Cohesion in Rome

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TAL TECH

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Toposes

> [. . .] vi el Aleph, desde todos los puntos, vi en el Aleph la tierra, y en la tierra otra vez el Aleph y en el Aleph la tierra, vi mi cara y mis vísceras, vi tu cara, y sentí vértigo y lloré. . .
>
> JLB

Topos theory is a cornerstone of category theory linking together algebra, geometry and logic.

In each topos it is possible to re-enact Mathematics; today we focus on

- Logic (better said, a fragment of dependent type theory)
- Differential geometry (better said, iterated tangent bundles)
- (Secretly, algebraic topology)
- . . .

Sheaves on spaces

Let (X, τ) be a topological space; a *sheaf on X* is a functor $F : \tau^{\text{op}} \to \underline{\text{Set}}$ such that for every $U \in \tau$ and every covering $\{U_i\}$ of U one has

- if $s, t \in FU$ are such that $s|_i = t|_i$ in FU_i for every $i \in I$, then $s = t$ in FU.
- if $s_i \in FU_i$ is a family of elements such that $s_i|_{ij} = s_j|_{ij}$, then there exists a $s \in FU$ such that $s|_i = s_i$.[1]

[1]We denote $s|_i$ the image of $s \in FU$ under the nameless map $FU \to FU_i$ induced by the inclusion $U_i \subseteq U$.

Examples of sheaves

Every construction in Mathematics that exhibits a local character is a sheaf:

- sending $U \mapsto CU$, continuous functions with domain U (similarly, differentiable, C^∞, C^ω, holomorphic...)
- sending $U \mapsto \Omega^p U$, differential forms supported on U (similarly: distributions, test functions...)
- ...sending $U \mapsto \{f : U \to \mathbb{R} \mid f \text{ has property } P \text{ locally}\}$ for some P.

Every construction that does involve global properties, is not a sheaf:

- sending $U \mapsto \{\text{bounded functions } f : U \to \mathbb{R}\}$
- sending $U \mapsto \{L^1 \text{ functions } f : U \to \mathbb{R}\}$
- ...

Grothendieck topologies

A sieve on an object X of a category $\mathcal{C}$ is a subobject S of the hom functor $yX = \mathcal{C}(\_, X)$;

A Grothendieck topology on a category amounts to the choice of a family of covering sieves for every object $X \in \mathcal{C}$; this family of sieves is chosen in such a way that

[list of axioms abstracting the fact that

- if $\{U_i\}$ covers U, then for every $V \subseteq U$ $V \cap U_i$ covers V;
- if $\{U_i\}$ covers U and $\{V_{ij}\}$ covers U_i, then V_{ij} covers U;
- $\{U\}$ covers U.

]

Grothendieck topologies

A Grothendieck site is a category with a Grothendieck topology, i.e. a function j that assigns to every object a family of covering sieves.

We denote a site as the pair $(\mathcal{C}, j)$.

A sheaf on a small site $\mathcal{C}$ is a functor $F : \mathcal{C}^{\mathrm{op}} \to \underline{\mathrm{Set}}$ such that for every covering sieve $R \to yU$ and every diagram

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "m"'] & F \\ yU \arrow[ur, dotted] & \end{tikzcd}$$

there is a unique dotted extension $yU \Rightarrow F$ (by the Yoneda lemma, this consists of a unique element $s \in FU$: exercise, derive the sheaf axioms from this).

The full subcategory of sheaves on a site $(\mathcal{C}, j)$ is denoted $\mathrm{Sh}(\mathcal{C}, j)$.

Giraud Theorem

By general facts on locally presentable categories, the subcategory of sheaves on a site is reflective via a functor

$$r : \mathrm{Cat}(\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{Set}}) \to \mathrm{Sh}(\mathcal{C}, j)$$

called *sheafification* of a presheaf $F : \mathcal{C}^{\mathrm{op}} \to \underline{\mathrm{Set}}$.

Grothendieck was the first to note that in every topos of sheaves the **internal language** is sufficiently expressive to concoct **higher-order logic** and he strived to advertise his intuitions to an audience of logicians.

