Electronic Supplementary Material Appendices

A Filtering algorithms: Subtraction and Multiplication

A.1 Subtraction

Here we deal with constraints of the form $x = y \boxminus_S z$.

Assume $X = [x_\ell, x_u]$, $Y = [y_\ell, y_u]$ and $Z = [z_\ell, z_u]$.

Again, thanks to Proposition 2 we need not be concerned with sets of rounding modes, as any such set $S \subseteq R$ can always be mapped to a pair of "worst-case rounding modes" which, in addition are never round-to-zero.

Direct Propagation. For direct propagation, we use Algorithm 6 and functions ds_ℓ and ds_u, as defined in Figure 12.

Algorithm 6 Direct projection for subtraction constraints.

Require: $x = y \boxminus_S z$, $x \in X = [x_\ell, x_u]$, $y \in Y = [y_\ell, y_u]$ and $z \in Z = [z_\ell, z_u]$.

Ensure: $X' \subseteq X$ and $\forall r \in S, x \in X, y \in Y, z \in Z : x = y \boxminus_r z \implies x \in X'$ and $\forall X'' \subset X, \exists r \in S, y \in Y, z \in Z : y \boxminus_r z \notin X''$.

1: $r_\ell := r_\ell(S, y_\ell, \boxminus, z_u)$; $r_u := r_u(S, y_u, \boxminus, z_\ell)$;
2: $x'_\ell := \mathrm{ds}_\ell(y_\ell, z_u, r_\ell)$; $x'_u := \mathrm{ds}_u(y_u, z_\ell, r_u)$;
3: $X' := X \cap [x'_\ell, x'_u]$;

$\mathrm{ds}_\ell(y_\ell, z_u, r_\ell)$	$-\infty$	$\mathbb{R}_-$	-0	$+0$	$\mathbb{R}_+$	$+\infty$
$-\infty$	$+\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\mathbb{R}_-$	$+\infty$	$y_\ell \boxminus_{r_\ell} z_u$	y_ℓ	y_ℓ	$y_\ell \boxminus_{r_\ell} z_u$	$-\infty$
-0	$+\infty$	$-z_u$	a_1	-0	$-z_u$	$-\infty$
$+0$	$+\infty$	$-z_u$	$+0$	a_1	$-z_u$	$-\infty$
$\mathbb{R}_+$	$+\infty$	$y_\ell \boxminus_{r_\ell} z_u$	y_ℓ	y_ℓ	$y_\ell \boxminus_{r_\ell} z_u$	$-\infty$
$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$

$$a_1 = \begin{cases} -0, & \text{if } r_\ell = \downarrow, \\ +0, & \text{otherwise;} \end{cases}$$

$\mathrm{ds}_u(y_u, z_\ell, r_u)$	$-\infty$	$\mathbb{R}_-$	-0	$+0$	$\mathbb{R}_+$	$+\infty$
$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\mathbb{R}_-$	$+\infty$	$y_u \boxminus_{r_u} z_\ell$	y_u	y_u	$y_u \boxminus_{r_u} z_\ell$	$-\infty$
-0	$+\infty$	$-z_\ell$	a_2	-0	$-z_\ell$	$-\infty$
$+0$	$+\infty$	$-z_\ell$	$+0$	a_2	$-z_\ell$	$-\infty$
$\mathbb{R}_+$	$+\infty$	$y_u \boxminus_{r_u} z_\ell$	y_u	y_u	$y_u \boxminus_{r_u} z_\ell$	$-\infty$
$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$-\infty$

$$a_2 = \begin{cases} -0, & \text{if } r_u = \downarrow, \\ +0, & \text{otherwise.} \end{cases}$$

Fig. 12 Direct projection of subtraction: function ds_ℓ (resp., ds_u); values for y_ℓ (resp., y_u) on rows, values for z_u (resp., z_ℓ) on columns.

Theorem 6 Algorithm 6 *satisfies its contract.*

$\mathrm{is}^f_\ell(x_\ell, z_\ell, \bar{r}_\ell)$	$-\infty$	$\mathbb{R}_-$	-0	$+0$	$\mathbb{R}_+$	$+\infty$
$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\mathbb{R}_-$	$-f_{\max}$	a_3	x_ℓ	x_ℓ	a_3	unsat.
-0	$-f_{\max}$	z_ℓ	-0	-0	z_ℓ	unsat.
$+0$	$-f_{\max}$	a_4	a_5	a_4	a_4	unsat.
$\mathbb{R}_+$	$-f_{\max}$	a_3	x_ℓ	x_ℓ	a_3	unsat.
$+\infty$	$-f_{\max}$	a_6	$+\infty$	$+\infty$	$+\infty$	unsat.

$$e_\ell \equiv x_\ell + \nabla_2^{\mathrm{n}-}(x_\ell)/2 + z_\ell;$$

$$a_3 = \begin{cases} -0, & \text{if } \bar{r}_\ell = \mathrm{n},\ \nabla_2^{\mathrm{n}-}(x_\ell) = -f_{\min} \text{ and } x_\ell = -z_\ell; \\ x_\ell \boxplus_\uparrow z_\ell, & \text{if } \bar{r}_\ell = \mathrm{n},\ \nabla_2^{\mathrm{n}-}(x_\ell) = -f_{\min} \text{ and } x_\ell \neq -z_\ell; \\ [\![e_\ell]\!]_\uparrow, & \text{if } \bar{r}_\ell = \mathrm{n},\ \mathrm{even}(x_\ell),\ \nabla_2^{\mathrm{n}-}(x_\ell) \neq -f_{\min} \text{ and } [\![e_\ell]\!]_\uparrow = [e_\ell]_\uparrow; \\ [\![e_\ell]\!]_\downarrow, & \text{if } \bar{r}_\ell = \mathrm{n},\ \mathrm{even}(x_\ell),\ \nabla_2^{\mathrm{n}-}(x_\ell) \neq -f_{\min} \text{ and } [\![e_\ell]\!]_\uparrow > [e_\ell]_\uparrow; \\ \mathrm{succ}\big([\![e_\ell]\!]_\downarrow\big), & \text{if } \bar{r}_\ell = \mathrm{n}, \text{ otherwise}; \\ -0, & \text{if } \bar{r}_\ell = \downarrow \text{ and } x_\ell = -z_\ell; \\ x_\ell \boxplus_\uparrow z_\ell, & \text{if } \bar{r}_\ell = \downarrow \text{ and } x_\ell \neq -z_\ell; \\ \mathrm{succ}\big(\mathrm{pred}(x_\ell) \boxplus_\downarrow z_\ell\big), & \text{if } \bar{r}_\ell = \uparrow; \end{cases}$$

$$(a_4, a_5) = \begin{cases} (\mathrm{succ}(z_\ell), +0), & \bar{r}_\ell = \downarrow; \\ (z_\ell, -0), & \text{otherwise}; \end{cases}$$

$$a_6 = \begin{cases} +\infty, & \bar{r}_\ell = \downarrow; \\ \mathrm{succ}(f_{\max} \boxplus_\downarrow z_\ell), & \bar{r}_\ell = \uparrow; \\ f_{\max} \boxplus_\uparrow \big(\nabla_2^{\mathrm{n}+}(f_{\max})/2 \boxplus_\uparrow z_\ell\big), & \text{otherwise}. \end{cases}$$

Fig. 13 First inverse projection of subtraction: function is^f_ℓ.

Inverse Propagation. For inverse propagation, we have to deal with two different cases depending on which variable we are computing: the first inverse projection on y or the second inverse projection on z.

The first inverse projection of subtraction is somehow similar to the direct projection of addition. In this case we define Algorithm 7 and functions is^f_ℓ and is^f_u, as defined in Figure 13 and 14 respectively.

Algorithm 7 First inverse projection for subtraction constraints.

Require: $x = y \boxminus_S z$, $x \in X = [x_\ell, x_u]$, $y \in Y = [y_\ell, y_u]$ and $z \in Z = [z_\ell, z_u]$.
Ensure: $Y' \subseteq Y$ and $\forall r \in S, x \in X, y \in Y, z \in Z : x = y \boxminus_r z \implies y \in Y'$.
1: $\bar{r}_\ell := \bar{r}^\ell_\ell(S, x_\ell, \boxminus, z_\ell)$; $\bar{r}_u := \bar{r}^\ell_u(S, x_u, \boxminus, z_u)$;
2: $y'_\ell := \mathrm{is}^f_\ell(x_\ell, z_\ell, \bar{r}_\ell)$; $y'_u := \mathrm{is}^f_u(x_u, z_u, \bar{r}_u)$;
3: **if** $y'_\ell \in \mathbb{F}$ and $y'_u \in \mathbb{F}$ **then**
4: $\quad Y' := Y \cap [y'_\ell, y'_u]$;
5: **else**
6: $\quad Y' := \varnothing$;
7: **end if**

Theorem 7 Algorithm 7 *satisfies its contract.*

The second inverse projection of subtraction is quite similar to the case of direct projection of subtraction. Here we define Algorithm 8 and functions is^s_ℓ and is^s_u, as defined in Figures 15 and 16 respectively.

Theorem 8 Algorithm 8 *is correct.*

Since subtraction is very closely related to addition, the proofs of Theorems 7 and 8 can be obtained by reasoning in the same way as for the projections of addition. Moreover, it is worth noting that in order to obtain more precise results, inverse projections for subtraction need to be intersected with maximum ULP filtering [5], as in the case of addition.

$\mathrm{is}_u^f(x_u, z_u, \bar{r}_u)$	$-\infty$	$\mathbb{R}_-$	-0	$+0$	$\mathbb{R}_+$	$+\infty$
$-\infty$	unsat.	$-\infty$	$-\infty$	$-\infty$	a_9	$f_{\max}$
$\mathbb{R}_-$	unsat.	a_7	x_u	x_u	a_7	$f_{\max}$
-0	unsat.	a_8	a_8	a_8	a_8	$f_{\max}$
$+0$	unsat.	z_u	$+0$	$+0$	z_u	$f_{\max}$
$\mathbb{R}_+$	unsat.	a_7	x_u	x_u	a_7	$f_{\max}$
$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$

$$e_u \equiv x_u + \nabla_2^{\mathrm{n}+}(x_u)/2 + z_u;$$

$$a_7 = \begin{cases} +0, & \text{if } \bar{r}_u = \mathrm{n},\ \nabla_2^{\mathrm{n}+}(x_u) = f_{\min} \text{ and } x_u = -z_u; \\ x_u \boxplus_\downarrow z_u, & \text{if } \bar{r}_u = \mathrm{n},\ \nabla_2^{\mathrm{n}+}(x_u) = f_{\min} \text{ and } x_u \neq -z_u; \\ [\![e_u]\!]_\downarrow, & \text{if } \bar{r}_u = \mathrm{n},\ \mathrm{even}(x_u),\ \nabla_2^{\mathrm{n}+}(x_u) \neq f_{\min} \text{ and } [\![e_u]\!]_\downarrow = [e_u]_\downarrow; \\ [\![e_u]\!]_\uparrow, & \text{if } \bar{r}_u = \mathrm{n},\ \mathrm{even}(x_u),\ \nabla_2^{\mathrm{n}+}(x_u) \neq f_{\min} \text{ and } [\![e_u]\!]_\downarrow < [e_u]_\downarrow; \\ \mathrm{pred}\left([\![e_u]\!]_\uparrow\right), & \text{if } \bar{r}_u = \mathrm{n}, \text{ otherwise}; \\ \mathrm{pred}\left(\mathrm{succ}(x_u) \boxplus_\uparrow z_u\right), & \text{if } \bar{r}_u = \downarrow; \\ +0, & \text{if } \bar{r}_u = \uparrow \text{ and } x_u = -z_u; \\ x_u \boxplus_\downarrow z_u, & \text{if } \bar{r}_u = \uparrow \text{ and } x_u \neq -z_u; \end{cases}$$

$$a_8 = \begin{cases} z_u, & \text{if } \bar{r}_u = \downarrow; \\ \mathrm{pred}(z_u), & \text{otherwise}; \end{cases}$$

$$a_9 = \begin{cases} -\infty, & \text{if } \bar{r}_u = \uparrow; \\ \mathrm{pred}(z_u \boxplus_\uparrow -f_{\max}), & \text{if } \bar{r}_u = \downarrow; \\ -f_{\max} \boxplus_\downarrow \left(\nabla_2^{\mathrm{n}-}(-f_{\max})/2 \boxplus_\downarrow z_u\right), & \text{otherwise.} \end{cases}$$

Fig. 14 First inverse projection of subtraction: function is_u^f.

Algorithm 8 Second inverse projection for subtraction constraints.

Require: $x = y \boxminus_S z$, $x \in X = [x_\ell, x_u]$, $y \in Y = [y_\ell, y_u]$ and $z \in Z = [z_\ell, z_u]$.
Ensure: $Z' \subseteq Z$ and $\forall r \in S, x \in X, y \in Y, z \in Z : x = y \boxminus_r z \implies z \in Z'$.

$\bar{r}_\ell := \bar{r}_\ell^r(S, x_u, \boxminus, y_\ell)$; $\bar{r}_u := \bar{r}_u^r(S, x_\ell, \boxminus, y_u)$;
$z'_\ell := \mathrm{is}_\ell^s(y_\ell, x_u, \bar{r}_\ell)$; $z'_u := \mathrm{is}_u^s(y_u, x_\ell, \bar{r}_u)$;
if $z'_\ell \in \mathbb{F}$ and $z'_u \in \mathbb{F}$ **then**
    $Z' := Z \cap [z'_\ell, z'_u]$;
else
    $Z' := \varnothing$;
end if

A.2 Multiplication

Here we deal with constraints of the form $x = y \boxdot_S z$. As usual, assume $X = [x_\ell, x_u]$, $Y = [y_\ell, y_u]$ and $Z = [z_\ell, z_u]$.

Direct Propagation. For direct propagation, a case analysis is performed in order to select the interval extrema y_L and z_L (resp., y_U and z_U) to be used to compute the new lower (resp., upper) bound for x.

Firstly, whenever $\mathrm{sgn}(y_\ell) \neq \mathrm{sgn}(y_u)$ and $\mathrm{sgn}(z_\ell) \neq \mathrm{sgn}(z_u)$, there is no unique choice for y_L and z_L (resp., y_U and z_U); therefore we need to compute the two candidate lower (and upper) bounds for x and then choose the minumum (the maximum, resp).

The choice is instead unique in all cases where the signs of one among y and z, or both of them, are constant over the respective intervals. Function σ of Figure 7 determines the extrema of y and z useful to compute the new lower (resp., upper) bound for y when the sign of z is constant. When the sign of y is constant, the appropriate choice for the extrema of y and z can be determined by swapping the role of y and z in function σ.

Once the extrema (y_L, y_U, z_L, z_U) have been selected, functions dm_ℓ and dm_u of Figure 17 are used to find new bounds for x. It is worth noting that it is not necessary to compute new values of r_ℓ and r_u for the application of functions dm_ℓ and dm_u at line 6 of Algorithm 9. This is true because, by Definition 7, the

$\mathrm{is}^s_\ell(y_\ell, x_u, \bar{r}_\ell)$	$-\infty$	$\mathbb{R}_-$	-0	$+0$	$\mathbb{R}_+$	$+\infty$
$-\infty$	$-f_{\max}$	$-f_{\max}$	$-f_{\max}$	$-f_{\max}$	$-f_{\max}$	$-\infty$
$\mathbb{R}_-$	a_{13}	a_{10}	a_{11}	y_ℓ	a_{10}	$-\infty$
-0	$+\infty$	$-x_u$	a_{12}	-0	$-x_u$	$-\infty$
$+0$	$+\infty$	$-x_u$	a_{11}	-0	$-x_u$	$-\infty$
$\mathbb{R}_+$	$+\infty$	a_{10}	a_{11}	y_ℓ	a_{10}	$-\infty$
$+\infty$	unsat.	unsat.	unsat.	unsat.	unsat.	$-\infty$

$$e_\ell \equiv y_\ell - \big(x_u + \nabla_2^{\mathrm{n}+}(x_u)/2\big);$$

$$a_{10} = \begin{cases} -0, & \text{if } \bar{r}_\ell = \mathrm{n},\ \nabla_2^{\mathrm{n}+}(x_u) = f_{\min} \text{ and } x_u = y_\ell; \\ y_\ell \boxminus_\uparrow x_u, & \text{if } \bar{r}_\ell = \mathrm{n},\ \nabla_2^{\mathrm{n}+}(x_u) = f_{\min} \text{ and } x_u \neq y_\ell; \\ [\![e_\ell]\!]_\uparrow, & \text{if } \bar{r}_\ell = \mathrm{n},\ \mathrm{even}(x_u),\ \nabla_2^{\mathrm{n}+}(x_u) \neq f_{\min} \text{ and } [\![e_\ell]\!]_\uparrow = [e_\ell]_\uparrow; \\ [\![e_\ell]\!]_\downarrow, & \text{if } \bar{r}_\ell = \mathrm{n},\ \mathrm{even}(x_u),\ \nabla_2^{\mathrm{n}+}(x_u) \neq f_{\min} \text{ and } [\![e_\ell]\!]_\uparrow > [e_\ell]_\uparrow; \\ \mathrm{succ}\big([\![e_\ell]\!]_\downarrow\big), & \text{if } \bar{r}_\ell = \mathrm{n}, \text{ otherwise}; \\ -0, & \text{if } \bar{r}_\ell = \uparrow \text{ and } x_u = y_\ell; \\ y_\ell \boxminus_\uparrow x_u, & \text{if } \bar{r}_\ell = \uparrow \text{ and } x_u \neq y_\ell; \\ \mathrm{succ}\big(y_\ell \boxminus_\downarrow \mathrm{succ}(x_u)\big), & \text{if } \bar{r}_\ell = \downarrow; \end{cases}$$

$$(a_{11}, a_{12}) = \begin{cases} (y_\ell, -0), & \text{if } \bar{r}_\ell = \downarrow; \\ (\mathrm{succ}(y_\ell), +0), & \text{otherwise}; \end{cases}$$

$$a_{13} = \begin{cases} +\infty, & \text{if } \bar{r}_\ell = \uparrow; \\ \mathrm{succ}(y_\ell \boxplus_\downarrow f_{\max}), & \text{if } \bar{r}_\ell = \downarrow; \\ f_{\max} \boxplus_\uparrow \big(\nabla_2^{\mathrm{n}+}(f_{\max})/2 \boxplus_\uparrow y_\ell\big), & \text{otherwise.} \end{cases}$$

Fig. 15 Second inverse projection of subtraction: function is^s_ℓ.

