HOMOTOPICAL TOPOS THEORY

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IN THE MOUNTAINS

There is WINTER.

Then there is the melting time.

Then there is summer.

Then there is the waiting time.

Then there is WINTER.

ABSTRACT

The purpose of this book is two fold.

(1) To give a systematic introduction to topos theory from a purely categorical point of view, thus ignoring all logical and algebraic issues.

(2) To give an account of the homotopy theory of the simplicial objects in a Grothendieck topos.

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EDITORIAL COMMENT I have always found the traditional homotopical treatments to be somewhat contrived and ad hoc. There is, however, a way out: Use Cisinski's "localizer theory". For then the classical results are mere instances of the output of this powerful machine which has the effect of sweeping all before it.

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CONTENTS

§1. *PARTIAL ORDERS*

Let X be a class -- then a binary relation $\leq$ on X is said to be a <u>preorder</u> if

- $\leq$ is reflexive: $\forall\, x \in X,\ x \leq x$;
- $\leq$ is transitive: $\forall\, x,y,z \in X: x \leq y \ \&\ y \leq z \Rightarrow x \leq z$.

A preorder is a <u>partial order</u> if in addition

$$\forall\, x,x' \in X, \begin{bmatrix} x \leq x' \\ x' \leq x \end{bmatrix} \Rightarrow x = x'.$$

Every preorder $(X,\leq)$ gives rise to a category $\underline{C}(X,\leq)$: The objects of $\underline{C}(X,\leq)$ are the elements of X and

$$\mathrm{Mor}(x,y) = \begin{bmatrix} \{(x,y)\} \text{ if } x \leq y \\ \emptyset \text{ otherwise,} \end{bmatrix} \qquad \mathrm{id}_x = (x,x),$$

and

$$(y,z) \circ (x,y) = (x,z).$$

1.1 LEMMA Let $(X,\leq)$ be a preorder -- then every arrow in $\underline{C}(X,\leq)$ is both a monomorphism and an epimorphism.

1.2 LEMMA Let $(X,\leq)$ be a partial order -- then the only isomorphisms in $\underline{C}(X,\leq)$ are the identities.

1.3 DEFINITION A <u>poset</u> is a set X equipped with a partial order.

If $(X,\leq)$, $(Y,\leq)$ are posets, then a functor $f:\underline{C}(X,\leq) \to \underline{C}(Y,\leq)$ is simply a function $f:X \to Y$ which is <u>monotonic</u>, i.e.,

$$x \le x' \text{ in } X \Rightarrow f(x) \le f(x') \text{ in } Y.$$

1.4 LEMMA Let $(X,\le)$, $(Y,\le)$ be posets and let

$$\begin{cases} f:\underline{C}(X,\le) \to \underline{C}(Y,\le) \\ g:\underline{C}(Y,\le) \to \underline{C}(X,\le) \end{cases}$$

be functors -- then f is a left adjoint for g if for all $x \in X$ and $y \in Y$,

$$f(x) \le y \Leftrightarrow x \le g(y).$$

1.5 DEFINITION Suppose that $(X,\le)$ is a poset -- then $(X,\le)$ is a <u>lattice</u> if $\underline{C}(X,\le)$ has binary products and binary coproducts, written

$$\begin{cases} x \wedge y \equiv x \times y \\ x \vee y \equiv x \coprod y. \end{cases}$$

[Note: Accordingly,

$$\begin{cases} x \wedge y \le x \\ x \wedge y \le y \end{cases} \quad \& \quad \begin{cases} z \le x \\ z \le y \end{cases} \Rightarrow z \le x \wedge y$$

and

$$\begin{cases} x \le x \vee y \\ y \le x \vee y \end{cases} \quad \& \quad \begin{cases} x \le z \\ y \le z \end{cases} \Rightarrow x \vee y \le z.]$$

1.6 DEFINITION Suppose that $(X,\le)$ is a lattice -- then $(X,\le)$ is said to be <u>bounded</u> if $\underline{C}(X,\le)$ admits a final object, denoted by 1, and an initial object, denoted by 0.

[Note: So, $\forall\ x \in X$, $0 \leq x \leq 1$ and $\begin{cases} x \wedge 1 = x \\ 0 \vee x = x \end{cases}$.]

1.7 LEMMA Let $(X,\leq)$ be a preorder -- then a commutative diagram

$$\begin{tikzcd} w \arrow[r] \arrow[d] & y \arrow[d] \\ x \arrow[r] & z \end{tikzcd}$$

in $\underline{C}(X,\leq)$ is a pullback square iff w is a product of x and y or is a pushout square iff z is a coproduct of x and y.

1.8 RAPPEL Let $\underline{C}$ be a category -- then $\underline{C}$ is finitely complete iff $\underline{C}$ has pullbacks and a final object and $\underline{C}$ is finitely cocomplete iff $\underline{C}$ has pushouts and an initial object.

1.9 SCHOLIUM If $(X,\leq)$ is a bounded lattice, then $\underline{C}(X,\leq)$ is finitely complete and finitely cocomplete.

1.10 REMARK Suppose that $(X,\leq)$ is a bounded lattice -- then $\underline{C}(X,\leq)$ has products iff it has coproducts. Therefore $\underline{C}(X,\leq)$ is complete iff it is cocomplete.

Let $(X,\leq)$ be a bounded lattice.

- $(X,\leq)$ is <u>distributive</u> if $\forall\ x,y,z \in X$:

$$\begin{cases} x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \\ x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z). \end{cases}$$

- $(X,\leq)$ is <u>complemented</u> if $\forall\ x \in X$, $\exists\ \neg x \in X$:

$$x \wedge \neg x = 0 \text{ and } x \vee \neg x = 1.$$

[Note: In a distributive lattice, a complement $\neg x$ of x, if it exists, is unique.]

1.11 DEFINITION A <u>boolean algebra</u> is a bounded lattice $(X,\leq)$ which is both distributive and complemented.

<u>N.B.</u> In a boolean algebra $(X,\leq)$, $\forall\ x \in X$, $\neg\neg x = x$.

[For

$$\begin{cases} \neg x \wedge \neg\neg x = 0 \\ \neg x \vee \neg\neg x = 1 \end{cases}$$

and complements are unique.]

1.12 LEMMA Let $(X,\leq)$ be a boolean algebra — then $\forall\ x,y \in X$,

$$\begin{cases} \neg(x \vee y) = \neg x \wedge \neg y \\ \neg(x \wedge y) = \neg x \vee \neg y. \end{cases}$$

[Note: These relations are called the laws of de Morgan.]

1.13 EXAMPLE If S is a set, then its power set PS is a boolean algebra.

§2. *SUBOBJECTS*

Given a category $\underline{C}$ and an object X in $\underline{C}$, let $M(X)$ be the class of all pairs (Y,f), where $f:Y \to X$ is a monomorphism -- then $M(X)$ is the object class of a full subcategory $\underline{M}(X)$ of $\underline{C}/X$.

Given (Y,f), (Z,g) in $M(X)$, write $(Y,f) \leq_X (Z,g)$ if there exists a morphism $h:Y \to Z$ such that $f = g \circ h$, i.e., if there exists

$$h \in \mathrm{Mor}_{\underline{C}/X} (Y \xrightarrow{f} X,\ Z \xrightarrow{g} X).$$

[Note: h is necessarily unique and is itself a monomorphism.]

2.1 LEMMA The binary relation $\leq_X$ is a preorder on $M(X)$.

<u>N.B.</u> So, in the notation of §1,

$$\underline{M}(X) = \underline{C}(M(X), \leq_X).$$

2.2 DEFINITION Two elements (Y,f) and (Z,g) of $M(X)$ are deemed <u>equivalent</u>, written $(Y,f) \sim_X (Z,g)$, if there exists an isomorphism $\phi:Y \to Z$ such that $f = g \circ \phi$.

2.3 LEMMA The binary relation $\sim_X$ is an equivalence relation on $M(X)$.

2.4 DEFINITION A <u>subobject</u> of X is an equivalence class of monomorphisms under $\sim_X$.

2.5 REMARK In practice, people tend to blur the distinction between a monomorphism $f:Y \to X$ and its associated subobject, a potentially confusing abuse of the language.

Let $\text{Sub}_{\underline{C}}\ X$ stand for $M(X)/\sim_X$, let [ ] denote an equivalence class, and let $[f] \leq_X [g]$ have the obvious connotation -- then the preorder on $\text{Sub}_{\underline{C}}\ X$ is a partial order. In fact,

$$\begin{cases} (Y,f) \leq_X (Z,g) \\ (Z,g) \leq_X (Y,f) \end{cases}$$

imply that $(Y,f) \sim_X (Z,g)$ or still, $[f] = [g]$.

2.6 EXAMPLE Let $(X,\leq)$ be a bounded lattice and take for $\underline{C}$ the category $\underline{C}(X,\leq)$ -- then

$$\text{Sub}_{\underline{C}(X,\leq)}\ 1 \longleftrightarrow X.$$

2.7 EXAMPLE Let X be a topological space and take for $\underline{C}$ the category $\underline{Sh}(X)$ (the sheaves of sets on X) -- then

$$\text{Sub}_{\underline{Sh}(X)}\ h_X \longleftrightarrow \tau_X.$$

[Note: τ_X is the topology on X and the correspondence $\leftarrow$ assigns to $U \in \tau_X$, the sheaf h_U, where $h_U V = \begin{cases} 1 \text{ if } V \subset U \\ \emptyset \text{ if } V \not\subset U \end{cases}$.]

2.8 DEFINITION A <u>representative class of monomorphisms</u> in $M(X)$ is a subclass of $M(X)$ which is a system of representatives for $\sim_X$.

2.9 EXAMPLE Suppose that $\underline{C}$ has an initial object $\emptyset_{\underline{C}}$. Let $f:Y \rightarrow \emptyset_{\underline{C}}$ be an element of $M(\emptyset_{\underline{C}})$ -- then f is an isomorphism, hence $f \sim_{\emptyset_{\underline{C}}} \mathrm{id}_{\emptyset_{\underline{C}}}$. Therefore

$$\mathrm{Sub}_{\underline{C}}\, \emptyset_{\underline{C}} = [\mathrm{id}_{\emptyset_{\underline{C}}}].$$

2.10 RAPPEL A category $\underline{C}$ is said to be wellpowered provided that each of its objects has a representative class of monomorphisms that can be indexed by a set.

2.11 EXAMPLE Take $\underline{C} = \underline{SET}$ and fix X -- then a subobject of X is an equivalence class of injective maps.

- Every subobject of X contains exactly one inclusion of a subset of X into X and that subset is the image of every element in the subobject.

- The subsets of X together with their inclusion maps form a representative set of monomorphisms in $M(X)$.

[Note: Therefore $\underline{SET}$ is wellpowered.]

2.12 EXAMPLE $\underline{TOP}$ is wellpowered.

[Let (X,τ_X) be a topological space -- then a representative set of monomorphisms in $M(X,\tau_X)$ are the pairs $((Y,\tau_Y),i_Y)$, where Y is a subset of X, τ_Y is a topology on Y finer than $\tau_X|Y$, and $i_Y:Y \to X$ is the (continuous) inclusion.]

2.13 CRITERION If $\underline{C}$ is a small category and if $\underline{D}$ is a finitely complete, wellpowered category, then the functor category $[\underline{C},\underline{D}]$ is wellpowered.

2.14 EXAMPLE If $\underline{C}$ is a small category, then the presheaf category

$$\hat{\underline{C}} = [\underline{C}^{OP},\underline{SET}]$$

is wellpowered. In particular:

$$\underline{\text{SISET}} = [\underline{\Delta}^{\text{OP}}, \underline{\text{SET}}]$$

is wellpowered.

2.15 RAPPEL Consider a pullback square

$$\begin{tikzcd} P \arrow[r, "\eta"] \arrow[d, "\xi"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

in a category $\underline{C}$. Assume: f is a monomorphism -- then η is a monomorphism.

2.16 DEFINITION Let $\underline{C}$ be a category with pullbacks. Given an object X in $\underline{C}$, suppose that $\left[\begin{array}{l} f_1:Y_1 \to X \\ f_2:Y_2 \to X \end{array}\right. \in M(X)$ -- then their <u>intersection</u> is the pair $(Y_1 \cap Y_2, \Delta_{1,2}) \in M(X)$, where $Y_1 \cap Y_2$ is defined by the pullback square

$$\begin{tikzcd} Y_1 \cap Y_2 \arrow[r] \arrow[d] & Y_2 \arrow[d, "f_2"] \\ Y_1 \arrow[r, "f_1"'] & X \end{tikzcd}$$

and

$$\Delta_{1,2}:Y_1 \cap Y_2 \to X$$

is the corner arrow.

2.17 SCHOLIUM If $\underline{C}$ is wellpowered and has pullbacks, then $\forall X \in \mathrm{Ob}\ \underline{C}$, the category $\underline{C}(\mathrm{Sub}_{\underline{C}}\ X, \leq_X)$ associated with the poset $(\mathrm{Sub}_{\underline{C}}\ X, \leq_X)$ has binary products.

2.18 DEFINITION Let $\underline{C}$ be a category. Given an object X in $\underline{C}$, suppose that $\{(Y_i, f_i): i \in I\}$ is a set-indexed collection of elements of $M(X)$ -- then an element $(Y,f) \in M(X)$ is called an <u>intersection</u> of the (Y_i, f_i) provided that

$$\forall\ i,\ (Y,f) \leq_X (Y_i, f_i)$$

and for any object $U \xrightarrow{u} X$ in $\underline{C}/X$ such that

$$\forall\ i,\ \exists\ g_i \in \mathrm{Mor}_{\underline{C}/X}\ (U \xrightarrow{u} X,\ Y_i \xrightarrow{f_i} X),$$

there exists a

$$g \in \mathrm{Mor}_{\underline{C}/X}\ (U \xrightarrow{u} X,\ Y \xrightarrow{f} X).$$

[Note: If $I = \{1,2\}$, then matters reduce to that of 2.16 (universal property of pullbacks).]

<u>N.B.</u> Intersections are unique up to isomorphism and the intersection of the empty collection of monomorphisms with codomain X is $\mathrm{id}_X: X \rightarrow X$.

2.19 DEFINITION A category $\underline{C}$ is said to have <u>(finite) intersections</u> if for each $X \in \mathrm{Ob}\ \underline{C}$ and any (finite) set-indexed collection of elements of $M(X)$, there exists an intersection.

2.20 LEMMA If $\underline{C}$ is a finitely complete category, then $\underline{C}$ has finite intersections, and if $\underline{C}$ is a complete category, then $\underline{C}$ has intersections.

[Note: An intersection ("finite or infinite") is a multiple pullback and a multiple pullback is a limit.]

2.21 SCHOLIUM If $\underline{C}$ is wellpowered and (finitely) complete, then $\forall\ X \in \mathrm{Ob}\ \underline{C}$, the category $\underline{C}(\mathrm{Sub}_{\underline{C}}\ X, \leq_X)$ associated with the poset $(\mathrm{Sub}_{\underline{C}}\ X, \leq_X)$ has (finite) products.

§3. *DECOMPOSITIONS*

Let $\underset{\sim}{C}$ be a category, $f:X \to Y$ an epimorphism — then there are various restrictions that can be imposed on f.

(1) f is a coequalizer, i.e., $\exists\ Z \in \text{Ob}\ \underset{\sim}{C}$ and $u,v \in \text{Mor}(Z,X)$ such that $f = \text{coeq}(u,v)$.

(2) f has the left lifting property w.r.t. monomorphisms, i.e., every commutative diagram

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & A \arrow[d, "i"] \\ Y \arrow[r, "b"'] & B, \end{tikzcd}$$

where $i:A \to B$ is a monomorphism, admits a filler $w:Y \to A$ (thus $w \circ f = a$, $i \circ w = b$, and w is necessarily unique).

[Note: Epimorphisms with this property are closed under composition.]

(3) f is extremal, i.e., in any factorization $f = h \circ g$, if h is a monomorphism, then h is an isomorphism.

In general,

$$(1) \Rightarrow (2) \Rightarrow (3)$$

and none of the implications can be reversed.

3.1 LEMMA Suppose that $\underset{\sim}{C}$ is finitely complete — then an epimorphism $f:X \to Y$ satisfies (2) iff it satisfies (3).

3.2 EXAMPLE In <u>CAT</u>, there are extremal epimorphisms that are not coequalizers.

3.3 DEFINITION A finitely complete category $\underline{C}$ fulfills the <u>standard conditions</u> if $\underline{C}$ has coequalizers and the epimorphisms that are coequalizers are pullback stable.

3.4 EXAMPLE In <u>SET</u>, every epimorphism is a coequalizer and surjective functions are pullback stable. Therefore <u>SET</u> fulfills the standard conditions.

3.5 EXAMPLE In <u>TOP</u>, an epimorphism is extremal iff it is a quotient map, thus "(1) = (3)". Still, <u>TOP</u> does not fulfill the standard conditions since quotient maps are not pullback stable.

3.6 REMARK If $\underline{C}$ fulfills the standard conditions and if $\underline{I}$ is small, then the functor category $[\underline{I},\underline{C}]$ fulfills the standard conditions.

3.7 LEMMA Suppose that $\underline{C}$ fulfills the standard conditions -- then an epimorphism $f:X \to Y$ satisfies (1) iff it satisfies (2).

3.8 DEFINITION Let $f:X \to Y$ be an arrow in a category $\underline{C}$ -- then a <u>decomposition</u> of f is a pair of arrows $X \xrightarrow{k} M \xrightarrow{m} Y$ such that $f = m \circ k$, where k is an epimorphism and m is a monomorphism. The decomposition (k,m) of f is said to be <u>minimal</u> (and M is said to be the <u>image</u> of f, denoted $\text{im } f$) if for any other factorization $X \xrightarrow{\ell} N \xrightarrow{n} Y$ of f with n a monomorphism, there is an $h:M \to N$ such that $h \circ k = \ell$ and $n \circ h = m$ ($\Rightarrow$ $(M,m) \leq_Y (N,n)$).

3.9 LEMMA Suppose that $\underline{C}$ fulfills the standard conditions -- then every morphism $f:X \to Y$ in $\underline{C}$ admits a decomposition $f = m \circ k$, where k is an epimorphism satisfying "(1) = (2)" and m is a monomorphism.

PROOF Form the pullback square

$$\begin{tikzcd} P \arrow[r, "v"] \arrow[d, "u"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y . \end{tikzcd}$$

Then u and v are epimorphisms. Pass now to coeq(u,v):

$$\begin{tikzcd} P \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & X \arrow[r, "f"] \arrow[d, "k"'] & Y \\ & Z. & \end{tikzcd}$$

Since $f \circ u = f \circ v$, there is a unique $m:Z \to Y$ such that $f = m \circ k$ and the claim is that m is a monomorphism. To see this, form the pullback square

$$\begin{tikzcd} Q \arrow[r, "r"] \arrow[d, "s"'] & Z \arrow[d, "m"] \\ Z \arrow[r, "m"'] & Y . \end{tikzcd}$$

Then

$$m \circ k \circ u = m \circ k \circ v,$$

so there is a unique morphism $q:P \to Q$ such that

$$r \circ q = k \circ u, \ s \circ q = k \circ v.$$

But q is an epimorphism (cf. infra) and $k \circ u = k \circ v$, hence $r = s$ which implies that m is a monomorphism.

[Note: From the definitions

$$\left[\begin{array}{l} P = X \times_Y X \\ Q = Z \times_Y Z \end{array}\right.$$

and there is a commutative diagram

$$\begin{tikzcd}
X \times_Y X \arrow[r, "a"] \arrow[d, "c"'] & Z \times_Y X \arrow[r] \arrow[d, "b"] & X \arrow[d, "k"] \\
X \times_Y Z \arrow[r, "d"'] \arrow[d] & Z \times_Y Z \arrow[r] \arrow[d] & Z \arrow[d, "m"] \\
X \arrow[r, "k"'] & Z \arrow[r, "m"'] & Y
\end{tikzcd}$$

of pullback squares. Since $\underline{C}$ fulfills the standard conditions and k is a coequalizer, the arrows a,b,c,d are coequalizers as well. Therefore $q = b \circ a = d \circ c$ is an epimorphism.

3.10 THEOREM Suppose that $\underline{C}$ fulfills the standard conditions -- then every morphism $f:X \to Y$ in $\underline{C}$ admits a minimal decomposition $f = m \circ k$ unique up to isomorphism.

N.B. The decomposition of f secured by 3.9 turns out to be minimal but there are two points of detail that will have to be addressed before this can be established.

- Suppose given two decompositions of f per 3.9, hence $m \circ k = m' \circ k'$, where

$$\left[\begin{array}{l} X \xrightarrow{k} M \xrightarrow{m} Y \\ \\ X \xrightarrow{k'} M' \xrightarrow{m'} Y. \end{array}\right.$$

Then we claim that there exists an isomorphism $\phi:M \to M'$ such that

$$\phi \circ k = k' \text{ and } m = m' \circ \phi.$$

Thus consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{k} & M \\ {\scriptstyle k'}\downarrow & & \downarrow{\scriptstyle m} \\ M' & \xrightarrow[m']{} & Y \end{array} .$$

Then by the left lifting property w.r.t. monomorphisms.

$$\exists\ u:M \to M' \quad \text{st} \quad \begin{cases} u \circ k = k' \\ m' \circ u = m \end{cases}$$

and

$$\exists\ u':M' \to M \quad \text{st} \quad \begin{cases} u' \circ k' = k \\ m \circ u' = m'. \end{cases}$$

Accordingly,

$$\begin{cases} m \circ u' \circ u \circ k = m' \circ k' = m \circ k \Rightarrow u' \circ u = \mathrm{id}_M \\ m' \circ u \circ u' \circ k' = m \circ k = m' \circ k' \Rightarrow u \circ u' = \mathrm{id}_{M'}. \end{cases}$$

It remains only to take $\phi = u$.

[Note: This is what is meant by "unique up to isomorphism" in 3.10.]

- Suppose given a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{k} & M & \xrightarrow{m} & Y \\ {\scriptstyle u}\downarrow & & & & \downarrow{\scriptstyle v} \\ X' & \xrightarrow[k']{} & M' & \xrightarrow[m']{} & Y', \end{array}$$

where $\begin{cases} f = m \circ k \\ f' = m' \circ k' \end{cases}$ are decompositions per 3.9 — then there exists a unique

$w:M \to M'$ such that $\begin{cases} w \circ k = k' \circ u \\ m' \circ w = v \circ m . \end{cases}$ The uniqueness of w is, of course, clear.

As for the existence of w, use 3.9 again and write

$$\begin{cases} k' \circ u = n \circ \ell \\ v \circ m = n' \circ \ell', \end{cases}$$

say

$$\begin{cases} X \xrightarrow{\ell} N \xrightarrow{m' \circ n} Y' \\ X \xrightarrow{\ell' \circ k} N' \xrightarrow{n'} Y'. \end{cases}$$

Since

$$m' \circ k' \circ u = v \circ m \circ k$$

and since

$$\begin{cases} (m' \circ n) \circ \ell = m' \circ k' \circ u \\ n' \circ (\ell' \circ k) = v \circ m \circ k \quad , \end{cases}$$

it follows from what has been said above that there exists an isomorphism $\phi:N \to N'$ such that

$$\begin{cases} \phi \circ \ell = \ell' \circ k \\ m' \circ n = n' \circ \phi. \end{cases}$$

Now put

$$w = n \circ \phi^{-1} \circ \ell'.$$

Then

$$\begin{cases} w \circ k = n \circ \phi^{-1} \circ \ell' \circ k = n \circ \ell = k' \circ u \\ m' \circ w = m' \circ n \circ \phi^{-1} \circ \ell' = n' \circ \ell' = v \circ m, \end{cases}$$

as desired.

[Note:

$$(u,v) \in \mathrm{Mor}_{\underline{C}(\to)}(f,f')$$

and

$$\left[\begin{array}{l}(u,w) \in \mathrm{Mor}_{\underline{C}(\to)}(k,k') \\ \\ (w,v) \in \mathrm{Mor}_{\underline{C}(\to)}(m,m').]\end{array}\right.$$

<u>Proof of 3.10</u> Write $f = m \circ k$ per 3.9 -- then this decomposition is minimal. For suppose as in 3.8 that $f = n \circ \ell$ and using 3.9 once more, write $\ell = m' \circ k'$. Thanks to the preceding discussion, the commutative diagram

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, equal] & M \arrow[r, "m"] & Y \arrow[d, equal] \\ X \arrow[r, "k'"'] & M' \arrow[r, "n \circ m'"'] & Y \end{tikzcd}$$

gives rise to a unique $w: M \to M'$ such that

$$w \circ k = k' \text{ and } n \circ m' \circ w = m.$$

Put $h = m' \circ w$ -- then $h: M \to N$ and

$$\left[\begin{array}{l} h \circ k = m' \circ w \circ k = m' \circ k' = \ell \\ n \circ h = n \circ m' \circ w = m. \end{array}\right.$$

[Note: Such an h is unique. For $\left[\begin{array}{l} n \circ h = m \\ n \circ h' = m \end{array}\right. \Rightarrow h = h'$, n being a monomorphism.]

3.11 DEFINITION Let $\underline{C}$ be a category. Given an object X in $\underline{C}$, suppose that $\{(Y_i,f_i):i \in I\}$ is a set-indexed collection of elements of M(X) -- then an element $(Y,f) \in M(X)$ is called a <u>union</u> of the (Y_i,f_i) provided that

$$\forall\ i,\ (Y_i,f_i) \leq_X (Y,f)$$

and for any element $U \xrightarrow{u} X$ of M(X) such that

$$\forall\ i,\ \exists\ g_i \in \mathrm{Mor}_{\underline{C}/X}\ (Y_i \xrightarrow{f_i} X,\ U \xrightarrow{u} X),$$

there exists a

$$g \in \mathrm{Mor}_{\underline{C}/X}\ (Y \xrightarrow{f} X,\ U \xrightarrow{u} X).$$

[Note: The definition of union is not the exact analog of the definition of intersection (cf. 2.18).]

3.12 DEFINITION A category $\underline{C}$ is said to have <u>(finite) unions</u> if for each $X \in \mathrm{Ob}\ \underline{C}$ and any (finite) set-indexed collection of elements of M(X), there exists a union.

3.13 LEMMA Suppose that $\underline{C}$ fulfills the standard conditions and has finite coproducts -- then $\underline{C}$ has finite unions.

PROOF Fix $X \in \mathrm{Ob}\ \underline{C}$ and let $\{(Y_i,f_i):i \in I\}$ be a finite collection of objects of M(X) $(I \neq \emptyset)$. Denote by

$$\begin{cases} \mathrm{in}_i:Y_i \longrightarrow \coprod\limits_{i\in I} Y_i \\ f:\coprod\limits_{i\in I} Y_i \longrightarrow X \end{cases}$$

the canonical arrows. Write $f = m \circ k$ per 3.10, thus

$$\coprod_{i \in I} Y_i \xrightarrow{k} M \xrightarrow{m} X.$$

Then (M,m) is a union of the (Y_i, f_i). To begin with, $k \circ in_i : Y_i \to M$ and

$$f_i = f \circ in_i = m \circ k \circ in_i \Rightarrow (Y_i, f_i) \leq_X (M,m).$$

Assume next that $U \xrightarrow{u} X$ is an element of $M(X)$ and

$$\forall\, i,\ \exists\, g_i \in \mathrm{Mor}_{\underline{C}/X}\,(Y_i \xrightarrow{f_i} X,\ U \xrightarrow{u} X),$$

so $f_i = u \circ g_i$ -- then there exists a unique $g: \coprod_{i \in I} Y_i \to U$ such that $g \circ in_i = g_i$.

But

$$u \circ g \circ in_i = u \circ g_i = f_i = f \circ in_i$$

$$\Rightarrow u \circ g = f \quad \text{(definition of coproduct).}$$

Now display the data:

$$\begin{tikzcd} \coprod_{i \in I} Y_i \arrow[r, "k"] \arrow[d, equal] & M \arrow[r, "m"] & X \arrow[d, equal] \\ \coprod_{i \in I} Y_i \arrow[r, "g"'] & U \arrow[r, "u"'] & X\,. \end{tikzcd}$$

Since the decomposition $f = m \circ k$ is minimal and since u is a monomorphism, there is an $h: M \to U$ for which $u \circ h = m$, i.e.,

$$(M,m) \leq_X (U,u).$$

[Note: The union of the empty collection of monomorphisms with codomain X is initial in M(X).]

N.B. The same argument works for an arbitrary index set so long as $\underline{C}$ has coproducts.

3.14 SCHOLIUM If $\underline{C}$ is wellpowered, fulfills the standard conditions, and has (finite) coproducts, then the category $\underline{C}(\mathrm{Sub}_{\underline{C}}\ X, \leq_X)$ associated with the poset $(\mathrm{Sub}_{\underline{C}}\ X, \leq_X)$ has (finite) coproducts.

§4. *SLICES*

Let $\underline{C}$ be a category.

4.1 THEOREM If $\underline{C}$ is finitely complete, then so are the $\underline{C}/X$.

4.2 REMARK It can happen that the $\underline{C}/X$ are finitely complete, yet $\underline{C}$ itself is not finitely complete.

[Take $\underline{C} = \underline{\mathrm{TOP}}_{\underline{\mathrm{LH}}}$, the category whose objects are the topological spaces and whose morphisms are the local homeomorphisms -- then $\underline{\mathrm{TOP}}_{\underline{\mathrm{LH}}}$ has pullbacks but does not have a final object, hence is not finitely complete (cf. 1.8). On the other hand, the $\underline{\mathrm{TOP}}_{\underline{\mathrm{LH}}}/X$ are finitely complete.]

4.3 LEMMA If $\underline{C}$ has pullbacks, then the $\underline{C}/X$ have binary products.

PROOF Given objects $U \xrightarrow{u} X$ and $V \xrightarrow{v} X$ in $\underline{C}/X$, form the pullback square

$$\begin{array}{ccc} P & \xrightarrow{\eta} & V \\ {\scriptstyle\xi}\downarrow & & \downarrow{\scriptstyle v} \\ U & \xrightarrow[u]{} & X \end{array}$$

in $\underline{C}$ -- then the corner arrow $P \to X$ is a product of $U \xrightarrow{u} X$ and $V \xrightarrow{v} X$ in $\underline{C}/X$.

4.4 LEMMA If the $\underline{C}/X$ have binary products, then $\underline{C}$ has pullbacks.

PROOF Consider a 2-sink $U \xrightarrow{u} X \xleftarrow{v} V$ in $\underline{C}$, thus $\begin{bmatrix} U \xrightarrow{u} X \\ V \xrightarrow{v} X \end{bmatrix} \in \mathrm{Ob}\, \underline{C}/X$.

Let

$$P \xrightarrow{\pi} X = (U \xrightarrow{u} X) \times (V \xrightarrow{v} X).$$

Then there are commutative diagrams

$$\begin{tikzcd} P \arrow[r, "pr_U"] \arrow[d, "\pi"'] & U \arrow[d, "u"] \\ X \arrow[r, equal] & X \end{tikzcd}, \quad \begin{tikzcd} P \arrow[r, "pr_V"] \arrow[d, "\pi"'] & V \arrow[d, "v"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

or still, a commutative diagram

$$\begin{tikzcd} P \arrow[r, "pr_V"] \arrow[d, "pr_U"'] & V \arrow[d, "v"] \\ U \arrow[r, "u"'] & X \end{tikzcd}$$

which is a pullback square in $\underline{C}$.

Let $X,Y \in \mathrm{Ob}\,\underline{C}$ and let $f:X \to Y$ be a morphism — then f induces a functor $f_!:\underline{C}/X \to \underline{C}/Y$ via postcomposition.

4.5 LEMMA Suppose that $\underline{C}$ has pullbacks -- then $\forall\, f$, $f_!$ has a right adjoint f^*.

PROOF Given an object $U \xrightarrow{u} Y$ in $\underline{C}/Y$, form the pullback square

$$\begin{tikzcd} P \arrow[r] \arrow[d, "p"'] & U \arrow[d, "u"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

and let

$$f^*(U \xrightarrow{u} Y) = P \xrightarrow{p} X.$$

Then this prescription defines a functor $f^*:\underline{C}/Y \to \underline{C}/X$ and $(f_!,f^*)$ is an adjoint pair.

4.6 REMARK Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ -- then

$$\left[\begin{array}{l} \underline{C}/X \xrightarrow{f_!} \underline{C}/Y \xrightarrow{g_!} \underline{C}/Z \\ \\ \underline{C}/Z \xrightarrow{g^*} \underline{C}/Y \xrightarrow{f^*} \underline{C}/X. \end{array}\right.$$

And

$$(g \circ f)_! = g_! \circ f_!$$

but in general

$$f^* \circ g^* \neq (g \circ f)^*.$$

Given $X \in \mathrm{Ob}\,\underline{C}$, denote by i_X the inclusion $\underline{M}(X) \to \underline{C}/X$.

4.7 LEMMA Suppose that $\underline{C}$ fulfills the standard conditions -- then i_X has a left adjoint

$$\mathrm{im}_X:\underline{C}/X \to \underline{M}(X).$$

[Given $U \xrightarrow{u} X \in \mathrm{Ob}\,\underline{C}/X$, write $u = m \circ k$ per 3.10, so $U \xrightarrow{k} M \xrightarrow{m} X$. Put

$$\mathrm{im}_X(U \xrightarrow{u} X) = M \xrightarrow{m} X.]$$

If $\underline{C}$ has pullbacks and if $f:X \to Y$ is a morphism, then $f^*:\underline{C}/Y \to \underline{C}/X$ restricts to a functor $f^{-1}:\underline{M}(Y) \to \underline{M}(X)$ (cf. 2.15).

4.8 LEMMA Suppose that $\underline{C}$ fulfills the standard conditions -- then f^{-1} has a left adjoint

$$\exists_f : \underline{M}(X) \to \underline{M}(Y).$$

[Take for $\exists_f$ the composite

$$\underline{M}(X) \xrightarrow{i_X} \underline{C}/X \xrightarrow{f_!} \underline{C}/Y \xrightarrow{im_Y} \underline{M}(Y).]$$

4.9 REMARK If $\underline{C}$ fulfills the standard conditions, then so do the $\underline{C}/X$.

§5. *CARTESIAN CLOSED CATEGORIES*

Let $\underline{C}$ be a category with finite products.

5.1 DEFINITION $\underline{C}$ is cartesian closed provided that each of the functors $- \times Y : \underline{C} \to \underline{C}$ has a right adjoint $Z \to Z^Y$, so

$$\mathrm{Mor}(X \times Y, Z) \approx \mathrm{Mor}(X, Z^Y).$$

N.B. The property of being cartesian closed is invariant under equivalence.

5.2 EXAMPLE $\underline{SET}$ is cartesian closed but $\underline{SET}^{OP}$ is not cartesian closed. The full subcategory of $\underline{SET}$ whose objects are finite is cartesian closed. On the other hand, the full subcategory of $\underline{SET}$ whose objects are at most countable is not cartesian closed.

5.3 EXAMPLE $\underline{TOP}$ is not cartesian closed but does have full, cartesian closed subcategories, e.g., the category of compactly generated Hausdorff spaces.

5.4 EXAMPLE $\underline{CAT}$ is cartesian closed:

$$\mathrm{Mor}(\underline{C} \times \underline{D}, \underline{E}) \approx \mathrm{Mor}(\underline{C}, \underline{E}^{\underline{D}}),$$

where $\underline{E}^{\underline{D}} = [\underline{D}, \underline{E}]$.

5.5 EXAMPLE Suppose that $(X, \leq)$ is a boolean algebra. Put $z^y = \neg y \vee z$ -- then

$$x \wedge y \leq z \iff x \leq z^y.$$

E.g.: Given that $x \wedge y \leq z$, write

$$x = x \wedge 1 = x \wedge (\neg y \vee y)$$

$$= (x \wedge \neg y) \vee (x \wedge y)$$

$$\leq (x \wedge \neg y) \vee z$$

$$\leq \neg y \vee z = z^y.$$

Therefore

$$\mathrm{Mor}(x \wedge y, z) \approx \mathrm{Mor}(x, z^y) \quad \text{(cf. 1.4)},$$

hence $\underline{C}(X,\leq)$ is cartesian closed.

Let $\underline{C}$ be a cartesian closed category.

5.6 DEFINITION The object Z^Y is called an <u>exponential object</u>, the <u>evaluation morphism</u> $ev_{Y,Z}$ being the arrow

$$Z^Y \times Y \to Z$$

with the property that for every $f{:}X \times Y \to Z$ there is a unique $g{:}X \to Z^Y$ such that

$$f = ev_{Y,Z} \circ (g \times id_Y).$$

One may view the association $(Y,Z) \to Z^Y$ as a bifunctor, covariant in Z and contravariant in Y.

- The functor

$$(\;—\;)^Y{:}\underline{C} \to \underline{C}$$

is defined on objects Z by

$$(\;—\;)^Y Z = Z^Y$$

and on morphisms $A \xrightarrow{f} B$ by

$$(\;—\;)^Y(A \xrightarrow{f} B) = A^Y \xrightarrow{f^Y} B^Y,$$

where f^Y is the unique arrow rendering the diagram

$$\begin{tikzcd} A^Y \times Y \arrow[r, "ev"] \arrow[d, "f^Y \times id"'] & A \arrow[d, "f"] \\ B^Y \times Y \arrow[r, "ev"'] & B \end{tikzcd}$$

commutative.

- The functor

$$Z^{(\,—\,)}$$

is defined on objects Y by

$$Z^{(\,—\,)}Y = Z^Y$$

and on morphisms $A \xrightarrow{f} B$ by

$$Z^{(\,—\,)}(A \xrightarrow{f} B) = Z^B \xrightarrow{Z^f} Z^A,$$

where Z^f is the unique arrow rendering the diagram

$$\begin{tikzcd} Z^B \times A \arrow[r, "id \times f"] \arrow[d, "Z^f \times id"'] & Z^B \times B \arrow[d, "ev"] \\ Z^A \times A \arrow[r, "ev"'] & Z \end{tikzcd}$$

commutative.

5.7 LEMMA The functor

$$Z^{(\,—\,)}:\underline{C}^{OP} \rightarrow \underline{C}$$

admits a left adjoint, viz.

$$(Z^{(\,—\,)})^{OP}:\underline{C} \rightarrow \underline{C}^{OP}.$$

N.B. $(\text{---})^Y$ preserves limits while $Z^{(\text{---})}$ sends colimits to limits.

5.8 LEMMA In a cartesian closed category $\underset{\sim}{C}$,

$$(1)\quad X^{Y \times Z} \approx (X^Y)^Z; \quad (3)\quad X^{\coprod_i Y_i} \approx \prod_i (X^{Y_i});$$

$$(2)\quad (\prod_i X_i)^Y \approx \prod_i (X_i^Y); \quad (4)\quad X \times (\coprod_i Y_i) \approx \coprod_i (X \times Y_i).$$

5.9 LEMMA In a cartesian closed category $\underset{\sim}{C}$, finite products of epimorphisms are epimorphisms.

5.10 RAPPEL A full, isomorphism closed subcategory $\underset{\sim}{D}$ of a category $\underset{\sim}{C}$ is said to be a reflective subcategory of $\underset{\sim}{C}$ if the inclusion $\iota:\underset{\sim}{D} \to \underset{\sim}{C}$ has a left adjoint R, a reflector for $\underset{\sim}{D}$.

[Note: A reflective subcategory $\underset{\sim}{D}$ of a category $\underset{\sim}{C}$ is closed under the formation of limits in $\underset{\sim}{C}$.]

Let $\underset{\sim}{D}$ be a reflective subcategory of a category $\underset{\sim}{C}$, R a reflector for $\underset{\sim}{D}$ -- then one may attach to each $X \in \text{Ob } \underset{\sim}{C}$ a morphism $r_X:X \to RX$ in $\underset{\sim}{C}$ with the following property: Given any $Y \in \text{Ob } \underset{\sim}{D}$ and any morphism $f:X \to Y$ in $\underset{\sim}{C}$, there exists a unique morphism $g:RX \to Y$ in $\underset{\sim}{D}$ such that $f = g \circ r_X$.

N.B. Matters can always be arranged in such a way as to ensure that $R \circ \iota = \text{id}_{\underset{\sim}{D}}$.

5.11 LEMMA Suppose that $\underset{\sim}{C}$ is cartesian closed and let $\underset{\sim}{D}$ be a reflective subcategory of $\underset{\sim}{C}$. Assume: The reflector $R:\underset{\sim}{C} \to \underset{\sim}{D}$ preserves finite products -- then

$\underline{D}$ is cartesian closed.

[If $Y,Z \in \text{Ob } \underline{D}$, then Z^Y is isomorphic to an object in $\underline{D}$, hence $Z^Y \in \text{Ob } \underline{D}$.]

Let $\underline{C}$ be cartesian closed -- then for any final object $*_{\underline{C}}$, we have

$$(*_{\underline{C}})^X \approx *_{\underline{C}} \ \& \ X^{*_{\underline{C}}} \approx X.$$

5.12 DEFINITION Let $\underline{C}$ be a category with an initial object $\emptyset_{\underline{C}}$ -- then $\emptyset_{\underline{C}}$ is <u>strict</u> if every morphism $f:X \to \emptyset_{\underline{C}}$ with codomain $\emptyset_{\underline{C}}$ is an isomorphism.

[Note: Any morphism to an initial object is an epimorphism.]

