathbb{N}}$$

is uniformly integrable in $\mathbb{L}^1(\mathbb{P},\mathbb{R})$ due to the fact that $\frac{1+p_0}{2} < p_0$(where $p_0 > 1$). So it follows from

(3.60) that

$$\lim_{n\to\infty}\mathbb{E}\bigg[\bigg(\int_0^T\Big|f_n(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)\Big|ds\bigg)^{\frac{1+p_0}{2}}\bigg]=0. \tag{3.63}$$

Note that $\phi(x)\geqslant\frac{\lambda_0\gamma}{2}x^2$ and $\phi'(x)\geqslant\lambda_0\gamma x, x\in\mathbb{R}$, it follows from (3.49) and (3.52) that

$$\lim_{n\searrow\infty}\mathbb{E}[(\xi-\xi_n)^2]=0 \text{ and } \lim_{n\searrow\infty}\|L-L^n\|_{\mathbb{C}^{\frac{p_0}{p_0-1}}(\mathbb{P})}=\lim_{n\searrow\infty}\bigg\{\mathbb{E}^{\mathbb{P}}\bigg[\sup_{0\leqslant s\leqslant T}|(L_s-L_s^n)|^{\frac{p_0}{p_0-1}}\bigg]\bigg\}^{\frac{p_0-1}{p_0}}=0. \tag{3.64}$$

Moreover, for any $p\in[1,\infty)$, it follows from (3.35) and (3.51) that

$$\|Y^n\|^p_{\mathbb{C}^p(\mathbb{P})}\leqslant\mathbb{E}[(\mathscr{L}_*+\mathscr{Y}_*)^p]\leqslant C\mathbb{E}[e^{\frac{\lambda\lambda'}{\lambda+\lambda'}\gamma(\mathscr{L}_*+\mathscr{Y}_*)}]\leqslant C\Xi,\forall n\in\mathbb{N}. \tag{3.65}$$

Step 2. For any $m,n\in\mathbb{N}$ with $m\geqslant n$, using Itô's formula to the process $(Y_\cdot^{m,n})^2$, this gives that

$$\begin{aligned}(Y_t^{m,n})^2=&\xi_{m,n}^2+2\int_t^T Y_s^{m,n}[f_m(s,Y_s^m,Z_s^m)-f_n(s,Y_s^n,Z_s^n)]ds\\&-2\int_t^T Y_s^{m,n}(Z_s^{m,n}\cdot dX_s+dU_s^{m,n})-\langle\int_t^\cdot Z_s^{m,n}\cdot dX_s+dM_s^{m,n}\rangle_T\\\leqslant&\xi_{m,n}^2+2\int_t^T Y_s^{m,n}[f_m(s,Y_s^m,Z_s^m)-f_n(s,Y_s^n,Z_s^n)]ds\\&-2\int_t^T Y_s^{m,n}(Z_s^{m,n}\cdot dX_s+dU_s^{m,n}),t\in[0,T].\end{aligned} \tag{3.66}$$

By the flat-off condition of (Y^m,Z^m,M^m,K^m), we have that $\mathbb{P}$-a.s.,

$$\int_0^T Y_s^{m,n}dK_s^{m,n}\leqslant\begin{cases}\int_t^T(Y^m-L^n)dK_s^m=\int_t^T L_s^{m,n}dK_s^m\leqslant K_T^m\sup_{s\in[0,T]}|L_s^{m,n}|,t\in[0,T],(\text{ increasing });\\\int_t^T(Y^n-L^m)dK_s^n=-\int_t^T L_s^{m,n}dK_s^n\leqslant K_T^n\sup_{s\in[0,T]}|L_s^{m,n}|,t\in[0,T],(\text{ decreasing }).\end{cases}$$

Furthermore, Hölder's inequality, (3.36), Burkholder-Davis-Gundy inequality and (3.65) imply that

$$\begin{aligned}\mathbb{E}\bigg[\sup_{t\in[0,T]}|Y_t^{m,n}|^2\bigg]\leqslant&\mathbb{E}[\xi_{m,n}^2]+C\Xi^{\frac{p_0-1}{p_0+1}}\|f_m(\cdot,Y_\cdot^m,Z_\cdot^m)-f_n(\cdot,Y_\cdot^n,Z_\cdot^n)\|_{\mathbb{H}^{1,\frac{1+p_0}{2}}(\mathbb{P},\mathbb{R})}\\&+C\Xi^{\frac{1}{p_0}}\|L^{m,n}\|_{\mathbb{C}^{\frac{p_0}{p_0-1}}(\mathbb{P})}+C\Xi^{\frac12}\|Z^{m,n}\|_{\mathbb{H}^2(\mathbb{P},\mathbb{R}^d)}+C\Xi^{\frac{p_0-1}{p_0}}\|M^{m,n}\|_{\mathbb{M}^2(\mathbb{P})}.\end{aligned} \tag{3.67}$$

By (3.64), (3.63) and (3.55), we conclude that $\{Y^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}^2(\mathbb{P})$. Let Y be its limit in $\mathbb{C}^2(\mathbb{P})$, there is a subsequence $\{n_i\}_{i\in\mathbb{N}}$ s.t. $\lim_{i\searrow\infty}\sup_{t\in[0,T]}|Y_t^{n_i}-Y_t|=0,\mathbb{P}$-a.s., which together with (M3) implies that $\mathbb{P}(Y_t^{n_i} = Y_t, \forall t \in [0,T]) = 1$. So Y is a continuous process satisfying

$$\lim_{n \searrow \infty} \sup_{t \in [0,T]} |Y_t^n - Y_t| = 0, \mathbb{P}\text{-a.s.} \tag{3.68}$$

Step 3. Since $\mathbb{E}[e^{\lambda\gamma Y_*^-} + e^{\lambda'\gamma Y_*^+}] \leqslant \mathbb{E}[e^{\lambda\gamma \mathscr{L}_*} + e^{\lambda'\gamma \mathscr{Y}_*}] < \infty$ by (3.35), we see that $Y \in \mathbb{C}^{\exp(\lambda\gamma,\lambda'\gamma)}$. □

Lemma 1.C. *Under the setting of Theorem 3.3.6, $K \in \mathbb{K}(\mathbb{P})$.*

Proof. Here we define an $\overline{\mathbb{F}}^{\mathbb{P}}$-adapted, continuous process: for any $t \in [0,T]$,

$$K_t := Y_0 - Y_t - \int_0^t f_s(Y_s, Z_s)ds + (Z_s \cdot dX_s + dM_s). \tag{3.69}$$

It follows from (3.63) and (3.55) that $\{Y^n, Z^n, M^n\}_{n\in\mathbb{N}}$ has a subsequence (we still denote it by $\{Y^n, Z^n, M^n\}_{n\in\mathbb{N}}$) s.t. $\mathbb{P}$-a.s.,

$$\lim_{n\to\infty} \left\{ \int_0^T |f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)| ds + \sup_{t\in[0,T]} \left| \int_0^t (Z_s^n - Z_s) dX_s + d(M_s^n - M_s) \right| \right\} = 0. \tag{3.70}$$

The last two equations together with (3.68), we have that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} |K_t^n - K_t| = 0, \mathbb{P}\text{-a.s.}, \tag{3.71}$$

which shows that K is a finite bounded variation process, so $K \in \mathbb{K}(\mathbb{P})$. Here we summarize that $L_t, Y_t, \xi(= Y_T)$ and f are known in the beginning while K, Z, M are verified in this proof, so letting $n \to \infty$ in (3.35) as well as the RBSDE w.r.t. (Y^n, Z^n, M^n, K^n) gives that $\mathbb{P}$-a.s.

$$L_t \leqslant Y_t = \xi + \int_t^T f_s(Y_s, Z_s) ds - (Z_s \cdot dX_s + dU_s), t \in [0,T].$$

□

Lemma 1.D. *Theorem 3.3.6 is valid.*

Proof. For $\mathbb{P}$-a.s. $\omega \in \Omega$, since $\{(Y_\cdot^n, L_\cdot^n, K_\cdot^n)\}_{n\in\mathbb{N}}$ uniformly converges $\{(Y_\cdot, L_\cdot, K_\cdot)\}$ w.r.t. t by (3.50) and (3.71), we could follow the standard arguments (e.g., [BY12]) and the flat-off condition of each (Y_t^n, L_t^n, K_t^n) that

$$\lim_{n\to\infty}\int_0^T (Y_t(\omega) - L_t(\omega))dK_t(\omega) = \lim_{n\to\infty}\int_0^T (Y_t^n(\omega) - L_t^n(\omega))dK_t^n(\omega) = 0.$$

Together with the previous steps, we have shown that (Y, Z, M, K) is a solution of the quadratic RBSDE(f, ξ, L). Since $Y \in \mathbb{C}^{\exp(\lambda\gamma,\lambda'\gamma)}$, Proposition 3.3.4 shows that $(Z, M, K) \in \mathbb{H}^{2,2p}(\mathbb{P}, \mathbb{R}^d) \times \mathbb{M}^p(\mathbb{P}) \times \mathbb{K}^p(\mathbb{P})$ for any $p \in \left(1, \frac{\lambda\lambda'}{\lambda+\lambda'}\right)$. □

Theorem 3.3.7 (BSDE version). *For any* $n \in \mathbb{N}$, *let* $(Y^n, Z^n, M^n, K^n) \in \mathbb{C}^0(\mathbb{P}) \times \mathbb{H}^{2,loc}(\mathbb{P}; \mathbb{R}^d) \times \mathbb{M}(\mathbb{P}) \times \mathbb{I}(\mathbb{P})$ *be the corresponding solution of BSDE* (f^n, ξ_n), $n \in \mathbb{N}$ *s.t.*

(M1) All $f^n, n \in \mathbb{N}$ *satisfy the quadratic growth (i.e., (3) in Assumption 3.3.1) where they share the same constants* $\alpha, \beta \geqslant 0$ *and* $\gamma > 0$;

(M2) There exists a function $f : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d^{\geqslant 0} \to \mathbb{R}$ *s.t. for* $dt \otimes d\mathbb{P}-$*a.e.* $(t, \omega) \in [0,T] \times \Omega$, *the mapping* $f_t(\omega, \cdot, \cdot)$ *is continuous and* $f_t^n(\omega, y, z)$ *converges to* $f_t(\omega, y, z)$ *locally uniformly in* (y, z)*;*

(M3) For some $\mathbb{R}$*-valued,* $\overline{\mathbb{F}}^{\mathbb{P}}-$ *adapted process* Y, *it holds* $\mathbb{P}$*-a.s. that for any* $t \in [0,T], \{Y_t^n\}_{n\in\mathbb{N}}$ *is increasing (resp. decreasing) sequence w.r.t.* n *with* $\lim_{n\to\infty} \uparrow Y_t^n = Y_t$ *(resp.,* $\lim_{n\to\infty} \downarrow Y_t^n = Y_t$ *).*

Denote $\mathcal{Y}_t := (Y_t^1)^+ \vee Y_t^+, \forall t \in [0,T]$. *If* $\Xi := \mathbb{E}[e^{\lambda'\gamma\mathcal{Y}_*}] < \infty$ *for some* $\lambda, \lambda' > 6$ *with* $\frac{1}{\lambda} + \frac{1}{\lambda'} < \frac{1}{6}$, *then* $Y \in \mathbb{C}^{\exp(\lambda\gamma,\lambda'\gamma)}(\mathbb{P})$ *and there exists* $(Z, M, K) \in \cap_{p\in\left(1,\frac{\lambda\lambda'}{\lambda+\lambda'}\right)} \mathbb{H}^{2,2p}(\mathbb{P}; \mathbb{R}^d) \times \mathbb{M}^p(\mathbb{P}) \times \mathbb{I}^p(\mathbb{P})$ *s.t. the quadruplet* (Y, Z, M, K) *is a solution of the BSDE* (f, ξ) *with* $\xi := Y_T$.

3.3.4 Existence

Theorem 3.3.8. *Let* (f, ξ, L) *be a triple s.t.* f *satisfies the quadratic growth (i.e., (3) in Assumption 3.3.1) and that*

$$dt \otimes d\mathbb{P}\text{-a.e.}(t, \omega) \in [0,T] \times \Omega, \text{ the mapping } f(t, \omega, \cdot, \cdot) \text{ is continuous.} \tag{3.72}$$

If $\mathbb{E}[e^{\lambda\gamma L_*^-} + e^{\lambda'\gamma e^{\beta T}(\xi^+\vee L_*^+)}] < \infty$ for some $\lambda, \lambda' > 6$ with $\frac{1}{\lambda} + \frac{1}{\lambda'} < \frac{1}{6}$, then the quadratic RBSDE (f, ξ, L) has a solution $(Y, Z, M, K) \in \cap_{p\in\left(1, \frac{\lambda\lambda'}{\lambda+\lambda'}\right)} \mathbb{C}^{exp(\lambda\gamma, \lambda'\gamma)}(\mathbb{P}) \times \mathbb{H}^{2,2p}(\mathbb{P}; \mathbb{R}^d) \times \mathbb{M}^p(\mathbb{P}) \times \mathbb{I}^p(\mathbb{P})$ that satisfies (3.10). Besides, if $\xi^+ \vee L_* \in \mathbb{L}^{\exp}(\mathbb{P}, \mathbb{R})$, then this solution (Y, Z, M, K) belongs to $\mathbb{S}^p(\mathbb{P}) = \mathbb{C}^{exp(p)}(\mathbb{P}) \times \mathbb{H}^{2,2p}(\mathbb{P}, \mathbb{R}^d) \times \mathbb{M}^p(\mathbb{P}) \times \mathbb{I}^p(\mathbb{P})$ for all $p \in [1, \infty)$. To be precise, for any $p \in (1, \infty)$, we have the following a priori estimates:

$$\begin{aligned} &\mathbb{E}[e^{p\gamma Y_*}] \leqslant \mathbb{E}[p\gamma L_*^-] + c_p\mathbb{E}[e^{p\gamma e^{\beta T}(\xi^+\vee L_*^+)}] < \infty; \\ &\mathbb{E}\left[\left(\int_0^T \widehat{\sigma}_s^2|Z_s|^2 ds + d\langle M\rangle_s\right)^p + K_T^p\right] \leqslant c_p\mathbb{E}[e^{3p\gamma Y_*}] < \infty. \end{aligned} \tag{3.73}$$

Proof. **Step 1.** Introduce the 'cut-off' and validate the wellposedness of a maximal bounded solution for corresponding quadratic RBSDE. To be detailed, for any $x \in \mathbb{R}$ and $m, n \in \mathbb{N}$, we define $x^m := x \vee (-m)$ and $x^{m,n} := \big(x \vee (-m)\big) \wedge n$. Clearly, $(x^m)^- \vee (x^{m,n})^- \leqslant x^-$ and $(x^m)^+ \wedge (x^{m,n})^+ \leqslant x^+$. Kobylanski, Lepeltier, Quenez and Torres [KLQT02] gives the wellposedness of a maximal bounded solution $(Y^{m,n}, Z^{m,n}, M^{m,n}, K^{m,n}) \in \mathbb{C}^\infty(\mathbb{P}) \times \mathbb{H}^2(\mathbb{P}, \mathbb{R}) \times \mathbb{M}(\mathbb{P}) \times \mathbb{K}(\mathbb{P})$ for the quadratic RBSDE$(f, \xi^{m,n}, L^{m,n})$(note that here we state their result in the context of a more general filtration). Then we could obtain from Proposition 3.10 and the above discussion on cut-off functions that $\mathbb{P}$-a.s.,

$$-L_t^- \leqslant -(L_t^{m,n})^- \leqslant L_t^{m,n} \leqslant Y_t^{m,n} \leqslant c_0 + \frac{1}{\gamma}\ln\mathbb{E}\left[e^{\gamma e^{\beta T}\left((\xi^{m,n})^+\vee(L^{m,n})^+\right)}\Big|\overline{\mathcal{F}}_t^{\mathbb{P}}\right] \leqslant c_0 + \frac{1}{\gamma}B_t, t \in [0, T], \tag{3.74}$$

where $B_t := \mathbb{E}\left[e^{\gamma e^{\beta T}(\xi^+\vee L_*^+)}\Big|\overline{\mathcal{F}}_t^{\mathbb{P}}\right]$. Besides, we could follow Proposition A.1(Comparison theorem for quadratic RBSDEs with bounded obstacles) in [BY11] to show that $\mathbb{P}$-a.s.,

$$Y_t^{m+1,n} \leqslant Y_t^{m,n} \leqslant Y_t^{m,n+1}, t \in [0, T]. \tag{3.75}$$

Step 2-1($n \to +\infty$). As n, m go to $+\infty$ by order, we conclude in the following two parts the existence of solution for our target RBSDE as the limit of solutions for cut-off case verified in **Step 1**. Indeed, given $m \in \mathbb{N}$, we have $L^m \in \mathbb{C}^0(\mathbb{P})$ and $\{L_t^{m,n}\}_{n\in\mathbb{N}}$ is an increasing sequence w.r.t. n as well as $\lim_{n\to\infty} \uparrow L_t^{m,n} = L_t^m$ for any $t \in [0, T]$. By (3.74) and (3.75), we know that outside a $\mathbb{P}$-null set $\mathcal{N}_m$, $\{Y_t^{m,n}\}_{n\in\mathbb{N}}$ is an increasing sequence w.r.t. n and bounded in the upper sense $c_0 + \frac{1}{\gamma}\ln B_t$ for any $t \in [0,T]$. Next except on this $\mathbb{P}$-null set $\mathcal{N}_m$, we define $\overline{\mathbb{F}}^{\mathbb{P}}$-adapted process $Y_t^m(\omega) := \lim_{n\to\infty} \uparrow Y_t^{m,n}(\omega), (t,\omega) \in [0,T] \times \Omega$. In particular,

$$Y_T^m = \lim_{n\to\infty} \uparrow Y_T^{m,n} = \lim_{n\to\infty} \uparrow \xi^{m,n} = \xi^m. \tag{3.76}$$

We could obtain that $\mathbb{P}$-a.s.

$$-L_t^- \leqslant Y_t^m \leqslant c_0 + \frac{1}{\gamma} \ln B_t, t \in [0,T], \tag{3.77}$$

by letting n go to ∞ in the above (3.74). For any $t \in [0,T]$, we define $\mathscr{L}_t^m := (L_t^{m,1})^- \vee (L_t^m)^- \leqslant L_t^-$, and $\mathscr{Y}_t^m := (Y_t^{m,1})^+ \vee (Y_t^m)^+ \leqslant c_0 + \frac{1}{\gamma} \ln B_t$. Then we get from from Doob's martingale inequality that

$$\begin{aligned}\mathbb{E}\Big[e^{\lambda\gamma\mathscr{L}_*^m} + e^{\lambda'\gamma\mathscr{Y}_*^m}\Big] \leqslant& \mathbb{E}\Big[e^{\lambda\gamma L_*^-}\Big] + c_{\lambda'}\mathbb{E}\Big[B_*^{\lambda'}\Big] = \mathbb{E}\Big[e^{\lambda\gamma L_*^-}\Big] + c_{\lambda'}\mathbb{E}\Big[B_T^{\lambda'}\Big] \\ =& \mathbb{E}\Big[e^{\lambda\gamma L_*^-}\Big] + c_{\lambda'}\mathbb{E}\Big[e^{\lambda'\gamma e^{\beta T}(\xi^+ \vee L_*^+)}\Big] < \infty.\end{aligned} \tag{3.78}$$

Therefore, we can now use the monotone stability result Theorem 3.3.6 to claim that $Y^m \in \mathbb{C}^{\exp(\lambda\gamma,\lambda'\gamma)}(\mathbb{P})$ and that there is $(Z^m, M^m, K^m) \in \cap_{p\in(1,\frac{\lambda\lambda'}{\lambda+\lambda'})} \mathbb{H}^{2,2p}(\mathbb{P},\mathbb{R}^d) \times \mathbb{M}^p(\mathbb{P}) \times \mathbb{I}^p(\mathbb{P})$ s.t. (Y^m, Z^m, M^m, K^m) is a solution of the quadratic RBSDE(f, Y_T^m, L^m). We let n go to $+\infty$ in (3.75) and obtain that $\mathbb{P}$-a.s.

$$Y_t^{m+1} \leqslant Y_t^m, t \in [0,T]. \tag{3.79}$$

Step 2-2($m \to +\infty$). Firstly, we note that $\{L_t^m\}_{m\in\mathbb{N}}$ is a decreasing sequence w.r.t. m and $\lim_{m\to\infty} \downarrow L_t^m = L_t$ for any $t \in [0,T]$. From (3.77) and (3.79), we know that outside a $\mathbb{P}$-null set $\mathcal{N}$, $\{Y_t^m\}_{m\in\mathbb{N}}$ is a decreasing sequence w.r.t. m and bounded in the lower sense $-L_t^-$ for any $t \in [0,T]$. Therefore, except on this $\mathbb{P}$-null set $\mathcal{N}$, we define $\overline{\mathbb{F}}^{\mathbb{P}}$-adapted process $Y_t := \lim_{m\to\infty} \downarrow Y_t^m, (t,\omega) \in [0,T] \times \Omega$. As $m \to \infty$ in (3.76) and (3.77), we have $\mathbb{P}$-a.s. that

$$Y_T = \lim_{m\to\infty} \downarrow Y_T^m = \lim_{m\to\infty} \downarrow \xi^m = \xi, \tag{3.80}$$

and

$$-L_t^- \leqslant Y_t \leqslant c_0 + \frac{1}{\gamma} \ln B_t, t \in [0,T]. \tag{3.81}$$

For any $t \in [0,T]$, we define $\mathscr{L}_t := (L_t^1)^- \vee L_t^- \leqslant L_t^-$, and $\mathscr{Y}_t := (Y_t^1)^+ \vee Y_t^+ \leqslant c_0 + \frac{1}{\gamma}\ln B_t, \mathbb{P}$-a.s.. The same as in (3.78), we can have that $\mathbb{E}[e^{\lambda\gamma\mathscr{L}_*} + e^{\lambda'\gamma\mathscr{Y}_*}] \leqslant \mathbb{E}[e^{\lambda\gamma L_*^-}] + c_{\lambda'}\mathbb{E}[e^{\lambda'\gamma e^{\beta T}(\xi^+ \vee L_*^+)}] < \infty$. Once again, we use the monotone stability Theorem 3.3.6 and (3.80) to obtain that $Y \in \mathbb{C}^{\exp(\lambda\gamma,\lambda'\gamma)}(\mathbb{P})$ and there si $(Z,M,K) \in \cap_{p\in(1,\frac{\lambda\lambda'}{\lambda+\lambda'})}\mathbb{H}^{2,2p}(\mathbb{P},\mathbb{R})(\mathbb{P},\mathbb{R}^d) \times \mathbb{M}^p(\mathbb{P}) \times \mathbb{K}^p(\mathbb{P})$ s.t. (Y,Z,M,K) is a solution of the quadratic RBSDE(f,ξ,L).

