NEW APPROACHES AND RESULTS IN THE THEORY OF DISCRETE ANALYTIC FUNCTIONS

Thesis submitted for the Degree
"Doctor of Philosophy"

by

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Submitted to the Feinberg Graduate School of
The Weizmann Institute of Science, Rehovot, Israel.

1976

Acknowledgements

It is my pleasure to express my warm thanks and deep appreciation to my advisor, Harry Dym, for his guidance and constant friendly encouragement.

I would also like to thank Miriam Drory who didn't save any effort to decipher my hieroglyphic handwriting and turned it into a beautiful typescript.

PUBLICATIONS.

1. A new approach to the theory of Discrete Analytic Functions, J. of Math. Anal. and Appl. (to appear). [Chapter 2].

2. Further properties of discrete analytic functions, J. of Math. Anal. and Appl. (to appear). [Chapter 3].

3. A new basis for discrete analytic polynomials, J. of the Australian Math. Soc.. (to appear). [Chapter 4].

4. Discrete analytic functions of exponential growth. Transactions of the Am. Math. Soc. (to appear). [Chapter 5]

5. Uniqueness theorems for harmonic functions of exponential growth, Proc. of the Am. Math. Soc. (to appear). [Appendix A1].

6. Binary operations in the set of solutions of a partial difference equation, Submitted. [Appendix A2].

A TECHNICAL REMARK

"Formula (m,n)" means "Formula (m,n) in the present chapter". If referfence will be made to formulas from another chapter, then it will be preceded by the relevant chapter's number; e.g., 1 (3.2) means: formula (3.2) in chapter I.

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CHAPTER I.

INTRODUCTION

This thesis is concerned with discrete analytic functions. A function $f\colon Z^2 \to \mathbb{C}$ from the two dimensional lattice to the complex numbers is said to be discrete analytic if for every $(m,n) \in Z^2$

$$\frac{f(m+1,n+1) - f(m,n)}{i+1} = \frac{f(m,n+1) - f(m+1,n)}{i-1} \qquad (1)$$

If we embed Z^2 in $\mathbb{C}$ by identifying it with the set of Gaussian integers $\{m + in\}$ we observe that (1) is an 'analyticity' condition: the difference quotients along the two diagonals of each unit square are the same. Equivalently, discrete analytic functions can be characterized as the solutions of the homogeneous partial difference equation

$$f(m,n) + if(m+1,n) - f(m+1,n+1) - if(m,n+1) \equiv 0 \qquad (2)$$

The theory of discrete analytic functions was initiated by Jacquelline Ferrand-Lelong [13] and further developed by Duffin [6], Duffin and Duris [7], [8], Hayabara [16], Deeter and Lord [4], Duffin and Peterson [9] and others. Duffin [6] gave discrete analogues of: The function z^{-1}, the Cauchy integral formula, Louiville's theorem, Harnack's inequality, polynomial expansion and the Hilbert transform. However, the proofs of most of the classical theorems of Complex Aanalysis do not carry over directly to the discrete case because they rely heavily upon the fact that the class of analytic functions is closed under pointwise multiplication. This is not the case for the class of discrete analytic functions and consequently new techniques had to be developed. That is the subject of this research.

Two basic tools are used. Let us describe them briefly:

(i) Formal Power Series. With any discrete analytic function $f: Z^+ \times Z^+ \to \mathbb{C}$ on the upper right quarter lattice we can associate a formal power series

$$\underline{f}(X,Y) = \sum_{m,n=0}^{\infty} f(m,n)X^m Y^n .$$

in terms of which

$$f(X,Y) = \frac{(1+iX)\phi_f(X)+(1-iY)\psi_f(Y)-f(0,0)}{1+iX - XY - iY}$$

(in the algebra of formal powers series) where ϕ_f , ψ_f are the 'boundary' series

$$\phi_f(X) = \sum_{m=0}^{\infty} f(m,0)X^m \; ; \; \psi_f(Y) = \sum_{n=0}^{\infty} f(0,n)Y^n .$$

In terms of (ϕ_f,ψ_f) , Duffin's operations of 'integral', 'derivative','convolution', etc., take a simple form which can be used to advantage to give simpler and quicker proofs to earlier results. It is also helpful in proving a discrete Phragmén-Lindelöf principle and in the study of Taylor expansion for discrete analytic functions.

(ii) Duality Methods. Any discrete function $f: Z^2 \to \mathbb{C}$ induces a linear functional on the algebra $\mathcal{A}$ generated by the indeterminates $\{z, z^{-1}, w, w^{-1}\}$ which is determined by $T_f(z^m w^n) = f(m,n)$ and extended by linearity. f is discrete analytic iff T_f annihilates the ideal $(1+iz-zw-iw)\mathcal{A}$. If f satisfies an appropriate growth condition then T_f can be extended to a continuous linear functional on an appropriate topological vector space or Banach space. For example, if f is of polynomial growth: $|f(m,n)| \leq C(|m| + |n|)^k$, then T_f can be extended to be a continuous linear functional (alias a distribution) on $C^\infty(T^2)$. If f is of exponential growth: $|f(m,n)| \leq CR^{|m|}S^{|n|}$ then T_f can be extended to be a continuous linear functional on the Banach space of bounded analytic functions on the polyannulus $\{\frac{1}{r} < |z| < r\} \times \{\frac{1}{s} < |w| < s\}$ $(r > R, s > S)$. These ideas are used to prove discrete analogues of Liouville's theorem and of the Paley-Wiener theorems. As already pointed out, the classical proofs do not carry over. However, our duality methods also yield new 'fancy' proofs to classical continuous theorems. This is illustrated in detail in the 2nd section of Chapter III (for Liouville's theorem).

Duality methods apply in both the discrete and continuous cases because the dual groups T^2 of Z^2 and R^2 of R^2 can be embedded in $\mathbb{C}^2$.

This thesis deals with discrete analytic functions, that is solutions of partial difference equations (2). However, our duality methods can be used just as well to the study of solutions of general partial difference equations with constant coefficients and even to systems of such equations. A future plan is to give a general theory of partial difference equations with constant coefficients in the spirit of Ehrenpreis' [12] theory of partial differential equations with constant coefficients. A first step towards the general theory, demonstrating the power of duality methods, is given in the appendix.

Finally, let us present a summary of the contents of this thesis. In Chapter II we introduce formal power series and show how the notions of 'integral', 'derivative', 'polynomials' and 'convolution products' (defined by Duffin [6] and Duffin and Duris [7]) translate to the language of formal power series. The power of this mechanism is demonstrated by giving new short proofs to results of Duffin [6], Duffin and Duris [7] and Deeter and Lord [4].

In Chapter III we use distributions on T^2 (the two dimensional torus) to give a short proof of the discrete Liouville theorem, first proved by Duffin [6]. Then using ideas of the previous chapter, we derive a discrete Phragmén-Lindelöf principle and finally we use Fourier methods to give discrete analogues of the one-sided Paley-Wiener theorem and of a Paley-Winer-Schwartz theorem.

Chapter IV deals with the McClaurin expansion for discrete analytic functions. The McClaurin expansion given by Duffin and Peterson [9] is unsatisfactory because in terms of their basis $\{z^{(n)}\}$, the sum $\sum_{0}^{\infty} a_n z^{(n)}$ defines a discrete entire function only if $\overline{\lim}\, |n!\, a_n|^{1/n} < 2$. This is a much more stringent requirement than the condition $\overline{\lim}\, |a_n|^{1/n} = 0$ for the power series $\sum a_n z^n$ to define an entire function. In fact a complete analogue is impossible, i.e., there exists no basis $\{p_n(z)\}$ for the discrete analytic polynomials such $\sum_{n=0}^{\infty} a_n p_n(z)$ converges for every z whenever $\overline{\lim}\, |a_n|^{1/n} = 0$. However, we define a new basis $\{\pi_n(z)\}$ for which $\sum a_n \pi_n(z)$ converges on the upper right quarter lattice whenever $\overline{\lim}\, |a_n|^{1/n} = 0$. The chapter ends with a discussion of the limiting behavior of the expansion as the mesh size tends to zero.

The final Chapter V gives some discrete analogues to theorems on entire functions of exponential type (Boas [2] is the standard reference for the latter). We give a discrete analogue to the 'Borel transform' and the 'conjugate indicator diagram' and establish a discrete analogue to the celebrated two-sided Paley-Wiener theorem.

The two appendices apply ideas developed in the body of the thesis to problems outside the realm of discrete analytic functions.

The first gives uniqueness theorems for harmonic functions of exponential growth. It uses duality methods to generalize to R^n a theorem proved by Boas [3] for R^2 . The second appendix deals with the generation of new solutions to a partial difference equation from known ones: Given a partial difference operator with constant coefficients P we are interested in binary operations $(f,g) \to f*g$ such that if $Pf=0$ and $Pg=0$ then $P(f*g) = 0$. Duffin and Rohrer [10] gave one such binary operation. We give a whole class of such binary operations which both simplify and extend the work of Duffin and Rohrer [6], and at the same time generalize the results of Duffin and Duris [7].

CHAPTER II.

A FORMAL POWER SERIES APPROACH TO THE THEORY OF DISCRETE ANALYTIC FUNCTIONS.

1. INTRODUCTION.

In this chapter a formal power series approach to the theory of discrete analytic functions is given which besides giving new insight to the theory, makes many proofs much simpler and shorter. To illustrate the method, new proofs are given to most of the results in Duffin and Duris [7] and Deeter and Lord [4].

If a function is discrete analytic in a simple region (a finite union of unit squares which is simply connected) it can be discrete analytically continued to the whole plane. Until Section 9 we shall assume that our functions are defined and discrete in each unit square of the quarter plane

$$Z^+ \times Z^+ = \{(m,n)\ ;\ m,n \text{ integers}, \ m,n \geq 0\} .$$

Since functions defined and discrete analytic in the other quarter planes can receive a similar treatment, our assumption involves no loss of generality.

The key idea of this chapter is to associate with each function $f : Z^+ \times Z^+ \to \mathbb{C}$ the formal power series

$$\underline{f}(X,Y)=\sum_{\substack{m=0\\n=0}}^{\infty} f(m,n)X^mY^n \qquad (1.2)$$

2. THE RING OF FORMAL POWER SERIES IN TWO VARIABLES.

The class of formal power series

$$R_{XY}=\{\sum_{\substack{m=0\\n=0}}^{\infty} a_{mn}X^mY^n \ ; \ a_{mn}\in\mathbb{C}\}$$

endowed with the usual rules for addition and multiplication:

$$(\Sigma a_{mn}X^mY^n)+(\Sigma b_{mn}X^mY^n)=\Sigma(a_{mn}+b_{mn})X^mY^n$$

$$(\Sigma a_{mn}X^mY^n)(\Sigma b_{mn}X^mY^n)=\sum_{m,n=0}^{\infty}\left(\sum_{k=0}^{m}\sum_{r=0}^{n} a_{k,r}b_{m-k,n-r}\right)X^mY^m$$

is a ring with an additive identity zero 0 ($a_{mn}=0$ for each m and n) and a multiplicative identity $\underline{1}(a_{00}=1$, $a_{mn}=0$ otherwise).

Since the product of any two non-zero formal power series is non-zero, this ring is an integral domain. An element $F(X,Y)$ of R_{XY} has a multiplicative inverse iff $a_{00}=F(0,0)\neq 0$ and then

$$F(X,Y)^{-1}=(a_{00}[1-\frac{[a_{00}-F(X,Y)]}{a_{00}}])^{-1}=a_{00}^{-1}\sum_{n=0}^{\infty}\left(\frac{a_{00}-F(X,Y)}{a_{00}}\right)^n \ ;$$

the infinite sum on the right defines a formal power series since the coefficient of each term is a finite sum and there are no problems of convergence. Of course, the inverse, when it exists is unique, since

R_{XY} has no zero divisors.

Later we shall also consider the ring R_X of formal power series of one variable

$$\{\sum_{n=0}^{\infty} a_n X^n\} .$$

This is a subring of R_{XY} and also an integral domain.

The following lemma will be needed later:

Lemma 2.1: Let $\phi(X)=\sum_{m=0}^{\infty} a_m X^m$. The equation $\psi(X)^k=\phi(X)$ has a solution $\psi(X)\in R_X$ iff there exists an integer $n\geq 0$ such that the first non-zero coefficient of $\phi(X)$ is a_{nk} . In this case $\psi(X)$ is given by

$$\psi(X)=X^n(a_{nk})^{1/k}\{1+(\frac{a_{nk+1}}{a_{nk}}X+\frac{a_{nk+2}}{a_{nk}}X^2+ \ldots)\}^{1/k} \tag{2.1}$$

where the right hand side is developed according to Newton's binomial expansion

$$(1+\chi)^{1/k}=\sum_{n=0}^{\infty}\binom{1/k}{n}\chi^n$$

Proof: Verify formally that $\psi(X)^k=\phi(X)$ to prove sufficiency. The necessity is trivial.

3. REPRESENTATION OF DISCRETE ANALYTIC FUNCTIONS AS FORMAL POWER SERIES

Let $f : Z^+\times Z^+\to \mathbb{C}$ be any function and associate with it the formal power series

$$\underline{f}(X,Y)=\sum_{\substack{m=0\\n=0}}^{\infty} f(m,n)X^mY^n .$$

Then

$$(1+iX-XY-iY)\underline{f}(X,Y)=(1+iX)\phi_f(X)+(1-iY)\psi_f(Y)-f(0,0)-\sum_{\substack{m=0\\n=0}}^{\infty} Lf(m,n)X^{m+1}Y^{n+1}. \quad (3.1)$$

where $Lf(m,n) = f(m,n) + if(m+1,n) - f(m+1,n+1) - if(m,n+1)$,

$$\phi_f(X)= \sum_{m=0}^{\infty} f(m,0)X^m$$

$$\psi_f(Y)= \sum_{n=0}^{\infty} f(0,n)Y^n \quad .$$

Now the last term vanishes for discrete analytic functions and so for such f

$$(1+iX-XY-iY)\underline{f}(X,Y)=(1+iX)\phi_f+(1-iY)\psi_f-f(0,0)$$

Multiplying both sides by $(1+iX-XY-iY)^{-1}$:

$$\underline{f}(X,Y)=\frac{\phi_f(X)(1+iX)+\psi_f(Y)(1-iY)-f(0,0)}{1+iX-XY-iY} . \quad (3.2)$$

This confirms the self evident fact that a discrete analytic function is uniquely determined by its values on the axes. In fact, (3.2) is a condensed form of formula (7) in Duffin [6].

Now let

$$\phi_k(X)= \sum_{m=0}^{\infty} f(m,k)X^m$$

for $k=0,1,2,...$ so that $\phi_0(X)=\phi_f(X)$

and

$$\underline{f}(X,Y)= \sum_{k=0}^{\infty} \phi_k(X)Y^k \quad .$$

Introduce this notation into (3.2) and comparing coefficients of Y yields

$$\phi_1(X)=\phi_0(X)\frac{X+i}{1+iX}+\frac{f(0,1)-if(0,0)}{1+iX} \quad (3.3)$$

and by applying this formula to the function

$$f^k(m,n)=f(m,n+k)$$

we obtain

$$\phi_{k+1}(X)=\phi_k(X)\frac{X+i}{1+iX}+\frac{f(0,k+1)-if(0,k)}{1+iX} \quad . \tag{3.4}$$

Formula (3.4) gives a convenient way to evaluate inductively the values of f inside $Z^+\times Z^+$ from its values on the axes.

Since a discrete analytic function in $Z^+\times Z^+$ is uniquely determined by the pair (ϕ_f,ψ_f) and evidently each discrete analytic function determines such a pair, there is a (1-1) correspondence between discrete analytic functions and the elements of the set

$$\{(\phi(X),\psi(Y))\ ;\ \phi(X)\in R_X\ ,\ \psi(Y)\in R_Y\ ,\ \phi(0)=\psi(0)\}\ .$$

In the following, a discrete analytic function in $Z^+\times Z^+$, f , will be identified with the pair (ϕ_f,ψ_f) referred to as the "function" (ϕ_f,ψ_f) .

Example 3.1: The discrete analytic function $f(m,n)\equiv C$ (C constant) corresponds to the pair (ϕ_f,ψ_f) where

$$\phi_f=C\sum_{n=0}^{\infty}X^n=C(1-X)^{-1}$$

$$\psi_f=C\sum_{n=0}^{\infty}Y^n=C(1-Y)^{-1}$$

and

$$\underline{C}(X,Y)=C\sum_{\substack{m=0\\n=0}}^{\infty}X^mY^n=\frac{C}{(1-X)(1-Y)}$$

Example 3.2: The function $f(m,n)=C(-1)^{m+n}$ corresponds to the pair

$$\left(C\sum_{n=0}^{\infty}(-1)^nX^n,C\sum_{n=0}^{\infty}(-1)^nY^n\right)=\left(\frac{C}{1+X},\frac{C}{1+Y}\right)$$

and

$$\underline{f}(X,Y)=C\sum_{\substack{m=0\\n=0}}^{\infty}(-1)^{m+n}X^mY^n=\frac{C}{(1+X)(1+Y)} \quad .$$

Duffin [2] termed a function f which assumes the value e_1 on the odd lattice points and the value e_2 on the even lattice points, a biconstant. This function can be written as

$$(1/2)(e_2+e_1)+(-1)^{m+n}(1/2)(e_2-e_1) \quad .$$

Thus, the general form of a biconstant is

$$(1/2)(e_2+e_1)(\frac{1}{1-X},\frac{1}{1-Y})+(1/2)(e_2-e_1)(\frac{1}{1+X},\frac{1}{1+Y})=(\frac{e_2+e_1X}{(1-X)(1+X)},\frac{e_2+e_1Y}{(1-Y)(1+Y)}) .$$

4. INTEGRAL AND DERIVATIVE

Duffin [6] defined a "line integral" by the rule

$$\int_a^b f(z)\partial z=\sum_{n=1}^{m}(f_n+f_{n-1})(z_n-z_{n-1})/2 \qquad (4.1)$$

where $a=z_0,z_1,\ldots \; . \; z_m=b$ is a chain of lattice points (that is $|z_k-z_{k+1}|=1$ and $f_k=f(z_k)$) .