5.13 LEMMA Let $\underline{C}$ be a category with finite products and an initial object $\emptyset_{\underline{C}}$ -- then $\emptyset_{\underline{C}}$ is strict iff $\forall\ X \in \text{Ob } \underline{C}$,

$$X \times \emptyset_{\underline{C}} \approx \emptyset_{\underline{C}}.$$

PROOF If $\emptyset_{\underline{C}}$ is strict, then the projection $X \times \emptyset_{\underline{C}} \to \emptyset_{\underline{C}}$ is an isomorphism. Conversely, let $f:X \to \emptyset_{\underline{C}}$ be a morphism -- then there is a commutative diagram

$$\begin{tikzcd}
 & X \times \emptyset_{\underline{C}} \arrow[d, "\approx"'] & \\
X \arrow[d, equal] & \emptyset_{\underline{C}} \arrow[l, "!"'] \arrow[r, "\text{id}"] & \emptyset_{\underline{C}} \arrow[d, equal] \\
X & X \arrow[l, "\text{id}_X"] \arrow[u, "f"] \arrow[r, "f"'] & \emptyset_{\underline{C}}
\end{tikzcd},$$

from which it follows that f is a split monomorphism ($! \circ f = \text{id}_X$). But f is also an epimorphism. Therefore f is an isomorphism.

5.14 APPLICATION Let $\underline{C}$ be a cartesian closed category with an initial object $\emptyset_{\underline{C}}$ -- then $\emptyset_{\underline{C}}$ is strict.

[The functor — × X preserves colimits, in particular initial objects, so $\emptyset_{\underline{C}} \times X \approx \emptyset_{\underline{C}}$. And

$$\emptyset_{\underline{C}} \times X \approx X \times \emptyset_{\underline{C}}.]$$

5.15 EXAMPLE Under the preceding assumptions,

$$X^{\emptyset_{\underline{C}}} \approx *_{\underline{C}}.$$

[Given $A \in \text{Ob } \underline{C}$,

$$\begin{aligned} \text{Mor}(A,X^{\emptyset_{\underline{C}}}) &\approx \text{Mor}(A \times \emptyset_{\underline{C}},X) \\ &\approx \text{Mor}(\emptyset_{\underline{C}},X). \end{aligned}$$

But there is a unique arrow $\emptyset_{\underline{C}} \to X$, so there is a unique arrow $A \to X^{\emptyset_{\underline{C}}}$ and this means that $X^{\emptyset_{\underline{C}}}$ is a final object.]

5.16 LEMMA Let $\underline{C}$ be a cartesian closed category with an initial object $\emptyset_{\underline{C}}$ -- then $\forall\ X \in \text{Ob } \underline{C}$, the canonical arrow $\emptyset_{\underline{C}} \xrightarrow{!} X$ is a monomorphism, thus is an element of $M(X)$.

PROOF Suppose that $a,b:A \to \emptyset_{\underline{C}}$ are morphisms such that $! \circ a = ! \circ b$. Since A is initial ($\emptyset_{\underline{C}}$ being strict), $a = b$, hence $\emptyset_{\underline{C}} \xrightarrow{!} X$ is a monomorphism.

5.17 EXAMPLE Under the preceding assumptions

$$!^X \in M(*_{\underline{C}}).$$

[The functor $(\text{—})^X$ preserves limits, in particular monomorphisms. Therefore

$$(\emptyset_{\underline{C}})^X \xrightarrow{!^X} (*_{\underline{C}})^X$$

is a monomorphism. But

$$(*_{\underline{C}})^X \approx *_{\underline{C}},$$

so

$$!^X \in M(*_{\underline{C}}).]$$

[Note: $M(*_{\underline{C}})$ is an <u>exponential ideal</u> in the sense that if $Z \xrightarrow{!} *_{\underline{C}}$ is a monomorphism, then $\forall\ Y \in \text{Ob}\ \underline{C}$, $Z^Y \xrightarrow{!} *_{\underline{C}}$ is a monomorphism.]

5.18 RAPPEL An object in a category $\underline{C}$ is called a <u>zero object</u> if it is both an initial object and a final object.

5.19 LEMMA Suppose that $\underline{C}$ is cartesian closed -- then $\underline{C}$ has a zero object iff $\underline{C}$ is equivalent to $\underline{1}$.

5.20 EXAMPLE Neither $\underline{SET}_*$ nor $\underline{TOP}_*$ is cartesian closed.

5.21 THEOREM Let $\underline{C}$ be a small category -- then $\hat{\underline{C}}$ is cartesian closed.

PROOF Given $F,G \in \text{Ob}\ \hat{\underline{C}}$, define

$$G^F : \underline{C}^{OP} \to \underline{SET}$$

by the rule

$$G^F(X) = \mathrm{Nat}(h_X \times F, G) \quad (X \in \mathrm{Ob}\ \underline{C}).$$

5.22 EXAMPLE $\hat{\underline{\Delta}}$ = $\underline{\text{SISET}}$ is cartesian closed:

$$\mathrm{Nat}(X \times Y, Z) \approx \mathrm{Nat}(X, Z^Y),$$

where

$$Z^Y([n]) = \mathrm{Nat}(\Delta[n] \times Y, Z) \quad (\Delta[n] = h_{[n]}).$$

5.23 DEFINITION A category $\underline{C}$ is <u>locally cartesian closed</u> if $\forall\ X \in \mathrm{Ob}\ \underline{C}$, the category $\underline{C}/X$ is cartesian closed.

[Note: A locally cartesian closed category with a final object is cartesian closed.]

5.24 EXAMPLE $\underline{\text{SET}}$ is locally cartesian closed. Proof: $\underline{\text{SET}}/X$ is equivalent to $\underline{\text{SET}}^X$.

5.25 EXAMPLE $\underline{\text{CAT}}$ is cartesian closed but $\underline{\text{CAT}}$ is not locally cartesian closed.

5.26 EXAMPLE $\underline{\text{TOP}}_{\underline{\text{LH}}}$ is locally cartesian closed but $\underline{\text{TOP}}_{\underline{\text{LH}}}$ is not cartesian closed.

5.27 THEOREM Let $\underline{C}$ be a small category — then $\hat{\underline{C}}$ is locally cartesian closed.

PROOF Given $F \in \mathrm{Ob}\ \hat{\underline{C}}$, write C/F in place of $\mathrm{gro}_{\underline{C}}\ F$ — then the canonical arrow

$$\widehat{\underline{C}/F} \longrightarrow \hat{\underline{C}}/F$$

is an equivalence and $\widehat{\underline{C}/F}$ is cartesian closed (cf. 5.21).

5.28 THEOREM Let $\underline{C}$ be a category with pullbacks. Assume: $\forall$ f, f^* has a right adjoint f_* -- then $\underline{C}$ is locally cartesian closed.

PROOF Thanks to 4.3, $\underline{C}/X$ has binary products. Since $\underline{C}/X$ also admits a final object (viz. $\mathrm{id}_X : X \to X$), it follows that $\underline{C}/X$ has finite products. This said, fix objects $\begin{cases} u:U \to X \\ v:V \to X \end{cases}$ in $\underline{C}/X$ and realize $u \times v$ as the corner arrow $P \to X$ in the pullback square

$$\begin{tikzcd} P \arrow[r, "\eta"] \arrow[d, "\xi"'] & V \arrow[d, "v"] \\ U \arrow[r, "u"'] & X \end{tikzcd},$$

thus

$$u \times v = u \circ \xi = v \circ \eta = v_! v^* u.$$

Then for any $f:Y \to X$, we have

$$\begin{aligned} \mathrm{Mor}(u \times v, f) &= \mathrm{Mor}(v_! v^* u, f) \\ &\approx \mathrm{Mor}(v^* u, v^* f) \\ &\approx \mathrm{Mor}(u, v_* v^* f). \end{aligned}$$

Definition:

$$f^V = v_* v^* f.$$

Suppose that $\underline{C}$ is finitely complete. Given $X \in \mathrm{Ob}\,\underline{C}$, denote by

$$X_! : \underline{C}/X \to \underline{C}$$

the forgetful functor and by

$$X^* : \underline{C} \to \underline{C}/X$$

the functor that sends Y to $X \times Y \to X$.

5.29 CRITERION The functor $— \times X$ has a right adjoint iff the functor X^* has a right adjoint.

5.30 LEMMA If $\underline{C}$ is locally cartesian closed, then $\forall\, X \in \mathrm{Ob}\ \underline{C}$, the category $\underline{C}/X$ is locally cartesian closed.

PROOF For every object $A \to X$ of $\underline{C}/X$,

$$\underline{C}/X/A \to X \approx \underline{C}/A.$$

5.31 LEMMA If $\underline{C}$ is locally cartesian closed, then $\forall\, X \in \mathrm{Ob}\ \underline{C}$, the category $\underline{C}/X$ is finitely complete.

PROOF Since the $\underline{C}/X$ are cartesian closed, they have products, in particular binary products, hence $\underline{C}$ has pullbacks (cf. 4.4). So $\forall\, X \in \mathrm{Ob}\ \underline{C}$, $\underline{C}/X$ has pullbacks (pullbacks in $\underline{C}/X$ are computed as in $\underline{C}$ (cf. 4.1)). But $\underline{C}/X$ has a final object, thus $\underline{C}/X$ is finitely complete (cf. 1.8).

5.32 LEMMA If $\underline{C}$ is locally cartesian closed, then $\forall\, f$, $f_!$ has a right adjoint f^*.

[Because, as noted above, $\underline{C}$ has pullbacks.]

5.33 THEOREM If $\underline{C}$ is locally cartesian closed, then $\forall\, f$, f^* has a right adjoint f_*.

[A morphism $f:X \to Y$ is an object of $\underline{C}/Y$ and

$$\begin{tikzcd} X \arrow[r, "!"] \arrow[d, "f"'] & Y \arrow[d, "\mathrm{id}_Y"] \\ Y \arrow[r, equal] & Y \end{tikzcd}\quad.$$

Therefore 5.29 is applicable.]

<u>N.B.</u> f^* preserves exponential objects.

§6. *SUBOBJECT CLASSIFIERS*

Let $\underline{C}$ be a finitely complete category.

6.1 DEFINITION A <u>subobject classifier</u> for $\underline{C}$ is a pair (Ω,τ), where $\tau:*_{\underline{C}} \to \Omega$ is a monomorphism with the property that for each object X in $\underline{C}$ and every monomorphism $f:Y \to X$ there exists a unique morphism $\chi_f:X \to \Omega$ such that the diagram

$$\begin{tikzcd} Y \arrow[r, "!"] \arrow[d, "f"'] & *_{\underline{C}} \arrow[d, "\tau"] \\ X \arrow[r, "\chi_f"'] & \Omega \end{tikzcd}$$

is a pullback square.

[Note: The morphism $\chi_f:X \to \Omega$ is called the <u>classifying arrow</u> of (Y,f) in X.]

6.2 EXAMPLE id_Ω is the classifying arrow of $(*_{\underline{C}},\tau)$ in Ω.

6.3 LEMMA If (Ω,τ) and (Ω',τ') are subobject classifiers, then Ω and Ω' are isomorphic.

PROOF From the definitions, there are pullback squares

$$\begin{tikzcd} *_{\underline{C}} \arrow[r, equal] \arrow[d, "\tau'"'] & *_{\underline{C}} \arrow[d, "\tau"] \\ \Omega' \arrow[r, "\chi"'] & \Omega \end{tikzcd} \qquad \begin{tikzcd} *_{\underline{C}} \arrow[r, equal] \arrow[d, "\tau"'] & *_{\underline{C}} \arrow[d, "\tau'"] \\ \Omega \arrow[r, "\chi'"'] & \Omega' \end{tikzcd} .$$

Therefore $\chi' \circ \chi$ is the classifying arrow of $(*_{\underline{C}},\tau')$ in Ω':

$$\begin{tikzcd}
*_{\underline{C}} \arrow[r, equal] \arrow[d, "\tau'"'] & *_{\underline{C}} \arrow[d, "\tau'"] \\
\Omega' \arrow[r, "\chi' \circ \chi"'] & \Omega'
\end{tikzcd}.$$

So, by uniqueness, $\chi' \circ \chi = \mathrm{id}_{\Omega'}$. And, analogously, $\chi \circ \chi' = \mathrm{id}_{\Omega}$.

6.4 EXAMPLE Take $\underline{C} = \underline{SET}$, let $*_{\underline{C}} = \{1\}$, $\Omega = \{0,1\}$, and define $\tau:*_{\underline{C}} \to \Omega$ by sending 1 to 1. Given X, if Y is a subset of X and if $f:Y \to X$ is the inclusion, then there is a pullback square

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "f"'] & \{1\} \arrow[d] \\
X \arrow[r, "\chi_Y"'] & \{0,1\},
\end{tikzcd}$$

where χ_Y is the characteristic function of Y.

6.5 LEMMA Let (Ω, τ) be a subobject classifier -- then $\forall$ $X \in \mathrm{Ob}\ \underline{C}$,

$$\begin{tikzcd}
X \times *_{\underline{C}} \arrow[r] \arrow[d, "\mathrm{id}_X \times \tau"'] & X \\
X \times \Omega \arrow[r] & X
\end{tikzcd}$$

is a subobject classifier in $\underline{C}/X$.

[Note: Recall that $\underline{C}/X$ is finitely complete (cf. 4.1).]

6.6 RAPPEL A category $\underline{C}$ is <u>balanced</u> if every morphism that is simultaneously a monomorphism and an epimorphism is an isomorphism.

6.7 EXAMPLE $\underline{\text{SET}}$ is balanced but $\underline{\text{TOP}}$ is not balanced.

6.8 LEMMA Let $\underline{C}$ be a category and let $f:X \to Y$ be a morphism. Assume: f is an equalizer and an epimorphism -- then f is an isomorphism.

PROOF Suppose that $f = \text{eq}(u,v)$, hence $u \circ f = v \circ f$, so $u = v$ (f being an epimorphism). But the equalizer of $u = v$ is id_Y, hence there is a unique arrow $g:Y \to X$ such that $f \circ g = \text{id}_Y$:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle \text{id}_Y} \\ Y & = & Y \end{array} .$$

And then

$$f \circ g \circ f = \text{id}_Y \circ f = f = f \circ \text{id}_X$$

$$\Rightarrow$$

$$g \circ f = \text{id}_X.$$

Therefore f is an isomorphism.

6.9 LEMMA If $\underline{C}$ admits a subobject classifier (Ω, τ), then every monomorphism $f:Y \to X$ is an equalizer.

PROOF Consider the pullback square

$$\begin{array}{ccc} Y & \xrightarrow{!} & *_{\underline{C}} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \tau} \\ X & \xrightarrow[\chi_f]{} & \Omega \end{array} .$$

Then $\top$ is a split monomorphism, hence the same is true of f. And a split monomorphism is an equalizer.

6.10 SCHOLIUM A category with a subobject classifier is balanced.

Assume: $\underline{C}$ admits a subobject classifier $(\Omega,\top)$.

6.11 LEMMA Let (Y,f), (Z,g) be elements of $M(X)$ -- then $(Y,f) \sim_X (Z,g)$ iff $\chi_f = \chi_g$.

6.12 LEMMA Given $\chi \in \mathrm{Mor}(X,\Omega)$, form the pullback square

$$\begin{tikzcd} Y \arrow[r, "!"] \arrow[d, "f"'] & *_{\underline{C}} \arrow[d, "\top"] \\ X \arrow[r, "\chi"'] & \Omega \end{tikzcd} .$$

Then $\chi_f = \chi$.

6.13 THEOREM The map $[f] \to \chi_f$ is a bijection between the class $\mathrm{Sub}_{\underline{C}}\, X$ of subobjects of X and the set $\mathrm{Mor}(X,\Omega)$.

[Note: Therefore $\mathrm{Sub}_{\underline{C}}\, X$ "is a set", i.e., has a representative class of monomorphisms which is a set, thus $\underline{C}$ is wellpowered.]

Consider pullback squares

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "k'"'] & X \end{tikzcd} , \quad \begin{tikzcd} Z' \arrow[r] \arrow[d, "g'"'] & Z \arrow[d, "g"] \\ X' \arrow[r, "k'"'] & X \end{tikzcd} .$$

6.14 LEMMA If $(Y,f) \sim_X (Z,g)$, then $(Y',f') \sim_X (Z',g')$.

Therefore not only is a pullback of a monomorphism a monomorphism but a pullback of a subobject is a subobject.

Denote by $\mathrm{Sub}_{\underline{C}}$ the association $\underline{C}^{OP} \to \underline{SET}$ that sends X to $\mathrm{Sub}_{\underline{C}}\, X$ and $k':X' \to X$ to $\mathrm{Sub}_{\underline{C}}\, k'$, where

$$\mathrm{Sub}_{\underline{C}}\, k' : \mathrm{Sub}_{\underline{C}}\, X \to \mathrm{Sub}_{\underline{C}}\, X'$$

is the arrow $[f] \to [f']$.

6.15 LEMMA $\mathrm{Sub}_{\underline{C}}$ is a functor.

PROOF It is clear that $\mathrm{Sub}_{\underline{C}}$ sends the identity of X to the identity of $\mathrm{Sub}_{\underline{C}}\, X$. As for compositions, if

$$\left[\begin{array}{l} k':X' \to X \\ \\ k'':X'' \to X', \end{array}\right.$$

then the claim is that

$$\mathrm{Sub}_{\underline{C}}\,(k' \circ k'') = \mathrm{Sub}_{\underline{C}}\, k'' \circ \mathrm{Sub}_{\underline{C}}\, k'.$$

To see this, pass from the pullback squares

$$\begin{tikzcd} Y'' \arrow[r] \arrow[d, "f''"'] & Y' \arrow[d, "f'"] \\ X'' \arrow[r, "k''"'] & X' \end{tikzcd} \quad , \quad \begin{tikzcd} Y' \arrow[r] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "k'"'] & X \end{tikzcd}$$

to the pullback square

$$\begin{tikzcd} Y'' \arrow[r] \arrow[d, "f''"'] & Y \arrow[d, "f'"] \\ X' \arrow[r, "k' \circ k''"'] & X \end{tikzcd} .$$

6.16 THEOREM The presheaf $\text{Sub}_{\underline{C}}$ is represented by Ω: $\forall\ X \in \text{Ob}\ \underline{C}$,

$$\text{Sub}_{\underline{C}}\ X \approx \text{Mor}(X,\Omega).$$

[Note: The natural isomorphism

$$\text{Sub}_{\underline{C}} \to \text{Mor}(—,\Omega)$$

sends a subobject [f] of X to its classifying arrow χ_f.]

6.17 LEMMA Every monomorphism $f:\Omega \to \Omega$ is an isomorphism.

PROOF It suffices to show that $f \circ f = \text{id}_\Omega$. Form the pullback squares

$$\begin{tikzcd} U \arrow[r, "!"] \arrow[d, "g"'] & *_{\underline{C}} \arrow[d, "\top"] \\ \Omega \arrow[r, "f"'] & \Omega \end{tikzcd} \quad , \quad \begin{tikzcd} V \arrow[r] \arrow[d] & *_{\underline{C}} \arrow[d, "\top"] \\ U \arrow[r, "g"'] & \Omega \end{tikzcd} .$$

Since f is a monomorphism, the arrow $U \xrightarrow{!} *_{\underline{C}}$ is a monomorphism and since g is a monomorphism, the arrow $V \longrightarrow *_{\underline{C}}$ is a monomorphism, thus the squares in the diagram

$$\begin{tikzcd} U \cap V = V \arrow[r, "\text{id}_V"] \arrow[d] & V \arrow[r] \arrow[d] & U \arrow[r, "!"] \arrow[d, "g"'] & *_{\underline{C}} \arrow[d, "\top"] \\ U \arrow[r, "!"'] & *_{\underline{C}} \arrow[r, "\top"'] & \Omega \arrow[r, "f"'] & \Omega \end{tikzcd}$$

are pullback squares, so by uniqueness, $f \circ \top \circ ! = g$, which implies that

$$f \circ f \circ g = f \circ \top \circ ! = g = g \circ id_U$$

or still, that the square

$$\begin{tikzcd} U \arrow[r, "id_U"] \arrow[d, "g"'] & U \arrow[d, "g"] \\ \Omega \arrow[r, "f \circ f"'] & \Omega \end{tikzcd}$$

commutes. Working through the definitions and bearing in mind that $f \circ f$ is a monomorphism, it follows that this square is in fact a pullback square. Therefore the outer rectangle

$$\begin{tikzcd} U \arrow[r, "id_U"] \arrow[d, "g"'] & U \arrow[r, "!"] \arrow[d, "g"'] & *_{\underline{C}} \arrow[d, "\top"] \\ \Omega \arrow[r, "f \circ f"'] & \Omega \arrow[r, "f"'] & \Omega \end{tikzcd}$$

is a pullback square, hence by uniqueness,

$$f \circ f \circ f = f = f \circ id_\Omega \Rightarrow f \circ f = id_\Omega.$$

§7. *SIEVES*

Let $\underline{C}$ be a small category.

7.1 DEFINITION Let $X \in \text{Ob } \underline{C}$ -- then a <u>sieve</u> over X is a subset $\mathcal{S}$ of $\text{Ob } \underline{C}/X$ such that the composition $Z \xrightarrow{g} Y \xrightarrow{f} X$ belongs to $\mathcal{S}$ if $Y \xrightarrow{f} X$ belongs to $\mathcal{S}$.

7.2 DEFINITION A <u>subfunctor</u> of a functor $F:\underline{C}^{OP} \to \underline{SET}$ is a functor $G:\underline{C}^{OP} \to \underline{SET}$ such that $\forall\ X \in \text{Ob } \underline{C}$, GX is a subset of FX and the corresponding inclusions constitute a natural transformation $G \to F$, so $\forall\ f:Y \to X$ there is a commutative diagram

$$\begin{tikzcd}
GY \arrow[r, "i_Y"] & FY \\
GX \arrow[u, "Gf"] \arrow[r, "i_X"'] & FX \arrow[u, "Ff"']
\end{tikzcd} .$$

7.3 LEMMA Fix an object X in $\underline{C}$ -- then there is a one-to-one correspondence between the sieves over X and the subfunctors of h_X.

PROOF If $\mathcal{S}$ is a sieve over X, then the designation

$$GY = \{f:Y \to X \ \&\ f \in \mathcal{S}\}$$

defines a subfunctor of h_X (given $Z \xrightarrow{g} Y$, $Gg:GY \to GZ$ is the map $f \to f \circ g$). Conversely, if G is a subfunctor of h_X, then $GY \subset \text{Mor}(Y,X)$ and

$$\mathcal{S} = \bigcup_Y GY$$

is a sieve over X.

7.4 EXAMPLE The maximal sieve over X is $\mathcal{S}_{max} = \text{Ob } \underline{C}/X$ and the associated subfunctor of h_X is h_X itself. The minimal sieve over X is $\mathcal{S}_{min} = \emptyset$ and the associated subfunctor of h_X is $\emptyset_{\hat{\underline{C}}}$ (the initial object of $\hat{\underline{C}}$).

Consider now the functor category

$$\hat{\underline{C}} = [\underline{C}^{OP}, \underline{SET}].$$

N.B. $\hat{\underline{C}}$ is wellpowered (cf. 2.14).

7.5 LEMMA The monomorphisms in $\hat{\underline{C}}$ are levelwise, i.e., an arrow $\Xi:G \to F$ in $\hat{\underline{C}}$ is a monomorphism iff $\forall\, X \in \text{Ob } \underline{C}$,

$$\Xi_X:GX \to FX$$

is a monomorphism in $\underline{SET}$.

Suppose that $\Xi:G \to F$ is a monomorphism in $\hat{\underline{C}}$ -- then $(G,\Xi) \in M(F)$, so $\forall\, X \in \text{Ob } \underline{C}$,

$$(GX,\Xi_X) \in M(FX)$$

and

$$(GX,\Xi_X) \sim_{FX} (G'X,\Xi'_X),$$

where $G'X$ is a subset of FX and Ξ'_X is the inclusion $G'X \to FX$.

7.6 LEMMA G' is a subfunctor of F.

It follows that there is a one-to-one correspondence between the subobjects of F and the subfunctors of F.

7.7 THEOREM Let $\underline{C}$ be a small category -- then $\hat{\underline{C}}$ admits a subobject classifier.

Definition of Ω There are two ways to proceed.

- Define

$$\Omega:\underline{C}^{OP} \to \underline{SET}$$

on an object X by letting ΩX be the set of all subfunctors of h_X and on a morphism $f:Y \to X$ by letting $\Omega f:\Omega X \to \Omega Y$ operate via the pullback square

$$\begin{tikzcd} \Omega f(G) \arrow[r] \arrow[d] & G \arrow[d] \\ h_Y \arrow[r, "h_f"'] & h_X. \end{tikzcd}$$

- Define

$$\Omega:\underline{C}^{OP} \to \underline{SET}$$

on an object X by letting ΩX be the set of all sieves over X and on a morphism $f:Y \to X$ by letting $\Omega f:\Omega X \to \Omega Y$ be the rule $\mathcal{S} \to \mathcal{S} \cdot f$, where $\mathcal{S} \cdot f = \{g:f \circ g \in \mathcal{S}\}$.

Definition of $\tau:*_{\hat{\underline{C}}} \to \Omega$ In terms of subfunctors, $\tau_X(*) = h_X$ and in terms of sieves, $\tau_X(*) = \mathcal{S}_{max}$.

The claim then is that the pair (Ω,τ) is a subobject classifier for $\hat{\underline{C}}$ and for this we shall work with sieves, the details in the subfunctor picture being analogous. So let $\Xi:G \to F$ be a monomorphism, where w.l.o.g., G is a subfunctor of F -- then the classifying arrow $\chi_\Xi:F \to \Omega$ of (G,Ξ) in F at a given $X \in \text{Ob}\,\underline{C}$ is the map

$$(\chi_\Xi)_X:FX \to \Omega X$$

that sends $x \in FX$ to the sieve

$$(\chi_{\Xi})_X(x) = \{Y \xrightarrow{f} X : (Ff)x \in GY\}.$$

Since

$$(\chi_{\Xi})_X(x) = \mathcal{S}_{max} \iff x \in GX,$$

the diagram

$$\begin{tikzcd} GX \arrow[r] \arrow[d, "\Xi_X"'] & * \arrow[d, "T_X"] \\ FX \arrow[r, "(\chi_{\Xi})_X"'] & \Omega X \end{tikzcd}$$

is a pullback square in <u>SET</u>, thus the diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d, "\Xi"'] & *_{\hat{\underline{C}}} \arrow[d, "T"] \\ F \arrow[r, "\chi_{\Xi}"'] & \Omega \end{tikzcd}$$

is a pullback square in $\hat{\underline{C}}$. This completes the verification, modulo uniqueness, i.e., if

$$\begin{tikzcd} G \arrow[r] \arrow[d, "\Xi"'] & *_{\hat{\underline{C}}} \arrow[d, "T"] \\ F \arrow[r, "\chi"'] & \Omega \end{tikzcd}$$

is a pullback square, then $\chi = \chi_{\Xi}$ ... .

7.8 EXAMPLE Let G be a group, considered as a category $\underline{G}$ -- then the category of right G-sets is the functor category $[\underline{G}^{OP},\underline{SET}]$, thus is cartesian closed (cf. 5.21) and admits a subobject classifier (cf. 7.7).

Let $\underline{C}$ be a small category -- then

- $\hat{\underline{C}}$ fulfills the standard conditions (cf. 3.4 and 3.6);
- $\hat{\underline{C}}$ admits a subobject classifier (cf. 7.7).

7.9 LEMMA Every epimorphism in $\hat{\underline{C}}$ is a coequalizer.

PROOF Suppose that $\Xi:F \to G$ is an epimorphism. Write $\Xi = m \circ k$ per 3.9, thus m is a monomorphism and k is a coequalizer. But then m is necessarily an epimorphism and $\hat{\underline{C}}$ is balanced (cf. 6.10). Therefore m is an isomorphism, hence Ξ is a coequalizer.

§8. *HEYTING ALGEBRAS*

A bounded lattice $(X,\leq)$ is called a <u>Heyting algebra</u> if $\underline{C}(X,\leq)$ is cartesian closed (as a category with finite products).

<u>N.B.</u> If $x,y,z \in X$, then

$$x \wedge y \leq z \Leftrightarrow x \leq z^y \quad \text{(cf. 1.4)}.$$

So, e.g.,

$$y \leq z \Leftrightarrow z^y = 1.$$

In particular: $\forall\, x \in X,\ x^x = 1$. And

$$z^y \wedge y \leq z.$$

In particular: $\forall\, x \in X,\ x \wedge 0^x = 0$.

8.1 EXAMPLE Every boolean algebra is a Heyting algebra (cf. 5.5).

8.2 LEMMA Let $(X,\leq)$ be a poset which is linearly ordered ($\forall\, x,y \in X$, either $x \leq y$ or $y \leq x$) and with least and greatest elements 0 and 1 -- then $(X,\leq)$ is a bounded lattice and, as such, is a Heyting algebra.

PROOF $\underline{C}(X,\leq)$ has binary products:

$$x \wedge y = \begin{cases} x \text{ if } x \leq y \\ y \text{ if } y \leq x \end{cases}$$

and binary coproducts:

$$x \vee y = \begin{cases} y \text{ if } x \leq y \\ x \text{ if } y \leq x. \end{cases}$$

This said, the prescription

$$y^x = \begin{cases} 1 \text{ if } x \leq y \\ y \text{ if } y \leq x \ \& \ y \neq x \end{cases}$$

defines an exponential object, so $\underline{C}(X,\leq)$ is cartesian closed.

8.3 EXAMPLE The closed unit interval $[0,1] \subset \underline{R}$ in its usual ordering is a Heyting algebra (but not a boolean algebra).

8.4 LEMMA A Heyting algebra is necessarily a distributive lattice.

The difference between a boolean algebra and a Heyting algebra lies in the notion of complement.

8.5 DEFINITION Let $(X,\leq)$ be a Heyting algebra. Given $x \in X$, put $\neg x = 0^x$ -- then $\neg x$ is called the <u>pseudocomplement</u> of x.

<u>N.B.</u> In a boolean algebra $(X,\leq)$,

$$0^x = \neg x \vee 0 = \neg x \quad \text{(cf. 5.5)}.$$

8.6 LEMMA Let $(X,\leq)$ be a Heyting algebra -- then $\forall\ x \in X$,

$$\neg x = \vee\{y: x \wedge y = 0\}.$$

8.7 EXAMPLE Let S be an infinite set and let X be the subset of the power set PS consisting of all finite subsets of S together with S itself -- then $(X,\subseteq)$ is a distributive lattice but it is not a Heyting algebra.

[If $x \in X$ and $x \neq \emptyset$, then the set of $y \in S$ such that $x \cap y = \emptyset$ has no largest member.]

To recapitulate:

boolean algebra => Heyting algebra => distributive lattice and none of the implications are reversible.

8.8 RULES In a Heyting algebra $(X,\leq)$,

(1) $\neg 0 = 1$, $\neg 1 = 0$;

(2) $x \leq y \Rightarrow \neg y \leq \neg x$;

(3) $\neg x = \neg\neg\neg x$;

(4) $\neg(x \vee y) = \neg x \wedge \neg y$;

(5) $\neg x \vee y \leq y^x$;

(6) $x \leq y \Rightarrow \neg\neg x \leq \neg\neg y$;

(7) $x \leq \neg\neg x$;

(8) $\neg\neg\neg\neg x = \neg\neg x$;

(9) $\neg\neg(x \wedge y) = \neg\neg x \wedge \neg\neg y$;

(10) $\neg\neg(y^x) = (\neg\neg y)^{\neg\neg x}$.

[Note: This list is by no means exhaustive but suffices for our purposes (there is another list to the effect that any Heyting algebra satisfies the axioms of the intuitionistic propositional calculus).]

8.9 LEMMA Let $(X,\leq)$ be a Heyting algebra — then $(X,\leq)$ is a boolean algebra iff $\forall\, x \in X$, $x \vee \neg x = 1$.

[Note: In any Heyting algebra, $x \wedge \neg x = 0$.]

8.10 LEMMA Let $(X,\leq)$ be a Heyting algebra — then $(X,\leq)$ is a boolean algebra iff $\forall\, x \in X$, $\neg\neg x = x$.

8.11 EXAMPLE Given a topological space X, let O(X) be the set of open subsets of X, thus under the operations

$$U \leq V \Leftrightarrow U \subset V, \quad \begin{bmatrix} U \wedge V = U \cap V \\ \\ U \vee V = U \cup V \end{bmatrix}, \ 0 = \emptyset,\ 1 = X,$$

O(X) is a bounded lattice. Denote by $\underline{O}(X)$ the category underlying O(X) — then

$\underline{O}(X)$ is cartesian closed:

$$V^U = \cup\, W \ (W \cap U \subset V).$$

Therefore $O(X)$ is a Heyting algebra. Here

$$\neg\, U = \emptyset^U = \mathrm{int}(X - U) = X - c\ell\, U$$

$$\Rightarrow$$

$$\neg\, \neg\, U = \mathrm{int}\ c\ell\, U \supset U.$$

[Note: In general, $O(X)$ is not a boolean algebra (cf. 8.9 and 8.10).]

8.12 DEFINITION Let $(X,\leq)$ be a Heyting algebra -- then an $x \in X$ is <u>boolean</u> if $\neg\, \neg\, x = x$.

[Note: It is always the case that $x \leq \neg\, \neg\, x$.]

8.13 EXAMPLE In 8.11, an open set U is boolean iff it coincides with the interior of its closure.

8.14 NOTATION $(X_b,\leq)$ is the subposet of $(X,\leq)$ whose elements are the boolean elements of X.

8.15 THEOREM $(X_b,\leq)$ is a boolean algebra.

PROOF First,

$$\begin{cases} \neg\, \neg\, 0 = 0 \\ \neg\, \neg\, 1 = 1, \end{cases}$$

so 0 and 1 are boolean. Next, if $x,y \in X$ are boolean, then

$$\neg\, \neg\, (x \wedge y) = \neg\, \neg\, x \wedge \neg\, \neg\, y = x \wedge y,$$

thus $x \wedge y$ is boolean. On the other hand, $x \vee y$ is not necessarily boolean. To remedy this, put

$$x \veebar y = \neg\neg(x \vee y).$$

Then

$$\neg\neg(x \veebar y) = \neg\neg\neg\neg(x \vee y)$$

$$= \neg\neg(x \vee y) = x \veebar y.$$

So, with these definitions, $(X_b,\leq)$ is a bounded lattice (which, in general, is not a sublattice of $(X,\leq)$). There remains the claim that $(X_b,\leq)$ is distributive and complemented.

- $\forall\, x,y,z \in X_b$:

$$\begin{aligned} x \wedge (y \veebar z) &= x \wedge \neg\neg(y \vee z) \\ &= \neg\neg x \wedge \neg\neg(y \vee z) \\ &= \neg\neg(x \wedge (y \vee z)) \\ &= \neg\neg((x \wedge y) \vee (x \wedge z)) \\ &= (x \wedge y) \veebar (x \wedge z). \end{aligned}$$

Analogously,

$$x \veebar (y \wedge z) = (x \veebar y) \wedge (x \veebar z).$$

- $\forall\, x \in X_b$:

$$x \wedge \neg x = 0$$

and

$$\begin{aligned} x \veebar \neg x &= \neg\neg(x \vee \neg x) \\ &= \neg(\neg(x \vee \neg x)) \end{aligned}$$

$$= \neg(\neg x \wedge \neg\neg x)$$

$$= \neg(\neg x \wedge x)$$

$$= \neg 0$$

$$= 1.$$

8.16 THEOREM Let $\underline{C}$ be a small category — then $\forall\ F \in \mathrm{Ob}\ \hat{\underline{C}}$, the poset $\mathrm{Sub}_{\hat{\underline{C}}}\ F$ is a Heyting algebra.

PROOF Suppose that G_1, G_2 are subfunctors of F — then under the operations

$$\begin{cases} (G_1 \wedge G_2)X = G_1X \cap G_2X \\ (G_1 \vee G_2)X = G_1X \cup G_2X \end{cases}, \ 0X = \emptyset,\ 1X = FX,$$

$\mathrm{Sub}_{\hat{\underline{C}}}\ F$ is a bounded lattice. As for the exponential object $G_2^{G_1}$, take $(G_1^{G_2})X$ to be the set of $x \in FX$ which have the property that if $f:Y \to X$ and if $(Ff)x \in G_1Y$, then $(Ff)x \in G_2Y$.

[Note: So, if G is a subfunctor of F, then $(\neg G)X$ is the set of $x \in FX$ such that for all $f:Y \to X$, $(Ff)x \notin GY$.]

8.17 EXAMPLE Consider the functor category $[\underline{G}^{\mathrm{OP}}, \underline{\mathrm{SET}}]$ per 7.8 — then for every right G-set X, the Heyting algebra $\mathrm{Sub}_{\hat{\underline{G}}}\ X$ is actually a boolean algebra.

§9. *LOCALES*

A <u>locale</u> is a Heyting algebra $(X,\leq)$ for which the category $\underline{C}(X,\leq)$ is complete and cocomplete (cf. 1.10).

[Note: If $\underline{C}(X,\leq)$ is complete and cocomplete, then $\underline{C}(X_b,\leq)$ is complete and cocomplete, hence the boolean algebra $(X_b,\leq)$ (cf. 8.15) is also a locale.]

9.1 EXAMPLE The closed unit interval $[0,1] \subset \underline{R}$ in its usual ordering is a locale (cf. 8.3).

9.2 EXAMPLE If X is a topological space, then $O(X)$ is a locale (cf. 8.11).

[Here $\bigvee_{i \in I} U_i = \bigcup_{i \in I} U_i$ while $\bigwedge_{i \in I} U_i$ is the largest open set contained in all the U_i.]

9.3 EXAMPLE If $\underline{C}$ is a small category and if $F \in \text{Ob } \hat{\underline{C}}$, then $\text{Sub}_{\hat{\underline{C}}} F$ is a locale (cf. 8.16).

9.4 LEMMA Suppose that $(X,\leq)$ is a locale — then for any index set I,

$$x \wedge (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \wedge y_i).$$

[Recall that left adjoints preserve colimits.]

[Note: If $(X,\leq)$ is a bounded lattice for which the category $\underline{C}(X,\leq)$ is complete and cocomplete (cf. 1.10) and with the property that "arbitrary joins distribute over finite meets", i.e., the conclusion of 9.4, then $(X,\leq)$ is a Heyting algebra or still, is a locale. Proof: Put

$$z^y = \vee\{x : x \wedge y \leq z\}.]$$

Generically, locales are denoted by $L, M, \ldots$ and are to be regarded as categories.

9.5 LEMMA Let L be a locale. Given $x \in L$, put

$$\begin{cases} \uparrow x = \{y \in L : x \le y\} \\ \downarrow x = \{y \in L : y \le x\}. \end{cases}$$

Then the subposets $\begin{cases} \uparrow x \\ \downarrow x \end{cases}$ are locales.

9.6 DEFINITION Let L, M be locales -- then a <u>localic arrow</u> $f: L \to M$ is a pair of functors

$$\begin{cases} f_*: L \to M \\ f^*: M \to L \end{cases}$$

such that f^* is a left adjoint for f_* and f^* preserves finite products.

9.7 REMARK There is a one-to-one correspondence between the localic arrows $f: L \to M$ and the functors $f^*: M \to L$ such that

(1) $f^*(\bigvee_{i \in I} y_i) = \bigvee_{i \in I} f^*(y_i)$,

(2) $f^*(y \wedge y') = f^*(y) \wedge f^*(y')$,

(3) $f^*(1) = 1$,

for all indexing sets I and elements y_i, y, y' of M.

[If f^* satisfies these conditions, then by quoting the appropriate "adjoint functor theorem" one infers the existence of f_* (f_* is uniquely determined by f

(in a poset, the only isomorphisms are the identities (cf. 1.2))). Specifically:

$$f_*(x) = \vee\{y \in M : f^*(y) \leq x\} \quad \text{(cf. 1.4).]}$$

9.8 EXAMPLE Let X,Y be topological spaces and let $f:X \to Y$ be a continuous function -- then f induces a localic arrow $f:O(X) \to O(Y)$.

[Take $f^* = f^{-1}$, hence

$$f_*(U) = \cup\,\{V \in O(Y) : f^{-1}(V) \subset U\}$$

or still,

$$f_*(U) = Y - \overline{f(X-U)}.]$$

9.9 NOTATION <u>LOC</u> is the category whose objects are the locales and whose morphisms are the localic arrows.

9.10 THEOREM <u>LOC</u> is complete and cocomplete.

<u>N.B.</u> An initial object for <u>LOC</u> is $\{*\}$ and a final object for <u>LOC</u> is $\{0,1\}$.

[E.g.: Given L, a localic arrow $f:L \to \{0,1\}$ must have the property that $f^*(0) = 0$, $f^*(1) = 1$ implying thereby the uniqueness of f as well as its existence (cf. 9.7).]

9.11 DEFINITION A <u>point</u> of a locale L is a localic arrow $p:\{0,1\} \to L$.

9.12 DEFINITION An element x of a locale L is <u>prime</u> if $\forall\ a,b \in L$,

$$a \wedge b \leq x \Rightarrow a \leq x \text{ or } b \leq x.$$

9.13 LEMMA Let L be a locale -- then there is a bijection between the points of L and the prime elements of L.

PROOF Given a point p of L, put

$$x = \vee\{a \in L : p^*(a) = 0\}.$$

Then $p^*(x) = 0$, hence $x \neq 1$ $(p^*(1) = 1)$. And x is prime:

$$\begin{aligned} a \wedge b \leq x &\Rightarrow p^*(a \wedge b) = 0 \\ &\Rightarrow p^*(a) \wedge p^*(b) = 0 \\ &\Rightarrow p^*(a) = 0 \text{ or } p^*(b) = 0 \\ &\Rightarrow a \leq x \text{ or } b \leq x. \end{aligned}$$

Conversely, if $x \in L$ is prime, define $p^*:L \to \{0,1\}$ by

$$p^*(a) = \begin{cases} 0 \text{ if } a \leq x \\ 1 \text{ if } a \nleq x. \end{cases}$$

Then p^* satisfies (1), (2), (3) of 9.7, so p^* is the left adjoint constituent of a localic arrow $p:\{0,1\} \to L$.

