Appendix

Theorem 2

Given a KELPS framework $< \boldsymbol{R}, \boldsymbol{Aux}, \boldsymbol{C} >$, initial state S_0 and sequence $ext_1, \ldots, ext_i, \ldots$ of sets of external events, suppose that the OS generates the sequences of sets $acts_1, \ldots, acts_i, \ldots$ of actions and $S_1, \ldots, S_i, \ldots$ of states. Then $\boldsymbol{R} \cup C_{pre}$ is true in $\boldsymbol{I} = \boldsymbol{Aux} \cup \boldsymbol{S}^* \cup \boldsymbol{ev}^*$ if, for every goal tree that is added to a goal state $G_i, i \geqslant 0$, the goal clause *true* is added to the same goal tree in some goal state $G_j, j \geqslant i$.

Proof

To show C_{pre} is true in $\boldsymbol{Aux} \cup \boldsymbol{S}^* \cup \boldsymbol{ev}^*$, it suffices to show C_{pre} is true in each $\boldsymbol{Aux} \cup S_i^* \cup ev_i^* \cup ev_{i+1}^*$. But this is ensured by step 4 of the OS.

To show $\boldsymbol{R}$ is true in $\boldsymbol{Aux} \cup \boldsymbol{S}^* \cup \boldsymbol{ev}^*$, we need to show that for every rule of the form $\forall X$ [*antecedent* $\rightarrow \exists Y$ *consequent*] in $\boldsymbol{R}$, whenever some instance *antecedent* σ of the *antecedent* is true in $\boldsymbol{I}$ then the corresponding instance *consequent* σ of the *consequent* is also true in $\boldsymbol{I}$. But if *antecedent* σ is true in $\boldsymbol{I}$, then *antecedent* σ becomes true at some time i in $\boldsymbol{Aux} \cup S_0^* \cup \ldots \cup S_i^* \cup ev_0^* \cup \ldots \cup ev_i^*$, and *consequent* σ is added as the root of a new goal tree to the current goal state G_i. Each disjunct *consequent*$_j$ σ whose temporal constraints are satisfiable in $\boldsymbol{Aux}$ is added as a child of the root node.

Clearly, *consequent*$_j$ σ implies *consequent* σ. So if *true* $\rightarrow$ *consequent*$_j$ σ is true in $\boldsymbol{I}$, then *consequent* σ is true in $\boldsymbol{I}$. The truth of *true* $\rightarrow$ *consequent*$_j$ σ in $\boldsymbol{I}$ follows from the more general fact that if a goal clause C is added in step 2 as a child of a goal clause C', then $C \rightarrow C'$ is true in $\boldsymbol{I}$.

Therefore, the existence of a goal state G_j where $i \leqslant j$ and *true* is added to the same goal tree as *consequent* σ in G_j implies that *consequent* σ is true at time j, and therefore *consequent* σ is true in $\boldsymbol{I}$.

The proof of Theorem 3 uses Lemma 2, which is proved using Lemma 1:

Lemma 1

For $i \geqslant 0$, let r be a rule in R_i. Then there exists a rule in $\boldsymbol{R}$ of the form *ear* $\wedge$ *con* $\rightarrow$ *consequent* and a substitution σ that grounds all and only the variables in *ear* such that

$$ear\ \sigma \text{ is } true \text{ in } \boldsymbol{Aux} \cup S_0^* \cup ev_0^* \ldots \cup ev_i^*$$

$$ear\ \sigma < con\ \sigma$$

$$con\ \sigma \rightarrow consequent\ \sigma \text{ is } r.$$

Proof

Let n be the number of applications of step 1 in the derivation of r. The proof is by induction on n.

Base case $n = 0$: Because r was derived by 0 applications of step 1, it follows that $r \in \boldsymbol{R}$. Then r has the form *ear* $\wedge$ *con* $\rightarrow$ *consequent,* where *ear* is empty (equivalent to *true*). Let σ be the empty substitution. Then

$$ear\ \sigma \text{ is } true \text{ in } \textbf{\textit{Aux}} \cup S_0^* \cup \ldots \cup S_i^* \cup ev_0^* \ldots \cup ev_i^*$$
$$ear\ \sigma < con\ \sigma$$
$$con\ \sigma \rightarrow consequent\ \sigma \text{ is } r.$$

This proves the base case.