$\mathrm{is}^s_u(y_u, x_\ell, \bar{r}_u)$	$-\infty$	$\mathbb{R}_-$	-0	$+0$	$\mathbb{R}_+$	$+\infty$
$-\infty$	$+\infty$	unsat.	unsat.	unsat.	unsat.	unsat.
$\mathbb{R}_-$	$+\infty$	a_{14}	y_u	a_{15}	a_{14}	$-\infty$
-0	$+\infty$	$-x_\ell$	$+0$	a_{15}	$-x_\ell$	$-\infty$
$+0$	$+\infty$	$-x_\ell$	$+0$	a_{16}	$-x_\ell$	$-\infty$
$\mathbb{R}_+$	$+\infty$	a_{14}	y_u	a_{15}	a_{14}	a_{17}
$+\infty$	$+\infty$	$f_{\max}$	$f_{\max}$	$f_{\max}$	$f_{\max}$	$f_{\max}$

$$e_u \equiv y_u - \big(x_\ell + \nabla_2^{\mathrm{n}-}(x_u)/2\big);$$

$$a_{14} = \begin{cases} +0, & \text{if } \bar{r}_u = \mathrm{n},\ \nabla_2^{\mathrm{n}-}(x_\ell) = -f_{\min} \text{ and } x_\ell = y_u; \\ y_u \boxminus_\downarrow x_\ell, & \text{if } \bar{r}_u = \mathrm{n},\ \nabla_2^{\mathrm{n}-}(x_\ell) = -f_{\min} \text{ and } x_\ell \neq y_u; \\ [\![e_u]\!]_\downarrow, & \text{if } \bar{r}_u = \mathrm{n},\ \mathrm{even}(x_\ell),\ \nabla_2^{\mathrm{n}-}(x_\ell) \neq -f_{\min} \text{ and } [\![e_u]\!]_\downarrow = [e_u]_\downarrow; \\ [\![e_u]\!]_\uparrow, & \text{if } \bar{r}_u = \mathrm{n},\ \mathrm{even}(x_u),\ \nabla_2^{\mathrm{n}-}(x_u) \neq -f_{\min} \text{ and } [\![e_u]\!]_\downarrow < [e_u]_\downarrow; \\ \mathrm{pred}\big([\![e_u]\!]_\uparrow\big), & \text{if } \bar{r}_u = \mathrm{n}, \text{ otherwise}; \\ \mathrm{pred}\big(y_u \boxminus_\uparrow \mathrm{pred}(x_\ell)\big), & \text{if } \bar{r}_u = \uparrow; \\ +0, & \text{if } \bar{r}_u = \downarrow \text{ and } x_\ell = y_u; \\ y_u \boxminus_\downarrow x_\ell, & \text{if } \bar{r}_u = \downarrow \text{ and } x_\ell \neq y_u; \end{cases}$$

$$(a_{15}, a_{16}) = \begin{cases} (\mathrm{pred}(y_u), -0), & \text{if } \bar{r}_u = \downarrow; \\ (y_u, +0), & \text{otherwise}; \end{cases}$$

$$a_{17} = \begin{cases} -\infty, & \text{if } \bar{r}_u = \downarrow; \\ \mathrm{pred}(y_u \boxminus_\uparrow f_{\max}), & \text{if } \bar{r}_u = \uparrow; \\ -f_{\max} \boxplus_\downarrow \big(\nabla_2^{\mathrm{n}-}(-f_{\max})/2 \boxplus_\downarrow y_u\big), & \text{otherwise.} \end{cases}$$

Fig. 16 Second inverse projection of subtraction: function is^s_u.

choice of r_ℓ (of r_u, resp.) is driven by the sign of $y_L \boxdot z_L$ (of $y_U \boxdot z_U$, resp.) only. Since, in this case, the sign of $y_L \boxdot z_L$ (of $y_U \boxdot z_U$, resp.) as defined at line 2 and the sign of $y_L \boxdot z_L$ (of $y_U \boxdot z_U$, resp.) as defined at line 5 are the same, we do not need to compute r_ℓ and r_u another time.

Algorithm 9 Direct projection for multiplication constraints.

Require: $x = y \boxdot_S z$, $x \in X = [x_\ell, x_u]$, $y \in Y = [y_\ell, y_u]$ and $z \in Z = [z_\ell, z_u]$.
Ensure: $X' \subseteq X$ and $\forall r \in S, x \in X, y \in Y, z \in Z : x = y \boxdot_r z \implies x \in X'$ and $\forall X'' \subset X : \exists r \in S, y \in Y, z \in Z \,.\, y \boxdot_r z \notin X''$.

1: **if** $\mathrm{sgn}(y_\ell) \neq \mathrm{sgn}(y_u)$ **and** $\mathrm{sgn}(z_\ell) \neq \mathrm{sgn}(z_u)$ **then**
2: $\quad (y_L, y_U, z_L, z_U) := (y_\ell, y_\ell, z_u, z_\ell)$;
3: $\quad r_\ell := r_\ell(S, y_L, \boxdot, z_L)$; $r_u := r_u(S, y_U, \boxdot, z_U)$;
4: $\quad v_\ell := \mathrm{dm}_\ell(y_L, z_L, r_\ell)$; $v_u := \mathrm{dm}_u(y_U, z_U, r_u)$;
5: $\quad (y_L, y_U, z_L, z_U) := (y_u, y_u, z_\ell, z_u)$;
6: $\quad w_\ell := \mathrm{dm}_\ell(y_L, z_L, r_\ell)$; $w_u := \mathrm{dm}_u(y_U, z_U, r_u)$;
7: $\quad x'_\ell := \min\{v_\ell, w_\ell\}$; $x'_u := \max\{v_u, w_u\}$;
8: **else**
9: $\quad$ **if** $\mathrm{sgn}(y_\ell) = \mathrm{sgn}(y_u)$ **then**
10: $\quad\quad (y_L, y_U, z_L, z_U) := \sigma(y_\ell, y_u, z_\ell, z_u)$;
11: $\quad$ **else**
12: $\quad\quad (z_L, z_U, y_L, y_U) := \sigma(z_\ell, z_u, y_\ell, y_u)$;
13: $\quad$ **end if**
14: $\quad r_\ell := r_\ell(S, y_L, \boxdot, z_L)$; $r_u := r_u(S, y_U, \boxdot, z_U)$;
15: $\quad x'_\ell := \mathrm{dm}_\ell(y_L, z_L, r_\ell)$; $x'_u := \mathrm{dm}_u(y_U, z_U, r_u)$;
16: **end if**
17: $X' := X \cap [x'_\ell, x'_u]$;

Theorem 9 Algorithm 9 *satisfies its contract.*

Inverse Propagation. For inverse propagation, Algorithm 10 partitions interval Z into the sign-homogeneous intervals $Z_- \stackrel{\text{def}}{=} Z \cap [-\infty, -0]$ and $Z_+ \stackrel{\text{def}}{=} Z \cap [+0, +\infty]$. This is done because the sign of Z must be taken into account in order to derive correct bounds for Y. Hence, once Z has been partitioned into sign-homogeneous intervals, we use intervals X and Z_- to obtain interval $[y_\ell^-, y_u^-]$, and X and Z_+ to obtain $[y_\ell^+, y_u^+]$. To do so, the algorithm determines the appropriate extrema of intervals X and $W = Z_-$ or $W = Z_+$ to be used for constraint propagation. To this aim, function τ of Figure 5 is employed; note that the sign of W is, by construction, constant over the interval. The chosen extrema are then passed as parameters to functions im_ℓ of Figure 18 and im_u of Figure 19, that compute the new, refined bounds for y, by using the inverse operation of multiplication, i.e., division. The so obtained intervals $Y \cap [y_\ell^-, y_u^-]$ and $Y \cap [y_\ell^+, y_u^+]$ will be then joined with convex union, denoted by $\uplus$, to obtain Y'.

Theorem 10 Algorithm 10 *satisfies its contract.*

Of course, the refinement Z' of Z can be defined analogously.

$\mathrm{dm}_\ell(y_L, z_L)$	$-\infty$	$\mathbb{R}_-$	-0	$+0$	$\mathbb{R}_+$	$+\infty$
$-\infty$	$+\infty$	$+\infty$	$+\infty$	-0	$-\infty$	$-\infty$
$\mathbb{R}_-$	$+\infty$	$y_L \boxdot_{r_\ell} z_L$	$+0$	-0	$y_L \boxdot_{r_\ell} z_L$	$-\infty$
-0	$+\infty$	$+0$	$+0$	-0	-0	-0
$+0$	-0	-0	-0	$+0$	$+0$	$+\infty$
$\mathbb{R}_+$	$-\infty$	$y_L \boxdot_{r_\ell} z_L$	-0	$+0$	$y_L \boxdot_{r_\ell} z_L$	$+\infty$
$+\infty$	$-\infty$	$-\infty$	-0	$+\infty$	$+\infty$	$+\infty$

$\mathrm{dm}_u(y_U, z_U)$	$-\infty$	$\mathbb{R}_-$	-0	$+0$	$\mathbb{R}_+$	$+\infty$
$-\infty$	$+\infty$	$+\infty$	$+0$	$-\infty$	$-\infty$	$-\infty$
$\mathbb{R}_-$	$+\infty$	$y_U \boxdot_{r_u} z_U$	$+0$	-0	$y_U \boxdot_{r_u} z_U$	$-\infty$
-0	$+0$	$+0$	$+0$	-0	-0	$-\infty$
$+0$	$-\infty$	-0	-0	$+0$	$+0$	$+0$
$\mathbb{R}_+$	$-\infty$	$y_U \boxdot_{r_u} z_U$	-0	$+0$	$y_U \boxdot_{r_u} z_U$	$+\infty$
$+\infty$	$-\infty$	$-\infty$	$-\infty$	$+0$	$+\infty$	$+\infty$

Fig. 17 Direct projection of multiplication: functions dm_ℓ and dm_u.

Algorithm 10 Inverse projection for multiplication constraints.

Require: $x = y \boxdot_S z$, $x \in X = [x_\ell, x_u]$, $y \in Y = [y_\ell, y_u]$ and $z \in Z = [z_\ell, z_u]$.
Ensure: $Y' \subseteq Y$ and $\forall r \in S, x \in X, y \in Y, z \in Z : x = y \boxdot_r z \implies y \in Y'$.

$Z_- := Z \cap [-\infty, -0]$;
if $Z_- \neq \varnothing$ **then**
  $W := Z_-$;
  $(x_L, x_U, w_L, w_U) := \tau(x_\ell, x_u, w_\ell, w_u)$;
  $\bar{r}_\ell := \bar{r}^\ell_\ell(S, x_L, \boxdot, w_L)$; $\bar{r}_u := \bar{r}^\ell_u(S, x_U, \boxdot, w_U)$;
  $y^-_\ell := \mathrm{im}_\ell(x_L, w_L, \bar{r}_\ell)$; $y^-_u := \mathrm{im}_u(x_U, w_U, \bar{r}_u)$;
  if $y^-_\ell \in \mathbb{F}$ and $y^-_u \in \mathbb{F}$ **then**
    $Y'_- = Y \cap [y^-_\ell, y^-_u]$;
  else
    $Y'_- = \varnothing$;
  end if
else
  $Y'_- = \varnothing$;
end if
$Z_+ := Z \cap [+0, +\infty]$;
if $Z_+ \neq \varnothing$ **then**
  $W := Z_+$;
  $(x_L, x_U, w_L, w_U) := \tau(x_\ell, x_u, w_\ell, w_u)$;
  $\bar{r}_\ell := \bar{r}^\ell_\ell(S, x_L, \boxdot, w_L)$; $\bar{r}_u := \bar{r}^\ell_u(S, x_U, \boxdot, w_U)$;
  $y^+_\ell := \mathrm{im}_\ell(x_L, w_L, \bar{r}_\ell)$; $y^+_u := \mathrm{im}_u(x_U, w_U, \bar{r}_u)$;
  if $y^+_\ell \in \mathbb{F}$ and $y^+_u \in \mathbb{F}$ **then**
    $Y'_+ = Y \cap [y^+_\ell, y^+_u]$;
  else
    $Y'_+ = \varnothing$;
  end if
else
  $Y'_+ = \varnothing$;
end if
$Y' := Y'_- \uplus Y'_+$;

$\mathrm{im}_\ell(x_L, w_L)$	$-\infty$	$\mathbb{R}_-$	-0	$+0$	$\mathbb{R}_+$	$+\infty$
$-\infty$	$f_{\min}$	a_4	unsat.	$-\infty$	$-\infty$	$-\infty$
$\mathbb{R}_-$	$f_{\min}$	a_3^-	unsat.	$-f_{\max}$	a_3^+	$f_{\min}$
-0	$+0$	$+0$	$+0$	$-f_{\max}$	a_5	$f_{\min}$
$+0$	$f_{\min}$	a_6	$-f_{\max}$	$+0$	$+0$	$+0$
$\mathbb{R}_+$	$f_{\min}$	a_3^-	$-f_{\max}$	unsat.	a_3^+	$f_{\min}$
$+\infty$	$-\infty$	$-\infty$	$-\infty$	unsat.	a_7	$f_{\min}$

$$e_\ell^+ \equiv (x_L + \nabla_2^{\mathrm{n}-}(x_L)/2)/w_L;$$

$$a_3^+ = \begin{cases} [\![e_\ell^+]\!]_\uparrow, & \text{if } \bar{r}_\ell = \mathrm{n}, \text{even}(x_L) \text{ and } [\![e_\ell^+]\!]_\uparrow = [e_\ell^+]_\uparrow; \\ [\![e_\ell^+]\!]_\downarrow, & \text{if } \bar{r}_\ell = \mathrm{n}, \text{even}(x_L) \text{ and } [\![e_\ell^+]\!]_\uparrow > [e_\ell^+]_\uparrow; \\ \mathrm{succ}\big([\![e_\ell^+]\!]_\downarrow\big), & \text{if } \bar{r}_\ell = \mathrm{n}, \text{otherwise}; \\ x_L \boxslash_\uparrow w_L, & \text{if } \bar{r}_\ell = \downarrow; \\ \mathrm{succ}\big(\mathrm{pred}(x_L) \boxslash_\downarrow w_L\big), & \text{if } \bar{r}_\ell = \uparrow; \end{cases}$$

$$e_\ell^- \equiv (x_L + \nabla_2^{\mathrm{n}+}(x_L)/2)/w_L;$$

$$a_3^- = \begin{cases} [\![e_\ell^-]\!]_\uparrow, & \text{if } \bar{r}_\ell = \mathrm{n}, \text{even}(x_L) \text{ and } [\![e_\ell^+]\!]_\uparrow = [e_\ell^+]_\uparrow; \\ [\![e_\ell^-]\!]_\downarrow, & \text{if } \bar{r}_\ell = \mathrm{n}, \text{even}(x_L) \text{ and } [\![e_\ell^-]\!]_\uparrow > [e_\ell^-]_\uparrow; \\ \mathrm{succ}\big([\![e_\ell^-]\!]_\downarrow\big), & \text{if } \bar{r}_\ell = \mathrm{n}, \text{otherwise}; \\ x_L \boxslash_\uparrow w_L, & \text{if } \bar{r}_\ell = \uparrow; \\ \mathrm{succ}\big(\mathrm{succ}(x_L) \boxslash_\downarrow w_L\big), & \text{if } \bar{r}_\ell = \downarrow; \end{cases}$$

$$e_\ell^1 \equiv (-f_{\max} + \nabla_2^{\mathrm{n}-}(-f_{\max})/2)/w_L;$$

$$a_4 = \begin{cases} +\infty, & \text{if } \bar{r}_\ell = \uparrow; \\ \mathrm{succ}(-f_{\max} \boxslash_\downarrow w_L), & \text{if } \bar{r}_\ell = \downarrow; \\ [\![e_\ell^1]\!]_\uparrow, & \text{if } \bar{r}_\ell = \mathrm{n} \text{ and } [\![e_\ell^1]\!]_\uparrow = [e_\ell^1]_\uparrow; \\ [\![e_\ell^1]\!]_\downarrow, & \text{if } \bar{r}_\ell = \mathrm{n}, \text{otherwise}; \end{cases}$$

$$(a_5, a_6) = \begin{cases} (-0,\ \mathrm{succ}(f_{\min} \boxslash_\downarrow w_L)), & \text{if } \bar{r}_\ell = \downarrow; \\ (\mathrm{succ}(-f_{\max} \boxslash_\downarrow w_L),\ -0), & \text{if } \bar{r}_\ell = \uparrow; \\ (-f_{\min} \boxslash_\uparrow (2 \cdot w_L),\ f_{\min} \boxslash_\uparrow (2 \cdot w_L)), & \text{if } \bar{r}_\ell = \mathrm{n}; \end{cases}$$

$$e_\ell^2 \equiv (f_{\max} + \nabla_2^{\mathrm{n}+}(f_{\max})/2)/w_L;$$

$$a_7 = \begin{cases} +\infty, & \text{if } \bar{r}_\ell = \downarrow; \\ \mathrm{succ}(f_{\max} \boxslash_\downarrow w_L), & \text{if } \bar{r}_\ell = \uparrow; \\ [\![e_\ell^2]\!]_\uparrow, & \text{if } \bar{r}_\ell = \mathrm{n} \text{ and } [\![e_\ell^2]\!]_\uparrow = [e_\ell^2]_\uparrow; \\ [\![e_\ell^2]\!]_\downarrow, & \text{if } \bar{r}_\ell = \mathrm{n}, \text{otherwise.} \end{cases}$$

Fig. 18 Inverse projection of multiplication: function im_ℓ.

$\mathrm{im}_u(x_U, w_U)$	$-\infty$	$\mathbb{R}_-$	-0	$+0$	$\mathbb{R}_+$	$+\infty$
$-\infty$	$+\infty$	$+\infty$	$+\infty$	unsat.	a_9	$-f_{\min}$
$\mathbb{R}_-$	$-f_{\min}$	a_8^-	$f_{\max}$	unsat.	a_8^+	$-f_{\min}$
-0	$-f_{\min}$	a_{10}	$f_{\max}$	-0	-0	-0
$+0$	-0	-0	-0	$f_{\max}$	a_{11}	$-f_{\min}$
$\mathbb{R}_+$	$-f_{\min}$	a_8^-	unsat.	$f_{\max}$	a_8^+	$-f_{\min}$
$+\infty$	$-f_{\min}$	a_{12}	unsat.	$+\infty$	$+\infty$	$+\infty$

$$e_u^+ \equiv (x_U + \nabla_2^{\mathrm{n}+}(x_U)/2)/w_U;$$

$$a_8^+ = \begin{cases} [\![e_u^+]\!]_\downarrow, & \text{if } \bar{r}_u = \mathrm{n}, \text{ even}(x_U) \text{ and } [\![e_u^+]\!]_\uparrow = [e_u^+]_\uparrow; \\ [\![e_u^+]\!]_\uparrow, & \text{if } \bar{r}_u = \mathrm{n}, \text{ even}(x_U) \text{ and } [\![e_u^+]\!]_\uparrow > [e_u^+]_\uparrow; \\ \mathrm{pred}\big([\![e_u^+]\!]_\uparrow\big), & \text{if } \bar{r}_u = \mathrm{n}, \text{ otherwise}; \\ \mathrm{pred}\big(\mathrm{succ}(x_U) \boxslash_\uparrow w_U\big), & \text{if } \bar{r}_u = \downarrow; \\ x_U \boxslash_\downarrow w_U, & \text{if } \bar{r}_u = \uparrow; \end{cases}$$

$$e_u^- \equiv (x_U + \nabla_2^{\mathrm{n}-}(x_U)/2)/w_U;$$

$$a_8^- = \begin{cases} [\![e_u^-]\!]_\downarrow, & \text{if } \bar{r}_u = \mathrm{n}, \text{ even}(x_U) \text{ and } [\![e_u^-]\!]_\uparrow = [e_u^-]_\uparrow; \\ [\![e_u^-]\!]_\uparrow, & \text{if } \bar{r}_u = \mathrm{n}, \text{ even}(x_U) \text{ and } [\![e_u^-]\!]_\uparrow > [e_u^-]_\uparrow; \\ \mathrm{pred}\big([\![e_u^-]\!]_\uparrow\big), & \text{if } \bar{r}_u = \mathrm{n}, \text{ otherwise}; \\ \mathrm{pred}\big(\mathrm{pred}(x_U) \boxslash_\uparrow w_U\big), & \text{if } \bar{r}_u = \uparrow; \\ x_U \boxslash_\downarrow w_U, & \text{if } \bar{r}_u = \downarrow; \end{cases}$$

$$e_u^1 \equiv (-f_{\max} + \nabla_2^{\mathrm{n}-}(-f_{\max})/2)/w_U;$$

$$a_9 = \begin{cases} -\infty, & \text{if } \bar{r}_u = \uparrow; \\ \mathrm{pred}(-f_{\max} \boxslash_\uparrow w_U), & \text{if } \bar{r}_u = \downarrow; \\ [\![e_u^1]\!]_\downarrow & \text{if } \bar{r}_u = \mathrm{n} \text{ and } [\![e_u^1]\!]_\downarrow = [e_u^1]_\downarrow; \\ [\![e_u^1]\!]_\uparrow & \text{if } \bar{r}_u = \mathrm{n}, \text{ otherwise}; \end{cases}$$

$$(a_{10}, a_{11}) = \begin{cases} (+0,\ \mathrm{pred}(f_{\min} \boxslash_\uparrow w_U)), & \text{if } \bar{r}_u = \downarrow; \\ (\mathrm{pred}(-f_{\min} \boxslash_\uparrow w_U),\ +0), & \text{if } \bar{r}_u = \uparrow; \\ (-f_{\min} \boxslash_\downarrow (2 \cdot w_U),\ f_{\min} \boxslash_\downarrow (2 \cdot w_U)), & \text{if } \bar{r}_u = \mathrm{n}; \end{cases}$$

$$e_u^2 \equiv (f_{\max} + \nabla_2^{\mathrm{n}+}(f_{\max})/2)/w_U;$$

$$a_{12} = \begin{cases} -\infty, & \text{if } \bar{r}_u = \downarrow; \\ \mathrm{pred}(f_{\max} \boxslash_\uparrow w_U), & \text{if } \bar{r}_u = \uparrow; \\ [\![e_u^2]\!]_\downarrow & \text{if } \bar{r}_u = \mathrm{n} \text{ and } [\![e_u^2]\!]_\downarrow = [e_u^2]_\downarrow; \\ [\![e_u^2]\!]_\uparrow & \text{if } \bar{r}_u = \mathrm{n}, \text{ otherwise}. \end{cases}$$

Fig. 19 Inverse projection of multiplication: function im_u.