He showed that if f is discrete analytic in a region then the sum is independent of the particular chain connecting a to b and hence (4.1) is well defined. He defined the indefinite integral F of f ,

$$F(z)=\int_a^z f(z)\partial z \quad . \qquad (4.2)$$

Since the starting point of the integral is arbitrary, $F(z)$ is only defined up to an additive constant. Duffin also showed that if $f(z)$ is discrete analytic in a simple region then so is $F(z)$.

Now, suppose $f=(\phi_f,\psi_f)$ and $F=(\phi_F,\psi_F)$. We would then like to find ϕ_F in terms of ϕ_f and ψ_F in terms of ψ_f .

By (4.1) with $a=0$ we have,

$$2F(m,0)=f(0,0)+2f(1,0)+ \dots +2f(m-1,0)+f(m,0)$$
$$=2[f(0,0)+ \dots +f(m,0)]-(f(0,0)+f(m,0)) .$$

Thus

$$2\Sigma F(m,0)X^m=2\frac{1}{1-X}\phi_f(X)-\phi_f(X)-\frac{f(0,0)}{1-X}$$

and we get

$$\phi_F=\frac{1}{2}\,\frac{1+X}{1-X}\phi_f-\frac{f(0,0)}{2(1-X)} \qquad (4.3a)$$

Similarly,

$$\psi_F=\frac{i}{2}\,\frac{1+Y}{1-Y}\psi_f-\frac{if(0,0)}{2(1-Y)} \qquad (4.3b)$$

Thus, the operation of integration is,

$$(\phi(X),\psi(Y))\to\frac{1}{2}\left(\frac{1+X}{1-X}\phi(X)-\frac{f(0,0)}{1-X},i\frac{1+Y}{1-Y}\psi(Y)-\frac{if(0,0)}{1-Y}\right)+C\left(\frac{1}{1-X},\frac{1}{1-Y}\right) \qquad (4.4)$$

Where C is an arbitrary constant. If the starting point of integration in (4.2) $a=0$ then $F(z)=\int_0^z f(z)\partial z$ and in (4.4) $C=0$.

Duffin also defined the dual f^- of discrete functions by the rule

$$f^-(m,n)=(-1)^{m+n}f^*(m,n) \quad .$$

Thus

$$\underline{f}^-(X,Y)=\underline{f}^*(-X,-Y)$$

and

$$(\phi(X),\psi(Y))^-=(\phi^*(-X),\psi^*(-Y)) \quad .$$

We are now in a position to give another proof of the following result which was first proved in Duffin [6], p. 341.

<u>Lemma 4.1</u>:

Let $F(z)$ be a given discrete analytic function. Let a and b be points of $Z^+\times Z^+$ and let k be an arbitrary constant.

Then

$$f(z)=(4\int_b^z F^{-}\partial z+k)^{-} \tag{4.5}$$

is analytic in $Z^{+}\times Z^{+}$ and

$$F(z)=\int_a^z f(z)\partial z+F(a) \tag{4.6}$$

<u>Proof</u>: If $F=(\Phi,\Psi)$

$$F^{-}=(\Phi^{*}(-X),\Psi^{*}(-Y)) .$$

By (4.4),

$$4\int_b^z F^{-}\partial z+k=2(\Phi^{*}(-X)\frac{1+X}{1-X}-\frac{\Phi^{*}(0)}{1-X},i\Psi^{*}(-Y)\frac{1+Y}{1-Y}-\frac{i\Psi^{*}(0)}{1-Y})+k_1(\frac{1}{1-X},\frac{1}{1-Y})$$

(k_1 some other constant).

So,

$$f(z)=(4\int_b^z F^{-}\partial z+k)^{-}=2(\Phi(X)\frac{1-X}{1+X}-\frac{\Phi(0)}{1+X},-i\Psi(Y)\frac{1-Y}{1+Y}+\frac{i\Psi(0)}{1+Y})+k_1^{*}(\frac{1}{1+X},\frac{1}{1+Y}) \tag{4.7}$$

Finally,

$$\int_0^z f(z)\partial z=[(\Phi(X)\frac{1-X}{1+X}-\frac{\Phi(0)}{1+X})\frac{1+X}{1-X},-i(\Psi(Y)\frac{1-Y}{1+Y}-\frac{\Psi(0)}{1+Y})\frac{i(1+Y)}{(1-Y)}]$$

$$+\underbrace{k^{*}(\frac{1}{1+X}\cdot\frac{1+X}{1-X}-\frac{1}{1-X},\frac{i}{1+Y}\frac{1+Y}{1-Y}-\frac{1}{1-Y})}_{0}$$

$$=(\Phi(X)-\frac{F(0,0)}{1-X},\Psi(Y)-\frac{F(0,0)}{1-Y}) .$$

Thus,

$$F(z)=\int_0^z f(z)\partial z+F(0,0) .$$

Duffin [6] used formula (4.5) to defined f as the derivative of F. Formula (4.7) says that the action of taking derivative is

$$(\phi(X),\psi(Y))\to 2(\phi(X)\frac{1-X}{1+X}-\frac{\phi(0)}{1+X},-i(\psi(Y)\frac{1-Y}{1+Y}-\frac{\psi(0)}{1+Y}))+k(\frac{1}{1+X},\frac{1}{1+Y}) \tag{4.7}$$

(k arbitrary constant).

So the derivative is unique up to addition by a constant multiple of $(-1)^{m+n}$ (example 3.2).

5. POLYNOMIALS*

In Duffin [6] (Section 5), polynomials which are discrete analytic everywhere were considered and it was shown that if f is a discrete analytic polynomial than the integral F is a discrete analytic polynomial. A sequence of discrete analytic polynomials was defined by the relations

$$z^{(n+1)} = (n+1)\int_0^z z^{(n)}\partial z \; ; \; z^{(0)} \equiv 1 .$$

So, $z^{(0)} = (\frac{1}{1-X}, \frac{1}{1-Y})$

$$z^{(1)} \doteq 1/2\left(\frac{1+X}{1-X}\cdot\frac{1}{1-X}-\frac{1}{1-X}, \frac{i(1+Y)}{(1-Y)}\cdot\frac{1}{1-Y}-\frac{i}{1-Y}\right)=\left(\frac{X}{(1-X)^2}, \frac{iY}{(1-Y)^2}\right) .$$

Since $z^{(k)}(0) = 0$, $k=1,2,3,\ldots$ one gets

$$z^{(n)} = \left(\frac{n!}{2^{n-1}}\frac{X(1+X)^{n-1}}{(1-X)^{n+1}}, \frac{(i)^n n!}{2^{n-1}}\frac{Y(1+Y)^{n-1}}{(1-Y)^{n-1}}\right) \quad n=1,2,3\ldots \qquad (5.2)$$

The discrete analytic exponential function

$$e(z,t) = \left(\frac{2+t}{2-t}\right)^x \left(\frac{2+it}{2-it}\right)^y$$

was introduced by Ferrand [1] and it is seen that

$$e(z,t) = \left(\frac{1}{1-\frac{2+t}{2-t}X}, \frac{1}{1-\frac{2+it}{2-it}Y}\right)$$

and

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* By $f(z)$ we mean $f(x,y)$, where $z = x+iy$, $(x,y) \in Z \times Z$.

$$\underline{e}(z,t)=\frac{\left(\frac{1+iX}{1-\frac{2+t}{2-t}X}+\frac{1-iY}{1-\frac{2+it}{2-it}}-1\right)}{1+iX-XY-iY}=\frac{1}{(1-\frac{2+t}{2-t}X)(1-\frac{2+it}{2-it}Y)}$$

By using (5.2) one can reprove (Duffin [6], formula (139)),

$$e(z,t)=\sum_{n=0}^{\infty}\frac{z^{(n)}t^n}{n!} \quad (|t|<2) .$$

6. A CONVOLUTION PRODUCT FOR DISCRETE FUNCTION THEORY

In Duffin and Duris [7] three types of convolution products were defined for discrete analytic functions.

The convolution of f,g is defined as

$$f*g=\int_0^z f(z-t):g(t)\partial t \tag{6.1}$$

where

$$\int_a^b f(z):g(z)\partial z=\sum_{n=1}^{m}\tfrac{1}{4}[f(z_n)+f(z_{n-1})]\cdot[g(z_n)+g(z_{n-1})]\cdot(z_n-z_{n-1})$$

where $a=z_0,z_1,\ldots,z_n=b$ is a chain connecting a and b.

It was shown in [7] that if f,g are discrete analytic then so is $f*g$.

For $\phi(X)\in R_X$ define

$$\overline{\phi(X)} = \frac{(1+X)\phi(X)-\phi(0)}{X}$$

Then, in terms of $f=(\phi_f,\psi_f)$, $g=(\phi_g,\psi_g)$; $f*g=(\phi_{f*g},\psi_{f*g})$ is given by

$$\phi_{f*g}=\tfrac{1}{4}X\overline{\phi}_f\overline{\phi}_g , \quad \psi_{f*g}=\tfrac{1}{4}iY\overline{\psi}_f\overline{\psi}_g . \tag{6.2}$$

Also, $\Phi_{f*g}(0)=0$ so

$\overline{\phi}_{f*g}=\frac{1}{4}(1+X)\overline{\phi}_f\overline{\phi}_g$ and if h is discrete analytic

$$\phi_{(f*g)*h}=\frac{X}{4}\overline{\phi}_{f*g}\overline{\phi}_h=\frac{X(1+X)}{16}\overline{\phi}_f\overline{\phi}_g\overline{\phi}_h=\phi_{f*(g*h)} .$$

Similarly,

$$\psi_{(f*g)*h}=\psi_{f*(g*h)}$$

and we obtained a simple proof of Duffin and Duris [7]'s results that the convolution product is associative: $(f*g)*h=f*(g*h)$.

Invoking (5.2),

$$\phi_{\frac{z^{(n)}}{n!}*\frac{z^{(m)}}{m!}}=\frac{1}{4}\ \frac{1}{2^{n-1}}\cdot\frac{1}{2^{m-1}}\ \frac{X(1+X)^{n-1}(1+X)X(1+X)^{m-1}(1+X)}{(1-X)^{n+1}(1-X)^{m+1}X}\quad (n,m\geq 1)$$

$$=\frac{1}{2^{n+m}}\ \frac{X(1+X)^{n+m}}{(1-X)^{n+m+2}}=\phi_{\frac{z^{(n+m+1)}}{(n+m+1)!}} .$$

Similarly; $\psi_{\frac{z^{(n)}}{n!}*\frac{z^{(m)}}{m!}}=\psi_{\frac{z^{(n+m+1)}}{(n+m+1)!}}$

Thus, (ref. [7],p.205)

$$\frac{z^{(n)}}{n!}*\frac{z^{(m)}}{m!}=\frac{z^{(n+m+1)}}{(n+m+1)!}$$

The prime convolution product

The Prime convolution product of $f(z)$ and $g(z)$ was defined in ref. [7] to be

$$f*'g=\int_0^z f(z-t):g'(t)\partial t+f(z)g(0) \qquad (6.3)$$

where

$$\int_a^b f:g'\partial z=\frac{1}{2}\sum_{n=1}^{\infty}(f(z_n)+f(z_{n-1}))(g(z_n)-g(z_{n-1}))$$

and it was shown there that if f,g are discrete analytic, so is $f*'g$.

The coefficient of X^{n-1} in $\phi_{f*'g}$ is

$$\tfrac{1}{2}[f(n)+f(n-1)][g(1)-g(0)]+\tfrac{1}{2}[f(n-1)+f(n-2)]\cdot[g(2)-g(1)]+\ldots$$

$$+[f(1)+f(0)][g(n)-g(n-1)]+f(n-1)g(0)\quad.$$

Thus,

$$\phi_{f*'g}=\frac{1}{2}\,\frac{(1+X)\phi_f-\phi_f(0)}{X}\cdot\frac{(1-X)\phi_g-\phi_g(0)}{X}\cdot X+\phi_g(0)\phi_f$$

$$=\tfrac{1}{2}X\overline{\phi}_f(\overline{\phi}_g-2\phi_g)+\phi_g(0)\phi_f=\tfrac{1}{2}X\overline{\phi}_f\overline{\phi}_g-X\phi_g\overline{\phi}_f+\phi_g(0)\phi_f$$

$$=\tfrac{1}{2}X\overline{\phi}_f\overline{\phi}_g-X\phi_g\overline{\phi}_f+\phi_g(0)\frac{X\overline{\phi}_f+\phi_f(0)}{1+X}=\tfrac{1}{2}X\overline{\phi}_f\overline{\phi}_g-X\overline{\phi}_f[\phi_g-\frac{\phi_g(0)}{1+X}]+\frac{\phi_g(0)\phi_f(0)}{1+X}$$

$$=\tfrac{1}{2}X\overline{\phi}_f\overline{\phi}_g-X\overline{\phi}_f\frac{X\overline{\phi}_g}{1+X}+\frac{\phi_f(0)\phi_g(0)}{1+X}=\frac{X(1-X)}{2(1+X)}\overline{\phi}_f\overline{\phi}_g+\frac{\phi_f(0)\phi_g(0)}{1+X}\quad.$$

Similarly,

$$\psi_{f*'g}=\frac{Y(1-Y)}{2(1+Y)}\overline{\psi}_f\overline{\psi}_g+\frac{\psi_f(0)\psi_g(0)}{1+Y}\quad.$$

Thus,

$$\overline{\phi_{f*'g}}=\frac{1-X}{2}\overline{\phi}_f\overline{\phi}_g \quad\text{and}\quad \overline{\psi_{f*'g}}=\frac{1-Y}{2}\overline{\psi}_f\overline{\psi}_g \tag{6.4}$$

Now, $\phi(X),\psi(X)\in R_X$, $\overline{\phi(X)}=\overline{\psi(X)}\Rightarrow\phi(X)-\psi(X)=\text{constant}$. Since $f*'g(0)=f(0)g(0)=g*'f(0)$ and

$$\overline{\phi_{f*'g}}=(\frac{1-X}{2})\overline{\phi}_f\overline{\phi}_g=\overline{\phi_{g*'f}}$$

(and similarly $\overline{\psi_{f*'g}}=\overline{\psi_{g*'f}}$)

it is seen that the prime convolution product is commutative.

Also, from (6.4),

$$\overline{\phi_{(f*'g)*'h}}=\left(\frac{1-X}{2}\right)^2\overline{\phi_f}\,\overline{\phi_g}\,\overline{\phi_h}=\overline{\phi_{f*'(g*'h)}}$$

and $[(f*'g)*'h](0)=f(0)g(0)h(0)=[f*'(g*'h)](0)$ it follows that

$$(f*'g)*'h=f*'(g*'h)$$

and the associativity of the prime convolution product is proved.

Let us prove that

$$\frac{z^{(n)}}{n!}*'\frac{z^{(m)}}{m!}=\frac{z^{(n+m)}}{(n+m)!} \qquad (n,m\geq 1) \tag{6.5}$$

$$\phi_{\frac{z^{(n)}}{n!}}=\frac{1}{2^{n-1}}\,\frac{(1+X)^n}{(1-X)^{n+1}} \qquad (n\geq 1)$$

So,

$$\overline{\phi_{\frac{z^{(n)}}{n!}*'\frac{z^{(m)}}{m!}}}=\frac{1}{2^{n-1}}\,\frac{(1+X)^n}{(1-X)^{n+1}}\cdot\frac{1}{2^{m-1}}\,\frac{(1+X)^m}{(1-X)^{m+1}}\cdot\frac{(1-X)}{2}=\frac{1}{2^{n+m-1}}\cdot\frac{(1+X)^{n+m}}{(1-X)^{n+m+1}}=\overline{\phi_{\frac{z^{(n+m)}}{(n+m)!}}}$$

Similarly,

$$\overline{\psi_{\frac{z^{(n)}}{n!}*'\frac{z^{(m)}}{m!}}}=\overline{\psi_{\frac{z^{(n+m)}}{(n+m)!}}}$$

Of course,

$$\frac{z^{(n)}}{n!}*'\frac{z^{(m)}}{m!}(0)=\frac{z^{(n+m)}}{(n+m)!}(0) \quad ,$$

obtaining (6.5)

<u>The double prime convolution product</u>

The double prime product was defined in [7] by

$$f *'' g = \int_0^z \frac{\partial f(z-t)}{\partial z} \cdot \frac{\partial g(t)}{\partial t} \partial t$$

where $\frac{\partial f}{\partial z}$ is the discrete analytic derivative of f, defined in Section 4. Invoking formula (4.7) and (6.2.) we get

$$\phi_{f*''g} = \left(\frac{(1-X)\phi_f - \phi_f(0)}{4X}\right) \cdot \left(\frac{(1-X)\phi_g - \phi_g(0)}{X}\right) \cdot X$$

Let for $\phi \in R_X$

$$\tilde{\phi} = \frac{(1-X)\phi - \phi(0)}{X}$$

Then,

$$\phi_{f*''g} = \frac{X}{4} \tilde{\phi}_f \tilde{\phi}_g \qquad \psi_{f*''g} = \frac{iY}{4} \tilde{\psi}_f \tilde{\psi}_g \tag{6.6}$$

from which the commutativity of the double prime product is seen. Also,

$$\phi_{(f*''g)*''h} = \frac{X}{4} \tilde{\phi}_f \tilde{\phi}_g \frac{(1-X)}{4} \frac{\tilde{\phi}_h X}{4} = \frac{X(1-X)}{16} \tilde{\phi}_f \tilde{\phi}_g \tilde{\phi}_h = f*''(g*''h) .$$

Similary for the ψ's ; thereofre

$$(f*''g)*''h = f*''(g*''h) .$$

7. New Proofs to some results in Deeter and Lord [4].

In this section it will be shown that Theorem 1, Lemma 2 and Theorem 7 in Deeter and Lord [4] (here propositions 7.1, 7.2, 7.3 respectively) are immediate consequences of formula (6.4).

Deeter and Lord [4] defined the mean of the function on the positive x-axis and y-axis respectively by,

$$F(m,0) = \frac{1}{2} [f(m,0) + f(m-1,0)] \quad m=1,2,...$$

$$F(0,m) = \frac{1}{2} [f(0,m) + f(0,m-1)] \quad m=1,2,...$$

So, in our notation,

$$\sum_{m=1}^{\infty} \overline{f}(m,0)X^m = X\overline{\phi}_f \qquad \sum_{m=1}^{\infty} \overline{f}(0,m)Y^m = Y\overline{\psi}_f \quad . \tag{7.1}$$

If f has mean zero on the $x(y)$ axis then $\overline{\phi}_f(\overline{\psi}_f) \equiv 0$.