● Start with a point p, form the prime element x as above, and consider the point q associated with x. Given $a \in L$,

$$q^*(a) = 0 \Leftrightarrow a \leq x \Leftrightarrow p^*(a) = 0.$$

Therefore $q^* = p^*$ or still, $q = p$.

● Start with a prime element x, pass to the point p corresponding to x, thence to the prime element y corresponding to p. Given $a \in L$,

$$a \leq x \Leftrightarrow p^*(a) = 0 \Leftrightarrow a \leq y.$$

Therefore $x = y$.

9.14 EXAMPLE Let X be a topological space — then each $x \in X$ determines a point $p_x:\{0,1\} \to O(X)$, thus

$$p_x^*(U) = 0 \iff x \notin U,$$

the prime element per p_x being $X - \overline{\{x\}}$.

9.15 NOTATION Given a locale L, let

$$\mathrm{pt}(L) = \mathrm{Mor}(\{0,1\},L),$$

the set of points of L.

[Note: It can happen that $\mathrm{pt}(L) = \emptyset$. E.g., take the real line $\underline{R}$ in its usual topology and let

$$L = (O(\underline{R})_b, \subseteq).$$

Then L has no prime element, thus $\mathrm{pt}(L) = \emptyset$ (cf. 9.13).]

9.16 LEMMA Let L be a locale. Given $x \in L$, put

$$U_x = \{p \in \mathrm{pt}(L) : p^*(x) = 1\}.$$

Then the collection $\{U_x : x \in L\}$ is a topology on $\mathrm{pt}(L)$.

[Note: We have

$$\begin{cases} U_0 = \emptyset \\ \qquad\qquad \bigcup\limits_{i \in I} U_{x_i} = U_{\bigvee\limits_{i \in I} x_i}, \ U_x \cap U_y = U_{x \wedge y}.] \\ U_1 = \mathrm{pt}(L), \end{cases}$$

<u>N.B.</u> If $f:L \to M$ is a localic arrow, then postcomposition

$$\mathrm{pt}(f):\mathrm{pt}(L) \to \mathrm{pt}(M) \quad (p \to f \circ p)$$

is continuous.

[In fact,

$$\mathrm{pt}(f)^{-1}(U_x) = U_{f^*(x)}.]$$

Therefore these definitions give rise to a functor

$$\mathrm{pt}:\underline{\mathrm{LOC}} \to \underline{\mathrm{TOP}}.$$

In the other direction, let

$$\ell oc:\underline{\mathrm{TOP}} \to \underline{\mathrm{LOC}}$$

be the functor that sends X to O(X) and $f:X \to Y$ to its associated localic arrow $f:O(X) \to O(Y)$ (cf. 9.8).

9.17 THEOREM The functor pt is a right adjoint for the functor ℓoc.

[Note: The arrows of adjunction

$$\begin{cases} \mu \in \mathrm{Nat}(\mathrm{id}_{\underline{\mathrm{TOP}}}, \mathrm{pt} \circ \ell oc) \\ \nu \in \mathrm{Nat}(\ell oc \circ \mathrm{pt}, \mathrm{id}_{\underline{\mathrm{LOC}}}) \end{cases}$$

are

- Given a topological space X,

$$\mu_X:X \longrightarrow \mathrm{pt}(O(X))$$

sends $x \in X$ to p_x (cf. 9.14);

- Given a locale L, the left adjoint part of

$$\nu_L:O(\mathrm{pt}(L)) \longrightarrow L$$

is the functor

$$\nu_L^*:L \longrightarrow O(\mathrm{pt}(L))$$

that sends $x \in L$ to U_x.]

9.18 RAPPEL Let X be a topological space -- then a nonempty closed subset $S \subset X$ is <u>irreducible</u> if for all closed subsets S_1, S_2 of X,

$$S \subset S_1 \cup S_2 \Rightarrow S \subset S_1 \text{ or } S \subset S_2,$$

i.e., if $X - S \in O(X)$ is prime. E.g.: $\forall\, x \in X$, $\overline{\{x\}}$ is an irreducible closed subset of X.

[Note: The only irreducible closed subsets of a Hausdorff space are singletons.]

9.19 DEFINITION A topological space X is <u>sober</u> provided that every irreducible closed subset S of X is the closure of a unique point $x \in X: S = \overline{\{x\}}$.

[Note: Consider the map $x \to \overline{\{x\}}$ from the points of X to the irreducible closed subsets of X -- then X is T_0 iff this map is injective and X is sober iff this map is bijective.]

9.20 EXAMPLE The spectrum of a commutative ring with unit in its Zariski topology is sober.

9.21 CRITERION A topological space X is sober iff the arrow of adjunction

$$\mu_X : X \to \text{pt}(O(X))$$

is bijective.

9.22 LEMMA Let L be a locale -- then $\text{pt}(L)$ is a sober topological space.

PROOF It is a question of applying 9.21 when $X = \text{pt}(L)$. So let

$$Q:\{0,1\} \to O(\text{pt}(L))$$

be an element of $\text{pt}(O(\text{pt}(L)))$ -- then there is a unique point $q \in \text{pt}(L)$ such that

$p_q = Q$ (here

$$p_q^*(U_x) = 0 \Leftrightarrow q \notin U_x \quad \text{(cf. 9.14)).}$$

To see this, let

$$y = \vee\{x \in L : Q^*(U_x) = 0\}.$$

Then $Q^*(U_y) = 0$, hence $y \neq 1$ $(Q^*(U_1) = Q^*(\mathrm{pt}(L)) = 1)$ and it is immediate that y is prime. Let now $q \in \mathrm{pt}(L)$ be the point corresponding to y, thus

$$q^*(x) = \begin{cases} 0 \text{ if } x \leq y \\ \\ 1 \text{ if } x \not\leq y \end{cases} \quad \text{(cf. 9.13).}$$

Claim: $p_q = Q$. Proof: $\forall\, x \in L$,

$$\begin{aligned} p_q^*(U_x) = 0 &\Leftrightarrow q \notin U_x \\ &\Leftrightarrow q^*(x) = 0 \\ &\Leftrightarrow x \leq y \\ &\Leftrightarrow Q^*(U_x) = 0. \end{aligned}$$

That q is unique can be established by a similar calculation.

9.23 DEFINITION A locale L is <u>spatial</u> if $U_x = U_y \Rightarrow x = y$.

<u>N.B.</u> In other words, L is spatial if

$$\nu_L^*: L \to O(\mathrm{pt}(L))$$

is injective (it is surjective by definition).

9.24 EXAMPLE Let X be a topological space -- then the locale $O(X)$ is spatial.

[Given $U \in O(X)$,

$$\nu^*_{O(X)}(U) = \{p \in pt(O(X)) : p^*(U) = 1\}.$$

And

$$p_x \in \nu^*_{O(X)}(U) \iff x \in U.$$

Therefore

$$\nu^*_{O(X)} : O(X) \to O(pt(O(X)))$$

is injective.]

The reason for introducing "sober topological spaces" and "spatial locales" is the following easy consequence of 9.17.

9.25 THEOREM The category of sober topological spaces is equivalent to the category of spatial locales.

Details:

- A topological space X is sober iff the arrow of adjunction

$$\mu_X : X \to pt(O(X))$$

is a homeomorphism.

[If X is a topological space, then μ_X is continuous (being a morphism in <u>TOP</u>) and if in addition X is sober, then μ_X is bijective (cf. 9.21), hence open:

$$\mu_X(U) = U_U \ldots .]$$

- A locale L is spatial iff the arrow of adjunction

$$\nu_L : O(pt(L)) \to L$$

is an isomorphism of locales.

[If L is a spatial locale, then ν_L^* is bijective. Moreover, ν_L^* preserves the poset structure (clear) and reflects it:

$$U_x \subset U_y \Rightarrow U_{x \wedge y} = U_x \cap U_y = U_x,$$

so by injectivity, $x \wedge y = x$ or still, $x \leq y$.]

Turning to 9.25, the image of the functor pt is contained in the full subcategory of <u>TOP</u> whose objects are the sober topological spaces (cf. 9.22) and the image of the functor ℓoc is contained in the full subcategory of <u>LOC</u> whose objects are the spatial locales (cf. 9.24). Therefore the adjunction (ℓoc, pt) restricts to an adjunction on these smaller subcategories and by the above observations, the restricted arrows of adjunction are natural isomorphisms.

9.26 SCHOLIUM Let X be a topological space -- then the locale O(X) is isomorphic to the locale of open subsets of a sober topological space.

[For O(X) is spatial (cf. 9.24), hence

$$\nu_{O(X)}:O(pt(O(X))) \rightarrow O(X)$$

is an isomorphism of locales. But pt(O(X)) is sober (cf. 9.22).]

§10. *SITES*

Let $\underline{C}$ be a small category.

10.1 NOTATION Given a sieve $\mathcal{S}$ over X and a morphism $f:Y \to X$, put

$$f^*\mathcal{S} = \{g:\text{cod } g = Y \ \&\ f \circ g \in \mathcal{S}\}.$$

Then $f^*\mathcal{S}$ is a sieve over Y.

10.2 DEFINITION A <u>Grothendieck topology</u> on $\underline{C}$ is a function τ that assigns to each $X \in \text{Ob } \underline{C}$ a set τ_X of sieves over X subject to the following assumptions.

(1) The maximal sieve $\mathcal{S}_{max} \in \tau_X$.

(2) If $\mathcal{S} \in \tau_X$ and if $f:Y \to X$ is a morphism, then $f^*\mathcal{S} \in \tau_Y$.

(3) If $\mathcal{S} \in \tau_X$ and if $\mathcal{S}'$ is a sieve over X such that $f^*\mathcal{S}' \in \tau_Y$ for all $f:Y \to X$ in $\mathcal{S}$, then $\mathcal{S}' \in \tau_X$.

10.3 DEFINITION A <u>site</u> is a pair $(\underline{C},\tau)$, where $\underline{C}$ is a small category and τ is a Grothendieck topology on $\underline{C}$.

10.4 EXAMPLE Let L be a locale. Given $x \in L$, a sieve over x is a subset $\mathcal{S}$ of $\downarrow x$ (cf. 9.5) which is hereditary in the sense that

$$\forall\, s \in \mathcal{S},\ \forall\, a \in L,\ a \le s \Rightarrow a \in \mathcal{S}.$$

One then says that $\mathcal{S}$ <u>covers</u> x if $x = \vee\, \mathcal{S}$. Denoting by τ_x the set of all such $\mathcal{S}$, the assignment $x \to \tau_x$ is a Grothendieck topology τ on L.

[It is straightforward to check (1), (2), and (3).

<u>Ad (1)</u> Here $\mathcal{S}_{max} = \downarrow x$ and it is obvious that

$$\bigvee_{y \le x} y = \downarrow x.$$

Ad (2) If $\mathcal{S} \in \tau_x$ and if $y \leq x$ $(f:y \rightarrow x)$, then

$$f^*\mathcal{S} = \{s \leq y : s \in \mathcal{S}\} = \{s \wedge y : s \in \mathcal{S}\}$$

and the claim is that $f^*\mathcal{S} \in \tau_y$. In fact,

$$y = x \wedge y = (\vee\mathcal{S}) \wedge y = \vee\{s \wedge y : s \in \mathcal{S}\} = \vee f^*\mathcal{S}.$$

Ad (3) Given $\mathcal{S}'$, suppose that

$$y = \vee\{s' \wedge y : s' \in \mathcal{S}'\} \quad (y \in \mathcal{S}).$$

Then

$$x = \vee\mathcal{S} = \bigvee_{s\in\mathcal{S}} s = \bigvee_{s\in\mathcal{S}} \bigvee_{s'\in\mathcal{S}'} s' \wedge s = \bigvee_{s'\in\mathcal{S}'} \bigvee_{s\in\mathcal{S}} s' \wedge s$$

$$= \bigvee_{s'\in\mathcal{S}'} (s' \wedge (\bigvee_{s\in\mathcal{S}} s)) = \bigvee_{s'\in\mathcal{S}'} s' \wedge x = \bigvee_{s'\in\mathcal{S}'} s'.$$

Therefore $\mathcal{S}' \in \tau_x$.]

N.B. Take $L = O(X)$, where X is a topological space -- then a sieve $\mathcal{S}$ over an open subset U of X is a set of open subsets $V \subset U$ such that $V' \subset V \in \mathcal{S} \Rightarrow V' \in \mathcal{S}$. And

$$\mathcal{S} \in \tau_U \iff \bigcup_{V\in\mathcal{S}} V = U.$$

10.5 LEMMA Let $(\underline{C},\tau)$ be a site -- then $\forall\, X \in \text{Ob}\,\underline{C}$,

$$\mathcal{S} \in \tau_X \;\&\; \mathcal{S} \subset \mathcal{S}' \Rightarrow \mathcal{S}' \in \tau_X$$

and

$$\mathcal{S},\mathcal{S}' \in \tau_X \Rightarrow \mathcal{S} \cap \mathcal{S}' \in \tau_X.$$

10.6 REMARK Suppose that we have an assignment $X \rightarrow \tau_X$ satisfying (1), (2) of

10.2 and for which

$$\mathcal{S} \in \tau_X \ \& \ \mathcal{S} \subset \mathcal{S}' \Rightarrow \mathcal{S}' \in \tau_X.$$

Then to check (3) of 10.2, it suffices to consider those $\mathcal{S}'$ such that $\mathcal{S}' \subset \mathcal{S}$.

Let $\underline{C}$ be a small category -- then by $\tau_{\underline{C}}$ we shall understand the set of Grothendieck topologies on $\underline{C}$.

10.7 EXAMPLE Take $\underline{C} = \underline{1}$ -- then $\underline{C}$ has two Grothendieck topologies: $\{\mathcal{S}_{max}\}$ and $\{\mathcal{S}_{min}, \mathcal{S}_{max}\}$.

10.8 DEFINITION

- The <u>minimal Grothendieck topology</u> on $\underline{C}$ is the assignment $X \to \{\mathcal{S}_{max}\}$.

- The <u>maximal Grothendieck topology</u> on $\underline{C}$ is the assignment $X \to \{\mathcal{S}\}$, where $\mathcal{S}$ runs through all the sieves over X.

Given $\tau, \tau' \in \tau_{\underline{C}}$, write $\tau \leq \tau'$ if $\forall\ X \in \text{Ob}\ \underline{C}$, $\tau_X \subset \tau'_X$.

10.9 LEMMA The poset $\tau_{\underline{C}}$ is a bounded lattice.

PROOF If $\tau, \tau' \in \tau_{\underline{C}}$, let $\tau \wedge \tau'$ be their set theoretical intersection and let $\tau \vee \tau'$ be the smallest Grothendieck topology containing their set theoretical union. As for 0 and 1, take 0 to be the minimal Grothendieck topology and 1 to be the maximal Grothendieck topology.

10.10 THEOREM The bounded lattice $\tau_{\underline{C}}$ is a locale.

Let $\underline{C}$ be a small category with pullbacks.

10.11 DEFINITION A <u>coverage</u> on $\underset{\sim}{C}$ is a function K that assigns to each $X \in \text{Ob}\, \underset{\sim}{C}$ a set K_X of subsets of Ob $\underset{\sim}{C}/X$ subject to the following assumptions.

(1) If $f:Y \to X$ is an isomorphism, then $\{f:Y \to X\}$ is in K_X.

(2) If $\{f_i:Y_i \to X\ (i \in I)\}$ is in K_X, then for any morphism $g:Z \to X$,

$$\{Y_i \times_X Z \xrightarrow{pr_Z} Z\ (i \in I)\}$$

is in K_Z.

[Note: Here

$$\begin{tikzcd} Y_i \times_X Z \arrow[r, "pr_Z"] \arrow[d] & Z \arrow[d, "g"] \\ Y_i \arrow[r, "f_i"'] & X \end{tikzcd}$$

is a pullback square.]

(3) If $\{f_i:Y_i \to X\ (i \in I)\}$ is in K_X and if $\forall\, i \in I$, $\{g_{ij}:Z_{ij} \to Y_i\ (j \in I_i)\}$ is in K_{Y_i}, then

$$\{f_i \circ g_{ij}:Z_{ij} \to X\ (i \in I,\ j \in I_i)\}$$

is in K_X.

10.12 EXAMPLE Let L be a locale. Given $x \in L$, let K_x be the set of all subsets of $\downarrow x$ consisting of those set indexed collections $\{x_i:i \in I\}$ such that $\bigvee_{i\in I} x_i = x$ -- then the assignment $x \to K_x$ is a coverage K on L.

10.13 DEFINITION Let K be a coverage on $\underline{C}$ -- then the Grothendieck topology τ on $\underline{C}$ <u>generated</u> by K is the prescription

$$\mathcal{S} \in \tau_X \Leftrightarrow \exists\ \mathcal{R} \in K_X : \mathcal{R} \subset \mathcal{S}.$$

10.14 EXAMPLE Let L be a locale -- then the Grothendieck topology on L per 10.4 is generated by the coverage on L per 10.12.

10.15 REMARK Suppose still that $\underline{C}$ is a small category with pullbacks. Let τ be a Grothendieck topology on $\underline{C}$ -- then there is a coverage K that generates τ, viz.

$$\mathcal{R} \in K_X \Leftrightarrow \langle \mathcal{R} \rangle \in \tau_X,$$

where

$$\langle \mathcal{R} \rangle = \{f \circ g : f \in \mathcal{R},\ \mathrm{dom}\ f = \mathrm{cod}\ g\}.$$

§11. *SHEAVES*

Let $\underline{C}$ be a small category.

11.1 RAPPEL For any $X \in \text{Ob } \underline{C}$, the sieves over X are in a one-to-one correspondence with the subfunctors of h_X (cf. 7.3).

Because of this, the notion of Grothendieck topology can be reformulated.

11.2 NOTATION Given a subfunctor G of h_X and a morphism $f:Y \to X$, define f^*G by the pullback square

$$\begin{tikzcd} f^*G \arrow[r] \arrow[d, "i_{f^*G}"'] & G \arrow[d, "i_G"] \\ h_Y \arrow[r, "h_f"'] & h_X \end{tikzcd}$$

in $\hat{\underline{C}}$ -- then f^*G is a subfunctor of h_Y.

11.3 DEFINITION A <u>Grothendieck topology</u> on $\underline{C}$ is a function τ that assigns to each $X \in \text{Ob } \underline{C}$ a set τ_X of subfunctors of h_X subject to the following assumptions.

(1) The subfunctor $h_X \in \tau_X$.

(2) If $G \in \tau_X$ and if $f:Y \to X$ is a morphism, then $f^*G \in \tau_Y$.

(3) If $G \in \tau_X$ and if G' is a subfunctor of h_X such that $f^*G' \in \tau_Y$ for all $f \in GY$, then $G' \in \tau_X$.

[Note: For use below, observe that 10.5 and 10.6 can be stated in terms of subfunctors instead of sieves.]

Suppose that $\underline{S}$ is a reflective subcategory of $\hat{\underline{C}}$. Denote the reflector by $\underline{a}$ -- then there is an adjoint pair $(\underline{a},\iota)$, $\iota:\underline{S} \to \hat{\underline{C}}$ the inclusion.

Assume: $\underline{a}$ preserves finite limits.

[Note: It is automatic that $\underline{a}$ preserves colimits.]

11.4 THEOREM Given $X \in \text{Ob } \underline{C}$, let τ_X be the set of those subfunctors $G \xrightarrow{i_G} h_X$ such that $\underline{a}i_G$ is an isomorphism -- then the assignment $X \to \tau_X$ is a Grothendieck topology τ on $\underline{C}$.

PROOF Since

$$\underline{a}(\text{id}_{h_X}) = \text{id}_{\underline{a}h_X},$$

it follows that $h_X \in \tau_X$, hence (1) is satisfied. As for (2), by assumption $\underline{a}$ preserves finite limits, so in particular $\underline{a}$ preserves pullbacks, thus

$$\begin{tikzcd}
\underline{a}f^*G \arrow[r] \arrow[d, "\underline{a}i_{f^*G}"'] & \underline{a}G \arrow[d, "\underline{a}i_G"] \\
\underline{a}h_Y \arrow[r, "\underline{a}h_f"'] & \underline{a}h_X
\end{tikzcd}$$

is a pullback square in $\underline{S}$. But $\underline{a}i_G$ is an isomorphism. Therefore $\underline{a}i_{f^*G}$ is an isomorphism, i.e., $f^*G \in \tau_Y$. The verification of (3), however, is more complicated.

- Suppose that $G \in \tau_X$ and G is a subfunctor of G':

$$\left[\begin{array}{l} i_G : G \to h_X \\ \\ i_{G'} : G' \to h_X \end{array}\right., \quad i : G \to G'.$$

Then

$$i_G = i_{G'} \circ i \Rightarrow \underline{a}i_G = \underline{a}i_{G'} \circ \underline{a}i.$$

But $\underline{a}i_G$ is an isomorphism, hence

$$\mathrm{id} = \underline{a}i_{G'} \circ \underline{a}i \circ (\underline{a}i_G)^{-1},$$

which implies that $\underline{a}i_{G'}$ is a split epimorphism. On the other hand, $\underline{a}$ preserves monomorphisms, hence $\underline{a}i_{G'}$ is a monomorphism. Therefore $\underline{a}i_{G'}$ is an isomorphism, i.e., $G' \in \tau_X$.

● It remains to establish (3) under the restriction that G' is a subfunctor of G. Using the Yoneda lemma, identify each $f \in GY$ with $f \in \mathrm{Nat}(h_Y, G)$ and display the data in the diagram

$$\begin{tikzcd} h_Y \times_G G' \arrow[r] \arrow[d, "i_f"'] & G' \arrow[r, equal] \arrow[d, "i"] & G' \arrow[d, "i_{G'}"] \\ h_Y \arrow[r, "f"'] & G \arrow[r, "i_G"'] & h_X \end{tikzcd} .$$

There is one such diagram for each Y and each $f \in GY$, so upon consolidation we have

$$\begin{tikzcd} \coprod\limits_Y \coprod\limits_f h_Y \times_G G' \arrow[r, "\Pi_{G'}"] \arrow[d, "\coprod \coprod i_f"'] & G' \arrow[d, "i"] \\ \coprod\limits_Y \coprod\limits_f h_Y \arrow[r, "\Gamma_G"'] & G \end{tikzcd} .$$

Now i is an equalizer (all monomorphisms in $\hat{\underline{C}}$ are equalizers), thus $\underline{a}i$ is an equalizer (by the assumption on $\underline{a}$). But the assumption on G' is that $\forall$ Y and $\forall$ $f \in GY$, $\underline{a}i_f$ is an isomorphism, thus $\underline{a}i$ is an epimorphism (see 11.8 below). And this means that $\underline{a}i$ is an isomorphism (cf. 6.8). Finally,

$$i_{G'} = i_G \circ i \Rightarrow \underline{a}i_{G'} = \underline{a}i_G \circ \underline{a}i.$$

Therefore $\underline{a}i_{G'}$ is an isomorphism, i.e., $G' \in \tau_X$.

11.5 RAPPEL Given a category $\underline{C}$, a set $\mathcal{U}$ of objects in $\underline{C}$ is said to be a <u>separating set</u> if for every pair $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ of distinct morphisms, there exists a $U \in \mathcal{U}$ and a morphism $\sigma: U \to X$ such that $f \circ \sigma \neq g \circ \sigma$.

11.6 EXAMPLE Suppose that $\underline{C}$ is small -- then the h_Y ($Y \in \text{Ob } \underline{C}$) are a separating set for $\hat{\underline{C}}$.

11.7 LEMMA Let $\underline{C}$ be a category with coproducts and let $\mathcal{U}$ be a separating set -- then $\forall$ $X \in \text{Ob } \underline{C}$, the unique morphism

$$\coprod_{U \in \mathcal{U}} \ \coprod_{f \in \text{Mor}(U,X)} \text{dom } f \xrightarrow{\Gamma_X} X$$

such that $\forall$ f, $\Gamma_X \circ in_f = f$ is an epimorphism.

11.8 APPLICATION Suppose that $\underline{C}$ is small. Working with $\hat{\underline{C}}$, take X = G in 11.7 -- then

$$\coprod_{Y} \coprod_{f} h_Y \xrightarrow[\Gamma_G]{} G$$

is an epimorphism.

[Note: To finish the argument that $\underline{a}i$ is an epimorphism, start with the relation

$$\Gamma_G \circ \coprod \coprod i_f = i \circ \Pi_{G'}.$$

Then

$$\underline{a}\Gamma_G \circ \underline{a}(\coprod \coprod i_f) = \underline{a}i \circ \underline{a}\Pi_{G'}.$$

Since Γ_G is an epimorphism, the same is true of $\underline{a}\Gamma_G$ (left adjoints preserve epimorphisms). And

$$\underline{a}(\coprod \coprod i_f) = \coprod \coprod \underline{a}i_f$$

is an isomorphism, call it Φ, hence

$$\underline{a}\Gamma_G = \underline{a}i \circ (\underline{a}\Pi_{G'} \circ \Phi^{-1}).$$

Therefore $\underline{a}i$ is an epimorphism.]

11.9 NOTATION Denote by $\underline{S}_{\underline{C}}$ the "set" of reflective subcategories $\underline{S}$ of $\hat{\underline{C}}$ with the property that the inclusion $\iota:\underline{S} \to \hat{\underline{C}}$ has a left adjoint $\underline{a}:\hat{\underline{C}} \to \underline{S}$ that preserves finite limits.

11.10 DEFINITION Fix a Grothendieck topology $\tau \in \tau_{\underline{C}}$ -- then a presheaf $F \in \text{Ob } \hat{\underline{C}}$ is called a <u>τ-sheaf</u> if $\forall\ X \in \text{Ob } \underline{C}$ and $\forall\ G \in \tau_X$, the precomposition map

$$i_G^*:\text{Nat}(h_X,F) \to \text{Nat}(G,F)$$

is bijective.

Write $\underline{\text{Sh}}_\tau(\underline{C})$ for the full subcategory of $\hat{\underline{C}}$ whose objects are the τ-sheaves.

11.11 EXAMPLE Take for τ the minimal Grothendieck topology on $\underline{C}$ -- then $\underline{\text{Sh}}_\tau(\underline{C}) = \hat{\underline{C}}$.

[Note: In particular, $\underline{\mathrm{Sh}}_\tau(\underline{1}) = \hat{\underline{1}} \approx \underline{\mathrm{SET}}$.]

11.12 EXAMPLE Take for τ the maximal Grothendieck topology on $\underline{C}$ -- then the objects of $\underline{\mathrm{Sh}}_\tau(\underline{C})$ are the final objects in $\hat{\underline{C}}$.

[First, $\forall\ X \in \mathrm{Ob}\ \underline{C}$, $\emptyset_{\hat{\underline{C}}} \to h_X$. But $\emptyset_{\hat{\underline{C}}}$ is initial, thus the condition that F be a τ-sheaf amounts to the existence for each X of a unique morphism $h_X \to F$. Meanwhile, by Yoneda, $\mathrm{Nat}(h_X,F) \approx FX$.]

11.13 EXAMPLE Given $\tau \in \tau_{\underline{C}}$, define 0_τ by the rule

$$0_\tau(X) = \begin{cases} \{0\} \text{ if } \emptyset_{\hat{\underline{C}}} \in \tau_X \\ \\ \emptyset \text{ if } \emptyset_{\hat{\underline{C}}} \notin \tau_X. \end{cases}$$

Then 0_τ is a τ-sheaf and, moreover, is an initial object in $\underline{\mathrm{Sh}}_\tau(\underline{C})$.

11.14 THEOREM The inclusion $\iota_\tau:\underline{\mathrm{Sh}}_\tau(\underline{C}) \to \hat{\underline{C}}$ admits a left adjoint $\underline{a}_\tau:\hat{\underline{C}} \to \underline{\mathrm{Sh}}_\tau(\underline{C})$ that preserves finite limits.

[Note: We can and will assume that $\underline{a}_\tau \circ \iota_\tau$ is the identity.]

Various categorical generalities can then be specialized to the situation at hand.

11.15 DEFINITION A morphism $f:A \to B$ and an object X in a category $\underline{C}$ are said to be <u>orthogonal</u> ($f \perp X$) if the precomposition map $f^*:\mathrm{Mor}(B,X) \to \mathrm{Mor}(A,X)$ is bijective.

11.16 RAPPEL Let $\underline{D}$ be a reflective subcategory of a category $\underline{C}$, R a reflector for $\underline{D}$ (cf. 5.10). Let $W_{\underline{D}}$ be the class of morphisms in $\underline{C}$ rendered invertible by R.

- Let $X \in \text{Ob } \underline{C}$ -- then $X \in \text{Ob } \underline{D}$ iff $\forall\ f \in W_{\underline{D}}$, $f \perp X$.
- Let $f \in \text{Mor } \underline{C}$ -- then $f \in W_{\underline{D}}$ iff $\forall\ X \in \text{Ob } \underline{D}$, $f \perp X$.

11.17 NOTATION Let W_τ be the class of morphisms in $\hat{\underline{C}}$ rendered invertible by $\underline{a}_\tau$.

11.18 EXAMPLE If $F \in \text{Ob } \hat{\underline{C}}$, then F is a τ-sheaf iff $\forall\ \Xi \in W_\tau$, $\Xi \perp F$.

11.19 EXAMPLE If $\Xi \in \text{Mor } \hat{\underline{C}}$, then $\Xi \in W_\tau$ iff for every τ-sheaf F, $\Xi \perp F$.

[Note: If $X \in \text{Ob } \underline{C}$ and if $G \in \tau_X$, then for every τ-sheaf F, $i_G \perp F$, thus $i_G \in W_\tau$.]

11.20 RAPPEL Let $\underline{D}$ be a reflective subcategory of a category $\underline{C}$, R a reflector for $\underline{D}$ (cf. 5.10) -- then the localization $W_{\underline{D}}^{-1}\underline{C}$ is equivalent to $\underline{D}$.

11.21 APPLICATION The localization $W_\tau^{-1}\hat{\underline{C}}$ is equivalent to $\underline{\text{Sh}}_\tau(\underline{C})$.

11.22 RAPPEL Let $\underline{D}$ be a reflective subcategory of a finitely complete category $\underline{C}$, R a reflector for $\underline{D}$ (cf. 5.10) -- then R preserves finite limits iff $W_{\underline{D}}$ is pullback stable.

[Note: When this is the case, $W_{\underline{D}}$ is saturated (i.e., $f \in W_{\underline{D}}$ iff Rf is an isomorphism).]

11.23 APPLICATION Since $\underline{a}_\tau:\hat{\underline{C}} \to \underline{Sh}_\tau(\underline{C})$ preserves finite limits, it follows that W_τ is pullback stable (and saturated).

11.24 EXAMPLE Take $\underline{C} = \underline{1}$, so $\hat{\underline{1}} \approx \underline{SET}$ -- then $\#\tau_{\underline{1}} = 2$. On the other hand, $\underline{SET}$ has precisely 3 reflective subcategories: $\underline{SET}$ itself, the full subcategory of final objects, and the full subcategory of final objects plus the empty set ($\#RX = 1$ if $X \neq \emptyset$, $R\emptyset = \emptyset$). In terms of Grothendieck topologies, the first two are accounted for by 11.11 and 11.12. But the third cannot be a category of sheaves per a Grothendieck topology on $\underline{C} = \underline{1}$. To see this, note that the class of morphisms rendered invertible by R consists of all functions $f:X \to Y$ with $X \neq \emptyset$ as well as the function $\emptyset \to \emptyset$ (thus the arrows $\emptyset \to X$ $(X \neq \emptyset)$ are excluded). Suppose now that Z is a nonempty set and X,Y are nonempty subsets of Z with an empty intersection. Consider the pullback square

$$\begin{tikzcd}
\emptyset = X \cap Y \arrow[r, "\tilde{i}_X"] \arrow[d, "\tilde{i}_Y"'] & Y \arrow[d, "i_Y"] \\
X \arrow[r, "i_X"'] & Z
\end{tikzcd},$$

where i_X, i_Y are the inclusions -- then Ri_Y is an isomorphism but $R\tilde{i}_Y$ is not an isomorphism. Therefore the class of morphisms rendered invertible by R is not pullback stable.

11.25 NOTATION Let $F \in \text{Ob } \hat{\underline{C}}$ be a presheaf. Given $X \in \text{Ob } \underline{C}$, let $\tau_X(F)$ be the set of subfunctors $i_G:G \to h_X$ such that for any morphism $f:Y \to X$, the precomposition arrow

$$(i_{f^*G})^*:\mathrm{Nat}(h_Y,F) \to \mathrm{Nat}(f^*G,F)$$

is bijective.

11.26 LEMMA The assignment $X \to \tau_X(F)$ is a Grothendieck topology $\tau(F)$ on $\underline{C}$.

N.B. $\tau(F)$ is the largest Grothendieck topology in which F is a sheaf.

11.27 SCHOLIUM For any class $\mathcal{F}$ of presheaves, there exists a largest Grothendieck topology $\tau(\mathcal{F})$ on $\underline{C}$ in which the $F \in \mathcal{F}$ are sheaves.

11.28 DEFINITION The canonical Grothendieck topology τ_{can} on $\underline{C}$ is the largest Grothendieck topology on $\underline{C}$ in which the $h_X (X \in \mathrm{Ob}\ \underline{C})$ are sheaves.

[Note: Let $\tau \in \tau_{\underline{C}}$ -- then τ is said to be subcanonical if the h_X $(X \in \mathrm{Ob}\ \underline{C})$ are τ-sheaves.]

11.29 EXAMPLE Let L be a locale -- then the Grothendieck topology τ on L defined in 10.4 is the canonical Grothendieck topology.

[Note: This applies in particular to the locale $O(X)$, where X is a topological space, $\underline{\mathrm{Sh}}_\tau(O(X))$ being the traditional sheaves of sets on X, i.e., $\underline{\mathrm{Sh}}(X)$.]

11.30 EXAMPLE Take for X the Sierpinski space (so $X = \{0,1\}$ with topology $\{X,\emptyset,\{0\}\}$) -- then $\underline{\mathrm{Sh}}(X)$ (cf. 11.29) is the arrow category $\underline{\mathrm{SET}}(\to)$.

§12. *LOCAL ISOMORPHISMS*

Let $\underline{C}$ be a small category.

12.1 RAPPEL $\hat{\underline{C}}$ fulfills the standard conditions (cf. 3.4 and 3.6) and is balanced (cf. 6.10 and 7.7).

Let $H,K \in \mathrm{Ob}\,\hat{\underline{C}}$ be presheaves and let $\Xi \in \mathrm{Nat}(H,K)$. Form the pullback square

$$\begin{array}{ccc} H \times_K H & \xrightarrow{q} & H \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \Xi} \\ H & \xrightarrow[\Xi]{} & K \end{array} .$$

Then p and q are epimorphisms.

12.2 NOTATION $\delta_H : H \to H \times_K H$ is the canonical arrow associated with id_H, thus $p \circ \delta_H = id_H = q \circ \delta_H$.

<u>N.B.</u> δ_H is a monomorphism.

12.3 LEMMA Ξ is a monomorphism iff δ_H is an epimorphism.

[Note: Consequently, if Ξ is a monomorphism, then δ_H is an isomorphism.]

Fix a Grothendieck topology $\tau \in \tau_{\underline{C}}$.

12.4 DEFINITION Let $H,K \in \mathrm{Ob}\,\hat{\underline{C}}$ be presheaves and let $\Xi \in \mathrm{Nat}(H,K)$. Factor Ξ per 3.9:

$$H \xrightarrow{k} M \xrightarrow{m} K.$$

Then Ξ is a <u>τ-local epimorphism</u> if for any $f:h_Y \to K$, the subfunctor f^*M of h_Y defined by the pullback square

$$\begin{array}{ccc} f^*M & \longrightarrow & M \\ {\scriptstyle i_{f^*M}}\big\downarrow & & \big\downarrow{\scriptstyle m} \\ h_Y & \xrightarrow[f]{} & K \end{array}$$

is in τ_Y.

12.5 LEMMA Every epimorphism in $\hat{\underline{C}}$ is a τ-local epimorphism.

12.6 DEFINITION Let $H,K \in \text{Ob}\,\hat{\underline{C}}$ be presheaves and let $\Xi \in \text{Nat}(H,K)$ -- then Ξ is a <u>τ-local monomorphism</u> if δ_H is a τ-local epimorphism (cf. 12.3).

12.7 LEMMA Every monomorphism in $\hat{\underline{C}}$ is a τ-local monomorphism.

12.8 DEFINITION Let $H,K \in \text{Ob}\,\hat{\underline{C}}$ be presheaves and let $\Xi \in \text{Nat}(H,K)$ -- then Ξ is a <u>τ-local isomorphism</u> if Ξ is both a τ-local epimorphism and a τ-local monomorphism.

12.9 EXAMPLE If $G \in \tau_X$, then $i_G:G \to h_X$ is a τ-local isomorphism.

[For any $f:Y \to X$, there is a pullback square

$$\begin{array}{ccc} f^*G & \longrightarrow & G \\ {\scriptstyle i_{f^*G}}\big\downarrow & & \big\downarrow{\scriptstyle i_G} \\ h_Y & \xrightarrow[h_f]{} & h_X \end{array}$$

in $\hat{\underline{C}}$ and $f^*G \in \tau_Y$ (cf. 11.3), thus i_G is a τ-local epimorphism. On the other hand,

i_G is a monomorphism, hence i_G is a τ-local monomorphism (cf. 12.7).]

12.10 THEOREM W_τ is the class of τ-local isomorphisms.

12.11 APPLICATION Let $H \in \text{Ob}\,\hat{\underline{C}}$ -- then the canonical arrow

$$H \longrightarrow \iota_\tau \underline{a}_\tau H$$

is a τ-local isomorphism.

12.12 APPLICATION Let $G \in \tau_X$ -- then $\underline{a}_\tau i_G$ is an isomorphism (cf. 11.19).

[Note: Suppose that $i_G : G \to h_X$ is a subfunctor -- then i_G is a monomorphism, hence i_G is a τ-local monomorphism (cf. 12.7). Assume in addition that i_G is a τ-local epimorphism. Claim: $G \in \tau_X$. Proof: Take $f = \text{id}_X$ and consider

$$\begin{tikzcd} G \arrow[r, equal] \arrow[d, "i_G"'] & G \arrow[d, "i_G"] \\ h_X \arrow[r, equal] & h_X \end{tikzcd} .]$$

We shall now proceed to establish the "fundamental correspondence".

12.13 THEOREM The arrows

$$\begin{cases} \underline{S}_{\underline{C}} \longrightarrow \tau_{\underline{C}} & \text{(cf. 11.4)} \\ \tau_{\underline{C}} \longrightarrow \underline{S}_{\underline{C}} & \text{(cf. 11.14)} \end{cases}$$

are mutually inverse.

To dispatch the second of these, consider the composite

$$\tau_{\underline{C}} \longrightarrow \underline{S}_{\underline{C}} \longrightarrow \tau_{\underline{C}}.$$

Take a $\tau \in \tau_{\underline{C}}$ and pass to $\underline{Sh}_\tau(\underline{C})$ -- then the Grothendieck topology on $\underline{C}$ determined by $\underline{Sh}_\tau(\underline{C})$ via 11.4 assigns to each $X \in \text{Ob } \underline{C}$ the set of those subfunctors $i_G:G \to h_X$ such that $\underline{a}_\tau i_G$ is an isomorphism or, equivalently, those subfunctors $i_G:G \to h_X$ such that i_G is a τ-local isomorphism (cf. 12.10). But, as has been seen above, the subfunctors of h_X with this property are precisely the elements of τ_X. Therefore the composite

$$\tau_{\underline{C}} \longrightarrow \underline{S}_{\underline{C}} \longrightarrow \tau_{\underline{C}}$$

is the identity map.

It remains to prove that the composite

$$\underline{S}_{\underline{C}} \longrightarrow \tau_{\underline{C}} \longrightarrow \underline{S}_{\underline{C}}$$

is the identity map. So take an $\underline{S} \in \underline{S}_{\underline{C}}$, produce a Grothendieck topology τ on $\underline{C}$ per 11.4, and pass to $\underline{Sh}_\tau(\underline{C})$ -- then $\underline{S} \subset \underline{Sh}_\tau(\underline{C})$. Thus let $F \in \text{Ob } \underline{S}$, the claim being that $F \in \text{Ob } \underline{Sh}_\tau(\underline{C})$ or still, that F is a τ-sheaf, or still, that $\forall\, X \in \text{Ob } \underline{C}$ and $\forall\, G \in \tau_X$, $i_G \perp F$, which is clear since $i_G \in W_\tau$ (cf. 11.19). To reverse matters and deduce that $\underline{Sh}_\tau(\underline{C}) \subset \underline{S}$, one has only to show that if $\Xi:H \to K$ is a morphism in $\hat{\underline{C}}$ and if $\underline{a}\Xi$ is an isomorphism, then $\underline{a}_\tau\Xi$ is an isomorphism. To this end, factor Ξ per 3.9:

$$H \xrightarrow{k} M \xrightarrow{m} K.$$

Then $\underline{a}\Xi = \underline{a}m \circ \underline{a}k$. But $\underline{a}\Xi$ is an isomorphism and $\underline{a}m$ is a monomorphism ($\underline{a}$ preserves finite limits). Therefore $\underline{a}k$ is a monomorphism. But $\underline{a}k$ is a coequalizer ($\underline{a}$ is a left adjoint), thus $\underline{a}k$ is an isomorphism (cf. 6.8). And then $\underline{a}m$ is an isomorphism as well.

• Assume that $\underline{a}\Xi$ is an isomorphism, where Ξ is a monomorphism -- then $\underline{a}_\tau\Xi$ is an isomorphism.