Step 3. If in addition we have the assumption that $\xi \vee L_* \in \mathbb{L}^{\exp}(\mathbb{P},\mathbb{R})$, then for any $p \in (1,\infty)$ we have

$$\begin{aligned}\mathbb{E}[e^{p\gamma Y_*}] \leqslant&\mathbb{E}[e^{p\gamma Y_*^-} + e^{p\gamma Y_*^+}] \leqslant \mathbb{E}[e^{p\gamma L_*^-}] + c_p\mathbb{E}[M_*^p] = \mathbb{E}[e^{p\gamma L_*^-}] + c_p\mathbb{E}[M_T^p] \\ =&\mathbb{E}[e^{p\gamma L_*^-}] + c_p\mathbb{E}[e^{p\gamma e^{\beta T}(\xi^+\vee L_*^+)}] \leqslant c_p\mathbb{E}[e^{p\gamma e^{\beta T}(\xi^+\vee L_*)}] < \infty,\end{aligned} \tag{3.82}$$

which gives that $Y \in \mathbb{C}^{\exp(p\gamma)}(\mathbb{P})$(of course in $\mathbb{C}^{\exp(p)}(\mathbb{P})$ as $\gamma \in \mathbb{R}_+$). Besides, we could obtain (3.73) by applying Proposition 3.3.4 with $\lambda = \lambda' = 3p$. □

Theorem 3.3.9 (BSDE version)**.** *Let (f,ξ) be a pair s.t. f satisfies the quadratic growth (i.e., (3) in Assumption 3.3.1) and that*

$$dt \otimes d\mathbb{P}\text{-a.e.}(t,\omega) \in [0,T]\times\Omega,\ \text{the mapping } f(t,\omega,\cdot,\cdot) \text{ is continuous.} \tag{3.83}$$

If $\mathbb{E}[e^{\lambda'\gamma e^{\beta T}\xi^+}] < \infty$ for some $\lambda,\lambda' > 6$ with $\frac{1}{\lambda} + \frac{1}{\lambda'} < \frac{1}{6}$, then the quadratic BSDE (f,ξ) has a solution $(Y,Z,M,K) \in \cap_{p\in\left(1,\frac{\lambda\lambda'}{\lambda+\lambda'}\right)}\mathbb{C}^{\lambda\gamma,\lambda'\gamma}(\mathbb{P}) \times \mathbb{H}^{2,2p}(\mathbb{P};\mathbb{R}^d) \times \mathbb{M}^p(\mathbb{P}) \times \mathbb{I}^p(\mathbb{P})$ that satisfies (3.27). Besides, if $\xi^+ \in \mathbb{L}^{\exp}(\mathbb{P},\mathbb{R})$, then this solution (Y,Z,M,K) belongs to $\mathbb{S}^p(\mathbb{P}) = \mathbb{C}^{\exp(p)}(\mathbb{P}) \times \mathbb{H}^{2,2p}(\mathbb{P},\mathbb{R}^d) \times \mathbb{M}^p(\mathbb{P}) \times \mathbb{I}^p(\mathbb{P})$ for all $p \in [1,\infty)$. To be precise, for any $p \in (1,\infty)$, we have the following a priori estimates:

$$\begin{aligned}&\mathbb{E}[e^{p\gamma Y_*}] \leqslant c_p + c_p\mathbb{E}[e^{p\gamma e^{\beta T}\xi^+}] < \infty; \\ &\mathbb{E}\left[\left(\int_0^T \widehat{\sigma}_s^2|Z_s|^2ds + d\langle M\rangle_s\right)^p + K_T^p\right] \leqslant c_p\mathbb{E}[e^{3p\gamma Y_*}] < \infty.\end{aligned} \tag{3.84}$$

3.3.5 Comparison

Theorem 3.3.10. *Let $(f^1,\xi^1,L^1),(f^2,\xi^2,L^2)$ be two triples and let (Y^1,Z^1,M^1,K^1) (resp. (Y^2,Z^2,M^2,K^2)) be a solution of RBSDE (f^1,ξ^1,L^1) (resp. RBSDE (f^2,ξ^2,L^2)) s.t.*

(C1) $\xi^1 \leqslant \xi^2$ and for any $t \in [0,T], L_t^1 \leqslant L_t^2, \mathbb{P}$-a.s.;

(C2) $\mathbb{E}[e^{\lambda Y_*^{1,+}} + e^{\lambda Y_*^{2,-}}] < \infty$ *for all* $\lambda \in (1,\infty)$ *and* $K^1 \in \mathbb{I}^p(\mathbb{P})$ *for some* $p \in (1,\infty)$*;*

(C3) For $\alpha, \beta, \kappa \geqslant 0, \gamma > 0$, f^1 *(resp.,* f^2*) satisfies (2), (3) in Assumption 3.3.1,* f^1 *(resp.,* f^2*) is concave in* z*, and* $\Delta f(t) := f_t^1(Y_t^2, Z_t^2) - f_t^2(Y_t^2, Z_t^2) \leqslant 0, dt \otimes d\mathbb{P}$*-a.e. (resp.,* $\Delta f(t) := f_t^1(Y_t^1, Z_t^1) - f_t^2(Y_t^1, Z_t^1) \leqslant 0, dt \otimes d\mathbb{P}$*-a.e.);*

then $Y_t^1 \leqslant Y_t^2$ *for any* $t \in [0,T], \mathbb{P}$*-a.s..*

Proof. **Step 1.** Following the classical procedure for comparison of quadratic BSDE with deterministic horizon(e.g., [BH08]), we choose some $\theta \in (0,1)$ and define $\mathfrak{Y} := \theta Y^1 - Y^2, \mathfrak{Z} := \theta Z^1 - Z^2, \mathfrak{M} := \theta M^1 - M^2, \mathfrak{K} := \theta K^1 - K^2$ as well as an $\overline{\mathbb{F}}^{\mathbb{P}}$-progressively measurable process

$$\begin{aligned} a_t :=& \mathbf{1}_{\{Y^1 \geqslant 0\}} \left(\mathbf{1}_{\{\mathfrak{Y}_t \neq 0\}} \frac{f^1(t, \theta Y_t^1, Z_t^2) - f^1(t, Y_t^2, Z_t^2)}{\mathfrak{Y}_t} - \kappa \mathbf{1}_{\{\mathfrak{Y}_t = 0\}} \right) - \kappa \mathbf{1}_{\{Y_t^1 < 0 \leqslant Y_t^2\}} \\ &+ \mathbf{1}_{\{Y^1 \vee Y^2 < 0\}} \left(\mathbf{1}_{\{Y^1 \neq Y^2\}} \frac{f^1(t, Y_t^1, Z_t^1) - f^1(t, Y_t^2, Z_t^1)}{Y_t^1 - Y_t^2} - \kappa \mathbf{1}_{\{Y^1 = Y^2\}} \right), t \in [0,T]. \end{aligned} \tag{3.85}$$

Note that here we only consider the first case in (C3), and for the second case in (C3), we just need to modify most parts related to f^1 in this proof to f^2, so we omit it here. Then $A_t := \int_0^t a_s ds, t \in [0,T]$ is an $\overline{\mathbb{F}}^{\mathbb{P}}$-adapted process. By Lipschitz continuity (2) in Assumption 3.3.1, we have $dt \otimes d\mathbb{P}$-a.e. that $|a_t| \leqslant \kappa, t \in [0,T]$, thus $A_* := \sup_{t \in [0,T]} |A_t| \leqslant \int_0^T |a_s| ds \leqslant \kappa T, \mathbb{P}$-a.s. By concavity (4) and quadratic growth (3) in Assumption 3.3.1, we have $dt \otimes d\mathbb{P}$-a.e. that

$$\begin{aligned} f^1(t, y, Z_t^2) \geqslant& \theta f^1(t, y, Z_t^1) + (1-\theta) f^1(t, y, \frac{-\mathfrak{Z}_t}{1-\theta}) \\ \geqslant& \theta f^1(t, y, Z_t^1) - (1-\theta)(\alpha + \beta |y|) - \frac{\gamma}{2(1-\theta)} \widehat{\sigma}_t^2 |\mathfrak{Z}_t|^2, \forall y \in \mathbb{R}, \end{aligned} \tag{3.86}$$

where we used $Z_t^2 = \theta Z_t^1 + (1-\theta) \frac{Z_t^2 - \theta Z_t^1}{1-\theta} = \theta Z_t^1 + (1-\theta) \frac{-\mathfrak{Z}_t}{1-\theta}$ in the first inequality. Next, for any $n \in \mathbb{N}$, define an $\overline{\mathbb{F}}^{\mathbb{P}}$-stopping time

$$\tau_n := \inf \left\{ t \in [0,T] : \int_0^t \widehat{\sigma}_s^2 (|Z_s^1|^2 + |Z_s^2|^2) ds + \langle M^1 \rangle_t + \langle M^2 \rangle_t > n \right\} \wedge T. \tag{3.87}$$

Clearly, $\lim_{n \to \infty} \tau_n = T, \mathbb{P}$-a.s.. Besides, we here define $\zeta_{\theta,T} := \frac{\gamma e^{\kappa T}}{1-\theta}$. Using Itô's formula to the process $\Gamma_s := \exp\{\zeta_{\theta,T} e^{A_s} \mathfrak{Y}_s\}, s \in [0,T]$, similar to the classical case(e.g., [BH08]). For readers' ease, we here write down this detail:

$$
\begin{aligned}
d\Gamma_s = d(e^{\zeta_{\theta,T}e^{A_s}\mathfrak{Y}_s}) &= \zeta_{\theta,T}\Gamma_s d(e^{A_s}\mathfrak{Y}_s) + \frac{1}{2}\zeta_{\theta,T}^2\Gamma_s d\langle e^{A_\cdot}\mathfrak{Y}_\cdot\rangle_s \\
=&\zeta_{\theta,T}\Gamma_s\Big(e^{A_s}d\mathfrak{Y}_s + e^{A_s}\mathfrak{Y}_s dA_s\Big) + \frac{1}{2}\zeta_{\theta,T}^2\Gamma_s e^{2A_s}d\langle\mathfrak{Y}_\cdot\rangle_s \\
=&\zeta_{\theta,T}\Gamma_s e^{A_s}\Big\{-\Big(\theta f_s^1 - f_s^2\Big)ds + (\theta Z_s^1 - Z_s^2)\cdot dX_s + d(\theta U_s^1 - U_s^2)\Big\} + \zeta_{\theta,T}\Gamma_s e^{A_s}\mathfrak{Y}_s a_s ds \\
&+ \frac{1}{2}\zeta_{\theta,T}^2\Gamma_s e^{2A_s}\Big[(\theta Z_s^1 - Z_s^2)^2\widehat{\sigma}_s^2 ds + d\langle\theta M_\cdot^1 - M_\cdot^2\rangle_s\Big],
\end{aligned}
\tag{3.88}
$$

then we write the above differential form as the integration form to obtain that

$$
\begin{aligned}
\Gamma_{\tau_n\wedge t} =&\Gamma_{\tau_n} + \zeta_{\theta,T}\int_{\tau_n\wedge t}^{\tau_n}\Gamma_s e^{A_s}\Big\{[(\theta f^1(s,Y_s^1,Z_s^1) - f^2(s,Y_s^2,Z_s^2)) - a_s\mathfrak{Y}_s]ds \\
&- \frac{1}{2}\zeta_{\theta,T}e^{A_s}d\langle\mathfrak{Z}_\cdot\cdot X + \mathfrak{M}_\cdot\rangle_s - \zeta_{\theta,T}\int_{\tau_n\wedge t}^{\tau_n}\Gamma_s e^{A_s}\Big(\mathfrak{Z}_s\cdot dX_s + d\mathfrak{M}_s\Big) \\
=&\Gamma_{\tau_n} + \int_{\tau_n\wedge t}^{\tau_n}G_s ds + \zeta_{\theta,T}\int_{\tau_n\wedge t}^{\tau_n}\Gamma_s e^{A_s}d\mathfrak{K}_s - \zeta_{\theta,T}\int_{\tau_n\wedge t}^{\tau_n}\Gamma_s e^{A_s}\Big(\mathfrak{Z}_s\cdot dX_s + d\mathfrak{M}_s\Big), t\in[0,T],
\end{aligned}
\tag{3.89}
$$

where we introduced the notation

$$
G_s ds :=\zeta_{\theta,T}\Gamma_s e^{A_s}\Big\{[(\theta f^1(s,Y_s^1,Z_s^1) - f^2(s,Y_s^2,Z_s^2)) - a_s\mathfrak{Y}_s]ds - \frac{1}{2}\zeta_{\theta,T}e^{A_s}d\langle\mathfrak{Z}_\cdot\cdot X_\cdot + \mathfrak{M}_\cdot\rangle_s\Big\}.
$$

Then by (C3) we can obtain that $ds\otimes d\mathbb{P}$-a.e.,

$$
G_s ds \leqslant \zeta_{\theta,T}\Gamma_s e^{A_s}\Big\{[\Big(\theta f^1(s,Y_s^1,Z_s^1) - f^1(s,Y_s^2,Z_s^2)\Big) - a_s\mathfrak{Y}_s]ds - \frac{\gamma}{2(1-\theta)}d\langle\mathfrak{Z}_\cdot\cdot X_\cdot + \mathfrak{M}_\cdot\rangle_s\Big\}, \tag{3.90}
$$

where we used in the last term that $\zeta_{\theta,T}e^{A_s} = \frac{\gamma}{1-\theta}e^{\kappa T + A_s} \geqslant \frac{\gamma}{1-\theta}$ due to the fact $A_* \leqslant \kappa T$.

Step 2. Below, we will verify by 3 cases whether there exist a bound not related to θ for G_s, precisely we will prove that

$$
G_s \leqslant \gamma e^{\kappa T + A_s}\Gamma_s\Big[\alpha + (\beta+\kappa)(Y_s^{1,+} + Y_s^{2,-})\Big], ds\otimes d\mathbb{P}\text{ a.e.}, \tag{3.91}
$$

where $Y_s^{1,+} := Y_s^1 \vee 0, Y_s^{2,-} := -(Y_s^2\wedge 0)$, for $s\in[0,T]$.

1) For $ds\otimes d\mathbb{P}$-a.e. $(s,\omega)\in\{Y_s^1(\omega)\geqslant 0\}$, using the definition of a and (3.86) with $y = Y_s^1$(where we used concavity and quadratic growth), i.e., $f^1(s,Y_s^1,Z_s^2) \geqslant \theta f^1(s,Y_s^1,Z_s^1) - (1-\theta)(\alpha + \beta|Y_s^1|) - \frac{\gamma}{2(1-\theta)}\widehat{\sigma}_s^2|\mathfrak{Z}_s|^2$,

we can then replace $\theta f^1(s,Y_s^1,Z_s^1)$ in (3.90) by this estimate to get that,

$$\begin{aligned} G_s ds \leqslant & \zeta_{\theta,T}\Gamma_s e^{A_s}\Big\{\Big[\theta f^1(s,Y_s^1,Z_s^1)-f^1(s,\theta Y_s^1,Z_s^2)\Big]ds-\frac{\gamma}{2(1-\theta)}d\langle \mathfrak{Z}.\cdot X.+\mathfrak{M}.\rangle_s\Big\} \\ = & \zeta_{\theta,T}\Gamma_s e^{A_s}\Big\{\Big(f^1(s,Y_s^1,Z_s^2)-f^1(s,\theta Y_s^1,Z_s^2)+(1-\theta)(\alpha+\beta|Y_s^1|)\Big)ds \\ & +\frac{\gamma}{2(1-\theta)}\Big(\widehat{\sigma}_s^2|\mathfrak{Z}_s|^2ds-d\langle \mathfrak{Z}.\cdot X.+\mathfrak{M}.\rangle_s\Big)\Big\}; \end{aligned} \tag{3.92}$$

besides, by using Lipschitz continuity, i.e. (2) in Assumption 3.3.1, we have

$$|f^1(s,Y_s^1,Z_s^2)-f^1(s,\theta Y_s^1,Z_s^2)|\leqslant \kappa(1-\theta)|Y_s^1|;$$

then to conclude the above calculus, we obtain that

$$\begin{aligned} G_s ds \leqslant & \zeta_{\theta,T}\Gamma_s e^{A_s}\Big\{\Big[\kappa(1-\theta)|Y_s^1|+(1-\theta)(\alpha+\beta|Y_s^1|)\Big]ds+\frac{\gamma}{2(1-\theta)}\Big(\widehat{\sigma}_s^2|\mathfrak{Z}_s|^2ds-d\langle \mathfrak{Z}.\cdot X+\mathfrak{M}.\rangle_s\Big)\Big\} \\ \leqslant & \gamma e^{\kappa T+A_s}\Gamma_s\Big[\alpha+(\beta+\kappa)Y_s^{1,+}\Big]ds. \end{aligned} \tag{3.93}$$

2) For $ds\otimes d\mathbb{P}$-a.e. $(s,\omega)\in\{Y_s^1(\omega)<0\leqslant Y_s^2(\omega)\}$, inserting '$-f^1(s,0,Z_s^1)+f^1(s,0,Z_s^1)$'$(\equiv 0)$ in (3.90), we have

$$\begin{aligned} G_s ds \leqslant & \zeta_{\theta,T}\Gamma_s e^{A_s}\Big\{[\Big(\theta|f^1(s,Y_s^1,Z_s^1)-f^1(s,0,Z_s^1)|+\theta f^1(s,0,Z_s^1)-f^1(s,Y_s^2,Z_s^2)\Big) \\ & -a_s\mathfrak{Y}_s]ds-\frac{\gamma}{2(1-\theta)}d\langle \mathfrak{Z}.\cdot X+\mathfrak{M}.\rangle_s\Big\} \\ \leqslant & \zeta_{\theta,T}\Gamma_s e^{A_s}\Big\{\Big[-\theta\kappa Y_s^1+\Big(f^1(s,0,Z_s^2)+(1-\theta)\alpha+\frac{\gamma}{2(1-\theta)}\widehat{\sigma}_s^2|\mathfrak{Z}_s|^2-f^1(s,Y_s^2,Z_s^2)\Big)+\kappa\mathfrak{Y}_s\Big]ds \\ & -\frac{\gamma}{2(1-\theta)}d\langle \mathfrak{Z}.\cdot X+\mathfrak{M}.\rangle_s\Big\} \end{aligned} \tag{3.94}$$

where we use (3.85), i.e. $a_s=-\kappa$ for the case $Y_s^1(\omega)<0\leqslant Y_s^2(\omega)$ and apply (3.86) with $y\equiv 0$; once again by Lipschitz continuity, we have

$$f^1(s,0,Z_s^2)-f^1(s,Y_s^2,Z_s^2)\leqslant \kappa Y_s^2;$$

then to conclude the above calculus, we obtain that

$$\begin{aligned} G_s ds \leqslant & \zeta_{\theta,T}\Gamma_s e^{A_s}\left\{\left[-\theta\kappa Y_s^1+\kappa Y_s^2+(1-\theta)\alpha+\kappa\mathfrak{Y}_s\right]ds+\frac{\gamma}{2(1-\theta)}\Big(\widehat{\sigma}_s^2|\mathfrak{Z}_s|^2ds-d\langle\mathfrak{Z}.\cdot X+\mathfrak{M}.\rangle_s\Big)\right\} \\ \leqslant & \zeta_{\theta,T}\Gamma_s e^{A_s}\Big[(1-\theta)\alpha\Big]ds\leqslant\alpha\gamma e^{\kappa T+A_s}\Gamma_s ds. \end{aligned} \tag{3.95}$$

3) For $ds\otimes d\mathbb{P}$-a.e. $(s,\omega)\in\{Y_s^1\vee Y_s^2<0\}$, we use (3.86) with $y=Y_s^2$ to get that

$$\begin{aligned} G_s ds = & \zeta_{\theta,T}\Gamma_s e^{A_s}\left\{\left[\theta f^1(s,Y_s^1,Z_s^1)-\theta f^1(s,Y_s^2,Z_s^1)-a_s\mathfrak{Y}_s+(1-\theta)(\alpha+\beta|Y_s^2|)+\frac{\gamma}{2(1-\theta)}\widehat{\sigma}_s^2|\mathfrak{Z}_s|^2\right]ds\right. \\ & \left.-\frac{\gamma}{2(1-\theta)}d\langle\mathfrak{Z}.\cdot X+\mathfrak{M}.\rangle_s\right\} \\ \leqslant & \zeta_{\theta,T}\Gamma_s e^{A_s}\Big\{\theta f^1(s,Y_s^1,Z_s^1)-\theta f^1(s,Y_s^2,Z_s^1)-a_s\mathfrak{Y}_s+(1-\theta)(\alpha+\beta|Y_s^2|)ds\Big\}; \end{aligned} \tag{3.96}$$

by the definition of a, we have

$$\theta f^1(s,Y_s^1,Z_s^1)-\theta f^1(s,Y_s^2,Z_s^1)-a_s\mathfrak{Y}_s=(1-\theta)a_sY_s^2;$$

then to conclude the above calculus, we obtain that

$$\begin{aligned} G_s ds \leqslant & \zeta_{\theta,T}\Gamma_s e^{A_s}\Big\{(1-\theta)a_sY_s^2+(1-\theta)(\alpha+\beta|Y_s^2|)ds\Big\} \\ \leqslant & \gamma e^{\kappa T+A_s}\Gamma_s\Big[-\kappa Y_s^2+\alpha+\beta|Y_s^2|\Big]ds=\gamma e^{\kappa T+A_s}\Gamma_s\Big[\alpha+(\kappa+\beta)Y_s^{2,-}\Big]ds. \end{aligned} \tag{3.97}$$

To conclude, we have obtained three bounds in above, namely $\gamma e^{\kappa b+A_s}\Gamma_s\Big[\alpha+(\beta+\kappa)Y_s^{1,+}\Big]$, $\alpha\gamma e^{\kappa b+A_s}\Gamma_s$ and $\gamma e^{\kappa b+A_s}\Gamma_s\Big[\alpha+(\kappa+\beta)Y_s^{2,-}\Big]$, then (3.91) follows.

Step 3. Now, we define a process $D_t:=\exp\Big\{\gamma e^{2\kappa T}\int_0^t\Big[\alpha+(\beta+\kappa)(Y_s^{1,+}+Y_s^{2,-})\Big]ds\Big\}, t\in[0,T]$.