Proposition 7.1 (Theorem 1 in [4])

If two discrete analytic functions are such that the mean of either function is zero on an axis then the mean of their (prime convolution) product is zero on that axis.

Proof: Immediate from formula (6.4) .

Proposition 7.2 (Lemma 2 in [4])

Let f,g be discrete analytic and satisfy,

$$\overline{f}(1,0) = \ldots = \overline{f}(n-1,0) = 0, \ \overline{f}(n,0) \neq 0$$

$$\overline{g}(1,0) = \ldots = \overline{g}(m-1,0) = 0, \ \overline{g}(m,0) \neq 0$$

then $\overline{f *' g}(1,0) = \ldots = \overline{f *' g}(m+n-2) = 0$

and $\overline{f *' g}(m+n-1,0) = \overline{f}(n,0)\overline{g}(m,0) \neq 0$.

Proof: From (7.1), the leading term of $\overline{\phi}_f$ is $\overline{f}(n,0)X^{n-1}$, the leading term of $\overline{\phi}_g$ is $\overline{g}(m,0)X^{m-1}$ and thus the leading term of

$$X\overline{\phi_{f*'g}} = X\frac{1-X}{2}\overline{\phi}_f\overline{\phi}_g$$

is $\overline{f}(n,0)\overline{g}(m,0)X^{n+m-1}$ and the conclusion follows from (7.1).

Proposition 7.3 (Theorem 7 in [4]).

Let f be discrete analytic, a necessary and sufficient condition for the existence of a solution of the equation

$$g*\,{}^{k}=f$$

is that there exist nonnegative integers m and n such that

$$\overline{f}(1,0)=\ldots=\overline{f}(km,0)=0\ ,\ \overline{f}(km+1,0)\neq 0$$

$$\overline{f}(0,1)=\ldots=\overline{f}(0,kn)=0\ ,\ \overline{f}(0,kn+1)\neq 0\ .$$

Proof: Since

$$\overline{\phi_{g*\,k}}=\left(\frac{1-X}{2}\right)^{k-1}(\overline{\phi}_g)^k$$

$$\overline{\psi_{g*\,k}}=\left(\frac{1-Y}{2}\right)^{k-1}(\overline{\psi}_g)^k$$

the conclusion follows from Lemma 2.1.

8. DISCRETE VOLTERRA INTEGRAL EQUATIONS

Let $f(z)$ and $k(z)$ be discrete analytic functions in a rectangular region R which (without loss of generality) will be assumed to be $Z^+\times Z^+$. Duffin and Duris [7] considered the problem of finding a discrete analytic solution $u(z)$ to the equation

$$u(z)=f(z)+\lambda\int_0^z k(z-t):u(t)\partial t\ . \tag{8.1}$$

Where λ is an arbitrary constant. Translated to our language (8.1) reads,

$$\phi_u=\phi_f+\frac{\lambda}{4}X\overline{\phi}_k\overline{\phi}_u \tag{8.2a}$$

$$\psi_u=\psi_f+\frac{\lambda}{4}Yi\overline{\psi}_k\overline{\psi}_u \tag{8.2b}$$

Thus

$$\phi_u=\phi_f+\frac{\lambda\overline{\phi}_k}{4}[\psi_u(1+X)-\phi_u(0)] \tag{8.3a}$$

$$\psi_u=\psi_f+\frac{\lambda Yi}{4}\overline{\psi}_k[\psi_u(1+Y)-\psi_u(0)] \quad . \tag{8.3b}$$

Now, if a solution $u(z)$ to (8.1) exists, $\phi_u(0)=\psi_u(0)=f(0)$ by (8.1). So,

$$\phi_u[1-\frac{\lambda}{4}\overline{\phi}_k\cdot(1+X)]=\phi_f-\frac{\lambda\overline{\phi}_k}{4}f(0) \tag{8.4a}$$

$$\psi_u[1-\frac{i\lambda}{4}\overline{\psi}_k\cdot(1+Y)]=\psi_f-\frac{i\lambda\overline{\psi}_k}{4}f(0) \quad . \tag{8.4b}$$

Theorem 8.1 (Theorem 5.2 in ref. [7])

Let $f(z)$ and $k(z)$ be discrete analytic functions in $Z^+\times Z^+$. Then there exists a unique function $u(z)$ discrete analytic in $Z^+\times Z^+$ such that

$$u(z)=f(z)+\lambda\int_0^z k(z-t):u(t)\partial t \tag{8.1}$$

for all values except possibly $\lambda=1/h[k(0)+k(h)]$ where h equals $+1$, $+i$. (This is not exactly the original wording but it is equivalent to it).

Proof: A solution of (8.1) exists iff there is a solution of (8.4a), (8.4b) simultaneously. A solution of (8.4) exists (and then is unique) if the coefficient terms of both

$$1-\frac{\lambda}{4}\overline{\phi}_k(1+X) \quad \text{and} \quad 1-\frac{i\lambda}{4}\overline{\psi}_k(1+Y)$$

are not zero; i.e., if

$$1\neq\frac{\lambda}{4}(k(0)+k(1))$$

$$1\neq\frac{i\lambda}{4}(k(0)+k(i))$$

and in this case the unique solution $u=(\phi_u,\psi_u)$ is given by:

$$\phi_u=(\phi_f-\frac{\lambda\overline{\phi}_k}{4}f(0))[1-\frac{\lambda}{4}\overline{\phi}_k(1+X)]^{-1}$$

$$\psi_u=(\psi_f-\frac{i\lambda\overline{\psi}_k}{4}f(0))[1-\frac{i\lambda}{4}\overline{\psi}_k(1+Y)]^{-1} .$$

If the condition on λ is not satisfied, i.e.,

$$\lambda=4/[h\cdot(k(0)+k(h))]$$

for either $h=1$ or $h=i$. Then a solution may or may not exist according to the leading term of the r.h.s. of (8.4). The solution is not unique iff

$$1-\frac{\lambda}{4}\overline{\phi}_k(1+X)\equiv 0 \quad \text{or} \quad 1-\frac{i\lambda}{4}\overline{\psi}_k\cdot(1+Y)\equiv 0$$

i.e.,

$$\phi_k=\frac{4}{\lambda}\frac{X}{(1+X)^2}+\frac{\phi_k(0)}{1+X}$$

or

$$\psi_k=\frac{4}{i\lambda}\frac{Y}{(1+Y)^2}+\frac{\psi_k(0)}{1+Y} .$$

It is also possible to prove, by the method of this chapter, most of the results in Duffin and Duris [8].

9. A REPRESENTATION FORMULA FOR THE HALF PLANE

Consider the abelian group T_{xy} of all formal power series

$$\sum_{m,n=-\infty}^{\infty} a_{mn}X^mY^n \qquad (9.1)$$

(note that we allow here also negative powers).

Define, again,

$$\Sigma a_{mn}X^mY^n+\Sigma b_{mn}X^mY^n=\Sigma(a_{mn}+b_{mn})X^mY^n .$$

Let,

$$A=\Sigma a_{mn}X^mY^n \quad ; \quad B=\Sigma b_{mn}X^mY^n$$

$C=AB$ is said to exist if

$$C_{mn}=\sum_{\substack{k=-\infty \\ r=-\infty}}^{\infty} b_{r,k}a_{m-r,n-k}$$

converges absolutely for each m,n integers.

The following lemma is trivial

Lemma 9.1: If $A,B,C\in T_{XY}$ and B has only a finite number of non-zero terms and if both $(AB)C$ and $A(BC)$ exist, then $(AB)C=A(BC)$.

Now we can reprove the following representation formula from Duffin [6] (p. 347).

Theorem 9.2: Let $q(z)$ be a discrete function such that

$$Lq(z)=0 \quad , \quad z\neq 0 \qquad\qquad Lq(0)=1 \tag{9.2}$$

and let $f(z)$ be a function which is analytic in every unit square of the upper half plane; suppose that for each fixed z_0

$$f(z)q(z-z_0)=o|z|^{-1} \quad \operatorname{Im}z\geq 0 \tag{9.3}$$

Then if

$$\theta(z)=q(-z)+iq(1-z) \tag{9.4}$$

we have

$$f(z_0)=\sum_{m=-\infty}^{\infty} f(m)\theta(z_0-m) \quad , \quad \operatorname{Im}z_0\geq 0 \tag{9.5}$$

and the r.h.s. is zero for $\operatorname{Im}z_0<0$.

Proof: Let $\underline{q}=\sum_{m,n=-\infty}^{\infty} q(m,n)X^mY^n$, from (9.2)

$$(1+iX^{-1}-X^{-1}Y^{-1}-iY^{-1})\underline{q}=1 \quad .$$

Let $\tilde{q}(z)=q(-z)$ then

$$\underline{\tilde{q}}=\underline{q}(X^{-1},Y^{-1}) \quad .$$

Thus,

$$(1+iX-XY-iY)\underline{\tilde{q}}=1 \quad .$$

Now

$$\theta(z)=q(-z)+iq(1-z)=\tilde{q}(z)+i\tilde{q}(-1+z)$$

$$\underline{\theta}=(1+iX)\underline{\tilde{q}} \quad .$$

So,

$$(1+iX-XY-iX)\underline{\theta}=1+iX \tag{9.6}$$

Let,

$$\underline{f}(X,Y)=\sum_{n=o}^{\infty}\ \sum_{m=-\infty}^{\infty} f(in+m)X^mY^n$$

$$\phi_0(X)=\sum_{n=-\infty}^{\infty} f(n)X^n \quad .$$

Since f is discrete analytic in the upper half plane, similar considerations as in section 3 show that

$$\underline{f}(X,Y)(1+iX-XY-iY)=\phi_0(X)(1+iX) \tag{9.7}$$

Multiply both sides by $\underline{\theta}(X,Y)$ (The product exists by virtue of condition (9.3))

$$[\underline{f}(X,Y)(1+iX-XY-iX)]\underline{\theta}(X,Y)=[\phi_0(X)(1+iX)]\underline{\theta}(X,Y) \quad .$$

By Lemma 9.1

$$\underline{f}(X,Y)[(1+iX-XY-iY)\underline{\theta}(X,Y)] = \phi_0(X)[(1+iX)\underline{\theta}(X,Y)]$$

since all products involved exist. By (9.6)

$$\underline{f}(X,Y)(1+iX)=\phi_0(X)[(1+iX)\underline{\theta}(X,Y)]=\phi_0(X)[\underline{\theta}(X,Y)(1+iX)]=[\phi_0(X)\underline{\theta}(X,Y)](1+iX)$$

Let $\underline{F}(X,Y)=\underline{f}(X,Y)-\phi_0(X)\underline{\theta}(X,Y)$. (9.8)

We want to show $F(X,Y)\equiv 0$ and then it would follow that

$$\underline{f}(X,Y)=\phi_0(X)\underline{\theta}(X,Y) \quad (9.9)$$

which is the same as (9.5) .

From (9.8)

$$\underline{F}(X,Y)(1+iX)\equiv 0 \quad (9.10)$$

But

$$\underline{F}(X,Y)(1+iX-XY-iY)=\underline{f}(X,Y)(1+iX-XY-iY)-[\phi_0(X)\underline{\theta}(X,Y)](1+iX-XY-iY)=$$

$$=\phi_0(X)(1+iX)-\phi_0(X)[\underline{\theta}(X,Y)(1+iX-XY-iY)]=\phi_0(X)(1+iX)-\phi_0(X)(1+iX)=0$$

So

$$\underline{F}(X,Y)(1+iX-XY-iY)\equiv 0 \quad . \quad (9.11)$$

Multiply equation (9.10) by $(1+iY)$ and subtract from (9.11) to get

$$\underline{F}(X,Y)2iY\equiv 0$$

and consequently $\underline{F}(X,Y)\equiv 0$.

CHAPTER III.

SOME NEW PROPERTIES OF DISCRETE ANALYTIC FUNCTIONS.

1. Introduction.

In this chapter we prove discrete analogues of the classical theorems of Liouville, Phragmén-Lindelöf and Paley-Wiener.

2. Discrete analytic functions of polynomial growth.

Duffin [6] defined a bipolynomial to be a discrete analytic function which assumes the values of one polynomial on the even* lattice points and the values of another (possibly the same) polynomial on the odd lattice points.

Theorem 1. Every discrete analytic function F of polynomial growth is a bipolynomial.

Proof: Assume $h = 1$, (the proof for general h is similar) and let $F(m,n)$ be a discrete analytic function of polynomial growth: $|F(m,n)| \leq C(|m| + |n|)^k$ for some constants C and k. Then (Edwards [11], Chapter 12)

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* The lattice point (mh,nh) is said to be even [odd] if $m+n$ is even [odd].

F is the Fourier transform of a distribution D on the 2 dimensional torus $T^2 (= \hat{Z}^2)$,

$$F(m,n) = D(e^{imt+ins})$$

Substituting this into I(2) one gets

$$0 = F(m,n)+iF(m+1,n)-F(m+1,n+1)-iF(m,n+1)$$

$$=D(e^{imt+ins})+iD(e^{i(m+1)t+ins})-D(e^{i(m+1)t+i(n+1)s})-iD(e^{imt+i(n+1)s})$$

$$= D(e^{imt+ins}+ie^{i(m+1)t+ins} - e^{i(m+1)t+i(n+1)s} - ie^{imt+i(n+1)s})$$

$$= D((1+ie^{it}-e^{it+is}-ie^{is})e^{imt+ins}) = 0$$

for every point $(m,n) \in Z^2$. Thus

$$(1+ie^{it} - e^{it+is} - ie^{is}) D \equiv 0 .$$

The only roots of $1+ie^{it} - e^{it+is} -ie^{is} = 0$ are the points $(0,0)$ and (π,π), which implies that D is supported in these points. So if δ denotes the Dirac measure and $\delta_{(\pi,\pi)}$ denotes the Dirac measure translated by (π,π), D can be written (Donoghue [5] p. 103) as a finite sum of derivatives of δ and $\delta_{(\pi,\pi)}$:

$$D = \sum_{\substack{k=0\\ \ell=0}}^{K,L} a_{k\ell} \frac{\partial^{k+\ell}}{\partial^k \partial^\ell} \delta + \sum_{\substack{k=0\\ \ell=0}}^{K,L} b_{k\ell} \frac{\partial^{k+\ell}}{\partial^k \partial^\ell} \delta_{(\pi,\pi)} .$$

So

$$F(m,n) = D(e^{imt+ins}) = \sum_{\substack{k=0\\ \ell=0}}^{K,L} a_{k\ell}(-1)^{k+\ell}(im)^k(in)^\ell$$

$$+ \sum_{\substack{k=0\\ \ell=0}}^{K,L} b_{k\ell}(-1)^{k+\ell}(im)^k(in)^\ell \quad e^{im\pi+in\pi} =$$

$$P(m,n)+(-1)^{m+n}Q(m,n) = \begin{cases} P(m,n)+Q(m,n), & m+n \text{ even} \\ P(m,n)-Q(m,n), & m+n \text{ odd} \end{cases}$$

where P Q are the polynomials

$$P(m,n) = \sum_{\substack{k=0 \\ \ell=0}}^{K,L} (-i)^{k+\ell} a_{k\ell}\, m^k n^\ell$$

$$Q(m,n) = \sum_{\substack{k=0 \\ \ell=0}}^{K,L} (-i)^{k+\ell} b_{k\ell}\, m^k n^\ell .$$

In the algebra $C^\infty(T^2)$ the discrete analytic functions of polynomial growth are exactly the Fourier transforms of distributions which annihilate the ideal $(1+ie^{it} - e^{it+is} - ie^{is})\, C^\infty(T^2)$. If the mesh size of the lattice is h instead of 1 , then the discrete analytic functions of polynomial growth are the Fourier transforms of distributions on $\frac{T}{h} \times \frac{T}{h}$ which annihilate the ideal

$$a_h(t,s)C^\infty\left(\frac{T}{h} \times \frac{T}{h}\right) ,$$

where $a_h(t,s) = \dfrac{1+ie^{iht} - e^{iht+ihs} - ie^{ihs}}{-(1+i)h}$.

Now

$$-(1+i)ha_h(t,s) = 1+ie^{iht} - e^{iht+ihs} - ie^{ihs} = 1+i(1+iht+O(h^2))$$
$$-(1+iht+ihs+O(h^2)) - i(1+ihs+O(h^2)) = -(1+i)h[(t+is)+O(h)].$$

So $a_h(t,s) = (t+is) + O(h)$.

Now let $f(z) = f(x+iy)$ be a (continuous) entire function of polynomial growth. Then $f(x,y) = \hat{D}$ for some temperate distribution* D , and by the Cauchy-Riemann equation

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\hat{D} \equiv 0 \quad \text{or, via the Fourier transform}$$

$$(t+is)\, D \equiv 0 \quad .$$

So (continuous) entire functions of polynomial growth are exactly the Fourier transforms of temperate distributions which annihilate the ideal $(t+is)\, C^{\infty}_{\downarrow}(R \times R) = a_o(t,s)\, C^{\infty}_{\downarrow}(R \times R)$. But $(t+is)\, D \equiv 0$ implies that D is supported at the point $(0,0)$ and therefore is a finite sum of derivatives of the Dirac measure δ , and the familiar Liouville theorem drops out: an entire function of polynomial growth is a polynomial. Since $a_o(t,s) = t+is$ vanishes just at one point (namely $(0,0) \in R^2$) while $a_h(t,s)$ vanishes at two points $((0,0)$ and $(\frac{\pi}{h}, \frac{\pi}{h}) \in \frac{T}{h} \times \frac{T}{h})$ it is clear why, in the discrete theory, we encounter bipolynomials and not just polynomials.

3. A Phragmén - Lindelöf principle for discrete analytic functions

In the classical theory of analytic functions there are a number of theorems, associated with the names of Phragmén and Lindelöf, which compare the growth of an analytic function inside a sector, or a strip, with the growth of the function on the boundary.

* A temperate distribution is a continuous linear functional on the Frechet space $C^{\infty}_{\downarrow}(R^2)$, the space of rapidly decreasing functions, cf. Donoghue [5], p. 134.

The only sectors that can be treated conveniently in the discrete theory are, evidently, the ones bounded by the axes. For the sake of definiteness we choose to consider

$Z^+ \times Z^+ = \{(m,n) \; ; \; m \text{ and } n \text{ integers}, \; m \geq 0 \; , \; n \geq 0\}$.