[Bearing in mind that here H = M, consider a pullback square

$$\begin{tikzcd} f^*H \arrow[r] \arrow[d, "i_{f^*H}"'] & H \arrow[d, "\Xi"] \\ h_Y \arrow[r, "f"'] & K \end{tikzcd} .$$

Then the assumption that $\underline{a}\Xi$ is an isomorphism implies that $\underline{a}i_{f^*H}$ is an isomorphism which in turn implies that $i_{f^*H} \in \tau_Y$. Therefore Ξ is a τ-local epimorphism or still, Ξ is a τ-local isomorphism, hence $\Xi \in \mathcal{W}_\tau$ (cf. 12.10), so $\underline{a}_\tau\Xi$ is an isomorphism.]

• Assume that $\underline{a}\Xi$ is an isomorphism, where Ξ is a coequalizer -- then $\underline{a}_\tau\Xi$ is an isomorphism.

[Because $\underline{a}_\tau\Xi$ is a coequalizer, to conclude that $\underline{a}_\tau\Xi$ is an isomorphism, it suffices to verify that $\underline{a}_\tau\Xi$ is a monomorphism (cf. 6.8). For this purpose, consider the pullback square

$$\begin{tikzcd} H \times_K H \arrow[r, "q"] \arrow[d, "p"'] & H \arrow[d, "\Xi"] \\ H \arrow[r, "\Xi"'] & K \end{tikzcd} .$$

Then δ_H is a monomorphism and there are pullback squares

$$\begin{array}{ccc} \underline{a}H \times_{\underline{a}K} \underline{a}H & \xrightarrow{\underline{a}q} & \underline{a}H \\ {\scriptstyle \underline{a}p}\downarrow & & \downarrow{\scriptstyle \underline{a}\Xi} \\ \underline{a}H & \xrightarrow[\underline{a}\Xi]{} & \underline{a}K \end{array}, \qquad \begin{array}{ccc} \underline{a}_\tau H \times_{\underline{a}_\tau K} \underline{a}_\tau H & \xrightarrow{\underline{a}_\tau q} & \underline{a}_\tau H \\ {\scriptstyle \underline{a}_\tau p}\downarrow & & \downarrow{\scriptstyle \underline{a}_\tau \Xi} \\ \underline{a}_\tau H & \xrightarrow[\underline{a}_\tau \Xi]{} & \underline{a}_\tau K \end{array}.$$

But $\underline{a}\delta_H = \delta_{\underline{a}H}$ is an isomorphism (cf. 12.3), thus $\underline{a}_\tau\delta_H = \delta_{\underline{a}_\tau H}$ is an isomorphism (cf. supra), so $\underline{a}_\tau\Xi$ is a monomorphism (cf. 12.3).]

12.14 THEOREM Let $H,K \in \text{Ob}\ \hat{\underline{C}}$ be presheaves and let $\Xi \in \text{Nat}(H,K)$ -- then $\underline{a}_\tau\Xi:\underline{a}_\tau H \to \underline{a}_\tau K$ is an epimorphism in $\underline{\text{Sh}}_\tau(\underline{C})$ iff Ξ is a τ-local epimorphism.

12.15 APPLICATION The epimorphisms in $\underline{\text{Sh}}_\tau(\underline{C})$ are pullback stable.

[The class of τ-local epimorphisms is pullback stable.]

§13. SORITES

The category $\underline{\mathrm{Sh}}_\tau(\underline{C})$ associated with a site $(\underline{C},\tau)$ has a number of properties that will be cataloged below.

13.1 LEMMA $\underline{\mathrm{Sh}}_\tau(\underline{C})$ is complete and cocomplete.

[This is because $\underline{\mathrm{Sh}}_\tau(\underline{C})$ is a reflective subcategory of $\hat{\underline{C}}$ which is both complete and cocomplete. Accordingly, limits in $\underline{\mathrm{Sh}}_\tau(\underline{C})$ are computed as in $\hat{\underline{C}}$ while colimits in $\underline{\mathrm{Sh}}_\tau(\underline{C})$ are computed by applying $\underline{a}_\tau$ to the corresponding colimits in $\hat{\underline{C}}$.]

13.2 LEMMA $\underline{\mathrm{Sh}}_\tau(\underline{C})$ is cartesian closed.

[Since $\underline{a}_\tau:\hat{\underline{C}} \to \underline{\mathrm{Sh}}_\tau(\underline{C})$ preserves finite limits, it preserves finite products so one can quote 5.11.]

[Note: Recall that $\hat{\underline{C}}$ is cartesian closed (cf. 5.21).]

13.3 LEMMA $\underline{\mathrm{Sh}}_\tau(\underline{C})$ admits a subobject classifier.

[Note: Therefore $\underline{\mathrm{Sh}}_\tau(\underline{C})$ is wellpowered (cf. 6.13).]

The proof of this result will be broken up into several steps (tacitly employing the license provided by 7.6).

<u>Step 1</u> Given $F \in \mathrm{Ob}\,\hat{\underline{C}}$ and a subfunctor $i:G \to F$, define a subfunctor $\bar{i}:\bar{G} \to F$ by the pullback square

$$\begin{array}{ccc}
\bar{G} & \longrightarrow & \iota_\tau \underline{a}_\tau G \\
{\scriptstyle \bar{i}}\downarrow & & \downarrow{\scriptstyle \iota_\tau \underline{a}_\tau i} \\
F & \longrightarrow & \iota_\tau \underline{a}_\tau F
\end{array} .$$

<u>Step 2</u> There is a commutative diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d, "i"'] & \iota_\tau \underline{a}_\tau G \arrow[d, "\iota_\tau \underline{a}_\tau i"] \\ F \arrow[r] & \iota_\tau \underline{a}_\tau F \end{tikzcd}$$

from which an arrow $\gamma : G \to \bar{G}$ such that the diagrams

$$\begin{tikzcd} G \arrow[r, "\gamma"] \arrow[d, "i"'] & \bar{G} \arrow[d, "\bar{i}"] \\ F \arrow[r, equal] & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r] \arrow[d, "\gamma"'] & \iota_\tau \underline{a}_\tau G \arrow[d, equal] \\ \bar{G} \arrow[r] & \iota_\tau \underline{a}_\tau G \end{tikzcd}$$

commute.

<u>Step 3</u> Definition: G is <u>closed</u> if $G = \bar{G}$. We have

(1) $G \subset \bar{G}$;

(2) $G \subset H \Rightarrow \bar{G} \subset \bar{H}$;

(3) $\bar{\bar{G}} = \bar{G}$.

In addition, closed subfunctors are stable under pullbacks.

[Note: To make the last point precise, suppose given an arrow $f : F' \to F$ in $\hat{\underline{C}}$. Define G' by the pullback square

$$\begin{tikzcd} G' \arrow[r] \arrow[d] & G \arrow[d, "i"] \\ F' \arrow[r, "f"'] & F \end{tikzcd}$$

and define $\bar{G}'$ by the pullback square

$$\begin{tikzcd} \bar{G}' \arrow[r] \arrow[d] & \bar{G} \arrow[d, "\bar{i}"] \\ F' \arrow[r, "f"'] & F . \end{tikzcd}$$

Then $\overline{G'} = \bar{G}'$, so

$$G = \bar{G} \Rightarrow G' = \bar{G}' = \overline{G'}.]$$

<u>Step 4</u> $\forall\, F \in \mathrm{Ob}\, \hat{\underline{C}}$,

$$\bar{F} = F.$$

In particular: $\forall\, X \in \mathrm{Ob}\, \underline{C}$,

$$\overline{h_X} = h_X.$$

<u>Step 5</u> Let (Ω, τ) be the subobject classifier for $\hat{\underline{C}}$ (cf. 7.7). Define

$$\Omega^{c\ell} : \underline{C}^{OP} \to \underline{SET}$$

on an object X by letting $\Omega^{c\ell}X$ be the set of all closed subfunctors of h_X and on a morphism $f:Y \to X$ by letting $\Omega^{c\ell}f : \Omega^{c\ell}X \to \Omega^{c\ell}Y$ operate via the pullback square

$$\begin{tikzcd} \Omega^{c\ell}f(G) \arrow[r] \arrow[d] & G \arrow[d] \\ h_Y \arrow[r, "h_f"'] & h_X \end{tikzcd}$$

and define

$$\tau^{c\ell} : *_{\hat{\underline{C}}} \to \Omega^{c\ell}$$

by factoring

$$\top : *_{\hat{\underline{C}}} \to \Omega$$

through $\Omega^{c\ell}$ (which makes sense since $\overline{h_X} = h_X$). With these agreements, $\Omega^{c\ell}$ is a subfunctor of Ω, say $i^{c\ell}:\Omega^{c\ell} \to \Omega$.

Step 6 Consider the pullback square

$$\begin{tikzcd} G \arrow[r, "!"] \arrow[d, "i"'] & *_{\hat{\underline{C}}} \arrow[d, "\top"] \\ F \arrow[r, "\chi_i"'] & \Omega \end{tikzcd} .$$

Then the classifying arrow χ_i factors through $\Omega^{c\ell}$ iff G is closed.

Step 7 If F is a τ-sheaf, then it and its τ-subsheaves G are closed. This said, consider the commutative diagram

$$\begin{tikzcd} G \arrow[r, "!"] \arrow[d, "i"'] & *_{\hat{\underline{C}}} \arrow[r, equal] \arrow[d, "\top^{c\ell}"] & *_{\hat{\underline{C}}} \arrow[d, "\top"] \\ F \arrow[r, "\chi_i^{c\ell}"'] & \Omega^{c\ell} \arrow[r, "i^{c\ell}"'] & \Omega \end{tikzcd} .$$

Here $\chi_i = i^{c\ell} \circ \chi_i^{c\ell}$ and both squares are pullbacks. If $\chi:F \to \Omega^{c\ell}$ is another morphism and if

$$\begin{tikzcd} G \arrow[r, "!"] \arrow[d, "i"'] & *_{\hat{\underline{C}}} \arrow[d, "\tau^{c\ell}"] \\ F \arrow[r, "\chi"'] & \Omega^{c\ell} \end{tikzcd}$$

is a pullback square, then $i^{c\ell} \circ \chi$ is a classifying arrow of (G,i) in F, so $i^{c\ell} \circ \chi = \chi_i = i^{c\ell} \circ \chi_i^{c\ell}$, hence $\chi = \chi_i^{c\ell}$.

<u>Step 8</u> $*_{\hat{\underline{C}}}$ is a τ-sheaf (obvious) and $\Omega^{c\ell}$ is a τ-sheaf (...). Therefore the pair $(\Omega^{c\ell}, \tau^{c\ell})$ is a subobject classifier for $\underline{\mathrm{Sh}}_\tau(\underline{C})$.

13.4 LEMMA $\underline{\mathrm{Sh}}_\tau(\underline{C})$ is balanced.

[Taking into account 13.3, one has only to cite 6.10.]

13.5 LEMMA Every monomorphism in $\underline{\mathrm{Sh}}_\tau(\underline{C})$ is an equalizer.

[In view of 13.3, this is a special case of 6.9.]

[Note: It is easy to proceed directly. Thus let $\Xi : F \to G$ be a monomorphism in $\underline{\mathrm{Sh}}_\tau(\underline{C})$ -- then $\iota_\tau \Xi : \iota_\tau F \to \iota_\tau G$ is a monomorphism in $\hat{\underline{C}}$, hence is an equalizer. But $\underline{a}_\tau$ preserves equalizers (since it preserves finite limits).]

<u>N.B.</u> Monomorphisms in $\underline{\mathrm{Sh}}_\tau(\underline{C})$ are pushout stable.

13.6 LEMMA Every epimorphism in $\underline{\mathrm{Sh}}_\tau(\underline{C})$ is a coequalizer.

PROOF Given an epimorphism $\Xi : F \to G$ in $\underline{\mathrm{Sh}}_\tau(\underline{C})$, form the pullback square

$$\begin{tikzcd}
P \arrow[r, "v"] \arrow[d, "u"'] & F \arrow[d, "\Xi"] \\
F \arrow[r, "\Xi"'] & G
\end{tikzcd}$$

in $\underline{\mathrm{Sh}}_\tau(\underline{C})$ -- then

$$\begin{tikzcd}
\iota_\tau P \arrow[r, "\iota_\tau v"] \arrow[d, "\iota_\tau u"'] & \iota_\tau F \arrow[d, "\iota_\tau \Xi"] \\
\iota_\tau F \arrow[r, "\iota_\tau \Xi"'] & \iota_\tau G
\end{tikzcd}$$

is a pullback square in $\hat{\underline{C}}$. Factor $\iota_\tau\Xi$ per 3.9:

$$\iota_\tau F \xrightarrow{k} M \xrightarrow{m} \iota_\tau G.$$

Then by construction there is a coequalizer diagram

$$\begin{tikzcd}
\iota_\tau P \arrow[r, shift left, "\iota_\tau u"] \arrow[r, shift right, "\iota_\tau v"'] & \iota_\tau F \arrow[r, "\iota_\tau \Xi"] \arrow[d, "k"'] & \iota_\tau G \\
& M \arrow[r, equal] & M \arrow[u, "m"']
\end{tikzcd}$$

in $\hat{\underline{C}}$. Now apply $\underline{a}_\tau$ to get a coequalizer diagram

$$\begin{tikzcd} P \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & F \arrow[r, "\Xi"] \arrow[d, "\underline{a}_\tau k"'] & G \\ & \underline{a}_\tau M \arrow[r, equal] & \underline{a}_\tau M \arrow[u, "\underline{a}_\tau m"'] \end{tikzcd}$$

in $\underline{\mathrm{Sh}}_\tau(\underline{C})$. Since

$$\Xi = \underline{a}_\tau m \circ \underline{a}_\tau k$$

and since Ξ is an epimorphism, it follows that $\underline{a}_\tau m$ is an epimorphism. But $\underline{a}_\tau m$ is also a monomorphism. Therefore $\underline{a}_\tau m$ is an isomorphism (cf. 13.4) and Ξ is a coequalizer, thus being the case of $\underline{a}_\tau k$.

13.7 LEMMA $\underline{\mathrm{Sh}}_\tau(\underline{C})$ fulfills the standard conditions.

[Epimorphisms in $\underline{\mathrm{Sh}}_\tau(\underline{C})$ are pullback stable (cf. 12.15) and every epimorphism in $\underline{\mathrm{Sh}}_\tau(\underline{C})$ is a coequalizer (cf. 13.6).]

13.8 LEMMA In $\underline{\mathrm{Sh}}_\tau(\underline{C})$, filtered colimits commute with finite limits.

13.9 RAPPEL Coproducts in $\hat{\underline{C}}$ are disjoint.

[In other words, if $F = \coprod_{i \in I} F_i$ is a coproduct of a set of presheaves F_i, then $\forall\, i \in I$, $in_i : F_i \to F$ is a monomorphism and $\forall\, i,j \in I$ $(i \neq j)$, the pullback $F_i \times_F F_j$ is the initial object in $\hat{\underline{C}}$.]

13.10 LEMMA Coproducts in $\underline{\mathrm{Sh}}_\tau(\underline{C})$ are disjoint.

13.11 RAPPEL Coproducts in $\hat{\underline{C}}$ are pullback stable.

[In other words, if $F = \coprod_{i \in I} F_i$ is a coproduct of a set of presheaves F_i, then for every arrow $F' \to F$,

$$\coprod_{i \in I} F' \times_F F_i \approx F'.]$$

13.12 LEMMA Coproducts in $\underline{\mathrm{Sh}}_\tau(\underline{C})$ are pullback stable.

13.13 DEFINITION Let $\underline{C}$ be a category which fulfills the standard conditions. Suppose that $R \underset{v}{\overset{u}{\rightrightarrows}} X$ is an equivalence relation on an object X in $\underline{C}$. Consider the coequalizer diagram

$$R \underset{v}{\overset{u}{\rightrightarrows}} X \xrightarrow{\pi} X/R \equiv \mathrm{coeq}(u,v).$$

Then there is a commutative diagram

$$\begin{tikzcd} R \arrow[r, "v"] \arrow[d, "u"'] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"'] & X/R \end{tikzcd}$$

and a pullback square

$$\begin{tikzcd} X \times_{X/R} X \arrow[r, "q"] \arrow[d, "p"'] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"'] & X/R \end{tikzcd}$$

One then says that R is <u>effective</u> if the canonical arrow

$$R \longrightarrow X \times_{X/R} X$$

is an isomorphism (it is always a monomorphism).

[Note: $\underline{C}$ has <u>effective equivalence relations</u> if every equivalence relation is effective.]

13.14 LEMMA Equivalence relations in $\underline{\mathrm{Sh}}_\tau(\underline{C})$ are effective.

[The usual methods apply: Equivalence relations in $\underline{\mathrm{SET}}$ are effective, hence equivalence relations in $\hat{\underline{C}}$ are effective etc.]

13.15 LEMMA The $\underline{a}_\tau h_X$ ($X \in \mathrm{Ob}\ \underline{C}$) are a separating set for $\underline{\mathrm{Sh}}_\tau(\underline{C})$.

PROOF Let $\Xi,\Xi':F \to G$ be distinct arrows in $\underline{\mathrm{Sh}}_\tau(\underline{C})$ — then the claim is that $\exists\ X \in \mathrm{Ob}\ \underline{C}$ and $\sigma:\underline{a}_\tau h_X \to F$ such that $\Xi \circ \sigma \neq \Xi' \circ \sigma$. But $\Xi \neq \Xi'$ implies that $\Xi_X \neq \Xi'_X$ ($\exists\ X \in \mathrm{Ob}\ \underline{C}$) which implies that $\Xi_X x \neq \Xi'_X x$ ($\exists\ x \in FX$). Owing to the Yoneda lemma, $FX \approx \mathrm{Nat}(h_X,F)$, so x corresponds to a $\sigma' \in \mathrm{Nat}(h_X,F)$, thus $\Xi \circ \sigma' \neq \Xi' \circ \sigma'$. Determine $\sigma:\underline{a}_\tau h_X \to F$ by the diagram

$$\begin{tikzcd} h_X \arrow[r] \arrow[d, "\sigma'"'] & \underline{a}_\tau h_X \arrow[d, "\sigma"] \\ F \arrow[r, equal] & F \end{tikzcd}$$

Then $\Xi \circ \sigma \neq \Xi' \circ \sigma$.

N.B. All epimorphisms in $\underline{\mathrm{Sh}}_\tau(\underline{C})$ are coequalizers (cf. 13.6). So, for every τ-sheaf F, the epimorphism Γ_F of 11.7 is automatically a coequalizer. Therefore the $\underline{a}_\tau h_X$ ($X \in \mathrm{Ob}\ \underline{C}$) are a "strong" separating set for $\underline{\mathrm{Sh}}_\tau(\underline{C})$.

[Note: This baroque technicality is used implicitly in 13.16 below.]

A summary of the theory of presentable categories can be found in the Appendix to CHT and will not be repeated here.

[Note: As a point of terminology, let $\underline{C}$ be a cocomplete category and let κ be a regular cardinal -- then an object $X \in \mathrm{Ob}\ \underline{C}$ is κ-definite if $\mathrm{Mor}(X,—)$ preserves κ-filtered colimits.]

13.16 LEMMA $\underline{\mathrm{Sh}}_\tau(\underline{C})$ is presentable.

PROOF Fix a regular cardinal $\kappa > \#\mathrm{Mor}\ \underline{C}$ -- then $\forall\ X \in \mathrm{Ob}\ \underline{C}$, $h_X \in \mathrm{Ob}\ \hat{\underline{C}}$ is κ-definite, the contention being that $\forall\ X \in \mathrm{Ob}\ \underline{C}$, $\underline{a}_\tau h_X \in \mathrm{Ob}\ \underline{\mathrm{Sh}}_\tau(\underline{C})$ is κ-definite, which suffices (cf. 13.15). To see this, note first that a κ-filtered colimit of τ-sheaves can be computed levelwise, i.e., its κ-filtered colimit per $\hat{\underline{C}}$ is a τ-sheaf. Now fix a κ-filtered category $\underline{I}$ and let $\Delta:\underline{I} \to \underline{\mathrm{Sh}}_\tau(\underline{C})$ be a diagram — then

$$\begin{aligned}
\mathrm{Nat}(\underline{a}_\tau h_X, \mathrm{colim}_{\underline{I}}\ \Delta_i) &\approx \mathrm{Nat}(\underline{a}_\tau h_X, \mathrm{colim}_{\underline{I}}\ \iota_\tau \Delta_i) \\
&\approx \mathrm{Nat}(h_X, \mathrm{colim}_{\underline{I}}\ \iota_\tau \Delta_i) \\
&\approx \mathrm{colim}_{\underline{I}}\ \mathrm{Nat}(h_X, \iota_\tau \Delta_i) \\
&\approx \mathrm{colim}_{\underline{I}}\ \mathrm{Nat}(\underline{a}_\tau h_X, \Delta_i).
\end{aligned}$$

13.17 REMARK It is a fact that a presentable category is complete and cocomplete, wellpowered and cowellpowered.

§14. TOPOS THEORY: FORMALITIES

Let $\underline{E}$ be a category.

14.1 DEFINITION $\underline{E}$ is a <u>topos</u> if

- $\underline{E}$ is finitely complete;
- $\underline{E}$ is cartesian closed;
- $\underline{E}$ has a subobject classifier (Ω,τ).

[Note: The defining properties of a topos are invariant under equivalence.]

<u>N.B.</u> Every topos is wellpowered.

14.2 EXAMPLE $\underline{SET}$ is a topos.

[Note: The full subcategory of $\underline{SET}$ whose objects are finite is a topos. On the other hand, the full subcategory of $\underline{SET}$ whose objects are at most countable has a subobject classifier but is not cartesian closed, hence is not a topos.]

14.3 EXAMPLE Let $\underline{C}$ be a small category -- then $\hat{\underline{C}}$ is a topos (cf. 5.21 and 7.7).

14.4 EXAMPLE Let $(\underline{C},\tau)$ be a site -- then $\underline{Sh}_\tau(\underline{C})$ is a topos (cf. 13.2 and 13.3).

14.5 THEOREM Every topos is finitely cocomplete.

14.6 THEOREM Every topos fulfills the standard conditions.

14.7 LEMMA Let $\underline{E}$ be a topos.

(1) Every monomorphism in $\underline{E}$ is an equalizer.

(2) Every epimorphism in $\underline{E}$ is a coequalizer.

(3) Every morphism in $\underline{E}$ which is both a monomorphism and an epimorphism is an isomorphism.

(4) Every morphism in $\underline{E}$ admits a minimal decomposition unique up to isomorphism.

14.8 EXAMPLE Not all monomorphisms in $\underline{CAT}$ are equalizers and not all epimorphisms in $\underline{CAT}$ are coequalizers. Therefore $\underline{CAT}$ is not a topos.

14.9 LEMMA Every topos has effective equivalence relations.

14.10 EXAMPLE In $\underline{POS}$ (the category whose objects are the posets and whose morphisms are the order preserving maps), not all equivalence relations are effective.

14.11 CRITERION In a topos $\underline{E}$, consider a pushout square

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\eta"] \\ X \arrow[r, "\xi"'] & P \end{tikzcd} .$$

Assume: f is a monomorphism -- then η is a monomorphism and the square is a pullback.

14.12 LEMMA In a topos $\underline{E}$, finite coproducts are disjoint.

PROOF Let $A,B \in \text{Ob } \underline{E}$ -- then on general grounds, there is a pushout square

$$\begin{tikzcd} \emptyset_{\underline{E}} \arrow[r, "b"] \arrow[d, "a"'] & B \arrow[d, "in_B"] \\ A \arrow[r, "in_A"'] & A \coprod B . \end{tikzcd}$$

On the other hand, a and b are monomorphisms (cf. 5.16). Therefore in_A and in_B are monomorphisms and the square is a pullback (cf. 14.11).

14.13 LEMMA In a topos $\underline{E}$, finite coproducts are pullback stable.

[Note: Finiteness is not needed provided that the coproducts in question exist. Thus suppose that $\{A_i \xrightarrow{f_i} A : i \in I\}$ is a coproduct diagram in $\underline{E}$. Let $B \xrightarrow{f} A$ and for each $i \in I$, define B_i by the pullback square

$$\begin{tikzcd} B_i \arrow[r] \arrow[d, "g_i"'] & A_i \arrow[d, "f_i"] \\ B \arrow[r, "f"'] & A . \end{tikzcd}$$

Then $\{B_i \xrightarrow{g_i} B : i \in I\}$ is a coproduct diagram in $\underline{E}$. To see this, use 15.3: Consider the composition

$$\underline{E} \xrightarrow{A^*} \underline{E}/A \xrightarrow{f^*} \underline{E}/B \xrightarrow{B_!} \underline{E}.$$

Each of the functors A^*, f^*, $B_!$ has a right adjoint, hence preserve colimits, in particular coproducts. On the other hand, given an arrow $X \to A$, define an arrow

$B \times_A X \to B$ by forming the pullback square

$$\begin{tikzcd} B \times_A X \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r, "f"'] & A. \end{tikzcd}$$

Then

$$B_! \circ f^* \circ A^*(X \to A) = B \times_A X \to B.]$$

Let $\underline{E}$ be a topos.

14.14 NOTATION Given $A \in \text{Ob } \underline{E}$, let $\delta_A : A \to A \times A$ be the diagonal -- then δ_A is a monomorphism, so there is a pullback square

$$\begin{tikzcd} A \arrow[r, "!"] \arrow[d, "\delta_A"'] & *_{\underline{E}} \arrow[d, "T"] \\ A \times A \arrow[r, "\chi_{\delta_A}"'] & \Omega. \end{tikzcd}$$

Abbreviate χ_{δ_A} to $=_A$.

We have

$$\text{Mor}(A \times A, \Omega) \approx \text{Mor}(A, \Omega^A).$$

Therefore

$$=_A \in \text{Mor}(A \times A, \Omega)$$

corresponds to an element

$$\{\cdot\}_A \in \text{Mor}(A, \Omega^A),$$

the <u>singleton</u> on A.

14.15 LEMMA $\{\cdot\}_A$ is a monomorphism, hence

$$(A,\{\cdot\}_A) \in M(\Omega^A).$$

14.16 EXAMPLE Take $\underline{E} = \underline{SET}$ -- then $\{\cdot\}_A:A \to \Omega^A$ sends $a \in A$ to the characteristic function of $\{a\}$ (cf. 6.4). Identifying Ω^A with PA (the power set of A), it follows that $\{\cdot\}_A:A \to \Omega^A$ sends a to $\{a\}$.

14.17 RAPPEL Given a category $\underline{C}$, an object Q in $\underline{C}$ is said to be <u>injective</u> if for each monomorphism $f:X \to Y$ and each morphism $\phi:X \to Q$, there exists a morphism $g:Y \to Q$ such that $g \circ f = \phi$.

14.18 LEMMA In a topos $\underline{E}$, the object Ω is injective.

PROOF Let $f:X \to Y$ be a monomorphism and let $\chi:X \to \Omega$ be a morphism. Define $(\tilde{X},\tilde{f}) \in M(X)$ by the pullback square

$$\begin{tikzcd}
\tilde{X} \arrow[r, "!"] \arrow[d, "\tilde{f}"'] & *_{\underline{E}} \arrow[d, "T"] \\
X \arrow[r, "\chi"'] & \Omega
\end{tikzcd}.$$

Then $\chi_{\tilde{f}} = \chi$ (cf. 6.12). Consider now the commutative diagram

$$\begin{tikzcd}
\tilde{X} \arrow[r, equal] \arrow[d, "\tilde{f}"'] & \tilde{X} \arrow[r, "!"] \arrow[d] & *_{\underline{E}} \arrow[d, "T"] \\
X \arrow[r, "f"'] & Y \arrow[r, "\chi_{f \circ \tilde{f}}"'] & \Omega
\end{tikzcd}.$$

Put $g = \chi_{f \circ \tilde{f}}$. Since the squares are pullbacks, the commutative diagram

$$\begin{array}{ccc}
\tilde{X} & \xrightarrow{\ !\ } & *_{\underline{E}} \\
{\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle T} \\
X & \xrightarrow[g \circ f]{} & \Omega
\end{array}$$

is a pullback square, so $\chi_{\tilde{f}} = g \circ f$. But

$$\chi_{\tilde{f}} = \chi \Rightarrow g \circ f = \chi.$$

14.19 LEMMA In a topos $\underline{E}$, the object Ω^A $(A \in \text{Ob } \underline{E})$ is injective.

PROOF Let $f:X \to Y$ be a monomorphism and let $\phi:X \to \Omega^A$ be a morphism -- then there is a commutative diagram

$$\begin{array}{ccc}
\text{Mor}(Y,\Omega^A) & \longrightarrow & \text{Mor}(X,\Omega^A) \\
\approx \| & & \| \approx \\
\text{Mor}(Y \times A,\Omega) & \longrightarrow & \text{Mor}(X \times A,\Omega).
\end{array}$$

Since Ω is injective, the bottom map is surjective, thus the same is true of the top map.

14.20 RAPPEL A category $\underline{C}$ has <u>enough injectives</u> provided that for any $X \in \text{Ob } \underline{C}$, there is a monomorphism $X \to Q$ with Q injective.

14.21 LEMMA A topos $\underline{E}$ has enough injectives.

PROOF If $A \in \text{Ob } \underline{E}$, then Ω^A is injective and $\{\cdot\}_A : A \to \Omega^A$ is a monomorphism (cf. 14.15).

14.22 LEMMA The injective objects in $\underline{E}$ are the retracts of the Ω^A $(A \in \text{Ob } \underline{E})$.

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§15. TOPOS THEORY: SLICES AND SUBOBJECTS

Let $\underline{E}$ be a topos.

15.1 THEOREM For every $A \in \text{Ob } \underline{E}$, the category $\underline{E}/A$ is a topos.

[Since $\underline{E}$ is finitely complete, the same is true of $\underline{E}/A$ (cf. 4.1). Let τ_A be the composition $A \xrightarrow{!} *_{\underline{E}} \xrightarrow{\tau} \Omega$. Bearing in mind that $\text{id}_A : A \to A$ is a final object in $\underline{E}/A$, define

$$\langle \text{id}_A, \tau_A \rangle : (\text{id}_A : A \to A) \to (\text{pr}_A : A \times \Omega \to A)$$

by consideration of

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "\text{id}_A"'] & A \arrow[r, equal] \arrow[d, dotted] & A \arrow[d, "\tau_A"] \\
A & A \times \Omega \arrow[l, "\text{pr}_A"] \arrow[r, "\text{pr}_\Omega"'] & \Omega
\end{tikzcd}.$$

Then $\langle \text{id}_A, \tau_A \rangle$ is a monomorphism (its domain being a final object in $\underline{E}/A$) and the pair

$$(\text{pr}_A : A \times \Omega \to A, \langle \text{id}_A, \tau_A \rangle)$$

is a subobject classifier for $\underline{E}/A$. The crux is therefore to establish that $\underline{E}/A$ is cartesian closed.]

In particular: $\underline{E}$ is locally cartesian closed (cf. 5.23).

15.2 EXAMPLE $\forall$ X, $\underline{\text{TOP}_{\underline{\text{LH}}}}/X$ is a topos but $\underline{\text{TOP}_{\underline{\text{LH}}}}$ is not a topos (recall that

$\underline{\text{TOP}}_{\underline{\text{LH}}}$ is not finitely complete (cf. 4.2)).

15.3 THEOREM Suppose that $f:A \to B$ is a morphism in $\underline{E}$ -- then $f^*:\underline{E}/B \to \underline{E}/A$ has a left adjoint $f_!:\underline{E}/A \to \underline{E}/B$ and a right adjoint $f_*:\underline{E}/A \to \underline{E}/B$.

[This is a special case of 5.32 and 5.33.]

[Note: f^* preserves exponential objects and subobject classifiers.]

15.4 LEMMA Let $A \in \text{Ob } \underline{E}$ -- then the poset $\text{Sub}_{\underline{E}}\ A$ is a bounded lattice.

[Simply apply 2.21 and 3.14. However, for the record, suppose that

$$\begin{cases} S \xrightarrow{\sigma} A \\ T \xrightarrow{\tau} A \end{cases}$$

are monomorphisms. Definition:

$$\begin{cases} S \wedge T = S \cap T \\ S \vee T = S \cup T. \end{cases}$$

To complete the picture, let

$$\begin{cases} 1 = (\text{id}_A:A \to A) \\ 0 = (!:\emptyset_{\underline{E}} \to A) \qquad \text{(cf. 5.14 and 5.16).} \end{cases}$$

15.5 REMARK The square

$$\begin{tikzcd} S \cap T \arrow[r] \arrow[d] & T \arrow[d] \\ S \arrow[r] & S \cup T \end{tikzcd}$$

is both a pullback and a pushout.

15.6 THEOREM Let $A \in \mathrm{Ob}\,\underline{E}$ -- then the bounded lattice $\mathrm{Sub}_{\underline{E}}\,A$ is a Heyting algebra.

PROOF Given monomorphisms

$$\begin{cases} S \xrightarrow{\sigma} A \\ T \xrightarrow{\tau} A, \end{cases}$$

define T^S as the equalizer

$$T^S \longrightarrow A \rightrightarrows \Omega$$

of χ_σ and χ_θ (where $S \cap T \xrightarrow{\theta} A$ is the corner arrow). Let $R \xrightarrow{\rho} A$ be a monomorphism -- then, from the properties of an equalizer,

$$R \leq_A T^S \iff \chi_\sigma \circ \rho = \chi_\theta \circ \rho.$$

But

$$\chi_\sigma \circ \rho = \chi_\theta \circ \rho \iff R \cap S \leq_A T.$$

[Note: There is a pullback square

$$\begin{array}{ccc} R \cap S & \longrightarrow & S \\ \downarrow & & \downarrow \sigma \\ R & \xrightarrow[\rho]{} & A, \end{array}$$

the classifying arrow of the monomorphism $R \cap S \to A$ being $\chi_\sigma \circ \rho$, and there is a pullback square

$$\begin{tikzcd} R \cap (S \cap T) \arrow[r] \arrow[d] & S \cap T \arrow[d, "\theta"] \\ R \arrow[r, "\rho"'] & A, \end{tikzcd}$$

the classifying arrow of the monomorphism $R \cap (S \cap T) \to A$ being $\chi_\theta \circ \rho$.]

15.7 REMARK If $(\underline{C},\tau)$ is a site and if $\underline{E} = \underline{\mathrm{Sh}}_\tau(\underline{C})$, then $\mathrm{Sub}_{\underline{E}}\, A$ is a locale.

15.8 NOTATION

- Define a monomorphism

$$<\top,\top> : *_{\underline{E}} \to \Omega \times \Omega$$

by consideration of the diagram

$$\begin{tikzcd} *_{\underline{E}} \arrow[r, equal] \arrow[d, "\top"'] & *_{\underline{E}} \arrow[r, equal] \arrow[d, dotted] & *_{\underline{E}} \arrow[d, "\top"] \\ \Omega & \Omega \times \Omega \arrow[l] \arrow[r] & \Omega \end{tikzcd}$$

and denote its classifying arrow by $\cap$, thus

$$\cap : \Omega \times \Omega \to \Omega.$$

- Let $\top_\Omega$ be the composition $\Omega \xrightarrow{!} *_{\underline{E}} \xrightarrow{\top} \Omega$ -- then there is a pullback

square

$$\begin{tikzcd} \Omega \arrow[r, "!"] \arrow[d, "\mathrm{id}_\Omega"'] & *_{\underline{E}} \arrow[d, "\top"] \\ \Omega \arrow[r, "\top_\Omega"'] & \Omega, \end{tikzcd}$$

so $\chi_{id_\Omega} = \top_\Omega$.

- Define a morphism

$$\langle \top_\Omega, id_\Omega\rangle \coprod \langle id_\Omega\, \top_\Omega\rangle : \Omega \coprod \Omega \to \Omega \times \Omega$$

by consideration of the diagram

$$\begin{tikzcd}
\Omega \arrow[r] \arrow[d, "{\langle \top_\Omega, id_\Omega \rangle}"'] & \Omega \coprod \Omega \arrow[d, dotted] & \Omega \arrow[l] \arrow[d, "{\langle id_\Omega, \top_\Omega \rangle}"] \\
\Omega \times \Omega \arrow[r, equal] & \Omega \times \Omega \arrow[r, equal] & \Omega \times \Omega,
\end{tikzcd}$$

factor it per 3.9, hence

$$\Omega \coprod \Omega \xrightarrow{k} M \xrightarrow{m} \Omega \times \Omega,$$

and put $\cup = \chi_m$:

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "m"'] & {}^*\underline{E} \arrow[d, "\top"] \\
\Omega \times \Omega \arrow[r, "\cup"'] & \Omega .
\end{tikzcd}$$

Given monomorphisms

$$\left[\begin{array}{l} S \xrightarrow{\sigma} A \\ T \xrightarrow{\tau} A, \end{array}\right.$$

define a morphism

$$\langle \chi_\sigma, \chi_\tau \rangle : A \to \Omega \times \Omega$$

by consideration of the diagram

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "\chi_\sigma"'] & A \arrow[r, equal] \arrow[d, dotted] & A \arrow[d, "\chi_\tau"] \\ \Omega & \Omega \times \Omega \arrow[l] \arrow[r] & \Omega \end{tikzcd} .$$

15.9 LEMMA Form the pullback square

$$\begin{tikzcd} S \cap T \arrow[r, "!"] \arrow[d, "\sigma \cap \tau"'] & *_{\underline{E}} \arrow[d, "\mathrm{T}"] \\ A \arrow[r, "\chi_{\sigma \cap \tau}"'] & \Omega \end{tikzcd} .$$

Then

$$\chi_{\sigma \cap \tau} = \cap \circ \langle \chi_\sigma, \chi_\tau \rangle.$$

15.10 LEMMA Form the pullback square

$$\begin{tikzcd} S \cup T \arrow[r, "!"] \arrow[d, "\sigma \cup \tau"'] & *_{\underline{E}} \arrow[d, "\mathrm{T}"] \\ A \arrow[r, "\chi_{\sigma \cup \tau}"'] & \Omega \end{tikzcd} .$$

Then

$$\chi_{\sigma \cup \tau} = \cup \circ \langle \chi_\sigma, \chi_\tau \rangle.$$

15.11 NOTATION Let $(\leq_\Omega, e_\Omega)$ be the equalizer of

$$\Omega \times \Omega \underset{pr_1}{\overset{\cap}{\rightrightarrows}} \Omega,$$

thus

$$\leq_\Omega \xrightarrow{e_\Omega} \Omega \times \Omega \underset{pr_1}{\overset{\cap}{\rightrightarrows}} \Omega,$$

and let $\Rightarrow : \Omega \times \Omega \to \Omega$ be its classifying arrow, thus

$$\begin{tikzcd}
\leq_\Omega \arrow[r, "!"] \arrow[d, "e_\Omega"'] & *_{\underline{E}} \arrow[d, "\top"] \\
\Omega \times \Omega \arrow[r, "\Rightarrow"'] & \Omega .
\end{tikzcd}$$

15.12 LEMMA Form the pullback square

$$\begin{tikzcd}
T^S \arrow[r, "!"] \arrow[d, "\tau^\sigma"'] & *_{\underline{E}} \arrow[d, "\top"] \\
A \arrow[r, "\chi_{\tau^\sigma}"'] & \Omega
\end{tikzcd}$$

Then

$$\chi_{\tau^\sigma} = \Rightarrow \circ \langle \chi_\sigma, \chi_\tau \rangle.$$

PROOF Consider the diagram

$$\begin{tikzcd}
P \arrow[r, "v"] \arrow[d, "u"'] & \leq_\Omega \arrow[r, "!"] \arrow[d] & *_{\underline{E}} \arrow[d, "T"] \\
A \arrow[r, "<\chi_\sigma{,}\chi_\tau>"'] & \Omega \times \Omega \arrow[r, "=>"'] \arrow[d, shift right=2, "pr_1"'] \arrow[d, shift left=2, "\cap"] & \Omega, \\
& \Omega &
\end{tikzcd}$$

where the squares are pullbacks and

$$\left[\begin{aligned} pr_1 \circ <\chi_\sigma,\chi_\tau> &= \chi_\sigma \\ \cap \circ <\chi_\sigma,\chi_\tau> &= \chi_{\sigma\cap\tau}. \end{aligned}\right.$$

By construction, the classifying arrow of u is $=> \circ <\chi_\sigma,\chi_\tau>$ and the claim is that $P = T^S$ (cf. 15.6) or still, that u is the equalizer of χ_σ and $\chi_{\sigma\cap\tau}$ or still, that u is the equalizer of $pr_1 \circ <\chi_\sigma,\chi_\tau>$ and $\cap \circ <\chi_\sigma,\chi_\tau>$. But

$$\begin{aligned} pr_1 \circ <\chi_\sigma,\chi_\tau> \circ u &= pr_1 \circ e_\Omega \circ v \\ &= \cap \circ e_\Omega \circ v \\ &= \cap \circ <\chi_\sigma\ \chi_\tau> \circ u. \end{aligned}$$

And if

$$pr_1 \circ <\chi_\sigma,\chi_\tau> \circ x = \cap \circ <\chi_\sigma,\chi_\tau> \circ x \quad (x:X \to P),$$

then

$$\langle \chi_\sigma, \chi_\tau \rangle \circ x = e_\Omega \circ y \qquad (y : X \to \leq_\Omega)$$

from which a unique $z:X \to P$ such that $\begin{cases} x = u \circ z \\ y = v \circ z. \end{cases}$

15.13 NOTATION

- Denote the classifying arrow of the monomorphism $\emptyset_{\underline{E}} \xrightarrow{!} *_{\underline{E}}$ by $\perp$.

Schematically:

$$\begin{tikzcd} \emptyset_{\underline{E}} \arrow[r, "!"] \arrow[d, "!"'] & *_{\underline{E}} \arrow[d, "\top"] \\ *_{\underline{E}} \arrow[r, "\perp"'] & \Omega \end{tikzcd}$$

- Denote the classifying arrow of the monomorphism $*_{\underline{E}} \xrightarrow{\perp} \Omega$ by $\neg$.