B Proofs of Results

B.1 Proofs of Results in Section 2

Proof (of Proposition 1) In order to prove (5), we first prove that $[x]_\downarrow \leq x$. To this aim, consider the following cases on $x \in \mathbb{R} \setminus \{0\}$:

$-f_{\max} \leq x < 0 \lor f_{\min} \leq x$: by (2) we have $[x]_\downarrow = \max\{ z \in \mathbb{F} \mid z \leq x \}$, hence $[x]_\downarrow \leq x$;
$0 < x < f_{\min}$: by (2) we have $[x]_\downarrow = -0 \leq x$;
$x < -f_{\max}$: by (2) we have $[x]_\downarrow = -\infty \leq x$.

We now prove that $x \leq [x]_\uparrow$. Consider the following cases on $x \in \mathbb{R} \setminus \{0\}$:

$x > f_{\max}$: by (1) we have $[x]_\uparrow = +\infty$ and thus $x \leq [x]_\uparrow$ holds;
$x \leq -f_{\min} \lor 0 < x \leq f_{\max}$: by (1) we have $[x]_\uparrow = \min\{ z \in \mathbb{F} \mid z \geq x \}$, hence $x \leq [x]_\uparrow$ holds;
$-f_{\min} < x < 0$: by (1) we have $[x]_\uparrow = -0$ hence $x \leq [x]_\uparrow$ holds.

In order to prove (6), consider the following cases on $x \in \mathbb{R} \setminus \{0\}$:

$x > 0$: by (3) we have $[x]_0 = [x]_\downarrow \leq [x]_\uparrow$;
$x < 0$: by (3) we have $[x]_\downarrow \leq [x]_\uparrow = [x]_0$.

In order to prove (7), consider the following cases on $x \in \mathbb{R} \setminus \{0\}$:

$-f_{\max} \leq x \leq f_{\max}$: we have the following cases
- $\big|[x]_\downarrow - x\big| < \big|[x]_\uparrow - x\big| \lor \big(\big|[x]_\downarrow - x\big| = \big|[x]_\uparrow - x\big|\big) \land \mathrm{even}\big([x]_\downarrow\big)$: by (4), we have $[x]_\mathrm{n} = [x]_\downarrow \leq [x]_\uparrow$;
- $\big|[x]_\downarrow - x\big| > \big|[x]_\uparrow - x\big| \lor \big(\big|[x]_\downarrow - x\big| = \big|[x]_\uparrow - x\big| \land \neg\mathrm{even}\big([x]_\downarrow\big)\big)$: by (4) we have $[x]_\downarrow \leq [x]_\uparrow = [x]_\mathrm{n}$.

$-f_{\max} > x$: we have the following cases
- $-2^{e_{\max}}(2 - 2^{-p}) < x < -f_{\max}$: by (4) we have $[x]_\downarrow \leq [x]_\uparrow = [x]_\mathrm{n}$.
- $x \leq -2^{e_{\max}}(2 - 2^{-p})$: by (4) we have $[x]_\mathrm{n} = [x]_\downarrow \leq [x]_\uparrow$;

$f_{\max} < x$: we have the following cases
- $2^{e_{\max}}(2 - 2^{-p}) > x > f_{\max}$: by (4) we have $[x]_\mathrm{n} = [x]_\downarrow \leq [x]_\uparrow$;
- $x \geq 2^{e_{\max}}(2 - 2^{-p})$: by (4) we have $[x]_\downarrow \leq [x]_\uparrow = [x]_\mathrm{n}$.

In order to prove (8), let us compute $-[-x]_\uparrow$. There are the following cases:

$-x > f_{\max}$: this implies that $x < -f_{\max}$ and, by (1), $[-x]_\uparrow = +\infty$; hence, by (2), $-[-x]_\uparrow = -\infty = [x]_\downarrow$;
$-x \leq -f_{\min} \lor 0 < -x \leq f_{\max}$: this implies that $x \geq f_{\min} \lor -f_{\max} \geq x > 0$ and, by (1), we have $[-x]_\uparrow = \min\{ z \in \mathbb{F} \mid z \geq -x \}$; therefore, by (2), $-[-x]_\uparrow = -\min\{ z \in \mathbb{F} \mid z \geq -x \} = \max\{ z \in \mathbb{F} \mid z \leq x \} = [x]_\downarrow$,
$-f_{\min} < -x < 0$: this implies that $0 < x < f_{\min}$ and, by (1), $[-x]_\uparrow = -0$; hence, by (2), $-[-x]_\uparrow = +0 = [x]_\downarrow$.

B.2 Proofs of Results in Section 3

Proof (Rest of the proof of Proposition 2) We prove the second part of Proposition 2, regarding rounding mode selectors for inverse propagators. Before doing so, we need to prove the following result. Let $\boxdot \in \{\boxplus, \boxminus, \boxdot, \boxslash\}$, and let r and s be two IEEE 754 rounding modes, such that for any $a, b \in \mathbb{F}$,

$$a \boxdot_r b \preccurlyeq a \boxdot_s b.$$

Moreover, let $x, z \in \mathbb{F}$, and let $\bar{y}_\mathrm{s}$ be the minimum $y_\mathrm{s} \in \mathbb{F}$ such that $x = y_\mathrm{s} \boxdot_\mathrm{s} z$. Then, for any $y_\mathrm{r} \in \mathbb{F}$ such that $x = y_\mathrm{r} \boxdot_\mathrm{r} z$ we have

$$\begin{aligned} \bar{y}_\mathrm{s} \boxdot_\mathrm{r} z &\preccurlyeq \bar{y}_\mathrm{s} \boxdot_\mathrm{s} z \\ &= x \\ &= y_\mathrm{r} \boxdot_\mathrm{r} z. \end{aligned}$$

This leads us to write

$$[\bar{y}_\mathrm{s} \circ z]_\mathrm{r} \preccurlyeq [y_\mathrm{r} \circ z]_\mathrm{r}$$

which, due to the isotonicity of all IEEE 754 rounding modes, implies

$$\bar{y}_\mathrm{s} \circ z \preccurlyeq y_\mathrm{r} \circ z.$$

Finally, if operator '$\circ$' is isotone we have

$$\bar{y}_\mathrm{s} \preccurlyeq y_\mathrm{r},$$

which implies that $\bar{y}_s$ is the minimum $y \in \mathbb{F}$ such that $x = y \boxdot_r z$ or $x = y \boxdot_s z$. On the other hand, if '$\circ$' is antitone we have

$$\bar{y}_s \succcurlyeq y_r,$$

and $\bar{y}_s$ is the maximum $y \in \mathbb{F}$ such that $x = y \boxdot_r z$ or $x = y \boxdot_s z$. An analogous result can be proved regarding the upper bound for y in case the operator is isotone, and regarding the lower bound for y in case it is antitone.

The above claim allows us to prove the following. Assume first that $\boxdot$ is isotone with respect to y in $x = y \boxdot z$. Let $\hat{y}_\uparrow$ be the minimum $y_\uparrow \in \mathbb{F}$ such that $x = y_\uparrow \boxdot_\uparrow z = [y_\uparrow \boxdot z]_\uparrow$, let $\hat{y}_n$ be the minimum $y_n \in \mathbb{F}$ such that $x = y_n \boxdot_n z = [y_n \boxdot z]_n$ and, finally, let $\hat{y}_\downarrow$ be the minimum $y_\downarrow \in \mathbb{F}$ such that $x = y_\downarrow \boxdot_\downarrow z = [y_\downarrow \boxdot z]_\downarrow$. We will prove that

$$\hat{y}_\uparrow \preccurlyeq \hat{y}_n \preccurlyeq \hat{y}_\downarrow.$$

Since we assumed that $\boxdot$ is isotone with respect to y in $x = y \boxdot z$, the rounding mode that gives the minimal y solution of $x = [y \boxdot z]_r$ is the one that yields a bigger (w.r.t. $\preccurlyeq$ order) floating point number, as we proved before. We must now separately treat the following cases:

$y \boxdot z \neq 0$: By (7), we have $[y \boxdot z]_\downarrow \leq [y \boxdot z]_n \leq [y \boxdot z]_\uparrow$. Since in this case $y \boxdot z \neq 0$, we have that $[y \boxdot z]_\downarrow \preccurlyeq [y \boxdot z]_n \preccurlyeq [y \boxdot z]_\uparrow$. This implies $\hat{y}_\uparrow \preccurlyeq \hat{y}_n \preccurlyeq \hat{y}_\downarrow$.

$y \boxdot z = 0$: In this case, $[y \boxdot z]_\downarrow \preccurlyeq [y \boxdot z]_n = [y \boxdot z]_\uparrow$. This implies $\hat{y}_\uparrow \preccurlyeq \hat{y}_n \preccurlyeq \hat{y}_\downarrow$.

Moreover, let $\tilde{y}_\uparrow$ be the maximum $y_\uparrow \in \mathbb{F}$ such that $x = y_\uparrow \boxdot_\uparrow z = [y_\uparrow \boxdot z]_\uparrow$, let $\tilde{y}_n$ be the maximum $y_n \in \mathbb{F}$ such that $x = y_n \boxdot_n z = [y_n \boxdot z]_n$ and, finally, let $\tilde{y}_\downarrow$ be the maximum $y_\downarrow \in \mathbb{F}$ such that $x = y_\downarrow \boxdot_\downarrow z = [y_\downarrow \boxdot z]_\downarrow$. We will prove the fact that

$$\tilde{y}_\uparrow \preccurlyeq \tilde{y}_n \preccurlyeq \tilde{y}_\downarrow.$$

Since we assumed that $\boxdot$ is isotone with respect to y in $x = y \boxdot z$, the rounding mode that gives a maximum y solution of $x = y \boxdot z_r$ is the one that gives a smaller (w.r.t. $\preccurlyeq$ order) floating point number. We must now deal with the following cases:

$y \boxdot z \neq 0$: By (7), we have $[y \boxdot z]_\downarrow \leq [y \boxdot z]_n \leq [y \boxdot z]_\uparrow$. Since in this case $y \boxdot z \neq 0$, we have $[y \boxdot z]_\downarrow \preccurlyeq [y \boxdot z]_n \preccurlyeq [y \boxdot z]_\uparrow$. This implies $\tilde{y}_\uparrow \preccurlyeq \tilde{y}_n \preccurlyeq \tilde{y}_\downarrow$.

$y \boxdot z = 0$: In this case $[y \boxdot z]_\downarrow \preccurlyeq [y \boxdot z]_n = [y \boxdot z]_\uparrow$. This implies $\tilde{y}_\uparrow \preccurlyeq \tilde{y}_n \preccurlyeq \tilde{y}_\downarrow$.

The inequalities $\hat{y}_\uparrow \preccurlyeq \hat{y}_n \preccurlyeq \hat{y}_\downarrow$ and $\tilde{y}_\uparrow \preccurlyeq \tilde{y}_n \preccurlyeq \tilde{y}_\downarrow$ allow us to claim that the rounding mode selectors $\hat{r}_\ell(S, \boxdot, b)$ and $\hat{r}_u(S, b)$ are correct when $\boxdot$ is isotone with respect to y. In a similar way it is possible to prove that, in case $\boxdot$ is antitone with respect to argument y, the above-mentioned rounding mode selectors can be exchanged: $\hat{r}_u(S, b)$ can be used to obtain the lower bound for y, while $\hat{r}_\ell(S, \boxdot, b)$ can be used to obtain the upper bound.

Note that, in general, the roundTowardZero rounding mode is equivalent to roundTowardPositive if the result of the rounded operation is negative, and to roundTowardNegative if it is positive. The only case in which this is not true is when the result is $+0$ and the operation is a sum or a subtraction: this value can come from the rounding toward negative infinity of a strictly positive exact result, or the sum of $+0$ and -0, which behaves like roundTowardPositive, yielding $+0$. This case must be treated separately, and it is significant only in $\hat{r}_\ell(S, \boxdot, b)$, which is used when seeking for the lowest possible value of the variable to be refined that yields $+0$.

Definition 8 also contains selectors that can choose between rounding mode selectors $\hat{r}_\ell(S, b)$ and $\hat{r}_u(S, b)$ by distinguishing whether the operator is isotone or antitone with respect to the operand y to be derived by propagation; they take the result of the operation b and the known operand a into account. In particular, $\bar{r}^\ell_\ell(S, b, \boxdot, a)$, $\bar{r}^\ell_u(S, b, \boxdot, a)$ choose the appropriate selector for the leftmost operand, and $\bar{r}^r_\ell(S, b, \boxdot, a)$, $\bar{r}^r_u(S, b, \boxdot, a)$ are valid for the rightmost one.

Proof (of Proposition 3) We first prove (15). By Definition 9, we have the following cases:

$x_\ell = -f_{\max}$: Then,

$$\begin{aligned}
x_\ell + \nabla_2^{n-}(x_\ell)/2 &= -f_{\max} + \big(-f_{\max} - \mathrm{succ}(-f_{\max})\big)/2 \\
&= -2^{e_{\max}}(2 - 2^{1-p}) + \big(-2^{e_{\max}}(2 - 2^{1-p}) + 2^{e_{\max}}(2 - 2^{1-p} - 2^{1-p})\big)/2 \\
&= -2^{e_{\max}}(2 - 2^{1-p} + 1 - 2^{-p} - 1 + 2^{1-p}) \\
&= -2^{e_{\max}}(2 - 2^{-p})
\end{aligned}$$

On the other hand, consider any x such that $x_\ell < x \leq x_u$. Since $x \in \mathbb{F}$, this implies that $\mathrm{succ}(-f_{\max}) \leq x \leq x_u$. In this case

$$x + \nabla_2^{n-}(x)/2 = (x + \mathrm{pred}(x))/2.$$

Since 'pred' is monotone, the minimum can be found when $x = \mathrm{succ}(-f_{\max})$. In this case, we have that

$$\begin{aligned}(x+\mathrm{pred}(x))/2 &= (\mathrm{succ}(-f_{\max}) - f_{\max})/2\\ &= \bigl(-2^{e_{\max}}(2-2^{1-p}-2^{1-p}) - 2^{e_{\max}}(2-2^{1-p})\bigr)/2\\ &= \bigl(-2^{e_{\max}}(2-2^{1-p}-2^{1-p}+2-2^{1-p})\bigr)/2\\ &= -2^{e_{\max}}(2-3\cdot 2^{-p})\\ &> x_\ell + \nabla_2^{\mathrm{n}-}(x_\ell)/2\\ &= -2^{e_{\max}}(2-2^{-p}).\end{aligned}$$

Hence we can conclude that $\min_{x_\ell \le x \le x_u}\bigl(x+\nabla_2^{\mathrm{n}-}(x)/2\bigr) = x_\ell + \nabla_2^{\mathrm{n}-}(x_\ell)/2$.

$x_\ell > -f_{\max}$: In this case

$$x+\nabla_2^{\mathrm{n}-}(x)/2 = (x+\mathrm{pred}(x))/2.$$

Since 'pred' is monotone, $\min_{x_\ell \le x \le x_u}\bigl(x+\nabla_2^{\mathrm{n}-}(x_\ell)/2\bigr) = x_\ell + \nabla_2^{\mathrm{n}-}(x_\ell)/2$.

We now prove (16). By Definition 9, we have the following cases:

$x_u = f_{\max}$: Then,

$$\begin{aligned}x_u+\nabla_2^{\mathrm{n}+}(x_u)/2 &= f_{\max} + \bigl(f_{\max} - \mathrm{pred}(f_{\max})\bigr)/2\\ &= 2^{e_{\max}}(2-2^{1-p}) + \bigl(2^{e_{\max}}(2-2^{1-p}) - 2^{e_{\max}}(2-2^{1-p}-2^{1-p})\bigr)/2\\ &= 2^{e_{\max}}(2-2^{1-p}+1-2^{-p}-1+2^{1-p})\\ &= 2^{e_{\max}}(2-2^{-p}).\end{aligned}$$

Now, consider any x such that $x_\ell \le x < x_u$. Since $x \in \mathbb{F}$, this implies that $x_\ell \le x \le \mathrm{pred}(f_{\max})$. In this case

$$x+\nabla_2^{\mathrm{n}+}(x)/2 = (x+\mathrm{succ}(x))/2.$$

Since 'succ' is monotone, the maximum can be found when $x = \mathrm{pred}(f_{\max})$. In this case, we have that

$$\begin{aligned}(x+\mathrm{succ}(x))/2 &= (\mathrm{pred}(f_{\max}) + f_{\max})/2\\ &= \bigl(2^{e_{\max}}(2-2^{1-p}-2^{1-p}) + 2^{e_{\max}}(2-2^{1-p})\bigr)/2\\ &= \bigl(2^{e_{\max}}(2-2^{1-p}-2^{1-p}+2-2^{1-p})\bigr)/2\\ &= 2^{e_{\max}}(2-3\cdot 2^{-p})\\ &> x_u + \nabla_2^{\mathrm{n}+}(x_u)/2\\ &= 2^{e_{\max}}(2-2^{-p}).\end{aligned}$$

Hence we can conclude that $\max_{x_\ell \le x \le x_u}\bigl(x+\nabla_2^{\mathrm{n}+}(x)/2\bigr) = x_u + \nabla_2^{\mathrm{n}+}(x_u)/2$.

$x_u < f_{\max}$: In this case

$$x+\nabla_2^{\mathrm{n}+}(x)/2 = (x+\mathrm{succ}(x))/2.$$

Since 'succ' is monotone, $\max_{x_\ell \le x \le x_u}\bigl(x+\nabla_2^{\mathrm{n}+}(x)/2\bigr) = x_u + \nabla_2^{\mathrm{n}+}(x_u)/2$.

We now introduce and prove Proposition 6, which contains properties of the rounding error functions that are only needed in the proof of Proposition 4.