The integration by parts with the bound (3.91) obtained for G_s gives that $\mathbb{P}$-a.s.

$$\begin{aligned}D_{\tau_n}\Gamma_{\tau_n} =& D_{\tau_n\wedge t}\Gamma_{\tau_n\wedge t}+\int_{\tau_n\wedge t}^{\tau_n} D_s d\Gamma_s+\Gamma_s dD_s\\ =& D_{\tau_n\wedge t}\Gamma_{\tau_n\wedge t}+\int_{\tau_n\wedge t}^{\tau_n} D_s\Big\{-\zeta_{\theta,T}\Gamma_s e^{A_s}\Big[[(\theta f^1(s,Y_s^1,Z_s^1)-f^2(s,Y_s^2,Z_s^2))\\ &-a_s\mathfrak{Y}_s]ds-\frac{1}{2}\zeta_{\theta,T}e^{A_s}d\langle\mathfrak{Z}_\cdot\cdot X+\mathfrak{M}_\cdot\rangle_s\Big]-\zeta_{\theta,T}\Gamma_s e^{A_s}d\mathfrak{K}_s+\zeta_{\theta,T}\Gamma_s e^{A_s}(\mathfrak{Z}_s\cdot dX_s+d\mathfrak{M}_s)\Big\}\\ &+\int_{\tau_n\wedge t}^{\tau_n}\Gamma_s D_s\gamma e^{2\kappa T}\Big[\alpha+(\beta+\kappa)(Y_s^{1,+}+Y_s^{2,-})\Big]ds\\ =& D_{\tau_n\wedge t}\Gamma_{\tau_n\wedge t}+\int_{\tau_n\wedge t}^{\tau_n} D_s\Big\{-G_s ds-\zeta_{\theta,T}\Gamma_s e^{A_s}d\mathfrak{K}_s+\zeta_{\theta,T}\Gamma_s e^{A_s}(\mathfrak{Z}_s\cdot dX_s+d\mathfrak{M}_s)\Big\}\\ &+\int_{\tau_n\wedge t}^{\tau_n}\Gamma_s D_s\gamma e^{2\kappa T}\Big[\alpha+(\beta+\kappa)(Y_s^{1,+}+Y_s^{2,-})\Big]ds,\end{aligned}\tag{3.98}$$

so we have

$$\Gamma_{\tau_n\wedge t}\leqslant D_{\tau_n\wedge t}\Gamma_{\tau_n\wedge t}\leqslant D_{\tau_n}\Gamma_{\tau_n}+\zeta_{\theta,T}\int_{\tau_n\wedge t}^{\tau_n}D_s\Gamma_s e^{A_s}dK_s^1-\zeta_{\theta,T}\int_{\tau_n\wedge t}^{\tau_n}D_s\Gamma_s e^{A_s}(\mathfrak{Z}_s\cdot dX_s+d\mathfrak{M}_s).\tag{3.99}$$

Together with that $\mathbb{P}$-a.s., $L_s^1\leqslant L_s^2\leqslant Y_s^2$ for any $s\in[0,T]$, the flat-off condition of (Y^1,Z^1,M^1,K^1) shows that $\mathbb{P}$-a.s.,

$$\int_0^T D_s\Gamma_s dK_s^1=\int_0^T\mathbf{1}_{\{Y_s^1=L_s^1\}}D_s\Gamma_s dK_s^1\leqslant\int_0^T\mathbf{1}_{\{Y_s^1\leqslant Y_s^2\}}D_s\Gamma_s dK_s^1\leqslant\int_0^T D_s e^{\gamma e^{\kappa T+A_s}Y_s^{2,-}}dK_s^1\leqslant\varsigma K_T^1,\tag{3.100}$$

where we used

$$\begin{aligned}\mathbf{1}_{\{Y_s^1\leqslant Y_s^2\}}\Gamma_s =&\mathbf{1}_{\{Y_s^1\leqslant Y_s^2\}}\exp\left\{\frac{\gamma e^{\kappa T}}{1-\theta}e^{A_s}\mathfrak{Y}_s\right\}\leqslant\exp\left\{\frac{\gamma e^{\kappa T}}{1-\theta}e^{A_s}(-1+\theta)Y_s^2\right\}\\ &\leqslant\exp\Big\{-\gamma e^{\kappa T+A_s}Y_s^2\Big\}\leqslant\exp\Big\{\gamma e^{\kappa T+A_s}Y_s^{2,-}\Big\},\end{aligned}$$

and

$$\begin{aligned}D_s e^{\gamma e^{\kappa T+A_s}Y_s^{2,-}}&=\exp\left\{\gamma e^{2\kappa T}\int_0^s\Big[\alpha+(\beta+\kappa)(Y_r^{1,+}+Y_r^{2,-})\Big]dr\right\}\cdot e^{\gamma e^{2\kappa T}Y_s^{2,-}}\\ &=\exp\left\{\gamma e^{2\kappa T}\left[\int_0^s\Big[\alpha+(\beta+\kappa)(Y_r^{1,+}+Y_r^{2,-})\Big]dr+Y_s^{2,-}\right]\right\}\\ &\leqslant\exp\left\{\gamma e^{2\kappa T}\Big[\alpha T+\Big((\beta+\kappa)T+1\Big)Y_*^{2,-}+(\beta+\kappa)TY_*^{1,+}\Big]\right\}=:\varsigma.\end{aligned}\tag{3.101}$$

By Hölder's inequality and the solution space, we have, except on a $\mathbb{P}$-null set $\mathfrak{N}^{\mathbb{P}}$, that

$$\begin{aligned}
&\mathbb{E}[\varsigma(1+K_T)] \leqslant \Big(\mathbb{E}[|\varsigma|^{\frac{p}{p-1}}]\Big)^{\frac{p-1}{p}} \Big(\mathbb{E}[(1+K_T)^p]\Big)^{\frac{1}{p}} \leqslant \|\varsigma\|_{\mathbb{L}^{\frac{p}{p-1}}(\mathbb{P},\mathbb{R})} \cdot C\Big(1+\|K_T\|_{\mathbb{L}^p(\mathbb{P},\mathbb{R})}\Big) < \infty, \\
&\text{and } \mathbb{E}[D_T\Gamma_*] \leqslant \mathbb{E}\bigg\{\exp\Big[\gamma e^{2\kappa T}\big[\alpha+(\beta+\kappa)(Y_*^{1,+}+Y_*^{2,-})\big]T\Big]\cdot \exp\Big[\zeta_{\theta,T}e^{\kappa T}(Y_*^{1,+}+Y_*^{2,-})\Big]\bigg\} \\
&\qquad\qquad\qquad \leqslant \mathbb{E}\bigg\{\exp\Big[\gamma e^{2\kappa T}\alpha T + \gamma e^{2\kappa T}\Big[(\beta+\kappa)T + \frac{1}{1-\theta}\Big](Y_*^{1,+}+Y_*^{2,-})\Big]\bigg\} < \infty,
\end{aligned} \tag{3.102}$$

where the last two inequalities are guaranteed by condition (C3). Then Burkholder-Davis-Gundy inequality shows that

$$\begin{aligned}
&\mathbb{E}\bigg[\sup_{t\in[0,T]}\bigg|\int_0^{\tau_n\wedge t} D_s\Gamma_s e^{A_s}(\mathfrak{Z}_s dX_s + d\mathfrak{M}_s)\bigg|\bigg] \\
&\leqslant C\mathbb{E}\bigg[D_T\Gamma_* e^{A_*}\bigg(\int_0^{\tau_n}\widehat{\sigma}_s^2|\mathfrak{Z}_s|^2 ds + d\langle\mathfrak{M}.\rangle_s\bigg)^{\frac{1}{2}}\bigg] \\
&\leqslant C\sqrt{n}\mathbb{E}[D_T\Gamma_*] < \infty,
\end{aligned} \tag{3.103}$$

so $\int_0^{\tau_n\wedge\cdot} D_s\Gamma_s e^{A_s}(\mathfrak{Z}_s dX_s + d\mathfrak{M}_s)$ is a uniformly integrable martingale.

Taking $\mathbb{E}[\cdot|\overline{\mathcal{F}}_{\tau_n\wedge t}]$ in (3.99), where the 3rd part in RHS disappears because it's local martingale and the 2nd part is

$$\mathbb{E}\bigg[\zeta_{\theta,T}\int_{\tau_n\wedge t}^{\tau_n} D_s\Gamma_s e^{A_s}dK_s^1\bigg|\overline{\mathcal{F}}_{\tau_n\wedge t}\bigg] \leqslant C\zeta_{\theta,T}\mathbb{E}\bigg[\int_{\tau_n\wedge t}^{\tau_n} D_s\Gamma_s dK_s^1\bigg|\overline{\mathcal{F}}_{\tau_n\wedge t}\bigg] \leqslant C\zeta_{\theta,T}\mathbb{E}\bigg[\varsigma K_T^1\bigg|\overline{\mathcal{F}}_{\tau_n\wedge t}\bigg]$$

by (3.100), then we get that $\mathbb{P}$-a.s.

$$\begin{aligned}
\Gamma_{\tau_n\wedge t} &\leqslant \mathbb{E}[D_{\tau_n}\Gamma_{\tau_n}|\overline{\mathcal{F}}_{\tau_n\wedge t}] + C\zeta_{\theta,T}\mathbb{E}[\varsigma K_T^1|\overline{\mathcal{F}}_{\tau_n\wedge t}] \\
&\leqslant \mathbf{1}_{\{\tau_n<t\}}D_{\tau_n}\Gamma_{\tau_n} + \mathbf{1}_{\{\tau_n\geqslant t\}}\mathbb{E}[D_{\tau_n}\Gamma_{\tau_n}|\overline{\mathcal{F}}_t] + C\zeta_{\theta,T}\mathbb{E}[\varsigma K_T^1|\overline{\mathcal{F}}_{\tau_n\wedge t}].
\end{aligned} \tag{3.104}$$

Letting $n\to\infty$, by dominated convergence theorem, we have $\mathbb{P}$-a.s.

$$\begin{aligned}
\Gamma_t \leqslant \mathbb{E}[D_T\Gamma_T|\overline{\mathcal{F}}_t] + C\zeta_{\theta,T}\mathbb{E}[\varsigma K_T^1|\overline{\mathcal{F}}_t] &\leqslant \mathbb{E}[D_T e^{\gamma e^{2\kappa T}\xi^{2,-}}|\overline{\mathcal{F}}_t] + C\zeta_{\theta,T}\mathbb{E}[\varsigma K_T^1|\overline{\mathcal{F}}_t] \\
&\leqslant C(1\vee\zeta_{\theta,T})\mathbb{E}[\varsigma(1+K_T^1)|\overline{\mathcal{F}}_t],
\end{aligned} \tag{3.105}$$

where we used two inequalities in (3.101) which correspond respectively to the penultimate and last inequality here. Taking the logarithmic in above leads to,

$$\theta Y_t^1 - Y_t^2 \leqslant \frac{1-\theta}{\gamma}\ln\left(1 \vee \frac{\gamma e^{\kappa T}}{1-\theta}\right)e^{-\kappa T - A_t} + \frac{1-\theta}{\gamma}(C + \ln \mathbb{E}[\varsigma(1+K_T^1)|\overline{\mathcal{F}}_t])e^{-\kappa T - A_t}, \mathbb{P}\text{-a.s.}. \quad (3.106)$$

Letting $\theta \to 1$,we have that $Y_t^1 - Y_t^2 \leqslant 0, \mathbb{P}$-a.s. □

Theorem 3.3.11 (BSDE version)**.** *Given $(f,\xi),(f',\xi')$, let (Y,Z,M,K) (resp. (Y',Z',M',K')) be a solution of BSDE (f,ξ) (resp. BSDE (f',ξ')) s.t.*

(C1) $\xi \leqslant \xi', \mathbb{P}$-a.s.;

(C2) $\mathbb{E}[e^{\lambda Y_*^+} + e^{\lambda Y_*^{\prime -}}] < \infty$ for all $\lambda \in (1,\infty)$ and $K \in \mathbb{I}^p(\mathbb{P})$ for some $p \in (1,\infty)$;

(C3) For $\alpha,\beta,\kappa \geqslant 0, \gamma > 0$, f (resp., f') satisfies (2), (3) in Assumption 3.3.1, f (resp., f') is concave in z, and $\Delta f(t) := f_t(Y_t', Z_t') - f_t'(Y_t', Z_t') \leqslant 0, dt \otimes d\mathbb{P}$-a.e. (resp., $\Delta f(t) := f_t(Y_t, Z_t) - f_t'(Y_t, Z_t) \leqslant 0, dt \otimes d\mathbb{P}$-a.e.);

then $Y_t \leqslant Y_t'$ for any $t \in [0,T], \mathbb{P}$-a.s..

Remark 3.3.12. *Directly, we could obtain the uniqueness of corresponding RBSDE and BSDE from the above comparison results.*

3.3.6 Stability

Theorem 3.3.13. *Let $\{(f^m, \xi_m, L^m)\}_{m\in\mathbb{N}\cup\{0\}}$ be a sequence of triples s.t.*

(S1) For the same constants $\alpha,\beta,\kappa \geqslant 0$ and $\gamma > 0$, f^0 satisfies the Lipschitz condition in y (i.e., (2) in Assumption 3.3.1) and $\{f^n\}_{n\in\mathbb{N}}$ satisfies quadratic in z, Lipschitz in y (i.e., (3), (2) of Assumption 3.3.1) and concave in z (i.e., (4) in Assumption 3.3.1);

(S2) It holds $\mathbb{P}$-a.s. that ξ_n converges to ξ_0 and that L_t^n converges to L_t^0 uniformly in $t \in [0,T]$;

(S3) $\Xi(p) := \sup_{m\in\mathbb{N}\cup\{0\}} \mathbb{E}[e^{p(\xi_m^+ \vee L_*^m)}] < \infty$ for all $p \in (1,\infty)$.

Taking $(Y^0, Z^0, M^0, K^0) \in \cap_{p\in[1,\infty)}\mathbb{S}^p(\mathbb{P}) = \cap_{p\in[1,\infty)}\mathbb{C}^{exp(p)}(\mathbb{P}) \times \mathbb{H}^{2,2p}(\mathbb{P};\mathbb{R}^d) \times \mathbb{M}^p(\mathbb{P}) \times \mathbb{I}^p(\mathbb{P})$ as a solution of the quadratic RBSDE (f^0, ξ_0, L^0), and for any $n \in \mathbb{N}$, we take (Y^n, Z^n, M^n, K^n) as the *unique solution of the quadratic RBSDE* $(f^n, \xi_n, L^n) \in \cap_{p\in[1,\infty)}\mathbb{S}^p(\mathbb{P})$. *If* $f^n_t(Y^0_t, Z^0_t)$ *converges* $dt \otimes d\mathbb{P}$-*a.e. to* $f^0_t(Y^0_t, Z^0_t)$, *then for any* $p \in [1,\infty)$, $\{\sup_{t\in[0,T]} |Y^n_t - Y^0_t|^p\}_{n\in\mathbb{N}}$ *is a uniformly integrable sequence in* $\mathbb{L}^1(\mathbb{P}, \mathbb{R})$ *and*

$$\lim_{n\to\infty} \mathbb{E}[\sup_{t\in[0,T]} |Y^n_t - Y^0_t|^p] = \lim_{n\to\infty} \mathbb{E}\left[\left(\int_0^T |Z^n_s - Z_s|^2 \hat{\sigma}^2_s ds + d\langle M^n_\cdot - M_\cdot\rangle_s\right)^p\right] = 0.$$

Moreover, if it holds $dt \otimes d\mathbb{P}$-*a.e. that* $f^n_t(\omega, y, z)$ *converges to* $f^0_t(\omega, y, z)$ *locally uniformly in* (y, z), *then up to a subsequence, we have further* $\lim_{n\to\infty} \mathbb{E}[\sup_{t\in[0,T]} |K^n_t - K^0_t|^p] = 0, \forall p \in [1,\infty)$.

Below we divide the proof into four steps.

Lemma 2.A. *Given* $n \in \mathbb{N}, \theta \in (0,1)$ *and* $\epsilon > 0$. *Then we have that* $\mathbb{P}$-*a.s.*

$$|Y^n_t - Y^0_t| \leqslant (1-\theta)(|Y^0_t| + |Y^n_t|) + \frac{1-\theta}{\gamma} \ln\left(\sum_{i=1}^4 I^{n,i}_t\right), t \in [0,T], \tag{3.107}$$

where $I^{n,i}_t := \mathbb{E}[I^{n,i}_T | \overline{\mathcal{F}}_t]$ *for* $i = 1,2,3,4$ *s.t.*

$$\begin{aligned}
I^{n,1}_T :=& D_T \eta_n \text{ with } D_t := \exp\left\{\gamma e^{2\kappa T} \int_0^t [\alpha + (\beta+\kappa)|Y^0_s|] ds\right\}, t \in [0,T] \text{ and} \\
&\eta_n := \exp\{\zeta_\theta e^{\kappa T}(|\xi_n - \theta\xi_0| \vee |\xi_0 - \theta\xi_n|)\}; \\
I^{n,2}_T :=& \zeta_\theta e^{\kappa T} D_T \Upsilon_n \int_0^T |\Delta_n f(s)| ds \text{ with } \zeta_\theta := \frac{\gamma e^{\kappa T}}{1-\theta}, \Upsilon_n := \exp\{\zeta_\theta e^{\kappa T}(Y^n_* + Y^0_*)\} \\
&\text{and } \Delta_n f(t) := f^n(t, Y^0_t, Z^0_t) - f^0(t, Y^0_t, Z^0_t), t \in [0,T]; \\
I^{n,3}_T :=& \Big(1 + \zeta_\theta \exp\{\kappa T + \epsilon\zeta_\theta e^{\kappa T}\}\Big)\Big\{1 + D_T \exp\Big[\gamma e^{2\kappa T}(Y^0_* + Y^n_*)\Big](K^0_T + K^n_T)\Big\}; \\
I^{n,4}_T :=& \frac{\zeta_\theta}{\epsilon} e^{\kappa T} D_T \Upsilon_n \Big[\sup_{t\in[0,T]} |L^n_t - L^0_t|\Big](K^0_T + K^n_T).
\end{aligned} \tag{3.108}$$

Proof. Define $\mathfrak{Y}^n := \theta Y^0 - Y^n, \mathfrak{Z}^n := \theta Z^0 - Z^n, \mathfrak{M}^n := \theta M^0 - M^n, \mathfrak{K}^n := \theta K^0 - K^n$, and define two processes

$$a^n_t := \mathbf{1}_{\{\mathfrak{Y}^n_t \neq 0\}} \frac{f^n(t, \theta Y^0_t, Z^n_t) - f^n(t, Y^n_t, Z^n_t)}{\mathfrak{Y}^n_t} - \kappa \mathbf{1}_{\{\mathfrak{Y}^n_t = 0\}}, \quad A^n_t := \int_0^t a^n_s ds, \quad t \in [0,T]. \tag{3.109}$$

As in (3.89)(Here we use ζ_θ to denote $\zeta_{\theta,T}$), using Itô's formula to the process $\Gamma^n_t := \exp\{\zeta_\theta e^{A^n_t} \mathfrak{Y}^n_t\}, t \in$

$[0,T]$ gives that

$$\begin{aligned}\Gamma^n_t =&\Gamma^n_T+\zeta_\theta\int_t^T\Gamma^n_se^{A^n_s}\Big\{[(\theta f^0(s,Y^0_s,Z^0_s)-f^n(s,Y^n_s,Z^n_s))-a^n_s\mathfrak{Y}^n_s]ds\\&-\frac{1}{2}\zeta_\theta e^{A^n_s}d\langle\mathfrak{Z}^n\cdot X+\theta M^0_\cdot-M^n_\cdot\rangle_s\Big\}+\zeta_\theta\int_t^T\Gamma^n_se^{A^n_s}(\theta dK^0_s-dK^n_s)\\&-\zeta_\theta\int_t^T\Gamma^n_se^{A^n_s}\Big[\mathfrak{Z}^n_s\cdot dX_s+d(\theta M^0_s-M^n_s)\Big],t\in[0,T]\\=&\Gamma^n_T+\zeta_\theta\int_t^T\Gamma^n_se^{A^n_s}\Big\{[(\theta f^0(s,Y^0_s,Z^0_s)-f^n(s,Y^n_s,Z^n_s))-a^n_s\mathfrak{Y}^n_s]ds\\&-\frac{1}{2}\zeta_\theta e^{A^n_s}d\langle\mathfrak{Z}^n\cdot X+\mathfrak{M}^n_\cdot\rangle_s\Big\}+\zeta_\theta\int_t^T\Gamma^n_se^{A^n_s}d\mathfrak{K}^n_s-\zeta_\theta\int_t^T\Gamma^n_se^{A^n_s}\Big[\mathfrak{Z}^n_s\cdot dX_s+d\mathfrak{M}^n_s\Big],t\in[0,T],\end{aligned}\tag{3.110}$$

and here we could define

$$\begin{aligned}G^n_sds:=&\zeta_\theta\Gamma^n_se^{A^n_s}\Big\{[\theta f^0(s,Y^0_s,Z^0_s)-f^n(s,Y^n_s,Z^n_s)-a^n_s\mathfrak{Y}^n_s]ds-\frac{1}{2}\zeta_\theta e^{A^n_s}d\langle\mathfrak{Z}^n\cdot X+\theta M^0_\cdot-M^n_\cdot\rangle_s\Big\}\\=&\zeta_\theta\Gamma^n_se^{A^n_s}\Big\{[\theta f^0(s,Y^0_s,Z^0_s)-f^n(s,Y^n_s,Z^n_s)-a^n_s\mathfrak{Y}^n_s]ds-\frac{1}{2}\zeta_\theta e^{A^n_s}d\langle\mathfrak{Z}^n\cdot X+\mathfrak{M}^n_\cdot\rangle_s\Big\}.\end{aligned}\tag{3.111}$$

By (2) in Assumption 3.3.1 and the fact that f_n is concave in z, we have that $dt\otimes d\mathbb{P}$-a.e. for any $y\in\mathbb{R}$,

$$\begin{aligned}f^n(s,y,Z^n_s)\geqslant&\theta f^n(s,y,Z^0_s)+(1-\theta)f^n\left(s,y,\frac{-\mathfrak{Z}^n_s}{1-\theta}\right)\\\geqslant&\theta f^n(s,y,Z^0_s)-(1-\theta)(\alpha+\beta|y|)-\frac{\gamma}{2(1-\theta)}\widehat{\sigma}^2_s|\mathfrak{Z}^n_s|^2.\end{aligned}\tag{3.112}$$

Then in the definition (3.111) , we use the definition of $\Delta_nf(t):=f^n(t,Y^0_t,Z^0_t)-f^0(t,Y^0_t,Z^0_t)$ to obtain that,

$$\begin{aligned}G^n_sds=&\zeta_\theta\Gamma^n_se^{A^n_s}\Big\{\Big[\Big[-\theta\Delta_nf(s)+\theta f^n(s,Y^0_s,Z^0_s)-f^n(s,Y^n_s,Z^n_s)\Big]-a^n_s\mathfrak{Y}^n_s\Big]ds-\frac{1}{2}\zeta_\theta e^{A^n_s}d\langle\mathfrak{Z}^n\cdot X+\mathfrak{M}^n_\cdot\rangle_s\Big\}\\\leqslant&\zeta_\theta\Gamma^n_se^{A^n_s}\Big\{\Big[\Big[|\Delta_nf(s)|+\theta f^n(s,Y^0_s,Z^0_s)-f^n(s,Y^0_s,Z^n_s)\\&+|f^n(s,Y^0_s,Z^n_s)-f^n(s,\theta Y^0_s,Z^n_s)|\Big]ds\Big]-\frac{1}{2}\zeta_\theta e^{A^n_s}d\langle\mathfrak{Z}^n\cdot X+\mathfrak{M}^n_\cdot\rangle_s\Big\},\end{aligned}\tag{3.113}$$

where we use the fact that '$-f^n(s,\theta Y^0_s,Z^n_s)+f^n(s,\theta Y^0_s,Z^n_s)\equiv0$' and the definition of a_s that