Inductive step $n > 0$: Let r be added to some R_k by an application of step 1 of the OS to some rule r' in R_k, where $k \leqslant i$. By step 1 of the OS:
r' has the form $current \wedge later \rightarrow consequent$, where $current\ \theta < later\ \theta$,
r has the form $later\ \theta \rightarrow consequent\ \theta$,
$current\ \theta$ is *true* in $\textbf{\textit{Aux}} \cup S_k^* \cup ev_k^*$,
θ instantiates all and only the variables in *current*, and
θ instantiates all the timestamp variables in FOL conditions in *current* to k.
By the inductive hypothesis applied to r', there exists a rule r^* in $\textbf{\textit{R}}$ of the form $earlier \wedge curr \wedge rest \rightarrow conseq$ and a substitution σ
that grounds all and only the variables in *earlier* such that
$earlier\ \sigma$ is *true* in $\textbf{\textit{Aux}} \cup S_0^* \cup \ldots \cup S_k^* \cup ev_0^* \ldots \cup ev_k^*$,
$earlier\ \sigma < curr\ \sigma \wedge rest\ \sigma$,
current is $curr\ \sigma$ and *later* is $rest\ \sigma$.
Then
$earlier\ \sigma\ \theta \wedge curr\ \sigma\ \theta$ is *true* in $\textbf{\textit{Aux}} \cup S_0^* \cup \ldots \cup S_i^* \cup ev_0^* \ldots \cup ev_i^*$,
$earlier\ \sigma\ \theta \wedge curr\ \sigma\ \theta < rest\ \sigma\ \theta$,
$rest\ \sigma\ \theta \rightarrow conseq\ \sigma\ \theta$ is r. This proves the inductive step.

Lemma 2
For $i \geqslant 0$, let C be a goal clause in G_i. Then there exists a rule r in $\textbf{\textit{R}}$ of the form $antecedent \rightarrow [other \vee [earlier \wedge conds]]$ and a substitution σ that grounds all and only the variables in $antecedent \wedge earlier$ such that

$antecedent\ \sigma \wedge earlier\ \sigma$ is *true* in $\textbf{\textit{Aux}} \cup S_0^* \cup \ldots \cup S_i^* \cup ev_0^* \ldots \cup ev_i^*$,
$earlier\ \sigma < conds\ \sigma$ and,
$conds\ \sigma$ is C.

Proof
Let n be the number of applications of step 2 in the derivation of C. The proof is by induction on n, and is similar to that of Lemma 1.

Base case $n = 0$: If C is in G_0, then, by the definition of G_0, there exists a rule r of the form $true \rightarrow [other \vee [earlier \wedge C]]$ where *earlier* is empty, and r has the form required in the statement of the Lemma. If C is added in step 1 of the OS to $G_k, k \leqslant i$, then R_k contains a rule r of the form $true \rightarrow [other \vee C]$ where $other \vee C$ is a new root node added to G_k. As a consequence of Lemma 1, there exists a rule in $\textbf{\textit{R}}$ of the form $ear \wedge con \rightarrow consequent$ and a substitution σ that grounds all and only the variables in *ear* such that
$ear\ \sigma$ is *true* in $\textbf{\textit{Aux}} \cup S_0^* \cup \ldots \cup S_k^* \cup ev_0^* \ldots \cup ev_k^*$,
$ear\ \sigma < con\ \sigma$,

con $\sigma \rightarrow$ *consequent* σ is r. So
con σ is *true*, and *consequent* σ is *other* $\vee$ C.
Let *consequent* have the form [*alternatives* $\vee$ [*earlier* $\wedge$ *conds*]] where *earlier* is *true* and *conds* σ is C. Then σ grounds all and only the variables in *ear* $\wedge$ *con* $\wedge$ *earlier* and
ear $\sigma \wedge$ *con* $\sigma \wedge$ *earlier* σ is *true* in $\boldsymbol{Aux} \cup S_0^* \cup ... \cup S_i^* \cup ev_0^* \ldots \cup ev_i^*$
earlier $\sigma <$ *conds* σ
conds σ is C. This proves the base case.