Proposition 6 *For each $r \in \mathbb{R} \setminus \{0\}$ we have*

$$0 \le r - [r]_\downarrow < \nabla^\downarrow\bigl([r]_\downarrow\bigr) \tag{43}$$

$$\nabla^\uparrow\bigl([r]_\downarrow\bigr) < r - [r]_\uparrow \le 0 \tag{44}$$

$$\nabla_2^{\mathrm{n}-}\bigl([r]_\mathrm{n}\bigr)/2 \le r - [r]_\mathrm{n} \le \nabla_2^{\mathrm{n}+}\bigl([r]_\mathrm{n}\bigr)/2, \tag{45}$$

where the two inequalities of (45) *are strict if $[r]_\mathrm{n}$ is odd.*

Proof Suppose $r \in \mathbb{R}$ was rounded down to $x \in \mathbb{F}$. Then the error that was committed, $r - x$, is a nonnegative extended real that is strictly bounded from above by $\nabla^{\downarrow}(x) = \mathrm{succ}(x) - x$, that is, $0 \leq r - x < \mathrm{succ}(x) - x$, for otherwise we would have $r \geq \mathrm{succ}(x)$ or $r < x$ and, in both cases r would not have been rounded down to x. Note that $\nabla^{\downarrow}(f_{\max}) = +\infty$, coherently with the fact that the error is unbounded from above in this case.

Dually, if $r \in \mathbb{R}$ was rounded up to $x \in \mathbb{F}$ the error that was committed, $r - x$, is a nonpositive extended real that is strictly bounded from below by $\nabla^{\uparrow}(x) = \mathrm{pred}(x) - x$, that is, $\mathrm{pred}(x) - x < r - x \leq 0$ since, clearly, $\mathrm{pred}(x) < r \leq x$. Note that $\nabla^{\uparrow}(-f_{\max}) = -\infty$, coherently with the fact that the error is unbounded from below in this case.

Suppose now that $r \in \mathbb{R}$ was rounded-to-nearest to $x \in \mathbb{F}$. Then the error that was committed, $r - x$, is such that $\nabla_2^{\mathrm{n}-}(x)/2 \leq r - x \leq \nabla_2^{\mathrm{n}+}(x)/2$, where the two inequalities are strict if x is odd.

In fact, if $x \notin \{-\infty, -f_{\max}\}$, then $\nabla_2^{\mathrm{n}-}(x)/2 = \bigl(\mathrm{pred}(x) - x\bigr)/2 \leq r - x$, for otherwise r would be closer to $\mathrm{pred}(x)$. If $x = -\infty$, then $\nabla_2^{\mathrm{n}-}(x)/2 = +\infty$ and $r - x = +\infty$, so $\nabla_2^{\mathrm{n}-}(x)/2 \leq r - x$ holds. If $x = -f_{\max}$, then

$$\begin{aligned}
\nabla_2^{\mathrm{n}-}(x)/2 &= \bigl(-f_{\max} - \mathrm{succ}(-f_{\max})\bigr)/2 \\
&= \bigl(-2^{e_{\max}}(2 - 2^{1-p}) + 2^{e_{\max}}(2 - 2^{1-p} - 2^{1-p})\bigr)/2 \\
&= \bigl(-2^{e_{\max}}(2 - 2^{1-p} - 2 + 2^{1-p} + 2^{1-p})\bigr)/2 \\
&= -2^{e_{\max}} 2^{1-p}/2 \\
&= -2^{e_{\max}+1-p}/2 \\
&= -2^{e_{\max}-p}
\end{aligned}$$

and thus, considering that $-f_{\max}$ is odd, $\nabla_2^{\mathrm{n}-}(x)/2 < r - x$ is equivalent to

$$\begin{aligned}
\nabla_2^{\mathrm{n}-}(x)/2 + x &= -\bigl(2^{e_{\max}-p} + 2^{e_{\max}}(2 - 2^{1-p})\bigr) \\
&= -2^{e_{\max}}(2^{-p} + 2 - 2^{1-p}) \\
&= -2^{e_{\max}}(2 - 2^{-p}) \\
&< r,
\end{aligned}$$

which must hold, for otherwise r would have been rounded to $-\infty$ [24, Section 4.3.1].

Suppose now $x \notin \{+\infty, f_{\max}\}$: then $\nabla_2^{\mathrm{n}+}(x)/2 = \bigl(\mathrm{succ}(x) - x\bigr)/2 \geq r - x$, for otherwise r would be closer to $\mathrm{succ}(x)$. If $x = +\infty$, then $\nabla_2^{\mathrm{n}+}(x)/2 = -\infty$ and $r - x = -\infty$, and thus $\nabla_2^{\mathrm{n}+}(x)/2 \geq r - x$ holds. If $x = f_{\max}$, then

$$\begin{aligned}
\nabla_2^{\mathrm{n}+}(x)/2 &= \bigl(f_{\max} - \mathrm{pred}(f_{\max})\bigr)/2 \\
&= \bigl(2^{e_{\max}}(2 - 2^{1-p}) - 2^{e_{\max}}(2 - 2^{1-p} - 2^{1-p})\bigr)/2 \\
&= \bigl(2^{e_{\max}}(2 - 2^{1-p} - 2 + 2^{1-p} + 2^{1-p})\bigr)/2 \\
&= 2^{e_{\max}} 2^{1-p}/2 \\
&= 2^{e_{\max}+1-p}/2 \\
&= 2^{e_{\max}-p}
\end{aligned}$$

and thus, considering that $f_{\max}$ is odd, $\nabla_2^{\mathrm{n}+}(x)/2 > r - x$ is equivalent to

$$\begin{aligned}
\nabla_2^{\mathrm{n}+}(x)/2 + x &= \bigl(2^{e_{\max}-p} + 2^{e_{\max}}(2 - 2^{1-p})\bigr) \\
&= 2^{e_{\max}}(2^{-p} + 2 - 2^{1-p}) \\
&= 2^{e_{\max}}(2 - 2^{-p}) \\
&> r,
\end{aligned}$$

which must hold, for otherwise r would have been rounded to $+\infty$.

Proof (of Proposition 4) In order to prove (17), first observe that $x \preccurlyeq y \boxdot_{\downarrow} z$ implies that $x \leq y \boxdot_{\downarrow} z$. Assume first that $y \boxdot_{\downarrow} z \in \mathbb{R}_+ \cup \mathbb{R}_-$. In this case, $y \boxdot_{\downarrow} z = [y \circ z]_{\downarrow}$. By inequality (5) of Proposition 1, $y \boxdot_{\downarrow} z = [y \circ z]_{\downarrow} \leq y \circ z$. Therefore, $x \leq y \boxdot_{\downarrow} z = [y \circ z]_{\downarrow} \leq y \circ z$. Then, assume that $y \boxdot_{\downarrow} z = +\infty$. In this case, since the rounding towards minus infinity never rounds to $+\infty$, it follows that $y \boxdot_{\downarrow} z = y \circ z$. Hence, $x \leq y \circ z = +\infty$, holds. Assume now that $y \boxdot_{\downarrow} z = -\infty$. In this case it must be that $x = -\infty$ then $x \leq y \circ z$, holds. Finally, assume that

$y \boxdot_{\downarrow} z = +0$ or $y \boxdot_{\downarrow} z = -0$. In any case $x \preccurlyeq +0$ that implies $x \leq 0$. On the other hand, we have two cases, $y \circ z \neq 0$ or $y \circ z = 0$. For the first case, by Definition 5, $0 \leq y \circ z < f_{\min}$, then $x \leq y \circ z$, holds. For the second case, since $x \leq 0$ then $x \leq y \circ z$.

In order to prove (18), as before observe that $x \preccurlyeq y \boxdot_{\uparrow} z$ implies that $x \leq y \boxdot_{\uparrow} z$. Note that $x + \nabla^{\uparrow}(x) = \mathrm{pred}(x)$. So we are left to prove $\mathrm{pred}(x) < y \circ z$. Assume now that $0 < y \circ z \leq f_{\max}$ or $x \leq -f_{\min}$. Moreover, note that it cannot be the case that $\mathrm{pred}(x) \geq y \circ z$, otherwise, by Definition 5, $y \boxdot_{\uparrow} z \leq \mathrm{pred}(x)$ and, therefore, $x \leq y \boxdot_{\uparrow} z$ would not hold. Then, in this case, we can conclude $\mathrm{pred}(x) < y \circ z$. Now, assume that $-f_{\min} < y \circ z < 0$. In this case $y \boxdot_{\uparrow} z = -0$. Hence, $x \leq 0$. By Definition 4, $\mathrm{pred}(x) \leq -f_{\min}$. Hence, $\mathrm{pred}(x) \leq y \circ z$, holds. Next, assume $y \circ z > f_{\max}$. In this case $y \boxdot_{\uparrow} z = \infty$. Hence, $x \leq \infty$. By Definition 4, $\mathrm{pred}(x) \leq f_{\max}$. Hence $\mathrm{pred}(x) < y \circ z$, holds. Next assume $y \circ z = 0$. In this case $y \boxdot_{\uparrow} z = +0$ or $y \boxdot_{\uparrow} z = -0$. Hence, $x \leq 0$. By Definition 4, $\mathrm{pred}(x) \leq -f_{\min}$. Hence $\mathrm{pred}(x) < y \circ z$, holds. Finally assume $y \circ z = \infty$. In this case $y \boxdot_{\uparrow} z = \infty$. Hence $x \preccurlyeq \infty$ and therefore $x \leq \infty$. By Definition 4, $\mathrm{pred}(x) \leq f_{\max}$. Hence $\mathrm{pred}(x) < y \circ z$, holds.

In order to prove (19), as the previous two cases, note that $x \preccurlyeq y \boxdot_{\mathrm{n}} z$ implies that $x \leq y \boxdot_{\mathrm{n}} z$. First observe that for $x \neq -\infty$, $x + \nabla_2^{\mathrm{n}-}(x)/2 < x$. Indeed, assume first that $x \neq -f_{\max}$, then, by Definition 9, $\nabla_2^{\mathrm{n}-}(x) = x - \mathrm{succ}(x)$. Hence $x + \nabla_2^{\mathrm{n}-}(x)/2 = x + (x - \mathrm{succ}(x))/2 = (3x - \mathrm{succ}(x))/2$. Since $x < \mathrm{succ}(x)$, we can conclude that $x + \nabla_2^{\mathrm{n}-}(x)/2 < x$. Assume now that $x = -f_{\max}$. By Definition 9, $\nabla_2^{\mathrm{n}-}(x) = \mathrm{pred}(x) - x$. Hence $x + \nabla_2^{\mathrm{n}-}(x)/2 = x + (\mathrm{pred}(x) - x)/2 = (x + \mathrm{pred}(x))/2$. Since $x > \mathrm{pred}(x)$, we can conclude that $x + \nabla_2^{\mathrm{n}-}(x)/2 < x$.

Now, by Definition 5, we have to consider the following cases for $x \boxdot_{\mathrm{n}} y \in \mathbb{R}_+ \cup \mathbb{R}_-$:

$y \boxdot_{\mathrm{n}} z = [y \circ z]_{\downarrow}$. In this case, by inequality (5) of Proposition 1, $x + \nabla_2^{\mathrm{n}-}(x)/2 < x \leq y \boxdot_{\mathrm{n}} z = [y \circ z]_{\downarrow} \leq y \circ z$. Therefore, $x + \nabla_2^{\mathrm{n}-}(x)/2 < y \circ z$, holds.

$y \boxdot_{\mathrm{n}} z = [y \circ z]_{\uparrow}$. Assume first that $x < y \boxdot_{\mathrm{n}} z$. In this case, by Definition 5, since $x \in \mathbb{F}$ and $x < y \boxdot_{\mathrm{n}} z$, it must be the case that $x < y \circ z$. Then, we can conclude that $x + \nabla_2^{\mathrm{n}-}(x)/2 < x < y \circ z$. Therefore, $x + \nabla_2^{\mathrm{n}-}(x)/2 < y \circ z$, holds. Assume now that $x = y \boxdot_{\mathrm{n}} z$ and $\mathrm{even}(x)$. In this case, by Proposition 6, we have that $\nabla_2^{\mathrm{n}-}\big([y \circ z]_{\mathrm{n}}\big)/2 \leq (y \circ z) - [y \circ z]_{\mathrm{n}}$. Since, in this case $x = y \boxdot_{\mathrm{n}} z$, we obtain $\nabla_2^{\mathrm{n}-}(x)/2 \leq (y \circ z) - x$. Hence, $x + \nabla_2^{\mathrm{n}-}(x)/2 \leq y \circ z$. If $\mathrm{odd}(x)$, by Proposition 6, we have that $\nabla_2^{\mathrm{n}-}\big([y \circ z]_{\mathrm{n}}\big)/2 < (y \circ z) - [y \circ z]_{\mathrm{n}}$. Hence, $x + \nabla_2^{\mathrm{n}-}(x)/2 < y \circ z$.

Consider now the case that $y \boxdot_{\mathrm{n}} z = +0$ or $y \boxdot_{\mathrm{n}} z = -0$. If $y \circ z \neq 0$, then $y \boxdot_{\mathrm{n}} z = [y \circ z]_{\downarrow}$ or $y \boxdot_{\mathrm{n}} z = [y \circ z]_{\uparrow}$. In this case we can reason as above. Assume then that $y \circ z = 0$. Since $x \preccurlyeq +0$ or $x \preccurlyeq -0$ implies that $x \leq 0$. Therefore, we can conclude that $x + \nabla_2^{\mathrm{n}-}(x) < x \leq 0$ holds. Assume now that $y \boxdot_{\mathrm{n}} z = +\infty$. If $y \circ z \neq \infty$ then $y \boxdot_{\mathrm{n}} z = [y \circ z]_{\uparrow}$. In this case we can reason as above. On the other hand if $y \circ z = +\infty$ then $x + \nabla_2^{\mathrm{n}-}(x) \leq \infty$ holds.

In order to prove (20), remember that $x \succcurlyeq y \boxdot_{\downarrow} z$ implies that $x \geq y \boxdot_{\downarrow} z$. Note that $x + \nabla^{\downarrow}(x) = \mathrm{succ}(x)$. So we are left to prove $\mathrm{succ}(x) > y \circ z$. Assume now that $-f_{\max} < y \circ z < 0$ or $f_{\min} < y \circ z \leq f_{\max}$. Note that it cannot be the case that $\mathrm{succ}(x) \leq y \circ z$, otherwise, by Definition 5, $y \boxdot_{\downarrow} z \geq \mathrm{succ}(x)$ and $x \geq y \boxdot_{\downarrow} z$ would not hold. Then, in this case, we can conclude that $\mathrm{succ}(x) > y \circ z$. Next, assume that $0 < y \circ z < f_{\min}$. In this case $y \boxdot_{\downarrow} z = +0$. Hence, $x \geq 0$. By Definition 4, $\mathrm{succ}(x) \geq f_{\min}$. Hence $\mathrm{succ}(x) \geq y \circ z$, holds. Next, assume $y \circ z < -f_{\max}$. In this case $y \boxdot_{\downarrow} z = -\infty$. Hence $x \geq -\infty$. By Definition 4, $\mathrm{succ}(x) \geq -f_{\max}$. Hence $\mathrm{succ}(x) > y \circ z$, holds. Next assume $y \circ z = 0$. In this case $y \boxdot_{\downarrow} z = +0$ or $y \boxdot_{\downarrow} z = -0$. In any case, $x \geq 0$. By Definition 4, $\mathrm{succ}(x) \geq f_{\min}$. Hence $\mathrm{succ}(x) > y \circ z$, holds. Finally assume $y \circ z = -\infty$. In this case $y \boxdot_{\downarrow} z = -\infty$. Hence, since $x \succcurlyeq -\infty$, $x \geq -\infty$. By Definition 4, $\mathrm{succ}(x) \geq -f_{\max}$. Hence, $\mathrm{succ}(x) > y \circ z$, holds.

In order to prove (21), as before, observe that $x \succcurlyeq y \boxdot_{\uparrow} z$ implies that $x \geq y \boxdot_{\uparrow} z$. Assume first that $y \boxdot_{\uparrow} z \in \mathbb{R}_+ \cup \mathbb{R}_-$. In this case, $y \boxdot_{\uparrow} z = [y \circ z]_{\uparrow}$. By (5) from Proposition 1, $y \boxdot_{\uparrow} z = [y \circ z]_{\uparrow} \geq y \circ z$. Then, assume that $y \boxdot_{\uparrow} z = -\infty$. In this case, since the rounding towards plus infinity never rounds to $-\infty$, it follows that $y \boxdot_{\uparrow} z = y \circ z$. Hence, $x \geq y \circ z = -\infty$, holds. Assume now that $y \boxdot_{\uparrow} z = +\infty$. In this case, $x = +\infty$ then $x \geq y \circ z$, holds. Finally, assume that $y \boxdot_{\uparrow} z = +0$ or $y \boxdot_{\uparrow} z = -0$. In any case $x \succcurlyeq -0$ that implies $x \geq 0$. On the other hand, we have two cases, $y \circ z \neq 0$ or $y \circ z = 0$. For the first case, by Definition 5, $-f_{\min} < y \circ z < 0$, then $x \geq y \circ z$, holds. For the second case, since $x \geq 0$ then $x \geq y \circ z$.

In order to prove (22), note that $x \succcurlyeq y \boxdot_{\mathrm{n}} z$ implies that $x \geq y \boxdot_{\mathrm{n}} z$. First observe that for $x \neq +\infty$, $x + \nabla_2^{\mathrm{n}+}(x)/2 > x$. Indeed, assume first that $x \neq f_{\max}$, then, by Definition 9, $\nabla_2^{\mathrm{n}+}(x) = x - \mathrm{pred}(x)$. Hence $x + \nabla_2^{\mathrm{n}+}(x)/2 = x + (x - \mathrm{pred}(x))/2 = (3x - \mathrm{pred}(x))/2$. Since $x > \mathrm{pred}(x)$, we can conclude that $x + \nabla_2^{\mathrm{n}+}(x)/2 > x$. Assume now that $x = f_{\max}$. By Definition 9, $\nabla_2^{\mathrm{n}+}(x) = \mathrm{succ}(x) - x$. Hence $x + \nabla_2^{\mathrm{n}+}(x)/2 = x + (\mathrm{succ}(x) - x)/2 = (x + \mathrm{succ}(x))/2$. Since $x < \mathrm{succ}(x)$, we can conclude that $x + \nabla_2^{\mathrm{n}+}(x)/2 > x$.

By Definition 5, we have to consider the following cases for $x \boxdot_{\mathrm{n}} y \in \mathbb{R}_+ \cup \mathbb{R}_-$:

$y \boxdot_{\mathrm{n}} z = [y \circ z]_{\uparrow}$. In this case, by inequality (5) of Proposition 1, $x + \nabla_2^{\mathrm{n}+}(x)/2 > x \geq y \boxdot_{\mathrm{n}} z = [y \circ z]_{\uparrow} \geq y \circ z$. Therefore, $x + \nabla_2^{\mathrm{n}+}(x)/2 > y \circ z$, holds.