Theorem 2. Let $F(m,n)$ be a discrete analytic function in the quarter lattice $Z^+ \times Z^+$, and assume that there are constants $T > 1$, $S > 1$ and C_1 , C_2 such that

$$|F(m,0)| \leq C_1 T^m \quad , \quad m \geq 0 \tag{3.1a}$$

$$|F(0,n)| \leq C_2 S^n \quad , \quad n \geq 0 \quad . \tag{3.1b}$$

Then for every $T_1 > T$, $S_1 > \max\{S , \frac{T+1}{T-1}\}$ there exists a constant C such that $|F(m,n)| \leq C T_1^m S_1^n \;\; \forall (m,n) \in Z^+ \times Z^+$.

Proof. Consider the formal power series

$$\underline{F}(z,w) = \sum_{\substack{m=0 \\ n=0}}^{\infty} F(m,n) z^m w^n \quad .$$

Then

$$(1+iz-zw-iw)\underline{F}(z,w) = (1+iz)\phi_F(z)+(1-iw)\psi_F(w)-F(0,0)$$

$$- \sum_{\substack{m=0 \\ n=0}}^{\infty} [F(m,n)+iF(m+1,n)-F(m+1,n+1)-iF(m,n+1)]z^{m+1}w^{n+1} \tag{3.3}$$

where ϕ_F , ψ_F are the formal power series

$$\phi_F(z) = \sum_{m=0}^{\infty} F(m,0)z^m$$

$$\psi_F(w) = \sum_{n=0}^{\infty} F(0,n)w^n \quad .$$

Now, since $F(m,n)$ is discrete analytic in $Z^+ \times Z^+$, the last term on the r.h.s. of (3.3) is zero and consequently

$$(1+iz-zw-iw)\underline{F}(z,w) = (1+iz)\phi_F(z)+(1+iw)\psi_F(w)-F(0,0) \quad . \qquad (3.4)$$

Until now, ϕ_F, ψ_F and $\underline{F}$ were considered as formal power series, but by (3.1a), $\phi_F(z)$ is convergent in $\{|z| < \frac{1}{T}\}$ and represents an analytic function there. Similarly, by (3.1b) $\psi_F(w)$ represent an analytic function in $\{|w| < \frac{1}{S}\}$ so the r.h.s. of (3.4) is an analytic function of two complex variables in the polydisc $\{|z| < \frac{1}{T}\} \times \{|w| < \frac{1}{S}\}$. Thus

$$\underline{F}(z,w) = \frac{(1+iz)\phi_F(z)+(1-iw)\psi_F(w)-F(0,0)}{1+iz-zw-iw} ,$$

which was only defined a priori as formal power series, is a convergent power series in the polydisc $\{|z| < \frac{1}{T}\} \times \{|w| < \frac{1}{S'}\}$ where $S' = \max\{S, \frac{T+1}{T-1}\}$.

Finally, since $F(m,n)$ is the coefficient of $z^m w^n$ in the Taylor expansion of $\underline{F}(z,w)$ it follows by Cauchy's inequality that for every $T_1 > T$, $S_1 > \max\{S, \frac{T+1}{T-1}\}$ there exists a constant C such that

$$|F(m,n)| \leq CT_1^m S_1^n$$

for every point $(m,n) \in Z^+ \times Z^+$.

4. Discrete Fourier analysis and discrete analytic functions.

The chareacters of the group R can be identified as the class of functions $e^{isx} : s \in R = \hat{R}$. Clearly each character can be extended analytically to the whole complex plane as $e^{is(x+iy)} := e^{isx}e^{-sy}$. Now look at the group Z, with characters $e^{imt}(t \in T = \hat{Z})$. One may ask: What is the natural discrete analytic extension of $e^{imt}(m \in Z)$ to the whole discrete lattice Z^2 ? With the continuous example in mind, let us try for an extension of the form

$$e^{imt} \cdot \phi_t(n) \quad \text{with} \quad \phi_t(0) = 1 .$$

Substituting this into (1.2) we obtain $e^{imt}[\phi_t(n)+ie^{it}\phi_t(n) - e^{it}\phi_t(n+1) - i\phi_t(n+1)] = 0$. Therefore

$$(1+ie^{it})\phi_t(n) = (i+e^{it})\,\phi_t(n+1) .$$

If $t \neq \pm\frac{\pi}{2}$ one gets

$$\phi_t(n) = \left(\frac{1+ie^{it}}{e^{it}+i}\right)^n \qquad n \in Z .$$

So the natural analogue to the exponential function e^{isz} $e^{isz} = e^{isx}\,e^{-sy}$ $(s \in R)$ is

$$\mathscr{E}(it;m+in) = e^{imt}\left(\frac{1+ie^{it}}{i+e^{it}}\right)^n \qquad (t \in T\ ,\ t \neq \pm\tfrac{\pi}{2}) \qquad (4.1)$$

which we shall call the discrete exponential function. This coincides with the discrete exponential function introduced by Ferrand-Lelong [13]:

$$e(m+in;s) = \left(\frac{2+s}{2-s}\right)^m \left(\frac{2+is}{2-is}\right)^n$$

if $\frac{2+s}{2-s} = e^{it}$. As the above considerations showed, our exponential function seems to be a more natural analogue of the continuous exponential function, at least for the purpose of doing Fourier analysis. In fact, the main theme of this chapter is that continuous analytic function theory (on $R^2 = \mathbb{C}$) is what it is because of the dual group of R : $\hat{R} = R$, and discrete analytic function theory on Z_h^2 is what it is because of the dual group of Z_h : $\hat{Z}_h = \frac{T}{h}$. Notice that if $t = \frac{\pi}{2}$ [$= -\frac{\pi}{2}$] , then (4.1) still defines a meaningful exponential function for $n \geq 0$ [$n \leq 0$]. The analogue on the lattice $Z_h \times Z_h$ is

$$\tilde{e}_h(it;mh+inh) = e^{imth}\left(\frac{1+ie^{ith}}{e^{ith}+i}\right)^n .$$

Notice that for any fixed $t \in R$

$$\tilde{e}_h(it;x+iy) = e^{itx}\left[\left(\frac{1+ie^{ith}}{e^{ith}+i}\right)^{\frac{1}{h}}\right]^y \longrightarrow e^{it(x+iy)}$$

as $h \downarrow 0$, since

$$\left(\frac{1+ie^{ith}}{e^{ith}+i}\right)^{\frac{1}{h}} \longrightarrow e^{-t} .$$

Let us return to the case $h = 1$. Immitating the notation in the continuous case we let Z^{2+} denote the upper half lattice $\{(m,n);n \geq 0\}$ and define the class $H^{2+}(Z)$ as follows:

<u>Definition.</u> A discrete analytic function F on Z^{2+} is said to belong to $H^{2+}(Z)$ if

$$\sup_{n\geq 0}\left(\sum_{m=-\infty}^{\infty}|F(m+in)|^2\right)^{1/2} < \infty \quad . \tag{4.2}$$

We are now in a position to give a discrete analogue to the famed one sided Paley-Wiener theorem.

<u>Theorem 3:</u> If F is discrete analytic on Z^{2+} and if $\sum_{-\infty}^{\infty}|F(m)|^2 < \infty$ then $F \in H^{2+}(Z)$ iff

$$F_o^v(t) = \sum_{-\infty}^{\infty} F(m)\, e^{-imt} = 0$$ a.e. in $(-\pi,0)$ and in that case F has the representation

$$F(m+in) = \frac{1}{2\pi}\int_0^{\pi} \mathscr{C}(it;m+in)F_o^v(t)dt \quad . \tag{4.3}$$

<u>Proof.</u> Suppose $F_o^v(t) = 0$ a.e. in $(-\pi,0)$ then (4.3) defines a discrete analytic extension of the starting sequence $F(m)$ to the upper half lattice Z^{2+} ; and

$$F_n(m) = F(m+in) = \frac{1}{2\pi}\int_0^{\pi} F_o^v(t)\left(\frac{1+ie^{it}}{i+e^{it}}\right)^n e^{imt}\,dt$$

implies, by Plancherel, that for $n \geq 0$

$$\sum_{m=-\infty}^{\infty}|F(m+in)|^2 = ||F_n||_{\ell^2}^2 = \frac{1}{2\pi}\int_0^{\pi}\left|\frac{1+ie^{it}}{i+e^{it}}\right|^{2n}|F_o^v(t)|^2dt$$

$$\leq \frac{1}{2\pi}\int_0^{\pi} F_o^v(t)\ ^2dt \quad .$$

Since $\left| \frac{1+ie^{it}}{i+e^{it}} \right| \leqslant 1$ for $t \in [0,\pi]$.

This proves $F \in H^{2+}(Z)$.

Conversely, if $F \in H^{2+}(Z)$ then $F_n \in \ell^2 = L^2(Z)$ for $n \geqslant 0$ and $F_n^v(t) = \sum_{m=-\infty}^{\infty} F(m+in)e^{-imt} \in L^2(T)$.

Now, using I(2) it is readily seen that

$$(1+ie^{it})F_n^v(t) = (i+e^{it})F_{n+1}^v(t) \qquad (n \geqslant 0) \quad .$$

Since the Fourier coefficients of the two sides match:

$$\frac{1}{2\pi} \int (1+ie^{it})F_n^v(t)e^{imt}dt = F_n(m)+i\,F_n(m+1)$$

$$= F(m+in) + iF(m+1+i\,n) = i\,F(m+i(n+1)) + F(m+1+i(n+1))$$

$$= iF_{n+1}(m) + F_{n+1}(m+1) = \frac{1}{2\pi} \int (i+e^{it})\,F_{n+1}^v(t)e^{imt}dt \ .$$

Therefore

$$F_n^v(t) = \left(\frac{1+ie^{it}}{i+e^{it}} \right)^n F_o^v(t) \qquad (n \geqslant 0) \quad . \qquad (4.4)$$

Now, suppose that F_o^v does not vanish a.e. in $(-\pi,0]$. Then there exists an interval $[\alpha,\beta] \subset (-\pi,0)$ such that $\int_\alpha^\beta |F_o^v|^2 \neq 0$, and so

$$\sum_m |F(m+in)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |F_n^v(t)|^2\,dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \left(\frac{1+ie^{it}}{i+e^{it}} \right)^n F_o^v(t) \right|^2 dt$$

$$\geqslant k^{2n}\ \frac{1}{2\pi} \int_\alpha^\beta |F_o^v(t)|^2\ dt$$

in which

$$k = \min \left\{ \left| \frac{1+ie^{it}}{i+e^{it}} \right| ; \alpha \leq t \leq \beta \right\} > 1$$

contradicting condition (4.2).

Hence $F_o^V(t) = \sum_{m=-\infty}^{\infty} F(m)e^{-imt} = 0$ a.e. in $(-\pi,0]$.
By (4.4), for $n \geq 0$

$$F(m+in) = F_n(m) = \frac{1}{2\pi} \int_0^{\pi} e^{imt} \left(\frac{1+ie^{it}}{i+e^{it}} \right)^n F_o^V(t)dt$$

$$= \frac{1}{2\pi} \int_0^{\pi} \tilde{e}(it;m+in)F_o^V(t)dt \quad ,$$

establishing (4.3).

By (4.3) a function $F(m+in)$ of class $H^{2+}(Z)$ is uniquely determined by its restriction to the discrete real line $n = 0$, so $H^{2+}(Z)$ can be viewed, in an obvious fashion, as a subset of $\ell^2 = L^2(Z)$ and Theorem 3 tells us that

$$L^2(Z) = L^2(-\pi,\pi)^\wedge \supset L^2(0,\pi)^\wedge = H^{2+}(Z)$$

which is in perfect analogy with the line (cf. Hoffman [17], p. 131)

$$L^2(R) = L^2(R)^\wedge \supset L^2(0,\infty)^\wedge = H^{2+}(R)$$

and the circle (cf. Hoffman [17], p. 39)

$$L^2(T) = L^2(Z)^\wedge \supset L^2(Z^+)^\wedge = H^{2+}(T) \quad .$$

Unfortunately, Beurling's elegant theory of invariant subspaces does not seem to have an analogue in the discrete theory, due to the fact that the dual group of Z , $\hat{Z} = T$ is not ordered (cf. Rudin [18], chapter 8, p. 210).

Define $H^{2-}(z)$ to be the class of discrete analytic function on the lower half plane $Z^{2-} = \{(m,n);n \leq 0\}$ satisfying

$$\sup_{n\leq 0} \sum_{m=-\infty}^{\infty} |F(m+in)|^2 < \infty \quad .$$

It is now readily checked that

$H^{2-}(Z) = L^2(-\pi,0)^{\wedge}$, so one has the orthogonal decomposition

$$\ell^2 = L^2(Z) = H^{2+}(Z) \oplus H^{2-}(Z) \quad .$$

Let $\chi_{[0,\pi]}$ be the characteristic function of $[0,\pi]$ and let

$$\theta(m,n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_{[0,\pi]} \mathscr{E}(it;\, m+in)dt = \frac{1}{2\pi} \int_0^{\pi} \left(\frac{i+ie^{it}}{i+eit}\right)^n e^{imt}dt$$

$$= \frac{1}{2\pi} \int_0^{\pi} \tan^n\left(-\frac{t}{2} + \frac{\pi}{4}\right) e^{imt}dt$$

which turns out to be Duffin's ([2], p. 349) discrete Cauchy kernel. Now, if $F(m,n) \in H^{2+}$ then $F_o^{\vee}(t)\chi_{[0,\pi]}(t) = F_o^{\vee}(t)$ so by (4.4)

$$F_n(m) = \left[F_o^{\vee}(t)\chi_{[0,\pi]}(t) \left(\frac{1+ie^{it}}{i+e^{it}}\right)^n\right]^{\wedge} (m) =$$

$$F_o * \left[\chi_{[0,\pi]} \left(\frac{1+ie^{it}}{i+e^{it}}\right)^n\right]^{\wedge} = F_o * \theta_n$$

obtaining the following representation formula for H^{2+} functions:

$$F(m+in) = \sum_{m=-\infty}^{\infty} F(k)\ \theta(m-k+in) \tag{4.5}$$

(Compare Duffin [6], p. 347 formula 52).

Another consequence of (4.4) is

$$\sum_{m=-\infty}^{\infty} |F(m+i(n+1))|^2 = \int_0^{\pi} \left| \frac{1+ie^{it}}{i+e^{it}} \right|^{2n+2} |F_o^v(t)|^2\ dt$$

$$\leq \int_0^{\pi} \left| \frac{1+ie^{it}}{i+e^{it}} \right|^{2n} |F_o^v(t)|^2 dt = \sum_{m=-\infty}^{\infty} |F(m+in)|^2 \ .$$

Thus $||F_n||^2 = \sum_{m=-\infty}^{\infty} |F(m+in)|^2 \downarrow$ and

$$\sup_{n\geq 0} \left(\sum_{m=-\infty}^{\infty} |F(m+in)|^2 \right)^{1/2} = \left(\sum_{m=-\infty}^{\infty} |F(m)|^2 \right)^{1/2}$$

and we have proved

Corollary. $H^{2+}(Z)$ is a Hilbert space with norm

$$||F|| = \sup_{n\geq 0} \left(\sum_{m=-\infty}^{\infty} |F(m+in)|^2 \right)^{1/2} = \left(\sum_{m=-\infty}^{\infty} |F(m)|^2 \right)^{1/2}$$

and reproducing kernel $\theta(m+in-k)$.

Finally, let us remark that if we chose to consider $Z_h \times Z_h$ instead of $Z \times Z$ we would have obtained, instead of (4.3), the representation formula

$$F(x+iy) = \int_0^{\pi/h} \tilde{e}_h(it,x+iy) F_o^v(t) dt$$

which, on letting $h \downarrow 0$ "tends" to the classical Paley-Wiener representation formula:

$$F(x+iy) = \int_0^{\infty} e^{it(x+iy)}\ F_o^v(t) dt \ .$$

5. Discrete Paley-Wiener-Schwartz theorems.

Let us recall that a distribution D on the real line R, with compact support, has a Fourier transform $\hat{D}(\xi) = D(e^{ix\xi})$ which can be extended to an entire function $\hat{D}(\zeta) = D(e^{ix\zeta}) = D(e^{ix\xi}e^{-x\eta})$, $\zeta = \xi+i\eta$; and one has the following results (Donoghue [5], p. 210-213):

a) Let D be a distribution supported in $[-a,a]$ then the Fourier transform $\hat{D}(\zeta)$ satisfies an inequality of the form

$$|\hat{D}(\zeta)| \leq C(1+|\zeta|)^N e^{a|\eta|}$$

where $\zeta = \xi+i\eta$ and N is the order of D.

b) The Fourier transform $\hat{\phi}(\zeta)$ of a testfunction ϕ supported in $[-a,a]$ is an entire function. For each integer k there exists a constant C_k such that

$$|\phi(\zeta)| \leq C_k(1+|\zeta|)^{-k} e^{a|\eta|}$$

c) (converse to b)) Let $F(\zeta)$ be an entire function with the property that for every integer $k \geq 0$ there exists a constant C_k such that

$$|F(\zeta)| \leq C_k(1+|\zeta|)^{-k} e^{|\eta|a}$$

where $\zeta = \xi+i\eta$; then there exists a testfunction ϕ supported in $[-a,a]$ such that $\hat{\phi}(\zeta) = F(\zeta)$.

d) (converse to a)) Let $F(\zeta)$ be an entire function which satisfies an inequality of the form

$$|F(\zeta)| \leq C(1+|\zeta|)^N e^{a|\eta|} \quad ; \text{ then } F(\zeta)$$

is the Fourier transform of a distribution supported in $[-a,a]$.

We were able to prove discrete analogues for a) and c) by translating their proofs to the language of the discrete case. However, the proofs of b) and d) do not carry over due to the fact that the discrete exponential function is not as nice as the continuous one (in the case of b)) and to the fact that the multiplication of two discrete analytic functions is not, in general, discrete analytic (in the case of d)).