$f^n(s, Y_s^n, Z_s^n) + a_s^n \mathfrak{Y}_s^n = f^n(s, \theta Y_s^0, Z_s^n)$; besides, we could apply (3.112) with $y = Y_s^0$ to get

$$\theta f^n(s, Y_s^0, Z_s^0) - f^n(s, Y_s^0, Z_s^n) \leqslant \alpha + \beta |Y_s^0|$$

and use Lipschitz continuity((2) in Assumption 3.3.1) to obtain

$$|f^n(s, Y_s^0, Z_s^n) - f^n(s, \theta Y_s^0, Z_s^n)| \leqslant \kappa(1-\theta)|Y_s^0|;$$

to conclude the above calculus, we have that

$$\begin{aligned} G_s^n ds \leqslant & \zeta_\theta \Gamma_s^n e^{A_s^n} \bigg\{ \bigg[|\Delta_n f(s)| + (\alpha + \beta |Y_s^0|) + \frac{\gamma}{2(1-\theta)} \widehat{\sigma}_s^2 |\mathfrak{Z}_s^n|^2 \\ & + \kappa(1-\theta)|Y_s^0| \bigg] ds - \frac{1}{2} \zeta_\theta e^{A_s^n} d\langle \mathfrak{Z}^n \cdot X + \mathfrak{M}_\cdot^n \rangle_s \bigg\} \\ \leqslant & \bigg\{ \gamma e^{2\kappa T} \Gamma_s^n \Big[\alpha + (\beta + \kappa)|Y_s^0| \Big] + \zeta_\theta e^{\kappa T} \Gamma_s^n |\Delta_n f(s)| \bigg\} ds. \end{aligned} \tag{3.114}$$

As in (3.99), integration by parts gives that

$$\begin{aligned} \Gamma_t^n \leqslant & D_t \Gamma_t^n \leqslant D_T \Gamma_T^n + \zeta_\theta e^{\kappa T} \int_t^T D_s \Gamma_s^n |\Delta_n f(s)| ds + \zeta_\theta \int_t^T D_s \Gamma_s^n e^{A_s^n} dK_s^0 - \zeta_\theta \int_t^T D_s \Gamma_s^n e^{A_s^n} (\mathfrak{Z}_s^n \cdot dX_s + d\mathfrak{M}_s^n) \\ \leqslant & I_T^{n,1} + I_T^{n,2} + \zeta_\theta e^{\kappa T} D_T \int_0^T \Gamma_s^n dK_s^0 - \zeta_\theta \int_t^T D_s \Gamma_s^n e^{A_s^n} (\mathfrak{Z}_s^n \cdot dX_s + d\mathfrak{M}_s^n), t \in [0, T]. \end{aligned} \tag{3.115}$$

By the flat-off condition of (Y^0, Z^0, M^0, K^0), we have that

$$\begin{aligned} \int_0^T \Gamma_s^n dK_s^0 &= \int_0^T \mathbf{1}_{\{Y_s^0 = L_s^0\}} \Gamma_s^n dK_s^0 = \int_0^T \mathbf{1}_{\{Y_s^0 = L_s^0 \leqslant L_s^n + \epsilon\}} \Gamma_s^n dK_s^0 + \int_0^T \mathbf{1}_{\{Y_s^0 = L_s^0 > L_s^n + \epsilon\}} \Gamma_s^n dK_s^0 \\ &\leqslant \int_0^T \mathbf{1}_{\{Y_s^0 \leqslant Y_s^n + \epsilon\}} \exp\{\gamma e^{2\kappa T} |Y_s^n| + \epsilon \zeta_\theta e^{\kappa T}\} dK_s^0 + \Upsilon_n \int_0^T \mathbf{1}_{\{|L_s^n - L_s^0| > \epsilon\}} dK_s^0 \\ &\leqslant \exp\{\gamma e^{2\kappa T} Y_*^n + \epsilon \zeta_\theta e^{\kappa T}\} K_T^0 + \frac{1}{\epsilon} \Upsilon_n \Big(\sup_{t \in [0,T]} |L_t^n - L_t^0| \Big) K_T^0, \mathbb{P}\text{-a.s.}, \end{aligned} \tag{3.116}$$

where in the penultimate inequality with the indicative function $\mathbf{1}_{\{Y_s^0 = L_s^0 \leqslant L_s^n + \epsilon \leqslant Y_s^n + \epsilon\}}$, we use

$$\begin{aligned} \Gamma_s^n = \exp\{\zeta_\theta e^{A_s^n} \mathfrak{Y}_s^n\} &= \exp\left\{ \frac{\gamma e^{\kappa T}}{1-\theta} e^{A_s^n} (\theta Y_s^0 - Y_s^n) \right\} = \exp\left\{ \frac{\gamma e^{\kappa T} e^{A_s^n}}{1-\theta} \theta Y_s^0 - \frac{\gamma e^{\kappa T} e^{A_s^n}}{1-\theta} Y_s^n \right\} \\ &\leqslant \exp\left\{ \frac{\gamma e^{\kappa T} e^{A_s^n}}{1-\theta} \theta \epsilon + \frac{\gamma e^{\kappa T} e^{A_s^n}}{1-\theta} (\theta - 1) Y_s^n \right\} \quad (Y_s^0 \leqslant Y_s^n + \epsilon) \\ &\leqslant \exp\{\zeta_\theta e^{A_s^n \theta \epsilon} + \gamma e^{2\kappa T} |Y_s^n|\} \leqslant \exp\{\zeta_\theta e^{\kappa T \theta \epsilon} + \gamma e^{2\kappa T} |Y_s^n|\}. \end{aligned} \tag{3.117}$$

For each $p\in(1,\infty)$, (3.73) and (S3) lead to

$$\begin{aligned}\sup_{n'\in\mathbb{N}}\mathbb{E}\Big[e^{p\gamma Y^{n'}_*}+\langle\int_0^T Z^{n'}_\cdot\cdot dX_\cdot+dM^{n'}_\cdot\rangle^p_s+(K^{n'}_T)^p\Big]\leqslant& c_p\sup_{n'\in\mathbb{N}}\mathbb{E}[e^{3p\gamma e^{\beta T}(\xi^+_{n'}\vee L^{n'}_*)}]\\ \leqslant& c_p\Xi\Big(3p\gamma e^{\beta T}\Big).\end{aligned}\tag{3.118}$$

Then, it holds that

$$\begin{aligned}&\sup_{m\in\mathbb{N}\cup\{0\}}\mathbb{E}\Big[e^{p\gamma Y^m_*}+\langle\int_0^\cdot Z^m_s\cdot dX_s+dM^m_s\rangle^p_T+(K^m_T)^p\Big]\\ \leqslant& c_p\Xi(3p\gamma e^{\beta T})+\mathbb{E}\Big[e^{p\gamma Y^0_*}+\langle\int_0^\cdot Z^0_s\cdot dX_s+dM^0_s\rangle^p_T+(K^0_T)^p\Big]:=\widetilde{\Xi}(p),\end{aligned}\tag{3.119}$$

which combining with (S1) gives that

$$\mathbb{E}[\eta^p_n]\leqslant\mathbb{E}[e^{p\zeta_\theta e^{\kappa T}(|\xi_n|+|\xi_0|)}]\leqslant\frac{1}{2}\mathbb{E}\Big[e^{2p\zeta_\theta e^{\kappa T}(\xi^+_n\vee L^n_*)}+e^{2p\zeta_\theta e^{\kappa T}(\xi^+_0\vee L^0_*)}\Big]\leqslant\Xi\Big(2p\zeta_\theta e^{\kappa T}\Big),\tag{3.120}$$

$$\mathbb{E}\Upsilon^p_n\leqslant\frac{1}{2}\mathbb{E}\Big[e^{2p\zeta_\theta e^{\kappa T}Y^n_*}+e^{2p\zeta_\theta e^{\kappa T}Y^0_*}\Big]\leqslant\widetilde{\Xi}\left(\frac{2p}{1-\theta}e^{2\kappa T}\right),\tag{3.121}$$

$$\begin{aligned}&\mathbb{E}\left[\left(\int_0^T|\Delta_n f(s)|ds\right)^p\right]\leqslant\mathbb{E}\left[\left(2T(\alpha+\beta Y^0_*)+\gamma\int_0^T|\widehat{\sigma}_s Z^0_s|^2ds\right)^p\right]\\ \leqslant& c_p\mathbb{E}\left[e^{p\gamma Y^0_*}+\left(\int_0^T|\widehat{\sigma}_s Z^0_s|^2ds\right)^p\right],\quad(\text{ Note that } p\in(1,\infty))\end{aligned}\tag{3.122}$$

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|L^n_t-L^0_t|^p\Big]\leqslant c_p\mathbb{E}\Big[(L^n_*)^p+(L^0_*)^p\Big]\leqslant c_p\mathbb{E}\Big[e^{pL^n_*}+e^{pL^0_*}\Big]\leqslant c_p\Xi(p).\tag{3.123}$$

By definition of D_t given in this theorem , we have $D_T\leqslant c_0\exp\{\gamma e^{2\kappa T}(\beta+\kappa)Y^0_*\}$, and then we could also see here that $D_T\in\mathbb{L}^p(\mathbb{P},\mathbb{R})$. So we can get from Young's inequality and (3.119)-(3.123) that the r.v.s $I^{n,i}_T, i=1,2,3,4$ are all integrable. Besides, Burkholder-Davis-Gundy inequality, definition of Υ_n and Hölder's inequality imply that

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t D_s\Gamma^n_s e^{A^n_s}(\mathfrak{Z}^n_s\cdot dX_s+d\mathfrak{M}_s)\right|\right]\leqslant& c_0\mathbb{E}\left[\left(\int_0^T(D_s\Gamma^n_s)^2e^{2A^n_s}(|\widehat{\sigma}_s\mathfrak{Z}^n_s|^2ds+d\langle\mathfrak{M}_\cdot\rangle_s\right)^{\frac{1}{2}}\right]\\ \leqslant& c_0\mathbb{E}\left[D_T\Upsilon_n\left(\int_0^T|\widehat{\sigma}_s\mathfrak{Z}^n_s|^2ds+d\langle\mathfrak{M}^n_\cdot\rangle_s\right)^{\frac{1}{2}}\right]\\ \leqslant& c_0\|D_T\|_{\mathbb{L}^4(\mathbb{P},\mathbb{R})}\|\Upsilon_n\|_{\mathbb{L}^4(\mathbb{P},\mathbb{R})}\mathbb{E}\left[\left(\int_0^T|\widehat{\sigma}_s\mathfrak{Z}^n_s|^2ds+d\langle\mathfrak{M}^n_\cdot\rangle_s\right)^{\frac{1}{2}}\right]<\infty,\end{aligned}\tag{3.124}$$

i.e. $\int_0^\cdot D_s\Gamma_s^n e^{A_s^n}(\mathfrak{Z}_s^n \cdot dX_s + d\mathfrak{M}_s)$ is a uniformly integrable martingale.

For any $t \in [0,T]$, taking $\mathbb{E}[\cdot|\overline{\mathcal{F}}_t^{\mathbb{P}}]$ in (3.115) and (3.116) gives that $\Gamma_t^n \leqslant \sum_{i=1}^4 I_t^{n,i}$, $\mathbb{P}$-a.s., which leads to that

$$Y_t^0 - Y_t^n \leqslant (1-\theta)|Y_t^0| + \theta Y_t^0 - Y_t^n \leqslant (1-\theta)|Y_t^0| + \frac{1-\theta}{\gamma}\ln\left(\sum_{i=1}^4 I_t^{n,i}\right), \mathbb{P}\text{-a.s.} \tag{3.125}$$

The other half of (3.107) is similar to the above proof, so we omit here. □

Lemma 2.B. *For any $p \in [1,\infty)$, $\exp\{p\gamma \sup_{t\in[0,T]}|Y_t^n - Y_t^0|\}$ converges to 1 in probability.*

Proof. Given $\delta > 0$, we use (by order) (3.107) and twice Doob's martingale inequality to get that

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[0,T]}|Y_t^n - Y_t^0| \geqslant \delta\right)\\
\leqslant&\mathbb{P}\left((1-\theta)(Y_*^0 + Y_*^n) \geqslant \frac{\delta}{2}\right) + \mathbb{P}\left(\frac{1-\theta}{\gamma}\ln\left(\sum_{i=1}^4 I_*^{n,i}\right) \geqslant \frac{\delta}{2}\right)\\
\leqslant&2\frac{1-\theta}{\delta}\mathbb{E}[Y_*^0 + Y_*^n] + \sum_{i=1}^4 \mathbb{P}\left(I_*^{n,i} \geqslant \frac{1}{4}e^{\frac{\delta\gamma}{2(1-\theta)}}\right);
\end{aligned} \tag{3.126}$$

by an adjustment on the estimate for the last term above, i.e.

$$2\frac{1-\theta}{\delta}\mathbb{E}[Y_*^0 + Y_*^n] + \sum_{i=1}^4 \mathbb{P}\left(I_*^{n,i} \geqslant \frac{1}{4}e^{\frac{\delta\gamma}{2(1-\theta)}}\right) \leqslant \frac{1-\theta}{\delta\gamma}\mathbb{E}[e^{2\gamma Y_*^0} + e^{2\gamma Y_*^n}] + 4e^{\frac{-\delta\gamma}{2(1-\theta)}}\sum_{i=1}^4 \mathbb{E}[I_T^{n,i}],$$

we know that $\mathbb{E}[e^{2\gamma Y_*^0} + e^{2\gamma Y_*^n}] \leqslant 2\widetilde{\Xi}(2)$ and

$$\begin{aligned}
\sum_{i=1}^4 \mathbb{E}[I_T^{n,i}] \leqslant& e^{\kappa T}C\Bigg\{\|\eta_n\|_{\mathbb{L}^2(\mathbb{P},\mathbb{R})} + \zeta_\theta\left[\widetilde{\Xi}\left(\frac{8}{1-\theta}e^{2\kappa T}\right)\right]^{\frac{1}{4}}\\
&\times\left\|\int_0^T|\Delta_n f(s)|ds\right\|_{\mathbb{L}^4(\mathbb{P},\mathbb{R})} + 1 + \zeta_\theta e^{\epsilon\zeta_\theta e^{\kappa T}} + \frac{\zeta_\theta}{\epsilon}\left[\widetilde{\Xi}\left(\frac{8}{1-\theta}e^{2\kappa T}\right)\right]^{\frac{1}{4}}\|L^n - L^0\|_{\mathbb{C}^4(\mathbb{P})}\Bigg\},
\end{aligned}$$

where $C := 1 + \|D_T\|_{\mathbb{L}^2(\mathbb{P},\mathbb{R})} + \sup_{n\in\mathbb{N}}\left\{\mathbb{E}\left[D_T e^{\gamma e^{2\kappa T}(Y_*^0+Y_*^n)}(K_T^0 + K_T^n)\right] + \|D_T(K_T^0 + K_T^n)\|_{\mathbb{L}^2(\mathbb{P},\mathbb{R})}\right\}$ and note that in the last inequality we have applied (3.119), (3.121) and Hölder's inequality; then to conclude the above calculus, we obtain

$$\mathbb{P}\bigg(\sup_{t\in[0,T]}|Y^n_t-Y^0_t|\geqslant\delta\bigg)\leqslant 2\frac{1-\theta}{\delta\gamma}\widetilde{\Xi}(2)+4e^{\kappa T}e^{\frac{-\delta\gamma}{2(1-\theta)}}C\bigg\{\|\eta_n\|_{\mathbb{L}^2(\mathbb{P},\mathbb{R})}+\zeta_\theta\bigg[\widetilde{\Xi}\bigg(\frac{8}{1-\theta}e^{2\kappa T}\bigg)\bigg]^{\frac14}$$
$$\times\bigg\|\int_0^T|\Delta_n f(s)|ds\bigg\|_{\mathbb{L}^4(\mathbb{P},\mathbb{R})}+1+\zeta_\theta e^{\epsilon\zeta_\theta e^{\kappa T}}+\frac{\zeta_\theta}{\epsilon}\bigg[\widetilde{\Xi}\bigg(\frac{8}{1-\theta}e^{2\kappa T}\bigg)\bigg]^{\frac14}\|L^n-L^0\|_{\mathbb{C}^4(\mathbb{P})}\bigg\}. \tag{3.127}$$

Note that by Hölder's inequality and (3.119), we can show that C is a finite constant. The fact that $\Delta_n f$ converges to 0 and (S1) lead to that $dt\otimes d\mathbb{P}$-a.e.

$$\lim_{n\to\infty}\Delta_n f(t,\omega)=0 \text{ and } |\Delta_n f(t,\omega)|\leqslant 2\alpha+2\beta Y^0_*(\omega)+\gamma\widehat{\sigma}^2_t|Z^0_t(\omega)|^2,\forall n\in\mathbb{N}. \tag{3.128}$$

For $\mathbb{P}$-a.s. $\omega\in\Omega$, we assume that (3.128) holds for a.e. $t\in[0,T]$, and that $Y^0_*+\int_0^T\widehat{\sigma}^2_s|Z^0_s(\omega)|ds\leqslant<\infty$. Then dominated convergence theorem gives that $\lim_{n\to\infty}\int_0^T|\Delta_n f(s,\omega)|ds=0$. By (S2), we also have $\mathbb{P}$-a.s. that $\lim_{n\to\infty}\eta_n=e^{\gamma e^{2\kappa T}|\xi_0|}$ and $\lim_{n\to\infty}\Big(\sup_{t\in[0,T]}|L^n_t-L^0_t|\Big)=0$. Applying (3.120), (3.122) and (3.123) with any $p>4$, we have that $\{\eta_n^2\}_{n\in\mathbb{N}},\{(\int_0^T|\Delta_n f(s)|ds)^4\}_{n\in\mathbb{N}}$ and $\Big\{\sup_{t\in[0,T]}|L^n_t-L^0_t|^4\Big\}_{n\in\mathbb{N}}$ are all uniformly integrable sequences in $\mathbb{L}^1(\mathbb{P},\mathbb{R})$, which gives that $\lim_{n\to\infty}\mathbb{E}[\eta_n^2]=\mathbb{E}[e^{2\gamma e^{2\kappa T}|\xi_0|}]$ and $\lim_{n\to\infty}\mathbb{E}\Big[(\int_0^T|\Delta_n f(s)|ds)^4+\sup_{t\in[0,T]}|L^n_t-L^0_t|^4\Big]=0$. Letting $n\to\infty$ in (3.126) and next letting $\epsilon\to 0$ yields that

$$\limsup_{n\to\infty}\mathbb{P}\bigg(\sup_{t\in[0,T]}|Y^n_t-Y^0_t|\geqslant\delta\bigg)\leqslant 2\frac{1-\theta}{\delta\gamma}\widetilde{\Xi}(2)+4e^{\kappa T}e^{\frac{-\delta\gamma}{2(1-\theta)}}C\bigg(1+\|e^{\gamma e^{2\kappa T}|\xi_0|}\|_{\mathbb{L}^2(\mathbb{P},\mathbb{R})}+\frac{\gamma e^{\kappa T}}{1-\theta}\bigg). \tag{3.129}$$

As $\theta\to 1$, it follows that $\lim_{n\to\infty}\mathbb{P}\Big(\sup_{t\in[0,T]}|Y^n_t-Y^0_t|\geqslant\delta\Big)=0$, which gives that for any $p\in[1,\infty),\exp\{p\gamma\sup_{t\in[0,T]}|Y^n_t-Y^0_t|\}$ converges to 1 in probability. □

Lemma 2.C. $\lim_{n\to\infty}\mathbb{E}\Big[\langle\int_0^T(Z^n_s-Z^0_s)\cdot dX_s+d(M^n_s-M^0_s)\rangle^p\Big]=0.$

Proof. Fix $p\in[1,\infty)$. Since $\mathbb{E}\Big\{\exp\Big[2p\gamma\sup_{t\in[0,T]}|Y^n_t-Y^0_t|\Big]\Big\}\leqslant\frac12\mathbb{E}[e^{4p\gamma Y^n_*}+e^{4p\gamma Y^0_*}]\leqslant\widetilde{\Xi}(4p)$ holds for any $n\in\mathbb{N}$ by (3.119), it follows that $\Big\{\exp\Big[p\gamma\sup_{t\in[0,T]}|Y^n_t-Y^0_t|\Big]\Big\}_{n\in\mathbb{N}}$ is a uniformly integrable sequence in $\mathbb{L}^1(\mathbb{P},\mathbb{R})$, which gives that $\lim_{n\to\infty}\mathbb{E}\Big\{\Big\{\exp\Big[p\gamma\sup_{t\in[0,T]}|Y^n_t-Y^0_t|\Big]\Big\}\Big\}=1$. Particularly, we have that $\Big\{\sup_{t\in[0,T]}|Y^n_t-Y^0_t|^p\Big\}_{n\in\mathbb{N}}$ is a uniformly integrable sequence in $\mathbb{L}^1(\mathbb{P},\mathbb{R})$ and that

$$\lim_{n\to\infty}\mathbb{E}\bigg[\sup_{t\in[0,T]}|Y^n_t-Y^0_t|^p\bigg]=0. \tag{3.130}$$

Given $n \in \mathbb{N}$, using Itô's formula to the process $|Y_t^n - Y_t^0|^2$, we can obtain from (S1) that

$$\begin{aligned}
&\int_0^T |Z_s^n - Z_s^0|^2 \widehat{\sigma}_s^2 ds + d\langle M_\cdot^n - M_\cdot^0\rangle_s \\
\leqslant & |\xi_n - \xi_0|^2 - |Y_0^n - Y_0^0|^2 + 2\int_0^T (Y_s^n - Y_s^0)[(f^n(s, Y_s^n, Z_s^n) - f^0(s, Y_s^0, Z_s^0))] ds \\
& + 2\int_0^T (Y_s^n - Y_s^0)(dK_s^n - dK_s^0) - 2\int_0^T (Y_s^n - Y_s^0)[(Z_s^n - Z_s^0)\cdot dX_s + d(M_s^n - M_s^0)] \\
\leqslant & 2 \sup_{t\in[0,T]} |Y_s^n - Y_s^0| \Bigg(2\alpha T + \beta T (Y_*^n + Y_*^0) \\
& + \frac{\gamma}{2} \int_0^T \widehat{\sigma}_s^2 (|Z_s^n|^2 + |Z_s^0|^2) ds + d\langle M_\cdot^n\rangle_s + d\langle M_\cdot^0\rangle_s + K_T^n + K_T^0 \Bigg) \\
& + \sup_{t\in[0,T]} |Y_t^n - Y_t^0|^2 + 2\left| \int_0^T (Y_t^n - Y_t^0)[(Z_s^n - Z_s^0)\cdot dX_s + d(M_s^n - M_s^0)] \right|, \mathbb{P}\text{-a.s.}
\end{aligned} \tag{3.131}$$