$y \boxdot_{\mathrm{n}} z = [y \circ z]_{\downarrow}$. Assume first that $x > y \boxdot_{\mathrm{n}} z$. In this case, by Definition 5, since $x \in \mathbb{F}$ and $x > y \boxdot_{\mathrm{n}} z$, it must be the case that $x > y \circ z$. Hence, as in the previous case, by inequality (5) Proposition 1, $x + \nabla_2^{\mathrm{n}+}(x)/2 >$

$x > y \circ z$. Therefore, $x + \nabla_2^{\mathrm{n}+}(x)/2 > y \circ z$, holds. Assume now that $x = y \boxdot_{\mathrm{n}} z$ and $\mathrm{even}(x)$. In this case, by Proposition 6, we have that $\nabla_2^{\mathrm{n}+}\big([y \circ z]_{\mathrm{n}}\big)/2 \geq (y \circ z) - [y \circ z]_{\mathrm{n}}$. Since, in this case $x = y \boxdot_{\mathrm{n}} z$, we obtain $\nabla_2^{\mathrm{n}+}(x)/2 \geq (y \circ z) - x$. Hence, $x + \nabla_2^{\mathrm{n}+}(x)/2 \geq y \circ z$. If $\mathrm{odd}(x)$, by Proposition 6, we have that $\nabla_2^{\mathrm{n}+}\big([y \circ z]_{\mathrm{n}}\big)/2 > y \circ z - [y \circ z]_{\mathrm{n}}$. Hence, $x + \nabla_2^{\mathrm{n}+}(x)/2 > y \circ z$.

Consider now the case that $y \boxdot_{\mathrm{n}} z = +0$ or $y \boxdot_{\mathrm{n}} z = -0$. If $y \circ z \neq 0$, then $y \boxdot_{\mathrm{n}} z = [y \circ z]_{\downarrow}$ or $y \boxdot_{\mathrm{n}} z = [y \circ z]_{\uparrow}$. In this case we can reason as above. Assume now that $y \circ z = 0$. Since $x \succcurlyeq +0$ or $x \succcurlyeq -0$ implies that $x \geq 0$, we can conclude that $x + \nabla_2^{\mathrm{n}+}(x)/2 > x \geq 0$ holds. Assume now that $y \boxdot_{\mathrm{n}} z = -\infty$. If $y \circ z \neq -\infty$ then $y \boxdot_{\mathrm{n}} z = [y \circ z]_{\downarrow}$. In this case we can reason as above. On the other hand if $y \circ z = -\infty$ then $x + \nabla_2^{\mathrm{n}+}(x)/2 \geq -\infty$ holds.

Proof (of Proposition 5) We first prove (25). By inequality (5) from Proposition 1, $e \geq [e]_{\downarrow}$. Hence, $x \geq [e]_{\downarrow}$. Since by hypothesis, $e \in E_{\mathbb{F}}$ is an expression that evaluates on $\overline{\mathbb{R}}$ to a nonzero value, we have three cases:

$[e]_{\downarrow} \neq 0$ and $x \neq 0$: In this case $x \geq [e]_{\downarrow}$ implies $x \succcurlyeq [e]_{\downarrow}$.
$[e]_{\downarrow} = +0$: In this case, $0 < e < f_{\mathrm{min}}$. Then, it must be the case that $x > 0$. Therefore $x \succcurlyeq [e]_{\downarrow}$ holds.
$x = 0$: In this case x must be strictly greater than e since $e \in E_{\mathbb{F}}$ evaluates to a nonzero value. Therefore, $e < 0$. Hence, by Definition 5, $[e]_{\downarrow} \leq -f_{\mathrm{min}}$. Then $x \succcurlyeq [e]_{\downarrow}$ holds.

In all cases, we have that $x \succcurlyeq [e]_{\downarrow}$. By Definition 10, we conclude that $x \succcurlyeq [\![e]\!]_{\downarrow}$.

We now prove (26). By inequality (5) from Proposition 1, as in the previous case, $e \geq [e]_{\downarrow}$. Hence, $x > [e]_{\downarrow}$. Since by hypothesis $e \in E_{\mathbb{F}}$ is an expression that evaluates on $\overline{\mathbb{R}}$ to a nonzero value, we have three cases:

$[e]_{\downarrow} \neq 0$ and $x \neq 0$: In this case $x > [e]_{\downarrow}$ implies $x \succ [e]_{\downarrow}$.
$[e]_{\downarrow} = +0$: In this case, $0 < e < f_{\mathrm{min}}$. Hence, $x > 0$. Therefore $x \succ [e]_{\downarrow}$ holds.
$x = 0$: In this case x must be strictly greater than e since $e \in E_{\mathbb{F}}$ evaluates to a nonzero value. Therefore, $e < 0$. Hence, by Definition 5, $[e]_{\downarrow} \leq -f_{\mathrm{min}}$. Then $x \succ [e]_{\downarrow}$ holds.

In all cases, we have that $x \succ [e]_{\downarrow}$. By Definition 10, we conclude that $x \succ [\![e]\!]_{\downarrow}$. Then, by Definition 4, we have the following cases on $[\![e]\!]_{\downarrow}$:

$[\![e]\!]_{\downarrow} = f_{\mathrm{max}}$: In this case $\mathrm{succ}([\![e]\!]_{\downarrow}) = +\infty$. Since $x \succ [\![e]\!]_{\downarrow}$, this implies that $x = +\infty$. Then $x \succcurlyeq \mathrm{succ}([\![e]\!]_{\downarrow})$, holds.
$-f_{\mathrm{max}} \leq [\![e]\!]_{\downarrow} < -f_{\mathrm{min}}$ or $f_{\mathrm{min}} \leq [\![e]\!]_{\downarrow} < f_{\mathrm{max}}$: In this case $\mathrm{succ}([\![e]\!]_{\downarrow}) = \min\{\, y \in \mathbb{F} \mid y > [\![e]\!]_{\downarrow} \,\}$. Since $x > [\![e]\!]_{\downarrow}$, $x \in \{\, y \in \mathbb{F} \mid y > [\![e]\!]_{\downarrow} \,\}$. Hence, $x \succcurlyeq \mathrm{succ}([\![e]\!]_{\downarrow})$, holds.
$[\![e]\!]_{\downarrow} = +0$ or $[\![e]\!]_{\downarrow} = -0$: In this case $\mathrm{succ}([\![e]\!]_{\downarrow}) = f_{\mathrm{min}}$. Since $x > [\![e]\!]_{\downarrow}$, this implies that $x \geq f_{\mathrm{min}}$. Hence, $x \succcurlyeq \mathrm{succ}([\![e]\!]_{\downarrow})$, holds.
$[\![e]\!]_{\downarrow} = -f_{\mathrm{min}}$: In this case $\mathrm{succ}([\![e]\!]_{\downarrow}) = -0$. Since $x > [\![e]\!]_{\downarrow} = -f_{\mathrm{min}}$, $x \succcurlyeq -0$. Hence, $x \succcurlyeq \mathrm{succ}([\![e]\!]_{\downarrow})$, holds.
$[\![e]\!]_{\downarrow} = -\infty$: In this case $\mathrm{succ}([\![e]\!]_{\downarrow}) = -f_{\mathrm{max}}$. Since $x > [\![e]\!]_{\downarrow} = -\infty$, $x \succcurlyeq -f_{\mathrm{max}}$. Hence, $x \succcurlyeq \mathrm{succ}([\![e]\!]_{\downarrow})$, holds.

We now prove (27). By inequality (5) from Proposition 1, $e \leq [e]_{\uparrow}$. Hence, like before, $x \leq [e]_{\uparrow}$. Since by hypothesis $e \in E_{\mathbb{F}}$ is an expression that evaluates on $\overline{\mathbb{R}}$ to a nonzero value, we have three cases:

$[e]_{\uparrow} \neq 0$ and $x \neq 0$: In this case $x \leq [e]_{\uparrow}$ implies $x \preccurlyeq [e]_{\uparrow}$.
$[e]_{\uparrow} = -0$: In this case, $-f_{\mathrm{min}} < e < 0$. Hence, $x < 0$. Therefore $x \preccurlyeq [e]_{\uparrow}$ holds.
$x = 0$: In this case it must be the case that x is strictly smaller than e, since $e \in E_{\mathbb{F}}$ evaluates to a nonzero value. Therefore, $e > 0$. Hence, by Definition 5, $[e]_{\uparrow} \geq f_{\mathrm{min}}$. Then $x \preccurlyeq [e]_{\uparrow}$ holds.

In any case, $x \preccurlyeq [e]_{\uparrow}$ holds. By Definition 10, we conclude that $x \preccurlyeq [\![e]\!]_{\uparrow}$.

Next we prove (28). By, again, inequality (5) from Proposition 1, $e \leq [e]_{\uparrow}$. Hence, $x < [e]_{\uparrow}$. Since by hypothesis $e \in E_{\mathbb{F}}$ is an expression that evaluates on $\overline{\mathbb{R}}$ to a nonzero value, we have three cases:

$[e]_{\uparrow} \neq 0$ and $x \neq 0$: In this case $x < [e]_{\uparrow}$ implies $x \prec [e]_{\uparrow}$.
$[e]_{\uparrow} = -0$: In this case, $-f_{\mathrm{min}} < e < 0$. Hence, $x < 0$. Therefore $x \prec [e]_{\uparrow}$ holds.
$x = 0$: In this case it must be the case that x is strictly smaller than e, since $e \in E_{\mathbb{F}}$ evaluates to a nonzero value. Therefore, $e > 0$. Hence, by Definition 5, $[e]_{\uparrow} \geq f_{\mathrm{min}}$. Then $x \prec [e]_{\uparrow}$ holds.

In any case, $x \prec [e]_{\uparrow}$ holds. By Definition 10, we conclude that $x \prec [\![e]\!]_{\uparrow}$. By Definition 4, we have the following cases on $[\![e]\!]_{\uparrow}$:

$[\![e]\!]_{\uparrow} = -f_{\mathrm{max}}$: In this case $\mathrm{pred}([\![e]\!]_{\uparrow}) = -\infty$. Since $x \prec [\![e]\!]_{\uparrow}$, this implies that $x = -\infty$. Then $x \preccurlyeq \mathrm{pred}([\![e]\!]_{\uparrow})$, holds.
$f_{\mathrm{min}} < [\![e]\!]_{\uparrow} \leq f_{\mathrm{max}}$ or $-f_{\mathrm{max}} < [\![e]\!]_{\uparrow} \leq -f_{\mathrm{min}}$: In this case $\mathrm{pred}([\![e]\!]_{\uparrow}) = \max\{\, y \in \mathbb{F} \mid y < [\![e]\!]_{\uparrow} \,\}$. Since $x < [\![e]\!]_{\uparrow}$, $x \in \{\, y \in \mathbb{F} \mid y < [\![e]\!]_{\uparrow} \,\}$. Hence, $x \preccurlyeq \mathrm{pred}([\![e]\!]_{\uparrow})$, holds.
$[\![e]\!]_{\uparrow} = +0$ or $[\![e]\!]_{\uparrow} = -0$: In this case $\mathrm{pred}([\![e]\!]_{\uparrow}) = -f_{\mathrm{min}}$. Since $x < [\![e]\!]_{\uparrow}$, this implies that $x \leq -f_{\mathrm{min}}$. Hence, $x \preccurlyeq \mathrm{pred}([\![e]\!]_{\uparrow})$, holds.

$[\![e]\!]_\uparrow = f_{\min}$: In this case $\mathrm{pred}([\![e]\!]_\uparrow) = +0$. Since $x < [\![e]\!]_\uparrow = f_{\min}$, $x \preccurlyeq +0$. Hence, $x \preccurlyeq \mathrm{pred}([\![e]\!]_\uparrow)$, holds.
$[\![e]\!]_\downarrow = +\infty$: In this case $\mathrm{pred}([\![e]\!]_\uparrow) = f_{\max}$. Since $x < [\![e]\!]_\uparrow = \infty$, $x \preccurlyeq f_{\max}$. Hence, $x \preccurlyeq \mathrm{pred}([\![e]\!]_\uparrow)$, holds.

In order to prove (29) we first want to prove that $x \succcurlyeq [e]_\uparrow$. To this aim consider the following cases for e:

$e > f_{\max}$: In this case $[e]_\uparrow = +\infty$. On the hand, $x \geq e > f_{\max}$. Since $x \in \mathbb{F}$ implies that $x = +\infty$. Hence $x \succcurlyeq [e]_\uparrow$.
$e \leq -f_{\min}$ or $0 < e \leq f_{\max}$: In this case $[e]_\uparrow = \min\{\, z \in \mathbb{F} \mid z \geq e \,\}$. Since $x \geq e$, $x \in \{\, z \in \mathbb{F} \mid z \geq e \,\}$. Hence, $x \geq [e]_\uparrow$, holds and also $x \succcurlyeq [e]_\uparrow$.
$-f_{\min} < e < 0$: In this case $[e]_\uparrow = -0$. Since $x \geq e$ and $x \in \mathbb{F}$, $x \succcurlyeq -0$, holds.
$e = -\infty$: In this case $[e]_\uparrow = -\infty$ and $x \succcurlyeq -\infty$ holds.

Since by hypothesis $[e]_\uparrow = [\![e]\!]_\uparrow$, we can conclude that $x \succcurlyeq [\![e]\!]_\uparrow$ holds.

In order to prove (30) we first want to prove that $x \preccurlyeq [e]_\downarrow$. To this aim consider the following cases for e:

$e < -f_{\max}$: In this case $[e]_\downarrow = -\infty$. On the hand, $x \leq e < -f_{\max}$. Since $x \in \mathbb{F}$ implies that $x = -\infty$. Hence $x \preccurlyeq [e]_\downarrow$.
$e \geq f_{\min}$ or $-f_{\max} \leq e < 0$: In this case $[e]_\downarrow = \max\{\, z \in \mathbb{F} \mid z \leq e \,\}$. Since $x \leq e$, $x \in \{\, z \in \mathbb{F} \mid z \leq e \,\}$. Hence, $x \leq [e]_\downarrow$, holds and also $x \preccurlyeq [e]_\downarrow$.
$0 < e < f_{\min}$: In this case $[e]_\downarrow = +0$. Since $x \leq e$ and $x \in \mathbb{F}$, $x \preccurlyeq +0$, holds.
$e = +\infty$: In this case $[e]_\downarrow = \infty$ and $x \preccurlyeq +\infty$ holds.

Since by hypothesis $[e]_\downarrow = [\![e]\!]_\downarrow$, we can conclude that $x \preccurlyeq [\![e]\!]_\downarrow$ holds.

B.3 Proofs of Results in Section 4.3

Proof (of Theorem 3) Given the constraint $x = y \boxslash_S z$ with $x \in X = [x_\ell, x_u]$, $y \in Y = [y_\ell, y_u]$ and $z \in Z = [z_\ell, z_u]$, Algorithm 3 computes a new refining interval X' for variable x. Note that $X' = [x'_\ell, x'_u] \cap X$, which assures us that $X' \subseteq X$.

As for the proof of Theorem 10, it is easy to verify that y_L and w_L (resp., y_U and w_U) computed using function τ of Figure 5, are the boundaries of Y and W upon which x touches its minimum (resp., maximum). Moreover, remember that by Proposition 2, following the same reasoning of the proofs of the previous theorems, we can focus on finding a lower bound for $y_L \boxslash_{r_\ell} w_L$ and an upper bound for $y_U \boxslash_{r_u} w_U$.

We will now comment only on the most critical entries of function dd_ℓ of Figure 6: let us briefly discuss the cases in which $y_L = -\infty$ and $w_L = \pm\infty$.

$w_L = -\infty$. In this case, by function τ of Figure 5 (see the first three cases), we have $y_L = y_u = -\infty$, while either $w_L = w_\ell$ or $w_L = w_u$. Since by the IEEE 754 Standard [24] dividing $\pm\infty$ by $\pm\infty$ is an invalid operation, we are left to consider the case $w_L = w_\ell$. In this case, recall that by the IEEE 754 Standard [24], dividing $-\infty$ by a finite negative number yields $+\infty$. Hence, we can conclude $x_\ell = +\infty$.
$w_L = +\infty$. By function τ of Figure 5 (see the fourth and last case), we have $y_L = y_\ell = -\infty$, while $w_L = w_\ell = +\infty$. Hence, $x_\ell = -0$, since dividing a negative finite number by $+\infty$ gives -0.

A similar reasoning applies for the cases $y_L = +\infty$, $w_L = \pm\infty$. Dually, the only critical entries of function dd_u of Figure 6 are those in which $y_U = \pm\infty$ and $w_U = \pm\infty$ and can be handled analogously.

We are left to prove that $\forall X'' \subset X, \exists r \in S, y \in Y, z \in Z : y \boxslash_r z \notin X''$. Let us focus on the lower bound x_ℓ^+ proving that, if $[x_\ell^+, x_\ell^+] \neq \varnothing$, then there exist $r \in S, y \in Y, z \in Z$ such that $y \boxslash_r z = x_\ell^+$. Consider the particular values y_L, $z_\ell = w_L$ and r_ℓ that correspond to x_ℓ^+ in Algorithm 3, i.e. y_L and w_L and r_ℓ are such that $\mathrm{dd}_\ell(y_L, w_L, r_\ell) = x_\ell^+$. By Algorithm 3, such y_L and w_L must exist. First consider the cases in which $y_L \notin (\mathbb{R}_- \cup \mathbb{R}_+)$ or $w_L \notin (\mathbb{R}_- \cup \mathbb{R}_+)$. A brute-force verification was successfully conducted, in this cases, to prove that $y_L \boxslash_{r_\ell} w_L = x_\ell^+$. For the cases in which $y_L \in (\mathbb{R}_- \cup \mathbb{R}_+)$ and $w_L \in (\mathbb{R}_- \cup \mathbb{R}_+)$ we have, by definition of dd_ℓ of Figure 6, that $x_\ell^+ = y_L \boxslash_{r_\ell} w_L$. Remember that, by Proposition 2, there exists $r \in S$ such that $y_L \boxslash_{\mathrm{r}_\ell} w_L = y_L \boxslash_{\mathrm{r}} w_L$. Since $y_L \in Y$ and $w_L \in Z$, we can conclude that $x_\ell^+ \notin X''$ implies that $y_L \boxslash_{\mathrm{r}} w_L \notin X''$, for any $X'' \subseteq X'$. An analogous reasoning applies to x_ℓ^-, to x_u^+ and x_u^-. This allows us to prove the optimality claim.

Proof (of Theorem 4) Given the constraint $x = y \boxslash_S z$ with $x \in X = [x_\ell, x_u]$, $y \in Y = [y_\ell, y_u]$ and $z \in Z = [z_\ell, z_u]$, Algorithm 4 computes a new, refining interval Y' for variable y. It returns either $Y' := (Y \cap [y_\ell^-, y_u^-]) \uplus (Y \cap [y_\ell^+, y_u^+])$ or $Y' = \varnothing$: hence, in both cases, we are sure that $Y' \subseteq Y$.

By Proposition 2, we can focus on finding a lower bound for $y \in Y$ by exploiting the constraint $y \boxslash_{\bar{r}_\ell} z = x$ and an upper bound for y by exploiting the constraint $y \boxslash_{\bar{r}_u} z = x$.

In order to compute correct bounds for y, Algorithm 5 first splits the interval of z into the sign-homogeneous intervals Z_- and Z_+, since knowing the sign of z is crucial to determine correct bounds for y. Hence, for

$W = Z_-$ (and, analogously, for $W = Z_+$), it calls function σ of Figure 7 to determine the appropriate extrema of intervals X and W to be used to compute the new lower and upper bounds for y. As we did in the proof of Theorem 9, it is easy to verify that x_L and w_L (resp., x_U and w_U), computed using function σ of Figure 7, are the boundaries of X and W upon which y touches its minimum (resp., maximum). Functions id_ℓ^f of Figure 8 and id_u^f of Figure 9 are then used to find the new bounds for y. The so obtained intervals for y will be eventually joined using convex union to obtain the refining interval for y.