Let us consider the exponential $e^{ix\zeta}$ as an entire function of ζ and let x vary along the extended real line $\tilde{R}$; we see that $e^{ix\zeta}$ defines an entire function for each $x \in \tilde{R} \setminus \{\infty,-\infty\}$ $(=R)$ and for each fixed ζ , $e^{ix\zeta}$ behaves nicely as long as one stays away from ∞ and $-\infty$. Now, the discrete exponential function $e(it,m+in)$, $t \in T$ is singular only at $t = \frac{\pi}{2}$ (if $n < 0$) or $t = -\frac{\pi}{2}$ (if $n > 0$) so, the pair of points $\{\frac{\pi}{2}, -\frac{\pi}{2}\}$ plays the role of the pair $\{\infty,-\infty\}$ in the continuous case. Therefore, compact subsets of $R = \tilde{R} \setminus \{-\infty,\infty\}$ will be replaced by compact subsets of $T \setminus \{-\frac{\pi}{2}, \frac{\pi}{2}\}$. Indeed, if D is a distribution on T whose support is a compact subset of $T \setminus \{-\frac{\pi}{2}, \frac{\pi}{2}\}$ then $\hat{D}(m) = D(e^{imt}) = D(\mathscr{e}(it;m+i0))$ can be discrete analytically continued to the whole lattice by

$$\hat{D}(m+in) = D(\mathscr{e}(it,m+in)) . \qquad (5.1)$$

This follows from the fact that D is linear and $e(it;m+in)$ is discrete analytic for each t in the support of D .

Let us now turn to the statement and proof of the disctete analogue of a).

Theorem 4. Let D be distribution on T whose support is a compact subset of $T \setminus \{-\frac{\pi}{2}, \frac{\pi}{2}\}$ and let it be contained in $\{|t| \leq \alpha\} \cup \{|t-\pi| \leq \alpha\}$, $(0 < \alpha < \frac{\pi}{2})$; then $\hat{D}(m+in)$ given by (5.1), satisfies an inequality of the form

$$|\hat{D}(m+in)| \leq K(1+|n|+|m|)^k C_\alpha^{|n|} \tag{5.2}$$

where k is the order of D, K is a constant depending only upon D and

$$C_\alpha = \tilde{e}(-i\alpha;0+i) = \frac{1+ie^{-i\alpha}}{i+e^{-i\alpha}} .$$

Proof. The proof is similar to the proof of a) as given in Donoghue [5], p.211, only that instead of the nice formula

$$\frac{d}{dt} e^{it(m+in)} = (im-n)e^{it(m+in)} = im\, e^{it(m+in)} - n\, e^{it(m+in)}$$

you have a somewhat more involved equality

$$\frac{d}{dt} \tilde{e}(it;m+in) = im\tilde{e}(it;m+in) - \frac{n}{2}[\tilde{e}(it;m+i(n+1)) + \tilde{e}(it;m+i(n-1))]$$

from which $\frac{d^k}{dt^k} \tilde{e}(it;m+in)$ can be computed inductively. Beside this minor technical complication the proof is the same.

Let F and G be functions on Z^2 and let $\Gamma: a = z_0, z_1, \ldots, z_m = b$ denote a discrete contour $(|z_{i+1} - z_i| = 1$, $0 \leq i \leq m-1)$. Duffin [6] defined the contour integral

$$\int_\Gamma F : G\, \partial z = \sum_{n=1}^{m} (F(Z_n)+F(z_{n-1}))(G(z_n)+G(z_{n-1}))\left(\frac{z_n - z_{n-1}}{4}\right) \tag{5.3}$$

and showed that if F and G are discrete analytic in a region containing Γ, and Γ is a closed contour then

$$\int_{\Gamma} F : G \, \partial z = 0 \quad . \tag{5.4}$$

Let us turn to the proof of the disctete analogue of c).

<u>Theorem 5.</u> Let $F(m+in)$ be a discrete entire function with the property that for every integer $k \geq 0$ there is a constant K_k such that

$$|F(m+in)| \leq K_k(1+|n|+|m|)^{-k} \, C_\alpha^{|n|} \tag{5.5}$$

where $C_\alpha = \tilde{e}(-i\alpha;0+i) = \dfrac{1+ie^{-i\alpha}}{i+e^{-i\alpha}}$,

then there exists a C^∞ function ϕ supported in $A_\alpha = \{|t| < \alpha\} \cup \{|\pi-t| < \alpha\}$ such that

$$F(m+in) = \hat{\phi}(m+in) = \int_{A_\alpha} \phi(t) \, e^{imt} \left(\frac{1+ie^{it}}{i+e^{it}}\right)^n dt \quad . \tag{5.6}$$

<u>Proof.</u> For each n, $F_n(m) = F(m+in)$ decreases faster than any power of $\frac{1}{|m|}$ and thus

$$F_n^{v}(t) = \sum_{m=-\infty}^{\infty} F_n(m) \, e^{-imt}$$

is a C^∞ function on T for every $n \in Z$, and it is easily checked just as in the proof of Theorem 3 that

$$F_n^{v}(t) = \left(\frac{1+ie^{it}}{i+e^{it}}\right)^n F_o^{v}(t) \qquad \forall\, n \in Z$$

and thus

$$F(m+in) = \int_{-\pi}^{\pi} F_o^{v}(t) \, e^{imt} \left(\frac{1+ie^{it}}{i+e^{it}}\right)^n dt \quad .$$

It remains to show that $F_0^{\vee}(t)$ vanishes outside $A_\alpha = \{|t| \leq \alpha\} \cup \{|\pi-t| \leq \alpha\}$ i.e. inside $\{|t-\frac{\pi}{2}| < \frac{\pi}{2} - \alpha\} \cup \{|t + \frac{\pi}{2}| < \frac{\pi}{2} - \alpha\}$. Take $\beta \in \{|t-\frac{\pi}{2}| < \frac{\pi}{2} - \alpha\}$ and consider the discrete contour integral

$$\int_{C_R} \tilde{e}(i\beta;m+in):F(m+in) \tag{5.7}$$

where the discrete contour is taken to be the boundary of the rectangle $-R \leq m \leq R$, $0 \leq n \leq R$. Since both $e(i\beta;m+in)$ and $F(m+in)$ are discrete entire and C_R is a closed contour it follows that the contour integral (5.7) vanishes. But by the Definition (5.3)

$$0 = \int_{C_R} \tilde{e}(i\beta;m+in):F(m+in) =$$

$$\frac{1}{4}\sum_{m=-R}^{R} (e^{i\beta m} + e^{i\beta(m+1)})(F(m)+F(m+1)+ \int_{C'_R} \tilde{e}(i\beta;m+in):F(m+in)$$

where C'_R is the part of C_R which lies in the "open" half lattice $n > 0$.

By (5.5)

$$|e(i\beta;m+in)|\cdot|F(m+in)| \leq K_k(1+|n|+|m|)^{-k}\left(\frac{C_\alpha}{C_\beta}\right)^n .$$

Since $|\beta-\frac{\pi}{2}| \leq \frac{\pi}{2} - \alpha$, $C_\beta = e(-i\beta;0+i) > C_\alpha$ and $\int_{C'_R} \tilde{e}(i\beta;m+in):F(m+in)$ tends to zero as $R \to \infty$.

Consequently,

$$\lim_{R\to\infty} \sum_{m=-R}^{R} (e^{i\beta m} + e^{i\beta(m+1)})(F(m)+F(m+1)) = 0 \tag{5.8}$$

But $\phi(-\beta) = \overset{\vee}{F_o}(-\beta) = \sum_{-\infty}^{\infty} F(m)e^{i\beta m}$.

and (5.8) implies that

$(1+e^{i\beta})^2 \sum_{m=-\infty}^{\infty} F(m)e^{i\beta m} = 0$. Since $\beta \neq -\frac{\pi}{2}$ it follows that $\phi(-\beta) = 0$ for every β in $|t - \frac{\pi}{2}| \leq \frac{\pi}{2} - \alpha$ i.e. ϕ vanishes in $|t + \frac{\pi}{2}| \leq \frac{\pi}{2} - \alpha$. If C_R is chosen in the lower half lattice you get that ϕ vanishes in $|t - \frac{\pi}{2}| \leq \frac{\pi}{2} - \alpha$ and thus $\overset{\vee}{F_o}(t) = \phi(t)$ is supported in $A_\alpha = \{|t| \leq \alpha\} \cup \{|t-\pi| \leq \alpha\}$ and (5.6) follows.

CHAPTER IV

A TAYLOR EXPANSION FOR DISCRETE ANALYTIC FUNCTIONS

1. Introduction

Duffin [6] introduced the following basis for discrete analytic polynomials

$$\rho_k(z) = \frac{d^k}{dt^k} \left\{ \left(\frac{2+t}{2-t}\right)^x \left(\frac{2+it}{2-it}\right)^y \right\} \Bigg|_{t=0}$$

$(z = x+iy)$, which he called pseudo-powers.

Each $\rho_k(z)$ is a discrete entire functions and a polynomial of degree k in (x,y) . Duffin [6] showed that every discrete analytic polynomial can be expressed as a linear combination of these pseudo-powers.

Duffin and Peterson [9] introduced an analogue of the McClaurim series in terms of these pseudo-powers. However, their analogue has the disadvantage that the convergence of $\sum_0^\infty a_n \xi^n$ on $\mathbb{C}$ does not ensure the convergence of $\sum_0^\infty a_n \rho_n(z)$ on $Z \times Z$. In fact they showed that $\sum_0^\infty a_n \rho_n(z)$ converges on the whole lattice $Z \times Z$ only if

$$\overline{\lim}\, (|a_n| n!)^{1/n} < 2 .$$

In Section 2 other "reasonable" bases for discrete analytic polynomials will be considered. These will be called <u>systems of pseudo-powers,</u> and it will be shown that the above drawback of Duffin's basis $\{\rho_n(z)\}$ as regards the convergence of $\sum a_n \rho_n(z)$ cannot be removed by using other systems of pseudo-powers.

On the other hand, we shall construct a system of pseudo-powers $\{\pi_k(z)\}_0^\infty$ such that $\sum_0^\infty a_k \pi_k(z)$ converges absolutely on the quarter lattice $Z^+ \times Z^+ = \{(x+iy)\ ;\ x$ and y integers, $x \geq 0$, $y \geq 0\}$ whenever $\sum_0^\infty a_k \xi^k$ converges on the entire plane. (The divergence of $\sum_0^\infty \frac{2^n}{n!} \rho_n(1,0)$ shows that this property is not enjoyed by the Duffin-Peterson series.)

In Section 3 we shall consider the existence and uniqueness of the expansion $\sum_0^\infty a_k \pi_k(z)$. The discrete analogue of 'multiplication by z' corresponding to the above basis will also be dealt with.

In Section 4, we discuss the lattice

$Z_h^+ \times Z_h^+$ where

$Z_h^+ = hZ^+$, $h > 0$ and show that if $\{\pi_k^h(z)\}_0^\infty$ is the corresponding basis then

$$\sum_0^\infty a_k \pi_k^h(z) \to \sum_0^\infty a_k z^k$$

when $h \downarrow 0$ along a sequence for which $z \in Z_h \times Z_h$, provided $\sum_0^\infty a_k \xi^k$ is an entire function of exponential type.

The analogous problem of representing monodiffric functions (that in functions satisfying $(i-1)\, f(x,y) - if(x+1,y) + f(x,y+1) \equiv 0$) by a series of polynomials was considered by Atadžanov [1].

Definition: A basis $\{p_n(z)\}_0^\infty$ for the discrete analytic polynomials is called a system of pseudo-powers if the following properties are satisfied:

(A1) $p_n(0) = 0$ for every $n > 0$

(A2) $\{p_n(z)\}$ satisfies the Binomial identity

$$p_n(z_1+z_2) = \sum_{k=0}^{n} \binom{n}{k} p_k(z_1)p_{n-k}(z_2)$$

(A3) $p_0 \equiv 1$ and for $n \geq 0$ $p_n(z) = z^n + \tilde{p}_{n-1}(x,y)$

where $\tilde{p}_{n-1}$ is a polynomial of degree $\leq n-1$.

It is readily checked that Duffin's basis $\{\rho_n(z)\}$ constitutes a system of pseudo-powers. On the other hand, Duffin's basis fails to satisfy the following:

(*) $\sum_0^\infty a_n p_n(z)$ converges absolutely for every $z \in Z \times Z$ if $\sum_0^\infty a_n \xi^n$ converges in the whole ξ-plane.

One may ask: Does there exist a system of pseudo-powers satisfying (*)? That no such system exists follows from the next lemma.

Lemma 2.1: Let $\{p_k\}$ be any system of pseudo-powers. Then there exists a point z_0 in the half lattice $\{x+iy , y \geq 0\}$ and a complex number ζ_0 such that

$\sum_0^\infty \frac{\zeta_0^k}{k!} p_k(z_0)$ fails to converge absolutely.

<u>Proof:</u> Suppose that the statement is false, i.e., there exists a system of pseudo-powers $\{p_k\}$ such that

$$e(\zeta;z) = \sum_{k=0}^{\infty} \frac{\zeta^k}{k!} p_k(z) \quad \text{converges}$$

absolutely for every point in the half lattice and for every complex number ζ. Then, for every such z, $e(\zeta,z)$ is an entire function in ζ and by (A2)

$$e(\zeta;z_1)e(\zeta;z_2) = (\sum_0^{\infty} \frac{\zeta^k}{k!} p_k(z_1))(\sum_0^{\infty} \frac{\zeta^n}{r!} p_r(z_2))$$

$$= \sum_0^{\infty} \frac{\zeta^n}{n!} [\sum_0^{n} \binom{n}{k} p_k(z_1)p_{n-k}(z_2)] = \sum_0^{\infty} \frac{\zeta^n}{n!} p_n(z_1+z_2) = e(\zeta;z_1+z_2) .$$

Thus

$$e(\zeta;x+iy) = f(\zeta)^x \, g(\zeta)^y$$

where $f(\zeta) = e(\zeta;1)$, $g(\zeta) = e(\zeta;i)$.

Since $e(\zeta;z)$ is discrete analytic in the upper half lattice I(2) must be satisfied there:

$$f(\zeta)^x g(\zeta)^y \{1+if(\zeta) - f(\zeta)g(\zeta)-ig(\zeta)\} = 0 .$$

Thus

$$g(\zeta) = \frac{1+if(\zeta)}{f(\zeta)+i}$$

and

$$e(\zeta;x+iy) = f(\zeta)^x\left(\frac{1+if(\zeta)}{f(\zeta)+i}\right)^y .$$

Since $e(\zeta;z)$ is entire in ζ for each fixed z in the half lattice and in particular for $z = 1,-1,i$ we see that $f(\zeta)$, $1/f(\zeta)$ and $\frac{1+if(\zeta)}{f(\zeta)+i}$ are entire. But this implies that $f(\zeta)$ is entire and excludes the values 0 and $-i$. By the "little" Picard theorem (Rudin [19], p.324) this is too much to ask from a non-constant entire function. Evidently $f(\zeta)$ cannot be constant and so we arrive at a contradiction and the lemma is proved.

We saw that there is no system of pseudo-powers satisfying (*). The next theorem will demonstrate a system of pseudo-powers satisfying the following weaker property.

(A4) $\sum_0^\infty a_n p_n(z)$ converges absolutely for every $z \in Z^+ \times Z^+ = \{x+iy \; ; \; x \geq 0 \; , \; y \geq 0\}$ if $\sum_0^\infty a_n \xi^n$ converges in the whole ξ-plane.

The divergence of $\sum \frac{2^n}{n!} \rho_n(1)$ shows that Duffin's basis does not satisfy (A4).

<u>Theorem 2.2:</u> The sequence of functions

$$\pi_k(x,y) = \frac{d^k}{d\zeta^k}\{[(1+i)e^{\frac{\zeta}{1+1}} - i]^x \, [(1-i)e^{\frac{-\zeta}{1+i}} + i]^y\}\Big|_{\zeta=0} \qquad (2.1)$$

$k=0,1,2,\ldots$. Consitutes a system of pseudo-powers satisfying (A4).

Proof: The discrete analyticity of $\pi_k(x,y)$ is readily checked. (A1) is trivial, while (A2) follows from Leibnitz' formula. Also, by a straightforward computation

$$\pi_{k+1}(x,y) = \frac{1}{1+i}\{(x-y)\pi_k(x,y) + ix\pi_k(x-1,y)+iy\pi_k(x,y-1)\} \qquad (2.2)$$

Since $\pi_0(x,y) \equiv 1$ it follows by induction that each $\pi_k(x,y)$ is a polynomial of degree k and that (A3) holds. Since Duffin [6] showed that the dimension of the space of discrete analytic polynomials of degree $\leq k$ is $k+1$, it follows that $\{\pi_r\}_0^k$ is a basis for the discrete analytic polynomials of degree $\leq k$ and consequently that $\{\pi_k\}_0^\infty$ is a basis for the discrete analytic polynomials. Thus $\{\pi_k\}$ is a system of pseudo-powers.

Now, let us note that for a fixed $z = x+iy \in Z^+ \times Z^+$

$$e(\zeta;x+iy) = \sum_0^\infty \pi_k(x,y)\frac{\zeta^k}{k!} = [(1+i)e^{\frac{\zeta}{1+1}}-i]^x\,[(1-i)e^{\frac{-\zeta}{1+i}}+i]^y \quad .$$

Since x and y are non-negative integers, the right hand side is an entire function of exponential type and the Taylor coefficients being $\frac{\pi_k(x,y)}{k!}$ you have (Boas [2], p.11) that there exist constants C and T (depending on (x,y)) such that

$$|\pi_k(x,y)| \leq CT^k \; .$$

Thus $\sum_{0}^{\infty} a_k \pi_k(x,y)$ converges absolutely whenever $\overline{\lim} |a_k|^{1/k} = 0$, since $\sum_{0}^{\infty} a_k T^k$ does. This holds for every $(x,y) \in Z^+ \times Z^+$ and it follows that $\{\pi_k\}$ is a system of pseudo-powers satysfing (A4).

By Theorem (2.2) it follows that whenever $\sum_{0}^{\infty} a_k \xi^k$ is an entire function, i.e., whenver $\overline{\lim} |a_k|^{1/k} = 0$, then $\sum_{0}^{\infty} a_k \pi_k(z)$ converges to a discrete analytic function in $Z^+ \times Z^+$ (substitute in (1.1) and rearrange terms, using the fact that each $\pi_k(z)$ is discrete analytic).