Then we use (by order) Burkholder-Davis-Gundy inequality, Hölder's inequality, and (3.119) to give that

$$\begin{aligned}
&\mathbb{E}\Bigg[\Bigg(\int_0^T |Z_s^n - Z_s^0|^2 \widehat{\sigma}_s^2 ds + d\langle M_s^n - M_s^0\rangle_s \Bigg)^p \Bigg] \\
\leqslant & c_p \mathbb{E}\Bigg[\sup_{s\in[0,T]} |Y_s^n - Y_s^0|^{2p} \Bigg] + c_p \mathbb{E}\Bigg[\sup_{s\in[0,T]} |Y_s^n - Y_s^0|^p \cdot \langle \int_0^\cdot (Z_s^n - Z_s^0)\cdot dX_s + d(M_s^n - M_s^0)\rangle_T^{\frac{p}{2}} \Bigg] \\
& + c_p \Bigg\{ \mathbb{E}\Bigg[\sup_{s\in[0,T]} |Y_s^n - Y_s^0|^{2p} \Bigg] \Bigg\}^{\frac{1}{2}} \times \Bigg\{ \sup_{m\in\mathbb{N}\cup\{0\}} \mathbb{E}\Bigg[e^{2p\gamma Y_*^m} + \Bigg(\int_0^T |Z_s^m|^2 \widehat{\sigma}_s^2 ds + d\langle M_\cdot^m\rangle_s \Bigg)^{2p} + (K_T^m)^{2p} \Bigg] \Bigg\}^{\frac{1}{2}} \\
\leqslant & c_p \mathbb{E}\Bigg[\sup_{s\in[0,T]} |Y_s^n - Y_s^0|^{2p} \Bigg] + \frac{1}{2} \mathbb{E}\Bigg[\langle \int_0^\cdot (Z_s^n - Z_s^0)\cdot dX_s + d(M_s^n - M_s^0)\rangle_T^{\frac{p}{T}} \Bigg] \\
& + c_p \sqrt{\widetilde{\Xi}(2p)} \Bigg\{ \mathbb{E}\Bigg[\sup_{s\in[0,T]} |Y_s^n - Y_s^0|^{2p} \Bigg] \Bigg\}^{\frac{1}{2}}.
\end{aligned} \tag{3.132}$$

Recalling that $\mathbb{E}\Big[\Big(\int_0^T |Z_s^n - Z_s^0|^2 ds \Big)^{\frac{2p}{2}} \Big] < \infty$ as $Z^n, Z^0 \in \mathbb{H}^{2,2p}(\mathbb{P}, \mathbb{R}^d)$, then we have that

$$\begin{aligned}
\mathbb{E}\Bigg[\langle \int_0^T (Z_s^n - Z_s^0)\cdot dX_s + d(M_s^n - M_s^0)\rangle^p \Bigg] \leqslant & c_p \mathbb{E}\Bigg[\sup_{s\in[0,T]} |Y_s^n - Y_s^0|^{2p} \Bigg] \\
& + c_p \sqrt{\widetilde{\Xi}(2p)} \Bigg\{ \mathbb{E}\Bigg[\sup_{s\in[0,T]} |Y_s^n - Y_s^0|^{2p} \Bigg] \Bigg\}^{\frac{1}{2}}.
\end{aligned} \tag{3.133}$$

As $n \to \infty$, (3.130) gives that

$$\lim_{n\to\infty} \mathbb{E}\Bigg[\Bigg(\int_0^T (Z_s^n - Z_s^0)\cdot dX_s + d(M_s^n - M_s^0) \Bigg)^p \Bigg] = 0. \tag{3.134}$$

□

Lemma 2.D. *Theorem 3.3.13 is valid.*

Proof. Here we assume additionally that $dt \otimes d\mathbb{P}$-a.e., $f^n(t,\omega,y,z)$ converges to $f^0(t,\omega,y,z)$ locally uniformly in (y,z). By (3.130) and (3.134) with $p=1, \{(Y^n,Z^n,M^n)\}_{n\in\mathbb{N}}$ has a subsequence(we still denote it by $\{(Y^n,Z^n,M^n)\}_{n\in\mathbb{N}}$) s.t. $\lim_{n\to\infty}\sup_{t\in[0,T]}|Y_t^n-Y_t^0|=0,\mathbb{P}$-a.s. and $\lim_{n\to\infty}Z_t^n=Z_t^0$ as well as $\lim_{n\to\infty}M_t^n=M_t^0, dt\otimes d\mathbb{P}$-a.e. In fact, we can select this subsequence s.t. $\sup_{n\in\mathbb{N}}|Z^n|\in\mathbb{H}^2(\mathbb{P},\mathbb{R}^d)$(see e.g., [LM97] or [Kob00](Lemma 2.5)). Fix $p\in[1,\infty)$, by similar arguments to those leading to (3.63), it follows from (S1) and (3.119) that

$$\lim_{n\to\infty}\mathbb{E}\bigg[\bigg(\int_0^t\Big[f_s^0(Y_s^0,Z_s^0)-f_s^n(Y_s^n,Z_s^n)\Big]ds\bigg)^p\bigg]=0. \tag{3.135}$$

For any $n\in\mathbb{N}$, we have $\mathbb{P}$-a.s. that

$$\begin{aligned}K_t^n-K_t^0 =&Y_0^n-Y_0^0-(Y_t^n-Y_t^0)+\int_0^t\Big[f_s^0(Y_s^0,Z_s^0)-f_s^n(Y_s^n,Z_s^n)\Big]ds\\ &-\int_0^t(dM_s^0-dM_s^n)-\int_0^t(Z_s^0-Z_s^n)\cdot dX_s, t\in[0,T].\end{aligned} \tag{3.136}$$

Then Burkholder-Davis-Gundy inequality gives that

$$\begin{aligned}&\mathbb{E}\bigg[\sup_{t\in[0,T]}|K_t^n-K_t^0|^p\bigg]\\ \leqslant&c_p\mathbb{E}\bigg[\sup_{t\in[0,T]}|Y_t^n-Y_t^0|^p+\bigg(\int_0^t\Big[f_s^0(Y_s^0,Z_s^0)-f_s^n(Y_s^n,Z_s^n)\Big]ds\bigg)^p\\ &+\bigg(\int_0^T\hat{\sigma}_s^2|Z_s^0-Z_s^n|^2ds+d\langle M_\cdot^0-M_\cdot^n\rangle_s\bigg)^{\frac{p}{2}}\bigg],\end{aligned} \tag{3.137}$$

where Hölder's inequality shows that $\mathbb{E}\bigg[\bigg(\int_0^T\hat{\sigma}_s^2|Z_s^0-Z_s^n|^2ds+d\langle M_\cdot^0-M_\cdot^n\rangle_s\bigg)^{\frac{p}{2}}\bigg]\leqslant\bigg\{\mathbb{E}\bigg(\int_0^T\hat{\sigma}_s^2|Z_s^0-Z_s^n|^2ds+d\langle M_\cdot^0-M_\cdot^n\rangle_s\bigg)^p\bigg]\bigg\}^{\frac{1}{2}}$. Letting $n\to\infty$, (3.130)-(3.135) give that $\lim_{n\to\infty}\mathbb{E}\bigg[\sup_{t\in[0,T]}|K_t^n-K_t^0|^p\bigg]=0$. □

Theorem 3.3.14 (BSDE version)**.** *Let $\{(f^m,\xi_m)\}_{m\in\mathbb{N}\cup\{0\}}$ be a sequence of pairs s.t.*

(S1) For the same constants $\alpha,\beta,\kappa\geqslant0$ and $\gamma>0, f^0$ satisfies the Lipschitz condition in y (i.e., (2) in

Assumption 3.3.1) and $\{f^n\}_{n\in\mathbb{N}}$ *satisfies quadratic in* z*, Lipschitz in* y *(i.e., (3), (2) of Assumption 3.3.1) and concave in* z *(i.e., (4) in Assumption 3.3.1);*

(S2) It holds $\mathbb{P}$*-a.s. that* ξ_n *converges to* ξ_0*;*

(S3) $\Xi(p) := \sup_{m\in\mathbb{N}\cup\{0\}} \mathbb{E}[e^{p\xi_m^+}] < \infty$ *for all* $p \in (1,\infty)$*.*

Taking $(Y^0, Z^0, M^0, K^0) \in \cap_{p\in[1,\infty)}\mathbb{S}^p(\mathbb{P}) = \cap_{p\in[1,\infty)}\mathbb{C}^{exp(p)}(\mathbb{P}) \times \mathbb{H}^{2,2p}(\mathbb{P};\mathbb{R}^d) \times \mathbb{M}^p(\mathbb{P}) \times \mathbb{I}^p(\mathbb{P})$ *as a solution of the quadratic BSDE* (f^0, ξ_0)*, and for any* $n \in \mathbb{N}$*, we take* (Y^n, Z^n, M^n, K^n) *as the unique solution of the quadratic BSDE* $(f^n, \xi_n) \in \cap_{p\in[1,\infty)}\mathbb{S}^p(\mathbb{P})$*. If* $f^n_t(Y^0_t, Z^0_t)$ *converges* $dt \otimes d\mathbb{P}$*-a.e. to* $f^0_t(Y^0_t, Z^0_t)$*, then for any* $p \in [1,\infty)$*,* $\{\sup_{t\in[0,T]} |Y^n_t - Y^0_t|^p\}_{n\in\mathbb{N}}$ *is a uniformly integrable sequence in* $\mathbb{L}^1(\mathbb{P},\mathbb{R})$ *and*

$$\lim_{n\to\infty} \mathbb{E}[\sup_{t\in[0,T]} |Y^n_t - Y^0_t|^p] = \lim_{n\to\infty} \mathbb{E}\left[\left(\int_0^T |Z^n_s - Z_s|^2 \widehat{\sigma}^2_s ds + d\langle M^n_\cdot - M_\cdot\rangle_s\right)^p\right] = 0.$$

Moreover, if it holds $dt \otimes d\mathbb{P}$*-a.e. that* $f^n_t(\omega, y, z)$ *converges to* $f^0_t(\omega, y, z)$ *locally uniformly in* (y, z)*, then up to a subsequence, we have further* $\lim_{n\to\infty} \mathbb{E}[\sup_{t\in[0,T]} |K^n_t - K^0_t|^p] = 0, \forall p \in [1,\infty)$*.*

3.4 Random horizon backward SDEs

3.4.1 Survey

The theory of backward stochastic differential equations(BSDE, here after) in finite terminal time was introduced by Pardoux and Peng [PP90] with square integrable terminal condition ξ and uniformly Lipschitz continuity of the generator f in both y and z components as well as an integrability $\mathbb{E}[\int_0^T |f_t(0,0)|^2 dt] < \infty$ for some deterministic terminal horizon $T > 0$. Barles, Buckdahn and Pardoux [BBP97] also give a proof of this wellposedness based on a fixed point argument under the context of BSDE w.r.t. both Brownian motion and a Poisson random measure. Some structures other than Lipschitz continuity are explored as in Darling [Dar95] which verifies the wellposedness under local Lipschitz and convexity of its generator.

Peng [Pen91] studies the probabilistic interpretation for the systems of second order quasilinear parabolic partial differential equation by introducing certain type BSDE associated with some classical Itô forward

SDEs, where he also describes how the solution Y of BSDE with an unbounded random terminal time is related to the semilinear elliptic PDE and obtains the existence and uniqueness of this BSDE under quite strong assumption. Darling and Pardoux [DP97] firstly establish an existence and uniqueness result for a 'classical' BSDE(in the sense that the terminal condition is given at a fixed, i.e., non random, terminal time) with generator not necessarily Lipschitz w.r.t. both variables y and z; and also obtain the wellposedness for BSDE with random terminal time by using a weaker assumption compared with the one used in [Pen91]. Precisely, relying on both monotonicity in y with parameter $a \in \mathbb{R}$ and Lipschitz continuity in z with parameter $b \in \mathbb{R}_+$ for the generator $f_t(y,z)$, together with some integrability condition w.r.t. terminal condition ξ and $f_\cdot(0,0)$ as well as the random horizon τ, they give the wellposedness(existence and uniqueness, Proposition 3.2-3.3, Theorem 3.4) of such BSDEs with random horizon in some Hilbert space depending on both the a.s. finite stopping time τ and the particular number $b^2 - 2a$. And their integrability condition requires also some parameter bigger than the number $b^2 - 2a$. One technique frequently referred later on by other research is their respective treatment of the definition for Y and Z parts on two subintervals, namely $[t \wedge \tau, n \wedge \tau]$ and $[n \wedge \tau, \tau]$. Furthermore, Fuhrman and Tessitore [FT04] extend this result to the case of infinite stopping time with the modification of a and b to be $b^2 - 2a < 0$ and the boundedness of $f_t(0,0)$.

Removal of the dependence of this particular number $b^2 - 2a$ had attracted several corresponding research for a while. In order to explain the main idea for progress made in this direction, let's modify the definition RBSDE(3.9) to the context of random horizon as below,

$$\begin{aligned} Y_{t\wedge\tau} =& Y_{T\wedge\tau} + \int_{t\wedge\tau}^{T\wedge\tau} [f_s(Y_s,0) + b_s \cdot Z_s]ds - \int_{t\wedge\tau}^{T\wedge\tau} Z_s \cdot dW_s + \int_{t\wedge\tau}^{T\wedge\tau} dK_s \\ =& Y_{T\wedge\tau} + \int_{t\wedge\tau}^{T\wedge\tau} f_s(Y_s,0)ds - \int_{t\wedge\tau}^{T\wedge\tau} Z_s \cdot d\widetilde{W}_s + \int_{t\wedge\tau}^{T\wedge\tau} dK_s, \end{aligned} \tag{3.138}$$

where we define $\widetilde{W}_s := W_s - \int_0^s b_r dr$ and the process b as $b_s := \frac{f_s(Y_s,Z_s) - f_s(Y_s,0)}{|Z_s|^2} Z_s \mathbf{1}_{\{|Z_s|>0\}}$. If we can remove the dependence of the generator f w.r.t. z, this will help to avoid the above mentioned particular number $b^2 - 2a$. Indeed, if f is Lipschitz w.r.t. z, then the process b is bounded and $\mathcal{E}_t := \exp(\int_0^t b_s dW_s - \frac{1}{2}\int_0^t |b_s|^2 ds)$ is a uniformly integrable martingale in the interval $t \in [0,T]$. We denote the original probability as $\mathbb{P}$ under which W is a Wiener process, then we take $\mathcal{E}_T$ as a density of a new probability measure $\mathbb{Q}_T$ w.r.t. to the restriction $\mathbb{P}_T$ (which is the probability measure $\mathbb{P}$ restricted to $\mathcal{F}_T$). This new probability $\mathbb{Q}_T$ is equivalent to $\mathbb{P}_T$ and $\{\widetilde{W}_t\}_{0\leqslant t\leqslant T}$ is a Wiener process under $\mathbb{Q}_T$. Then we can study the equation(3.138) under the new probability measure $\mathbb{Q}_T$, for which there is no more dependence w.r.t z for its generator. This explains the main idea how some research used later on to get rid of the particular number $b^2 - 2a$, like Briand and Hu [BH98], Royer [Roy14], Hu and Tessitore [HT07], Lin Ren Touzi and Yang [LRTY20].

Precisely, for the case where the process Y takes value in the real set $\mathbb{R}$, we could refer to [BH98] and [Roy14] for related extension on generator without quadratic growth w.r.t. z. Both of them no longer need the dependence on the number $b^2 - 2a$ of the generator, instead they require the boundedness of $f_t(0,0)$(p.468 Assumption (A6.3) and p.478 Assumption (A7) in [BH98], p.283 Assumption (H4) and p.289 Assumption (H4') in [Roy14]) and strict positiveness of monotone coefficient $a \in \mathbb{R}_+$(no longer valide for $a \in \mathbb{R}_-$) as well as use the Girsanov transform to get the wellposedness for which Y is a continuous bounded process. The difference between these two is that [BH98] requires uniform Lipschitz continuity on y and z while [Roy14] requires only Lipschitz continuity on z and adds another growth condition on the y part(p.283 Assumption (H2) in [Roy14]). Meanwhile, [Roy14] also investigates the case where $a \equiv 0$.

Particularly, we would like to mention the most recent result [LRTY20], they require a similar integrability condition as [DP97] with a modification on the discount coefficient and consider an integrability classes under dominated nonlinear expectation $\mathcal{E}^{\mathbb{P}}$ instead of $\mathbb{E}^{\mathbb{P}}$. Their Example 3.5(page 6) illustrates this relevance of their assumption in a simple case where a linear generator is considered. For most works which generalize [DP97], like [BH98] and [Roy14] etc., it is always assumed that $a > 0$, i.e., the generator is strictly monotone, and the coefficients ξ, f_0 are bounded. And all these have been included as a special case of Assumption 3.2(page 5) in [LRTY20]. For $a = 0$, i.e. the generator f is monotone, [Roy14] provides the existence and uniqueness under assumptions that the generator f depending only on z is bounded and ξ is bounded as we mentioned above. This result was later generalized by [HT07], [BC08] and Papapantoleon et al. [PPS18] to a more general setting. And theorem 3.4(page 5) of [LRTY20] generalizes these previous results by allowing for $a \leqslant 0$, thanks to the new norms under which they set up the wellposedness result.

Precisely, [HT07] consider the same argument in the case of a cylindrical Wiener process with values in a Hilbert space, where they consider the mild solution of certain type of elliptic partial differential equation in some Hilbert space and the main technical point there is the discussion on differentiability of such bounded solution of the backward equation in some Markovian forward-backward system of equations w.r.t. the initial datum x of the forward equation.

For the quadratic case, Kobylanski [Kob00] gives the results for quadratic BSDEs with bounded or a.s. finite stopping time by means of a Hopf-Cole transformation and approximation of the generator f.

Later on, Briand and Confortola [BC08] extends to the infinite stopping time with a bounded terminal condition under two additional assumptions made on the generator $f_t(y,z)$: local Lipschitz on z and strictly monotone w.r.t. y. Precisely, they assume that for $\mathbb{P}$-a.s. and for any $t \geqslant 0$, there exist constants $C \geqslant 0$ and $\lambda > 0$, s.t.

- (BC08-A.1.(i)) for any $y \in \mathbb{R}$ and

$$z, z' \in \mathbb{R}^d, |f_t(y,z) - f_t(y,z')| \leqslant C(1 + |z| + |z'|)|z - z'|;$$

- (BC08-A.1.(ii)) f is strictly monotone w.r.t. y: for any $z \in \mathbb{R}^d$,

$$\forall y, y' \in \mathbb{R}, (y - y')(f_t(y,z) - f_t(y',z)) \leqslant -\lambda |y - y'|^2.$$

Denote $L^2_{\text{loc}}(\mathbb{R}^d)$ as the space of equivalence classes of progressively measurable processes $\psi : [0,\infty) \times \Omega \to \mathbb{R}^d$ s.t. for any $t > 0, \mathbb{E}[\int_0^t |\psi_r|^2 dr] < \infty$ and $M^{2,\epsilon}(\mathbb{R}^d)$ the set of $\{\mathcal{F}_t\}_{t\geqslant 0}$(filtration generated by standard Brownian motion)-progressively measurable processes $\{\psi_t\}_{t\geqslant 0}$ with values in $\mathbb{R}^d$ s.t. $\mathbb{E}[\int_0^{+\infty} e^{-2\epsilon s}|\psi_s|^2 ds] < \infty$, then with a bounded terminal condition ξ and quadratic growth of $f_t(y,z)$ w.r.t. z, they give the existence and uniqueness of its solution, for which Y is a bounded process and Z belongs to $L^2_{\text{loc}}(\mathbb{R}^d)$ moreover $M^{2,\epsilon}(\mathbb{R}^d)$ for all $\epsilon > 0$. The main idea of the proof: for any $n \in \mathbb{N}$, define

(Y^n, Z^n) as the unique solution to the BSDE

$$Y_t^n = \xi \mathbf{1}_{\{\tau \leqslant n\}} + \int_t^n \mathbf{1}_{\{s \leqslant \tau\}} f_s(Y_s^n, Z_s^n) ds - \int_t^n Z_s^n dW_s, 0 \leqslant t \leqslant n. \tag{3.139}$$

Under the above Assumption (BC08-A.1.(i)) and (BC08-A.1.(ii)), the BSDE(3.139) has a unique bounded solution according to [Kob00], moreover $Y_t^n = Y_{t\wedge\tau}^n, Z_t^n \mathbf{1}_{t>\tau} = 0$ by [Roy14]. Furthermore, it's necessary to extend this solution to the whole time axis by setting for any $t > n, Y_t^n = Y_n^n = \xi \mathbf{1}_{\tau \leqslant n}, Z_t^n = 0.$ Then the proof follows a routine procedure with necessary technique modification: firstly to prove that Y^n is bounded by a constant independent of n; then to verify that the convergence of the sequence $(Y^n)_{n\geqslant 0}$ and it converges to Y in $M^{2,\epsilon}(\mathbb{R})$ for all $\epsilon > 0$; next to study that the sequence $(Z^n)_{n\geqslant 0}$ is a Cauchy sequence in $M^{2,\epsilon}(\mathbb{R}^d)$, for all $\epsilon > 0$; and finally to show that the process (Y, Z) satisfies the BSDE

$$-dY_t = \mathbf{1}_{\{t \leqslant \tau\}} f_t(Y_t, Z_t) dt - Z_t dW_t, Y_\tau = \xi \text{ on } \{\tau < \infty\}. \tag{3.140}$$

[BC08] looks for a probabilistic representation for the solution of some elliptic PDE, for which this kind of Feynman-Kac formula involves Markovian BSDEs with infinite horizon. Similar to [HT07], their main technical point is to prove the differentiability of the bounded solution of corresponding BSDE with infinite horizon w.r.t. the initial datum x of some forward equation. Generally the proof is based on an a priori bound for suitable approximations of equations for the gradient of Y w.r.t. x and to this end they also require certain dissipative structure and the diffusion parameter in corresponding forward equation valued in $\mathbb{R}$. The difference between [BC08] and [HT07]: [HT07] use this strategy for the same elliptic PDE with the generator having sublinear growth w.r.t. the gradient; [BC08] require only the monotonicity constant of f to be positive.

3.4.2 Future research

One open question is the wellposedness of quadratic RBSDE with random terminal time(not necessary to be bounded or finite in certain cases) and exponential integrable terminal condition(not necessary to be bounded). Based on the wellposedness of quadratic BSDEs on determinal terminal horizon in [BH06] and Briand and Hu [BH08], we know that this type of equation calls for exclusive technique, exponential transformation on the original BSDE/RBSDE, other than those we mentioned in above. Besides, we should also adapt properly the original BSDE to the required solution spaces in advance with certain discount coefficient concerning the random horizon before we do the exponential transformation. According to our trial, this will unfortunately make the tools used in [BH06], [BH08] and [BY12], like the wellposedness of RODE, no longer valid for the new discussion. And it often calls for the study of a new random PDE.

For example, we take the a priori estimates and comparison as an example to illustrate this particularity.

About the a priori estimates: two classical steps are concerned in [BY12]. **1.** First of all, quadratic structure of $f_t(y,z)$ are needed to get a 'bound generator' H which possesses better properties(e.g. monotonicity) and try to solve some corresponding random ODE which takes this H as its generator. The solution of this random ODE should a nice bound due to its explicit expression and the behaviour of H. **2.** Then, the usage of Gronwall's inequality is a routine procedure here to construct the relation between the solution w.r.t. generator $f_t(y,z)$ required in the a priori estimate theorem and the solution w.r.t. the 'bound generator' H, and next the nice bound obtained for the latter one in step **1** could be transferred to apply on the former case(the solution w.r.t. generator $f_t(y,z)$), which gives the a priori estimate in need. However under the context of quandratic RBSDE with random terminal time and unbounded terminal condition, RODE and Gronwall's inequality are no longer applicable.

About the comparison principle, instead of estimating the difference of two solutions, e.g., Y and Y', here the classical treatment is the '$1-\theta$' method to construct the term $Y-\theta Y'$ for any $\theta \in (0,1)$ for which the convexity of the generator plays an important role. To this end, the construction of this variable $\zeta_\theta := \frac{\gamma e^{\kappa T}}{1-\theta}$, concerning the deterministic terminal horizon $T>0$ and Lipschitz coefficient κ, brings great convenience for related discussion like Itô's formula and estimates. However, this procedure becomes seemingly difficult to handle when the terminal time is random τ.