We will now prove the non-trivial parts of the definitions of functions id_ℓ^f and id_u^f. Concerning the case analysis of id_ℓ^f (Fig 8) marked as a_4, the result changes depending on the selected rounding mode:

$\bar{r}_\ell = \uparrow$: we clearly must have $y = +\infty$, according to the IEEE 754 Standard [24];

$\bar{r}_\ell = \downarrow$: it must be $y/w_L < -f_{\max}$ and thus, since w_L is negative, $y > -f_{\max} \cdot w_L$ and, by (26) of Proposition 5, $y \succcurlyeq \mathrm{succ}(-f_{\max} \boxdot_\downarrow w_L)$.

$\bar{r}_\ell = \mathrm{n}$: since $\mathrm{odd}(f_{\max})$, for $w_L = -\infty$ we need y to be greater than or equal to $\bigl(-f_{\max} + \nabla_2^{\mathrm{n}-}(-f_{\max})/2\bigr) \cdot w_L$. If $[\![\bigl(-f_{\max} + \nabla_2^{\mathrm{n}-}(-f_{\max})/2\bigr) \cdot w_L]\!]_\uparrow = [\bigl(-f_{\max} + \nabla_2^{\mathrm{n}-}(-f_{\max})/2\bigr) \cdot w_L]_\uparrow$, by (29) of Proposition 5, we can conclude $y \succcurlyeq [\![\bigl(-f_{\max} + \nabla_2^{\mathrm{n}-}(-f_{\max})/2\bigr) \cdot w_L]\!]_\uparrow$. On the other hand, if $[\![\bigl(-f_{\max} + \nabla_2^{\mathrm{n}-}(-f_{\max})/2\bigr) \cdot w_L]\!]_\uparrow \neq [\bigl(-f_{\max} + \nabla_2^{\mathrm{n}-}(-f_{\max})/2\bigr) \cdot w_L]_\uparrow$, then we can only apply (25) of Proposition 5, obtaining $y \succcurlyeq [\![\bigl(-f_{\max} + \nabla_2^{\mathrm{n}-}(-f_{\max})/2\bigr) \cdot w_L]\!]_\downarrow$.

The case analysis of id_ℓ^f (Fig 8) marked as a_5 can be explained as follows:

$\bar{r}_\ell = \downarrow$: we must have $y = +\infty$, according to the IEEE 754 Standard [24];

$\bar{r}_\ell = \uparrow$: inequality $y/w_L > f_{\max}$ must hold and thus, since w_L is positive, $y > f_{\max} \cdot w_L$ and, by (26) of Proposition 5, $y \succcurlyeq \mathrm{succ}(f_{\max} \boxdot_\downarrow w_L)$.

$\bar{r}_\ell = \mathrm{n}$: since $\mathrm{odd}(f_{\max})$, for $x_L = +\infty$ we need y to be greater than or equal to $\bigl(f_{\max} + \nabla_2^{\mathrm{n}+}(f_{\max})/2\bigr) \cdot w_L$. If $[\![\bigl(f_{\max} + \nabla_2^{\mathrm{n}+}(f_{\max})/2\bigr) \cdot w_L]\!]_\uparrow = [\bigl(f_{\max} + \nabla_2^{\mathrm{n}+}(f_{\max})/2\bigr) \cdot w_L]_\uparrow$, by (29) of Proposition 5, we can conclude $y \succcurlyeq [\![\bigl(f_{\max} + \nabla_2^{\mathrm{n}+}(f_{\max})/2\bigr) \cdot w_L]\!]_\uparrow$. On the other hand, if $[\![\bigl(f_{\max} + \nabla_2^{\mathrm{n}+}(f_{\max})/2\bigr) \cdot w_L]\!]_\uparrow \neq [\bigl(f_{\max} + \nabla_2^{\mathrm{n}+}(f_{\max})/2\bigr) \cdot w_L]_\uparrow$ then, we can only apply (25) of Proposition 5, obtaining $y \succcurlyeq [\![\bigl(f_{\max} + \nabla_2^{\mathrm{n}+}(f_{\max})/2\bigr) \cdot w_L]\!]_\downarrow$.

The explanation for the case analysis of id_ℓ^f (Fig 8) marked as a_6 is the following:

$\bar{r}_\ell = \uparrow$: the lowest value of y that yields $x_L = +0$ with $w_L \in \mathbb{R}_-$ is clearly $y = -0$;

$\bar{r}_\ell = \downarrow$: inequality $y/w_L < f_{\min}$ should hold and thus, since w_L is negative, $y > f_{\min} \cdot w_L$ and, by (26) of Proposition 5, $y \succcurlyeq \mathrm{succ}(f_{\min} \boxdot_\downarrow w_L)$.

$\bar{r}_\ell = \mathrm{n}$: since $\mathrm{odd}(f_{\min})$, for $x_L = +0$ we need y to be greater than or equal to $(f_{\min} \cdot w_L)/2$. Since in this case $[\![(f_{\min} \cdot w_L)/2]\!]_\uparrow = [(f_{\min} \cdot w_L)/2]_\uparrow = (f_{\min} \boxdot_\uparrow w_L)/2$, by (29) of Proposition 5, we can conclude $y \succcurlyeq (f_{\min} \boxdot_\uparrow w_L)/2$.

Concerning the case analysis of id_ℓ^f (Fig 8) marked as a_7, we must distinguish between the following cases:

$\bar{r}_\ell = \downarrow$: considering $x_L = -0$ and $w_L \in \mathbb{R}_+$, we clearly must have $y = -0$;

$\bar{r}_\ell = \uparrow$: it should be $y/w_L > -f_{\min}$ and thus, since w_L is positive, $y > -f_{\min} \cdot w_L$ and, by (26) of Proposition 5, $y \succcurlyeq \mathrm{succ}(-f_{\min} \boxdot_\downarrow w_L)$.

$\bar{r}_\ell = \mathrm{n}$: since $\mathrm{odd}(f_{\min})$, for $x_L = -0$ we need y be to greater than or equal to $\bigl(-f_{\min} \cdot w_L\bigr)/2$. Since in this case $[\![\bigl(-f_{\min} \cdot w_L\bigr)/2]\!]_\uparrow = [\bigl(-f_{\min} \cdot w_L\bigr)/2]_\uparrow = (-f_{\min} \boxdot_\uparrow w_L)/2$, by (29) of Proposition 5, we can conclude $y \succcurlyeq (-f_{\min} \boxdot_\uparrow w_L)/2$.

Similar arguments can be used to prove the case analyses of id_u^f of Fig 9 marked as a_9, a_{10}, a_{11} and a_{12}.

We will now analyze the case analyses of id_ℓ^f of Fig 8 marked as a_3^- and a_3^+, and the ones of id_u^f of Fig 9 marked as a_8^- and a_8^+. We can assume, of course, $X = [x_\ell, x_u]$, $Y = [y_\ell, y_u]$ and $Z = [w_\ell, w_u]$, where $x_\ell, x_u, w_\ell, w_u \in \mathbb{F} \cap \mathbb{R}$, $x_\ell \leq x_u$, $w_\ell \leq w_u$ and $\mathrm{sgn}(w_\ell) = \mathrm{sgn}(w_u)$. Exploiting $x \preccurlyeq y \boxslash z$ and $x \succcurlyeq y \boxslash z$, by Proposition 4, we have

$$
y/z \begin{cases} \geq x, & \text{if } \bar{r}_\ell = \downarrow; \\ > x + \nabla^{\uparrow}(x) = \mathrm{pred}(x), & \text{if } \bar{r}_\ell = \uparrow; \\ \geq x + \nabla_2^{\mathrm{n}-}(x)/2, & \text{if } \bar{r}_\ell = \mathrm{n} \text{ and } \mathrm{even}(x); \\ > x + \nabla_2^{\mathrm{n}-}(x)/2, & \text{if } \bar{r}_\ell = \mathrm{n} \text{ and } \mathrm{odd}(x). \end{cases} \tag{46}
$$

$$
y/z \begin{cases} < x + \nabla^{\downarrow}(x) = \mathrm{succ}(x), & \text{if } \bar{r}_u = \downarrow; \\ \leq x, & \text{if } \bar{r}_u = \uparrow; \\ \leq x + \nabla_2^{\mathrm{n}+}(x)/2, & \text{if } \bar{r}_u = \mathrm{n} \text{ and } \mathrm{even}(x); \\ < x + \nabla_2^{\mathrm{n}+}(x)/2, & \text{if } \bar{r}_u = \mathrm{n} \text{ and } \mathrm{odd}(x). \end{cases} \tag{47}
$$

Since the case $z = 0$ is handled separately by id_ℓ^f of Fig 8 and by id_u^f of Fig 9, we can assume $z \neq 0$. Thanks to the split of Z into a positive and a negative part, the sign of z is determinate. In the following, we will prove the case analyses marked as a_3^+ and a_8^+, hence assuming $z > 0$. From the previous case analysis we can derive

$$y \begin{cases} \geq x \cdot z, & \text{if } \bar{r}_\ell = \downarrow; \\ > \mathrm{pred}(x) \cdot z, & \text{if } \bar{r}_\ell = \uparrow \\ \geq \big(x + \nabla_2^{\mathrm{n}-}(x)/2\big) \cdot z, & \text{if } \bar{r}_\ell = \mathrm{n} \text{ and even}(x); \\ > \big(x + \nabla_2^{\mathrm{n}-}(x)/2\big) \cdot z, & \text{if } \bar{r}_\ell = \mathrm{n} \text{ and odd}(x); \end{cases} \tag{48}$$

$$y \begin{cases} < \mathrm{succ}(x) \cdot z, & \text{if } \bar{r}_u = \downarrow; \\ \leq x \cdot z, & \text{if } \bar{r}_u = \uparrow; \\ \leq \big(x + \nabla_2^{\mathrm{n}+}(x)/2\big) \cdot z, & \text{if } \bar{r}_u = \mathrm{n} \text{ and even}(x); \\ < \big(x + \nabla_2^{\mathrm{n}+}(x)/2\big) \cdot z, & \text{if } \bar{r}_u = \mathrm{n} \text{ and odd}(x). \end{cases} \tag{49}$$

Note that the members of the product are independent. Therefore, we can find the minimum of the product by minimizing each member of the product. Since we are analyzing the case in which $W = Z_+$, let (x_L, x_U, w_L, w_U) as defined in function σ of Figure 7, replacing the role of y with z and the role of z with x. Hence, by Proposition 3 and the monotonicity of 'pred' and 'succ' we obtain

$$y \begin{cases} \geq x_L \cdot w_L, & \text{if } \bar{r}_\ell = \downarrow; \\ > \mathrm{pred}(x_L) \cdot w_L, & \text{if } \bar{r}_\ell = \uparrow \\ \geq \big(x_L + \nabla_2^{\mathrm{n}-}(x_L)/2\big) \cdot w_L, & \text{if } \bar{r}_\ell = \mathrm{n} \text{ and even}(x); \\ > \big(x_L + \nabla_2^{\mathrm{n}-}(x_L)/2\big) \cdot w_L, & \text{if } \bar{r}_\ell = \mathrm{n} \text{ and odd}(x); \end{cases} \tag{50}$$

$$y \begin{cases} < \mathrm{succ}(x_U) \cdot w_U, & \text{if } \bar{r}_u = \downarrow; \\ \leq x_U \cdot w_U, & \text{if } \bar{r}_u = \uparrow; \\ \leq \big(x_U + \nabla_2^{\mathrm{n}+}(x_U)/2\big) \cdot w_U, & \text{if } \bar{r}_u = \mathrm{n} \text{ and even}(x); \\ < \big(x_U + \nabla_2^{\mathrm{n}+}(x_U)/2\big) \cdot w_U, & \text{if } \bar{r}_u = \mathrm{n} \text{ and odd}(x). \end{cases} \tag{51}$$

We can now exploit Proposition 5 and obtain:

$$y'_\ell \stackrel{\text{def}}{=} \begin{cases} x_L \boxdot_\uparrow w_L, & \text{if } \bar{r}_\ell = \downarrow; \\ \mathrm{succ}\big(\mathrm{pred}(x_L) \boxdot_\downarrow w_L\big), & \text{if } \bar{r}_\ell = \uparrow; \end{cases} \tag{52}$$

$$y'_u \stackrel{\text{def}}{=} \begin{cases} \mathrm{pred}\big(\mathrm{succ}(x_U) \boxdot_\uparrow w_U\big), & \text{if } \bar{r}_u = \downarrow; \\ x_U \boxdot_\downarrow w_U, & \text{if } \bar{r}_u = \uparrow. \end{cases} \tag{53}$$

Indeed, if $\bar{r}_\ell = \uparrow$ and $x_L \neq 0$, then part (29) of Proposition 5 applies and we have $y \succcurlyeq x_L \boxdot_\uparrow w_L$. On the other hand, if $x_L = 0$, since by hypothesis $z > 0$ implies $w_L > 0$, according to IEEE 754 [24, Section 6.3], we have $x_L \boxdot_\uparrow w_L = \mathrm{sgn}(x_L) \cdot 0$ and, indeed, for each non-NaN, nonzero and finite $w \in \mathbb{F} \cap [+0, +\infty]$, $\mathrm{sgn}(x_L) \cdot 0$ is the least value for y that satisfies $\mathrm{sgn}(x_L) \cdot 0 = y \boxslash_\downarrow w$.

Analogously, if $\bar{r}_\ell = \uparrow$ and $x_L \neq f_{\min}$, then Proposition 5 applies and we have $\mathrm{succ}\big(\mathrm{pred}(x_L) \boxdot_\downarrow w_L\big)$. On the other hand, if $x_L = f_{\min}$, in this case, $\mathrm{succ}\big(\mathrm{pred}(x_L) \boxdot_\downarrow w_L\big) = f_{\min}$ which is consistent with the fact that, for each non-NaN, nonzero and finite $w \in \mathbb{F} \cap [+0, +\infty]$, $f_{\min}$ is the lowest value for y that satisfies $f_{\min} = y \boxslash_\uparrow w$.

A symmetric argument justifies (53).

As before, we need to approximate the values of the expressions $e_\ell^+ = \big(x_L + \nabla_2^{\mathrm{n}-}(x_L)/2\big) \cdot w_L$ and $e_u^+ = \big(x_U + \nabla_2^{\mathrm{n}+}(x_U)/2\big) \cdot w_U$. We leave this as an implementation choice, thus taking into account the case $[\![e_\ell^+]\!]_\uparrow = [e_\ell^+]_\uparrow$ and $[\![e_u^+]\!]_\downarrow = [e_u^+]_\downarrow$ as well as $[\![e_\ell^+]\!]_\uparrow > [e_\ell^+]_\uparrow$ and $[\![e_u^+]\!]_\downarrow < [e_u^+]_\downarrow$. Therefore, when $[\![e_u^+]\!]_\downarrow < [e_u^+]_\downarrow$ by (51) and (27) of Proposition 5 we obtain $y \preccurlyeq [\![e_u^+]\!]_\uparrow$, while, when $[\![e_\ell^+]\!]_\downarrow > [e_\ell^+]_\downarrow$ by (51) and (25) of Proposition 5 we obtain $y \succcurlyeq [\![e_\ell^+]\!]_\downarrow$.

Thus, for the case in which $\bar{r}_\ell = \mathrm{n}$, since $e_u^+ \neq 0$ and $e_\ell^+ \neq 0$, by Proposition 5, we have

$$y'_\ell \stackrel{\text{def}}{=} \begin{cases} [\![e_\ell^+]\!]_\uparrow, & \text{if even}(x_L) \text{ and } [\![e_\ell^+]\!]_\uparrow = [e_\ell^+]_\uparrow; \\ [\![e_\ell^+]\!]_\downarrow, & \text{if even}(x_L) \text{ and } [\![e_\ell^+]\!]_\uparrow \neq [e_\ell^+]_\uparrow; \\ \mathrm{succ}\big([\![e_\ell^+]\!]_\downarrow\big), & \text{otherwise}; \end{cases} \tag{54}$$

whereas, for the case in which $\bar{r}_u = \mathrm{n}$,

$$y'_u \stackrel{\text{def}}{=} \begin{cases} [\![e_u^+]\!]_\downarrow, & \text{if } \mathrm{even}(x_U) \text{ and } [\![e_u^+]\!]_\downarrow = [e_u^+]_\downarrow; \\ [\![e_u^+]\!]_\uparrow, & \text{if } \mathrm{even}(x_U) \text{ and } [\![e_u^+]\!]_\downarrow \neq [e_u^+]_\downarrow; \\ \mathrm{pred}\big([\![e_u^+]\!]_\uparrow\big), & \text{otherwise.} \end{cases} \tag{55}$$

An analogous reasoning, but with $z < 0$, allows us to obtain the case analyses marked as a_3^- and a_8^-.

Proof (of Theorem 5) Given the constraint $x = y \boxslash_S z$ with $x \in X = [x_\ell, x_u]$, $y \in Y = [y_\ell, y_u]$ and $z \in Z = [z_\ell, z_u]$, Algorithm 5 finds a new, refined interval Z' for variable z.

Since it assigns either $Z' := (Z \cap [z_\ell^-, z_u^-]) \uplus (Z \cap [z_\ell^+, z_u^+])$ or $Z' = \varnothing$, in both cases we are sure that $Z' \subseteq Z$. By Proposition 2, as in the previous proofs, we can focus on finding a lower bound for $z \in Z$ by exploiting the constraint $y \boxslash_{\bar{r}_\ell} z = x$ and an upper bound for z by exploiting the constraint $y \boxslash_{\bar{r}_u} z = x$.

We first need to split interval X into the sign-homogeneous intervals X_- and X_+, because knowing the sign of x is crucial for determining correct bounds for z. Hence, for $V = X_-$ (and, analogously, for $V = X_+$) function τ of Figure 5 determines the appropriate interval extrema of Y and V to be used to compute the new lower and upper bounds for z. As in the previous proofs (see, for example, proof of Theorem 10), it is easy to verify that y_L and v_L (resp., y_U and v_U) computed using function τ of Figure 5 are the boundaries of Y and V upon which z touches its minimum (resp., maximum). Functions id_ℓ^s of Figure 10 and id_u^s of Figure 11 are then used to find the new bounds for z. The so obtained intervals for z will be then joined with convex union in order to obtain the refining interval for z.

We will prove the most important parts of the definitions of id_ℓ^s (Figure 10) and id_u^s (Figure 11) only, starting with the case analysis marked as a_4. Depending on the rounding mode in effect, the following arguments are given:

$\bar{r}_\ell = \downarrow$: in this case, the only possible way to obtain -0 as the result of the division is having $z = +\infty$ (with $y \in \mathbb{R}_-$);

$\bar{r}_\ell = \uparrow$: it should be $y_L/z > -f_{\min}$ and thus, since y_L and x_L are negative, we can conclude that z is positive. Thus, $y_L > -f_{\min} \cdot z$ implies $y_L/-f_{\min} < z$, and by (26) of Proposition 5, $z \succcurlyeq \mathrm{succ}(z_L \boxslash_\downarrow -f_{\min})$.

$\bar{r}_\ell = \mathrm{n}$: since $\mathrm{odd}(-f_{\min})$, for $v_L = -0$ we need $y_L/z \geq (-f_{\min} + \nabla_2^{\mathrm{n}+}(-f_{\min})/2) = (-f_{\min} + f_{\min}/2) = -f_{\min}/2$. As before, since y_L and v_L are negative, we can conclude that z is positive: hence $y_L \geq (-f_{\min}/2) \cdot z$. Therefore, $z \geq y_L/(-f_{\min}/2) = z \geq (y_L/-f_{\min}) \cdot 2$. Since in this case $[\![(y_L/-f_{\min}) \cdot 2]\!]_\uparrow = [(y_L/-f_{\min}) \cdot 2]_\uparrow = (y_L \boxslash_\uparrow -f_{\min}) \cdot 2$, by (29) of Proposition 5, we can conclude $y \succcurlyeq (y_L \boxslash_\uparrow -f_{\min}) \cdot 2$.