Schematically:

$$\begin{tikzcd} *_{\underline{E}} \arrow[r, "!"] \arrow[d, "\perp"'] & *_{\underline{E}} \arrow[d, "\top"] \\ \Omega \arrow[r, "\neg"'] & \Omega \end{tikzcd}$$

15.14 LEMMA Given a monomorphism $S \xrightarrow{\sigma} A$, form the pullback square

$$\begin{tikzcd}
\neg S \arrow[r, "!"] \arrow[d, "\neg\sigma"'] & *_{\underline{E}} \arrow[d, "\top"] \\
A \arrow[r, "\chi_{\neg\sigma}"'] & \Omega .
\end{tikzcd}$$

Then

$$\chi_{\neg\sigma} = \neg \circ \chi_\sigma.$$

[Note: The monomorphism $\neg S \xrightarrow{\neg\sigma} A$ represents the pseudocomplement of $[\sigma]$ in the Heyting algebra $\mathrm{Sub}_{\underline{E}}\, A$. E.g.: Take $A = \Omega$, $S = *_{\underline{E}}$, $\sigma = \top$ -- then

$$\chi_{\neg\top} = \neg \circ \chi_\top = \neg \circ \mathrm{id}_\Omega = \neg = \chi_\perp.$$

Therefore $\perp$ is the pseudocomplement of $\top$ in $\mathrm{Sub}_{\underline{E}}\, \Omega$.]

15.15 DEFINITION A topos $\underline{E}$ is a <u>boolean topos</u> if for every $A \in \mathrm{Ob}\, \underline{E}$, the Heyting algebra $\mathrm{Sub}_{\underline{E}}\, A$ is a boolean algebra.

15.16 THEOREM A topos $\underline{E}$ is a boolean topos iff $\mathrm{Sub}_{\underline{E}}\, \Omega$ is a boolean algebra.

15.17 REMARK If $\underline{E}$ is a boolean topos, then for every $A \in \mathrm{Ob}\, \underline{E}$, the topos $\underline{E}/A$ (cf. 15.1) is a boolean topos.

15.18 LEMMA A topos $\underline{E}$ is a boolean topos iff $\neg \circ \neg = \mathrm{id}_\Omega$.

[To see that the condition is sufficient, consider a monomorphism $S \xrightarrow{\sigma} A$ --

then

$$\chi_{\neg\neg\sigma} = \neg \circ \neg \circ \chi_\sigma = \chi_\sigma \quad \text{(cf. 15.14)},$$

so

$$\neg\neg\sigma \sim_A \sigma \quad \text{(cf. 6.11)}.$$

Therefore $\mathrm{Sub}_{\underline{E}}\, A$ is a boolean algebra (cf. 8.12 and 8.15).

15.19 LEMMA A topos $\underline{E}$ is a boolean topos iff the pair

$$(*_{\underline{E}} \coprod *_{\underline{E}}, \mathrm{in}_1)$$

is a subobject classifier.

[To see that the condition is sufficient, define an isomorphism

$$\top \coprod \bot : *_{\underline{E}} \coprod *_{\underline{E}} \longrightarrow \Omega$$

by consideration of the diagram

$$\begin{tikzcd}
*_{\underline{E}} \arrow[r, "\mathrm{in}_1"] \arrow[d, "\top"'] & *_{\underline{E}} \coprod *_{\underline{E}} \arrow[d, dotted] & *_{\underline{E}} \arrow[l, "\mathrm{in}_2"'] \arrow[d, "\bot"] \\
\Omega \arrow[r, equal] & \Omega \arrow[r, equal] & \Omega
\end{tikzcd}$$

Then the arrow $\neg : \Omega \to \Omega$ corresponds to the involution which interchanges the factors of $*_{\underline{E}} \coprod *_{\underline{E}}$.]

15.20 EXAMPLE Let $\underline{C}$ be a small category -- then the topos $\hat{\underline{C}}$ is a boolean topos iff $\underline{C}$ is a groupoid (in particular, $\underline{SET} \approx \hat{\underline{1}}$ is a boolean topos).

[Note: Let G be a group -- then the category of right G-sets is a boolean topos (cf. 7.8).]

15.21 EXAMPLE Let X be a topological space and take $\underline{Sh}(X)$ per 11.29 -- then $\underline{Sh}(X)$ is a boolean topos iff every open subset of X is closed.

[In fact, $\underline{Sh}(X)$ is a boolean topos iff $\forall\ U \in O(X)$, $U \cup \neg U = X$. But $\neg U = \mathrm{int}(X - U)$ (cf. 8.11), thus $\underline{Sh}(X)$ is a boolean topos iff $\forall\ U \in O(X)$, $X - U = \mathrm{int}(X - U)$ or still, iff $\forall\ U \in O(X)$, $X - U \in O(X)$.]

[Note: This condition is met if X is discrete, the converse being true if X is in addition T_0. For if every open set is closed, then every closed set is open, so $X:T_0 \Rightarrow X:T_2$. But then every subset is a union of closed subsets, hence is a union of open subsets, hence is open.]

15.22 DEFINITION A topos $\underline{E}$ is said to satisfy the axiom of choice if every epimorphism in $\underline{E}$ has a section.

15.23 REMARK If $\underline{E}$ satisfies the axiom of choice, then for every $A \in \mathrm{Ob}\ \underline{E}$, the topos $\underline{E}/A$ (cf. 15.1) satifies the axiom of choice.

15.24 THEOREM Let $\underline{E}$ be a topos. Assume: $\underline{E}$ satisfies the axiom of choice -- then $\underline{E}$ is a boolean topos.

15.25 EXAMPLE Let G be a group -- then the category of right G-sets is a boolean topos (cf. 15.20) but it satisfies the axiom of choice iff G is trivial.

[Suppose that G is nontrivial and view G as operating to the right on itself.

Let $\{*\}$ be the final right G-set -- then $G \xrightarrow{!} \{*\}$ is an epimorphism but there is no morphism $\{*\} \to G$ of right G-sets.]

15.26 EXAMPLE Let L be a locale and take $\underline{\mathrm{Sh}}(L)$ per 11.29 -- then the following conditions are equivalent.

(1) $\underline{\mathrm{Sh}}(L)$ satisfies the axiom of choice.

(2) $\underline{\mathrm{Sh}}(L)$ is a boolean topos.

(3) L is a boolean algebra.

[Note: Recall that by definition L is a Heyting algebra whose underlying category is complete and cocomplete.

15.27 DEFINITION Let $\underline{C}$ be a category with a final object $*_{\underline{C}}$ -- then an object X is said to be <u>subfinal</u> if the arrow $X \xrightarrow{!} *_{\underline{C}}$ is a monomorphism.

15.28 LEMMA Suppose that the topos $\underline{E}$ satisfies the axiom of choice -- then there is a set of subfinal objects of $*_{\underline{E}}$ which constitute a separating set for $\underline{E}$.

§16. *TOPOLOGIES*

Let $\underline{E}$ be a topos, (Ω,τ) its subobject classifier.

16.1 DEFINITION A Lawvere-Tierney topology on $\underline{E}$ is a morphism $j:\Omega \to \Omega$ in $\underline{E}$ with the following properties.

(1) $j \circ \tau = \tau$.

(2) $j \circ j = j$.

(3) $j \circ \cap = \cap \circ (j \times j)$.

16.2 EXAMPLE $\mathrm{id}_\Omega:\Omega \to \Omega$ is a Lawvere-Tierney topology on $\underline{E}$.

16.3 EXAMPLE $\tau_\Omega:\Omega \to \Omega$ is a Lawvere-Tierney topology on $\underline{E}$.

16.4 EXAMPLE $\neg \circ \neg:\Omega \to \Omega$ is a Lawvere-Tierney topology on $\underline{E}$.

16.5 THEOREM Let $\underline{C}$ be a small category -- then there is a one-to-one correspondence between the set of Grothendieck topologies on $\underline{C}$ and the set of Lawvere-Tierney topologies on $\hat{\underline{C}}$:

$$\left[\begin{array}{l} \tau \longrightarrow j_\tau \\ \\ j \longrightarrow \tau_j. \end{array}\right.$$

PROOF Recall from 7.7 that

$$\Omega:\underline{C}^{OP} \to \underline{SET}$$

is defined on an object X by letting ΩX be the set of all subfunctors of h_X and on a morphism $f:Y \to X$ by letting $\Omega f:\Omega X \to \Omega Y$ operate via the pullback square

$$\begin{tikzcd} \Omega f(G) \arrow[r] \arrow[d] & G \arrow[d] \\ h_Y \arrow[r, "h_f"'] & h_X. \end{tikzcd}$$

- If τ is a Grothendieck topology on $\underline{C}$, then $\tau \in M(\Omega)$ and if $j_\tau = \chi_\tau$, then j_τ is a Lawvere-Tierney topology on $\hat{\underline{C}}$.

- If $j:\Omega \to \Omega$ is a Lawvere-Tierney topology on $\hat{\underline{C}}$ and if

$$\begin{tikzcd} \tau_j \arrow[r, "!"] \arrow[d] & *_{\underline{E}} \arrow[d, "T"] \\ \Omega \arrow[r, "j"'] & \Omega \end{tikzcd} \qquad \text{(cf. 6.12)},$$

then τ_j is a Grothendieck topology on $\underline{C}$.

[Note: These constructions are mutually inverse.]

16.6 EXAMPLE Let L be a locale -- then Ωx is the set of all subfunctors of h_x or still, Ωx is the set of all sieves over x. Let $x \to \tau_x$ be the Grothendieck topology τ on L determined by the sieves that cover x (cf. 10.4) -- then $j_\tau:\Omega \to \Omega$ is the natural transformation

$$(j_\tau)_x:\Omega x \to \Omega x,$$

where

$$(j_\tau)_x \mathcal{S} = \{y \leq x : y = \bigvee_{s \in \mathcal{S}} (y \wedge s)\}.$$

16.7 DEFINITION Suppose that $j:\Omega \to \Omega$ is a Lawvere-Tierney topology on $\underline{E}$. Let $(B,f) \in M(A)$ — then (B,f) is <u>j-dense in A</u> if $j \circ \chi_f = \tau_A$.

16.8 EXAMPLE Let $(\underline{C},\tau)$ be a site and let G be a subfunctor of h_X — then (G,i_G) is j_τ-dense in h_X iff $G \in \tau_X$.

16.9 DEFINITION Suppose that $j:\Omega \to \Omega$ is a Lawvere-Tierney topology on $\underline{E}$ — then an $A \in \text{Ob } \underline{E}$ is a <u>j-sheaf</u> if for every $B \in \text{Ob } \underline{E}$, for every j-dense (S,s) in B, and for every $f \in \text{Mor}(S,A)$, there exists a unique $g \in \text{Mor}(B,A)$ such that $g \circ s = f$:

$$\begin{tikzcd} S \arrow[r, "s"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ A \arrow[r, equal] & A \end{tikzcd}$$

I.e.: The precomposition map

$$s^*:\text{Mor}(B,A) \to \text{Mor}(S,A)$$

is bijective.

16.10 EXAMPLE Since j is idempotent and $\underline{E}$ is finitely complete, j splits: $j = i \circ r$ $(r \circ i = \text{id})$, where

$$\Omega_j = \text{eq}(j,\text{id}_\Omega) \text{ and } \left[\begin{array}{l} i:\Omega_j \to \Omega \\ \\ r:\Omega \to \Omega_j. \end{array}\right.$$

But Ω is injective (cf. 14.18), thus Ω_j is injective (being a retract of Ω), and the claim is that Ω_j is a j-sheaf. In fact, the existence of the relevant liftings is then immediate which leaves the uniqueness... .

Write $\underline{\mathrm{Sh}}_j(\underline{E})$ for the full subcategory of $\underline{E}$ whose objects are the j-sheaves.

16.11 EXAMPLE Take $j = \mathrm{id}_\Omega$ -- then $\underline{\mathrm{Sh}}_j(\underline{E}) = \underline{E}$.

16.12 EXAMPLE Take $j = \tau_\Omega$ -- then $\underline{\mathrm{Sh}}_j(\underline{E})$ is the full subcategory of $\underline{E}$ whose objects are the final objects.

16.13 THEOREM Fix a Lawvere-Tierney topology $j:\Omega \to \Omega$ on $\underline{E}$ — then the inclusion $\iota_j:\underline{\mathrm{Sh}}_j(\underline{E}) \to \underline{E}$ admits a left adjoint $\underline{a}_j:\underline{E} \to \underline{\mathrm{Sh}}_j(\underline{E})$ that preserves finite limits.

<u>N.B.</u> Let $\mathcal{W}_j$ be the class of morphisms in $\underline{E}$ rendered invertible by $\underline{a}_j$ -- then the localization $\mathcal{W}_j^{-1}\underline{E}$ is equivalent to $\underline{\mathrm{Sh}}_j(\underline{E})$ (cf. 11.20).

16.14 LEMMA Let $f:B \to A$ be a monomorphism -- then (B,f) is j-dense in A iff $\underline{a}_j f$ is an isomorphism.

16.15 SCHOLIUM Let $\underline{C}$ be a small category. Suppose that $j:\Omega \to \Omega$ is a Lawvere-Tierney topology on $\hat{\underline{C}}$ and let τ_j be the associated Grothendieck topology on $\underline{C}$ (cf. 16.5) -- then

$$\underline{\mathrm{Sh}}_j(\hat{\underline{C}}) = \underline{\mathrm{Sh}}_{\tau_j}(\underline{C}).$$

[Viewing $\underline{\mathrm{Sh}}_j(\hat{\underline{C}})$ as an element $\underline{S}$ of $\underline{S}_{\underline{C}}$ (cf. 11.9), introduce $\tau \in \tau_{\underline{C}}$ per 11.4, thus τ_X is the set of those subfunctors $G \xrightarrow{i_G} h_X$ such that $\underline{a}_j i_G$ is an isomorphism or still, those subfunctors $G \xrightarrow{i_G} h_X$ such that (G,i_G) is j-dense in h_X (cf. 16.14).

On the other hand, a subfunctor $G \xrightarrow{i_G} h_X$ is j_{τ_j}-dense in h_X iff $G \in (\tau_j)_X$ (cf. 16.8). But $j_{\tau_j} = j$, hence $\tau_X = (\tau_j)_X$, and therefore $\tau = \tau_j$. Since

$$\underline{Sh}_j(\hat{\underline{C}}) = \underline{Sh}_\tau(\underline{C}) \quad \text{(cf. 12.13)},$$

it follows that

$$\underline{Sh}_j(\hat{\underline{C}}) = \underline{Sh}_{\tau_j}(\underline{C}).$$

[Note: Consequently, $\forall\ \tau \in \tau_{\underline{C}}$,

$$\underline{Sh}_\tau(\underline{C}) = \underline{Sh}_{\tau_{j_\tau}}(\underline{C}) = \underline{Sh}_{j_\tau}(\hat{\underline{C}}).]$$

16.16 REMARK Let $\underline{E}$ be a topos -- then it can be shown that the Lawvere-Tierney topologies on $\underline{E}$ are in a one-to-one correspondence with the reflective subcategories of $\underline{E}$ whose reflector preserves finite limits (cf. 12.13).

16.17 THEOREM Fix a Lawvere-Tierney topology $j:\Omega \to \Omega$ on $\underline{E}$ -- then $\underline{Sh}_j(\underline{E})$ is a topos.

[Note: The pair $(\Omega_j, \top_j)$ is a subobject classifier for $\underline{Sh}_j(\underline{E})$. Here (cf. 16.10)

$$\begin{tikzcd}
\Omega_j \arrow[r, "i"] & \Omega \arrow[r, shift left, "j"] \arrow[r, shift right, "\mathrm{id}_\Omega"'] & \Omega \\
{}^*\underline{E} \arrow[u, "\top_j"] \arrow[r, equal] & {}^*\underline{E} \arrow[u, "\top"'] &
\end{tikzcd} \qquad (j \circ \top = \top).]$$

16.18 EXAMPLE Take $j = \neg \circ \neg$ -- then $\underline{Sh}_{\neg \circ \neg}(\underline{E})$ is a boolean topos.

§17. *GEOMETRIC MORPHISMS*

Let $\underline{C}$, $\underline{D}$ be finitely complete categories.

17.1 DEFINITION A <u>geometric morphism</u> $f:\underline{C} \to \underline{D}$ is a pair (f^*,f_*), where

$$\left[\begin{array}{l} f^*:\underline{D} \to \underline{C} \\ \\ f_*:\underline{C} \to \underline{D} \end{array}\right.$$

are functors and

$$\left[\begin{array}{l} f^* \text{ is a left adjoint for } f_* \\ \\ f^* \text{ preserves finite limits.} \end{array}\right.$$

[Note: The second condition on f^* is automatic if f^* is a right adjoint.]

17.2 EXAMPLE Let X,Y be topological spaces and let $f:X \to Y$ be a continuous function -- then f induces a geometric morphism $f:\underline{Sh}(X) \to \underline{Sh}(Y)$, where $f_*:\underline{Sh}(X) \to \underline{Sh}(Y)$ is "direct image" and $f^*:\underline{Sh}(Y) \to \underline{Sh}(X)$ is "inverse image".

[Note: Here $\underline{Sh}(X)$, $\underline{Sh}(Y)$ are taken per the canonical Grothendieck topology on $O(X)$, $O(Y)$ (cf. 11.29).]

17.3 EXAMPLE Let G,H be groups and let $\phi:G \to H$ be a homomorphism -- then ϕ induces a geometric morphism ϕ from right G-sets to right H-sets, i.e.,

$$\phi:[\underline{G}^{OP},\underline{SET}] \to [\underline{H}^{OP},\underline{SET}] \qquad \text{(cf. 7.8).}$$

[There are three functors

$$\begin{tikzcd}
{[\underline{G}^{OP},\underline{SET}]} \arrow[r, shift left=6, "\phi_!"] \arrow[r, shift right=6, "\phi_*"'] & {[\underline{H}^{OP},\underline{SET}]}, \arrow[l, "\phi^*"']
\end{tikzcd}$$

where

$$\phi_! \dashv \phi^* \dashv \phi_*.$$

- Definition of ϕ^*: Given a right H-set Y, $\phi^*(Y) = Y$ with the right G-action induced by ϕ.

- Definition of ϕ_*: Given a right G-set X, $\phi_*(X) = \mathrm{Hom}_G\,(H,X)$, the set of G-equivariant functions $H \to X$.

- Definition of $\phi_!$: Given a right G-set X, $\phi_!(X) = X \otimes_G H$, the cartesian product $X \times H$ modulo the equivalence relation $(x\cdot g,h) \sim (x,\phi(g)\cdot h)$.]

17.4 EXAMPLE Take $\underline{C} = \underline{\mathrm{SISET}}$, $\underline{D} = \underline{\mathrm{CGH}}$ and consider the adjoint pair $(|\ |,\sin)$:

$$\begin{cases} |\ |:\underline{\mathrm{SISET}} \to \underline{\mathrm{CGH}} \\ \sin:\underline{\mathrm{CGH}} \to \underline{\mathrm{SISET}}. \end{cases}$$

Then $|\ |$ preserves finite limits, hence $(|\ |,\sin)$ is a geometric morphism $\underline{\mathrm{SISET}} \to \underline{\mathrm{CGH}}$.

17.5 EXAMPLE Let $\underline{E}$ be a topos that has arbitrary copowers of $*_{\underline{E}}$. Define a functor $\Gamma_*:\underline{E} \to \underline{\mathrm{SET}}$ by stipulating that

$$\Gamma_*A = \mathrm{Mor}(*_{\underline{E}},A)$$

and define a functor $\Gamma^*:\underline{\mathrm{SET}} \to \underline{E}$ by stipulating that

$$\Gamma^*S = \coprod_{s\in S} *_{\underline{E}}.$$

Then (Γ^*,Γ_*) is an adjoint pair and Γ^* preserves finite limits (cf. 18.2). Therefore (Γ^*,Γ_*) is a geometric morphism $\underline{E} \to \underline{\mathrm{SET}}$.

17.6 EXAMPLE Let $(\underline{C},\tau)$ be a site -- then the adjoint pair $(\underline{a}_\tau,\iota_\tau)$ is a geometric morphism $\underline{Sh}_\tau(\underline{C}) \to \hat{\underline{C}}$ (cf. 11.14).

17.7 EXAMPLE Let $\underline{E}$ be a topos, $j:\Omega \to \Omega$ a Lawvere-Tierney topology on $\underline{E}$ -- then the adjoint pair $(\underline{a}_j,\iota_j)$ is a geometric morphism $\underline{Sh}_j(\underline{E}) \to \underline{E}$.

17.8 EXAMPLE Let $\underline{E}$ be a topos. Suppose that $f:A \to B$ is a morphism in $\underline{E}$ -- then $f^*:\underline{E}/B \to \underline{E}/A$ has a left adjoint $f_!:\underline{E}/A \to \underline{E}/B$ and a right adjoint $f_*:\underline{E}/A \to \underline{E}/B$ (cf. 15.3). Therefore the adjoint pair (f^*,f_*) is a geometric morphism $\underline{E}/A \to \underline{E}/B$.

17.9 EXAMPLE Let $\underline{I},\underline{J}$ be small categories and let $\underline{S}$ be a complete and cocomplete category. Suppose that $F:\underline{I} \to \underline{J}$ is a functor -- then by the theory of Kan extensions,

$$F^*:[\underline{J},\underline{S}] \to [\underline{I},\underline{S}]$$

has a right adjoint

$$F_*:[\underline{I},\underline{S}] \to [\underline{J},\underline{S}]$$

and a left adjoint

$$F_!:[\underline{I},\underline{S}] \to [\underline{J},\underline{S}].$$

Therefore F^* preserves limits and the adjoint pair (F^*,F_*) is a geometric morphism $[\underline{I},\underline{S}] \to [\underline{J},\underline{S}]$.

17.10 EXAMPLE Let L,M be locales and let $f:L \to M$ be a localic arrow (cf. 9.6) -- then f induces a geometric morphism $\underline{Sh}(L) \to \underline{Sh}(M)$ (taken per the canonical Grothendieck topology on L,M (cf. 11.29)), call it $\bar{f}$ to forgo any possibility of confusion.

[Proceed as follows. The functor $f^*:M \to L$ gives rise to a functor $f^{**}:\hat{L} \to \hat{M}$ (technically, $f^{**} = ((f^*)^{OP})^*$), which then restricts to a functor $\bar{f}_*:\underline{Sh}(L) \to \underline{Sh}(M)$.

On the other hand, f^{**} has a left adjoint $f_!^*:\hat{M} \to \hat{L}$ (take $\underline{S} = \underline{SET}$ in 17.9). Accordingly, denote the composite

$$\underline{Sh}(M) \xrightarrow{\iota_\tau} \hat{M} \xrightarrow{f_!^*} \hat{L} \xrightarrow{\underline{a}_\tau} \underline{Sh}(L)$$

by f^* -- then f^* is a left adjoint for f_*. Proof: Given $F \in \text{Ob}\ \underline{Sh}(L)$, $G \in \text{Ob}\ \underline{Sh}(M)$,

$$\begin{aligned}
\text{Mor}(f^*G,F) &\approx \text{Mor}(\underline{a}_\tau f_!^* \iota_\tau G,F) \\
&\approx \text{Mor}(f_!^* \iota_\tau G,\iota_\tau F) \\
&\approx \text{Mor}(\iota_\tau G,f^{**}\iota_\tau F) \\
&\approx \text{Mor}(\iota_\tau G,\iota_\tau f_* F) \\
&\approx \text{Mor}(G,f_* F).
\end{aligned}$$

The final point is that f^* preserves finite limits. Since this is true of ι_τ and $\underline{a}_\tau$, matters reduce to verifying it for $f_!^*$ (which is not an a priori property of Kan extensions...).]

17.11 DEFINITION Let $f,g:\underline{C} \to \underline{D}$ be geometric morphisms -- then a geometric transformation $\xi:f \to g$ is a natural transformation $f^* \to g^*$.

[Note: Since

$$\left[\begin{array}{l} f^* \dashv f_* \\ \\ g^* \dashv g_*, \end{array}\right.$$

natural transformations $f^* \to g^*$ correspond bijectively to natural transformations $g_* \to f_*$.]

§18. *GROTHENDIECK TOPOSES*

Let $\underline{E}$ be a topos.

18.1 DEFINITION $\underline{E}$ is said to be <u>defined over SET</u> if $\underline{E}$ admits a geometric morphism $\underline{E} \to \underline{SET}$.

18.2 THEOREM $\underline{E}$ is defined over $\underline{SET}$ iff $\underline{E}$ has arbitrary copowers of $*_{\underline{E}}$.

PROOF If $f:\underline{E} \to \underline{SET}$ is a geometric morphism, then f^* preserves finite limits, thus in particular $f^* * \approx *_{\underline{E}}$. Therefore, since f^* preserves colimits, for any set S,

$$f^*S \approx f^* \coprod_S * \approx \coprod_S f^* * \approx \coprod_S *_{\underline{E}}.$$

Turning to the converse, define $\Gamma_*:\underline{E} \to \underline{SET}$ by

$$\Gamma_* A = \mathrm{Mor}(*_{\underline{E}}, A)$$

and define $\Gamma^*:\underline{SET} \to \underline{E}$ by

$$\Gamma^* S = \coprod_S *_{\underline{E}} \quad (\Gamma^*\emptyset \approx \emptyset_{\underline{E}}).$$

Here $\Gamma^*\phi$ $(\phi:S \to T)$ is the unique arrow in $\underline{E}$ such that $\forall\ s \in S$, $\Gamma^*\phi \circ \mathrm{in}_s = \mathrm{in}_{\phi(s)}$:

$$\begin{tikzcd}
*_{\underline{E}} \arrow[r, "\mathrm{in}_s"] \arrow[rr, bend right, "\mathrm{in}_{\phi(s)}"'] & \coprod_S *_{\underline{E}} \arrow[r, "\Gamma^*\phi"] & \coprod_T *_{\underline{E}}
\end{tikzcd}.$$

It is clear that (Γ^*, Γ_*) is an adjoint pair, so the issue is whether Γ^* preserves finite limits and for this one need only show that Γ^* preserves finite products and equalizers.

• By construction, Γ^* sends final objects to final objects. Suppose now that S and T are sets. Distinguish two cases: (1) S is empty or T is empty; (2) S is not empty and T is not empty. If S is empty, then $S \times T = \emptyset \times T = \emptyset$ and $\Gamma^*(\emptyset \times T) = \Gamma^*\emptyset \approx \emptyset_{\underline{E}}$, while $\Gamma^*\emptyset \times \Gamma^*T \approx \emptyset_{\underline{E}} \times \Gamma^*T \approx \emptyset_{\underline{E}}$ (cf. 5.13 and 5.14). If neither S nor T is empty, then

$$\Gamma^*(S \times T) = \coprod_{S \times T} *_{\underline{E}}.$$

On the other hand,

$$\begin{aligned}\Gamma^*S \times \Gamma^*T &= \coprod_S *_{\underline{E}} \times \coprod_T *_{\underline{E}} \\ &\approx \coprod_S \left(\coprod_T *_{\underline{E}}\right) \\ &\approx \coprod_{S \times T} *_{\underline{E}}.\end{aligned}$$

• Let $S \underset{\psi}{\overset{\phi}{\rightrightarrows}} T$ be arrows in $\underline{\text{SET}}$ and let $K = \text{eq}(\phi,\psi)$, so $K \xrightarrow{\kappa} S \underset{\psi}{\overset{\phi}{\rightrightarrows}} T$.

Put $A = \Gamma^*S$, $B = \Gamma^*T$, $C = \Gamma^*K$, $f = \Gamma^*\phi$, $g = \Gamma^*\psi$, $k = \Gamma^*\kappa$ -- then the claim is that

$$C \xrightarrow{k} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

is an equalizer in $\underline{E}$. Thus consider a morphism $u:E \to A$ and $\forall\ s \in S$, define E_s by the pullback square

$$\begin{tikzcd} E_s \arrow[r, "!"] \arrow[d, "i_s"'] & *_{\underline{E}} \arrow[d, "in_s"] \\ E \arrow[r, "u"'] & A \end{tikzcd} .$$

Then i_s is a monomorphism (this being the case of in_s) and since $\{*_{\underline{E}} \xrightarrow{in_s} A : s \in S\}$ is a coproduct diagram in $\underline{E}$, the same is true of $\{E_s \xrightarrow{i_s} E : s \in S\}$ (cf. 14.13). I.e.:

$$E \approx \coprod_{s \in S} E_s.$$

If u equalizes f and g ($\Rightarrow f \circ u = g \circ u$), then this time

$$E \approx \coprod_{t \in T} E_t.$$

And there are monomorphisms

$$\left[\begin{array}{l} E_s \longrightarrow E_{\phi(s)} \\ \\ E_s \longrightarrow E_{\psi(s)} \end{array}\right. \qquad (s \in S).$$

E.g.: Given the situation

$$\begin{tikzcd} E_s \arrow[rrd, "!"] \arrow[rd, "i_s"'] & & \\ & E_{\phi(s)} \arrow[r, "!"] \arrow[d, "i_{\phi(s)}"'] & *_{\underline{E}} \arrow[d, "in_{\phi(s)}"] \\ & E \arrow[r, "f \circ u"'] & B \end{tikzcd} ,$$

$$f \circ u \circ i_s = f \circ in_s \circ ! = in_{\phi(s)} \circ !,$$

from which a unique arrow $\lambda_s : E_s \to E_{\phi(s)}$ such that $i_s = i_{\phi(s)} \circ \lambda_s$. Moreover, λ_s is a monomorphism (because i_s is a monomorphism). Proceeding, the intersection $E_{\phi(s)} \cap E_{\psi(s)}$ is officially defined by the pullback square

$$\begin{tikzcd} E_{\phi(s)} \cap E_{\psi(s)} \arrow[r] \arrow[d] & E_{\psi(s)} \arrow[d] \\ E_{\phi(s)} \arrow[r] & E \end{tikzcd} \qquad \text{(cf. 2.16)}$$

but the answer is the same if instead we use the pullback square

$$\begin{tikzcd} E_{\phi(s)} \cap E_{\psi(s)} \arrow[r] \arrow[d] & E_{\psi(s)} \arrow[d] \\ E_{\phi(s)} \arrow[r] & E_{\phi(s)} \coprod E_{\psi(s)}. \end{tikzcd}$$

The data provides us with a monomorphism

$$E_s \to E_{\phi(s)} \cap E_{\psi(s)} \quad (s \in S)$$

and if $\phi(s) \neq \psi(s)$, then $E_{\phi(s)} \cap E_{\psi(s)} \approx \emptyset_{\underline{E}}$, hence $E_s \approx \emptyset_{\underline{E}}$. Consequently,

$$E \approx \coprod_{s \in K} E_s$$

and $u : E \to A$ factors through k (uniquely).

[Note: The geometric morphism (Γ^*, Γ_*) extends to a geometric morphism

$$\underline{SIE} = [\underline{\Delta}^{OP}, \underline{E}] \to [\underline{\Delta}^{OP}, \underline{SET}] = \underline{SISET}$$

denoted by the same symbol.

- Define

$$\Gamma^*:\underline{\mathrm{SISET}} \to \underline{\mathrm{SI}E}$$

by

$$(\Gamma^*K)_n = \coprod_{K_n} *_{\underline{E}}\,.$$

- Define

$$\Gamma_*:\underline{\mathrm{SI}E} \to \underline{\mathrm{SISET}}$$

by

$$(\Gamma_*X)_n = \mathrm{Mor}(*_{\underline{E}},X_n).]$$

18.3 LEMMA Suppose that $\underline{E}$ has arbitrary copowers of $*_{\underline{E}}$. Let $A \in \mathrm{Ob}\,\underline{E}$ and let $\{B_i \xrightarrow{f_i} A : i \in I\} \subset M(A)$ -- then $\coprod_{i\in I} B_i$ exists.

PROOF First of all, the copower $\coprod_I A$ exists. In fact,

$$A \times \coprod_I *_{\underline{E}} \approx \coprod_I A \times *_{\underline{E}} \approx \coprod_I A.$$

Next, for each $i \in I$, let χ_i be the classifying arrow of (B_i,f_i) in A:

$$\begin{tikzcd} B_i \arrow[r, "!"] \arrow[d, "f_i"'] & *_{\underline{E}} \arrow[d, "\top"] \\ A \arrow[r, "\chi_i"'] & \Omega \end{tikzcd}$$

Determine $\chi:\coprod_I A \to \Omega$ via the χ_i $(\chi \circ \mathrm{in}_i = \chi_i)$ and form the pullback square

$$\begin{tikzcd} B \arrow[r, "!"] \arrow[d, "f"'] & *_{\underline{E}} \arrow[d, "\top"] \\ \coprod_I A \arrow[r, "\chi"'] & \Omega \end{tikzcd}$$ (cf. 6.12).

Then for each $i \in I$, there is a unique arrow $g_i : B_i \to B$ such that the diagram

$$\begin{tikzcd} B_i \arrow[r, "g_i"] \arrow[d, "f_i"'] & B \arrow[r, "!"] \arrow[d, "f"] & *_{\underline{E}} \arrow[d, "\top"] \\ A \arrow[r, "in_i"'] & \coprod_I A \arrow[r, "\chi"'] & \Omega \end{tikzcd}$$

commutes (so g_i is necessarily a monomorphism). Inspection of the rectangle and the right hand square then implies that the left hand square

$$\begin{tikzcd} B_i \arrow[r, "g_i"] \arrow[d, "f_i"'] & B \arrow[d, "f"] \\ A \arrow[r, "in_i"'] & \coprod_I A \end{tikzcd}$$

is a pullback. Since $\{A \xrightarrow{in_i} \coprod_I A : i \in I\}$ is a coproduct diagram, the same is true of $\{B_i \xrightarrow{g_i} B : i \in I\}$ (cf. 14.13), hence $\coprod_{i \in I} B_i$ exists.

18.4 APPLICATION Under the preceding hypotheses, the copower $\coprod_I A$ exists (sic), as does the power $\prod_I A$:

$$A^{\coprod_I *_{\underline{E}}} \approx \prod_I A^{*_{\underline{E}}} \approx \prod_I A.$$

18.5 EXAMPLE Suppose that $\underline{E}$ has arbitrary copowers of $*_{\underline{E}}$ -- then it does not follow that $\underline{E}$ has coproducts.

[Let $\underline{E}$ be the full subcategory of $[Z^{OP}, \underline{SET}]$ whose objects are the right Z-sets S with the property that multiplication by n is the identity on S for some positive integer n -- then $\underline{E}$ is a topos and has arbitrary copowers of $*_{\underline{E}}$ but $\underline{E}$ does not have coproducts (e.g., one cannot construct $\coprod_{n\geq 1} Z/nZ$).]

18.6 DEFINITION Let $\underline{E}$ be a topos -- then $\underline{E}$ is said to be a <u>Grothendieck topos</u> if $\underline{E}$ is cocomplete and has a separating set.

[Note: In general, a cocomplete topos need not admit a separating set.]

18.7 EXAMPLE Let $(\underline{C},\tau)$ be a site -- then the topos $\underline{Sh}_\tau(\underline{C})$ (cf. 14.4) is a Grothendieck topos (cf. 13.1 and 13.15).

18.8 DEFINITION Let $\underline{E}$ be a topos -- then a <u>subseparator</u> is an object Γ in $\underline{E}$ with the property that $M(\Gamma)$ contains a separating set.

18.9 LEMMA Suppose that $\underline{E}$ is a Grothendieck topos -- then $\underline{E}$ has a subseparator.

PROOF If $\mathcal{U}$ is a separating set, let

$$\Gamma = \coprod_{U\in\mathcal{U}} U.$$

Then Γ is a subseparator.

18.10 RAPPEL An object X in a category $\underset{\sim}{C}$ is a <u>coseparator</u> if for every pair $f,g:A \to B$ of distinct morphisms in $\underset{\sim}{C}$, there exists a morphism $\sigma:B \to X$ such that $\sigma \circ f \neq \sigma \circ g$.

18.11 LEMMA Let $\underset{\sim}{E}$ be a topos. Assume: Γ is a subseparator — then Ω^Γ is a coseparator.

[Consider the simplest possibility, viz. when $\Gamma = *_{\underset{\sim}{E}}$ ($\Rightarrow \Omega^{*_{\underset{\sim}{E}}} \approx \Omega$). Let $f,g:A \to B$ be morphisms such that for any $\sigma:B \to \Omega$, $\sigma \circ f = \sigma \circ g$. Claim: $f = g$. To see this, let $e:E \to *_{\underset{\sim}{E}}$ be a subfinal object and given a morphism $\phi:E \to A$, pass to the pullback square

$$\begin{tikzcd} E \arrow[r, "e"] \arrow[d, "f \circ \phi"'] & *_{\underset{\sim}{E}} \arrow[d, "\top"] \\ B \arrow[r, "\chi_{f \circ \phi}"'] & \Omega \end{tikzcd} \qquad (f \circ \phi \in M(B)).$$

Since $\chi_{f \circ \phi} \in \mathrm{Mor}(B,\Omega)$, from the assumptions

$$\chi_{f \circ \phi} \circ f = \chi_{f \circ \phi} \circ g,$$

thus

$$\top_E = \chi_{f \circ \phi} \circ f \circ \phi = \chi_{f \circ \phi} \circ g \circ \phi,$$

so there exists a unique morphism $\varepsilon:E \to E$ rendering the diagram

$$\begin{tikzcd} E \arrow[r, "\varepsilon"] \arrow[d, "\phi"'] \arrow[rr, bend left, "e"] & E \arrow[r, "e"] \arrow[d, "f \circ \phi"] & *_{\underline{E}} \arrow[d, "\top"] \\ A \arrow[r, "g"'] & B \arrow[r, "\chi_{f \circ \phi}"'] & \Omega \end{tikzcd}$$

commutative. But $\mathrm{Mor}(E,E) = \{\mathrm{id}_E\}$, hence $\varepsilon = \mathrm{id}_E$, which implies that $f \circ \phi = g \circ \phi$. Therefore $f = g$ (E and ϕ being arbitrary).]

[Note: In general, Ω is not a coseparator but if Ω is a coseparator, it does not follow that $*_{\underline{E}}$ is a subseparator.]

18.12 REMARK Let $\underline{E}$ be a Grothendieck topos -- then $\underline{E}$ satisfies the axiom of choice iff $\underline{E}$ is a boolean topos and $*_{\underline{E}}$ is a subseparator.

[E.g.: If $\underline{E}$ satisfies the axiom of choice, then $\underline{E}$ is a boolean topos (cf. 15.24) and $*_{\underline{E}}$ is a subseparator (cf. 15.28).]

18.13 LEMMA A topos $\underline{E}$ is a Grothendicek topos iff it is defined over $\underline{SET}$ and has a subseparator.

PROOF That the conditions are necessary is implied by 18.2 and 18.9. As for the sufficiency, since a topos is finitely cocomplete (cf. 14.5), to finish the proof it suffices to show that $\underline{E}$ has coproducts. For this purpose, note first that $\underline{E}$ has arbitrary powers of objects (cf. 18.4) and has a coseparator, call if X (cf. 18.11). Suppose now that $\{A_i : i \in I\}$ is a set-indexed collection of objects of $\underline{E}$.

Choose a set S such that $\forall\ i \in I$, $\mathrm{Mor}(A_i,X) \subset S$ and put $B = \prod_S X$ -- then the monomorphism

$$A_i \longrightarrow \prod_{\mathrm{Mor}(A_i,X)} X$$

leads to a monomorphism $A_i \to B$. Therefore $\coprod_{i \in I} A_i$ can be constructed as an element of $M(\coprod_I B)$.

18.14 LEMMA Every Grothendieck topos $\underline{E}$ is complete.

PROOF Given a set-indexed collection of objects $\{A_i : i \in I\}$ of $\underline{E}$, define P_i by the pullback square

$$\begin{array}{ccc} P_i & \longrightarrow & \prod_I \coprod_{i \in I} A_i \\ \downarrow & & \downarrow \mathrm{pr}_i \\ A_i & \xrightarrow[\mathrm{in}_i]{} & \coprod_{i \in I} A_i \end{array} .$$

Then

$$\bigcap_{i \in I} P_i = \prod_{i \in I} A_i.$$

18.15 LEMMA If $\underline{E}$ is a Grothendieck topos, then $\forall\ A \in \mathrm{Ob}\ \underline{E}$, the topos $\underline{E}/A$ (cf. 15.1) is a Grothendieck topos.

PROOF As a category, $\underline{E}/A$ is cocomplete ($\underline{E}$ being cocomplete). This said, let $\mathcal{U} = \{U\}$ be a separating set (per $\underline{E}$) and put

$$\mathcal{U}/A = \{f : U \to A,\ U \in \mathcal{U}\}.$$

Then $\mathcal{U}/A$ is a separating set (per $\underline{E}/A$).

18.16 THEOREM If $\underline{E}$ is a cocomplete topos, then for any small category $\underline{I}$, the functor category $[\underline{I},\underline{E}]$ is a cocomplete topos.

[Note: If $\underline{E}$ is a topos (hence finitely cocomplete (cf. 14.5), then for any finite category $\underline{I}$, the functor category $[\underline{I},\underline{E}]$ is a topos.]

18.17 LEMMA If $\underline{E}$ is a Grothendieck topos, then for any small category $\underline{I}$, the functor category $[\underline{I},\underline{E}]$ is a Grothendieck topos.

PROOF If $\mathcal{U} = \{U\}$ is a separating set for $\underline{E}$, then

$$\{F_{U,i}: U \in \mathcal{U},\ i \in \text{Ob } \underline{I}\}$$

is a separating set for $[\underline{I},\underline{E}]$, where

$$F_{U,i}(j) = \coprod_{\text{Mor}(i,j)} U \quad (j \in \text{Ob } \underline{I}).$$

Let $\underline{E}$ be a Grothendieck topos, $\underline{I}$ a small category, and $\Delta:\underline{I} \to \underline{E}$ a functor. Put $B = \text{colim}_{\underline{I}}\Delta$ and let $A \to B$ be a morphism -- then $\forall$ $i \in \text{Ob } \underline{I}$, there is a pullback square

$$\begin{tikzcd} A \times_B \Delta_i \arrow[r] \arrow[d] & \Delta_i \arrow[d] \\ A \arrow[r] & B. \end{tikzcd}$$

18.18 LEMMA The canonical arrow

$$\text{colim}_{\underline{I}} (i \to A \times_B \Delta_i) \to A$$

is an isomorphism.