But it wasn't until Lawvere devised the notion of **elementary topos** that the community agreed on the potential of this theory.

Elementary toposes

An *elementary topos* is a category $\mathcal{E}$ that

- it has finite limits (products, equalizers, pullbacks);
- is cartesian closed (every $A \times \_$ has a right adjoint);
- has a *subobject classifier*, i.e. an object $\Omega \in \mathcal{E}$ such that the functor $\mathrm{Sub} : \mathcal{E}^{\mathrm{op}} \to \underline{\mathrm{Set}}$ sending A into the set of isomorphism classes of monomorphisms $\begin{smallmatrix} U \\ \downarrow \\ A \end{smallmatrix}$ is representable by the object Ω.

Elementary toposes

The natural bijection $\mathcal{E}(A, \Omega) \cong \mathrm{Sub}(A)$ is obtained pulling back a "characteristic arrow" $\chi_U : A \to \Omega$ along a universal arrow $t : 1 \to \Omega$ to obtain the monic U, as in the diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner"] & 1 \arrow[d, "t"] \\
A \arrow[r, "\chi_m"'] & \Omega
\end{tikzcd}$$

The bijection is induced by the maps

- $\chi_- : \begin{bmatrix} U \\ \downarrow \\ A \end{bmatrix} \mapsto \chi_m$ and
- $- \times_\Omega t : \chi_U \mapsto \chi_U \times_\Omega t$.

Grothendieck $\subset$ elementary

> En los libros herméticos está escrito que lo que hay abajo es igual a lo que hay arriba, y lo que hay arriba, igual a lo que hay abajo; en el Zohar, que el mundo inferior es reflejo del superior.†
>
> JLB

- Every Grothendieck topos is elementary;
- An elementary topos is Grothendieck if and only if it is a locally finitely presentable category.

Giraud theorem characterises Grothendieck toposes as such elementary toposes.

†Microcosm principle: a topos, i.e. a place where subobjects are well-behaved, is but a well-behaved subobject in the 2-category of presheaf categories.

Axiomatic Cohesion

What is cohesion

Cohesion is the mutual attraction of molecules sticking together to form *droplets*, caused by mild electrical attraction between them.

Figure 1: Droplets of mercury "exhibiting cohesion"

What is cohesion

Classes of geometric spaces exhibit similar coagulation properties, similar to internal forces leading them to adhere and form coherent conglomerates.

This behaviour is typical of smooth spaces.

Example

Smooth manifolds can be probed via smooth open balls and every smooth space is a "coherent conglomerate" of *cohesive pieces*.

Question

Which axioms formalize this intuition? What is *axiomatic cohesion*?

Axioms to answer this question have been devised by Lawvere [Law1] (worth reading, but quite mystical!).

Desiderata

We would like to operate in a *category* (a topos) of “cohesive spaces”, such that

- there is a functor $\Pi \colon \mathcal{H} \to \underline{\mathrm{Set}}$ that sends every cohesive space $X \in \mathcal{H}$ into its set of connected components.
- Every set $S \in \underline{\mathrm{Set}}$ can be regarded as a cohesive space in two complementary ways:
    - *discretely*, with a functor $\underline{\mathrm{Set}} \to \mathcal{H}$ that regards every singleton of S as a cohesive droplet;
    - *codiscretely*, with a functor $\underline{\mathrm{Set}} \to \mathcal{H}$ that regards the whole S as an unseparable cohesive droplet.
- Discretely and codiscretely cohesive spaces embed in $\mathcal{H}$, with fully faithful functors.