Let $\mathcal{A}$ be the algebra of entire functions and let $\mathcal{D}$ be the set of discrete analytic functions on $Z^+ \times Z^+$. Define a mapping

$$T : \mathcal{A} \to \mathcal{D}$$

by

$$T\left(\sum_{0}^{\infty} a_n \xi^n\right) = \sum_{0}^{\infty} a_n \pi_n(z) .$$

Let $\mathcal{F} \subset \mathcal{D}$ be the range of T. $\mathcal{F}$ can be made into an algebra by requiring T to be a homomorphism:

$$\left(\sum_{0}^{\infty} a_k \pi_k(z)\right) \left(\sum_{0}^{\infty} b_k \pi_k(z)\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) \pi_n(z) .$$

Thus in our class $\mathcal{F}$, multiplication is defined for every pair $f,g \in \mathcal{F}$. This is an improvement on the multiplication in the Duffin-Peterson class,

$$\mathcal{J}_{DP} = \{\sum_0^\infty a_n\rho_n(z) \; ; \; \overline{\lim} \, (|a_n|n!)^{1/n} < 2\}$$

which is only defined on a subset of $\mathcal{J}_{DP} \times \mathcal{J}_{DP}$. In particular exp f is well defined in our class:

$$\exp\,(\sum_0^\infty a_k\pi_k(z)) = T(\exp\,(\sum_0^\infty a_k\xi^k)) \; .$$

3. Existence and uniqueness of Taylor expansion.

Formula (2.2) motivates the following analogue for the continuous "multiplication by z"

$$\mathfrak{z}f(x,y) = \frac{1}{1+i}\{(x-y)f(x,y) + ixf(x-1,y) + iyf(x,y-1)\} . \qquad (3.1)$$

It is readily checked that if f is discrete analytic, then so is $\mathfrak{z}f$ and, by (2.2)

$$\mathfrak{z}\pi_k = \pi_{k+1}$$

$$\mathfrak{z}e(\xi;x+iy) = \frac{d}{d\xi}e(\xi;x+iy)$$

Let us restrict attention to $\mathcal{D}$, the class of discrete analytic functions on $Z^+ \times Z^+$. It was shown in Chapter II that each $f \in \mathcal{D}$ is uniquely determined by the pair of formal power series (ϕ_f,ψ_f) where

$$\phi_f(X) = \sum_{x=0}^{\infty} f(x,0)X^x$$

$$\psi_f(Y) = \sum_{y=0}^{\infty} f(0,y)Y^y$$

and we write $f = (\phi_f,\psi_f)$.

Since $\mathfrak{z}f(x,0) = \frac{1}{1+i}\{xf(x,0) + ixf(x-1,0)\}$

$$\sum_{x=0}^{\infty} \mathfrak{Z}f(x,0)X^x = \frac{1}{1+i} \sum_{x=0}^{\infty} x(f(x,0) + if(x-1,0))X^x$$

$$= \frac{X}{1+i} \frac{d}{dX} [(1+iX)\phi_f(X)] \quad .$$

Similarly

$$\sum_{y=0}^{\infty} \mathfrak{Z}f(0,y)Y^y = \frac{Y}{1+i} \frac{d}{dY} [(iY-1)\psi_f(Y)] \ .$$

So the operation of $\mathfrak{Z}$ in terms of formal power series is

$$(\phi_f,\psi_f) \to \frac{1}{1+i} \left(X \frac{d}{dX} [(1+iX)\phi_f(X)] \ , \ Y\frac{d}{dY} [(iY-1)\psi_f(Y)]\right) \ . \qquad (3.2)$$

Thus $\mathfrak{Z}f \equiv 0$ iff

$$\phi_f(X) = \frac{c}{1+iX} \quad ; \quad \psi_f(Y) = \frac{c}{1-iY} \quad .$$

(The constants agree since $\phi_f(0) = f(0,0) = \psi_f(0)$) . So, unfortunately, $\mathfrak{Z}$ has a non-trivial kernel.

Clearly, $\mathfrak{Z}f(0) = 0$ for every function f discrete analytic in $Z^+ \times Z^+$. Let $g \in \mathcal{D}$, $g(0) = 0$ then $f \in \mathcal{D}$ given by

$$\phi_f(X) = \frac{1+i}{1+iX} \int \frac{\phi_g(X)}{X} dX = \frac{1+i}{1+iX} \left[\sum_{1}^{\infty} \frac{g(x,0)X^x}{x} + c\right]$$

$$\psi_f(Y) = \frac{1+i}{1-iY} \int \frac{\phi_g(Y)}{Y} dY = \frac{1+i}{1-iY} \left[\sum_{1}^{\infty} \frac{g(0,y)Y^y}{y} + c\right]$$

solves $\mathfrak{Z}f = g$.

We have thus obtained

<u>Theorem 3.1:</u> The operator

$$\mathfrak{z} : \mathcal{D} \to \mathcal{D}$$

has range $\{f \in \mathcal{D} \; ; \; f(0,0) = 0\}$ and kernel

$$\{Cf_0\}$$

where $f_0 \in \mathcal{D}$ is given by

$$\phi_{f_0} = \frac{1}{1+iX} \; ; \; \psi_{f_0} = \frac{1}{1-iY} \; .$$

Let us consider the class $\mathcal{J} \subset \mathcal{D}$ defined at the end of Section 2. It is not yet known whether the inclusion $\mathcal{J} \subset \mathcal{D}$ is proper or not; i.e., whether every discrete analytic function on $Z^+ \times Z^+$ possesses a discrete Taylor expansion

$$f(z) = \sum_{k=0}^{\infty} a_k \pi_k(z) \tag{3.3}$$

Theorem (3.1) implies that even if such a representation exists it need not be unique. However if attention is restricted to the class

$$\mathcal{J}_e = \{\sum_0^{\infty} a_k \pi_k(z) \; ; \; \overline{\lim} \, (k! |a_k|)^{1/k} < \infty\}$$

then the representation (3.3) is unique, as follows from the following

Theorem 3.2: If $\sum_{0}^{\infty} a_k \pi_k(z) \equiv 0$ in $Z^+ \times Z^+$ and $\overline{\lim}\, (k!|a_k|)^{1/k} < \infty$ then $a_k = 0$ for every k.

Proof: By definition (2.1)

$$\pi_k(x,y) = \frac{k!}{2\pi i} \int_{\Gamma} \frac{[(1+i)e^{\frac{\zeta}{1+i}}-i]^x[(1-i)e^{\frac{-\zeta}{1+i}}+i]^y}{\zeta^{k+1}}\, d\zeta$$

where Γ is any contour surrounding 0. So,

$$f(z) = \sum_{0}^{\infty} a_k \pi_k(x,y) = \frac{1}{2\pi i} \int_{\Gamma} \left(\sum \frac{k!a_k}{\zeta^{k+1}}\right)[(1+i)e^{\frac{\zeta}{1+i}}-i]^x[(1-i)e^{\frac{-\zeta}{1+i}}+i]^y\, d\zeta$$

for any contour Γ for which

$$f_B(\zeta) = \sum_{k=0}^{\infty} \frac{k!a_k}{\zeta^{k+1}}$$

is defined. $f_B(\zeta)$ is the Borel transform of

$$f^c(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^k$$

and $f_B(\zeta)$ converges for $|\zeta| \geq$ type f^c (see Boas [2], p.73). Thus

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} f_B(\zeta)[(1+i)e^{\frac{\zeta}{1+i}}-i]^x[(1-i)e^{\frac{-\zeta}{1+i}}+i]^y d\zeta$$

and for some constant M

$|f(x,0)| \leq CM^x$

and $\phi_f(t) = \sum_0^\infty f(x,0)t^x$ converges in the disc $|t| < \frac{1}{M}$.
We have then

$$\phi_f(t) = \sum_{x=0}^\infty f(x,0)t^x = \frac{1}{2\pi i} \int_\Gamma f_B(\zeta) \sum_0^\infty [(1+i)e^{\frac{\zeta}{1+i}} - i]^x t^x$$

$$= \frac{1}{2\pi i} \int_\Gamma \frac{f_B(\zeta)d\zeta}{1-[(1+i)e^{\frac{\zeta}{1+i}} - i]t} .$$

The right hand side defines an analytic function in any region in the t-plane for which the denominator of the integrand does not vanish in a neighborhood of Γ in the ζ-plane. In particular, this includes a neighborhood of the point i in the t-plane. Thus for any discrete analytic function of class $\mathcal{J}_e$

$$\phi_f(t) = \sum_{x=0}^\infty f(x,0)t^x$$

whose radius of convergence is in general smaller than 1 , can be analytically continued through the boundary of the circle of convergence to a neighborhood of $t = i$.

Now $\sum_0^\infty a_k \pi_k(z) \equiv 0$ implies $a_0 = 0$ and

$$\partial(\sum_1^\infty a_k \pi_{k-1}(z)) = 0 .$$

Let $g_1(z) = \sum_1^\infty a_k \pi_{k-1}(z)$. Then $g_1 \in \mathcal{J}_e$ and hence $\phi_{g_1}(t)$ can be analytically continued to a neighborhood of $t = i$. But $\partial g_1 \equiv 0$ implies, by Theorem 3.1, that $\phi_{g_1}(t) = \frac{C}{1+it}$ for some constant C . This forces C = 0 for, otherwise ϕ_{g_1} would have a pole at $t = i$. Thus,

$$g_1(z) = \sum_1^\infty a_k \pi_{k-1}(z) \equiv 0 \quad \text{and} \quad a_1 = 0 .$$

Continuing inductively we get that $a_k = 0$ for every k and the theorem is proved.

4. Limiting behavior as $h \downarrow 0$.

Let $h > 0$. For the lattice of mesh size h

$$Z_h \times Z_h = \{(hm, hn); m, n \in Z\}$$

discrete analyticity is defined by

$$F(x,y) + iF(x+h,y) - F(x+h,y+h) - iF(x,y+h) = 0 \quad . \tag{4.1}$$

The above discussion carries over to discrete analytic functions for such lattices (all it amounts to is a change of scale). Now we have the basis

$$\pi_k^h(x,y) = \frac{d^k}{d\zeta^k}\left\{\left[(1+i)e^{\frac{\zeta h}{1+i}} - i\right]^{\frac{x}{h}}\left[(1-i)e^{\frac{-\zeta h}{1+i}} + i\right]^{\frac{y}{h}}\right\}\Bigg|_{\zeta=0} \tag{4.2}$$

And for discrete analytic functions on the lattice $Z_h \times Z_h$ the exponential function is

$$e_h(x,y) = \sum_{k=0}^{\infty} \pi_k^h(x,y)\frac{\zeta^k}{k!} = \left[(1+i)e^{\frac{\zeta h}{1+i}}-i\right]^{\frac{x}{h}}\left[(1-i)e^{\frac{-\zeta h}{1+i}} + i\right]^{\frac{y}{h}} .$$

Now as $h \downarrow 0$

$$\left[(1+i)e^{\frac{\zeta h}{1+i}} - i\right]^{\frac{1}{h}} \to e^{\zeta}$$

$$\left[(1-i)e^{\frac{-\zeta h}{1+i}} + i\right]^{\frac{1}{h}} \to e^{i\zeta} .$$

So $e_h(x,y) \to e^{\zeta(x+iy)}$ and consequently

$$\pi_k^h(z) \to z^k \quad \text{as} \quad h \downarrow 0 .$$

Suppose $|a_n| \leq C \frac{\zeta_0^n}{n!}$ for some constants C and ζ_0, by dominated convergence

$$f^h(z) = \sum_0^\infty a_k \pi_k^h(z) \to \sum_0^\infty a_k z^k$$

as $h \downarrow 0$. We obtained

<u>Lemma 4.1:</u> If $\overline{\lim}\, (|a_k| k!)^{1/k} < \infty$ then

$$f^h(z) \to f^c(z) = \sum_0^\infty a_k z^k$$

along a sequence $h \downarrow 0$ for which $z \in Z_h^+ \times Z_h^+$.

CHAPTER V

DISCRETE ANALYTIC FUNCTIONS OF EXPONENTIAL GROWTH

1. Introduction.

In this chapter we shall prove theorems on discrete analytic functions of exponential growth which are analogous to certain classical theorems about entire functions of exponential type (Boas [2] is the standard reference for the latter). Perhaps the main result of this chapter is a proof of the discrete analogue of the (two-sided) Paley-Wiener theorem (Theorem 3.4). Our methods, which are completely different from the ones used in the classical theory, use duality arguments on certain Banach spaces of analytic functions of two complex variables. In essence the trick is to translate into discrete language a 'continuous' idea due to Ehrenpreis [12] (see the preface of the latter). Ehrenpreis deals with the solutions of partial differential equations, whereas present interest focuses on solutions of the simple partial difference equation $Lf = 0$, based upon the Duffin operator L introduced in I(2). Since the discrete case is, by its nature, simpler than the continuous one, no explicit reference need be made to Ehrenpreis [12].

The key idea of this chapter is to associate with each discrete function $f: Z \times Z \to \mathbb{C}$ a linear functional T_f defined on the algebra

$$\mathcal{A} = \{ \sum_{m=-M}^{M} \sum_{n=-N}^{N} a_{mn} z^m w^n \; ; \; a_{mn} \in \mathbb{C} \; , M,N \text{ integers} \}$$

of polynomials in z, z^{-1}, w, w^{-1} which is given by

$$T_f\left(\sum_{-M}^{M} \sum_{-N}^{N} a_{mn} z^m w^n \right) = \sum_{-M}^{M} \sum_{-N}^{N} a_{mn} \, f(m,n)$$

and using the fact that (1.1) holds iff

$$T_f((1+iz-zw-iw)z^m w^m) = 0 \;\; \forall \; (m,n) \in Z \times Z$$

we get that $f(m,n)$ is discrete entire iff T_f annihilates the ideal $(1+iz-zw-iw)\mathcal{A}$.

We shall first consider, in Section 2, discrete analytic functions of exponential growth defined only on the upper right quarter lattice $Z^+ \times Z^+ = \{(m,n) \; ; \; m \geq 0 \; , \; n \geq 0\}$.

2. Discrete analytic functions of exponential growth on the upper right quarter lattice.

Let $\mathcal{A}_+$ be the algebra of polynomials

$$\{ \sum_{m=0}^{M} \sum_{n=0}^{N} a_{m,n} z^m w^n \; ; \; a_{mn} \in \mathbb{C} \; , M,N \text{ integers} \} ,$$

Any discrete function $f:Z^+ \times Z^+ \to \mathbb{C}$ induces a linear functional T_f on $\mathcal{A}_+$ given by

$$T_f\left(\sum_{m=0}^{M}\sum_{n=0}^{N} a_{mn} z^m w^n\right) = \sum_{m=0}^{M}\sum_{n=0}^{N} a_{mn}\, f(m,n) \qquad (2.1)$$

and for any linear functional T on $\mathcal{A}_+$, $T = T_g$ where $g(m,n) = T(z^m w^n)$.

Let r,s be any positive numbers and consider the polydisc $\{|z| < r\} \times \{|w| < s\}$ in $\mathbb{C}^2$. Let $H(r,s)$ be the class of functions holomorphic on this polydisc and continuous on its closure. This is a Banach space with norm

$$||u||_\infty = \sup_{\substack{|z|<r \\ |w|<s}} |u(z,w)|$$

(see Rudin [20], p.3).

Evidently $\mathcal{A}_+ \subset H(r,s)$ and in fact $\mathcal{A}_+$ is dense in $H(r,s)$. We now make the following

<u>Definition:</u> A discrete function $f:Z^+ \times Z^+ \to \mathbb{C}$ is said to be of exponential growth (R,S) if there exists a constant C such that

$$|f(m,n)| \leq CR^m S^n \quad \text{for every} \quad (m,n) \in Z^+ \times Z^+ .$$

Later we shall need the following

<u>Lemma 2.1:</u> Let $f: Z^+ \times Z^+ \to \mathbb{C}$ be of exponential growth (R,S) and let $r > R$, $s > S$; then T_f, defined on $\mathcal{A}_+$ by (2.1) can be extended continuously to the Banach space $H(r,s)$.

<u>Proof:</u>

$$F(z,w) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f(m,n) z^{-(m+1)} w^{-(n+1)}$$

is defined and holomorphic in $\{|z| > R\} \times \{|w| > S\}$. Let $u(z,w) \in \mathcal{A}_+$, then

$$T_f(u) = \frac{1}{(2\pi i)^2} \int_\Gamma F(z,w) u(z,w) dz dw$$

for some poly-contour Γ in the poly-annulus $\{R < |z| < r\} \times \{S < |w| < s\}$. Thus

$$|T_f(u)| \leq C(F) ||u||_\infty$$

for some constant $C(F)$ depending only on F (and hence on f). Since $\mathcal{A}_+$ is dense in $H(r,s)$, the lemma is proved.

The "typical" discrete function of exponential growth (R,S) is $f(m,n) = z_0^m w_0^n$ for some complex constants z_0, w_0 for which $|z_0| = R$, $|w_0| = S$ and the induced linear functional T_f is the "point evaluator" at (z_0,w_0), $J_{(z_0,w_0)}$, $T_f(u) = u(z_0,w_0) = J_{(z_0,w_0)}(u)$. If we require that $z_0^m w_0^n$ be discrete analytic then $w_0 = \frac{1+iz_0}{z_0+i}$ and so the "typical" discrete analytic function of exponential growth is

$$e(z;m+in) = z^m\left(\frac{1+iz}{z+i}\right)^n$$

which is of exponential growth $\left(|z|, \left|\frac{z-i}{z+i}\right|\right)$.

The next theorem tells us that every discrete analytic function of exponential growth is in some sense a "linear combination" of (discrete) exponentials $e(z;m+in)$.

Theorem 2.2: Let f be discrete analytic in the quarter lattice $Z^+ \times Z^+$ and let it be of exponential growth (R,S) there. Then there exists a plane measure $d\mu(z)$ supported in the region

$$A_{R,S}=\{z \in \mathbb{C} \; ; \; |z| < R \; , \; \left|\frac{z-i}{z+i}\right| < S\}$$

for which

$$f(m,n) = \int e(z;m+in)d\mu(z) \tag{2.2}$$

Proof: We proceed by steps.

Step 1: T_f annihilates the principal ideal $(1+iz-zw-iw)\mathcal{A}_+$.

Proof: Since $f(m,n)$ is discrete analytic in $Z^+ \times Z^+$

$$T_f((1+iz-zw-iw)z^m w^n) = f(m,n) + if(m+1,n) - f(m+1,n+1) - if(m,n+1)$$
$$\equiv Lf(m,n) = 0$$

for every $(m,n) \in Z^+ \times Z^+$.