Given a set $\{X_i : i \in I\}$ of objects in $\underline{E}$, put

$$X = \coprod_{i \in I} X_i.$$

18.19 EXAMPLE Let $Y \to X$ be a morphism -- then the canonical arrow

$$\coprod_{i \in I} X_i \times_X Y \to Y$$

is an isomorphism.

18.20 EXAMPLE Let $Y \in \text{Ob } \underline{E}$ -- then

$$\coprod_{i \in I} (X_i \times Y) \approx X \times Y \quad (\text{cf. } 5.8).$$

[This is a special case of 18.19: Replace Y by $X \times Y$, consider the projection $X \times Y \to X$, and note that

$$X_i \times_X (X \times Y) \approx X_i \times Y.]$$

The following result is Giraud's "recognition principle".

18.21 THEOREM Suppose that $\underline{E}$ is a Grothendieck topos -- then there exists a site $(\underline{C}, \tau)$ such that $\underline{E}$ is equivalent to $\underline{Sh}_\tau(\underline{C})$.

[Here is a sketch of the proof. Take for $\underline{C}$ the small full subcategory of $\underline{E}$ whose objects are a separating set. Given $X \in \text{Ob } \underline{C}$, let τ_X be the set of subfunctors $G \to h_X$ such that the arrow

$$\coprod_{Y \in \text{Ob } \underline{C}} \coprod_{g \in GY} Y \to X$$

is an epimorphism -- then the assignment $X \to \tau_X$ defines a Grothendieck topology on $\underline{C}$.

Next, $\forall\ A \in \mathrm{Ob}\ \underline{E}$, the presheaf $h_A|\underline{C}^{OP}$ is a τ-sheaf ($h_A = \mathrm{Mor}(—,A)$) and the specification $A \rightarrow h_A|\underline{C}^{OP}$ defines a functor $\underline{E} \rightarrow \underline{\mathrm{Sh}}_\tau(\underline{C})$ which at length can be shown to be an equivalence of categories.]

[Note: Making a simple expansion, one can always arrange that $\underline{C}$ is finitely complete.]

18.22 REMARK The Grothendieck topology figuring in 18.21 is subcanonical. However, it is possible to enlarge $\underline{C}$ so as to replace "subcanonical" by "canonical". Thus let $\mathcal{U} = \{U\}$ be a separating set and for each $U \in \mathcal{U}$, let $\{U_i : i \in I_U\}$ be a set of representatives for $\mathrm{Sub}_{\underline{E}}\ U$ ($\underline{E}$ is wellpowered (cf. 6.13)). Perform the construction of 18.21 on the full subcategory of $\underline{E}$ generated by the U_i ($i \in I_U$, $U \in \mathcal{U}$) -- then the resulting "τ" is canonical.

18.23 LEMMA Every Grothendieck topos $\underline{E}$ is presentable (cf. 13.16).

18.24 LEMMA Every Grothendieck topos $\underline{E}$ is cowellpowered (cf. 13.17).

18.25 CRITERION Let $\underline{E},\underline{F}$ be Grothendieck toposes -- then any functor $\underline{F} \rightarrow \underline{E}$ which preserves colimits has a right adjoint $\underline{E} \rightarrow \underline{F}$.

[The categories involved are cocomplete, cowellpowered, and have separating sets. Now quote the appropriate "adjoint functor theorem".]

18.26 NOTATION Given Grothendieck toposes $\underline{E},\underline{F}$, write $[\underline{E},\underline{F}]_{\mathrm{gro}}$ for the metacategory whose objects are the geometric morphisms $\underline{E} \rightarrow \underline{F}$ and whose morphisms are the geometric transformations.

18.27 LEMMA Let $\underline{E},\underline{F}$ be Grothendieck toposes -- then $[\underline{E},\underline{F}]_{\text{geo}}$ is a category.

[In other words, if $f,g:\underline{E} \to \underline{F}$ are geometric morphisms, then there is but a set of natural transformations $f^* \to g^*$.]

18.28 LEMMA Let $\underline{E},\underline{F}$ be Grothendieck toposes and suppose that $f:\underline{E} \to \underline{F}$ is a geometric morphism -- then the following conditions are equivalent.

(1) f^* is faithful;

(2) f^* reflects isomorphisms;

(3) f^* reflects epimorphisms;

(4) f^* reflects monomorphisms.

18.29 THEOREM Let $\underline{E}$ be a Grothendieck topos -- then there is a Grothendieck topos $\underline{B}$ satisfying the axiom of choice and a geometric morphism $f:\underline{B} \to \underline{E}$ such that f^* is faithful.

§19. *POINTS*

Let $\underline{E}$ be a Grothendieck topos.

19.1 DEFINITION A point of $\underline{E}$ is a geometric morphism $f:\underline{SET} \to \underline{E}$.

N.B. Alternatively, a point of $\underline{E}$ is a functor $p:\underline{E} \to \underline{SET}$ which preserves colimits and finite limits (cf. 18.15).

19.2 EXAMPLE Let X be a nonempty topological space -- then each $x \in X$ determines a point $p_x:\underline{Sh}(X) \to \underline{SET}$, where $\underline{Sh}(X)$ is computed per the canonical Grothendieck topology on $O(X)$.

[Apply 17.2 to the continuous function $\{*\} \xrightarrow{x} X$, thus $p_x:\underline{Sh}(X) \to \underline{Sh}(\{*\}) = \underline{SET}$ sends F to its stalk F_x at x.]

[Note: If X is sober, then this construction is exhaustive, i.e., up to natural isomorphism, every point $\underline{Sh}(X) \to \underline{SET}$ is a "p_x". In general, the full subcategory of $\underline{TOP}$ whose objects are the sober topological spaces is reflective with arrow of reflection $X \to \text{sob } X$. But

$$O(X) \longleftrightarrow O(\text{sob } X) \quad (\text{cf. } 9.26),$$

hence

$$\underline{Sh}(X) \longleftrightarrow \underline{Sh}(\text{sob } X).$$

Therefore the points of sob X "parameterize" the points of $\underline{Sh}(X)$: If $f:\underline{SET} \to \underline{Sh}(X)$ is a point, let U be the union of all open $V \subset X$ such that $f^*V = \emptyset$ -- then $X - U$ is an irreducible closed subset of X, thus is a point of sob X. Conversely, ...]

19.3 REMARK If X is empty, then $\underline{Sh}(X)$ is the full subcategory of $\underline{SET}$ whose objects are the final objects so there is no functor $p:\underline{\mathrm{Sh}}(X) \rightarrow \underline{\mathrm{SET}}$ which preserves colimits and finite limits. Proof: All objects in $\underline{\mathrm{Sh}}(X)$ are both initial and final.

19.4 EXAMPLE Let X be a nonempty Hausdorff topological space in which no singletons are open -- then

$$\underline{\mathrm{Sh}}_{\neg \circ \neg}(\underline{\mathrm{Sh}}(X)) \quad \text{(cf. 16.18)}$$

has no points.

19.5 NOTATION Given a Grothendieck topos $\underline{E}$, let

$$\underline{\mathrm{PT}}(\underline{E}) = [\underline{\mathrm{SET}},\underline{E}]_{\mathrm{geo}} \quad \text{(cf. 18.26).}$$

<u>N.B.</u> $\underline{\mathrm{PT}}(\underline{E})$ is a category (cf. 18.27).

[Note: It is not necessarily true that $\underline{\mathrm{PT}}(\underline{E})$ is equivalent to a small category (e.g., there are $\underline{E}$ for which $\underline{\mathrm{PT}}(\underline{E})$ is equivalent to $\underline{\mathrm{SET}}$).]

19.6 RAPPEL Let $\underline{C}$ be a small category -- then the functor $Y^*_{\underline{C}}:[\hat{\underline{C}},\underline{\mathrm{SET}}] \rightarrow [\underline{C},\underline{\mathrm{SET}}]$ has a left adjoint that sends $T \in \mathrm{Ob}[\underline{C},\underline{\mathrm{SET}}]$ to $\Gamma_T \in \mathrm{Ob}[\hat{\underline{C}},\underline{\mathrm{SET}}]$.

[Note: Γ_T is the <u>realization functor</u>; it is a left adjoint for the <u>singular functor</u> $\mathrm{sin}_T:\underline{\mathrm{SET}} \rightarrow \hat{\underline{C}}$ which is defined by the prescription

$$(\mathrm{sin}_T\, Y)X = \mathrm{Mor}(TX,Y).]$$

19.7 LEMMA Let $\underline{C}$ be a small category. Suppose that $f:\underline{\mathrm{SET}} \rightarrow \hat{\underline{C}}$ is a point -- then there exists a functor $T:\underline{C} \rightarrow \underline{\mathrm{SET}}$ such that f^* is naturally isomorphic to Γ_T.

19.8 DEFINITION Let $\underline{C}$ be a small category — then a functor $T:\underline{C} \to \underline{SET}$ is said to be flat if Γ_T preserves finite limits.

So, if T is flat, then the adjoint pair $(\Gamma_T, \sin_T)$ is a geometric morphism $\underline{SET} \to \hat{\underline{C}}$, i.e., is a point of $\hat{\underline{C}}$. Moreover, up to natural isomorphism, all points of $\hat{\underline{C}}$ are of this form (cf. 19.7).

Write $[\underline{C},\underline{SET}]_{flat}$ for the full subcategory of $[\underline{C},\underline{SET}]$ whose objects are the flat functors.

19.9 THEOREM There is an equivalence

$$[\underline{C},\underline{SET}]_{flat} \longleftrightarrow \underline{PT}(\hat{\underline{C}})$$

of categories.

[Send T to $(\Gamma_T, \sin_T)$ and send f to $f^* \circ Y_{\underline{C}}$.]

19.10 REMARK Let τ be a Grothendieck topology on $\underline{C}$ -- then $\underline{PT}(\underline{Sh}_\tau(\underline{C}))$ is equivalent to the full subcategory of $\underline{PT}(\hat{\underline{C}})$ consisting of those points that factor through ι_τ.

19.11 DEFINITION Let $\underline{C}$ be a category. Suppose that the $\underline{C}_i$ are categories and the $F_i:\underline{C} \to \underline{C}_i$ are functors -- then $\{F_i\}$ is faithful if given distinct morphisms $f,g:X \to Y$ in $\underline{C}$, there exists an F_i such that $F_i f \neq F_i g$.

19.12 EXAMPLE Take $\underline{C} = \underline{Sh}(X)$ (X a nonempty topological space), let $\underline{C}_x = \underline{SET}$ ($x \in X$), and let $p_x:\underline{Sh}(X) \to \underline{SET}$ be as in 19.2 -- then $\{p_x\}$ is faithful.

19.13 DEFINITION Let $\underline{C}$ be a category. Suppose that the $\underline{C}_i$ are categories and the $F_i:\underline{C} \to \underline{C}_i$ are functors.

- $\{F_i\}$ <u>reflects isomorphisms</u> if any $f \in \text{Mor } \underline{C}$ with the property that $F_i f$ is an isomorphism for all F_i must itself be an isomorphism in $\underline{C}$.
- $\{F_i\}$ <u>reflects monomorphisms</u> if any $f \in \text{Mor } \underline{C}$ with the property that $F_i f$ is a monomorphism for all F_i must itself be a monomorphism in $\underline{C}$.
- $\{F_i\}$ <u>reflects epimorphisms</u> if any $f \in \text{Mor } \underline{C}$ with the property that $F_i f$ is an epimorphism for all F_i must itself be an epimorphism in $\underline{C}$.

Let $\mathcal{P} \subset \text{Ob } \underline{PT}(\underline{E})$ be a class of points.

19.14 LEMMA Suppose that $\mathcal{P}$ is faithful -- then $\mathcal{P}$ reflects isomorphisms.

PROOF It is immediate that $\mathcal{P}$ reflects monomorphisms and epimorphisms. But $\underline{E}$ is balanced (cf. 14.7).

19.15 LEMMA Suppose that $\mathcal{P}$ reflects isomorphisms -- then $\mathcal{P}$ is faithful.

PROOF Let $f,g:A \to B$ be morphisms in $\underline{E}$ and suppose that $pf = pg$ for all $p \in \mathcal{P}$. Form the equalizer diagram

$$\text{eq}(f,g) \xrightarrow{k} A \overset{f}{\underset{g}{\rightrightarrows}} B.$$

Since p preserves finite limits, it preserves equalizers:

$$p(\text{eq}(f,g)) \approx \text{eq}(pf,pg).$$

Therefore

$$p(\mathrm{eq}(f,g)) \xrightarrow{pk} pA \overset{pf}{\underset{pg}{\rightrightarrows}} pB$$

is an equalizer diagram. But pf = pg, thus

$$pA \xrightarrow{id_{pA}} pA \overset{pf}{\underset{pg}{\rightrightarrows}} pB$$

is also an equalizer diagram, which implies that pk is an isomorphism, hence k is an isomorphism, hence $f = g$ ($f \circ k = g \circ k$).

19.16 DEFINITION $\underline{E}$ is said to have enough points if the class of all points of $\underline{E}$ is faithful.

19.17 THEOREM If $\underline{E}$ has enough points, then $\underline{E}$ has a faithful set of points.

19.18 DEFINITION A weak point of $\underline{E}$ is a functor $p:\underline{E} \to \underline{SET}$ which preserves epimorphisms and finite limits.

N.B. Every point is a weak point.

19.19 LEMMA A class of weak points of $\underline{E}$ is faithful iff it reflects isomorphisms.

19.20 THEOREM The class of all weak points of $\underline{E}$ is faithful.

PROOF Take $\underline{B}$ and $f:\underline{B} \to \underline{E}$ as in 18.29 -- then every epimorphism of $\underline{B}$ has a section, thus $\forall\ B \in \mathrm{Ob}\ \underline{B}$, the functor $X \to \mathrm{Mor}(B,X)$ from $\underline{B}$ to $\underline{SET}$ is a weak point of $\underline{B}$, so $\forall\ B \in \underline{B}$, the functor $X \to \mathrm{Mor}(B,f^*X)$ from $\underline{E}$ to $\underline{SET}$ is a weak point of $\underline{E}$

(f^* preserves epimorphisms (being a left adjoint)). And: $\{p_B : B \in \text{Ob } \underset{\sim}{B}\}$ is a faithful class of weak points of $\underset{\sim}{E}$. Proof: Bearing in mind 19.19, suppose that $\phi : U \rightarrow V$ is a morphism in $\underset{\sim}{E}$ such that $\forall\ B \in \text{Ob } \underset{\sim}{B}$,

$$p_B\phi : \text{Mor}(B, f^*U) \rightarrow \text{Mor}(B, f^*V)$$

is bijective -- then $f^*\phi : f^*U \rightarrow f^*V$ is an isomorphism. But f^* reflects isomorphisms (cf. 18.28), hence ϕ is an isomorphism.

19.21 LEMMA Let $p : \underset{\sim}{E} \rightarrow \underline{\text{SET}}$ be a weak point. Given a morphism $f : A \rightarrow B$ in $\underset{\sim}{E}$, factor it per 3.9:

$$A \xrightarrow{k} M \xrightarrow{m} B \quad (f = m \circ k).$$

Then

$$pM \approx \text{im } pf$$

or still,

$$p(\text{im } f) \approx \text{im } pf.$$

PROOF Since p preserves epimorphisms and monomorphisms, pk is a surjection and pm is an injection:

$$\begin{array}{ccccc} pA & \xrightarrow{pk} & pM & \xrightarrow{pm} & pB \qquad (pf = pm \circ pk) \\ & & \approx \Big\| & & \\ & & \text{im } pf. & & \end{array}$$

19.22 LEMMA Suppose that $\{p\}$ is a faithful class of weak points of $\underset{\sim}{E}$ -- then $\{p\}$ reflects epimorphisms.

PROOF First, $f : A \rightarrow B$ is an epimorphism iff the canonical arrow $M \xrightarrow{m} B$ is an epimorphism, then $\forall$ p, pm is an isomorphism (cf. 19.21), hence m is an isomorphism (cf. 19.19).

19.23 SCHOLIUM A morphism f in $\underline{E}$ is an epimorphism iff $\forall$ weak point p, pf is an epimorphism.

19.24 LEMMA Suppose that R is an equivalence relation on X and $p:\underline{E} \to \underline{SET}$ is a weak point -- then pR is an equivalence relation on pX and

$$pX/pR \approx p(X/R).$$

19.25 APPLICATION Let $f,g \in \mathrm{Mor}(X,Y)$ and let

$$(f,g):X \to Y \times Y.$$

Suppose that im(f,g) is an equivalence relation on Y and $p:\underline{E} \to \underline{SET}$ is a weak point -- then p(im(f,g)) ($\approx$ im p(f,g) (cf. 19.21)) is an equivalence relation on pY and the canonical map

$$\mathrm{coker}(pf,pg) \to p(\mathrm{coker}(f,g))$$

is bijective.

19.26 LEMMA Let R be a relation on X. Assume: $\forall$ weak point $p:\underline{E} \to \underline{SET}$, pR is an equivalence relation on pX -- then R is an equivalence relation on X, hence

$$pX/pR \approx p(X/R).$$

19.27 APPLICATION Let $f,g \in \mathrm{Mor}(X,Y)$ and let

$$(f,g):X \to Y \times Y.$$

Assume: $\forall$ weak point $p:\underline{E} \to \underline{SET}$, p(im(f,g)) ($\approx$ im p(f,g) (cf. 19.21)) is an equivalence relation on pY -- then im(f,g) is an equivalence relation on Y and the canonical map

$$\mathrm{coker}(pf,pg) \to p(\mathrm{coker}(f,g))$$

is bijective.

§20. CISINSKI[†] THEORY

Let $\underline{E}$ be a Grothendieck topos — then the class $M \subset \text{Mor } \underline{E}$ of monomorphisms is retract stable and the pair $(M, \text{RLP}(M))$ is a w.f.s. on $\underline{E}$.

N.B. Elements of $\text{RLP}(M)$ are called trivial fibrations.

20.1 THEOREM There exists a set $\text{M} \subset M$ such that $M = \text{LLP}(\text{RLP}(\text{M}))$, hence $M = \text{cof M}$ ($\underline{E}$ being presentable (cf. 18.23)).

20.2 RAPPEL Let $\underline{C}$ be a category, $W \subset \text{Mor } \underline{C}$ a class of morphisms — then $(\underline{C}, W)$ is a category pair if W is closed under composition and contains the identities of $\underline{C}$.

20.3 DEFINITION Suppose that $(\underline{E}, \text{W})$ is a category pair — then W is an $\underline{E}$-localizer provided the following conditions are met.

(1) W satisfies the 2 out of 3 condition.

(2) W contains $\text{RLP}(M)$.

(3) $\text{W} \cap M$ is a stable class, i.e., is closed under the formation of pushouts and transfinite compositions.

Let $C \subset \text{Mor } \underline{E}$ — then the $\underline{E}$-localizer generated by C, denoted $\text{W}(C)$, is the intersection of all the $\underline{E}$-localizers containing C. The minimal $\underline{E}$-localizer is $\text{W}(\emptyset)$ ($\emptyset$ the empty set of morphisms).

[Note: Let $C_1, C_2 \subset \text{Mor } \underline{E}$ — then

$$\text{W}(C_1 \cup C_2) = \text{W}(\text{W}(C_1) \cup \text{W}(C_2)).]$$

20.4 DEFINITION An $\underline{E}$-localizer is admissible if it is generated by a set of

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[†] *Astérisque* 308 (2006); see also *Faisceaux Localement Asphériques* (2003)(preprint).

morphisms of $\underline{E}$.

20.5 EXAMPLE Mor $\underline{E}$ is an admissible $\underline{E}$-localizer. In fact,

$$W(\{\emptyset_{\underline{E}} \to *_{\underline{E}}\}) = \text{Mor } \underline{E}.$$

20.6 EXAMPLE Take $\underline{E}$ = $\underline{\text{SISET}}$ (= $\hat{\underline{\Delta}}$) and let W_∞ be the class of simplicial weak equivalences -- then W_∞ is a $\hat{\underline{\Delta}}$-localizer.

- W_∞ is generated by the projections

$$p_K:K \times \Delta[1] \to K \quad (K \in \text{Ob } \hat{\underline{\Delta}}).$$

- W_∞ is generated by the maps $\Delta[n] \to \Delta[0]$ $(n \geq 0)$.

<u>N.B.</u> It follows from the first description that W_∞ is closed under the formation of products of pairs of arrows and from the second description that W_∞ is admissible.

[Note: In <u>SISET</u>, a <u>simplicial weak equivalence</u> is a simplicial map $f:X \to Y$ such that $|f|:|X| \to |Y|$ is a homotopy equivalence.]

20.7 EXAMPLE Take $\underline{E}$ = $\underline{\text{SET}}$ -- then $W(\emptyset)$ is the class

$$\{\emptyset \to \emptyset\} \cup \{f:X \to Y \ (X \neq \emptyset)\}.$$

20.8 NOTATION Given $C \subset$ Mor $\underline{E}$, let cart C be the class of arrows of the form

$$f \times \text{id}_Z:X \times Z \to Y \times Z \quad (f \in C,\ Z \in \text{Ob } \underline{E}).$$

20.9 LEMMA The $\underline{E}$-localizer generated by cart C is closed under the formation of products of pairs of arrows and is admissible if C is a set.

20.10 APPLICATION The minimal $\underline{E}$-localizer $W(\emptyset)$ is closed under the formation of products of pairs of arrows.

[Note: This is one way to distinguish a generic $\underline{E}$-localizer W from $W(\emptyset)$.]

20.11 DEFINITION A cofibrantly generated model structure on $\underline{E}$ is said to be a <u>Cisinski structure</u> if the cofibrations are the monomorphisms.

[Note: The acyclic fibrations of a Cisinski structure are the trivial fibrations.]

20.12 THEOREM Suppose that $(\underline{E},W)$ is a category pair — then W is an admissible $\underline{E}$-localizer iff there exists a cofibrantly generated model structure on $\underline{E}$ whose class of weak equivalences are the elements of W and whose cofibrations are the monomorphisms.

20.13 SCHOLIUM The map

$$W \rightarrow W, M, \mathrm{RLP}(W \cap M)$$

induces a bijection between the class of admissible $\underline{E}$-localizers and the class of Cisinski structures on $\underline{E}$.

20.14 REMARK The stable class $W \cap M$ is retract stable. In addition, W is necessarily saturated, i.e., $W = \overline{W}$.

20.15 LEMMA Let W be an admissible $\underline{E}$-localizer — then the cofibrantly generated model structure on $\underline{E}$ determined by W is left proper.

20.16 EXAMPLE Take $\underline{E} = \underline{\mathrm{SISET}}$ and let W be the class of categorical weak equivalences — then W is a $\hat{\underline{\Delta}}$-localizer. As such, it is generated by the maps $I[n] \rightarrow \Delta[n]$ $(n \geq 0)$, hence W is admissible. The resulting cofibrantly generated model structure on $\underline{\text{SISET}}$ is the Joyal structure. It is left proper but not right proper.

[Note: In $\underline{\text{SISET}}$, a categorical weak equivalence is a simplicial map $f:X_1 \to X_2$ such that for every weak Kan complex Y, the arrow

$$c_0 \operatorname{map}(X_2,Y) \to c_0 \operatorname{map}(X_1,Y)$$

is bijective.]

N.B. Every categorical weak equivalence is a simplicial weak equivalence.

20.17 CRITERION Let $S \subset \operatorname{Mor} \underline{E}$ be a set -- then the cofibrantly generated model structure on $\underline{E}$ corresponding to $W(S)$ is right proper iff

- $\forall$ arrow $f:X \to Y$ in S,
- $\forall$ fibration $p:E \to B$ with B fibrant,
- $\forall$ arrow $u:Y \to B$,

the induced arrow

$$g:X \times_B E \to Y \times_B E$$

per

$$\begin{tikzcd}
X \times_B E \arrow[r, "g"] \arrow[d] & Y \times_B E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
X \arrow[r, "f"'] & Y \arrow[r, "u"'] & B
\end{tikzcd}$$

is in $W(S)$.

[Note: One can replace the set S by a class $\mathcal{C}$ provided that $W(\mathcal{C})$ is admissible.]

N.B. Take $S = \emptyset$ to see that the Cisinski structure on $\underline{E}$ corresponding to $W(\emptyset)$ is right proper.

20.18 LEMMA If X_i $(i \in I)$ is a set of objects of $\underline{E}$, then the $\underline{E}$-localizer generated by the projections $X_i \times Z \to Z$ for all i and Z is admissible (cf. 20.9) and the associated Cisinski structure is right proper (hence proper (cf. 20.15)).

[To infer right proper, apply 20.17 and consider

$$\begin{tikzcd}
(X_i \times Z) \times_B E \arrow[r] \arrow[d] & Z \times_B E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
X_i \times Z \arrow[r] & Z \arrow[r] & B
\end{tikzcd}$$

or still,

$$\begin{tikzcd}
X_i \times (Z \times_B E) \arrow[r] \arrow[d] & Z \times_B E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
X_i \times Z \arrow[r] & Z \arrow[r] & B .
\end{tikzcd}$$

But the arrow

$$X_i \times (Z \times_B E) \to Z \times_B E$$

is in our generating class.]

20.19 EXAMPLE Take $\underline{\text{SISET}}$ in its Kan structure -- then this model structure is proper.

[Since all objects are cofibrant, left proper is an application of standard generalities while classically, right proper lies deeper in that it uses the fact that the geometric realization of a Kan fibration is a Serre fibration. But, as has been noted in 20.6, W_∞ is generated by the projections

$$p_K : K \times \Delta[1] \to K \quad (K \in \text{Ob}\ \underline{\hat{\Delta}}).$$

Therefore right proper is immediate (cf. 20.18).

20.20 LEMMA Let $S_1, S_2 \subset \text{Mor } \underline{E}$ be sets. Suppose that the Cisinski structures corresponding to $W(S_1), W(S_2)$ are right proper -- then the Cisinski structure corresponding to $W(S_1 \cup S_2)$ is right proper.

[To infer right proper, apply 20.17, noting that every fibration per $W(S_1 \cup S_2)$ is a fibration per $W(S_1)$ and $W(S_2)$.]

20.21 NOTATION Given an admissible $\underline{E}$-localizer W and a small category $\underline{I}$, denote by $W_{\underline{I}} \subset \text{Mor}[\underline{I},\underline{E}]$ the class of morphisms $\Xi:F \to G$ such that $\forall\ i \in \text{Ob } \underline{I}$, $\Xi_i:Fi \to Gi$ is in W.

<u>N.B.</u> Recall that $[\underline{I},\underline{E}]$ is a Grothendieck topos (cf. 18.17).

20.22 LEMMA $W_{\underline{I}}$ is an admissible $[\underline{I},\underline{E}]$-localizer.

[Note: Therefore 20.12 is applicable with $\underline{E}$ replaced by $[\underline{I},\underline{E}]$ and W replaced by $W_{\underline{I}}$.]

APPENDIX

What follows is a summary of some basic facts from model category theory.

Let $\underline{C}$ be a model category.

DEFINITION $\underline{C}$ is <u>combinatorial</u> if $\underline{C}$ is cofibrantly generated and presentable.

EXAMPLE If W is an admissible $\underline{E}$-localizer, then $\underline{E}$ in the Cisinski structure corresponding to W is combinatorial (recall that $\underline{E}$ is presentable (cf. 18.23)).

Fix a small category $\underline{I}$.

DEFINITION Let $\underline{C}$ be a model category and suppose that $\Xi \in \text{Mor}[\underline{I},\underline{C}]$, say $\Xi:F \to G$.

- Ξ is a levelwise weak equivalence if $\forall\ i \in \text{Ob}\ \underline{I}$, $\Xi_i:Fi \to Gi$ is a weak equivalence in $\underline{C}$.

- Ξ is a levelwise fibration if $\forall\ i \in \text{Ob}\ \underline{I}$, $\Xi_i:Fi \to Gi$ is a fibration in $\underline{C}$.

- Ξ is a projective cofibration if it has the LLP w.r.t. those morphisms which are simultaneously a levelwise weak equivalence and a levelwise fibration.

DEFINITION The triple consisting of the classes of levelwise weak equivalences, levelwise fibrations, and projective cofibrations is called the projective structure on $[\underline{I},\underline{C}]$.

THEOREM Suppose that $\underline{C}$ is a combinatorial model category — then for every $\underline{I}$, the projective structure on $[\underline{I},\underline{C}]$ is a model structure that, moreover, is combinatorial.

DEFINITION Let $\underline{C}$ be a model category and suppose that $\Xi \in \text{Mor}[\underline{I},\underline{C}]$, say $\Xi:F \to G$.

- Ξ is a levelwise weak equivalence if $\forall\ i \in \text{Ob}\ \underline{I}$, $\Xi_i:Fi \to Gi$ is a weak equivalence in $\underline{C}$.

- Ξ is a levelwise cofibration if $\forall\ i \in \text{Ob}\ \underline{I}$, $\Xi_i:Fi \to Gi$ is a cofibration in $\underline{C}$.

• Ξ is an injective fibration if it has the RLP w.r.t. those morphisms which are simultaneously a levelwise weak equivalence and a levelwise cofibration.

DEFINITION The triple consisting of the classes of levelwise weak equivalences, levelwise cofibrations, and injective fibrations is called the injective structure on $[\underline{I},\underline{C}]$.

THEOREM Suppose that $\underline{C}$ is a combinatorial model category — then for every $\underline{I}$, the injective structure on $[\underline{I},\underline{C}]$ is a model structure that, moreover, is combinatorial.

REMARK

• Every projective cofibration is necessarily levelwise, hence is a cofibration in the injective structure.

• Every injective fibration is necessarily levelwise, hence is a fibration in the projective structure.

EXAMPLE If W is an admissible $\underline{E}$-localizer, then the Cisinski structure on $[\underline{I},\underline{E}]$ corresponding to $W_{\underline{I}}$ (cf. 20.22) is the injective structure (monomorphisms are levelwise).

[Note: Of course one can also equip $[\underline{I},\underline{E}]$ with its projective structure.]

LEMMA Suppose that $\underline{C}$ is combinatorial — then

$$\underline{C} \text{ left proper} \Rightarrow \left[\begin{array}{l} [\underline{I},\underline{C}] \text{ (Projective Structure)} \\ \\ [\underline{I},\underline{C}] \text{ (Injective Structure)} \end{array}\right. \quad \text{left proper}$$

and

$$\underline{C} \text{ right proper} \Rightarrow \left| \begin{array}{l} [\underline{I},\underline{C}] \text{ (Projective Structure)} \\ \\ [\underline{I},\underline{C}] \text{ (Injective Structure)} \end{array} \right. \quad \text{right proper.}$$

REMARK If W is an admissible $\underline{E}$-localizer, then the Cisinski structure on $[\underline{I},\underline{E}]$ corresponding to $W_{\underline{I}}$ (cf. 20.22) is left proper (cf. 20.15) and is right proper if the Cisinski structure on $\underline{E}$ corresponding to W is right proper.

Let $\underline{C}$ and $\underline{C}'$ be model categories.

DEFINITION A left adjoint functor $F:\underline{C} \to \underline{C}'$ is a <u>left model functor</u> if F preserves cofibrations and acyclic cofibrations.

DEFINITION A right adjoint functor $F':\underline{C}' \to \underline{C}$ is a <u>right model functor</u> if F' preserves fibrations and acyclic fibrations.

LEMMA Suppose that

$$\left| \begin{array}{l} F:\underline{C} \to \underline{C}' \\ F':\underline{C}' \to \underline{C} \end{array} \right.$$

are an adjoint pair — then F is a left model functor iff F' is a right model functor.

DEFINITION A <u>model pair</u> is an adjoint situation (F,F'), where F is a left model functor and F' is a right model functor.

LEMMA The adjoint situation (F,F') is a model pair iff F preserves cofibrations and F' preserves fibrations.

LEMMA The adjoint situation (F,F') is a model pair iff F preserves acyclic cofibrations and F' preserves acyclic fibrations.

REMARK If $\underline{C}$ and $\underline{C}'$ are combinatorial and if

$$\begin{tikzcd} \underline{C} \arrow[r, shift left, "F"] & \underline{C}' \arrow[l, shift left, "F'"] \end{tikzcd}$$

is a model pair, then composition with F and F' determines a model pair

$$\begin{tikzcd} {[\underline{I},\underline{C}]} \arrow[r, shift left, "F_*"] & {[\underline{I},\underline{C}']} \arrow[l, shift left, "F'_*"] \end{tikzcd}$$

w.r.t. either the projective structure or the injective structure.

If the adjoint situation (F,F') is a model pair, then the derived functors

$$\left[\begin{array}{l} LF:\underline{HC} \to \underline{HC}' \\ RF':\underline{HC}' \to \underline{HC} \end{array}\right.$$

exist and are an adjoint pair.

DEFINITION A model pair (F,F') is a <u>model equivalence</u> if the adjoint pair (LF,RF') is an adjoint equivalence of homotopy categories.

LEMMA Suppose that $\underline{C}$ is combinatorial and consider the setup

$$[\underline{I},\underline{C}]\ \text{(Projective Structure)}\ \underset{\mathrm{id}_{[\underline{I},\underline{C}]}}{\overset{\mathrm{id}_{[\underline{I},\underline{C}]}}{\rightleftarrows}}\ [\underline{I},\underline{C}]\ \text{(Injective Structure)}.$$

Then $(\mathrm{id}_{[\underline{I},\underline{C}]}, \mathrm{id}_{[\underline{I},\underline{C}]})$ is a model equivalence.

§21. *SIMPLICIAL MACHINERY*

Let $\underline{C}$ be a category.

21.1 NOTATION $\underline{\text{SIC}}$ is the functor category $[\underline{\Delta}^{\text{OP}},\underline{C}]$ and a <u>simplicial object</u> X in $\underline{C}$ is an object in $\underline{\text{SIC}}$.

21.2 RAPPEL Assume: $\underline{C}$ has coproducts. Define $X|\square|K$ by

$$(X|\square|K)_n = K_n \cdot X_n \quad (= \coprod_{K_n} X_n).$$

Then

$$|\square|:\underline{\text{SIC}} \times \underline{\text{SISET}} \to \underline{\text{SIC}}$$

is a simplicial action, the <u>canonical</u> simplicial action.

[Note: Therefore

$$X|\square|(K \times L) \approx (X|\square|K)|\square|L$$

and

$$X|\square|\Delta[0] \approx X,$$

subject to the usual assumptions.]

<u>N.B.</u> Take $\underline{C} = \underline{\text{SET}}$ -- then

$$X|\square|K \approx X \times K.$$

In fact,

$$(X \times K)_n \approx X_n \times K_n \approx K_n \times X_n \approx K_n \cdot X_n.$$

21.3 REMARK Thus there is an S-category $|\square|\underline{\text{SIC}}$ such that $\underline{\text{SIC}}$ is isomorphic to the underlying category $U|\square|\underline{\text{SIC}}$.

[Recall the construction: Put $O = \text{Ob}\ \underline{\text{SIC}}$ and assign to each ordered pair

$X,Y \in O$ the simplicial set $\mathrm{HOM}(X,Y)$ defined by

$$\mathrm{HOM}(X,Y)_n = \mathrm{Mor}(X|\overline{\underline{\ }}|\Delta[n],Y) \quad (n \geq 0).]$$

21.4 LEMMA Assume: $\underset{\sim}{C}$ has coproducts -- then $\forall\ X \in \mathrm{Ob}\ \underline{\mathrm{SIC}}$, the functor

$$X|\overline{\underline{\ }}|—:\underline{\mathrm{SISET}} \to \underline{\mathrm{SIC}}$$

has a right adjoint, viz. the functor

$$\mathrm{HOM}(X,—):\underline{\mathrm{SIC}} \to \underline{\mathrm{SISET}}.$$

21.5 LEMMA Assume: $\underset{\sim}{C}$ has coproducts and is complete -- then $\forall\ K \in \mathrm{Ob}\ \hat{\underline{\Delta}}$, the functor

$$—|\overline{\underline{\ }}|K:\underline{\mathrm{SIC}} \to \underline{\mathrm{SIC}}$$

has a right adjoint, denoted by

$$X \to \mathrm{hom}(K,X).$$

<u>N.B.</u> In terms of <u>SIC</u>,

$$\begin{cases} \mathrm{Mor}(X|\overline{\underline{\ }}|K,Y) \approx \mathrm{Mor}(K,\mathrm{HOM}(X,Y)) \\ \mathrm{Mor}(X|\overline{\underline{\ }}|K,Y) \approx \mathrm{Mor}(X,\mathrm{hom}(K,Y)), \end{cases}$$

and in terms of $|\overline{\underline{\ }}|$<u>SIC</u>,

$$\begin{cases} \mathrm{HOM}(X|\overline{\underline{\ }}|K,Y) \approx \mathrm{map}(K,\mathrm{HOM}(X,Y)) \\ \mathrm{HOM}(X|\overline{\underline{\ }}|K,Y) \approx \mathrm{HOM}(X,\mathrm{hom}(K,Y)). \end{cases}$$

[Note: Here is another point. On the one hand,

$$\mathrm{Mor}(X|\overline{\underline{\ }}|(K \times L),Y) \approx \mathrm{Mor}(X,\mathrm{hom}(K \times L,Y)),$$

while on the other hand,

$$\mathrm{Mor}(X|\overline{\underline{\ }}|(K \times L),Y) \approx \mathrm{Mor}((X|\overline{\underline{\ }}|K)|\overline{\underline{\ }}|L,Y)$$

$$\approx \mathrm{Mor}(X|\overline{\underline{\ }}|K,\mathrm{hom}(L,Y))$$

$$\approx \mathrm{Mor}(X,\mathrm{hom}(K,\mathrm{hom}(L,Y))).$$

Therefore

$$\mathrm{hom}(K \times L,Y) \approx \mathrm{hom}(K,\mathrm{hom}(L,Y)).]$$

21.6 LEMMA Assume: $\underset{\sim}{C}$ has coproducts and is complete. Suppose that $K \approx \mathrm{colim}_i\, K_i$ -- then $\forall\ X,Y \in \mathrm{Ob}\ \underline{\mathrm{SIC}}$,

$$\mathrm{Mor}(X,\mathrm{hom}(\mathrm{colim}_i\, K_i,Y)) \approx \lim_i\, \mathrm{Mor}(X,\mathrm{hom}(K_i,Y)).$$

PROOF

$$\mathrm{LHS} \approx \mathrm{Mor}(X|\overline{\underline{\ }}|\mathrm{colim}_i\, K_i,Y)$$

$$\approx \mathrm{Mor}(\mathrm{colim}_i\, X|\overline{\underline{\ }}|K_i,Y)$$

$$\approx \lim_i\, \mathrm{Mor}(X|\overline{\underline{\ }}|K_i,Y) \approx \mathrm{RHS}.$$

21.7 NOTATION Let $\underset{\sim}{C}$ be a complete category. Given a simplicial object X in $\underset{\sim}{C}$ and a simplicial set K, put

$$X \pitchfork K = \int_{[n]} (X_n)^{K_n},$$

an object in $\underset{\sim}{C}$.

21.8 EXAMPLE Take $K = \Delta[n]$ -- then it follows from the integral Yoneda lemma that

$$X \pitchfork \Delta[n] \approx X_n.$$

Let K be a simplicial set. Assume: $\underset{\sim}{C}$ has coproducts -- then K determines a functor

$$K \cdot \text{---} : \underset{\sim}{C} \rightarrow \underline{\mathrm{SIC}}$$

by writing

$$(K \cdot X)[n] = K_n \cdot X.$$

21.9 LEMMA Assume: $\underline{C}$ has coproducts and is complete -- then $K \cdot —$ is a left adjoint for

$$— \pitchfork K : \underline{SIC} \to \underline{C}.$$

21.10 LEMMA Assume: $\underline{C}$ has coproducts and is complete. Suppose that $K \approx \operatorname{colim}_i K_i$ -- then $\forall\ X \in \operatorname{Ob} \underline{SIC}$,

$$X \pitchfork K \approx \lim_i X \pitchfork K_i.$$

PROOF Given $A \in \operatorname{Ob} \underline{C}$, let $\underline{A} \in \operatorname{Ob} \underline{SIC}$ be the constant simplicial object determined by A, thus

$$\begin{aligned}
\operatorname{Mor}(A, X \pitchfork K) &\approx \operatorname{Mor}(K \cdot A, X) \\
&\approx \operatorname{Mor}(\underline{A} \,\square\, K, X) \\
&\approx \operatorname{Mor}(\operatorname{colim}_i \underline{A} \,\square\, K_i, X) \\
&\approx \lim_i \operatorname{Mor}(\underline{A} \,\square\, K_i, X) \\
&\approx \lim_i \operatorname{Mor}(K_i \cdot A, X) \\
&\approx \lim_i \operatorname{Mor}(A, X \pitchfork K_i) \\
&\approx \operatorname{Mor}(A, \lim_i X \pitchfork K_i).
\end{aligned}$$

21.11 LEMMA Assume: $\underline{C}$ has coproducts and is complete -- then $\forall\ X \in \operatorname{Ob} \underline{SIC}$,

$$\hom(K, X)_n \approx X \pitchfork (K \times \Delta[n]).$$

PROOF Write

$$K \times \Delta[n] \approx \operatorname{colim}_i \Delta[n_i].$$

Then

$$X \pitchfork (K \times \Delta[n]) \approx \lim_i X \pitchfork \Delta[n_i]$$

$$\approx \lim_i X_{n_i} \quad \text{(cf. 21.8)}$$

$$\approx \hom(K,X)_n.$$

[Note: The not so obvious final point is implicit in the proof of 21.5 (which was omitted).]