Axiomatic cohesion

An adjunction

$$\Pi \dashv \mathrm{disc} \dashv \Gamma \dashv \mathrm{codisc} : \mathcal{H} \begin{tikzcd} \arrow[r, shift left=3, "\Pi"] & \arrow[l, shift left=1, "\mathrm{disc}"] \arrow[l, phantom, "\perp"] \end{tikzcd} \ldots$$

$$\Pi \dashv \mathrm{disc} \dashv \Gamma \dashv \mathrm{codisc} : \begin{tikzcd}[column sep=huge] \mathcal{H} \arrow[r, shift left=4.5, "\Pi"] \arrow[r, shift left=4.5, phantom, shift right=3, "\perp"] \arrow[r, shift right=1.5, "\Gamma" description] & \underline{\mathrm{Set}} \arrow[l, shift right=1.5, "\mathrm{disc}" description] \arrow[l, shift left=4.5, "\mathrm{codisc}"] \end{tikzcd}$$

exhibits the cohesion of $\mathcal{H}$ over $\underline{\mathbf{Set}}$ if

- disc and codisc are fully faithful;
- the leftmost adjoint Π preserves finite products.

(Γ "forgets cohesion": it sends a space to its underlying set of points)

Formal fact. Every quadruple of adjoints induces a triple of adjoints.

- There is an adjoint triple of idempotent co/monads on $\mathcal{H}$, induced by the cohesion:

$$
\begin{tikzcd}[column sep=huge]
\mathcal{H} \arrow[r, shift left=3ex, "\Pi"] \arrow[r, phantom, shift left=1.5ex, "\perp"] \arrow[r, phantom, shift right=1.5ex, "\perp"] \arrow[r, shift right=3ex, "\Gamma"'] & \underline{\mathrm{Set}} \arrow[l, "\mathrm{disc}" description] \arrow[r, shift left=3ex, "\mathrm{disc}"] \arrow[r, phantom, shift left=1.5ex, "\perp"] \arrow[r, phantom, shift right=1.5ex, "\perp"] \arrow[r, shift right=3ex, "\mathrm{codisc}"'] & \mathcal{H} \arrow[l, "\Gamma" description]
\end{tikzcd}
$$

monad	comonad	monad
$\int$	$(-)^\flat$	$(-)^\sharp$
$\mathrm{disc} \circ \Pi$	$\mathrm{disc} \circ \Gamma$	$\mathrm{codisc} \circ \Gamma$
pron.: *shape*	pron.: *flat*	pron.: *sharp*

Modalities, pieces

The triple of adjoints

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\flat"] & \mathcal{H} \arrow[l, bend right, "\int"'] \arrow[l, bend left, "\sharp"] \end{tikzcd}$$

is called the shape, flat, sharp string of "co/modalities" (idempotent co/monads) for the cohesive topos $\mathcal{H}$.

The shape of $X \in \mathcal{H}$ is the discrete object on the "fundamental groupoid" of X.

Idea. The adjunction $\Pi \dashv \mathrm{disc}$ has something to do with (topological) Galois theory.

Modalities, pieces

1. The flat functor corresponds to the **object of flat connections** on $X \in \mathcal{H}$: if G is a group,

$$\left\{ \begin{smallmatrix} \text{principal} \\ \text{bundles on } X \end{smallmatrix} \right\} \cong \left\{ X \longrightarrow BG \right\} \qquad \left\{ \begin{smallmatrix} \text{flat con-} \\ \text{nections on } X \end{smallmatrix} \right\} \cong \left\{ \begin{tikzcd} & BG^\flat \arrow[d] \\ X \arrow[ur, dotted] \arrow[r] & BG \end{tikzcd} \right\}$$

(keep in mind this equivalences: they will reappear later)

2. sharp of X, $X^\sharp$, corresponds to the codiscrete object on the sets of points ΓX of X.
3. Co/discrete objects are precisely the objects for which $X^\flat \cong X$, resp. $Y^\sharp \cong Y$.

Every object fits in a "complex":

Definition

There is a canonical natural trasformation

$$\sharp X \xrightarrow{\epsilon_{(\mathsf{disc}\dashv\Gamma),X}} X \xrightarrow{\eta_{(\Pi\dashv\mathsf{disc}),X}} \int X$$

called the "points to pieces" map; this map comes from a natural transformation

$$\alpha : \Gamma \Rightarrow \Pi$$
$$\alpha_X : \Gamma X \to \Pi X$$

It is a "comparison" between the action of Γ (send X into its "sections" or "set of points") and Π (send X into its "pieces" or "components").