3.5 Backward SDEs with jumps

We focus on the backward SDE with jumps(BSDEJ here after). A concise review of the former research: under the setting of exponential integrable terminal condition ξ and quadratic generator g, El Karoui, Matoussi and Ngoupeyou [KMN18] have proved the existence of solution for BSDEJ driven by a random measure(the first part of p.23 Theorem 5.6); furthermore Jeanblanc, Matoussi and Ngoupeyou [JMN16] have verified the uniqueness for some special generator which has an explicit form, and their BSDEJ is driven by a continuous local martingale(e.g., Brownian motion) and Poisson process(p.7 Proposition 2). Note that their uniqueness result benefits from the special explicit generator which doesn't cover the general case. Based on this, one future research could be a downcrossing inequality for some g-supermartingale based on this well-defined speacial BSDEJs considered in [JMN16]. This is important to study the path regularity of g-supermartingales which will play an important role in the wellposedness of corresponding second-order backward SDEs with jumps.

Concerning the up-(resp., down-) crossing inequality for g-sub(resp., super-) martingale while terminal condition ξ satisfies certain type of integrability and generator g is Lipschitz continuous in (y, z) or quadratic in z for the related BSDE, a summation of former results is as following:

1. without jumps: there exists generally three previous results for $g(y, z)$. The first one is the case of square integrable ξ and Lipschitz continuous f in Chen and Peng [CP00](p.173 theorem 6) where they use the explicit form of y-part of the solution for related BSDE to construct a supermartingale in order to perform classical downcrossing inequality. The second one is the case of bounded ξ and quadratic f Ma and Yao [MY10](p.731 theorem 5.5), which use that $\int_0^{\cdot} Z_s \cdot dW_s$ is a BMO martingale and f is quadratic combined with Y is in $\mathbb{L}^\infty(\mathbb{P}, \mathbb{R})$ for some probability $\mathbb{P}$. And the third one is in Sheng [S.22b] where they don't have Lipschitz continuity jointly w.r.t. (y, z), and there is no more BMO martingale there due to the exponential integrability ξ in their setting.
2. with jumps: we here mention two results for $g(y, z, u)$. One is that Royer [Roy06] study the properties(including regularity) of related non-linear expectation where $g(y, z, u)$ is Lipschitz w.r.t. z, u,

continuous et monotonic w.r.t. y and ξ is square integrable. Particularly, they mentioned the Girsanov's theorem w.r.t. some local martingale whose continuous part corresponds to Brownian motion and discrete part relates to compensated Poisson random measure. This technique has also already been used in Xia [Xia00](Theorem 4.1) without pointed out obiviously. And this fact could easily be verified by using Corollary 12.15 and 12.16 in He, Wang and Yan [HWY92]. The other is that Kazi-Tani, Possamaï and Zhou[KTPZ16] extend the BSDE's case in [MY10] to the corresponding quadratic BSDEJ's case by means of BMO martingale.

Chapter 4

The Sannikov optimal contracting problem under defaultable output process

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After a first glimpse of Agent problem and Sannikov Principal problem originally studied by Possamaï and Touzi [PT20] and initially introduced by Sannikov [San08], we introduce a new scheme of delegation in a contract, i.e., the Principal problem under defaultable output process for which there exists an additional restriction on the termination time. Dynamic programming equations are given for both cases as well as their equivalent forms, and a comparison result is proved under the setting of a defaultable output process. Besides, we propose a Galerkin neural network numerical approximation for understanding the value function in Sannikov Principal problem with/without defaultable process.

4.1 Introduction

Holmström and Milgrom [HM87] reveal the linear form of an optimal contract in a finite horizon problem together with CARA utility functions for both parts(principal and agent), where only the drift part of the output process is influenced by the agent's effort. Generally we consider this paper as the first seminal result to highlight that optimal contracting problems seem to be easier to tackle with in the continuous–time setting.

This point-view has been followed and verified by a large amount of continuous–time literature. Precisely, the later extension of Holmström and Milgrom's result are Schättler and Sung [SS93], Sung [Sun95b, Sun95a], Müller [M9̈8, M0̈0], and Hellwig and Schmidt [Hel07, HS02], etc.. Compared with these papers sharing the common ground where all of them consider the continuous–time extensions of first–order approach arised from the contract theory literature in static cases(see for example Rogerson [Rog85]), Williams [Wil08, Wil11, Wil15] and Cvitanić, Wan, and Zhang [CWZ06, CWZ08, CWZ09] instead turn to utilize the stochastic maximum principle and forward–backward stochastic differential equations to characterise the optimal compensation for more general utility functions, see also the excellent monograph by Cvitanić and Zhang [CZ12].

Sannikov's work [San08, San13] show the innovation from different perspectives: one is the original idea to consider the dynamic continuation value of the agent as a state variable for the principal's problem; the other is considerable economic implications revealed by the infinite horizon setting in his research. For the first case, we should notice that an elegant solution approach by means of a representation result of the dynamic value function has been provided by its systematic implementation in continuous–time. Meanwhile, in the discrete–time case, Spear and Srivastava [SS87] show that this idea is already underlying in the setting of the problem. As for the second point, the main arguments of Sannikov's work are that the principal optimally retires the agent, offering him a Golden Parachute, that is to say a lifetime constant continuous stream of consumption, when his continuation utility reaches a sufficiently high level, and that an agent with small reservation utility possesses an informational rent, in the sense that he is offered a contract with strictly higher value.

Our discussion in this paper is fixed in the case where the principal and the agent share the same discount rates.

In addition to the methodological novelty introduced in Sannikov's article, his model highlights several remarkable economic results.

Firstly, his model justifies the optimality of a fixed annuity in the form of a golden parachute when the agent reaches a high enough utility continuation value. In other words, if the agent reaches a high enough hit, the principal prefers to terminate his contract and separate from it subject to the payment of compensation at the time of stopping the contract. A first motivation for this chapter is to examine whether this result continues to be true if the risk of bankruptcy of the production is taken into account by imposing an end-of-contract clause in the event of bankruptcy of the production.

The second remarkable economic result in Sannikov's work concerns the existence of an information rent phenomenon. This is a utility gain that the agent might realize due to the information asymmetry inherent in the problem of delegation: the principal cannot observe the effort of the agent and has access only to the realization of the value of the production. This information asymmetry is the origin of the moral hazard modeled in the delegation problem studied by Sannikov.

In Sannikov's optimal contract problem, the principal seeks the best contract by maximizing its utility, given the optimal response of the agent to the proposed contract, and under the acceptability constraint of the agent who claims a higher utility value at a reserve level. Below this level, the agent refuses to enter the game of delegation. The informational rent is precisely a situation where the optimal contract offered by the principal confers a utility upon the agent strictly higher than its level reserve. In other words, thanks to his informational advantage, the agent forces the principal to offer him an optimal contract which provides him with a utility strictly superior to his threshold of acceptability.

In Sannikov's model, the phenomenon of informational rent is less evident for fairly small acceptability threshold values. If the agent acceptability threshold is below a given value (y^*, the one that achieves the maximum utility of the principal), it is optimal for the principal to offer a contract that induces value utility for the agent to be equal to y^*.

In the model studied in this chapter where the end of the contract is conditioned by the bankruptcy of the production, our numerical results reveal that this rent information is greatly diminished. More precisely, if the value of the production is largely removed from the risk of bankruptcy, we find that the informational rent exists in the image of Sannikov's results. However, the more the value of the production decreases, the more this informational rent decreases, until it disappears when the production is very close to the level of bankruptcy.

As for numerical methods, we state some basic background for the classical treatment on PDEs(semilinear and fully nonlinear). For the first case-semilinear PDEs, the main challenge comes from the curse of dimensionality which makes the standard discretion impracticable when the dimension is greater than 3. In order to solve this problem, there are probabilistic mesh-free methods based on backward SDE, branching method, multilevel Picard methods. For the second case-fully nonlinear PDEs, normally there exists a nonlinearity part containing the second derivative term. Several classical methods, e.g., deterministic methods by finite difference method or finite element method, max-plus methods for HJB equations, are only developped for lower dimensions. We refer the readers to Germain, Pham and Warin [GPW20] for detailed description of related references.

Machine learning methods have developed quickly over the past few years. Among them, Beck, E and Jentzen [BEJ19] have proposed a global deep neural networks aiming at solving the high-dimensional nonlinear PDEs with the minimisation of an objective loss function based on the second order backward SDEs representation, they didn't give any concrete test on fully nonlinear example. Later on, [GPW20] focus on providing an efficient approximation of the Hessian, i.e. Γ-term of backward SDEs, to give 3 new algorithms, namely, second order explicit multi-step DBDP, second order multi-step DBDP and second order multi-step Malliavin DBDP.

Different to these machine learning trials on semilinear and fully nonlinear PDEs, we here will discuss a totally new Galerkin network numerical approximation to solve our ODE/PDE in interest.

4.2 Sannikov's contracting problem

The contracting problem consists of a delegation scheme between two parties, the Agent in charge of the management of the output process, and the Principal who sets up the terms of a contract indexed on the performance of the output process so as to incite the Agent to best serve her objective. The contract requires the Agent's approval at the initial time, and is then subject to the full commitment of both parties.

The contract stipulates the termination time of the contract, a continuous payment until termination, and a lump sum payment at termination. Our main objective in this paper is to explore the effect of restricting the termination time of the contract to occur before the default of the contract.

4.2.1 Output process under agent's effort

Let W^0 be a scalar Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P}^0)$. The formulation of the Sannikov optimal contracting problem starts from the output process X defined by

$$X_t := X_0 + \sigma W_t^0, \ t \geqslant 0,$$

where the initial data $X_0 \geqslant 0$ and the constant volatility coefficient $\sigma > 0$ are given. The information structure is defined by $\mathbb{F} = \{\mathcal{F}_t, t \geqslant 0\}$, the $\mathbb{P}^0$-augmentation of the natural filtration of X.

The Agent devotes an effort which affects the distribution of the output process. This is modelled by introducing the equivalent probability measure $\mathbb{P}^\alpha$ defined by the Radon-Nikodym density $\frac{d\mathbb{P}^\alpha}{d\mathbb{P}^0} = \exp\left(\int_0^T \frac{\alpha_s}{\sigma} dW_s^0 - \frac{1}{2}(\frac{\alpha_s}{\sigma})^2 ds\right)$ on $\mathcal{F}_t$, for all $t \geqslant 0$. Here the effort process α is $\mathbb{F}-$ progressively measurable with values in a compact subset A of $[0, \infty)$.

We shall denote by $\mathcal{A}$ the collection of all such effort processes. For all $\alpha \in \mathcal{A}$, it follows from the Girsanov theorem that the process $W_t^\alpha := W_t^0 - \int_0^t \frac{\alpha_s}{\sigma} ds$ is a $\mathbb{P}^\alpha-$ Brownian motion. We may then write equivalently the output process dynamics in terms of the $\mathbb{P}^\alpha-$ Brownian motion W^α as

$$dX_t = \alpha_t dt + \sigma dW_t^\alpha, \quad t \geqslant 0,$$

which highlights the effect of the effort process α on the $\mathbb{P}^\alpha-$ distribution of the output process.

4.2.2 The Agent's problem

The agent preferences are defined by a fixed discount rate $r > 0$, a utility function

$$u(x) := \gamma x^{\frac{1}{\gamma}}, \quad \text{for all} \quad x \geqslant 0,$$

for some parameter $\gamma > 1$, and a function $h : [0, \infty) \longrightarrow [0, \infty)$ representing the agent's cost of effort. We assume h to be increasing, strictly convex, and continuously differentiable, with $h(0) = 0$. In our numerical experiments, we shall take

$$h(x) := \beta x + cx^2, \; x \geqslant 0, \text{ for some constants } \beta, c > 0.$$

The contract proposed by the Principal at time zero consists of a triple $\mathbf{C} := (\tau, \xi, \pi)$, where τ is an $\mathbb{F}-$stopping time indicating the termination time, $\pi = \{\pi_t, t \geqslant 0\}$ is a non-negative scalar process representing the continuous payment until termination, and ξ is a non-negative $\mathcal{F}_\tau-$ measurable r.v. representing the payment at termination, or the Golden Parachute in the terminology of Sannikov. Such a contract $\mathbf{C}$ is admissible if it satisfies in addition the following integrability conditions:

$$\lim_{n\to\infty} \sup_{\alpha\in\mathcal{A}} \mathbb{P}^\alpha[\tau \geqslant n] = 0, \text{ and } \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{\mathbb{P}^\alpha}\Big[\Big(\mathrm{e}^{-r'\tau}|\xi|\Big)^\gamma + \int_0^\tau \Big(\mathrm{e}^{-r's}|\pi_s|\Big)^\gamma ds\Big] < \infty, \tag{4.1}$$

for some $r' \in (0, r \wedge 1)$. We denote by $\mathfrak{C}$ the collection of all admissible contracts.

Given a contract $\mathbf{C} := (\tau, \pi, \xi) \in \mathfrak{C}$, we denote

$$\zeta := u(\xi), \text{ and } \eta_t := u(\pi_t), \; t \geqslant 0,$$

and we introduce the Agent's problem as:

$$U^{\mathrm{A}}(\mathbf{C}) := \sup_{\alpha\in\mathcal{A}} J^{\mathrm{A}}(\mathbf{C}, \alpha), \text{ where } J^{\mathrm{A}}(\mathbf{C}, \alpha) := \mathbb{E}^{\mathbb{P}^\alpha}\Big[\mathrm{e}^{-r\tau}\zeta + \int_0^\tau r\mathrm{e}^{-rs}\Big(\eta_t - h(\alpha_t)\Big)dt\Big], \tag{4.2}$$

Notice that the last expectation is well-defined due to the non-negativity of the utility function u, and the boundedness of A.

Moreover, as the agent is allowed to choose zero effort $\alpha = 0$, and $\zeta = u(\xi) \geqslant 0$, $\eta = u(\pi) \geqslant 0$, we have

$$U^{\mathrm{A}}(\mathbf{C}) \geqslant J^{\mathrm{A}}(\mathbf{C}, 0) \geqslant 0. \tag{4.3}$$

We finally introduce the (possibly empty) set of agent's optimal responses

$$\mathcal{A}^\star(\mathbf{C}) := \Big\{\alpha \in \mathcal{A} : U^{\mathrm{A}}(\mathbf{C}) = J^{\mathrm{A}}(\mathbf{C}, \alpha)\Big\}.$$

4.2.3 The Sannikov Principal's problem

The principal has the same discount factor $r > 0$ as the agent, and aims at optimizing the following criterion:

$$J^{\mathrm{P}}(\mathbf{C},\alpha) \;:=\; \mathbb{E}^{\mathbb{P}^\alpha}\Big[-\mathrm{e}^{-r\tau}\xi + \int_0^\tau r\mathrm{e}^{-\rho s}(dX_s - \pi_s ds)\Big],$$

where the last expectation is well-defined in $\{-\infty\} \cup \mathbb{R}$, due to the boundedness of A and the non-negativity of ξ and π. Introducing the function

$$F(y) := -u^{-1}(y) = -\Big(\frac{y}{\gamma}\Big)^\gamma, \quad y \geqslant 0,$$

we rewrite the last problem as

$$J^{\mathrm{P}}(\mathbf{C},\alpha) \;=\; \mathbb{E}^{\mathbb{P}^\alpha}\Big[\mathrm{e}^{-\rho\tau}F(\zeta) + \int_0^\tau \rho\mathrm{e}^{-\rho s}(\alpha_t + F(\eta_t)dt\Big].$$

The Principal proposes a contract $\mathbf{C}$, the Agent considers the contract provided that it fulfills his participation constraint which states that it induces a utility value larger than $u(R)$, for some given $R > 0$. Then the agent is only willing to consider the contracts in the set

$$\mathfrak{C}^R := \Big\{\mathbf{C} \in \mathfrak{C} : U^{\mathrm{A}}(\mathbf{C}) \geqslant u(R)\Big\}.$$

The Principal anticipates the agent's optimal response, and chooses the contract which best serves her objective under the Agent's participation constraint

$$U^{\mathrm{P}} := \sup_{\mathbf{C}\in\mathfrak{C}^R}\; \sup_{\alpha\in\mathcal{A}^\star(\mathbf{C})} J^{\mathrm{P}}(\mathbf{C},\alpha). \tag{4.4}$$

4.2.4 The Principal problem under defaultable output process

Our main objective in this paper is to introduce the additional restriction that the termination time τ can not exceed the default time of the output process

$$\mathrm{T}_0 := \inf\Big\{t \geqslant 0 : X_t = 0\Big\}.$$

We then introduce the restricted set of contracts

$$\mathfrak{C}_0 := \Big\{ \mathbf{C} = (\tau, \pi, \xi) \in \mathfrak{C} : \tau \leqslant \mathrm{T}_0 \Big\} \text{ and } \mathfrak{C}_0^R := \Big\{ \mathbf{C} \in \mathfrak{C}_0 : U^A(\mathbf{C}) \geqslant u(R) \Big\}.$$

By restricting the Principal-proposed contracts to $\mathfrak{C}_0^R$, we introduce the Principal problem under defaultable output process:

$$V^{\mathrm{P}} := \sup_{\mathbf{C} \in \mathfrak{C}_0^R} \sup_{\alpha \in \mathcal{A}^\star(\mathbf{C})} J^{\mathrm{P}}(\mathbf{C}, \alpha). \tag{4.5}$$

This formulation seems to be more reasonable from the economic viewpoint. Indeed, in the Sannikov formulation, the Principal extracts the total effort devoted by the Agent, and this has no consequence on the state of the output process. This is largely subject to criticism as the Principal can still benefit from the project even if its value drops to deep non-positive values, as long as the Agent is willing to devote such an effort. In our formulation of V^{P} in (4.5), the Principal can extract the Agent's effort as long as the project is solvent, where we assume that the project is bankrupt when the value of its output hits the origin.

4.3 Dynamic programming equations

4.3.1 Agent's continuation utility and optimal response

In this section, we recall the dynamic programming representation of the Agent's value function which will serve for both problem U^{P} and V^{P}. Notice that the Hamiltonian of the Agent's problem is given by the convex conjugate of the cost of effort function

$$h^\star(z) := \sup_{a \in A} \{za - h(a)\}, \; z \in \mathbb{R}. \tag{4.6}$$

The sub-gradient of this convex function

$$\hat{A}(z) := \partial h^\star(z) = \{a \in A : h^\star(z) = za - h(a)\}$$

contains all possible maximizers of $h^\star(z)$, and will later serve as a parameterization of the Agent's optimal responses. As we claimed before, we will follow the example given in [San08] and [PT20], i.e., consider in our later numerical analysis the case where $h(a) := \frac{1}{2}ha^2 + \beta a, a \in A \subseteq [0, \infty)$, for some positive constants h and β. Therefore, $\hat{a} = \left(\frac{z-\beta}{h}\right)^+ \in \hat{A}(z)$, and $h^* = \frac{(z-\beta)^2}{2h}$.

Then, the lump-sum payment $\xi = -F(\zeta)$ promised by the principal at the termination time τ can be represented as

$$\zeta = Y_\tau^{Y_0,Z,\eta}, \quad Y_t^{Y_0,Z,\eta} = Y_0 + r\int_0^t \Big[Z_s \mathrm{d}X_s + \big(Y_s^{Y_0,Z,\pi} - h^\star(Z_s) - \eta_s\big)ds\Big], \tag{4.7}$$

where $Y^{Y_0,Z,\pi}$ represents the continuation utility of the Agent given a continuous consumption stream $\pi = -F(\eta)$ and Z satisfies the integrability condition

$$\sup_{\alpha\in\mathcal{A}} \mathbb{E}^{\mathbb{P}^\alpha}\left[\sup_{0\leqslant t\leqslant\tau} \left(\mathrm{e}^{-r't}|Y_t|\right)^p\right] < \infty, \text{ and } \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{\mathbb{P}^\alpha}\left[\left(\int_0^\tau (\mathrm{e}^{-r't}|Z_t|)^2 \mathrm{d}t\right)^{\frac{p}{2}}\right] < \infty. \tag{4.8}$$

Denote by $\mathcal{Z}$ the collection of all such processes. This representation plays a crucial role for the sequel, due to the fact that

$$\text{for } \mathbf{C} = (\tau, Y_\tau^{Y_0,Z,\pi}, \pi) \in \mathfrak{C}, \text{ we have}$$
$$Y_0 = U^A(\mathbf{C}) \text{ and } \mathcal{A}^\star\big(\mathbf{C}\big) = \Big\{\alpha\ \mathbb{F} - \text{prog. meas. with } \alpha \in \widehat{A}(Z),\ dt \otimes d\mathbb{P}^0 - \text{a.e.}\Big\}.$$

Moreover, for any admissible contract $(\tau, \xi, \pi) \in \mathfrak{C}$, we may find $Y_0 \in \mathbb{R}$ such that $\xi = Y_T^{Y_0,Z,\pi}$ for some pair (Y, Z) satisfying (4.8).

Consequently, we may reduce the Principal's objective to the following standard stochastic control problem

$$U^{\mathrm{P}} = \sup_{Y_0 \geqslant u(R)} U(Y_0), \text{ where } U(Y_0) := \sup_{\substack{(\tau,Z,\pi) \\ \hat{a}\in\hat{A}(Z)}} J(\tau, \pi, Z, \hat{a}), \tag{4.9}$$

and

$$J(\tau, \pi, Z, \hat{a}) := \mathbb{E}^{\mathbb{P}^{\hat{a}}}\Big[\mathrm{e}^{-r\tau}F\Big(Y_\tau^{Y_0,Z,\pi}\Big) + \int_0^\tau r\mathrm{e}^{-rt}\Big(\hat{a}_t + F(\eta_t)\Big)dt\Big]. \tag{4.10}$$

where the maximization is over $Z \in \mathcal{Z}$ and (τ, π) satisfying (4.1), together with the limited liability condition $Y^{Y_0,Z,\pi} \geqslant 0$ inherited from the dynamic version of (4.3).