As for the case analysis of id_ℓ^s (Figure 10) marked as a_5, we must distinguish between the following cases:

$\bar{r}_\ell = \uparrow$: we must have $z = +\infty$ in order to obtain $x = +0$;

$\bar{r}_\ell = \downarrow$: inequality $y_L/z < f_{\min}$ must hold and thus, since positive y_L and v_L imply a positive z, $z > y_L/f_{\min}$ and, by (26) of Proposition 5, $z \succcurlyeq \mathrm{succ}(y_L \boxslash_\downarrow f_{\min})$.

$\bar{r}_\ell = \mathrm{n}$: since $\mathrm{odd}(f_{\min})$, for $v_L = +0$ we need $y_L/z \leq f_{\min}/2$. As z is positive in this case, $(y_L/f_{\min}) \cdot 2 \leq z$. Since $[\![(y_L/f_{\min}) \cdot 2]\!]_\uparrow = [(y_L/f_{\min}) \cdot 2]_\uparrow = (y_L \boxslash_\uparrow f_{\min}) \cdot 2$, by (29) of Proposition 5, we can conclude $y \succcurlyeq (y_L \boxslash_\uparrow f_{\min}) \cdot 2$.

Concerning the case analysis of id_ℓ^s (Fig 10) marked as a_6, we must distinguish between the following cases:

$\bar{r}_\ell = \downarrow$: the lowest value of z that gives $x = +\infty$ with $y \in \mathbb{R}_-$ is $z = -0$;

$\bar{r}_\ell = \uparrow$: inequality $y_L/z > f_{\max}$ must hold; since y_L is negative and v_L is positive, z must be negative, and therefore $y_L < f_{\max} \cdot z$. Hence, $y_L/f_{\max} < z$. By (26) of Proposition 5, we obtain $z \succcurlyeq \mathrm{succ}(y_L \boxslash_\downarrow f_{\max})$.

$\bar{r}_\ell = \mathrm{n}$: since $\mathrm{odd}(f_{\max})$, for $v_L = +\infty$ we need $y_L/z \geq (f_{\max} + \nabla_2^{\mathrm{n}+}(f_{\max})/2)$. As before, since w_L is negative and v_L is positive, we can conclude that z is negative, and, therefore, $y_L \leq (f_{\max} + \nabla_2^{\mathrm{n}+}(f_{\max})/2) \cdot z$ holds. As a consequence, $y_L/(f_{\max} + \nabla_2^{\mathrm{n}+}(f_{\max})/2) \leq z$. If $[\![y_L/(f_{\max} + \nabla_2^{\mathrm{n}+}(f_{\max})/2)]\!]_\uparrow = [y_L/(f_{\max} + \nabla_2^{\mathrm{n}+}(f_{\max})/2)]_\uparrow$, by (29) of Proposition 5, we can conclude $z \succcurlyeq [\![y_L/(f_{\max} + \nabla_2^{\mathrm{n}+}(f_{\max})/2)]\!]_\uparrow$. On the other hand, if $[\![y_L/(f_{\max} + \nabla_2^{\mathrm{n}+}(f_{\max})/2)]\!]_\uparrow \neq [y_L/(f_{\max} + \nabla_2^{\mathrm{n}+}(f_{\max})/2)]_\uparrow$ then, we can only apply (25) of Proposition 5, obtaining $z \succcurlyeq [\![y_L/(f_{\max} + \nabla_2^{\mathrm{n}+}(f_{\max})/2)]\!]_\downarrow$.

Regarding the case analysis of id_ℓ^s (Fig 10) marked as a_7, we have the following cases:

$\bar{r}_\ell = \uparrow$: the lowest value of z that yields $x = -\infty$ with $y \in \mathbb{R}_+$ is $z = -0$;

$\bar{r}_\ell = \downarrow$: inequality $y_L/z < -f_{\max}$ must hold and thus, since a positive y_L and a negative v_L imply that the sign of z is negative, $y_L > -f_{\max} \cdot z$. Hence, $y_L/-f_{\max} < z$. By (26) of Proposition 5, $z \succcurlyeq \mathrm{succ}(y_L \boxslash_\downarrow -f_{\max})$.

$\bar{r}_\ell = \mathrm{n}:$ since $\mathrm{odd}(-f_{\max})$, for $v_L = -\infty$ we need $y_L/z \leq -f_{\max} + \nabla_2^{\mathrm{n}-}(-f_{\max})/2$. Since z in this case is negative, we obtain the inequality $z \geq y_L/(-f_{\max} + \nabla_2^{\mathrm{n}-}(-f_{\max})/2)$. If $[\![y_L/\big(-f_{\max} + \nabla_2^{\mathrm{n}-}(-f_{\max})/2\big)]\!]_\uparrow = [y_L/\big(-f_{\max} + \nabla_2^{\mathrm{n}-}(-f_{\max})/2\big)]_\uparrow$, by (29) of Proposition 5, we can conclude $y \succcurlyeq [\![y_L/\big(-f_{\max} + \nabla_2^{\mathrm{n}-}(-f_{\max})/2\big)]\!]_\uparrow$. On the other hand, if $[\![y_L/\big(-f_{\max} + \nabla_2^{\mathrm{n}-}(-f_{\max})/2\big)]\!]_\uparrow \neq [y_L/\big(-f_{\max} + \nabla_2^{\mathrm{n}-}(-f_{\max})/2\big)]_\uparrow$, then we can only apply (25) of Proposition 5, obtaining $y \succcurlyeq [\![y_L/\big(-f_{\max} + \nabla_2^{\mathrm{n}-}(-f_{\max})/2\big)]\!]_\downarrow$.

Similar arguments can be used to prove the case analyses of function id_u^s of Figure 11 marked as a_9, a_{10}, a_{11} and a_{12}.

We will now analyze the case analyses of id_ℓ^s of Figure 10 marked as a_3^- and a_3^+, and the ones of id_u^s of Figure 9 marked as a_8^- and a_8^+. In this proof, we can assume $y_L, v_L \in \mathbb{R}_- \cup \mathbb{R}_+$, $y_U, v_U \in \mathbb{R}_- \cup \mathbb{R}_+$ and $\mathrm{sgn}(v_L) = \mathrm{sgn}(v_U)$. First, note that the argument that leads to (48) and (49) starting from $x \preccurlyeq y \boxslash z$ and $x \succcurlyeq y \boxslash z$ is in common with the proof of Theorem 4.

Provided that interval X is split into intervals X_+ and X_-, it is worth discussing the reasons why it is not necessary to partition also Y directly in Algorithm 5. Assume $Y = [-a, b]$ with $a, b > 0$ and consider the partition of Y into two sign-homogeneus intervals $Y \cap [-\infty, -0]$ and $Y \cap [+0, +\infty]$, as usual. Note that the values $-0 \in Y \cap [-\infty, -0] = [-a, -0]$ and the values $+0 \in Y \cap [+0, +\infty] = [+0, b]$ can never be the boundaries of Y upon which z touches its minimum (resp., maximum). This is because y will be the numerator of fractions (see expressions (56) and (57)). Moreover, by the definition of functions id_ℓ^s of Fig 10 and id_u^s of Fig 11, it easy to verify that the partition of Y would not prevent the interval computed for y from being equal to the empty set. That is, if $\mathrm{id}_\ell^s(y_L, v_L, \bar{r}_\ell) = \mathrm{unsat.}$ or $\mathrm{id}_u^s(y_U, v_U, \bar{r}_u) = \mathrm{unsat.}$, then partitioning also Y into sign-homogeneus intervals and then applying the procedure of Algorithm 5 to the two distinct intervals results again into an empty refining interval for z.

Hence, to improve efficiency, Algorithm 5 does not split interval Y into sign-homogeneous intervals. However, in this proof it is necessary to partition Y into intervals Y_- and Y_+ in order to determine the correct formulas for lower and upper bounds for z. In the following, for the sake of simplicity, we will analyze the special case X_+ and $Y = Y_+$, so that Y does not need to be split because it is already a sign-homogeneous interval. The remaining cases in which Y is sign-homogeneous as well as those in which it is not can be derived analogously. To sum up, in this case we assume $x \geq 0$ and $y \geq 0$, and therefore $z > 0$.

Now, we need to prove the cases marked as a_3^+ and a_8^+. The case analysis of (46) and (47) yields (48) and (49). Remember that the case $x = \pm 0$ is handled separately by functions id_ℓ^s of Figure 10 and id_u^s of Figure 11, hence assuming $x > 0$, we obtain

$$z \begin{cases} \leq y/x, & \text{if } \bar{r}_u = \downarrow; \\ < y/\mathrm{pred}(x), & \text{if } \bar{r}_u = \uparrow \text{ and } x \neq f_{\min}; \\ \leq f_{\max}, & \text{if } \bar{r}_u = \uparrow \text{ and } x = f_{\min}; \\ \leq y/\big(x + \nabla_2^{\mathrm{n}-}(x)/2\big), & \text{if } \bar{r}_u = \mathrm{n} \text{ and } \mathrm{even}(x); \\ < y/\big(x + \nabla_2^{\mathrm{n}-}(x)/2\big), & \text{if } \bar{r}_u = \mathrm{n} \text{ and } \mathrm{odd}(x); \end{cases} \tag{56}$$

$$z \begin{cases} > y/\mathrm{succ}(x), & \text{if } \bar{r}_\ell = \downarrow \text{ and } x \neq -f_{\min}; \\ \geq -f_{\max}, & \text{if } \bar{r}_\ell = \downarrow \text{ and } x = -f_{\min}; \\ \geq y/x, & \text{if } \bar{r}_\ell = \uparrow; \\ \geq y/\big(x + \nabla_2^{\mathrm{n}+}(x)/2\big), & \text{if } \bar{r}_\ell = \mathrm{n} \text{ and } \mathrm{even}(x); \\ > y/\big(x + \nabla_2^{\mathrm{n}+}(x)/2\big), & \text{if } \bar{r}_\ell = \mathrm{n} \text{ and } \mathrm{odd}(x). \end{cases} \tag{57}$$

Since the members of the divisions are independent, we can find the minimum of said divisions by minimizing each one of their members. Let (y_L, y_U, v_L, v_U) be as returned by function τ of Figure 5. By Proposition 3 and the monotonicity of 'pred' and 'succ' we obtain

$$z \begin{cases} \leq y_U/v_U, & \text{if } \bar{r}_u = \downarrow; \\ < y_U/\mathrm{pred}(v_U), & \text{if } \bar{r}_u = \uparrow \text{ and } v_U \neq f_{\min}; \\ \leq f_{\max}, & \text{if } \bar{r}_u = \uparrow \text{ and } v_U = f_{\min}; \\ \leq y_U/\big(v_U + \nabla_2^{\mathrm{n}-}(v_U)/2\big), & \text{if } \bar{r}_u = \mathrm{n} \text{ and } \mathrm{even}(v_U); \\ < y_U/\big(v_U + \nabla_2^{\mathrm{n}-}(v_U)/2\big), & \text{if } \bar{r}_u = \mathrm{n} \text{ and } \mathrm{odd}(v_U); \end{cases} \tag{58}$$

$$z \begin{cases} > y_L/\mathrm{succ}(v_L), & \text{if } \bar{r}_\ell = \downarrow \text{ and } v_L \neq -f_{\min}; \\ \geq -f_{\max}, & \text{if } \bar{r}_\ell = \downarrow \text{ and } v_L = -f_{\min}; \\ \geq y_L/v_L, & \text{if } \bar{r}_\ell = \uparrow; \\ \geq y_L/\big(v_L + \nabla_2^{\mathrm{n}+}(v_L)/2\big), & \text{if } \bar{r}_\ell = \mathrm{n} \text{ and } \mathrm{even}(v_L); \\ > y_L/\big(v_L + \nabla_2^{\mathrm{n}+}(v_L)/2\big), & \text{if } \bar{r}_\ell = \mathrm{n} \text{ and } \mathrm{odd}(v_L). \end{cases} \tag{59}$$

We can now exploit Proposition 5 and obtain:

$$z'_\ell \stackrel{\text{def}}{=} \begin{cases} y_L \boxslash_\uparrow v_L, & \text{if } \bar{r}_\ell = \uparrow; \\ \operatorname{succ}\bigl(y_L \boxslash_\downarrow \operatorname{succ}(v_L)\bigr), & \text{if } \bar{r}_\ell = \downarrow \text{ and } v_L \neq -f_{\min}; \end{cases} \tag{60}$$

$$z'_u \stackrel{\text{def}}{=} \begin{cases} \operatorname{pred}\bigl(y_U \boxslash_\uparrow \operatorname{pred}(v_U)\bigr), & \text{if } \bar{r}_u = \uparrow \text{ and } v_U \neq f_{\min}; \\ y_U \boxslash_\downarrow v_U, & \text{if } \bar{r}_u = \downarrow. \end{cases} \tag{61}$$

Since $y_L \neq 0$, then $y_L / \operatorname{succ}(v_L) \neq 0$. Hence, Proposition 5 applies and we have $z \succcurlyeq y_L \boxslash_\uparrow v_L$ if $\bar{r}_\ell = \uparrow$ and $z \succcurlyeq \operatorname{succ}\bigl(y_L / \operatorname{succ}(v_L)\bigr)$ if $\bar{r}_\ell = \downarrow$ and $v_L \neq -f_{\min}$. Analogously, since $y_U \neq 0$, then $y_U / \operatorname{pred}(v_L) \neq 0$. Hence, by Proposition 5 we obtain (61).

Note that, since division by zero is not defined on real numbers, we had to separately address the case $\bar{r}_u = \uparrow$ and $x = f_{\min}$ in (56), and the case $\bar{r}_\ell = \downarrow$ and $x = -f_{\min}$ in (57). Division by zero is, however, defined on IEEE 754 floating-point numbers. Indeed, if we evaluate the second case of (60) with $v_L = -f_{\min}$, we obtain $\operatorname{succ}(y_L \boxslash_\downarrow \operatorname{succ}(-f_{\min})) = -f_{\max}$, which happens to be the correct value for z'_ℓ, provided $y_L > 0$. The same happens for (61). Therefore, there is no need for a separate treatment when variable x takes the values $\pm f_{\min}$.

As before, we need to approximate the values of the expressions $e_u^+ \stackrel{\text{def}}{=} y_U / \bigl(v_U + \nabla_2^{\mathrm{n}-}(v_U)/2\bigr)$ and $e_\ell^+ \stackrel{\text{def}}{=} y_L / \bigl(v_L + \nabla_2^{\mathrm{n}+}(v_L)/2\bigr)$. Thus, when $[\![e_u^+]\!]_\downarrow < [e_u^+]_\downarrow$ by (51) and (27) of Proposition 5 we obtain $y \preccurlyeq [\![e_u^+]\!]_\uparrow$, while, when $[\![e_\ell^+]\!]_\downarrow > [e_\ell^+]_\downarrow$ by (51) and (25) of Proposition 5 we obtain $y \succcurlyeq [\![e_\ell^+]\!]_\downarrow$. Thus, for the case where $\bar{r}_\ell = \mathrm{n}$, since $e_u^+ \neq 0$ and $e_\ell^+ \neq 0$, by Proposition 5, we have

$$y'_\ell \stackrel{\text{def}}{=} \begin{cases} [\![e_\ell^+]\!]_\uparrow, & \text{if } \operatorname{even}(v_L) \text{ and } [\![e_\ell^+]\!]_\uparrow = [e_\ell^+]_\uparrow; \\ [\![e_\ell^+]\!]_\downarrow, & \text{if } \operatorname{even}(v_L) \text{ and } [\![e_\ell^+]\!]_\uparrow \neq [e_\ell^+]_\uparrow; \\ \operatorname{succ}\bigl([\![e_\ell^+]\!]_\downarrow\bigr), & \text{otherwise}; \end{cases} \tag{62}$$

whereas, for the case in which $\bar{r}_u = \mathrm{n}$,

$$y'_u \stackrel{\text{def}}{=} \begin{cases} [\![e_u^+]\!]_\downarrow, & \text{if } \operatorname{even}(v_U) \text{ and } [\![e_u^+]\!]_\downarrow = [e_u^+]_\downarrow; \\ [\![e_u^+]\!]_\uparrow, & \text{if } \operatorname{even}(v_U) \text{ and } [\![e_u^+]\!]_\downarrow \neq [e_u^+]_\downarrow; \\ \operatorname{pred}\bigl([\![e_u^+]\!]_\uparrow\bigr), & \text{otherwise.} \end{cases} \tag{63}$$

An analogous reasoning allows us to prove the case analyses marked as a_3^- and a_8^-.

Proof (of Theorem 9) Given the constraint $x = y \boxdot_S z$ with $x \in X = [x_\ell, x_u]$, $y \in Y = [y_\ell, y_u]$ and $z \in Z = [z_\ell, z_u]$, then $X' = [x'_\ell, x'_u] \cap X$. Hence, we are sure that $X' \subseteq X$.

It should be immediate to verify that function σ of Figure 7, related to the case $\operatorname{sgn}(y_\ell) = \operatorname{sgn}(y_u)$, chooses the appropriate interval extrema y_L, y_U, z_L, z_U, necessary for computing bounds for x. Indeed, note that such choice is completely driven by the sign of the resulting product. Analogously, the correct interval extrema y_L, y_U, z_L, z_U related to the case $\operatorname{sgn}(z_\ell) = \operatorname{sgn}(z_u)$ can be determined by applying function σ of Figure 7, but swapping the role of y and z. Hence, if the sign of y or of z is constant (see the second part of Algorithm 9) function σ of Figure 7 finds the appropriate extrema for y and z to compute the bound for x.

Concerning the cases $\operatorname{sgn}(y_\ell) = \operatorname{sgn}(z_\ell) = -1$ and $\operatorname{sgn}(y_u) = \operatorname{sgn}(z_u) = 1$ (first part of Algorithm 9), note that we have only two possibilities for the interval extrema y_L and z_L, that are y_ℓ and z_u or y_u and z_ℓ. Since the product of y_L and z_L will have a negative sign in both cases, the right extrema for determining the lower bound x'_ℓ have to be chosen by selecting the smallest product of y_L and z_L. Analogously, for y_U and z_U there are two possibilities: y_ℓ and z_ℓ or y_u and z_u. Since the product of y_U and z_U will have a positive sign in both cases, the appropriate extrema for determining the upper bound x'_u have to be chosen as the biggest product of y_U and z_U.

Remember that by Proposition 2, following the same reasoning as in the previous proofs, it suffices to find a lower bound for $y_L \boxdot_{r_\ell} z_L$ and an upper bound for $y_U \boxdot_{r_u} z_U$.

We now comment on some critical case analyses of function dm_ℓ of Figure 17. Consider, for example, when $y_L = \pm\infty$ and $z_L = \pm 0$. In particular, we analyze the case in which $y_L = -\infty$ and $z_L = \pm 0$. Note that $y_L = -\infty$ implies $y_\ell = -\infty$. Assume, first, that $z_L = +0$. Recall that by the IEEE 754 Standard [24] $\pm\infty \boxdot \pm 0$ is an invalid operation. However, since $y_\ell = -\infty$, we have two cases:

$y_u \geq -f_{\max}$: note that, in this case, $-f_{\max} \boxdot +0 = -0$;

$y_u = -\infty$: in this case, z_L must correspond to z_u (see the last three cases of function σ). Since $-\infty \boxdot z$ for $z < 0$ results in $+\infty$, we can conclude that -0 is a correct lower bound for x.

A similar reasoning applies for the cases $y_L = +\infty$, $z_L = \pm 0$. Dually, the only critical entries of function dm_u of Figure 17 are those in which $y_U = \pm\infty$ and $z_U = \pm 0$. In these cases we can reason in a similar way, too.