- We say that **pieces have points** in the cohesive topos $\mathcal{H}$ (or that "$\mathcal{H}$ satisfies *Nullstellensatz*") if the points-to-pieces transformation $\alpha_X \colon \Gamma X \to \Pi X$ is surjective for all $X \in \mathcal{H}$.
- We say that **discrete is concrete** in $\mathcal{H}$ if natural transformation whose components are

$$\mathrm{disc}(S) \to \mathrm{codisc}(\Gamma(\mathrm{disc}(S))) \cong \mathrm{codisc}(S)$$

  is a monomorphism (discrete cohesion sits into codiscrete cohesion).
- We say that $\mathcal{H}$ **has contractible subobjects** or **has sufficient cohesion** if $\Pi(\Omega) \cong *$. This implies that for all $X \in \mathcal{H}$ also $\Pi(\Omega^X) \cong *$.
- ...and many others (see [Law]).

Example of modal truth

Definition

$\psi\colon S \hookrightarrow A$ in a cohesive topos $\mathcal{H}$ is a proposition of type A in the internal logic of $\mathcal{H}$. We say that ψ is *discretely true* if the pullback $\psi^*(S) \to A$

$$\begin{tikzcd}
\psi^*(S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \flat S \arrow[d, "\flat\psi"] \\
A \arrow[r, "\eta"'] & \flat A
\end{tikzcd}$$

is an isomorphism in $\mathcal{H}$, where $\eta : A \to \flat A$ is the $\flat$-unit of the flat monad.

Example of modal truth

- Discrete truth specifies a mode/modality in which a proposition can be true. Propositions true over all discrete objects (i.e., such that $\flat\psi$ is an iso) are discretely true.
- Let $\mathcal{H} = \mathrm{Sh}(\mathit{Cart}, J)$ be the topos of sheaves over cartesian spaces ($\mathrm{hom}(m, n) =$ smooth maps $\mathbb{R}^n \to \mathbb{R}^m$) is cohesive.
- Let $\psi \colon Z^p(U) \hookrightarrow \Omega^p(U)$ be the proposition in $\mathcal{H}$ given by "the p-form ω is closed on a neighbourhood $V_x \subseteq U$ of a point $x \in U$". Then ψ is discretely true ("every form is closed over a discrete space").

Examples

EX: The Sierpiński topos

Let $\mathcal{C} = \{0 \to 1\}$ be the interval category with a unique non-identity arrow.

The category $\mathcal{H} = \mathrm{Cat}(\mathcal{C}, \underline{\mathrm{Set}})$ exhibits cohesion: an object in $\mathcal{H}$ is an arrow in $\underline{\mathrm{Set}}$, and

- the functor Π sends an object $S \to I$ to its codomain I;
- the functor Γ sends an object $S \to I$ to its domain S;
- the functor disc sends a set K into the identity $1 \colon K \to K$;
- the functor codisc sends a set K into its terminal morphism $K \to *$.

EX: The Sierpiński topos

Evidently these functors form an adjunction ($\Pi \dashv \mathrm{disc} \dashv \Gamma \dashv \mathrm{codisc}$) so that $\mathcal{H}$ exhibits cohesion; this matches our intuition, in that

- The "points to pieces" transformation sends $f : S \to I$ into $S = \Gamma(f) \to \Pi(f) = I$;
- $\mathrm{disc}(K)$ "keeps all the pieces of K maximally distinguished" and
- $\mathrm{codisc}(K)$ "lumps all the pieces of K together".

EX: Pointed categories

Let $\mathcal{C}$ small and with a terminal object. Then there exists a triple

$$\begin{tikzcd}[\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{Set}}] \arrow[r, shift left=2, "\varinjlim"] \arrow[r, shift right=2, "\varprojlim"'] & \underline{\mathrm{Set}} \arrow[l, "\mathrm{const}" description]\end{tikzcd}$$

that extends to $\varprojlim \dashv \text{よ}$:

$$S \overset{\text{よ}}{\mapsto} \Big(c \mapsto \underline{\mathrm{Set}}(\mathcal{C}(*, c), S)\Big)$$

(Dually, if $\mathcal{C}$ has an initial object. . .)