Inductive step $n > 0$: Let C be added in step 2 of the OS to G_k as a child of a goal clause C', where C' is in G_k, $k \leqslant i$. By step 2 of the OS:
C' has the form *current* $\wedge$ *later*, where *current* $\theta <$ *later* θ ,
C has the form *later* θ,
current θ is *true* in $\boldsymbol{Aux} \cup S_k^* \cup ev_k^*$,
θ instantiates all and only the variables in *current*, and
θ instantiates all the timestamp variables in FOL conditions in *current* to k.
By the inductive hypothesis applied to C', there exists a rule r in **R** of the form
antecedent $\rightarrow$ [*other* $\vee$ [*earlier* $\wedge$ *curr* $\wedge$ *rest*]] and a substitution σ
that grounds all and only the variables in *antecedent* $\wedge$ *earlier* such that
antecedent $\sigma \wedge$ *earlier* σ is *true* in $\boldsymbol{Aux} \cup S_0^* \cup \ldots \cup S_k^* \cup ev_0^* \ldots \cup ev_k^*$,
earlier $\sigma <$ *curr* $\sigma \wedge$ *rest* σ,
current is *curr* σ and *later* is *rest* σ.
Then
antecedent $\sigma\ \theta \wedge$ *earlier* $\sigma\ \theta \wedge$ *curr* $\sigma\ \theta$,
is *true* in $\boldsymbol{Aux} \cup S_0^* \cup \ldots \cup S_i^* \cup ev_0^* \ldots \cup ev_i^*$,
earlier $\sigma\ \theta \wedge$ *curr* $\sigma\ \theta <$ *rest* $\sigma\ \theta$,
rest $\sigma\ \theta$ is C. This proves the inductive step.

Theorem 3
Given a range restricted KELPS framework $<\boldsymbol{R}, \boldsymbol{Aux}, \boldsymbol{C}>$, initial state S_0 and set of external events $\boldsymbol{ext}^*$, let $\boldsymbol{acts}^*$ be the set of actions generated by the OS, and $\boldsymbol{ev}^* = \boldsymbol{ext}^* \cup \boldsymbol{acts}^*$. Then $\boldsymbol{I} = \boldsymbol{Aux} \cup \boldsymbol{S}^* \cup \boldsymbol{ev}^*$ is a reactive interpretation.

Proof
Assume that, for $i \geqslant 0$, an action *action* τ is added to $candidate\text{-}acts_{i+1}$ in step 3 and included in $acts_{i+1}$ in step 4 of the OS at time i. It follows that there exists a sequencing *action* $\tau \leqslant$ *rest* τ of an instance of a goal clause *action* $\wedge$ *rest* in G_i, where τ instantiates only the timestamp variable in *action* to the time $i+1$.

By Lemma 2 there exists a rule r in $\boldsymbol{R}$ of the form *antecedent* $\rightarrow$ [*other* $\vee$ [*earlier* $\wedge$ *conds1* $\wedge$ *conds2*]] and a substitution σ that grounds all and only the variables in *antecedent* $\wedge$ *earlier* such that
antecedent $\sigma \wedge$ *earlier* σ is *true* in $\boldsymbol{Aux} \cup S_0^* \cup \ldots \cup S_i^* \cup ev_0^* \ldots \cup ev_i^*$,
conds1 σ is *action,*
conds2 σ is *rest,*
earlier $\sigma <$ *action* $\wedge$ *rest.*

It follows that $r\,\sigma\,\tau$ *supports action* τ, in the sense that:

(a) *action* τ is *conds1* $\sigma\ \tau$,

(b) *antecedent* $\sigma\ \tau\ \wedge$ *earlier* $\sigma\ \tau <$ *conds*1 $\sigma\ \tau \wedge$ *conds*2 $\sigma\ \tau$,

(c) *antecedent* $\sigma\ \tau\ \wedge$ *earlier* $\sigma\ \tau\ \wedge$ *conds*1 $\sigma\ \tau$ is *true* in $\boldsymbol{I}$.