Step 2: T_f extended to $H(r,s)$ as in Lemma 2.1, annihilates the ideal $(1+iz-zw-iw)\,H(r,s)$.

Proof: This follows immediately from Step 1 and the fact that $\mathcal{A}_+$ is dense in $H(r,s)$.

Step 3: Let

$$V_{r,s} = \{(z,w) \ ; \ |z| < r \ , \ |w| < s \ , \ 1+iz-zw-iw = 0\}$$

then

$$(1+iz-zw-iw)\ H(r,s) = \{u \in H(r,s) \ ; \ u|_{V_{r,s}} \equiv 0\}.$$

Proof: This is the famous Hilbert semi local nullstellenstaz for a very special case. Suppose $u|_{V_{r,s}} \equiv 0$ then $v(z,w) = \frac{u(z,w)}{1+iz-zw-iw}$ is holomorphic in $\{|z| < r\} \times \{|w| < s\} \sim V_{r,s}$ and locally bounded in $\{|z| < r\} \times \{|w| < s\}$, (Gunning and Rossi [14],p.19). By the Riemann Removable Singularity Theorem $v(z,w)$ can be extended to be holomorphic in $\{|z| < r\} \times \{|w| < s\}$ and is evidently continuous on its closure, i.e., $v(z,w) \in H(r,s)$. Thus $u(z,w) = (1+iz-zw-iw)\ v(z,w) \in (1+iz-zw-iw)H(r,s)$. The opposite inclusion is trivial.

Step 4: There exists a measure $d\tilde{\mu}(z,w)$ on $\mathbb{C}^2$ supported in $V_{r,s}$ such that

$$T_f(u) = \int_{V_{r,s}} u(z,w)\,d\tilde{\mu}(z,w) \ , \quad u \in H(r,s) \ .$$

Proof: Let $(z,w) \in \{|z| < r\} \times \{|w| < s\}$ and denote by $J_{(z,w)}$ the point evaluator at (z,w) :

$$J_{(z,w)}(u) = u(z,w)$$

By Step 3 and Step 2

$$J_{(z,w)}(u) = 0 \quad \forall (z,w) \in V_{r,s} \Rightarrow T_f(u) = 0 .$$

Thus the annihilator of span $\{J_{(z,w)} ; (z,w) \in V_{r,s}\}$ is contained in the annihilator of T_f. Since $H(r,s)$ is a Banach space, it follows (cf. Taylor [21], p. 225-226) that T_f is contained in the closed linear span of $\{J_{(z,w)}; (z,w) \in V_{r,s}\}$. Consequently there exists a sequence of atomic measures $\{d\mu_n\}$, supported in $V_{r,s}$ such that for every $u \in H(r,s)$

$$\int u(z,w)d\mu_n(z,w) \to T_f(u) .$$

By Helly's selection principle there is a measure $d\tilde{\mu}(z,w)$, supported in $V_{r,s}$ such that

$$\int u(z,w)d\mu_n \to \int u(z,w)d\tilde{\mu} \qquad \forall u \in H(r,s)$$

and we have

$$T_f(u) = \int_{V_{r,s}} u(z,w)d\tilde{\mu}(z,w) .$$

<u>Step 5:</u> is to complete the proof of the theorem. Let $d\mu(z)$ be the "projection" of $d\tilde{\mu}(z,w)$ on $\mathbb{C}$

$$\int_{\mathbb{C}} v(z)d\mu(z) = \int v(z)d\tilde{\mu}(z,w)$$

$d\mu(z)$ is supported in

$$A^{\circ}_{r,s} = \{z \in \mathbb{C} \; ; \; |z| < r \; , \; \left|\frac{z-i}{z+i}\right| < s\}$$

for every $r > R$, $s > S$ and hence in

$$A_{R,S} = \{z \in \mathbb{C} \; ; \; |z| \leq R \; , \; \left|\frac{z-i}{z+i}\right| \leq S\}$$

and finally

$$f(m,n) = T_f(z^m w^n) = \int_{V_{r,s}} z^m w^n d\tilde{\mu}(z,w) = \int_{A_{R,S}} z^m \left(\frac{1+iz}{z+i}\right)^n d\mu(z)$$

$$= \int_{A_{R,S}} e(z;m+in)d\mu(z)$$

Obviously, the knowledge of $\{f(m,0)\}_{m=0}^{\infty}$ and $\{f(0,n)\}_{n=0}^{\infty}$ uniquely determines the discrete analytic function $f(m,n)$ on the whole of $Z^+ \times Z^+$. The next theorem shows that if f satisfies an appropriate growth condition then the knowledge of f just on the m-axis, i.e., the sequence $\{f(m,0)\}_{m=0}^{\infty}$, determines f on all $Z^+ \times Z^+$.

Theorem 2.3: If f is discrete analytic on $Z^+ \times Z^+$ and of exponential growth (R,S) where $R > 1$, $S < \left|\frac{R+1}{R-1}\right|$ then the values $\{f(m,0)\}_{m=0}^{\infty}$ uniquely determine f .

Proof: By drawing a diagram it is easily seen that if $S < \left|\frac{R+1}{R-1}\right|$ then

$$A_{R,S} = \{z \in \mathbb{C} \; ; \; |z| \leq R \; , \; \left|\frac{z-i}{z+i}\right| \leq S\}$$

is simply connected. Let $r > R$, $s > S$ be such that

$$A^{\circ}_{r,s} = \{z \in \mathbb{C} \; ; \; |z| < r \; , \; \left|\frac{z-i}{z+i}\right| < s\}$$

is still simply connected. Then by Runge's theorem (Rudin [19], p. 258) each holomorphic function in $A^{\circ}_{r,s}$ can be approximated, uniformly on compact sets, by polynomials. For every $m \geq 0$ $\int_{A_{R,S}} z^m d\mu(z) = f(m,0)$ is known and hence $d\mu$ is determined on the polynomials. Since $A_{R,S}$ is a compact subset of $A^{\circ}_{r,s}$, $d\mu$ is uniquely determined by its restriction to the polynomials and hence f is uniquely determined by $\{f(m,0)\}_0^{\infty}$, the theorem is proved.

3. Discrete entire functions of exponential growth.

In this section we deal with discrete entire functions, that is functions $f: Z \times Z \to \mathbb{C}$ such that $Lf \equiv 0$ on $Z \times Z$.

Let $\mathcal{A}$ be the algebra generated by z, z^{-1}, w, w^{-1}

$$\mathcal{A} = \{ \sum_{-M}^{M} \sum_{-N}^{N} a_{mn} z^m w^n \; ; \; M,N \text{ integers}, \; a_{mn} \in \mathbb{C} \}$$

then as already mentioned in Section 1, each discrete function $f: Z \times Z \to \mathbb{C}$ induces a linear functional T_f on

$$T_f(\sum a_{mn} z^m w^n) = \sum a_{mn} f(m,n) \tag{3.1}$$

Moreover, if T is a linear functional on $\mathcal{A}$ and $f(m,n) = T(z^m w^n)$ then $T = T_f$.

It follows much as in Section 2 that $f(m,n)$ is discrete entire iff T_f annihilates the ideal $(1+iz-zw-iw)\mathcal{A}$.

We define

<u>Definition:</u> A discrete function $f: Z \times Z \to \mathbb{C}$ is of exponential growth (R,S) if there exists a constant C such that

$$|f(m,n)| \leq C\, R^{|m|}\, S^{|n|} \tag{3.2}$$

for every $(m,n) \in Z \times Z$.

Let $R > 1$, $S > 1$ and $U_{RS} = \{\frac{1}{R} < |z| < R\} \times \{\frac{1}{S} < |w| < S\}$. The class of functions continuous on $\bar{U}_{RS}$ and holomorphic in U_{RS} is a Banach space with sup norm which we shall denote by $\tilde{H}(R,S)$, and instead of Lemma (2.1) we have

<u>Lemma 3.1:</u> If $f(m,n)$ is of exponential growth (R,S) and $r > R$, $s > S$, then T_f defined on $\mathcal{A}$ by (3.1) can be extended to be a continuous linear functional on the Banach space $\tilde{H}(r,s)$.

There is an analogue to Theorem 2.2 also, which can be proved in much the same way.

Theorem 3.2.: Let $f(m,n)$ be discrete entire and of exponential growth (R,S), $(R > 1, S > 1)$. Then there exists a plane measure $d\mu(z)$ supported in

$$\tilde{A}_{R,S} = \{z \in \mathbb{C} \; ; \; \frac{1}{R} \leq |z| \leq R \; , \; \frac{1}{S} \leq |\frac{z-i}{z+i}| \leq S\}$$

such that

$$f(m,n) = \int_{\tilde{A}_{R,S}} e(z;m+in)d\mu(z)$$

for every $(m,n) \in Z \times Z$.

The measure $d\mu(z)$ in the above theorem is a continuous linear functional on the algebra of bounded holomorphic functions on the region $A^{o}_{r,s}$. At this point[†] the following theorem due to Havin [15] is useful.

Theorem: (Havin) Let G be an open set in $\mathbb{C}$ and let $\mathcal{O}(G)$ be the space of analytic functions on G. Put $F = C \sim G$ and assume $\infty \in F$. Then for every continuous linear functional ϕ on $\mathcal{O}(G)$ there exists a unique locally analytic function g_ϕ such that if g_k is analytic on some $G_k \supset F$ such that $g_k|_F = g$ then

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[†] The impatient reader may skip immediately to the Paley-Wiener theorem (Th. 3.4) the proof of which is independent of the present circle of ideas.

$$\phi(f) = \frac{1}{2\pi i} \int_{\Gamma} f(z)g_k(z)dz$$

where Γ is a contour in $G \cap G_k$.

Applying this theorem to our functional $d\mu(z)$ on the space of bounded holomorphic functions on the region $\tilde{A}^{\circ}_{r,s}$ we have the following

Theorem 3.3: Let $f(m,n)$ be discrete entire and of exponential growth (R,S) ; let $r > R$, $s > S$. Put $\tilde{B}_{rs} = \mathbb{C}^2 - \tilde{A}^{\circ}_{rs}$ then there exists a unique locally analytic function g such that if g_k is analytic on some $G_k \supset B_{r,s}$ such that $g_k|_{B_{rs}} = g$ then

$$f(m,n) = \frac{1}{2\pi i} \int_{\Gamma} e(z;m+in)g_k(z)dz \tag{3.3}$$

where $\Gamma \subset A^{\circ}_{r,s} \cap G_k$.

The above theorem can be viewed as the discrete analogue of the representation theorem for entire functions of exponential type (cf. Boas [2], p.74): "If $f(z)$ is an entire function of exponential type, D is its conjugate indicator diagram and C is a contour containing D in its interior, then

$$f(z) = \frac{1}{2\pi i} \int_{C} F(\omega)e^{z\omega}d\omega$$

where $F(\omega)$ is the so called Borel transform of $f(z)$."

Imitating continuous usage we may call the support of $d\mu$ the "conjugate indicator diagram" and the function g of Theorem 3.3 the "Borel transform" Notice that the support of $d\mu$ (the "conjugate indicator diagram") is not, in general, simply connected.

We shall finish this chapter with a discrete analogue to the celebrated two-sided Paley-Wiener theorem (Boas [2], p. 103):

Theorem (Paley-Wiener) : The entire function $f(z)$ is of exponential type τ and belongs to L^2 on the real axis iff

$$f(z) = \int_{-\tau}^{\tau} e^{izt}\phi(t)dt$$

where $\phi(t) \in L^2(-\tau,\tau)$.

In the following T will denote the unit circle $\{|z| = 1\}$.

Theorem 3.4: Let $f(m,n)$ be discrete entire and of exponential growth (R,S) where $S< |\frac{R+1}{R-1}|$ and suppose it belongs to L^2 on the discrete real line

$$\sum_{m=-\infty}^{\infty} |f(m,0)|^2 < \infty$$

then there exists a function $\phi \in L^2(T)$ whose support is a compact subset of $T \sim \{i,-i\} = \{z \in \mathbb{C} ; |z| = 1 , z \neq \pm i\}$ such that

$$f(m,n) = \frac{1}{2\pi} \int_T \phi(z)e(z;m+in)dz =$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} [i\phi(e^{it})e^{it}]e^{imt}\left(\frac{1+ie^{it}}{e^{it}+i}\right)^n dt .$$

Proof: By Theorem 4.2

$$f(m,n) = \int e(z;m+in)d\mu(z)$$

for some measure $d\mu$ supported in

$$\tilde{A}_{R,S} = \{z \in \mathbb{C} \; ; \; \frac{1}{R} \leq |z| \leq R \; , \; \frac{1}{S} \leq \left|\frac{z-i}{z+i}\right| \leq S\}$$

Since $S < \left|\frac{R+1}{R-1}\right|$ the complement of $\tilde{A}_{R,S}$ is connected ($\tilde{A}_{R,S}$ consists of two simply connected components, one containing $z = 1$ and the other $z = -1$).

Let $r > R$, $s > S$ be sufficiently close to R,S (respectively) to make the complement of

$$\tilde{A}^{\circ}_{r,s} = \{z \in \mathbb{C} \; ; \; \frac{1}{r} < |z| < r \; , \; \frac{1}{s} < \left|\frac{z-i}{z+i}\right| < s \}$$

connected. Then by Runge's theorem (Rudin [19], p. 258) every bounded holomorphic function on $\tilde{A}^{\circ}_{r,s}$ can be approximated uniformly on compact sets by polynomials. It follows that the values $\int z^m d\mu(z)$, $m = 0, \pm 1, \pm 2, \ldots$ determine $d\mu$. Also for every polynomial $u(z) = \sum_{-M}^{M} a_m z^m$

$$\int u(z)d\mu(z) = \sum_{-M}^{M} a_m f(m,0) \quad .$$

Since $\sum_{-\infty}^{\infty} |f(m,0)|^2 < \infty$, $d\mu$ can be extended to be a linear functional on $L^2(T \cap \tilde{A}_{R,S})$ and by Riesz' representation theorem there exists a function $\phi(z) \in L^2(T_{RS})$ (where $T_{RS} = T \cap \tilde{A}_{R,S}$), such that for every bounded holomorphic function $u(z)$ on $\tilde{A}^{\circ}{}_{r,s}$ (which automatically then belongs to $L^2(T_{RS})$) :

$$\int_{\tilde{A}_{R,S}} u(z)d\mu(z) = \frac{1}{2\pi} \int_{T_{RS}} \phi(z)u(z)dz$$

In particular

$$f(m,n) = \frac{1}{2\pi} \int_{T_{RS}} e(z;m+in)\phi(z)dz \quad .$$

$T_{RS} = T \cap \tilde{A}_{R,S}$ is a compact subset of $T \sim \{i, -i\}$ and evidently

$$ie^{it}\phi(e^{it}) = \sum_{-\infty}^{\infty} f(m,0)e^{-imt}$$

vanishes, a.e., outside T_{RS} .

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APPENDIX A1.

UNIQUENESS THEOREMS FOR HARMONIC FUNCTIONS OF EXPONENTIAL GROWTH

1. Introduction and statement of results.

The purpose of this appendix is to prove the following two theorems.

Theorem A. Let u be a real valued harmonic function in R^n astisfying $|u(x)| < Ce^{A|x|}$ where $A < \pi$, $|x| = \sum_{i=1}^{n} |x_i|$ and C is a constant. If u vanishes on the integer lattice points of the hyperplanes $x_n = 0$ and $x_n = a$ ($|a| \leq (\frac{1}{n-1})^{1/2}$) then it vanishes identically.

Theorem B. Let u be as above and suppose both u and $\frac{\partial u}{\partial x_n}$ vanish on the integer lattice points of $x_n = 0$, then u vanishes identically.

Theorem A is a generalization of a theorem of Boas [1] who proved if for $n = 2$.: Boas used the fact that in the two-dimensional case every real valued entire harmonic function is the real part of an entire (analytic) function. Evidently, this method does not generalize to higher dimensions. Our strategy will be, instead, to view u as a "distribution" (i.e., a continuous linear functional) on the test space of bounded analytic functions on the polystrip $\mathop{X}_{i=1}^{n} \{|Imt_i| < A''\} \subset \mathbb{C}^n$ for $A'' > A$.

2. Proof of the results.

We shall proceed by a sequence of lemmas.

Lemma 1. If u is harmonic in R^n and $|u(x)| < Ce^{A|x|}$ then any partial derivative of u enjoys the same properties.

Proof: For any $x_0 \in R^n$ look at the Poisson representation formula for the ball $||y-x_0|| < 1$, differentiate under the integral sign and estimate.

Let $\mathcal{A}$ be the class of analytic functions of n complex variables of the form

$$\hat{v}(t) = (2\pi)^{n/2} \int v(x)e^{ixt} \quad ; \quad (xt = \sum x_i t_i)$$

where $v \in C_0^\infty(R^n)$. All these functions are bounded in $K_{A''} = \overset{n}{\underset{i=1}{X}} \{|\mathrm{Im}\, t_i| < A''\}$ for every A''. Define a linear functional on $\mathcal{A}$ by

$$T_u \left((2\pi)^{n/2} \int_{R^n} v(x)e^{ixt} \right) = \int_{R^n} u(x)v(x) \tag{2.1}$$

The next lemma will show that T_u can be extended continuously to $\mathcal{H}_{A''}$, the Banach space of bounded holomorphic functions on $K_{A''}$, provided $A'' > A$.