Proposition
If $\mathcal{C}$ has both an initial and a terminal object (e.g. it is pointed) then $[\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{Set}}]$ exhibits cohesion with

$$(\varinjlim \dashv \mathrm{const} \dashv \varprojlim \dashv \text{よ}) \colon [\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{Set}}] \overset{\mathrm{const}}{\leftrightarrows} \underline{\mathrm{Set}}$$

EX: Simplicial sets

Proposition
Let $\mathbf{\Delta}$ be the simplex category having objects nonempty finite ordinals and morphisms monotone maps. The topos $\mathcal{H} = [\mathbf{\Delta}^{\mathrm{op}}, \underline{\mathrm{Set}}]$ exhibits cohesion, and in $\mathcal{H}$ pieces have points.

- $\Gamma = (-)_0$ sends a simplicial set X into its set of 0-simplices X_0
- $\Pi = \pi_0$ sends a simplicial set X into its set of connected components $\mathrm{coeq}\Big(X_1 \rightrightarrows X_0\Big)$.
- disc sends a set S into the constant simplicial set in S having constant set of simplices and identities as faces and degeneracies.
- codisc sends a set S into the simplicial set whose n-simplices are $(n+1)$-tuples of elements of S (and faces and degeneracies forget and add elements accordingly).

EX: Tangent cohesion

Consider the codomain fibration

$$\mathcal{C}^{\to} \xrightarrow{p} \mathcal{C}$$

of a finitely complete category $\mathcal{C}$, sending an arrow $f\colon X \to Y$ to its codomain. The fiber $p^{\leftarrow}(Y)$ is canonically isomorphic to the category $\mathcal{C}/Y$ of arrows over Y.

There exists a fibration $T\mathcal{C} \to \mathcal{C}$ having typical fiber the fiberwise abelianization of $\mathcal{C}/Y$, i.e. the category $\mathrm{Ab}(\mathcal{C}/Y)$ of abelian groups in $\mathcal{C}/Y$.

(hint: un/straighten the prestack $\mathcal{C} \to \mathrm{Cat}\colon Y \mapsto \mathrm{Ab}(\mathcal{C}/Y)$).

Proposition
If $\mathcal{C}$ is a topos over $\mathcal{S}$, then so is $T\mathcal{C}$; moreover, the projection $q\colon T\mathcal{C} \to \mathcal{C}$ creates co/limits.

Tangent cohesion

Proposition
Functor $\delta\colon T\mathcal{C} \to \mathcal{C}$ = domain projection. Has left adjoint the functor $\Omega\colon \mathcal{C} \to T\mathcal{C}$ that is also a *section* for q.

$\Omega(A)$ = the complex of differential forms on an internal abelian group $A \in \mathrm{Ab}(\mathcal{C}/X)$.

In classical differential geometry a leading theorem is that the co/tangent bundle to a smooth manifold is itself a smooth manifold. Here we can prove that

Fact
The tangent category to a cohesive topos is itself a cohesive topos.

Infinitesimal cohesion

Let $\mathcal{H}$ be cohesive. An infinitesimal thickening of $\mathcal{H}$ is a new cohesive topos $\widetilde{\mathcal{H}}$ linked to the previous by a quadruple of adjoints

$$\begin{tikzcd}
\mathcal{H} \arrow[d, hook, shift right=3, "(i_! \dashv i^* \dashv i_! \dashv i^!)"'] \arrow[d, hook, shift left=1] \\
\widetilde{\mathcal{H}} \arrow[u, shift right=1] \arrow[u, shift left=3]
\end{tikzcd}$$

such that $i_*, i_!$ are fully faithful and $i_!$ commutes with finite products.

If such a structure exists, $\mathcal{H}$ "exhibits **infinitesimal** cohesion".

Neighbourhoods of some spaces are "infintesimally extended around a single (global) point". Cohesive structure can be refined to capture this phenomenon.