Moreover, step 4 ensures that C_{pre} is *true* in $\boldsymbol{Aux} \cup S_i^* \cup ev_{i+1}^*$. Therefore, C_{pre} is *true* in $\boldsymbol{I}$. Therefore, $\boldsymbol{I}$ is reactive. End of proof.

Theorem 4

Given a range restricted KELPS framework $< \boldsymbol{R}, \boldsymbol{Aux}, \boldsymbol{C} >$, initial state S_0 and external events $\boldsymbol{ext}^*$, let $\boldsymbol{acts}^*$ be a set of actions such that $\boldsymbol{I} = \boldsymbol{Aux} \cup \boldsymbol{S}^* \cup \boldsymbol{ev}^*$, where $\boldsymbol{ev}^* = \boldsymbol{ext}^* \cup \boldsymbol{acts}^*$, is a reactive interpretation. Then there exist choices in steps 2, 3, and 4 such that the OS generates $\boldsymbol{acts}^*$ (and therefore generates $\boldsymbol{I}$).

Proof

Let $\boldsymbol{R^I} = \{(r, \sigma, t) \mid r\ \sigma$ supports an action act_t at time $t\}$. We show by induction on i that for all times $i \geqslant 0$, there exist choices in steps 2, 3, and 4 such that

(1) For all $(r,\ \sigma,\ t) \in \boldsymbol{R^I}$, if $i \leqslant t$ then, at the beginning of the OS cycle at time i, either (a) there exists a reactive rule $r_i \in R_i$ such that
- r has the form *earlier* $\wedge$ *later* $\rightarrow$ *consequent*,
- *earlier* σ is true in $\boldsymbol{Aux} \cup S_0^* \cup \ldots \cup S_{i\text{-}1}^* \cup ev_0^* \ldots \cup ev_{i\text{-}1}^*$,
- *later* $\sigma \rightarrow$ *consequent* σ is an instance of r_i and
- *earlier* $\sigma <$ *later* σ,

  or (b) there exists a goal clause C_i in G_i such that
- r has the form *antecedent* $\rightarrow$ [*other* $\vee$ [*early* $\wedge$ *late*]],
- *antecedent* $\sigma \wedge$ *early* σ is true in $\boldsymbol{Aux} \cup S_0^* \cup \ldots \cup S_{i\text{-}1}^* \cup ev_0^* \ldots \cup ev_{i\text{-}1}^*$,
- *late* σ is an instance of C_i and
- *antecedent* $\sigma \wedge$ *early* $\sigma <$ *late* σ.

(2) At the end of the OS cycle at time *i-1*, the OS has chosen in step 4 all and only the actions in $acts_i^*$. Clearly, (2) implies the statement of the Theorem.

Let $i = 0$ and $(r,\ \sigma,\ t) \in \boldsymbol{R^I}$. If r has the form *true* $\rightarrow$ [*other* $\vee$ [*earlier* $\wedge$ *act* $\wedge$ *rest*]], where $r\ \sigma$ supports *act* σ, then *early* $\wedge$ *earlier* $\wedge$ *act* $\wedge$ *rest*, where *early* is empty (i.e. *true*), is the desired goal clause C_0 in G_0. Otherwise, r has the form *later* $\rightarrow$ *consequent*, where *later* is not empty, which has the same form as *earlier* $\wedge$ *later* $\rightarrow$ *consequent,* where *earlier* is empty. This is the desired reactive rule $r_0 \in R_0$. So case (1a) holds. (2) also holds, because there are no actions before time 1.