4.3.2 Dynamic programming equation of the Sannikov problem

The reduction of the Sannikov optimal contracting problem to the control problem (4.9) opens the door for the standard dynamic programming approach. Notice that $Y^{Y_0,Z,\pi}$ is the only state variable here, and is subject to the state constraint $Y^{Y_0,Z,\pi} \geqslant 0$ together with the following dynamics under the Agent's optimal response:

$$dY_t^{Y_0,Z,\pi} = r\Big(Y_t^{Y_0,Z,\pi} + h(\hat{a}_t) - \eta_t\Big)\mathrm{d}t + rZ_t\sigma dW_t^{\hat{a}},\ \mathbb{P}^{\hat{a}}\text{–a.s., for all } \hat{a} \in \hat{A}(Z). \tag{4.11}$$

Given the reduced formulation of the principal problem in (4.9)-(4.10), the nonlinear differential operator corresponding to the stochastic control part of the problem is given by:

$$\mathbf{L}^0 v \ := \ v - yv' - \sup_{\eta\geqslant 0, z\in\mathbb{R}, \hat{a}\in\hat{A}(z)} \Big\{F(\eta) - \eta v' + \hat{a} + h(\hat{a})v' + \frac{1}{2}r\sigma^2 z^2 v''\Big\}.$$

Then, introducing the dual function

$$F^\star(p) := \inf_{\eta\geqslant 0}\{\eta p - F(\eta)\} = -(\gamma-1)(p^-)^{\frac{\gamma}{\gamma-1}}, \quad p\in\mathbb{R}, \tag{4.12}$$

together with the second order differential operator $\mathfrak{I}$ is given by

$$\begin{aligned}\mathfrak{I}(v',v'') \ &:= \ \sup_{z\in\mathbb{R},\ \hat{a}\in\partial h^\star(z)} \Big\{\hat{a} + h(\hat{a})v' + \frac{1}{2}r\sigma^2 z^2 v''\Big\} \\ &= \ \infty \mathbf{1}_{\{v''>0\}} + \mathbf{1}_{\{v''\leqslant 0\}} \sup_{z\geqslant h'(0),\ \hat{a}\in\hat{A}(z)} \Big\{\hat{a} + h(\hat{a})v' + \frac{1}{2}r\sigma^2 z^2 v''\Big\}^+,\end{aligned}$$

we may rewrite differential operator $\mathbf{L}^0$ as

$$\mathbf{L}^0 v \ = \ v - yv'(y) + F^\star\Big(v'\Big) - \mathfrak{I}\Big(v',v''\Big). \tag{4.13}$$

By standard stochastic control theory, the dynamic programming equation corresponding to the mixed control-and-stopping problem (4.9)-(4.10) is $\min\{v - F, \mathbf{L}^0 v\} = 0$ on $(0,\infty)$, together with the boundary condition $v(0) = 0$ induced by the state constraint $Y^{Y_0,Z} \geqslant 0$. As shown in Possamaï and Touzi [PT20], the obstacle part of this equation can be dismissed. Indeed, as $F^\star(v') \leqslant yv' - F(y)$ and $\mathfrak{I}(v',v'') \geqslant 0$, it follows that $\mathbf{L}^0 v \leqslant v - F$, and therefore the last dynamic programming equation is not altered by taking away the obstacle part. One of the main results of Possamaï and Touzi [PT20] is that the value function U^{P} is a C^2 solution of the equation

$$\mathbf{L}^0 U^{\mathrm{P}} = 0, \text{ on } (0,\infty), \quad \text{and} \quad U^{\mathrm{P}}(0) = 0, \tag{4.14}$$

and in fact, U^{P} is the unique viscosity solution of this equation in the class of functions with bounded $U^{\mathrm{P}} - F$.

4.3.3 The defaultable output case

We next turn to the problem V^{P} which differs from U^{P} by the restriction on the stopping time component of the contract $\tau \leqslant \mathrm{T}_0$. Following the same line of argument as in Section 4.3.1, we may again reduce the Principal's objective to the following standard stochastic control problem

$$V^{\mathrm{P}} = \sup_{Y_0 \geqslant u(R)} V(Y_0), \text{ with } V(Y_0) := \sup_{\substack{(\tau,Z,\pi) \\ \hat{a} \in \hat{A}(Z)}} J(\tau,\pi,Z,\hat{a}), \tag{4.15}$$

where the performance criterion $J(\tau,\pi,Z,\hat{a})$ is given by (4.10), and the maximization is over $Z \in \mathcal{Z}$ and (τ,π) satisfying (4.1), together with the limited liability condition $Y^{Y_0,Z,\pi} \geqslant 0$ inherited from the dynamic version of (4.3), and the defaultable output restriction on the termination time

$$\tau \leqslant \mathrm{T}_0 = \inf\Big\{t > 0 : X_t = 0\Big\}.$$

As this restriction involves explicitly the process X through its hitting time of the origin, we have to account for both state variables X and $Y = Y^{Y_0,Z,\pi}$, whose dynamics under the Agent's optimal response is given by:

$$\begin{aligned} dX_t &= \hat{a}_t dt + \sigma dW_t^{\hat{a}}, \\ dY_t &= r\Big(Y_t + h(\hat{a}_t) - \eta_t\Big)\mathrm{d}t + rZ_t\sigma dW_t^{\hat{a}}, \ \mathbb{P}^{\hat{a}}\text{–a.s..} \end{aligned} \tag{4.16}$$

The nonlinear differential operator induced by the optimal control part of the problem V^{P} in (4.15) is

$$\begin{aligned}\mathbf{L}^1 v &:= v - yv_y - \frac{\sigma^2}{2r}v_{xx} \\ &\quad - \sup_{\eta\geqslant 0, z\in\mathbb{R}, \hat{a}\in\hat{A}(z)}\Big\{F(\eta) - \eta v_y + \hat{a}(1+\frac{v_x}{r}) + h(\hat{a})v_y + \frac{1}{2}\sigma^2(rz^2 v_{yy} + 2zv_{xy})\Big\} \\ &= v - yv_y - \frac{\sigma^2}{2r}v_{xx} + F^\star(v_y) - \mathfrak{I}^1(v_x, v_y, v_{xx}, v_{xy}, v_{yy}),\end{aligned}$$

where we introduced the second order degenerate operator

$$\mathfrak{I}^1(v_x, v_y, v_{xx}, v_{xy}, v_{yy}) := \sup_{z\in\mathbb{R}, \hat{a}\in\hat{A}(z)}\Big\{\hat{a}(1+\frac{v_x}{r}) + h(\hat{a})v_y + \frac{1}{2}\sigma^2(rz^2 v_{yy} + 2zv_{xy})\Big\}.$$

By standard stochastic control theory, the function V is a viscosity solution of the dynamic programming equation corresponding to the mixed control-and-stopping problem (4.15):

$$\min\Big\{v - F, \mathbf{L}^1 v\Big\} = 0 \text{ on } (0,\infty)^2, \text{ with } v\Big|_{x=0} = F,\ v\Big|_{y=0} = 0. \tag{4.17}$$

Proposition 4.3.1. *Equation (4.17) is equivalent to*

$$\mathbf{L}^1 v = 0 \textit{ on } (0,\infty)^2, \quad \textit{with boundary conditions} \quad v\Big|_{x=0} = F,\ v\Big|_{y=0} = 0. \tag{4.18}$$

Proof. (4.17) $\Rightarrow$ (4.18). Define $S := \{v = F\}$, we have by (4.17) that $\mathbf{L}^1 v \geqslant 0$ on S and $\mathbf{L}^1 v = 0$ on S^c. Now if we want to prove $\mathbf{L}^1 v = 0$ on $(0,\infty)^2 = S \cup S^c$, we only need to show that $\mathbf{L}^1 v \leqslant 0$ on S, i.e., to show $\mathbf{L}^1 F \leqslant 0$ on S. To this end, note that

$$\begin{aligned}\mathbf{L}^1 F &= F - yF_y - \frac{\sigma^2}{2r}F_{xx} \\ &\quad - \sup_{\eta\geqslant 0, z\in\mathbb{R}, \hat{a}\in\hat{A}(z)}\Big\{F(\eta) - \eta F_y + \hat{a}(1+\frac{F_x}{r}) + h(\hat{a})F_y + \frac{1}{2}\sigma^2(rz^2 F_{yy} + 2zF_{xy})\Big\} \\ &= F - yF_y - \frac{\sigma^2}{2r}0 + F^\star(F_y) \\ &\quad - \sup_{z\in\mathbb{R}, \hat{a}\in\hat{A}(z)}\Big\{\hat{a}(1+\frac{0}{r}) + h(\hat{a})F_y + \frac{1}{2}\sigma^2(rz^2 F_{yy} + 2z0)\Big\} \\ &= F - yF_y + F^\star\Big(F_y\Big) - \mathfrak{I}\Big(F_y, F_{yy}\Big)^+ = \mathbf{L}^0 F,\end{aligned}$$

where the latter $\mathbf{L}^0 F \leqslant -\mathfrak{I}\Big(F_y, F_{yy}\Big)^+ \leqslant 0$ by the definition of F^* in (4.12) .

(4.18) $\Rightarrow$ (4.17). By definition of $\mathbf{L}^1 v$, we have

$$\begin{aligned}\mathbf{L}^1 v &= v - y v_y - \frac{\sigma^2}{2r} v_{xx} \\ &\quad - \sup_{\eta \geqslant 0, z \in \mathbb{R}, \hat{a} \in \hat{A}(z)} \Big\{ F(\eta) - \eta v_y + \hat{a}(1 + \frac{v_x}{r}) + h(\hat{a}) v_y + \frac{1}{2}\sigma^2 (r z^2 v_{yy} + 2 z v_{xy}) \Big\} \\ &\leqslant v - F - \frac{\sigma^2}{2r} v_{xx},\end{aligned}$$

by considering the particular values $\eta = y$ and $z = 0$. Then $\mathbf{L}^1 v = 0$ implies that $w = v - F$ is a supersolution of the equation $w - \frac{\sigma^2}{2r} w_{xx} \geqslant 0$ on $(0, \infty)^2$, and $w|_{x=0} = w|_{y=0} = 0$. As w is bounded, it follows from the comparison result that $w = v - F \geqslant 0$. Consequently, we have $\min \Big\{ v - F, \mathbf{L}^1 v \Big\} = 0$. □

Lemma 4.3.2. (Comparison) *Let u and v be respectively a viscosity sub-solution and a viscosity super-solution of* (4.18)*, such that for some constant $C > 0$:*

$$0 \leqslant \varphi - F \leqslant C, \text{ for } \mathbb{R}^2_+, \text{ for } \varphi \in \{u, v\}.$$

Then $u \leqslant v$ on $\mathbb{R}^2_+$.

Proof of Lemma 4.3.2. We rewrite the equation (4.18) as

$$v + G(x, y, Dv, D^2 v) = 0, \text{ on } \mathbb{R}^2_+, \tag{4.19}$$

where the nonlinearity G is given, for any $(y, p, q) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathcal{M}^2$ and $p := Dv, q := D^2 v$, by

$$G(x, y, Dv, D^2 v) \;:=\; -y v_y + F^\star(v_y) - \overline{\mathfrak{I}}^1(Dv, D^2 v),$$

with $\overline{\mathfrak{I}}^1(Dv, D^2 v) := \frac{\sigma^2}{2r} v_{xx} - \mathfrak{I}^1(v_x, v_y, v_{xx}, v_{xy}, v_{yy})$.

Step 1. Following the proof of Possamaï & Touzi [PT20] in the non-defaultable output setting, let $\mu := 2 \vee \gamma'$, for some $\gamma' > \gamma$, and we introduce for all $\alpha > 0$ and $\varepsilon > 0$:

$$M_{\alpha,\varepsilon} := \sup_{\theta, \theta' \in \mathbb{R}^2_+} \Big\{ u(\theta) - v(\theta') - \psi_{\alpha,\varepsilon}(\theta, \theta') \Big\}.$$

where, denoting $\mathbf{1} := (1,1)$,

$$\begin{aligned}\psi_{\alpha,\varepsilon}(\theta,\theta') &:= \frac{\alpha}{\mu}\Big|\theta-\theta'\Big|_\mu^\mu + \varepsilon\log(1+\mathbf{1}\cdot\theta)\\ &= \frac{\alpha}{\mu}\Big(|x-x'|^\mu + |y-y'|^\mu\Big) + \varepsilon\log(1+x+y),\ \theta=(x,y),\theta'\in\mathbb{R}^2_+.\end{aligned}$$

For all fixed $\varepsilon>0$, it follows from the growth assumptions on u and v that the supremum in the definition of $M_{\alpha,\varepsilon}$ may be confined to a compact subset $K_\varepsilon\subset\mathbb{R}^2_+\times\mathbb{R}^2_+$, with K_ε independent of α. Then,

$$M_{\alpha,\varepsilon} = u(\theta_{\alpha,\varepsilon}) - v(\theta'_{\alpha,\varepsilon}) - \psi_{\alpha,\varepsilon}(\theta_{\alpha,\varepsilon},\theta'_{\alpha,\varepsilon}),\ \text{for some } \theta_{\alpha,\varepsilon},\theta'_{\alpha,\varepsilon}\in K_\varepsilon.$$

Moreover by compactness of K_ε, we may find a sequence $(\theta_{\varepsilon,n},\theta'_{\varepsilon,n}) := (\theta_{\alpha_n,\varepsilon},\theta'_{\alpha_n,\varepsilon})$, converging to some $(\hat\theta_\varepsilon,\hat\theta'_\varepsilon)$. By standard arguments from viscosity solution theory, we have

$$\hat\theta_\varepsilon=\hat\theta'_\varepsilon,\ \lim_{n\to\infty}\alpha_n\Big|\theta_{\varepsilon,n}-\theta'_{\varepsilon,n}\Big|_\mu^\mu=0,\ M_\varepsilon := \lim_{n\to\infty} M_{\alpha_n,\varepsilon} = \sup_{\hat\theta_\epsilon\in\mathbb{R}^2_+}(u-v)(\hat\theta_\epsilon) - \varepsilon\log\Big(1+\mathbf{1}\cdot\hat\theta_\varepsilon\Big).$$

In order to prove the required comparison result, we assume that there is some $\theta_0=(x_0,y_0)\in(0,\infty)^2$ such that $\eta:=(u-v)(\theta_0)>0$, and work towards a contradiction.

Under this assumption, we have for sufficiently small ε that

$$0<\eta-\varepsilon\log(1+\mathbf{1}\cdot\theta_0)\leqslant M_{\alpha_n,\varepsilon}\longrightarrow(u-v)(\hat\theta_\varepsilon)-\varepsilon\log\Big(1+\mathbf{1}\cdot\hat\theta_\varepsilon\Big)\leqslant(u-v)(\hat\theta_\varepsilon).$$

As $u-v\leqslant0$ on $\partial\mathbb{R}^2_+$, it follows that $\hat\theta_\varepsilon\in(0,\infty)^2$ for sufficiently small ε. In particular, for n sufficiently large, we have that $\theta_{\varepsilon,n}$ and $\theta'_{\varepsilon,n}$ are both in $(0,\infty)^2$. Then denoting

$$p_{\varepsilon,n} := \alpha_n\begin{pmatrix}\Big|\delta x_{\varepsilon,n}\Big|^{\mu-1}\mathrm{sgn}(\delta x_{\varepsilon,n})\\ \Big|\delta y_{\varepsilon,n}\Big|^{\mu-1}\mathrm{sgn}(\delta y_{\varepsilon,n})\end{pmatrix},\ \text{with } (\delta x_{\varepsilon,n},\delta y_{\varepsilon,n}) := (x_{\varepsilon,n}-x'_{\varepsilon,n},y_{\varepsilon,n}-y'_{\varepsilon,n}),$$

it follows from the Crandall–Ishii lemma (M.G. Crandall, H. Ishii and P.-L. Lions) [CIL92, Theorem 3.2]) that there exists for all $n\geqslant1$, a pair $(X_{\varepsilon,n},X'_{\varepsilon,n})$ of $2\times2-$symmetric matrices such that

$$\Big(p_{\varepsilon,n},X_{\varepsilon,n}\Big)\in\overline{J}^{2,+}u(\theta_{\varepsilon,n}),\ \Big(p_{\varepsilon,n}-\frac{\varepsilon\mathbf{1}}{1+\mathbf{1}\cdot\theta'_{\varepsilon,n}},X'_{\varepsilon,n}\Big)\in\overline{J}^{2,-}v(\theta'_{\varepsilon,n}),$$

and

$$-\Big(\frac{1}{\lambda}+\|C_{\varepsilon,n}\|\Big)I_4 \leqslant \begin{pmatrix} X_{\varepsilon,n} & 0 \\ 0 & -X'_{\varepsilon,n}\end{pmatrix} \leqslant C_{\varepsilon,n}+\lambda C^2_{\varepsilon,n}, \quad \text{for all} \quad \lambda>0,$$

$$C_{\varepsilon,n} := D^2\psi_{\varepsilon,n}(\theta_{\varepsilon,n},\theta'_{\varepsilon,n}) = \left(\begin{array}{c|c} D_{\varepsilon,n} & -D_{\varepsilon,n} \\ \hline -D_{\varepsilon,n} & D_{\varepsilon,n}\end{array}\right) + b_{\varepsilon,n}\left(\begin{array}{c|c} J_2 & 0_2 \\ \hline 0_2 & 0_2\end{array}\right),$$

$$D_{\varepsilon,n} := \alpha_n(\mu-1)\mathrm{diag}\Big(|\delta x_{\varepsilon,n}|^{\mu-2},|\delta y_{\varepsilon,n}|^{\mu-2}\Big),\ b_{\varepsilon,n} := \frac{\varepsilon}{(1+\mathbf{1}\cdot\theta'_{\varepsilon,n})^2},$$

where I_4 are the identity matrices in $\mathbb{R}^2$ and $\mathbb{R}^4$, J_2 the 2×2 matrix with all entries equal to 1, 0_2 the 2×2 matrix with all entries equal to 0, and where we use the spectral norm for symmetric matrices. Take $\lambda = \|C_{\varepsilon,n}\|^{-1}$, we get

$$-2\|C_{\varepsilon,n}\|I_4 \leqslant \begin{pmatrix} X_{\varepsilon,n} & 0 \\ 0 & -X'_{\varepsilon,n}\end{pmatrix} \leqslant C_{\varepsilon,n}+\frac{C^2_{\varepsilon,n}}{\|C_{\varepsilon,n}\|}. \tag{4.20}$$

Multiplying the last inequality by $(\xi^\top,\xi^\top)$ to the left and $(\xi^\top,\xi^\top)^\top$ to the right, for an arbitrary $\xi\in\mathbb{R}^2$, it follows from the right hand side inequality of (4.20) that

$$X_{\varepsilon,n}-X'_{\varepsilon,n} \leqslant \Big(b_{\varepsilon,n}+\frac{2b^2_{\varepsilon,n}}{\|C_{\varepsilon,n}\|}\Big)J_2. \tag{4.21}$$

Step 2. By the sub-solution and super-solution properties of u and v, we have for any $n\in\mathbb{N}$

$$u(\theta_{\varepsilon,n})+G\Big(\theta_{\varepsilon,n},p_{\varepsilon,n},X_{\varepsilon,n}\Big) \leqslant 0 \leqslant v(\theta'_{\varepsilon,n})+G\bigg(\theta'_{\varepsilon,n},p_{\varepsilon,n}-\frac{\varepsilon\mathbf{1}}{1+\mathbf{1}\cdot\theta'_{\varepsilon,n}},X'_{\varepsilon,n}\bigg).$$

We deduce that

$$\begin{aligned}
\eta-\varepsilon\log(1+\mathbf{1}\cdot\theta_0) &\leqslant u(\theta_{\varepsilon,n})-v(\theta'_{\varepsilon,n})\\
&\leqslant G\Big(\theta'_{\varepsilon,n},p_{\varepsilon,n}-\frac{\varepsilon\mathbf{1}}{1+\mathbf{1}\cdot\theta'_{\varepsilon,n}},X'_{\varepsilon,n}\Big)-G\Big(\theta_{\varepsilon,n},p_{\varepsilon,n},X_{\varepsilon,n}\Big)\\
&= \alpha_n|\delta y_{\varepsilon,n}|^\mu+\frac{\varepsilon\, y'_{\varepsilon,n}}{1+x'_{\varepsilon,n}+y'_{\varepsilon,n}}\\
&\quad +F^\star\Big(\alpha_n|\delta y_{\varepsilon,n}|^{\mu-1}\mathrm{sgn}(\delta y_{\varepsilon,n})-\frac{\varepsilon}{1+\mathbf{1}\cdot\theta'_{\varepsilon,n}}\Big)-F^\star\Big(\alpha_n|\delta y_{\varepsilon,n}|^{\mu-1}\mathrm{sgn}(\delta y_{\varepsilon,n})\Big)\\
&\quad +\overline{\mathfrak{I}}^1\Big(p_{\varepsilon,n},X_{\varepsilon,n}\Big)-\overline{\mathfrak{I}}^1\Big(p_{\varepsilon,n}-\frac{\varepsilon\mathbf{1}}{1+\mathbf{1}\cdot\theta'_{\varepsilon,n}},X'_{\varepsilon,n}\Big)\\
&= \alpha_n|\delta y_{\varepsilon,n}|^\mu+\varepsilon+\overline{\mathfrak{I}}^1\Big(p_{\varepsilon,n},X_{\varepsilon,n}\Big)-\overline{\mathfrak{I}}^1\Big(p_{\varepsilon,n}-\frac{\varepsilon\mathbf{1}}{1+\mathbf{1}\cdot\theta'_{\varepsilon,n}},X'_{\varepsilon,n}\Big),
\end{aligned}$$

where the last inequality follows from the non-negativity of $x_{\varepsilon,n}, y_{\varepsilon,n}$ and the non-decrease of $F^\star$. We next use the fact that $\overline{\mathfrak{I}}^1$ is elliptic and c_0-Lipschitz for some $c_0>0$. Then, it follows from the last inequalities together with (4.21) that:

$$\eta-\varepsilon\log(1+\mathbf{1}\cdot\theta_0) \leqslant \alpha_n|\delta y_{\varepsilon,n}|^\mu+\varepsilon(1+c_0)+2c_0\Big(b_{\varepsilon,n}+\frac{2b^2_{\varepsilon,n}}{\|C_{\varepsilon,n}\|}\Big). \tag{4.22}$$

Step 3. We now send n to ∞, and distinguish two cases. First, if $\Big(\|C_{\varepsilon,n}\|\Big)_{n\in\mathbb{N}}$ is unbounded, then after possibly passing to a subsequence, we deduce by letting n go to ∞ that

$$\eta-\varepsilon\log(1+\mathbf{1}\cdot\theta_0)\leqslant(1+c_0)\varepsilon+c_0\frac{\varepsilon}{(1+\mathbf{1}\cdot\hat\theta_\varepsilon)^2}\longrightarrow 0,\ \text{as } \varepsilon\searrow 0,$$

which is the required contradiction as the left hand side converges to $\eta>0$.