Infinitesimal cohesion

- The cohesion exhibited by $\widetilde{\mathcal{H}}$ factors through that of $\mathcal{H}$, in that

$$(\Pi_{\widetilde{\mathcal{H}}} \dashv \mathrm{disc}_{\widetilde{\mathcal{H}}} \dashv \Gamma_{\widetilde{\mathcal{H}}})\colon\ \widetilde{\mathcal{H}} \begin{smallmatrix} \xrightarrow{i^*} \\ \xleftarrow{i_*} \\ \xrightarrow[i^!]{} \end{smallmatrix} \mathcal{H} \begin{smallmatrix} \xrightarrow{\Pi} \\ \xleftarrow{\mathrm{disc}} \\ \xrightarrow[\Gamma]{} \end{smallmatrix} \underline{\mathrm{Set}}$$

- **Infinitesimal** cohesion describes formally infinitesimally extended neighbourhoods: if the functor i^* is interpreted as a contraction of a fat point onto its singleton, then $X \in \widetilde{\mathcal{H}}$ is infinitesimal if $i^*(X) \cong *$. This motivates the fact that

$$\widetilde{\mathcal{H}}(*, X) \cong \widetilde{\mathcal{H}}(i_!(*), X) \cong \mathcal{H}(*, i^*(X)) \cong \mathcal{H}(*, *) \cong *$$

so that $\mathcal{H}$ sees X as a "small neighbourhood concentrated around a single point $*_X$".

Higher order cohesion: jet spaces

Most examples of infinitesimal cohesions come equipped with an infinite chain of thickening approximations.

Consider the infinitesimal shape modality $\Im := i_* i^*$
(it comes equipped with other two adjoints, $\Re \dashv \Im \dashv \&$)[2]

In several cases (like smooth manifolds) we have a **chain** of infinitesimal thickenings

$$\begin{tikzcd}
\widetilde{\mathcal{H}}_0 \arrow[r, hook, shift left=4.5] \arrow[r, hook, shift right=1.5, "i_{*,(0)}"'] & \widetilde{\mathcal{H}}_1 \arrow[l, shift right=1.5, "i^*_{(0)}"'] \arrow[l, shift left=-4.5] \arrow[r, hook, shift left=4.5] \arrow[r, hook, shift right=1.5, "i_{*,(1)}"'] & \widetilde{\mathcal{H}}_2 \arrow[l, shift right=1.5, "i^*_{(1)}"'] \arrow[l, shift left=-4.5] \arrow[r, hook, shift left=4.5] \arrow[r, hook, shift right=1.5, "i_{*,(2)}"'] & \cdots \arrow[l, shift right=1.5, "i^*_{(2)}"'] \arrow[l, shift left=-4.5] \arrow[r, hook, shift left=4.5] \arrow[r, hook, shift right=1.5, "i_{*,\#}"'] & \widetilde{\mathcal{H}}_\infty \arrow[l, shift right=1.5, "i^*_{\#}"'] \arrow[l, shift left=-4.5] \arrow[r, hook, shift left=4.5] \arrow[r, hook, shift right=1.5, "i_{*,(\infty)}"'] & \mathcal{H} \arrow[l, shift right=1.5, "i^*_{(\infty)}"'] \arrow[l, shift left=-4.5]
\end{tikzcd}$$

here we speak of a sequence of orders of differential structures.

[2] This is the same general fact inducing $\int \dashv \flat \dashv \sharp$ adjunction.

Higher order cohesion: jet spaces

Each of these approximations comes equipped with an *order k infinitesimal shape* modality $\Im^{(k)}X$ in a sequence

$$X \to \Im X = \Im^{(0)} X \to \Im^{(1)} X \to \Im^{(2)} X \to \cdots$$

Example: Every cohesive topos exhibits infinitesimal cohesion via its **tangent** cohesive topos. This cohesion extends to any order of differential structure ("cohesive jet spaces").