Let $i > 0$ and assume that (1) holds (at the beginning of the cycle at time *i-1*) and that (2) holds (at the end of cycle at time *i-2*). To show that (1) holds at time i, let $(r,\ \sigma,\ t) \in \boldsymbol{R^I}$ where $i \leqslant t$. By the induction hypothesis, either (1a) or (1b) holds for $(r,\ \sigma,\ t)$ at time *i-1*. Suppose first that (1a) holds at time *i-1*. Then there exists a reactive rule $r_{i\text{-}1} \in R_{i\text{-}1}$ such that

- r has the form *earlier* $\wedge$ *later* $\rightarrow$ *consequent*,
- *earlier* σ is true in $\boldsymbol{Aux} \cup S_0^* \cup \ldots \cup S_{i\text{-}2}^* \cup ev_0^* \ldots \cup ev_{i\text{-}2}^*$,
- *later* $\sigma \rightarrow$ *consequent* σ is an instance of $r_{i\text{-}1}$ and
- *earlier* $\sigma <$ *later* σ.

If no timestamp in *later* σ is equal to *i*-1, then $r_{i\text{-}1}$ persists until the end of the cycle, becomes the desired r_i at the beginning of the next cycle, and (1a) holds for (r, σ, t) at time i. Otherwise, *later* has the form *current* $\wedge$ *rest* where *current* σ is true in $\boldsymbol{Aux} \cup S^*_{i\text{-}1} \cup ev^*_{i\text{-}1}$ and *current* σ < *rest* σ. Then step 1 of the OS must evaluate the FOL conditions and temporal constraints in $r_{i\text{-}1}$ that have *current* σ as an instance, generating a rule $r_i \in R_{i\text{-}1}$ such that *rest* $\sigma \rightarrow$ *consequent* σ is an instance of r_i. Therefore, $r_i \in R_{i\text{-}1}$ is such that

- r has the form *earlier* $\wedge$ *current* $\wedge$ *rest* $\rightarrow$ *consequent*,
- *earlier* $\sigma \wedge$ *current* σ is true in $\boldsymbol{Aux} \cup S^*_0 \cup \ldots \cup S^*_{i\text{-}1} \cup ev^*_0 \ldots \cup ev^*_{i\text{-}1}$,
- *rest* $\sigma \rightarrow$ *consequent* σ is an instance of r_i and
- *earlier* $\sigma \wedge$ *current* σ < *rest* σ.

If *rest* σ is not empty, then r_i persists until the end of the cycle, becomes the desired r_i at beginning of the next cycle, and (1a) holds for (r, σ, t) at time i.

If *rest* σ is empty, then the OS deletes r_i from $R_{i\text{-}1}$ and adds a new goal tree to $G_{i\text{-}1}$ with root node having *consequent* σ as an instance. Because r σ supports some action act_t at time t where $i\text{-}1 \leqslant t$, then r has the form *antecedent* $\rightarrow$ [*other* $\vee$ [*conclusion*]] where act_t is a bare action conjunct of *conclusion*. Then the OS adds to $G_{i\text{-}1}$ a goal clause C as a child of the new root node such that

- r has the form *antecedent* $\rightarrow$ [*other* $\vee$ [*conclusion*]],
- *antecedent* σ is true in $\boldsymbol{Aux} \cup S^*_0 \cup \ldots \cup S^*_{i\text{-}1} \cup ev^*_0 \ldots \cup ev^*_{i\text{-}1}$,
- *conclusion* σ is an instance of C and
- *antecedent* $\sigma \leqslant$ *conclusion* σ.

If no timestamp in FOL conditions in *conclusion* σ is equal to *i*-1, then rewrite *conclusion* as *early* $\wedge$ *late* where *early* is empty. Then

- r has the form *antecedent* $\rightarrow$ [*other* $\vee$ [*early* $\wedge$ *late*]],
- *antecedent* σ $\wedge$ *early* σ is true in $\boldsymbol{Aux} \cup S^*_0 \cup \ldots \cup S^*_{i\text{-}1} \cup ev^*_0 \ldots \cup ev^*_{i\text{-}1}$,
- *late* σ is an instance of C and
- *antecedent* $\sigma \wedge$ *early* σ < *late* σ.

C persists until the end of the cycle, becomes the desired C_i at the beginning of the next cycle, and (1b) holds for (r, σ, t) at time i.