21.12 EXAMPLE Take $n = 0$ to get

$$\hom(K,X)_0 \approx X \pitchfork K$$

and then replace K by $\Delta[n]$ to get

$$\hom(\Delta[n],X)_0 \approx X \pitchfork \Delta[n] \approx X_n.$$

[Note: Accordingly,

$$\hom(K,X)_n \approx \hom(\Delta[n],\hom(K,X))_0$$

$$\approx \hom(K \times \Delta[n],X)_0.]$$

21.13 LEMMA Assume: $\underline{C}$ has coproducts and is complete -- then $\forall\ K,L \in \mathrm{Ob}\ \hat{\underline{\Delta}}$,

$$\hom(K,X) \pitchfork L \approx X \pitchfork (K \times L).$$

21.14 RAPPEL A simplicial set K is <u>finite</u> if it has a finite number of nondegenerate simplexes.

21.15 FACT Suppose that K is finite -- then there exists a finite category $\underline{I}$ and a functor $\Phi:\underline{I} \to \underline{\Delta}$ such that

$$K \approx \operatorname{colim} Y_{\underline{\Delta}} \circ \Phi$$

or still,

$$K \approx \operatorname{colim}_i \Delta[n_i] \quad (i \in \operatorname{Ob} \underline{I},\ \Phi i = \Delta[n_i]).$$

21.16 THEOREM Let $\underline{C}$, $\underline{C}'$ be categories. Assume: $\underline{C}$, $\underline{C}'$ have coproducts and are complete. Suppose that $F:\underline{C} \to \underline{C}'$ is a functor which preserves finite limits -- then

$$F_*:[\underline{\Delta}^{OP},\underline{C}] \to [\underline{\Delta}^{OP},\underline{C}']$$

and $\forall$ $X \in \operatorname{Ob} \underline{SIC}$ and every finite $K \in \operatorname{Ob} \hat{\underline{\Delta}}$, the canonical arrow

$$F_*\hom(K,X) \to \hom(K,F_*X)$$

is an isomorphism.

PROOF Since

$$\hom(K,X)_n \approx \hom(K \times \Delta[n],X)_0 \quad \text{(cf. 21.12)}$$

and since $K \times \Delta[n]$ is finite, it will be enough to verify that

$$(F_*\hom(K,X))_0 = F\hom(K,X)_0 \approx \hom(K,F_*X)_0.$$

Per 21.15, write

$$K \approx \operatorname{colim}_i \Delta[n_i].$$

Then

$$\begin{aligned}
F\hom(K,X)_0 &\approx F\hom(\operatorname{colim}_i \Delta[n_i],X)_0 \\
&\approx F(X \pitchfork \operatorname{colim}_i \Delta[n_i]) \\
&\approx F(\lim_i X \pitchfork \Delta[n_i]) \quad \text{(cf. 21.10)} \\
&\approx \lim_i F(X \pitchfork \Delta[n_i]) \\
&\approx \lim_i FX_{n_i} \quad \text{(cf. 21.8)} \\
&\approx \lim_i (F_*X)_{n_i}
\end{aligned}$$

$$\approx \lim_i F_*X \pitchfork \Delta[n_i]$$

$$\approx F_*X \pitchfork \operatorname{colim}_i \Delta[n_i]$$

$$\approx F_*X \pitchfork K$$

$$\approx \hom(K, F_*X)_0.$$

21.17 APPLICATION Let $\underline{E}$ be a Grothendieck topos. Suppose that $p:\underline{E} \to \underline{SET}$ is a weak point -- then for every simplicial object X in $\underline{E}$ and for every finite simplicial set K, the canonical arrow

$$p_*\hom(K,X) \to \hom(K,p_*X)$$

is an isomorphism.

§22. *LIFTING*

Let $\underline{E}$ be a Grothendieck topos.

[Note: $\underline{E}$ is cocomplete (by definition), hence has coproducts, and is complete (cf. 18.14). Therefore the technology developed in §21 is applicable.]

22.1 DEFINITION A <u>geometric family</u> is a class $\mathcal{G}$ of monomorphisms of finite simplicial sets.

22.2 EXAMPLE The inclusions

$$\dot{\Delta}[n] \to \Delta[n] \quad (n \geq 0)$$

constitute a geometric family.

22.3 EXAMPLE The inclusions

$$\Lambda[k,n] \to \Delta[n] \quad (0 \leq k \leq n, \ n \geq 1)$$

constitute a geometric family.

Given an element $i:K \to L$ of a geometric family $\mathcal{G}$ and a morphism $\Xi:X \to Y$ of simplicial objects in $\underline{E}$, there is a commutative diagram

$$\begin{tikzcd}
\hom(L,X) \arrow[r, "i^*"] \arrow[d, "\Xi_*"'] & \hom(K,X) \arrow[d, "\Xi_*"] \\
\hom(L,Y) \arrow[r, "i^*"'] & \hom(K,Y)
\end{tikzcd}$$

which then leads to an arrow

$$(\Xi_*, i^*):\hom(L,X) \to \hom(L,Y) \times_{\hom(K,Y)} \hom(K,X)$$

or, upon evaluating at 0, to an arrow

$$(\Xi_*, i^*)_0 : \mathrm{hom}(L,X)_0 \to \mathrm{hom}(L,Y)_0 \times_{\mathrm{hom}(K,Y)_0} \mathrm{hom}(K,X)_0.$$

22.4 DEFINITION $\Xi:X \to Y$ has the <u>local right lifting property w.r.t. Ч</u> if $\forall$ $i:K \to L$ in Ч, the arrow $(\Xi_*, i^*)_0$ is an epimorphism in $\underline{E}$.

22.5 EXAMPLE Take $\underline{E}$ = $\underline{\mathrm{SET}}$ -- then $\Xi:X \to Y$ has the local right lifting property w.r.t. Ч iff $\Xi:X \to Y$ has the right lifting property w.r.t. Ч.

[For simplicial sets A and B,

$$\mathrm{hom}(A,B) = \mathrm{map}(A,B) \Rightarrow \mathrm{hom}(A,B)_0 = \mathrm{Mor}(A,B).]$$

22.6 NOTATION Given a geometric family Ч, denote by $\mathrm{LOC}_{Ч}(\underline{E})$ the class of morphisms in $\underline{\mathrm{SIE}}$ that have the local right lifting property w.r.t. Ч.

22.7 LEMMA Let $\underline{E}$, $\underline{F}$ be Grothendieck toposes and let $f:\underline{E} \to \underline{F}$ be a geometric morphism -- then

$$(f^*)_* \mathrm{LOC}_{Ч}(\underline{F}) \subset \mathrm{LOC}_{Ч}(\underline{E}).$$

[Apply 21.16 (f* preserves finite limits).]

[Note: By definition, $f^*:\underline{F} \to \underline{E}$. Therefore

$$(f^*)_*: [\underline{\Delta}^{\mathrm{OP}}, \underline{F}] \to [\underline{\Delta}^{\mathrm{OP}}, \underline{E}].]$$

Let $\Xi:X \to Y$ be a morphism of simplicial objects in $\underline{E}$. Suppose that $p:\underline{E} \to \underline{\mathrm{SET}}$ is a weak point of $\underline{E}$ -- then the compositions

$$\begin{cases} pX:\underline{\Delta}^{OP} \xrightarrow{X} \underline{E} \xrightarrow{p} \underline{SET} \\ pY:\underline{\Delta}^{OP} \xrightarrow{Y} \underline{E} \xrightarrow{p} \underline{SET} \end{cases}$$

are simplicial sets and

$$p\Xi:pX \rightarrow pY$$

is a simplicial map.

[Note: Here, $\forall$ n

$$\begin{cases} (pX)_n = pX_n \\ (pY)_n = pY_n \end{cases}, \ \Xi_n:X_n \rightarrow Y_n,$$

and $(p\Xi)_n = p\Xi_n$, thus

$$pX_n \xrightarrow{p\Xi_n} pY_n.]$$

22.8 CRITERION $\Xi:X \rightarrow Y$ has the local right lifting property w.r.t. $\natural$ iff for every weak point $p:\underline{E} \rightarrow \underline{SET}$, $p\Xi:pX \rightarrow pY$ has the right lifting property w.r.t. $\natural$.

It is obvious that $LOC_\natural(\underline{E})$ contains the isomorphisms.

22.9 LEMMA The class $LOC_\natural(\underline{E})$ is composition stable, pullback stable, and closed under the formation of retracts.

Let $\underline{I}$ be a small category — then $[\underline{I},\underline{E}]$ is a Grothendieck topos (cf. 18.17) and epimorphisms are levelwise.

N.B. There is an identification

$$[\underline{\Delta}^{OP},[\underline{I},\underline{E}]] \approx [\underline{I},[\underline{\Delta}^{OP},\underline{E}]].$$

22.10 LEMMA Denote by $\mathrm{LOC}_{\text{Ч}}(\underline{E})_{\underline{I}}$ the class of morphisms $\Xi:F \to G$ such that $\forall\ i \in \mathrm{Ob}\ \underline{I}$, $\Xi_i:Fi \to Gi$ is in $\mathrm{LOC}_{\text{Ч}}(\underline{E})$ -- then

$$\mathrm{LOC}_{\text{Ч}}(\underline{E})_{\underline{I}} = \mathrm{LOC}_{\text{Ч}}([\underline{I},\underline{E}]).$$

22.11 LEMMA The class $\mathrm{LOC}_{\text{Ч}}(\underline{E})$ is closed under the formation of filtered colimits.

[If $\underline{I}$ is filtered, then the functor

$$\mathrm{colim}_{\underline{I}}:[\underline{I},\underline{E}] \to \underline{E}$$

preserves finite limits. But $\mathrm{colim}_{\underline{I}}$ has a right adjoint, viz. the constant diagram functor. In other words, the data provides us with a geometric morphism $\underline{E} \to [\underline{I},\underline{E}]$. Now quote 22.7 (modulo 22.10).]

22.12 LEMMA $\Xi:X \to Y$ has the local right lifting property w.r.t. Ч if it has the right lifting property w.r.t. the arrows

$$\mathrm{id}_{\underline{A}}|\square|i:\underline{A}|\square|K \to \underline{A}|\square|L,$$

where A runs through the objects of $\underline{E}$ and $i:K \to L$ runs through the elements of Ч, i.e., if every commutative diagram

$$\begin{tikzcd}
\underline{A}|\square|K \arrow[r] \arrow[d, "\mathrm{id}_{\underline{A}}|\square|i"'] & X \arrow[d, "\Xi"] \\
\underline{A}|\square|L \arrow[r] & Y
\end{tikzcd}$$

admits a filler.

N.B. The arrow

$$\underline{A}|\square|K \to \underline{A}|\square|L$$

is a monomorphism.

[From the definitions,

$$\begin{cases} (\underline{A}|\square|K)_n = \coprod_{K_n} A \\ \\ (\underline{A}|\square|L)_n = \coprod_{L_n} A, \end{cases}$$

and K_n injects into L_n.]

22.13 REMARK There is a characterization, namely $\Xi: X \to Y$ has the local right lifting property w.r.t. Ч iff for every $A \in \mathrm{Ob}\ \underline{E}$, for every $i: K \to L$ in Ч, and for every commutative diagram

$$\begin{tikzcd} \underline{A}|\square|K \arrow[r] \arrow[d, "\mathrm{id}_{\underline{A}}|\square|i"'] & X \arrow[d, "\Xi"] \\ \underline{A}|\square|L \arrow[r] & Y \end{tikzcd},$$

one can find an $A' \in \mathrm{Ob}\ \underline{E}$ and an epimorphism $\pi: A' \to A$ with the property that the commutative diagram

$$\begin{tikzcd} \underline{A}'|\square|K \arrow[r, "\pi|\square|\mathrm{id}_K"] \arrow[d, "\mathrm{id}_{\underline{A}'}|\square|K"'] & \underline{A}|\square|K \arrow[r] & X \arrow[d, "\Xi"] \\ \underline{A}'|\square|K \arrow[r, "\pi|\square|\mathrm{id}_L"'] & \underline{A}|\square|L \arrow[r] & Y \end{tikzcd}$$

admits a filler.

§23. *LOCALIZERS OF DESCENT*

Let $\underline{E}$ be a Grothendieck topos.

23.1 DEFINITION Let $\Xi:X \to Y$ be a morphism of simplicial objects in $\underline{E}$ -- then Ξ is said to be a <u>hypercovering</u> of <u>SI$\underline{E}$</u> if it has the local right lifting property w.r.t. the inclusions $\dot{\Delta}[n] \to \Delta[n]$ $(n \geq 0)$.

[Note: Recall that

$$\begin{cases} \hom(\Delta[n],X)_0 \approx X_n \\ \hom(\Delta[n],Y)_0 \approx Y_n \end{cases} \qquad \text{(cf. 21.12).}$$

On the other hand,

$$\begin{cases} \hom(\dot{\Delta}[n],X)_0 \approx X \pitchfork \dot{\Delta}[n] \\ \hom(\dot{\Delta}[n],Y)_0 \approx Y \pitchfork \dot{\Delta}[n] \end{cases} \qquad \text{(cf. 21.11)}$$

and

$$\begin{cases} X \pitchfork \dot{\Delta}[n] \approx M_nX \\ Y \pitchfork \dot{\Delta}[n] \approx M_nY, \end{cases}$$

the symbols on the right standing for the matching object of $\begin{cases} X \\ Y \end{cases}$ familiar from "Reedy theory", thus

$$\begin{cases} M_nX(= M_{[n]}X) = (\operatorname{cosk}^{(n-1)}X)_n \\ M_nY(= M_{[n]}Y) = (\operatorname{cosk}^{(n-1)}Y)_n, \end{cases}$$

the matching morphisms being the canonical arrows

$$\begin{cases} X_n \to M_nX \\ Y_n \to M_nY. \end{cases}$$

Therefore the demand is that $\forall\ n \geq 0$, the arrow

$$X_n \to Y_n \times_{M_n Y} M_n X$$

is an epimorphism in $\underline{E}$.]

23.2 NOTATION $HR(\underline{E})$ is the class of hypercoverings of $\underline{SIE}$, so

$$HR(\underline{E}) = LOC_{\{\dot{\Delta}[n] \to \Delta[n]\ (n \geq 0)\}}(\underline{E}).$$

[Note: The stability properties formulated in 22.9 are in force here.]

23.3 EXAMPLE Take $\underline{E} = \underline{SET}$ -- then in this situation, $HR(\underline{E})$ is the class of acyclic Kan fibrations (cf. 22.5).

23.4 LEMMA Every hypercovering $\Xi: X \to Y$ is an epimorphism.

PROOF Since epimorphisms in $\underline{SIE}$ are levelwise, it suffices to prove that $\forall\ n$, $\Xi_n: X_n \to Y_n$ is an epimorphism in $\underline{E}$. To this end, let $p: \underline{E} \to \underline{SET}$ be a weak point -- then $p\Xi: pX \to pY$ has the right lifting property w.r.t. the $\dot{\Delta}[n] \to \Delta[n]$ $(n \geq 0)$ (cf. 22.8), hence is an acyclic Kan fibration, hence is an epimorphism (see below). But $p\Xi_n = (p\Xi)_n$ is an epimorphism in $\underline{SET}$, thus one can quote 19.23.

[Note: In $\underline{SISET}$, all objects are cofibrant, so in the commutative diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & pX \arrow[d, "p\Xi"] \\ pY \arrow[r, equal] & pY \end{tikzcd},$$

there is an arrow $w: pY \to pX$ such that $p\Xi \circ w = \mathrm{id}_{pY}$, which implies that $p\Xi$ is an epimorphism.]

23.5 LEMMA The hypercoverings are closed under the formation of products of pairs of arrows.

PROOF Suppose that

$$\begin{cases} \Xi_1 : X_1 \to Y_1 \\ \Xi_2 : X_2 \to Y_2 \end{cases}$$

are hypercoverings -- then for any weak point $p:\underline{E} \to \underline{SET}$,

$$p(\Xi_1 \times \Xi_2) \approx p(\Xi_1) \times p(\Xi_2).$$

But $\begin{cases} p\Xi_1 \\ p\Xi_2 \end{cases}$ are acyclic Kan fibrations and the product of two acyclic Kan fibrations is an acyclic Kan fibration. Now apply 22.8.

23.6 DEFINITION The $\underline{SI\underline{E}}$-localizer of descent is the $\underline{SI\underline{E}}$-localizer generated by $HR(\underline{E})$, i.e.,

$$W(HR(\underline{E})).$$

N.B. The elements of $W(HR(\underline{E}))$ are called the weak equivalences of descent.

23.7 EXAMPLE Take $\underline{E} = \underline{SET}$ -- then

$$W(HR(\underline{E})) = W(\emptyset),$$

the minimal $\hat{\underline{\Delta}}$-localizer.

[Since $HR(\underline{E})$ is the class of acyclic Kan fibrations (cf. 23.3), if W is a $\hat{\underline{\Delta}}$-localizer, then

$$\begin{aligned} W \supset \text{RLP}(M) &= \text{RLP}(\{\dot{\Delta}[n] \to \Delta[n] \ (n \geq 0)\}) \\ &= HR(\underline{E}). \end{aligned}$$

Therefore

$$W \supset W(HR(\underline{E})).]$$

23.8 LEMMA $W(HR(\underline{E}))$ is admissible.

Consequently, $\underline{SIE}$ admits a cofibrantly generated model structure whose class of weak equivalences are the elements of $W(HR(\underline{E}))$ and whose cofibrations are the monomorphisms (cf. 20.12).

23.9 REMARK The foregoing model structure on $\underline{SIE}$ is left proper (cf. 20.15) and right proper (use 20.17 (the elements of $HR(\underline{E})$ are pullback stable)).

N.B. $W(HR(\underline{E}))$ is closed under the formation of products of pairs of arrows (use 20.9 (cf. 23.5)).

23.10 RAPPEL The geometric morphism (Γ^*,Γ_*) of 18.2 extends to a geometric morphism $\underline{SIE} \to \underline{SISET}$ denoted by the same symbol. In particular:

$$\Gamma^*:\underline{SISET} \to \underline{SIE}$$

is defined by the prescription

$$(\Gamma^*K)_n = \coprod_{K_n} *_{\underline{E}}.$$

So $\forall\ X \in \text{Ob}\ \underline{SIE}$,

$$\begin{aligned}(X \times \Gamma^*K)_n &= X_n \times (\Gamma^*K)_n \\ &= X_n \times (\coprod_{K_n} *_{\underline{E}}) \\ &\approx \coprod_{K_n} X_n \times *_{\underline{E}} \quad \text{(cf. 18.20)}\end{aligned}$$

$$= \coprod_{K_n} X_n = (X|\overline{}|K)_n \quad \text{(cf. 21.2)}.$$

Therefore

$$X|\overline{}|K \approx X \times \Gamma^*K.$$

23.11 NOTATION Given $X \in \text{Ob}\ \underline{E}$, $\underline{X}$ is the constant simplicial object in $\underline{\text{SIE}}$.

23.12 DEFINITION Let W be a $\hat{\underline{\Delta}}$-localizer — then the $\underline{\text{SIE}}$-localizer of W-descent, denoted $W_{\underline{E}}$, is the $\underline{\text{SIE}}$-localizer generated by $HR(\underline{E})$ and by the morphisms

$$\text{id}_{\underline{X}}|\overline{}|f:\underline{X}|\overline{}|K \to \underline{X}|\overline{}|L,$$

where $X \in \text{Ob}\ \underline{E}$ and $f:K \to L$ is an arrow in W.

N.B. The elements of $W_{\underline{E}}$ are called the weak equivalences of W-descent.

23.13 LEMMA Suppose that $W = W(C)$ $(C \subset \text{Mor}\ \hat{\underline{\Delta}})$ — then $W_{\underline{E}}$ is generated by $HR(\underline{E})$ and by the morphisms

$$\text{id}_{\underline{X}}|\overline{}|f:\underline{X}|\overline{}|K \to \underline{X}|\overline{}|L,$$

where $X \in \text{Ob}\ \underline{E}$ and $f:K \to L$ is an arrow in C.

PROOF Letting $W_{\underline{E},C}$ be the $\underline{\text{SIE}}$-localizer generated by the morphisms in question, it is clear that $W_{\underline{E},C} \subset W_{\underline{E}}$. To go the other way, given $X \in \text{Ob}\ \underline{E}$, let

$$F_X:\hat{\underline{\Delta}} \to \underline{\text{SIE}}$$

be the functor that sends K to $\underline{X}|\overline{}|K$ $(\approx \underline{X} \times \Gamma^*K)$ — then $F_X^{-1}W_{\underline{E},C}$ is a $\hat{\underline{\Delta}}$-localizer (cf. infra) and

$$C \subset F_X^{-1}W_{\underline{E},C} \Rightarrow W \subset F_X^{-1}W_{\underline{E},C}.$$

Since this is true of all $X \in \text{Ob}\ \underline{E}$, it follows that $W_{\underline{E}} \subset W_{\underline{E},C}$.

[Note: The claim is that $F_X^{-1} W_{\underline{E},C}$ satisfies the three conditions of 20.3. E.g., to check condition (2), let $f:K \to L$ be an acyclic Kan fibration -- then $\Gamma^*f:\Gamma^*K \to \Gamma^*L$ is a hypercovering (cf. 22.7), thus the same is true of

$$\mathrm{id}_{\underline{X}} \times \Gamma^*f:\underline{X} \times \Gamma^*K \to \underline{X} \times \Gamma^*L \quad \text{(cf. 23.5)}.$$

I.e.:

$$\mathrm{id}_{\underline{X}} \times \Gamma^*f \in HR(\underline{E}).$$

Therefore $F_X^{-1} W_{\underline{E},C}$ contains the class of acyclic Kan fibrations, as claimed.]

N.B. The $\underline{SIE}$-localizer of $W(\emptyset)$-descent is the $\underline{SIE}$-localizer of descent.

23.14 EXAMPLE Consider the $\underline{SIE}$-localizer generated by $HR(\underline{E})$ and by the morphisms

$$\mathrm{id}_{\underline{X}} |\overline{\underline{\ }}| p_K:\underline{X} |\overline{\underline{\ }}| (K \times \Delta[1]) \to \underline{X} |\overline{\underline{\ }}| K \quad (K \in \text{Ob}\ \hat{\underline{\Delta}}).$$

Then this is the $\underline{SIE}$-localizer of W_∞-descent (cf. 20.6).

23.15 LEMMA If W is admissible, then $W_{\underline{E}}$ is admissible.

23.16 THEOREM If W is admissible, then $\underline{SIE}$ admits a cofibrantly generated model structure whose class of weak equivalences are the elements of $W_{\underline{E}}$ and whose cofibrations are the monomorphisms (cf. 20.12).

[Note: If the Cisinski structure on $\hat{\underline{\Delta}}$ per W is proper, then the Cisinski structure on $\underline{SIE}$ per $W_{\underline{E}}$ is proper.]

23.17 SCHOLIUM $\underline{SIE}$ admits a cofibrantly generated proper model structure whose class of weak equivalences are the elements of $(W_\infty)_{\underline{E}}$ and whose cofibrations are the monomorphisms.

23.18 LEMMA Every trivial fibration $\Xi:X \to Y$ is a hypercovering.

PROOF By definition, $\Xi \in \mathrm{RLP}(M)$, where $M \subset \mathrm{Mor}\ \underline{\mathrm{SI}E}$ is the class of monomorphisms. Accordingly, every commutative diagram

$$\begin{array}{ccc} \underline{A}|\underline{\ }|\dot{\Delta}[n] & \longrightarrow & X \\ \downarrow & & \downarrow \Xi \\ \underline{A}|\underline{\ }|\Delta[n] & \longrightarrow & Y \end{array} \quad (A \in \mathrm{Ob}\ \underline{E},\ n \geq 0)$$

admits a filler. Therefore Ξ has the local right lifting property w.r.t. the inclusions $\dot{\Delta}[n] \to \Delta[n]$ $(n \geq 0)$ (cf. 22.12). And this just means that Ξ is a hypercovering.

Let $\underline{E}$, $\underline{F}$ be Grothendieck toposes and let $f:\underline{E} \to \underline{F}$ be a geometric morphism -- then f induces a geometric morphism si $f:\underline{\mathrm{SI}E} \to \underline{\mathrm{SI}F}$, thus there is an adjoint pair (si f^*, si f_*) and si f^* preserves finite limits.

[Note: si $f^* = (f^*)_*$ (cf. 22.7).]

23.19 LEMMA Suppose that W is admissible -- then

$$\mathrm{si}\ f^* W_{\underline{F}} \subset W_{\underline{E}}.$$

PROOF Applying 22.7 (and bearing in mind 23.18), it follows that $(\mathrm{si}\ f^*)^{-1} W_{\underline{E}}$ is a $\underline{\mathrm{SI}F}$-localizer which contains the hypercoverings. On the other hand, if $Y \in \mathrm{Ob}\ \underline{F}$ and $f:K \to L$ is an arrow in W, then

$$(\mathrm{si}\ f)^*(\mathrm{id}_{\underline{Y}}|\underline{\ }|f) \approx \mathrm{id}_{\underline{f^*Y}}|\underline{\ }|f.$$

Therefore

$$W_{\underline{F}} \subset (\text{si } f^*)^{-1} W_{\underline{E}}$$

or still,

$$\text{si } f^* W_{\underline{F}} \subset W_{\underline{E}}.$$

23.20 THEOREM Suppose that W is admissible -- then the adjoint situation

$$\begin{cases} \text{si } f^*: \underline{\text{SI}F} \to \underline{\text{SI}E} \\ \text{si } f_*: \underline{\text{SI}E} \to \underline{\text{SI}F} \end{cases}$$

is a model pair.

PROOF In fact, si f* preserves finite limits, hence preserves cofibrations (these being the monomorphisms). Meanwhile, thanks to 23.19, si f* sends weak equivalences to weak equivalences.

Let $\underline{I}$ be a small category -- then $[\underline{I},\underline{E}]$ is a Grothendieck topos (cf. 18.17) and

$$\underline{\text{SI}}[\underline{I},\underline{E}] = [\underline{\Delta}^{\text{OP}},[\underline{I},\underline{E}]] \approx [\underline{I},[\underline{\Delta}^{\text{OP}},\underline{E}]] = [\underline{I},\underline{\text{SI}E}].$$

Let W be an admissible $\hat{\underline{\Delta}}$-localizer -- then $W_{\underline{E}}$ is an admissible $\underline{\text{SI}E}$-localizer (cf. 23.15), so it makes sense to form $(W_{\underline{E}})_{\underline{I}}$ (cf. 20.21), which is an admissible $[\underline{I},\underline{\text{SI}E}]$-localizer (cf. 20.22).

23.21 LEMMA In $[\underline{I},\underline{\text{SI}E}]$,

$$W_{[\underline{I},\underline{E}]} = (W_{\underline{E}})_{\underline{I}}.$$

Therefore the Cisinski structure on $[\underline{I},\underline{\text{SI}E}]$ per $W_{[\underline{I},\underline{E}]}$ is the injective structure on $[\underline{I},\underline{\text{SI}E}]$ w.r.t. the Cisinski structure on $\underline{\text{SI}E}$ per $W_{\underline{E}}$.

§24. LOCAL FIBRATIONS AND LOCAL WEAK EQUIVALENCES

Let $\underline{E}$ be a Grothendieck topos.

24.1 DEFINITION Let $\Xi: X \to Y$ be a morphism of simplicial objects in $\underline{E}$ -- then Ξ is said to be a local fibration if it has the local right lifting property w.r.t. the inclusions $\Lambda[k,n] \to \Delta[n]$ $(0 \leq k \leq n,\ n \geq 1)$.

24.2 LEMMA $\Xi: X \to Y$ is a local fibration iff for every weak point $p:\underline{E} \to \underline{SET}$, $p\Xi: pX \to pY$ is a Kan fibration (cf. 22.8).

N.B. Therefore the hypercoverings are local fibrations.

24.3 LEMMA Let $\Xi: X \to Y$ be a local fibration and let $i: K \to L$ be a monomorphism of finite simplicial sets -- then the arrow

$$(\Xi_*, i^*): \hom(L,X) \to \hom(L,Y) \times_{\hom(K,Y)} \hom(K,X)$$

is a local fibration which is a hypercovering if Ξ is a hypercovering or i is a simplicial weak equivalence.

[Note: These conditions are reminiscent of those figuring in the definition of "simplicial model category".]

24.4 DEFINITION Consider $\underline{SIE}$ in its Cisinski structure per an admissible $W \subset \operatorname{Mor} \hat{\underline{\Delta}}$ (cf. 23.16) -- then the elements of

$$\mathrm{RLP}(W_{\underline{E}} \cap M)$$

are called the fibrations of W-descent.

24.5 EXAMPLE Take $W = W_\infty$ -- then every fibration $\Xi: X \to Y$ of W_∞-descent is a local fibration.

[In view of 22.12, it suffices to show that every commutative diagram

$$\begin{array}{ccc} \underline{A}|\square|\Lambda[k,n] & \longrightarrow & X \\ \Big| & & \Big\downarrow \Xi \\ \underline{A}|\square|\Delta[n] & \longrightarrow & Y \end{array} \qquad (A \in \mathrm{Ob}\ \underline{E},\ 0 \le k \le n,\ n \ge 1)$$

admits a filler. But this is plain: The arrow

$$\underline{A}|\square|\Lambda[k,n] \longrightarrow \underline{A}|\square|\Delta[n]$$

is both a weak equivalence of W_∞-descent and a monomorphism.]

24.6 REMARK Suppose that $\underline{E}$ satisfies the axiom of choice — then in this case, the fibrations of W_∞-descent are precisely the local fibrations (Rezk†).

24.7 DEFINITION A simplicial object X in $\underline{E}$ is said to be <u>locally fibrant</u> if the arrow $X \to *_{\underline{SIE}}$ is a local fibration.

24.8 LEMMA X is locally fibrant iff for every weak point $p:\underline{E} \to \underline{SET}$, pX is a Kan complex.

24.9 EXAMPLE If X is locally fibrant and if K is a finite simplicial set, then hom(K,X) is locally fibrant.

[In fact, $\forall$ weak point $p:\underline{E} \to \underline{SET}$,

$$\begin{aligned} p_* \mathrm{hom}(K,X) &\approx \mathrm{hom}(K,p_*X) \qquad \text{(cf. 21.17)} \\ &\Xi\ \mathrm{map}(K,p_*X) \end{aligned}$$

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† arXiv:math/9811038

or still, dropping the sub-*,

$$\mathrm{phom}(K,X) \approx \mathrm{map}(K,pX).$$

But

$$pX \text{ Kan} \Rightarrow \mathrm{map}(K,pX) \text{ Kan.]}$$

24.10 EXAMPLE If X is locally fibrant, then $\mathrm{hom}(\Delta[1],X)$ is locally fibrant and there is a local fibration

$$\mathrm{hom}(\Delta[1],X) \to X \times X.$$

[In 24.3, let $K = \Delta[0] \coprod \Delta[0]$, $L = \Delta[1]$.]

24.11 NOTATION Let $\underline{\mathrm{SI}E}_{\ell oc}$ be the full subcategory of $\underline{\mathrm{SI}E}$ whose objects are locally fibrant.

24.12 DEFINITION Let $\Xi:X \to Y$ be a morphism of locally fibrant simplicial objects in $\underline{E}$ -- then Ξ is said to be a <u>local weak equivalence</u> if for every weak point $p:\underline{E} \to \underline{\mathrm{SET}}$, $p\Xi:pX \to pY$ is a simplicial weak equivalence, i.e., $p\Xi \in W_\infty$.

[Note: Take $\underline{E} = \underline{\mathrm{SET}}$ -- then it is true but not obvious that "local weak equivalence" coincides with "simplicial weak equivalence" (cf. 24.23).]

24.13 RAPPEL Consider a triple $(\underline{C},W,\mathrm{fib})$, where $\underline{C}$ is a category with a final object $*$ and

$$\begin{cases} W \subset \mathrm{Mor}\ \underline{C} \\ \mathrm{fib} \subset \mathrm{Mor}\ \underline{C} \end{cases}$$

are two composition closed classes of morphisms termed

$$\begin{cases} \underline{\text{weak equivalences}} \\ \underline{\text{fibrations}}, \end{cases}$$

the <u>acyclic fibrations</u> being the elements of

$$W \cap \text{fib}.$$

Then $\underline{C}$ is said to be a category of fibrant objects provided that the following axioms are satisfied.

(FIB-1) For every object X in $\underline{C}$, the arrow $X \to *$ is a fibration.

(FIB-2) All isomorphisms are weak equivalences and all isomorphisms are fibrations.

(FIB-3) Given composable morphisms f,g, if any two of $f,g,g \circ f$ are weak equivalences, so is the third.

(FIB-4) Every 2-sink $X \xrightarrow{f} Z \xleftarrow{g} Y$, where g is a fibration (acyclic fibration), admits a pullback $X \xleftarrow{\xi} P \xrightarrow{\eta} Y$, where ξ is a fibration (acyclic fibration):

$$\begin{tikzcd} P \arrow[r, "\eta"] \arrow[d, "\xi"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z. \end{tikzcd}$$

(FIB-5) Every morphism in $\underline{C}$ can be written as the composite of a weak equivalence and a fibration.

24.14 THEOREM Take $\underline{C} = \underline{\mathrm{SI}\mathcal{E}}_{\ell oc}$ and let

$$\begin{cases} W = \text{the local weak equivalences} \\ \text{fib} = \text{the local fibrations.} \end{cases}$$

Then the triple $(\underline{C}, W, \text{fib})$ is a category of fibrant objects and the acyclic fibrations are the hypercoverings.

[Note: Given an arrow Ξ in $\underline{\mathrm{SIE}}_{\ell oc}$, one can write $\Xi = q \circ j$, where q is a local fibration and j is a local weak equivalence with the property that it has a left inverse r which is a hypercovering ($r \circ j = \mathrm{id}$).]

24.15 LEMMA Suppose that $\Xi:X \to Y$ is a local weak equivalence -- then Ξ is a weak equivalence of descent.

PROOF Write $\Xi = q \circ j$ per supra -- then q is a local weak equivalence (this being the case of Ξ and j). But q is also a local fibration, thus q is a hypercovering, thus q is a weak equivalence of descent. As for j, it too is a weak equivalence of descent. To see this, recall that $W(HR(\underline{E}))$ is the class of weak equivalences for a model structure on $\underline{\mathrm{SIE}}$, hence is saturated:

$$\overline{W(HR(\underline{E}))} = W(HR(\underline{E})).$$

Therefore any arrow whose image in the homotopy category is an isomorphism is necessarily in $W(HR(\underline{E}))$. But $r \circ j = \mathrm{id}$ and $r \in HR(\underline{E})$, hence is invertible in the homotopy category, hence the same holds for j, i.e., j is a weak equivalence of descent.

The functor $\underline{E} \to \underline{\mathrm{SIE}}$ that sends X to $\underline{X}$ (cf. 23.11) has a left adjoint $\pi_0:\underline{\mathrm{SIE}} \to \underline{E}$ that sends X to the coequalizer of the arrows

$$\begin{cases} d_0:X_1 \to X_0 \\ d_1:X_1 \to X_0, \end{cases}$$

so

$$X_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} X_0 \longrightarrow \pi_0 X.$$

[Note: Take $\underline{E} = \underline{\mathrm{SET}}$ -- then in the context of simplicial sets, π_0 preserves finite products and $\pi_0 X$ can be identified with the set of components of X.]

24.16 LEMMA Suppose that X is locally fibrant -- then for every weak point $p:\underline{E} \to \underline{\mathrm{SET}}$, the canonical map

$$\pi_0 pX \longrightarrow p\pi_0 X$$

is bijective.

PROOF Let R be the image of the arrow

$$(d_0,d_1):X_1 \to X_0 \times X_0.$$

Then R is a relation on X_0 and $\forall$ weak point $p:\underline{E} \to \underline{\mathrm{SET}}$, pX is a Kan complex and pR is an equivalence relation on pX_0. Therefore R is an equivalence relation on X_0 and the canonical map

$$\pi_0 pX \longrightarrow p\pi_0 X$$

is bijective (cf. 19.27).

24.17 RAPPEL The class of all weak points of $\underline{E}$ is faithful (cf. 19.20), hence reflects isomorphisms (cf. 19.19).

24.18 LEMMA The restriction of π_0 to $\underline{\mathrm{SI}E}_{\ell oc}$ preserves finite products.

PROOF To check that the canonical arrow

$$\pi_0(X \times Y) \longrightarrow \pi_0 X \times \pi_0 Y$$

is an isomorphism, let $p:\underline{E} \to \underline{\mathrm{SET}}$ be a weak point and note that

$$\begin{aligned} p\pi_0(X \times Y) &\approx \pi_0 p(X \times Y) \\ &\approx \pi_0(pX \times pY) \end{aligned}$$

$$\approx \pi_0 pX \times \pi_0 pY$$

$$\approx p\pi_0 X \times p\pi_0 Y$$

$$\approx p(\pi_0 X \times \pi_0 Y).$$

[Note: It is clear that π_0 preserves final objects.]

24.19 LEMMA Let $\Xi:X \to Y$ be a local weak equivalence — then $\pi_0\Xi:\pi_0 X \to \pi_0 Y$ is an isomorphism.

PROOF Take a weak point $p:\underline{E} \to \underline{SET}$ and consider the commutative diagram

$$\begin{tikzcd}
p\pi_0 X \arrow[r, "p\pi_0\Xi"] \arrow[d, "\approx"'] & p\pi_0 Y \arrow[d, "\approx"] \\
\pi_0 pX \arrow[r, "\pi_0 p\Xi"'] & \pi_0 pY.
\end{tikzcd}$$

Since $p\Xi$ is a simplicial weak equivalence, the arrow

$$\pi_0 p\Xi:\pi_0 pX \to \pi_0 pY$$

is bijective. Therefore the arrow

$$p\pi_0\Xi:p\pi_0 X \to p\pi_0 Y$$

is bijective.

The preceding considerations can be extended from π_0 to π_n $(n \geq 1)$ but before doing this it will be best to review how things go for simplicial sets (i.e., the case $\underline{E} = \underline{SET}$).

Thus given a Kan complex X, let

$$\pi_n X = \coprod_{x_0 \in X_0} \pi_n(X, x_0).$$

Then there is a map $c_n : \pi_n X \to X_0$ and $\pi_n X$ is a group object in $\underline{\mathrm{SET}}/X_0$ (abelian if $n \geq 2$).

[Note: The construction $X \to \pi_n X$ is functorial in X and natural w.r.t. c_n.]

N.B. Denote by $\Omega^n X$ the n^{th} loop space of X -- then $\Omega^n X$ is a Kan complex and

$$\pi_0 \Omega^n X = \pi_n X.$$

24.20 THEOREM Let X and Y be Kan complexes, $f: X \to Y$ a simplicial map -- then f is a simplicial weak equivalence iff $\pi_0 f : \pi_0 X \to \pi_0 Y$ is bijective and $\forall\ n \geq 1$, the commutative diagram

$$\begin{tikzcd} \pi_n X \arrow[r, "\pi_n f"] \arrow[d, "c_n"'] & \pi_n Y \arrow[d, "c_n"] \\ X_0 \arrow[r, "f_0"'] & Y_0 \end{tikzcd}$$

is a pullback square.

While I shall omit the particulars, the story for an arbitrary $\underline{E}$ is analogous: One can assign to each locally fibrant X its n^{th} loop space $\Omega^n X$, a locally fibrant simplicial object in $\underline{E}$, and

$$\pi_0 \Omega^n X = \pi_n X.$$

N.B. There is a map $c_n:\pi_n X \to X_0$ and for any $\Xi:X \to Y$, there is a commutative diagram

$$\begin{tikzcd} \pi_n X \arrow[r, "\pi_n \Xi"] \arrow[d, "c_n"'] & \pi_n Y \arrow[d, "c_n"] \\ X_0 \arrow[r, "\Xi_0"'] & Y_0. \end{tikzcd}$$

24.21 LEMMA Let $p:\underline{E} \to \underline{SET}$ be a weak point — then

$$p\Omega^n X \approx \Omega^n pX.$$

PROOF The formalities give rise to a pullback square

$$\begin{tikzcd} \Omega^n X \arrow[r] \arrow[d] & \hom(\Delta[1],\Omega^{n-1}X) \arrow[d] \\ \underline{X}_0 \arrow[r] & \Omega^{n-1}X \times \Omega^{n-1}X, \end{tikzcd}$$

the vertical arrow on the RHS being an instance of 24.10. Now apply p -- then the commutative diagram

$$\begin{tikzcd} p\Omega^n X \arrow[r] \arrow[d] & p\hom(\Delta[1],\Omega^{n-1}X) \arrow[d] \\ p\underline{X}_0 \arrow[r] & p\Omega^{n-1}X \times p\Omega^{n-1}X \end{tikzcd}$$

is a pullback square in $\underline{SISET}$. Proceeding inductively, it can be assumed that

$$p\Omega^{n-1}X \approx \Omega^{n-1}pX.$$

Here $p\underline{X}_0 = (p\underline{X})_0$ and

$$\begin{aligned} \mathrm{phom}(\Delta[1],\Omega^{n-1}X) &\approx \mathrm{hom}(\Delta[1],p\Omega^{n-1}X) \quad (\text{cf. } 21.17) \\ &\approx \mathrm{hom}(\Delta[1],\Omega^{n-1}pX). \end{aligned}$$

But the commutative diagram

$$\begin{tikzcd} \Omega^n pX \arrow[r] \arrow[d] & \mathrm{hom}(\Delta[1],\Omega^{n-1}pX) \arrow[d] \\ (p\underline{X})_0 \arrow[r] & \Omega^{n-1}pX \times \Omega^{n-1}pX \end{tikzcd}$$

is also a pullback square in SISET. Therefore

$$p\Omega^n X \approx \Omega^n pX.$$

[Note: If $n = 1$, then there is a pullback square

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & \mathrm{hom}(\Delta[1],X) \arrow[d] \\ \underline{X}_0 \arrow[r] & X \times X \end{tikzcd}$$

from which a pullback square

$$\begin{tikzcd} p\Omega X \arrow[r] \arrow[d] & \mathrm{phom}(\Delta[1],X) \arrow[d] \\ p\underline{X}_0 \arrow[r] & pX \times pX \end{tikzcd}$$

in SISET. But

$$\mathrm{phom}(\Delta[1],X) \approx \mathrm{hom}(\Delta[1],pX) \quad (\text{cf. } 21.17)$$

and the commutative diagram

$$\begin{tikzcd} \Omega pX \arrow[r] \arrow[d] & \mathrm{hom}(\Delta[1],pX) \arrow[d] \\ (p\underline{X})_0 \arrow[r] & pX \times pX \end{tikzcd}$$

is also a pullback square in SISET. Therefore

$$p\Omega X \approx \Omega pX.]$$

24.22 LEMMA Let $p:\underline{E} \to \underline{SET}$ be a weak point — then

$$\pi_n pX \approx p\pi_n X.$$

PROOF In fact,

$$\begin{aligned}\pi_n pX &= \pi_0 \Omega^n pX \\ &\approx \pi_0 p\Omega^n X \quad (\text{cf. } 24.21) \\ &\approx p\pi_0 \Omega^n X \quad (\text{cf. } 24.16) \\ &= p\pi_n X.\end{aligned}$$

24.23 THEOREM Let X and Y be Kan complexes, $f:X \to Y$ a simplicial map -- then f is a local weak equivalence iff f is a simplicial weak equivalence.