We are left to prove that $\forall X'' \subset X : \exists r \in S, y \in Y, z \in Z \,.\, y \boxdot_r z \notin X''$. Let us focus on the lower bound x'_ℓ, proving that there exist values $r \in S, y \in Y, z \in Z$ such that $y \boxdot_r z = x'_\ell$. Consider the particular values of y_L, z_L and r_ℓ that correspond to the value of x'_ℓ chosen by Algorithm 9, that is y_L, z_L and r_ℓ are such that $\mathrm{dm}_\ell(y_L, z_L, r_\ell) = x'_\ell$. By Algorithm 9, such values of y_L and z_L must exist. First, consider the cases in which $y_L \notin (\mathbb{R}_- \cup \mathbb{R}_+)$ or $z_L \notin (\mathbb{R}_- \cup \mathbb{R}_+)$. In these cases, a brute-force verification was successfully conducted to verify that $y \boxdot_{r_\ell} z = x'_\ell$. For the cases in which $y_L \in (\mathbb{R}_- \cup \mathbb{R}_+)$ and $z_L \in (\mathbb{R}_- \cup \mathbb{R}_+)$ we have, by definition of dm_ℓ of Figure 17, that $x'_\ell = y_L \boxdot_{r_\ell} z_L$. Remember that, by Proposition 2, there exist $r' \in S$ such that $y_L \boxdot_{\mathrm{r}_\ell} z_L = y_L \boxdot_{\mathrm{r}'} z_L$. Since $y_L \in Y$ and $z_L \in Z$, we can conclude that $x'_\ell \notin X''$ implies that $y'_L \boxdot_{\mathrm{r}'} z_L \notin X''$. An analogous reasoning allows us to conclude that $\exists r \in S$ for which the following holds: $x'_u \notin X''$ implies $y_U \boxdot_{\mathrm{r}} z_U \notin X''$.

Proof (of Theorem 10) Given the constraint $x = y \boxdot_S z$ with $x \in X = [x_\ell, x_u]$, $y \in Y = [y_\ell, y_u]$ and $z \in Z = [z_\ell, z_u]$, Algorithm 10 computes Y', a new and refined interval for variable y.

First, note that either $Y' := (Y \cap [y_\ell^-, y_u^-]) \uplus (Y \cap [y_\ell^+, y_u^+])$ or $Y' = \varnothing$, hence, in both cases, we are sure that $Y' \subseteq Y$ holds.

By Proposition 2, we can focus on finding a lower bound for $y \in Y$ by exploiting the constraint $y \boxdot_{\bar{r}_\ell} z = x$ and an upper bound for $y \in Y$ by exploiting the constraint $y \boxdot_{\bar{r}_u} z = x$.

Now, in order to compute correct bounds for y, we first need to split the interval of z into the sign-homogeneous intervals Z_- and Z_+, because it is crucial to be sure of the sign of z. As a consequence, for $W = Y_-$ (and, analogously, for $W = Y_+$) function τ of Figure 5 picks the appropriate interval extrema of X and W to be used to compute the new lower and upper bounds for y. It is easy to verify that the values of x_L and w_L (resp., x_U and w_U) computed using function τ of Figure 5 are the boundaries of X and W upon which y touches its minimum (resp., maximum). Functions im_ℓ of Figure 18 and im_u of Figure 19 are then employed to find the new bounds for y. The so obtained intervals for y are then joined using convex union between intervals, in order to obtain the refining interval for y.

Observe that functions im_ℓ of Fig 18 and im_u of Fig 19 are dual to each other: every row/column of one table can be found in the other table reversed and changed of sign. This is due to the fact that, for each $r \in R$ and each $D \subseteq \mathbb{F} \times \mathbb{F}$, we have

$$\begin{aligned} &\min\{\, y \in \mathbb{F} \mid (x,z) \in D, x = y \boxdot_r z \,\} \\ &= -\max\{\, y \in \mathbb{F} \mid (x,z) \in D, -x = y \boxdot_r z \,\} \\ &= -\max\{\, y \in \mathbb{F} \mid (x,z) \in D, x = y \boxdot_r -z \,\}. \end{aligned}$$

Concerning the case analysis of im_ℓ marked as a_4 of Fig 18, we must consider the following cases:

$\bar{r}_\ell = \uparrow$: we clearly must have $y = +\infty$ in this case;

$\bar{r}_\ell = \downarrow$: inequality $y \cdot w_L < -f_{\max}$ must hold and thus, since w_L is negative, $y > -f_{\max}/w_L$ and, by (26) of Proposition 5, $y \succcurlyeq \mathrm{succ}(-f_{\max} \boxslash_\downarrow w_L)$.

$\bar{r}_\ell = \mathrm{n}$: since $\mathrm{odd}(f_{\max})$, for $x_L = -\infty$ we need y to be greater or equal than $\big(-f_{\max} + \nabla_2^{\mathrm{n}-}(-f_{\max})/2\big)/w_L$. If $[\![\big(-f_{\max} + \nabla_2^{\mathrm{n}-}(-f_{\max})/2\big)/w_L]\!]_\uparrow = [-f_{\max} + \nabla_2^{\mathrm{n}-}(-f_{\max})/2\big)/w_L]_\uparrow$, by (29) of Proposition 5, we can conclude $y \succcurlyeq [\![\big(-f_{\max} + \nabla_2^{\mathrm{n}-}(-f_{\max})/2\big)/w_L]\!]_\uparrow$. On the other hand, if $[\![\big(-f_{\max} + \nabla_2^{\mathrm{n}-}(-f_{\max})/2\big)/w_L]\!]_\uparrow \neq [-f_{\max} + \nabla_2^{\mathrm{n}-}(-f_{\max})/2\big)/w_L]_\uparrow$, then we can only apply (25) of Proposition 5, obtaining $y \succcurlyeq [\![\big(-f_{\max} + \nabla_2^{\mathrm{n}-}(-f_{\max})/2\big)/w_L]\!]_\downarrow$.

Regarding the case analysis of im_ℓ marked a a_5 of Fig 18, we have the following cases:

$\bar{r}_\ell = \downarrow$: in this case, we must have $y = -0$;

$\bar{r}_\ell = \uparrow$: inequality $y \cdot w_L > -f_{\min}$ must hold and thus, since w_L is positive, $y > -f_{\min}/w_L$ and, by (28) of Proposition 5, $y \succcurlyeq \mathrm{succ}(-f_{\min} \boxslash_\downarrow w_L)$.

$\bar{r}_\ell = \mathrm{n}$: since $\mathrm{odd}(f_{\min})$, for $x_L = -0$ we need y to be greater or equal than $-f_{\min}/(2 \cdot w_L)$. Since in this case $[\![-f_{\min}/(2 \cdot w_L)]\!]_\uparrow = [-f_{\min}/(2 \cdot w_L)]_\uparrow = (-f_{\min}) \boxslash_\uparrow (2 \cdot w_L)$, by (29) of Proposition 5, we can conclude $y \succcurlyeq -f_{\min} \boxslash_\uparrow (2 \cdot w_L)$.

As for the case analysis of im_ℓ marked as a_6 of Figure 18, the following cases must be studied:

$\bar{r}_\ell = \uparrow$: we must have $y = +0$ in this case;

$\bar{r}_\ell = \downarrow$: it should be $y \cdot w_L < f_{\min}$ and thus, since w_L is negative, $y > f_{\min}/w_L$ and, by (28) of Proposition 5, $y \succcurlyeq \mathrm{succ}(-f_{\min} \boxslash_\downarrow w_L)$.

$\bar{r}_\ell = \mathrm{n}$: since $\mathrm{odd}(f_{\min})$, for $x_L = -0$ we need y to be greater than or equal to $\big(f_{\min}/(2 \cdot w_L)\big)$. Since in this case $[\![f_{\min}/(2 \cdot w_L)]\!]_\uparrow = [f_{\min}/(2 \cdot w_L)]_\uparrow = f_{\min} \boxslash_\uparrow (2 \cdot w_L)$, by (29) of Proposition 5 we can conclude $y \succcurlyeq f_{\min} \boxslash_\uparrow (2 \cdot w_L)$.

Finally, for the case analysis of im_ℓ marked as a_7 of Fig 18, the following cases must be considered:

$\bar{r}_\ell = \downarrow$: in this case we must have $y = +\infty$;
$\bar{r}_\ell = \uparrow$: it should be $y \cdot w_L > -f_{\max}$ and thus, since w_L is positive, $y > f_{\max}/w_L$ and, by (26) of Proposition 5, $y \succcurlyeq \mathrm{succ}(f_{\max} \boxslash_\downarrow w_L)$.
$\bar{r}_\ell = \mathrm{n}$: since $\mathrm{odd}(f_{\max})$, for $x_L = +\infty$ we need y to be greater than or equal to $\big(f_{\max} + \nabla_2^{\mathrm{n}+}(f_{\max})/2\big)/w_L$. If $[\![\big(f_{\max} + \nabla_2^{\mathrm{n}+}(f_{\max})/2\big)/w_L]\!]_\uparrow = [f_{\max} + \nabla_2^{\mathrm{n}+}(f_{\max})/2)/w_L]_\uparrow$, by (29) of Proposition 5, we can conclude $y \succcurlyeq [\![\big(f_{\max} + \nabla_2^{\mathrm{n}+}(f_{\max})/2\big)/w_L]\!]_\uparrow$. On the other hand, if $[\![\big(f_{\max} + \nabla_2^{\mathrm{n}+}(f_{\max})/2\big)/w_L]\!]_\uparrow \neq [f_{\max} + \nabla_2^{\mathrm{n}+}(f_{\max})/2\big)/w_L]_\uparrow$, then we can only apply (25) of Proposition 5, obtaining $y \succcurlyeq [\![\big(f_{\max} + \nabla_2^{\mathrm{n}+}(f_{\max})/2\big)/w_L]\!]_\downarrow$.

Similar arguments can be used to prove the case analyses of im_u of Figure 19 marked as a_9, a_{10}, a_{11} and a_{12}.

We now analyze the case analyses of im_ℓ of Fig 18 marked as a_3^- and a_3^+ and the ones of im_u of Fig 19 marked as a_8^- and a_8^+, for which we can assume $x_L, w_L \in \mathbb{F} \cap \mathbb{R}$ and $x_U, w_U \in \mathbb{F} \cap \mathbb{R}$, and $\mathrm{sgn}(w_\ell) = \mathrm{sgn}(w_u)$. Exploiting $x \preccurlyeq y \boxdot z$ and $x \succcurlyeq y \boxdot z$, by Proposition 4 we have

$$y \cdot z \begin{cases} \geq x, & \text{if } \bar{r}_\ell = \downarrow; \\ > x + \nabla^\uparrow(x) = \mathrm{pred}(x), & \text{if } \bar{r}_\ell = \uparrow; \\ \geq x + \nabla_2^{\mathrm{n}-}(x)/2, & \text{if } \bar{r}_\ell = \mathrm{n} \text{ and even}(x); \\ > x + \nabla_2^{\mathrm{n}-}(x)/2, & \text{if } \bar{r}_\ell = \mathrm{n} \text{ and odd}(x). \end{cases} \tag{64}$$

$$y \cdot z \begin{cases} < x + \nabla^\downarrow(x) = \mathrm{succ}(x), & \text{if } \bar{r}_u = \downarrow; \\ \leq x, & \text{if } \bar{r}_u = \uparrow; \\ \leq x + \nabla_2^{\mathrm{n}+}(x)/2, & \text{if } \bar{r}_u = \mathrm{n} \text{ and even}(x); \\ < x + \nabla_2^{\mathrm{n}+}(x)/2, & \text{if } \bar{r}_u = \mathrm{n} \text{ and odd}(x). \end{cases} \tag{65}$$

Since the case $z = 0$ is handled separately by im_ℓ of Figure 18 and by im_u of Figure 19, we can assume $z \neq 0$. Thanks to the splitting of Z into a positive and a negative part, the sign of z is determined. In the following, we will prove the case analyses marked as a_3^+ and a_8^+. Hence, assuming $z > 0$, the previous case analysis gives us

$$y \begin{cases} \geq x/z, & \text{if } \bar{r}_\ell = \downarrow; \\ > \mathrm{pred}(x)/z, & \text{if } \bar{r}_\ell = \uparrow \\ \geq \big(x + \nabla_2^{\mathrm{n}-}(x)/2\big)/z, & \text{if } \bar{r}_\ell = \mathrm{n} \text{ and even}(x); \\ > \big(x + \nabla_2^{\mathrm{n}-}(x)/2\big)/z, & \text{if } \bar{r}_\ell = \mathrm{n} \text{ and odd}(x); \end{cases} \tag{66}$$

$$y \begin{cases} < \mathrm{succ}(x)/z, & \text{if } \bar{r}_u = \downarrow; \\ \leq x/z, & \text{if } \bar{r}_u = \uparrow; \\ \leq \big(x + \nabla_2^{\mathrm{n}+}(x)/2\big)/z, & \text{if } \bar{r}_u = \mathrm{n} \text{ and even}(x); \\ < \big(x + \nabla_2^{\mathrm{n}+}(x)/2\big)/z, & \text{if } \bar{r}_u = \mathrm{n} \text{ and odd}(x). \end{cases} \tag{67}$$

Note that the numerator and the denominator of the previous fractions are independent. Therefore, we can find the minimum of the fractions by minimizing the numerator and maximizing the denominator. Since we are analyzing the case in which $W = Z_+$, let (x_L, w_L, x_U, w_U) as the result of function τ of Figure 5. Hence, by Proposition 3 and the monotonicity of 'pred' and 'succ we obtain

$$y \begin{cases} \geq x_L/w_L, & \text{if } \bar{r}_\ell = \downarrow; \\ > \mathrm{pred}(x_L)/w_L, & \text{if } \bar{r}_\ell = \uparrow \\ \geq \big(x_L + \nabla_2^{\mathrm{n}-}(x_L)/2\big)/w_L, & \text{if } \bar{r}_\ell = \mathrm{n} \text{ and even}(x); \\ > \big(x_L + \nabla_2^{\mathrm{n}-}(x_L)/2\big)/w_L, & \text{if } \bar{r}_\ell = \mathrm{n} \text{ and odd}(x); \end{cases} \tag{68}$$

$$y \begin{cases} < \mathrm{succ}(x_U)/w_U, & \text{if } \bar{r}_u = \downarrow; \\ \leq x_U/w_U, & \text{if } \bar{r}_u = \uparrow; \\ \leq \big(x_U + \nabla_2^{\mathrm{n}+}(x_U)/2\big)/w_U, & \text{if } \bar{r}_u = \mathrm{n} \text{ and even}(x); \\ < \big(x_U + \nabla_2^{\mathrm{n}+}(x_U)/2\big)/w_U, & \text{if } \bar{r}_u = \mathrm{n} \text{ and odd}(x). \end{cases} \tag{69}$$

We can now exploit Proposition 5 and obtain:

$$y'_\ell \stackrel{\text{def}}{=} \begin{cases} x_L \boxslash_\uparrow w_L, & \text{if } \bar{r}_\ell = \downarrow; \\ \operatorname{succ}\bigl(\operatorname{pred}(x_L) \boxslash_\downarrow w_L\bigr), & \text{if } \bar{r}_\ell = \uparrow; \end{cases} \tag{70}$$

$$y'_u \stackrel{\text{def}}{=} \begin{cases} \operatorname{pred}\bigl(\operatorname{succ}(x_U) \boxslash_\uparrow w_U\bigr), & \text{if } \bar{r}_u = \downarrow; \\ x_U \boxslash_\downarrow w_U, & \text{if } \bar{r}_u = \uparrow. \end{cases} \tag{71}$$

Indeed, if $x_L \neq 0$, then Proposition 5 applies and we have $y \succcurlyeq x_L \boxslash_\uparrow w_L$. On the other hand, if $x_L = 0$, since by hypothesis $z > 0$ implies $w_L > 0$, according to IEEE 754 [24, Section 6.3], we have $(x_L \boxslash_\uparrow w_L) = \operatorname{sgn}(x_L) \cdot 0$ and, indeed, for each non-NaN, nonzero and finite $w \in \mathbb{F} \cap [+0, +\infty]$, $\operatorname{sgn}(x_L) \cdot 0$ is the least value for y that satisfies $\operatorname{sgn}(x_L) \cdot 0 = y \boxdot_\downarrow w$.

Analogously, if $x_L \neq f_{\min}$, then Proposition 5 applies and we have $\operatorname{succ}\bigl(\operatorname{pred}(x_L) \boxslash_\downarrow w_L\bigr)$. On the other hand, if $x_L = f_{\min}$, $\operatorname{succ}\bigl(\operatorname{pred}(x_L) \boxslash_\downarrow w_L\bigr) = f_{\min}$, which is consistent with the fact that, for each non-NaN, nonzero and finite $w \in \mathbb{F} \cap [+0, +\infty]$, $f_{\min}$ is the lowest value of y that satisfies $f_{\min} = y \boxdot_\uparrow w$.

A symmetric argument justifies (71).

As before, we will consider both the cases $\llbracket e_\ell^+ \rrbracket_\uparrow = [e_\ell^+]_\uparrow$ and $\llbracket e_u^+ \rrbracket_\downarrow = [e_u^+]_\downarrow$ as well as $\llbracket e_\ell^+ \rrbracket_\uparrow > [e_\ell^+]_\uparrow$ and $\llbracket e_u^+ \rrbracket_\downarrow < [e_u^+]_\downarrow$. Thus, when $\llbracket e_u^+ \rrbracket_\downarrow < [e_u^+]_\downarrow$ by (69) and (27) of Proposition 5 we obtain $y \preccurlyeq \llbracket e_u^+ \rrbracket_\uparrow$. Instead, when $\llbracket e_\ell^+ \rrbracket_\downarrow > [e_\ell^+]_\downarrow$, by (69) and (25) of Proposition 5 we obtain $y \succcurlyeq \llbracket e_\ell^+ \rrbracket_\downarrow$. In conclusion, for the case in which $\bar{r}_\ell = \mathrm{n}$, since $e_u \neq 0$ and $e_\ell \neq 0$, by Proposition 5, we have

$$y'_\ell \stackrel{\text{def}}{=} \begin{cases} \llbracket e_\ell^+ \rrbracket_\uparrow, & \text{if } \operatorname{even}(x_L) \text{ and } \llbracket e_\ell^+ \rrbracket_\uparrow = [e_\ell^+]_\uparrow; \\ \llbracket e_\ell^+ \rrbracket_\downarrow, & \text{if } \operatorname{even}(x_L) \text{ and } \llbracket e_\ell^+ \rrbracket_\uparrow \neq [e_\ell^+]_\uparrow; \\ \operatorname{succ}\bigl(\llbracket e_\ell^+ \rrbracket_\downarrow\bigr), & \text{otherwise;} \end{cases} \tag{72}$$

whereas, for the case in which $\bar{r}_u = \mathrm{n}$,

$$y'_u \stackrel{\text{def}}{=} \begin{cases} \llbracket e_u^+ \rrbracket_\downarrow, & \text{if } \operatorname{even}(x_U) \text{ and } \llbracket e_u^+ \rrbracket_\downarrow = [e_u^+]_\downarrow; \\ \llbracket e_u^+ \rrbracket_\uparrow, & \text{if } \operatorname{even}(x_U) \text{ and } \llbracket e_u^+ \rrbracket_\downarrow \neq [e_u^+]_\downarrow; \\ \operatorname{pred}\bigl(\llbracket e_u^+ \rrbracket_\uparrow\bigr), & \text{otherwise.} \end{cases} \tag{73}$$

An analogous reasoning with $z < 0$ allows us to obtain the case analyses marked as a_3^- and a_8^-.