One can go way further, but the terminology becomes pretty dire:

Remark 2.2.13. The perspective of def. 2.2.12 has been highlighted in [Law91], where it is proposed (p. 7) that adjunctions of this form usefully formalize "many instances of the *Unity and Identity of Opposites*" that control Hegelian metaphysics [He1841].

[DCCT170811], 1040 pages of Hegel-ish mathematics

uses axiomatic cohesion of ∞-toposes to axiomatise string theory. With Aufhebung.

Supergeometry: rheonomy

We can speak of supergeometry and show that certain categories of supersmooth manifolds exhibit cohesion (but not over Set...):

$$\begin{tikzcd} \mathsf{SuperSmoothS} \arrow[d, shift right=3ex, "\Pi"'] \arrow[d, shift left=3ex, "\Gamma"] \\ \mathsf{SuperS} \arrow[u, shift right=1ex, "d"'] \arrow[u, shift left=5ex, "c"'] \end{tikzcd}$$

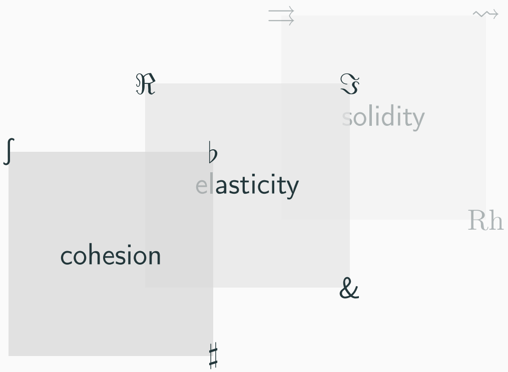
The quadruple of adjoints generates the triple

$$\rightrightarrows \quad \dashv \quad \rightsquigarrow \quad \dashv \quad \mathrm{Rh}$$

(in some sense “fermions” $\dashv$ “bosons”)

Supergeometry: rheonomy

There is a "quadruple-to-triple"pattern here:



de Rham cohomology in cohesion

- Let $\mathcal{H}$ be a cohesive topos, and $0 \to A$ a pointed object (e.g. an internal abelian group); then, A fits into a pullback square

$$\begin{tikzcd} \flat_{\mathrm{dR}} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \flat A \arrow[d] \\ 0 \arrow[r] & A \end{tikzcd}$$

where $\flat_{\mathrm{dR}} A$ is the object of coefficients for de Rham cohomology.

- Let $X \in \mathcal{H}$ any object; we define $\int_{\mathrm{dR}} X$ to be the pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ \int X \arrow[r] & \int_{\mathrm{dR}} X \end{tikzcd}$$

where $\int_{\mathrm{dR}} X$ is the de Rham object associated to X.

de Rham cohomology in cohesion

There is an adjunction

$$\ast/\mathcal{H} \underset{\int_{\mathsf{dR}}}{\overset{\flat_{\mathsf{dR}}}{\rightleftarrows}} \mathcal{H}$$

The mapping space $\ast/\mathcal{H}(\int_{\mathsf{dR}} X, A) \cong \mathcal{H}(X, \flat_{\mathsf{dR}} A)$ is called the de Rham space of X with coefficients in A and denoted $\mathbf{H}^0_{\mathsf{dR}}(X, A)$.

de Rham cohomology in cohesion

Consider the pullback defining $\flat_{\mathrm{dR}} A$ and apply the limit-preserving functor $\mathcal{H}(X, -)$: the square

$$\begin{tikzcd} \mathcal{H}(X, \flat_{\mathrm{dR}} A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{H}(X, \flat A) \arrow[d] \\ 0 \arrow[r] & \mathcal{H}(X, A) \end{tikzcd}$$

remains a pullback and the object $\mathcal{H}(X, \flat A)$ identifies to A-valued differential forms, and the maps $X \to \flat_{\mathrm{dR}} A$ are the flat ones: under the $\int \dashv \flat$ adjunction, a map $X \to \flat A$ mates to a smooth map $\int X \to A$, corresponding to a square $\begin{smallmatrix} X & \to & 0 \\ \downarrow & & \downarrow \\ \int X & \to & A \end{smallmatrix}$

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