PROOF The nontrivial claim is that if f is a simplicial weak equivalence, then for any weak point $p:\underline{SET} \to \underline{SET}$, $pf:pX \to pY$ is a simplicial weak equivalence, and to establish this, we shall apply 24.20.

- Consider the commutative diagram

$$\begin{tikzcd}
\pi_0 pX \arrow[r, "\pi_0 pf"] \arrow[d, "\approx"'] & \pi_0 pY \arrow[d, "\approx"] \\
p\pi_0 X \arrow[r, "p\pi_0 f"'] & p\pi_0 Y.
\end{tikzcd}$$

Then $\pi_0 f$ is bijective, hence $p\pi_0 f$ is bijective, hence $\pi_0 pf$ is bijective.

- The commutative diagram

$$\begin{tikzcd} \pi_n X \arrow[r, "\pi_n f"] \arrow[d, "c_n"'] & \pi_n Y \arrow[d, "c_n"] \\ X_0 \arrow[r, "f_0"'] & Y_0 \end{tikzcd}$$

is a pullback square, thus the commutative diagram

$$\begin{tikzcd} p\pi_n X \arrow[r, "p\pi_n f"] \arrow[d, "c_n"'] & p\pi_n Y \arrow[d, "c_n"] \\ pX_0 \arrow[r, "pf_0"'] & pY_0 \end{tikzcd}$$

is a pullback square. But

$$\begin{tikzcd} p\pi_n X \arrow[r, "p\pi_n f"] \arrow[d, "\approx"'] & p\pi_n Y \arrow[d, "\approx"] \\ \pi_n pX \arrow[r, "\pi_n pf"'] & \pi_n pY. \end{tikzcd}$$

Therefore the commutative diagram

$$\begin{tikzcd} \pi_n pX \arrow[r, "\pi_n pf"] \arrow[d, "c_n"'] & \pi_n pY \arrow[d, "c_n"] \\ pX_0 \arrow[r, "pf_0"'] & pY_0 \end{tikzcd}$$

is a pullback square.

24.24 THEOREM Let $\Xi:X \to Y$ be a morphism of locally fibrant simplicial objects in $\underline{E}$ -- then Ξ is a local weak equivalence iff $\pi_0\Xi:\pi_0 X \to \pi_0 Y$ is an isomorphism and $\forall\ n \geq 1$, the commutative diagram

$$\begin{tikzcd} \pi_n X \arrow[r, "\pi_n\Xi"] \arrow[d, "c_n"'] & \pi_n Y \arrow[d, "c_n"] \\ X_0 \arrow[r, "\Xi_0"'] & Y_0 \end{tikzcd}$$

is a pullback square.

Every local weak equivalence is a weak equivalence of descent (cf. 24.15), hence is a weak equivalence of W_∞-descent. When $\underline{E} = \underline{SET}$, this can be turned around: Every weak equivalence of W_∞-descent (a.k.a. simplicial weak equivalence) is a local weak equivalence (cf. 24.23), a conclusion that persists to an arbitrary $\underline{E}$.

24.25 LEMMA Let $\Xi:X \to Y$ be a morphism of locally fibrant simplicial objects in $\underline{E}$. Assume: Ξ is a weak equivalence of W_∞-descent -- then Ξ is a local weak equivalence.

[The full proof is lengthy and technical but here is the strategy. First treat the case when $Y = *$ and use it to treat the case when in addition the arrow $Y \to *$ is a fibration of W_∞-descent. This done, factor $Y \to *$ as

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "j"'] & * \arrow[d, equal] \\ Y' \arrow[r] & *, \end{tikzcd}$$

where j is an acyclic cofibration (thus a weak equivalence of W_∞-descent) and $Y' \to *$ is a fibration of W_∞-descent. Consider

$$X \xrightarrow{\Xi} Y \xrightarrow{j} Y'.$$

Then j is a local weak equivalence and $j \circ \Xi$ is a local weak equivalence. Therefore Ξ is a local weak equivalence.

[Note: Another approach is to use 24.6 and prove it initially under the assumption that $\underline{E}$ satisfies the axiom of choice. To proceed in general, take $f:\underline{B} \to \underline{E}$ as in 18.29 -- then si $f^*\Xi$ is a weak equivalence of W_∞-descent (cf. 23.19), hence is a local weak equivalence. And from there it is not difficult to see that Ξ is a local weak equivalence.]

Using standard methods, one can introduce a functor

$$\mathrm{Ex}^\infty:\underline{\mathrm{SI}E} \to \underline{\mathrm{SI}E}$$

and a natural transformation

$$e^\infty:\mathrm{id}_{\underline{\mathrm{SI}E}} \to \mathrm{Ex}^\infty$$

with the property that if X is a locally fibrant simplicial object in $\underline{E}$, then $\mathrm{Ex}^\infty X$ is a locally fibrant simplicial object in $\underline{E}$ and the arrow $e_X^\infty:X \to \mathrm{Ex}^\infty X$ is a local weak equivalence.

24.26 LEMMA If X is a locally fibrant simplicial object in $\underline{E}$, then the arrow $e_X^\infty:X \to \mathrm{Ex}^\infty X$ induces an isomorphism

$$\pi_0 X \to \pi_0 \mathrm{Ex}^\infty X \quad \text{(cf. 24.19)}$$

and $\forall\ n \geq 1$,

$$\pi_n X \approx \pi_n \mathrm{Ex}^\infty X.$$

PROOF The commutative diagram

$$\begin{array}{ccc}\pi_n X & \xrightarrow{\pi_n e_X^\infty} & \pi_n \mathrm{Ex}^\infty X \\ {\scriptstyle c_n}\downarrow & & \downarrow{\scriptstyle c_n} \\ X_0 & \xrightarrow[(e_X^\infty)_0]{} & (\mathrm{Ex}^\infty X)_0\end{array}$$

is a pullback square (cf. 24.24). But $(e_X^\infty)_0$ is an isomorphism and the pullback of an isomorphism is an isomorphism. Therefore $\pi_n e_X^\infty$ is an isomorphism.

24.27 LEMMA If X is a simplicial object in $\underline{E}$, then $\mathrm{Ex}^\infty X$ is a locally fibrant simplicial object in $\underline{E}$ and the arrow $e_X^\infty : X \to \mathrm{Ex}^\infty X$ is a weak equivalence of W_∞-descent.

24.28 DEFINITION Given $X \in \mathrm{Ob}\ \underline{SIE}$, put

$$\pi_n X = \pi_n \mathrm{Ex}^\infty X \quad (n \geq 1).$$

[Note: Up to isomorphism, matters are consistent when $X \in \mathrm{Ob}\ \underline{SIE}_{\ell oc}$ (cf. 24.26).]

24.29 THEOREM Let $\Xi : X \to Y$ be a morphism of simplicial objects in $\underline{E}$ -- then the following conditions are equivalent.

(1) Ξ is a weak equivalence of W_∞-descent.

(2) $\mathrm{Ex}^\infty \Xi$ is a weak equivalence of W_∞-descent.

(3) $\mathrm{Ex}^\infty \Xi$ is a local weak equivalence.

(4) $\pi_0 \Xi : \pi_0 X \to \pi_0 Y$ is an isomorphism and $\forall\ n \geq 1$, the commutative diagram

$$\begin{tikzcd} \pi_n X \arrow[r, "\pi_n \Xi"] \arrow[d, "c_n"'] & \pi_n Y \arrow[d, "c_n"] \\ X_0 \arrow[r, "\Xi_0"'] & Y_0 \end{tikzcd}$$

is a pullback square.

PROOF Taking into account 24.27, the equivalence of (1) and (2) results upon inspection of the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\Xi"] \arrow[d, "e_X^\infty"'] & Y \arrow[d, "e_Y^\infty"] \\ \mathrm{Ex}^\infty X \arrow[r, "\mathrm{Ex}^\infty \Xi"'] & \mathrm{Ex}^\infty Y. \end{tikzcd}$$

Next, since $\mathrm{Ex}^\infty X$ and $\mathrm{Ex}^\infty Y$ are locally fibrant, the equivalence of (2) and (3) follows from 24.25. Finally, in view of 24.24, the equivalence of (3) and (4) can be read off from consideration of

$$\begin{tikzcd} \pi_0 X \arrow[r, "\pi_0 \Xi"] \arrow[d, "\approx"'] & \pi_0 Y \arrow[d, "\approx"] \\ \pi_0 \mathrm{Ex}^\infty X \arrow[r, "\pi_0 \mathrm{Ex}^\infty \Xi"'] & \pi_0 \mathrm{Ex}^\infty Y \end{tikzcd}$$

and

$$\begin{array}{ccc}
\pi_n X = \pi_n \mathrm{Ex}^\infty X & \xrightarrow{\pi_n \mathrm{Ex}^\infty \Xi} & \pi_n \mathrm{Ex}^\infty Y = \pi_n Y \\
\Big\downarrow c_n & & \Big\downarrow c_n \\
X_0 \approx (\mathrm{Ex}^\infty X)_0 & \xrightarrow[(\mathrm{Ex}^\infty \Xi)_0]{} & (\mathrm{Ex}^\infty Y)_0 \approx Y_0.
\end{array}$$

Let

$$\left[\begin{array}{l}
\mathcal{W}_\infty = \text{the local weak equivalences} \\
(W_\infty)_{\underline{E}} = \text{the weak equivalences of } W_\infty\text{-descent.}
\end{array}\right.$$

24.30 LEMMA The arrow of inclusion

$$i_{\ell oc}:\underline{\mathrm{SI}E}_{\ell oc} \to \underline{\mathrm{SI}E}$$

is a morphism of category pairs (cf. 25.9) and the induced functor

$$\overline{i_{\ell oc}}:\mathcal{W}_\infty^{-1}\underline{\mathrm{SI}E}_{\ell oc} \to (W_\infty)_{\underline{E}}^{-1}\underline{\mathrm{SI}E}$$

is an equivalence of categories.

[Use Ex^∞ to construct a functor in the opposite direction.]

24.31 NOTATION Put

$$\underline{H_\infty \mathrm{SI}E} = (W_\infty)_{\underline{E}}^{-1}\underline{\mathrm{SI}E}.$$

24.32 LEMMA The arrow

$$\underline{E} \to \underline{H_\infty \mathrm{SI}E}$$

that sends X to the image of $\underline{X}$ in the homotopy category is fully faithful.

§25. *COMPARISON PRINCIPLES*

Let $\underline{C}$ be a small category -- then

$$\begin{aligned}\underline{\mathrm{SI}\hat{C}} &= [\underline{\Delta}^{\mathrm{OP}}, [\underline{C}^{\mathrm{OP}},\underline{\mathrm{SET}}]] \\ &\approx [\underline{C}^{\mathrm{OP}}, [\underline{\Delta}^{\mathrm{OP}},\underline{\mathrm{SET}}]] \\ &= [\underline{C}^{\mathrm{OP}},\underline{\mathrm{SISET}}].\end{aligned}$$

25.1 LEMMA Let W be an admissible $\hat{\underline{\Delta}}$-localizer — then the elements of $W_{\hat{\underline{C}}}$ are levelwise the elements of W.

PROOF In 23.21, let $\underline{I} = \underline{C}^{\mathrm{OP}}$ and $\underline{E} = \underline{\mathrm{SET}}$.

25.2 REMARK Since

$$\underline{\mathrm{SI}\hat{C}} \approx [\underline{C}^{\mathrm{OP}},\underline{\mathrm{SISET}}],$$

it follows that if W is an admissible $\hat{\underline{\Delta}}$-localizer and if the Cisinski structure on $\underline{\mathrm{SISET}}$ determined by W is proper, then the Cisinski structure on $\underline{\mathrm{SI}\hat{C}}$ determined by $W_{\hat{\underline{C}}}$ is proper.

Let $\underline{C}$ be a small category, τ a Grothendieck topology on $\underline{C}$.

25.3 RAPPEL The inclusion $\iota_\tau:\underline{\mathrm{Sh}}_\tau(\underline{C}) \to \hat{\underline{C}}$ admits a left adjoint $\underline{a}_\tau:\hat{\underline{C}} \to \underline{\mathrm{Sh}}_\tau(\underline{C})$ that preserves finite limits (cf. 11.14).

Abusing the notation, we shall use the same symbols $\begin{bmatrix} \underline{a}_\tau \\ \iota_\tau \end{bmatrix}$ for the induced adjoint pair

$$\begin{bmatrix} \underline{\mathrm{SI}\hat{C}} \longrightarrow \underline{\mathrm{SISh}}_\tau(\underline{C}) \\ \underline{\mathrm{SISh}}_\tau(\underline{C}) \longrightarrow \underline{\mathrm{SI}\hat{C}}. \end{bmatrix}$$

25.4 DEFINITION Let $\Xi:X \to Y$ be a morphism of simplicial objects in $\hat{\underline{C}}$ -- then Ξ is said to be a <u>τ-hypercovering</u> if its image $\underline{a}_\tau \Xi$ is a hypercovering of $\underline{\mathrm{SISh}}_\tau(\underline{C})$.

25.5 DEFINITION Let W be a $\hat{\underline{\Delta}}$-localizer -- then the <u>$\underline{\mathrm{SI}}\hat{\underline{C}}$-localizer of (W,τ)-descent</u>, denoted $W_{\hat{\underline{C}}}(\tau)$, is the $\underline{\mathrm{SI}}\hat{\underline{C}}$-localizer generated by the τ-hypercoverings and by the morphisms

$$\mathrm{id}_{\underline{X}} \,|\overline{\underline{\ }}|\, f:\underline{X}\,|\overline{\underline{\ }}|\,K \to \underline{X}\,|\overline{\underline{\ }}|\,L,$$

where $X \in \mathrm{Ob}\,\hat{\underline{C}}$ and $f:K \to L$ is an arrow in W.

<u>N.B.</u> The elements of $W_{\hat{\underline{C}}}(\tau)$ are called the <u>weak equivalences of (W,τ)-descent</u> and the elements of

$$\mathrm{RLP}(W_{\hat{\underline{C}}}(\tau) \cap M)$$

are called the <u>fibrations of (W,τ)-descent</u>.

25.6 EXAMPLE Take for τ the minimal Grothendieck topology on $\underline{C}$ (cf. 11.11) -- then $\underline{\mathrm{Sh}}_\tau(\underline{C}) = \hat{\underline{C}}$ and $W_{\hat{\underline{C}}}(\tau) = W_{\hat{\underline{C}}}$.

25.7 LEMMA If X is a simplicial object in $\hat{\underline{C}}$, then the canonical arrow $X \to \iota_\tau \underline{a}_\tau X$ is a weak equivalence of (W,τ)-descent.

25.8 THEOREM Let W be a $\hat{\underline{\Delta}}$-localizer -- then

$$\left[\begin{array}{l} \underline{a}_\tau^{-1} W_{\underline{\mathrm{Sh}}_\tau(\underline{C})} = W_{\hat{\underline{C}}}(\tau) \\[1em] \iota_\tau^{-1} W_{\hat{\underline{C}}}(\tau) = W_{\underline{\mathrm{Sh}}_\tau(\underline{C})}. \end{array}\right.$$

PROOF The pair $(\underline{a}_\tau, \iota_\tau)$ defines a geometric morphism $\underline{Sh}_\tau(\underline{C}) \to \hat{\underline{C}}$ and $\underline{a}_\tau^{-1} W_{\underline{Sh}_\tau(\underline{C})}$ is a $\underline{SI\hat{C}}$-localizer which contains $W_{\hat{\underline{C}}}$ (cf. 23.19). In particular: The

$$\mathrm{id}_{\underline{X}} |\square| f \in \underline{a}_\tau^{-1} W_{\underline{Sh}_\tau(\underline{C})}.$$

But the τ-hypercoverings are also in $\underline{a}_\tau^{-1} W_{\underline{Sh}_\tau(\underline{C})}$, thus

$$\underline{a}_\tau^{-1} W_{\underline{Sh}_\tau(\underline{C})} \supset W_{\hat{\underline{C}}}(\tau).$$

As for $\iota_\tau^{-1} W_{\hat{\underline{C}}}(\tau)$, it is a $\underline{SISh}_\tau(\underline{C})$-localizer and

$$\iota_\tau^{-1} W_{\hat{\underline{C}}}(\tau) \supset W_{\underline{Sh}_\tau(\underline{C})}.$$

- Let $\Xi: X \to Y$ be an element of $\underline{a}_\tau^{-1} W_{\underline{Sh}_\tau(\underline{C})}$ -- then the claim is that $\Xi \in W_{\hat{\underline{C}}}(\tau)$. To see this, consider the commutative diagram

$$\begin{tikzcd}
\iota_\tau \underline{a}_\tau X \arrow[r, "\iota_\tau \underline{a}_\tau \Xi"] & \iota_\tau \underline{a}_\tau Y \\
X \arrow[u] \arrow[r, "\Xi"'] & Y \arrow[u]
\end{tikzcd}.$$

Here

$$\underline{a}_\tau \Xi \in W_{\underline{Sh}_\tau(\underline{C})} \subset \iota_\tau^{-1} W_{\hat{\underline{C}}}(\tau)$$

$$\Rightarrow$$

$$\iota_\tau \underline{a}_\tau \Xi \in W_{\hat{\underline{C}}}(\tau).$$

On the other hand, the vertical arrows are weak equivalences of (W,τ)-descent (cf. 25.7). But $W_{\underline{\hat{C}}}(\tau)$ satisfies the 2 out of 3 condition. Therefore $\Xi \in W_{\underline{\hat{C}}}(\tau)$.

- Let $\Xi: X \to Y$ be an element of $\iota_\tau^{-1} W_{\underline{\hat{C}}}(\tau)$ -- then the claim is that $\Xi \in W_{\underline{Sh}_\tau}(\tau)$. Proof:

$$\iota_\tau \Xi \in W_{\underline{\hat{C}}}(\tau) \Rightarrow \underline{a}_\tau \iota_\tau \Xi \in W_{\underline{Sh}_\tau}(\underline{C})$$

$$\Rightarrow \Xi \in W_{\underline{Sh}_\tau}(\underline{C}) \quad (\underline{a}_\tau \circ \iota_\tau = \mathrm{id}).$$

25.9 RAPPEL A morphism

$$F: (\underline{C}_1, \mathcal{W}_1) \to (\underline{C}_2, \mathcal{W}_2)$$

of category pairs is a functor $F: \underline{C}_1 \to \underline{C}_2$ such that $F\mathcal{W}_1 \subset \mathcal{W}_2$, thus there is a unique functor $\bar{F}: \mathcal{W}_1^{-1}\underline{C}_1 \to \mathcal{W}_2^{-1}\underline{C}_2$ for which the diagram

$$\begin{tikzcd}
\underline{C}_1 \arrow[r, "F"] \arrow[d, "L_{\mathcal{W}_1}"'] & \underline{C}_2 \arrow[d, "L_{\mathcal{W}_2}"] \\
\mathcal{W}_2^{-1}\underline{C}_1 \arrow[r, "\bar{F}"'] & \mathcal{W}_2^{-1}\underline{C}_2
\end{tikzcd}$$

commutes.

- Take

$$\begin{bmatrix} \underline{C}_1 = \underline{SI\hat{C}} \\ \\ \underline{C}_2 = \underline{SISh}_\tau(\underline{C}) \end{bmatrix}, \quad \begin{bmatrix} \mathcal{W}_1 = W_{\underline{\hat{C}}}(\tau) \\ \\ \mathcal{W}_2 = W_{\underline{Sh}_\tau}(\underline{C}) \end{bmatrix}$$

and let

$$F = \underline{a}_\tau .$$

Then $\underline{a}_\tau : \underline{C}_1 \to \underline{C}_2$ is a morphism of category pairs, so

$$\overline{\underline{a}_\tau} : \mathcal{W}_1^{-1}\underline{C}_1 \to \mathcal{W}_2^{-1}\underline{C}_2 .$$

- Take

$$\left[\begin{array}{l} \underline{C}_1 = \underline{\mathbf{SISh}}_\tau(\underline{C}) \\ \\ \underline{C}_2 = \underline{\mathbf{SI}\hat{\underline{C}}} \end{array}\right. , \quad \left[\begin{array}{l} \mathcal{W}_1 = W_{\underline{\mathbf{Sh}}_\tau(\underline{C})} \\ \\ \mathcal{W}_2 = W_{\hat{\underline{C}}}(\tau) \end{array}\right.$$

and let

$$F = \iota_\tau .$$

Then $\iota_\tau : \underline{C}_1 \to \underline{C}_2$ is a morphism of category pairs, so

$$\overline{\iota_\tau} : \mathcal{W}_1^{-1}\underline{C}_1 \to \mathcal{W}_2^{-1}\underline{C}_2 .$$

25.10 THEOREM The functors $\left[\begin{array}{l} \overline{\underline{a}_\tau} \\ \\ \overline{\iota_\tau} \end{array}\right.$ are an adjoint pair and induce an adjoint equivalence of metacategories.

[The arrows of adjunction are natural isomorphisms.]

25.11 CRITERION Let $\underline{E}_1$, $\underline{E}_2$ be Grothendieck toposes, let $\Phi : \underline{E}_1 \to \underline{E}_2$ be a functor, and let W_2 be an admissible $\underline{E}_2$-localizer. Assume that Φ preserves colimits and finite limits and that $\Phi^{-1}W_2$ is an $\underline{E}_1$-localizer -- then $\Phi^{-1}W_2$ is admissible.

25.12 LEMMA If W is admissible, then $W_{\hat{\underline{C}}}(\tau)$ is admissible.

PROOF In 25.11, let $\underline{E}_1 = \underline{\mathrm{SI}\hat{C}}$, $\underline{E}_2 = \underline{\mathrm{SISh}}_\tau(\underline{C})$, $\Phi = \underline{a}_\tau$, $W_2 = W_{\underline{\mathrm{Sh}}_\tau(\underline{C})}$ — then $W_{\underline{\mathrm{Sh}}_\tau(\underline{C})}$ is admissible (cf. 23.15) and

$$\underline{a}_\tau^{-1} W_{\underline{\mathrm{Sh}}_\tau(\underline{C})} = W_{\hat{\underline{C}}}(\tau) \quad \text{(cf. 25.8).}$$

25.13 REMARK Since $W_{\hat{\underline{C}}}(\tau)$ is admissible if W is admissible, $\underline{\mathrm{SI}\hat{C}}$ admits a cofibrantly generated model structure whose class of weak equivalences are the elements of $W_{\hat{\underline{C}}}(\tau)$ and whose cofibrations are the monomorphisms (cf. 20.12). Accordingly, in 25.10, the data gives rise to an adjoint equivalence of homotopy categories.

[Note: If $\underline{C}$ is a model category, then $\underline{HC}$ $(= \mathcal{W}^{-1}\underline{C})$ is a category (and not just a metacategory).]

25.14 LEMMA Suppose that W is admissible and that the Cisinski structure on $\hat{\underline{\Delta}}$ per W is proper — then the Cisinski structure on $\underline{\mathrm{SIC}}$ per $W_{\hat{\underline{C}}}(\tau)$ is proper.

PROOF To begin with, this is the case if τ is the minimal Grothendieck topology on $\underline{C}$ (cf. 25.1 and 25.6). In general, there are two points.

(1) Since $\underline{a}_\tau$ preserves finite limits, hence preserves pullbacks, the τ-hypercoverings are pullback stable (cf. 22.9).

(2) Every fibration of W-descent per $W_{\hat{\underline{C}}}(\tau)$ is a fibration of W-descent per $W_{\hat{\underline{C}}}$.

Now quote 20.17.

[Note: As always, it is right proper which is at issue (cf. 20.15).]

25.15 LEMMA Suppose that W is admissible and that the Cisinski structure on $\hat{\underline{\Delta}}$ per W is proper -- then the Cisinski structure on $\underline{\mathrm{SISh}}_\tau(\underline{C})$ per $W_{\underline{\mathrm{Sh}}_\tau(\underline{C})}$ is proper.

PROOF Fibrations in $\underline{\mathrm{SISh}}_\tau(\underline{C})$ "are" fibrations in $\underline{\mathrm{SI}\hat{C}}$ and pullbacks in $\underline{\mathrm{SISh}}_\tau(\underline{C})$ "are" pullbacks in $\underline{\mathrm{SI}\hat{C}}$.

[To provide a modicum of detail, suppose that $g:Y \to Z$ is a fibration of W-descent per $\underline{\mathrm{SISh}}_\tau(\underline{C})$ -- then $\iota_\tau g$ is a fibration of W-descent per $\underline{\mathrm{SI}\hat{C}}$. Thus consider the lifting problem

$$\begin{array}{ccc} A & \xrightarrow{\quad u \quad} & \iota_\tau Y \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \iota_\tau g} \\ B & \xrightarrow[\quad v \quad]{} & \iota_\tau Z \end{array} ,$$

where f is an acyclic cofibration -- then

$$f \in W_{\hat{\underline{C}}}(\tau) \Rightarrow \underline{a}_\tau f \in W_{\underline{\mathrm{Sh}}_\tau(\underline{C})} \qquad \text{(cf. 25.8).}$$

But $\underline{a}_\tau$ preserves monomorphisms, hence

$$\underline{a}_\tau f:\underline{a}_\tau A \to \underline{a}_\tau B$$

is an acyclic cofibration. Therefore the commutative diagram

$$\begin{array}{ccc} \underline{a}_\tau A & \xrightarrow{\quad \underline{a}_\tau u \quad} & Y \\ {\scriptstyle \underline{a}_\tau f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \underline{a}_\tau B & \xrightarrow[\quad \underline{a}_\tau v \quad]{} & Z \end{array} \qquad (a_\tau \circ \iota_\tau = \mathrm{id})$$

has a filler $w:\underline{a}_\tau B \to Y$, i.e.,

$$\begin{cases} w \circ \underline{a}_\tau f = \underline{a}_\tau u \\ g \circ w = \underline{a}_\tau v. \end{cases}$$

Now form the commutative diagram

$$\begin{tikzcd}
 & A \arrow[r, "u"] \arrow[d, "\alpha"'] & \iota_\tau Y \arrow[d, equal] \\
A \arrow[r, "\alpha"] \arrow[d, "f"'] & \iota_\tau \underline{a}_\tau A \arrow[r, "\iota_\tau \underline{a}_\tau u"] \arrow[d, "\iota_\tau \underline{a}_\tau f"] & \iota_\tau Y \arrow[d, "\iota_\tau g"] \\
B \arrow[r, "\beta"'] & \iota_\tau \underline{a}_\tau B \arrow[r, "\iota_\tau \underline{a}_\tau v"'] & \iota_\tau Z \arrow[d, equal] \\
 & B \arrow[u, "\beta"] \arrow[r, "v"'] & \iota_\tau Z
\end{tikzcd}$$

Then $\iota_\tau w \circ \beta : B \to \iota_\tau Y$ is a solution to our lifting problem:

$$\begin{cases} \iota_\tau w \circ \beta \circ f = \iota_\tau w \circ \iota_\tau \underline{a}_\tau f \circ \alpha = \iota_\tau \underline{a}_\tau u \circ \alpha = u \\ \iota_\tau g \circ \iota_\tau w \circ \beta = \iota_\tau \underline{a}_\tau v \circ \beta = v.] \end{cases}$$

25.16 SCHOLIUM (cf. 23.17) Fix $\tau \in \tau_{\underline{C}}$ and take $W = W_\infty$ -- then

$$\begin{cases} \underline{\mathrm{SI}\hat{C}} \\ \underline{\mathrm{SISh}_\tau}(\underline{C}) \end{cases}$$

admit a cofibrantly generated proper model structure whose class of weak equivalences are the elements of

$$\left[\begin{array}{l} (W_\infty)_{\hat{\underline{C}}}(\tau) \\ \\ (W_\infty)_{\underline{Sh}_\tau(\underline{C})} \end{array}\right.$$

and whose cofibrations are the monomorphisms.

[Note: Here there is present an additional item of structure, viz. that these model categories are simplicial model categories.]

INTERNAL AFFAIRS

IA-1 NOTATION $\underline{GRD}$ is the full subcategory of $\underline{CAT}$ whose objects are the groupoids (the morphisms are functors).

IA-2 LEMMA Let $\underline{G},\underline{H} \in \text{Ob } \underline{GRD}$ and suppose that $F:\underline{G} \to \underline{H}$ is a functor.

- F is fully faithful iff the diagram

$$\begin{array}{ccc} \text{Mor } \underline{G} & \xrightarrow{F} & \text{Mor } \underline{H} \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ \text{Ob } \underline{G} \times \text{Ob } \underline{G} & \xrightarrow[F \times F]{} & \text{Ob } \underline{H} \times \text{Ob } \underline{H} \end{array}$$

is a pullback in $\underline{SET}$.

- F has a representative image iff the composite

$$\text{Ob } \underline{G} \times_{\text{Ob } \underline{H}} \text{Mor } \underline{H} \longrightarrow \text{Mor } \underline{H} \xrightarrow{s} \text{Ob } \underline{H}$$

is surjective.

[Note: Here

$$\begin{array}{ccccc} \text{Ob } \underline{G} \times_{\text{Ob } \underline{H}} \text{Mor } \underline{H} & \longrightarrow & \text{Mor } \underline{H} & \xrightarrow{s} & \text{Ob } \underline{H} \\ \downarrow & & \downarrow{\scriptstyle t} & & \\ \text{Ob } \underline{G} & \xrightarrow[F]{} & \text{Ob } \underline{H} & & \end{array}$$

N.B. These points characterize an equivalence between groupoids and provide the motivation for the notion of "internal equivalence" infra.

IA-3 THEOREM $\underline{\mathrm{GRD}}$ is a model category if weak equivalence = equivalence and the cofibrations are those functors $F:\underline{G} \to \underline{H}$ such that the map

$$\begin{bmatrix} \mathrm{Ob}\,\underline{G} \to \mathrm{Ob}\,\underline{H} \\ X \to FX \end{bmatrix}$$

is injective.

[Note: All objects are fibrant and cofibrant.]

IA-4 LEMMA Let $\underline{G},\underline{H} \in \mathrm{Ob}\,\underline{\mathrm{GRD}}$, $F:\underline{G} \to \underline{H}$ a functor -- then F is an equivalence iff the induced simplicial map ner F:ner $\underline{G} \to$ ner $\underline{H}$ of nerves is a simplicial weak equivalence.

IA-5 LEMMA Let $\underline{G},\underline{H} \in \mathrm{Ob}\,\underline{\mathrm{GRD}}$, $F:\underline{G} \to \underline{H}$ a functor -- then F is a fibration iff the induced simplicial map ner F:ner $\underline{G} \to$ ner $\underline{H}$ of nerves is a Kan fibration.

IA-6 LEMMA Let X,Y be simplicial sets and let $f:X \to Y$ be a simplicial map.

- If f is a simplicial weak equivalence, then the induced morphism $\Pi f:\Pi X \to \Pi Y$ of fundamental groupoids is an equivalence.

- If f is a cofibration, then the induced morphism $\Pi f:\Pi X \to \Pi Y$ of fundamental groupoids is injective on objects.

IA-7 REMARK Since

$$\Pi:\underline{\mathrm{SISET}} \to \underline{\mathrm{GRD}}$$

is a left adjoint for

$$\mathrm{ner}:\underline{\mathrm{GRD}} \to \underline{\mathrm{SISET}},$$

it follows from the lemmas that Π is a left model functor, i.e., preserves cofibrations and acyclic cofibrations, and ner is a right model functor, i.e., preserves fibrations and acyclic fibrations.

[Note: Here the underlying model structure on SISET is, of course, the Kan structure. To get a model equivalence, simply replace it by its truncation at level 1 (thus now the weak equivalences are the 1-equivalences (so the arrows are isomorphisms at π_0 and π_1)).]

Let $\underline{E}$ be a Grothendieck topos — then $\underline{E}$ is complete so the formalism of internal category theory is applicable. And, as will be seen below, the results outlined above for the case $\underline{E}$ = SET actually go through in general.

IA-8 NOTATION $\underline{\mathrm{GRD}}(\underline{E})$ is the full subcategory of $\underline{\mathrm{CAT}}(\underline{E})$ whose objects are the groupoids in $\underline{E}$ (the morphisms are internal functors).

[Note: Recall that an object $\underline{G}$ of $\underline{\mathrm{GRD}}(\underline{E})$ is a pair (G_0,G_1) of objects of $\underline{E}$ together with a battery of morphisms satisfying the usual axioms.]

IA-9 EXAMPLE Let $\underline{C}$ be a small category -- then

$$\underline{\mathrm{GRD}}(\hat{\underline{C}}) \approx [\underline{C}^{\mathrm{OP}},\underline{\mathrm{GRD}}].$$

IA-10 DEFINITION Let $\underline{G},\underline{H} \in \mathrm{Ob}\ \underline{\mathrm{GRD}}(\underline{E})$ and suppose that $F:\underline{G} \to \underline{H}$ is an internal functor, hence $F = (F_0,F_1)$, where

$$\begin{cases} F_0:G_0 \to H_0 \\ F_1:G_1 \to H_1 \end{cases}$$

are morphisms in $\underline{E}$ (subject to ...) -- then F is said to be an <u>internal equivalence</u> if

(1) The diagram

$$\begin{tikzcd}
G_1 \arrow[r, "F_1"] \arrow[d, "(d_0,d_1)"'] & H_1 \arrow[d, "(d_0,d_1)"] \\
G_0 \times G_0 \arrow[r, "F_0 \times F_0"'] & H_0 \times H_0
\end{tikzcd}$$

is a pullback in $\underline{E}$ and

(2) The composite

$$G_0 \times_{H_0} H_1 \longrightarrow H_1 \xrightarrow{d_0} H_0$$

is an epimorphism.

[Note: Here

$$\begin{tikzcd}
G_0 \times_{H_0} H_1 \arrow[r] \arrow[d] & H_1 \arrow[r, "d_0"] \arrow[d, "d_1"] & H_0 \\
G_0 \arrow[r, "F_0"'] & H_0 &
\end{tikzcd}$$

.]

IA-11 THEOREM $\underline{\mathrm{GRD}}(\underline{E})$ is a model category if weak equivalence = internal equivalence and the cofibrations are those internal functors $F:\underline{G} \to \underline{H}$ such that the arrow

$$F_0:G_0 \to H_0$$

is a monomorphism.

<u>N.B.</u> Take $\underline{E} = \underline{\mathrm{SET}}$ to recover IA-3.

IA-12 RAPPEL Every category $\underline{C}$ in $\underline{E}$ gives rise to a simplicial object ner $\underline{C}$ in $\underline{E}$ by letting $\text{ner}_0\underline{C} = \underline{C}_0$, $\text{ner}_1\underline{C} = \underline{C}_1$, and

$$\text{ner}_n\underline{C} = C_1 \times_{C_0} \cdots \times_{C_0} C_1 \text{ (n factors)}.$$

[Note: An internal functor $\underline{C} \rightarrow \underline{C}'$ induces a morphism ner $\underline{C} \rightarrow$ ner $\underline{C}'$ of simplicial objects.]

IA-13 LEMMA Let $\underline{G},\underline{H} \in \text{Ob } \underline{\text{GRD}}(\underline{E})$, $F:\underline{G} \rightarrow \underline{H}$ an internal functor — then F is an internal equivalence iff ner F:ner $\underline{G} \rightarrow$ ner $\underline{H}$ is a weak equivalence of W_∞-descent.

IA-14 REMARK The functor

$$\text{ner}:\underline{\text{GRD}}(\underline{E}) \rightarrow \underline{\text{SI}E}$$

has a left adjoint

$$\Pi:\underline{\text{SI}E} \rightarrow \underline{\text{GRD}}(\underline{E}).$$

Working with the model structure on $\underline{\text{SI}E}$ per 23.17 (the weak equivalences thus being the weak equivalences of W_∞-descent), what was said in IA-7 can be said again. In particular: If $\underline{G} \in \text{Ob } \underline{\text{GRD}}(\underline{E})$ is fibrant, then ner $\underline{G}$ is fibrant.

Let $\underline{C}$ be a small category, τ a Grothendieck topology on $\underline{C}$ -- then $\underline{\text{SI}\hat{C}}$ admits a cofibrantly generated proper model structure whose class of weak equivalences are the elements of

$$(W_\infty)_{\hat{\underline{C}}}(\tau)$$

and whose cofibrations are the monomorphisms (cf. 25.16).

[Note: If τ is the minimal Grothendieck topology on $\underline{C}$, then

$$(W_\infty)_{\hat{\underline{C}}}(\tau) = (W_\infty)_{\hat{\underline{C}}}$$

and the elements of $(W_\infty)_{\hat{\underline{C}}}$ are levelwise the elements of W_∞ (cf. 25.1). Therefore in this case the model structure on

$$\underline{\mathrm{SI}\hat{\mathrm{C}}} \approx [\underline{C}^{\mathrm{OP}}, \underline{\mathrm{SISET}}]$$

is the injective structure.]

<u>N.B.</u>

- If $G:\underline{C}^{\mathrm{OP}} \to \underline{\mathrm{GRD}}$, then

$$\mathrm{ner}\ G:\underline{C}^{\mathrm{OP}} \to \underline{\mathrm{SISET}}.$$

- If $G,H:\underline{C}^{\mathrm{OP}} \to \underline{\mathrm{GRD}}$ and if $\Xi:G \to H$, then

$$\mathrm{ner}\ \Xi:\mathrm{ner}\ G \to \mathrm{ner}\ H.$$

IA-15 THEOREM $[\underline{C}^{\mathrm{OP}}, \underline{\mathrm{GRD}}]$ is a model category if the weak equivalences are the $\Xi:G \to H$ such that ner Ξ is a weak equivalence of (W_∞,τ)-descent and the fibrations are the $\Xi:G \to H$ such that ner Ξ is a fibration of (W_∞,τ)-descent.

For ease of reference, call the objects of $[\underline{C}^{\mathrm{OP}}, \underline{\mathrm{SISET}}]$ <u>simplicial presheaves</u> and the objects of $[\underline{C}^{\mathrm{OP}}, \underline{\mathrm{GRD}}]$ <u>simplicial groupoids</u>.

IA-16 DEFINITION A <u>fibrant model</u> for a simplicial presheaf X is a fibrant simplicial presheaf X_f and a weak equivalence of (W_∞,τ)-descent $X \to X_f$.

IA-17 DEFINITION A simplicial presheaf X is said to <u>satisfy descent</u> if for some fibrant model X_f, the arrow

$$XU \to X_f U$$

is a simplicial weak equivalence $\forall\ U \in \mathrm{Ob}\ \underline{C}$.

IA-18 LEMMA If A and B are fibrant simplicial presheaves and if $f:A \to B$ is a weak equivalence of (W_∞,τ)-descent, then $\forall\ U \in \text{Ob}\ \underline{C}$, the arrow $AU \to BU$ is a simplicial weak equivalence.

IA-19 APPLICATION If X is a simplicial presheaf, if X_f and X_f' are fibrant models for X, and if $\forall\ U \in \text{Ob}\ \underline{C}$, the arrow

$$XU \to X_f U$$

is a simplicial weak equivalence, then $\forall\ U \in \text{Ob}\ \underline{C}$, the arrow

$$XU \to X_f' U$$

is a simplicial weak equivalence.

[Choose $\phi:X_f \to X_f'$ such that the diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d] & X \arrow[d] \\ X_f \arrow[r, "\phi"'] & X_f' \end{tikzcd}$$

commutes -- then ϕ is a weak equivalence of (W_∞,τ)-descent (by the 2 out of 3 condition), hence $\forall\ U \in \text{Ob}\ \underline{C}$, the arrow

$$X_f U \to X_f' U$$

is a simplicial weak equivalence, from which the assertion.]

Consequently, the notion of "descent" is independent of the choice of a fibrant model.

IA-20 DEFINITION Let G be a simplicial groupoid -- then G is said to be a

<u>stack</u> if ner $\underset{\sim}{G}$ satisfies descent.

IA-21 DEFINITION A <u>stack completion</u> of a presheaf of groupoids G is a weak equivalence $G \to G'$, where G' is a stack.

It is a fact that a stack completion for a given G always exists. E.g.: One possibility is to take $G' = G\text{-tors}_d$ (Jardine's "discrete G-torsors").

IA-22 REMARK The definition of stack is a moving target.