Otherwise, *conclusion* has the form *early* $\wedge$ *late* where *early* is not empty, *early* σ is true in $\boldsymbol{Aux} \cup S^*_{i\text{-}1} \cup ev^*_{i\text{-}1}$, and *early* σ < *late* σ. Let the OS in step 2 choose and evaluate the FOL conditions and temporal constraints in C that have *early* σ as an instance, generating a goal clause C_i in $G_{i\text{-}1}$ such that *late* σ is an instance of C_i. Then

- r has the form *antecedent* $\rightarrow$ [*other* $\vee$ [*early* $\wedge$ *late*]],
- *antecedent* σ $\wedge$ *early* σ is true in $\boldsymbol{Aux} \cup S^*_0 \cup \ldots \cup S^*_{i\text{-}1} \cup ev^*_0 \ldots \cup ev^*_{i\text{-}1}$,
- *late* σ is an instance of C_i and
- *antecedent* σ $\wedge$ *early* σ < *late* σ.

C_i persists until the end of the cycle, becomes the desired C_i at the beginning of the next cycle; and (1b) holds for (r, σ, t) at time i.

Suppose instead that the induction hypothesis holds for (1b). Then there exists a goal clause $C_{i\text{-}1}$ in $G_{i\text{-}1}$ such that

- r has the form *antecedent* → [*other* ∨ [*early* ∧ *late*]],
- *antecedent* σ ∧ *early* σ is true in $\boldsymbol{Aux} \cup S^*_0 \cup \ldots \cup S^*_{i\text{-}2} \cup ev^*_0 \ldots \cup ev^*_{i\text{-}2}$,
- *late* σ is an instance of $C_{i\text{-}1}$ and
- *antecedent* σ ∧ *early* σ < *late* σ.

If no timestamp in FOL conditions in *late* σ is equal to i-1, then $C_{i\text{-}1}$ persists until the end of the cycle, becomes the desired C_i at the beginning of the next cycle; and (1b) holds for (r, σ, t) at time i.

Otherwise *late* has the form *current* ∧ *rest* where *current* is not empty, *current* σ is true in $\boldsymbol{Aux} \cup S^*_{i\text{-}1} \cup ev^*_{i\text{-}1}$, and *current* σ < *rest* σ. Let the OS in step 2 choose and evaluate the FOL conditions and temporal constraints in $C_{i\text{-}1}$ that have *current* σ as an instance, generating a goal clause C_i in $G_{i\text{-}1}$ such that *rest* σ is an instance of C_i. Then

- r has the form *antecedent* → [*other* ∨ [*early* ∧ *current* ∧ *rest*]],
- *antecedent* σ ∧ *early* σ ∧ *current* σ is true in $\boldsymbol{Aux} \cup S^*_0 \cup \ldots \cup S^*_{i\text{-}1} \cup ev^*_0 \ldots \cup ev^*_{i\text{-}1}$,
- *rest* σ is an instance of C_i and
- *antecedent* σ ∧ *early* σ ∧ *current* σ < *rest* σ.

C_i persists until the end of the cycle, becomes the desired C_i at the beginning of the next cycle; and (1b) holds for (r, σ, t) at time i.

To show that (2) holds at time i, we need to ensure that steps 3 and 4 of the OS can choose act_i if $(r, \sigma, i) \in R^I$. But this follows from (1b), which ensures that if r has the form *antecedent* → [*other* ∨ [*earlier* ∧ *action* ∧ *rest*]] where *action* $\sigma = act_i$ and r σ *supports* act_i, then there exists a goal clause $C_{i\text{-}1}$ in $G_{i\text{-}1}$ such that *action* σ ∧ *rest* σ is an instance of $C_{i\text{-}1}$. It is easy to see that step 3 can include act_i in *candidate-acts*$_i$. Because C_{pre} is true in $\boldsymbol{I}$, step 4 of the OS can choose act_i among the actions generated at the end of the cycle. Moreover, for any other bare action atom *act* in a goal clause in $G_{i\text{-}1}$ (whether $i \leqslant t$ or $i > t$ for all $(r, \sigma, t) \in R^I$), whether or not step 3 chooses *act*, step 4 should not choose *act*; and this is possible because $\boldsymbol{I}$ satisfies C_{pre}.