Alternatively, if $\Big(\|C_{\varepsilon,n}\|\Big)_{n\in\mathbb{N}}$ is bounded, then after possibly passing to a subsequence, we have a converging subsequence, and notice that we then have

$$\|C_{\varepsilon,n}\|\longrightarrow\|C^\varepsilon\|:=\Big\|A^\varepsilon+b^\varepsilon B\Big\|,\ \text{where } A^\varepsilon:=\left(\begin{array}{c|c} D^\varepsilon & -D^\varepsilon\\ \hline -D^\varepsilon & D^\varepsilon\end{array}\right),\ D^\varepsilon:=\lim_{n\to\infty}D_{\varepsilon,n},$$

$$\text{and } \hat\theta_\varepsilon:=(\hat x^\varepsilon,\hat y^\varepsilon),\ b^\varepsilon:=\frac{\varepsilon}{(1+\hat x^\varepsilon+\hat y^\varepsilon)^2},\ B:=\left(\begin{array}{c|c} J_2 & 0_2\\ \hline 0_2 & 0_2\end{array}\right).$$

If the sequence $(D^\varepsilon)_{\varepsilon>0}$ is unbounded, we take again a subsequence and get a contradiction by letting ε go to 0 in

$$\eta - \varepsilon \log(1 + \mathbf{1} \cdot \theta_0) \leqslant (1 + c_0)\varepsilon + c_0 b^\varepsilon + c_0 \frac{(b^\varepsilon)^2}{\|C^\varepsilon\|}. \tag{4.23}$$

In the remaining case where the sequence $(D^\varepsilon)_{\varepsilon>0}$ is bounded, we also show by the same argument as in Possamaï & Touzi that inequality (4.22) leads to a contradiction. □

4.4 Galerkin neural network numerical approximation

Let $(\mathcal{L}_\ell)_{1\leqslant \ell \leqslant L}$ be a sequence of linear functions defined by:

$$\mathcal{L}_\ell : \mathbb{R}^{d_{\ell-1}} \to \mathbb{R}^{d_\ell}$$
$$x \mapsto \mathcal{W}_\ell x + b_\ell, \quad 1 \leqslant \ell \leqslant L.$$

in which $L \geqslant 2, d_\ell \geqslant 1$, $\mathcal{W}_\ell \in \mathcal{M}_{d_\ell, d_{\ell-1}}$ and $b_\ell \in \mathbb{R}^{d_\ell}$ for $1 \leqslant \ell \leqslant L$. We introduce $K : \mathbb{R} \mapsto \mathbb{R}$ a nonlinear function. For any $x = (x_1, \dots x_p) \in \mathbb{R}^p$ with $p \geqslant 1$, we denote by $K(x)$ the component-wise application of K, i.e.

$$K(x_1, \dots x_p) = (K(x_1), \dots K(x_p)).$$

A feedforward neural network is a function defined from $\mathbb{R}^{d_0}$ to $\mathbb{R}^{d_L}$ defined as the composition:

$$x \mapsto \mathcal{L}_L \circ K \circ \mathcal{L}_{L-1} \circ \dots \circ K \circ \mathcal{L}_1(x). \tag{4.24}$$

In our framework, $d_0 \geqslant 1$ is the dimension of the inputs whereas the dimension of the output is fixed at $d_L = 1$. We introduce $d = (d_\ell)_{1\leqslant \ell \leqslant L}$ the sequence of the dimensions of the neural network. We denote by θ its set of parameters:

$$\theta := (\mathcal{W}_\ell, b_\ell)_{1\leqslant \ell \leqslant L},$$

whose dimension is $n(d) := \sum_{\ell=1}^{L} (d_{\ell-1} + 1) d_\ell$ and can be seen as an element of $\mathbb{R}^{n(d)}$. We denote by $\mathbf{N}^K_{d_0,L}$ the collection of all neural networks defined in (4.24) for $\left\{d \in \{d_0\} \times (\mathbb{N}^*)^{L-2} \times \{1\}, \theta \in \mathbb{R}^{n(d)}\right\}$.

Remark 4.4.1. *The use of neural networks as a function approximator is justified by Hornik, Stinchcombe* *and White [HSW89] which states that $\mathbf{N}^K_{d_0,L}$ is dense in $L^2(\mu)$ for any finite measure μ on $\mathbb{R}^{d_0}$ whenever K is continuous and non-constant. Moreover, [HSW90] ensures that, if K in non-constant and C^k, then $\mathbf{N}^K_{d_0,2}$ approximates any C^k function and its derivatives up to order k, arbitrarily well on any compact set of $\mathbb{R}^{d_0}$.*

4.4.1 Deep learning for Sannikov's problem

Recall from Possamaï and Touzi [PT20] that equation (4.14) admits a unique solution in the class of functions that satisfy the growth condition

$$(U^{\mathrm{P}} - F)(y) \leqslant C \log(1 + \log(1 + y)). \tag{4.25}$$

and that U^{P} and F are asymptotic to each other

$$(U^{\mathrm{P}} - F)(y) \to 0 \quad \text{when} \quad y \to +\infty, \tag{4.26}$$

and even equal away from some large value y_{G}, for some values of the parameters. We shall use the property above to ensure that (4.25) holds and to get the right solution from (4.14). We recall that (4.26) is a consequence of the stronger estimate

$$(U^{\mathrm{P}} - F)(y) \leqslant (U^{\mathrm{FB}} - F)(y) \to 0 \quad \text{when} \quad y \to +\infty, \tag{4.27}$$

where U^{FB} is the value function of the first best formulation of the contracting problem defined by

$$U^{\mathrm{FB}}(y) := \sup \Big\{ J^{\mathrm{P}}(\mathbf{C}, \alpha) : \mathbf{C} \in \mathfrak{C},\ \alpha \in \mathcal{A} \text{ and } J^{\mathrm{A}}(\mathbf{C}, \alpha) \geqslant y \Big\}. \tag{4.28}$$

Our purpose is to solve numerically the equation (4.14)-(4.26) using a neural network N which belongs to $\mathbf{N}^K_{d_0,L}$ on $y \in [0, \overline{y}]$ for some $\overline{y} > 0$. We fix $\mathbf{Y} \subset [0, \overline{y}]$ a finite number of points in which we shall evaluate the accuracy of the neural network. The accuracy is linked to a loss function $h : [0, \overline{y}] \times C^2\left([0, \overline{y}]\right) \to \mathbb{R}_+$ which should have the form

$$h(\mathbf{Y}; N(\cdot, \theta)) = ||\mathbf{L}^0 N(\mathbf{Y}; \theta)||^{p_L}_{\mathbf{L}} + ||N(0; \theta)||^{p_B}_{\mathbf{B}} + ||N(\overline{y}; \theta) - F(\overline{y})||^{p_I}_{\mathbf{I}}, \qquad N \in \mathbf{N}^K_{d_0,L}, \tag{4.29}$$

in which $||.||_{\mathbf{L}}$, $||.||_{\mathbf{B}}$ and $||.||_{\mathbf{I}}$ are norms and $p_L, p_B, p_I \in \mathbb{N}^*$, and assuming that $\overline{y}$ is great enough such that $U^{\mathrm{P}}(\overline{y}) - F(\overline{y})$ is close to 0.

Nonetheless, this form is not accurate for a good minimization in order to estimate θ, since the loss function defined in (4.29) is a combination of three norms. An idea is to make the boundary condition $N(0;\theta) = 0$ and the limit condition $N(y;\theta) - F(y) \to 0$ when $y \to \overline{y}$ endogenous. Instead of using N as a numerical solution of equation (4.14)-(4.26), we introduce a function g defined by:

$$\begin{aligned} g(\cdot;\theta) : [0,\overline{y}] &\to \mathbb{R} \\ y &\mapsto f(y, N(y;\theta)), \end{aligned} \tag{4.30}$$

in which f is a smooth function from $\mathbb{R}^2$ to $\mathbb{R}$, linear in its second argument. We require g to satisfy

$$g(0;\theta) = 0 \quad \text{and} \quad g(\overline{y};\theta) = F(\overline{y}) \quad \forall N \in \mathbf{N}^K_{d_0,L}.$$

Such a function g satisfies the boundary condition of (4.14) and the limit condition (4.26) for all $N \in \mathbf{N}^K_{d_0,L}$, assuming that $\overline{y}$ is great enough. Applied to g, the loss function h is simply:

$$h(\mathbf{Y}; g(\cdot,\theta)) = ||\mathbf{L}^0 g(\cdot;\theta)(\mathbf{Y};\theta)||^{p_L}_{\mathbf{L}}, \qquad N \in \mathbf{N}^K_{d_0,L}. \tag{4.31}$$

The aim is to minimize $h(\mathbf{Y}; g(\cdot,\theta))$ over θ, and denote by $\hat{\theta}$ a corresponding numerical minimizer. The resulting $g(\cdot,\hat{\theta})$ is the numerical solution of (4.14)-(4.26).

4.4.2 Numerical results

We use the euclidean norm for $||.||_{\mathbf{L}}$ in the loss function h with $p_L := 2$. We introduce:

$$g(\cdot;\theta) : y \mapsto F(y) + y(\overline{y} - y)^2 N(y;\theta),$$

as an approximator of the solution of (4.14)-(4.26). We set the network N with the parameters $\mathbf{L} = 3$ and $d = (1, 32, 8, 1)$. Note that $d_0 = 1$ since the function is defined on a subset of $\mathbb{R}$ and d_L is the output, fixed at 1 since the values of N belongs to $\mathbb{R}$. The dimension of θ, which is number of parameters to estimate, is 337. The estimation of the parameter θ is done with the stochastic gradient descent of

Adam (see Kingma and Ba [KB14]).

Figure 4.1 shows the numerical solution obtained from the neural network estimation in which we take the parameters of [San08] and [PT20] (with $\gamma = 2, \eta = 0.05, h = 0.5, \beta = 0.4$, $\delta = 1$ and $\overline{y} = 1.75$). We get the archetypal case where a Golden Parachute exists.

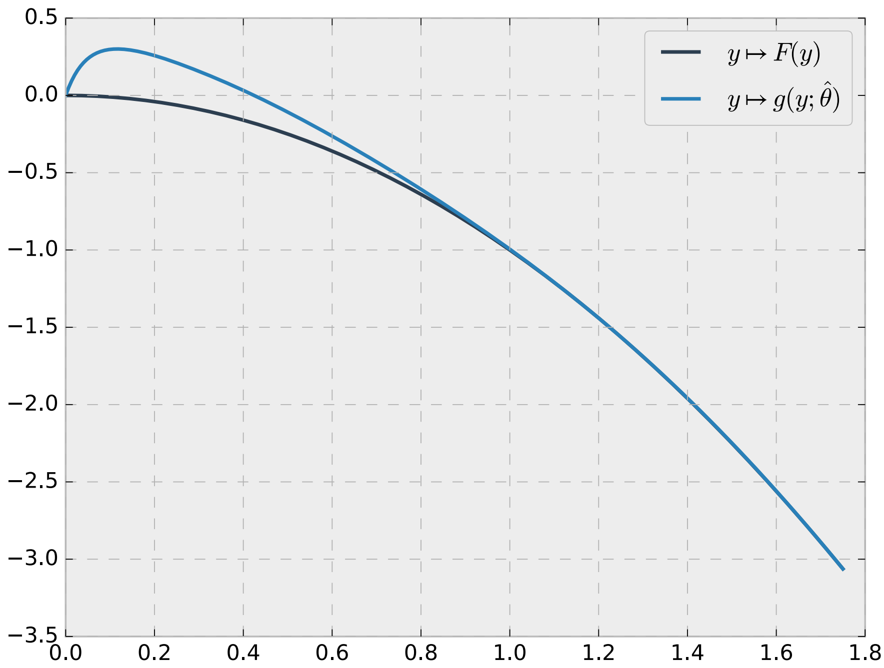

Figure 4.1: Final squared euclidean loss is 2.11×10^{-4}.

4.4.3 Deep learning the defaultable output case

Recall that as $F \leqslant V^{\mathrm{P}} \leqslant U^{\mathrm{P}}$, it follows from (4.27) that

$$0 \leqslant V^{\mathrm{P}}(x, y) - F(y) \leqslant (U^{\mathrm{FB}} - F)(y) \to 0 \quad \text{when} \quad y \to +\infty, \ x > 0. \tag{4.32}$$

Moreover, V^{P} is the unique viscosity solution of the equation (4.18)-(4.32) in the class of functions that satisfy the growth condition, for all $x > 0$,

$$V^{\mathrm{P}}(x, y) - F(y) \leqslant C \log(1 + \log(1 + y)). \tag{4.33}$$

We shall use the property above to ensure that (4.25) holds and to get the right solution from (4.18).

The purpose is to solve numerically the equation (4.18)-(4.32) using a neural network which belongs to $\mathbf{N}^K_{d_0,L}$ on $(x,y) \in [0,\overline{x}] \times [0,\overline{y}]$ for some $\overline{x} > 0$ and $\overline{y} > 0$. We fix $\mathbf{X} \subset [0,\overline{x}]$ and $\mathbf{Y} \subset [0,\overline{y}]$ two sets with a finite number of points in which we shall evaluate the accuracy of the neural network.

As in the Sannikov problem, we shall introduce a function g depending on the neural network and which integrates the boundary and limit conditions:

$$g(\cdot;\theta) : [0,\overline{x}] \times [0,\overline{y}] \longrightarrow \mathbb{R} \tag{4.34}$$

$$(x,y) \longmapsto f(x,y,N((x,y);\theta)), \tag{4.35}$$

in which f is a smooth function from $\mathbb{R}^3$ to $\mathbb{R}$, linear in its third argument.

We require g to integrate the boundary and limit conditions. For the limit condition, since it holds for $x > 0$, and is not continuous in x, we shall only impose it at $\overline{x}$. We propose the following g definition.

$$g((0,y);\theta) = F(y), g((x,0);\theta) = 0 \quad \text{and} \quad g(\overline{x},\overline{y};\theta) = F(\overline{y}) \quad x \in [0,\overline{x}], y \in [0,\overline{y}], N \in \mathbf{N}^K_{d_0,L}.$$

Such a function g satisfies the boundary condition of (4.18) and the limit condition (4.32) for all $N \in \mathbf{N}^K_{d_0,L}$, assuming that $\overline{y}$ is great enough. Applied to g, the loss function h can be simply:

$$h(\mathbf{X} \times \mathbf{Y}; g(\cdot,\theta)) = ||\mathbf{L}^0 g(\cdot;\theta)(\mathbf{X} \times \mathbf{Y};\theta)||^{p_L}_{\mathbf{L}}, \qquad N \in \mathbf{N}^K_{d_0,L}. \tag{4.36}$$

in which $||.||_{\mathbf{L}}$ is a norm and $p_L \in \mathbb{N}^*$. The aim is to minimize $h(\mathbf{X} \times \mathbf{Y}; g(\cdot,\theta))$ over θ, and denote by $\hat{\theta}$ a corresponding numerical minimizer. The $g(\cdot,\hat{\theta})$ is the numerical solution of (4.18)-(4.32).

4.4.4 Numerical results

We use the euclidean norm for $||.||_{\mathbf{L}}$ in the loss function h with $p_L := 2$. We introduce:

$$g(\cdot;\theta) : (x,y) \longmapsto F(y) + xy\left(\frac{k\overline{x} - x}{k\overline{x}} + \frac{k\overline{y} - y}{k\overline{y}}\right) N(x,y;\theta),$$

as an approximation of the solution of (4.18)-(4.32), with $k = 1.2$. We set the network N with the parameters $L = 2$ and $d = (2, 32, 8, 1)$. Note that $d_0 = 2$ since the function is defined on a subset of $\mathbb{R}^2$ and d_L is the output, fixed at 1 since the values of N belongs to $\mathbb{R}$. Here, the condition $g(\overline{x}, \overline{y}) = 0$ is replaced by $g(k\overline{x}, k\overline{y}) = 0$ to leave more flexibility for the estimation of the solution, and we choose $k = 1.2$. The dimension of θ, which is number of parameters to estimate, is 369. The estimation of the parameter θ is done with the stochastic gradient descent of Adam (see [KB14]).

Figure 4.2 shows the numerical solution obtained from the neural network estimation in which we take the same parameters of the one dimensional case (with $\gamma = 2, \eta = 0.05, h = 0.5, \beta = 0.4$, $\delta = 1$ and $(\overline{x}, \overline{y}) = (16, 1.75)$). We draw the function $y \mapsto g(x, y; \hat{\theta})$ for several values of $x \in [0, \overline{x}]$.

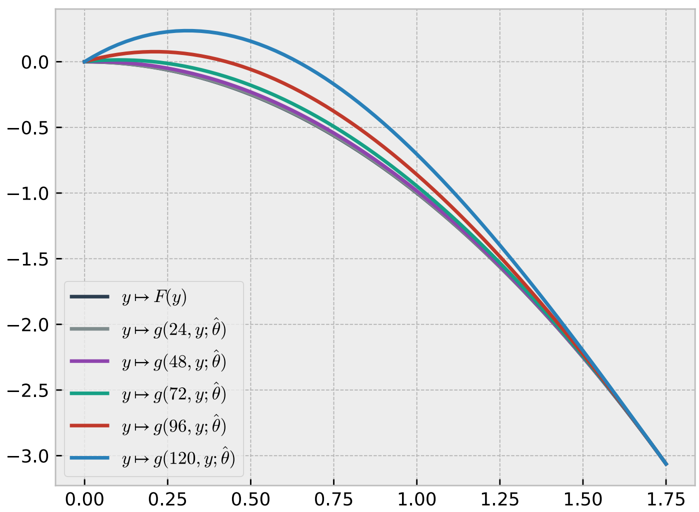

Figure 4.2: Final squared euclidean loss is 3.09×10^{-2}.

As expected, Figure 4.2 exhibits an increase of the Principal's value function V^{P} in terms of the output initial value X_0, with $V(0, y) = F(y)$, due to the fact that the production asset is at the bankruptcy level. Since F is decreasing, this induces the principal's value function for $X_0 = 0$ that

$$V^{\mathrm{P}}(0) = \sup_{y \geqslant R} V(0, y) = F(R).$$

For positive initial value X_0 of the output process, the value function $y \longmapsto V(X_0, y)$ exhibits a bump near the origin with a maximum value attained at some point $\bar{R}(X_0)$. Consequently, the principal's value function is given by

$$V^{\mathrm{P}}(X_0) = \sup_{y \geqslant R} V(X_0, y) = V(X_0, R \vee \bar{R}(X_0)).$$

In other words, if the participation level R of the agent is below $\bar{R}(X_0)$, it is optimal for the principal to award him with $\bar{R}(X_0) \geqslant R$ in compensation for the hidden action of the agent. This illustrates the so-called informational rent gained by the agent due to the information asymmetry. This effect is in agreement with the main finding of Sannikov in the non-defaultable production asset setting. We also observe that $V^{\mathrm{P}}(X_0)$ increases towards $U^{\mathrm{P}}(X_0)$ for large X_0.

Finally, Figure 4.2 shows that the map $X_0 \longmapsto \bar{R}(X_0)$ seems to be increasing. In other words, the farther the production asset is from bankruptcy, the more informational rent can be extracted by the agent, and also the larger is the utility value of the principal. This is more visible on the surface representation of $(x, y) \longmapsto g(x, y; \hat{\theta}) \approx V(x, y)$ in Figure 4.3.

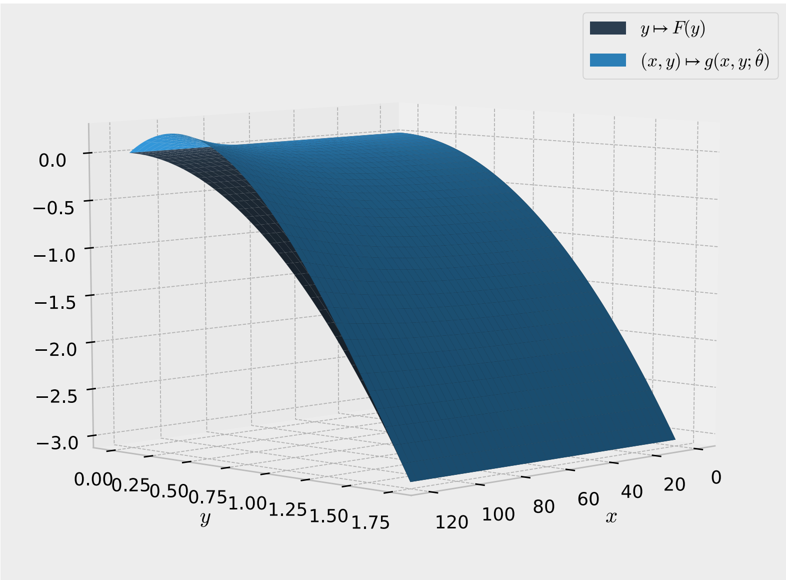

Figure 4.3: Surface view.

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Titre : Equations différentielles stochastiques rétrogrades de second ordre et l'analyse numérique du problème de Sannikov

Mots clés : EDS rétrogrades d'ordre 2 (EDS rétrogrades, EDS rétrogrades réfléchies) ; problème de Sannikov ; problème de Principal-Agent ; apprentissage profond ; contrôle stochastique et arrêt optimal

Résumé : Cette thèse se concentre principalement sur deux sujets. La première partie présente le problème bien-défini de solution d'équations différentielles stochastiques de second ordre (2EDSR en bref). Après un résumé concernant les résultats d'équations différentielles stochastiques réflexies à croissance quadratique, i.e., le problème bien-défini de solution (l'existence et l'unité), la comparaison, la stabilité, etc., on centralise essentiellement le problème bien-défini de 2EDSR correspondante. Hormis la croissance quadratique, on assume aussi que le generateur f soit concave de z avec une gradient linéaire. Celui-ci nous offre une inégalité nouvelle de déscentées de la fonction valeur(aussi une f-surmartingale) V. Après avoir obtenu la régularité de sa limite en t, on s'est appuyé sur les études de Soner, Touzi et Zhang [STZ12] pour établir une expression de solution et vérifier l'existence & l'unité. La deuxième partie traite à fond le problème de contrat optimal de Sannikov sous contrainte de faille du bien de production. Un modèle de mandantement fait partie du problème de contrat, où l'Agent s'occupe du processus de la production et le Principal définit les régles du contrat d'après la performance du processus de la production, en espérant inciter l'Agent à s'engager plus efficacement vers son objectif. Le contrat a besoin de l'accord de l'Agent au début et progresse en temps réel selon des efforts de deux cotés. Et puis il stipule le temps d'arrêt (deterministe ou aléatoire) du contrat et un versement continu du Principal à l'Agent jusqu'au terminal ainsi que un versement en une fois à la fin. Notre objectif pincipal est en étudier l'effet d'imposer des restrictions au temps d'arrêt du contrat avant qu'une faille du bien de production ait lieu. Dans ce but, on propose un nouveau rapprochement numérique, se fondant sur les réseaux neurons de Galerkin, pour examiner le comportement de la fonction valeur issue de problème correspondant du Principal.

Title : Quadratic second-order backward stochastic differential equation and numeric analysis for Sannikov's optimal contracting problem

Keywords : 2BSDE (BSDE,RBSDE) ; Sannikov's problem ; Principal-Agent problem ; deep learning ; stochastic control and optimal stopping

Abstract : This thesis works mainly on two subjects. The first part is about the wellposedness of quadratic second-order backward stochastic differential equation. After a summary of previous research results on the solutions of quadratic reflected backward SDE, i.e., wellposedness (existence and uniqueness), comparison principle, stability result, etc., we in this paper mainly consider the wellposedness of solutions for the corresponding second order backward SDE. Besides the quadratic growth, we also assume that the generator f is concave in z with linearly growing gradient. This leads to a new downcrossing inequality of the value function (also a f-supermartingale) V. Gaining the regularity of its right limit w.r.t. t, we follow Soner, Touzi and Zhang [STZ12] to get the representation of solutions and implement the proof of wellposedness.
The second part is about the Sannikov optimal contracting problem under defaultable output process. The contracting problem consists of a delegation scheme between two parties, the Agent in charge of the management of the output process, and the Principal who sets up the terms of a contract indexed on the performance of the output process so as to incite the Agent to best serve her objective. The contract requires the Agent approval at the initial time, and is then subject to the full commitment of both parties. The contract stipulates the termination time of the contract, a continuous payment until termination, and a lump sum payment at termination. Our main objective in this paper is to explore the effect of restricting the termination time of the contract to occur before the default of the contract. To this end, we propose a type of Galerkin neural network numerical approximation to investigate the behavior of the value function arised in the Principal's problem.

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