Lemma 2. Let u be harmonic in R^n and satisfy $|u(x)| < Ce^{A|x|}$. Let $A'' > A$, then T_u defined on $\mathcal{A}$ by (2.1) can be extended to be a continuous linear functional on the Banach space $\mathcal{H}_{A''}$, which consists of bounded analytic functions on $K_{A''}$ where the norm is given by

$$||f||_{A''} = \sup_{t\in K_{A''}} |f(t)|$$

<u>Proof:</u> Let $A < A' < A''$ and let $R_+ = [0,\infty)$, $R_- = (-\infty,0]$. Then

$$U_{\pm\pm\cdots\pm}(t) = (2\pi)^{-n/2} \int_{R_\pm \times R_\pm \times \cdots \times R_\pm} u(x)e^{-ixt}dx$$

belongs to $L^2\left(\overset{n}{\underset{i=1}{X}} \{\mathrm{Im}\, t_i = \mp A'\}\right)$ and for $\hat{v} \in \mathcal{A}$,

$$T_u(\hat{v}) = \sum_{\pm} \int_{R_\pm\times\cdots\times R_\pm} u(x)v(x) = \sum_{\pm} \int_{\overset{n}{\underset{i=1}{X}}\{\mathrm{Im}\, t_i = \mp A'\}} U_{\pm\cdots\pm}(t)\hat{v}(t)dt_1\cdots dt_n \quad (2.2)$$

The sums in (2.2) each contain 2^n terms, corresponding to all possible choices of sign. Let us consider the term in the sum on the right hand side of (2.2) involving $U_{--\cdots-}(t)$ and let us write, for the moment, $\Omega = R_- \times R_- \times \cdots \times R_-$. Then, by Green's formula

$$(2\pi)^{n/2}\, U_{--\cdots-}(t) = \int_\Omega u(x)e^{-ixt}dx = \int_\Omega u(x)\, \Delta\left(\frac{-e^{-ixt}}{t_1^2+\cdots+t_n^2}\right)$$

$$= -\int \Delta u(x) \frac{e^{-ixt}}{t_1^2+\cdots+t_n^2} + \int_{\partial\Omega} u(x)\frac{\partial}{\partial n}\left[\frac{-e^{-ixt}}{t_1^2+\cdots+t_n^2}\right] d\sigma$$

$$- \int_{\partial\Omega} \frac{\partial u}{\partial n} \frac{-e^{-ixt}}{t_1^2+\cdots+t_n^2}\, d\sigma \quad (2.3)$$

The first term on the right hand side of (2.3) vanishes since u is harmonic. Now $\partial\Omega$ consists of n pieces:

$$\partial\Omega = \bigcup_{i=1}^{n} \{x_i = 0\} \cap \Omega$$

Let us consider the contribution from the face $x_1 = 0$. Here $\frac{\partial}{\partial n} = \frac{\partial}{\partial x_1}$ and

$$\int_{\{x_1=0\}\cap\Omega} u(x)\frac{\partial}{\partial n}\left(\frac{-e^{-ixt}}{t_1^2+\dots+t_n^2}\right)dx_2\dots dx_n = \int_{\{x_1=0\}\cap\Omega} u(0,x_2,\dots,x_n)\,\frac{it_1 e^{-ixt}\,d\sigma}{t_1^2+\dots+t_n^2} \tag{2.4}$$

and

$$\int_{\{x_1=0\}\cap\Omega} \frac{\partial u}{\partial n}(x)\frac{e^{-ixt}d\sigma}{t_1^2+\dots+t_n^2} = \int_{\{x_1=0\}\cap\Omega} \frac{\partial u}{\partial x_1}(0,x_2,\dots,x_n) \times$$

$$- \frac{it_1 e^{-ix_2t_2-\dots-ix_nt_n}}{t_1^2+\dots+t_n^2}\, dx_2\dots dx_n \tag{2.5}$$

Now look at (2.2), the contribution from (2.4) is

$$\int_{\{\mathrm{Im}t_1=A'\}\times \mathop{X}_{2}^{n} \{\mathrm{Im}t_i=A'\}} \hat{v}(t)dt_1\dots dt_n \int_{\{x_1=0\}\cap\Omega} u(0,x_2,\dots x_n) \times$$

$$\times \frac{it_1 e^{-ix_2t_2-\dots-ix_nt_n}}{t_1^2+\dots+t_n^2}\, dx_2\dots dx_n .$$

But there is a similar contribution, with an opposite sign, from integration on $\{\mathrm{Im}t_1= -A'\} \times \mathop{X}_{2}^{n} \{\mathrm{Im}t_i=A'\}$. Let $\Gamma_{A'}$ be the rectangular contour in the t_1-plane with sides $\pm iA' \pm R$ then, as $R \to \infty$, the sum of these contributions is

$$\int_{\overset{n}{\underset{2}{X}}\{\operatorname{Im} t_i = A'\}} \phi(t_2,\dots,t_n)dt_2\dots dt_n \int_{\Gamma_{A'}} \frac{\hat{v}(t_1,\dots,t_n)it_1}{t_1^2+\dots+t_n^2}\,dt_1 \tag{2.6}$$

where

$$\phi(t_2,\dots,t_n) = \int_{\{x_1=0\}\cap\Omega} u(0,x_2,\dots,x_n)e^{-ix_2t_2-\dots-ix_nt_n}\,dx_2\dots dx_n \quad .$$

For fixed $t_2,\dots,t_n$

$$\int_{\Gamma_{A'}} \frac{\hat{v}(t_1,\dots,t_n)it_1}{t_1^2+\dots+t_n^2}\,dt_1 = \begin{cases} \pi i\{\hat{v}(\tau_1,t_2,\dots,t_n)+\hat{v}(-\tau_1,t_2,\dots,t_n)\} \\ \qquad \text{if } |\operatorname{Im}\tau_1| < A' \\ 0 \quad \text{if } |\operatorname{Im}\tau_1| > A' \end{cases}$$

where $\tau_1 = \tau_1(t_2,\dots,t_n)$ is given by $\tau_1^2+t_2^2+\dots+t_n^2 = 0$ i.e., $\tau_1 = i(t_2^2+\dots+t_n^2)^{1/2}$. Now

$$M_{A'} = \{(t_2,\dots,t_n) \in \mathbb{C}^{n-1} \;,\; \operatorname{Im}t_2 = A',\dots,\operatorname{Im}t_n = A' \;,\; |\operatorname{Im}\tau_1| < A'\}$$

is seen to be a compact subset of $\overset{n}{\underset{i=2}{X}} \{\operatorname{Im}t_i = \mp A'\}$ and we get that the contribution from the pair of boundary terms, (obtained in (2.3)) considered is

$$\pi i \int_{M_{A'}} \phi(t_2,\dots,t_n)[\hat{v}(\tau_1,t_2,\dots,t_n) + \hat{v}(-\tau_1,t_2,\dots,t_n)]dt_2\dots dt_n \tag{2.6'}$$

and its absolute value is $\leq$ constant $||\hat{v}||_{A''}$.

Similarly, if

$$\phi'(t_2,\ldots,t_n) = -\int_{\{x_1=0\}\cap R_-\times\ldots\times R_-} \frac{\partial u}{\partial x_1}(0,x_2,\ldots,x_n)$$

$$\times \quad e^{-ix_2t_2-\ldots-ix_nt_n}\, dx_2\ldots dx_n \quad .$$

The net contribution from the two terms in (2.2) involving $\phi'(t_2,\ldots,t_n)$ is

$$\int_{\{Imt_2=A'\}\times\ldots\times\{Imt_n=A'\}} \phi'(t_2,\ldots,t_n) \int_{\Gamma_{A'}} \frac{\hat{v}(t_1,\ldots,t_n)}{t_1^2+\ldots+t_n^2}\, dt_1 \qquad (2.7)$$

which is equal to

$$\pi\int_{M_{A'}} \phi'(t_2,\ldots,t_n) \cdot \frac{1}{\tau_1} [\hat{v}(\tau_1,t_2,\ldots,t_n)-\hat{v}(-\tau_1,t_2,\ldots,t_n)]dt_2\ldots dt_n$$

(2.7')

which, in absolute value is $\leq$ constant $||v||_{A''}$.

In a similar way we can consider all other terms of (2.2) and write it as a sum of $n2^{n-1}$ terms of the form (2.6') and $n2^{n-1}$ terms of the form (2.7') . The resulting formula defines $T_u(f)$ for every $f \in \mathcal{H}_{A''}$ and T_u is a bounded linear functional on $\mathcal{H}_{A''}$.

Lemma 3. For every $x \in R^n$, $T_u(e^{ixt}) = (2\pi)^{-n/2} u(x)$.

Proof: Let K_ε be a C^∞ compact support approximate identity, then $\int K_\varepsilon(y-x)e^{iyt}dt \to e^{ixt}$ in the topology of $\mathcal{H}_{A''}$ and

$$T_u(e^{ixt}) = \lim_{\varepsilon\to 0} T_u\left(\int K_\varepsilon(y-x)e^{iyt}dy\right) =$$

$$= \lim_{\varepsilon\to 0} (2\pi)^{-n/2} \int K_\varepsilon(y-x)u(y) = (2\pi)^{-n/2} \ u(x) .$$

Lemma 4. There exist measures $d\mu_1$, $d\mu_2$ on $\{t_n = 0\} = \mathbb{C}^{n-1}$, supported in the compact set

$$L_{A''} = \{(t_1,\ldots,t_{n-1}) \ ; \ |\mathrm{Im}t_1| < A'',\ldots,|\mathrm{Im}t_{n-1}| < A'' ,$$
$$|\mathrm{Re}(t_1^2 + \ldots + t_{n-1}^2)^{1/2}| < A'' \}$$

such that for every $f \in \mathcal{H}_{A''}$

$$T_u(f) = \int f(t_1,\ldots,t_{n-1} , i(t_1^2+\ldots+t_{n-1}^2)^{1/2})d\mu_1$$
$$+ \int f(t_1,\ldots,t_{n-1}, -i(t_1^2+\ldots+t_{n-1}^2)^{1/2}d\mu_2 . \qquad (2.8)$$

In particular

$$u(x) = \int e^{ix_1t_1+\ldots+ix_{n-1}t_{n-1}} e^{-(t_1^2+\ldots+t_{n-1}^2)^{1/2}x_n} d\mu_1$$
$$+ \int e^{ix_1t_1+\ldots+ix_{n-1}t_{n-1}} e^{(t_1^2+\ldots+t_{n-1}^2)^{1/2} x_n} d\mu_2 \qquad (2.9)$$

<u>Proof:</u> Let $V_{A''} = \{(t_1,\ldots,t_n) \in K_{A''}\ ;\ t_1^2 + t_2^2 + \ldots + t_n^2 = 0\}$. Then by the proof of Lemma 2, by adding all the terms like (2.6') and (2.7') we get that there exists a measure $d\nu$, supported in $V_{A''}$ such that for every $f \in \mathcal{H}_{A''}$

$$T_u(f) = \int f\, d\nu$$

Let $d\nu = d\nu_1 + d\nu_2$ where $d\nu_1$ is supported in $\{(t_1,t_2,\ldots,i(t_1^2+\ldots+t_{n-1}^2)^{1/2})\}$ and $d\nu_2$ is supported in $\{(t_1,t_2,\ldots,t_{n-1},-i(t_1^2+\ldots+t_{n-1}^2)^{1/2})\}$. Let $d\mu_1, d\mu_2$ be the projections of $d\nu_1, d\nu_2$ respectively on $t_n = 0$. Then the lemma follows since $d\mu_1, d\mu_2$ are supported in the projection of $V_{A''}$ on $t_n = 0$ which is $L_{A''}$.

Now we are in a position to prove the theorems.

<u>Proof of Theorem A.</u> Since $A < \pi$ we can choose $A < A'' < \pi$. It is easily seen that $L_{A''}$ is contained in $\mathop{X}_{i=1}^{n-1} \{|\mathrm{Im}\,t_i| < A''\} \times \{|\mathrm{Re}\,t_i| < A''\}$ and since $A'' < \pi$ the span of $\{e^{ixt}\ ;\ x \in Z^{n-1}\}$ where Z^{n-1} are the integer lattice points of R^{n-1} , is dense in the space of bounded holomorphic functions on $L_{A''}$. By (2.9)

$$d\mu_1 + d\mu_2 \equiv 0$$

and

$$e^{-a(t_1^2+\ldots+t_{n-1}^2)^{1/2}}\, d\mu_1 + e^{a(t_1^2+\ldots+t_{n-1}^2)^{1/2}}\, d\mu_2 \equiv 0 \ .$$

Since $a \leqslant (\frac{1}{n-1})^{1/2}$

it follows that $d\mu_1, d\mu_2$ are supported in $\{t_1^2+\dots t_{n-1}^2 = 0\}$ and by (2.9) $u(x)$ is identically zero. □

Proof of Theorem B. Applying $\frac{\partial}{\partial x_n}$ to (2.9) we get

$$\frac{\partial u}{\partial x_n}(x_1,\dots,x_{n-1},0) = -\int e^{ix_1t_1+\dots+ix_{n-1}t_{n-1}}(t_1^2+\dots+t_{n-1})^{1/2}d\mu_1 ,$$

$$+\int e^{ix_1t_1+\dots+ix_{n-1}t_{n-1}}(t_1^2+\dots+t_{n-1}^2)^{1/2}\, d\mu_2 .$$

As in the proof of the Theorem A we get that

$$d\mu_1 + d\mu_2 \equiv 0$$

$$(t_1^2+\dots+t_{n-1}^2)^{1/2}(d\mu_1 - d\mu_2) \equiv 0 .$$

Thus $d\mu_1 = -d\mu_2$ is supported at the set $\{t_1^2+\dots+t_{n-1}^2 \doteq 0\}$ and by using (2.9) it once again follows that u vanishes identically.

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BINARY OPERATIONS IN THE SET OF SOLUTIONS OF A PARTIAL DIFFERENCE EQUATION.

1. Introduction.

Let Z^n be the n-dimensional lattice and consider a partial difference operator on Z^n

$$\mathcal{P}f(m) = \sum_{|k|\leq N} C_k f(m+k) \quad ,$$

where $m,k \in Z^n$, $|k| = \sum_{i=1}^{N} |k_i|$, $k=(k_1,\ldots,k_n)$ and N is an integer. In this appendix we shall characterize all products $*$ of the form

$$(f*g)(m) = \sum_{\substack{r\in Z^n \\ k\in Z^n}} d^m_{kr} f(r)g(k) \tag{1.1}$$

(only a finite number of terms on the right hand side being non-zero) with the property that if $\mathcal{P}f \equiv 0$ and $\mathcal{P}g \equiv 0$ then $\mathcal{P}(f*g) \equiv 0$. The product of Duffin and Rohrer [1] falls in this category. The basic idea is to associate with every discrete function $f:Z^n \to \mathbb{C}$ a linear functional T_f on the algebra $\mathcal{A}_n$ generated by the indeterminates

$\{z_1,z_1^{-1},\ldots,z_n,z_n^{-1}\}$, given by

$$T_f(z_1^{k_1},\ldots,z_n^{k_n}) = f(k_1,\ldots,k_n) \tag{1.2}$$

for every $(k_1,\ldots,k_n) \in Z^n$ and extended by linearity. Conversely, (1.2) associates a discrete function $f:Z^n \to \mathbb{C}$ to every such linear functional.

2. BINARY OPERATIONS ON THE SET OF SOLUTIONS OF $\mathcal{P}u \equiv 0$.

Definition 2.1. Any operation $(f,g) \to f*g$ which maps pairs of functions on Z^n to another function on Z^n and is of the form (1.1) will be termed a Duffin product.

Lemma 2.2. Any Duffin product induces a linear mapping

$$\mathcal{F} : \mathcal{A}_n \to \mathcal{A}_{2n}$$

such that if $z = (z_1,\ldots,z_n)$, $t = (t_1,\ldots,t_n)$

$$T_{f*g}(u(z)) = T_f T_g(\mathcal{F}u(z,t)) \tag{2.1}$$

where $T_f T_g$ is the linear functional on $\mathcal{A}_{2n}$ defined by

$$T_f T_g(z^k t^r) = T_f(z^k) T_g(t^r) \tag{2.2}$$

and extended by linearity.

Proof: By (1.1)

$$T_{f*g}(z^m) = (f*g)(m) = \sum d^m_{kr} \, T_f(z^k) T_g(t^r)$$

$$= T_f T_g(\sum d^m_{kr} z^k t^r) \quad .$$

Define $\mathcal{F}(z^m) = \sum d^m_{kr} z^k t^r$ and extend by linearity. Obviously (2.1) defines a Duffin product for each such mapping.

Lemma 2.3: Let $\mathcal{P}$ be a partial difference operator with constant coefficients

$$\mathcal{P}f(m) = \sum C_k f(m+k) \quad ,$$

and let $P(z) \in \mathcal{A}_n$ be its symbol,

$$P(z) = \sum C_k z^k \ .$$

Then $\mathcal{P}f \equiv 0$ iff T_f annihilates the principal ideal $P(z)\mathcal{A}_n = \{P(z)u(z) \ ; \ u(z) \in \mathcal{A}_n\}$.

Proof: The statement is self-evident from the identity

$$T_f(P(z)z^m) = T_f(\sum C_k z^{m+k}) = \sum C_k f(m+k) \ .$$

Now we are in a position to prove our central result.

Theorem. A Duffin product induced by the mapping $\mathcal{F}: \mathcal{A}_n \to \mathcal{A}_{2n}$, given in Lemma 2.2, maps pairs of solutions of $\mathcal{P}u \equiv 0$ into another solution if $\mathcal{F}(P(z)\mathcal{A}_n)$ is contained in the ideal generated by $\{P(z),P(t)\}$, i.e., if for every $u(z) \in \mathcal{A}_m$ we can find $a(z,t)$, $b(z,t) \in \mathcal{A}_{2n}$ such that

$$\mathcal{F}(P(z)u(z)) = a(z,t)P(z) + b(z,t)P(t) \quad .$$

Proof: $\mathcal{P}(f*g) \equiv 0$ if $T_{f*g}(P(z)\mathcal{A}_n) = 0$.

Now

$$T_{f*g}(P(z)u(z) = T_f T_g(\mathcal{F}P(z)u(z)) = T_f T_g(a(z,t)P(z) + b(z,t)P(t)) = 0 .$$

3. APPLICATIONS.

The theorem makes very easy the verification that a given Duffin product preserves the property of being a solution of a given partiall difference equation with constant coefficients. This will be illustrated by the following two examples.

a) Duffin and Duris [2] introduced three kinds of 'convolution products' for solutions of the discrete Cauchy-Riemann equation.

$$f(m,n)+if(m+1,n) - f(m+1,n+1)-i\, f(m,n+1) \equiv 0 \qquad (3.1)$$

They denoted them by $f*g$, $f*'g$ and $f*''g$. An easy calculation, which is not reproduced here in order to save space, shows that the corresponding mappings $\mathcal{F}$, $\mathcal{F}'$, $\mathcal{F}''$: $\mathcal{A}_2 \to \mathcal{A}_4$ are (make the notational transformation $z = (z_1,z_2) = (z,w)$, $t = (t_1,t_2) = (t,s)$)

$$\mathcal{F} : \quad u(z,w) \to (1+t)(1+z)\,\frac{u(z,w)-u(t,w)}{z-t} + i(1+s)(1+w)\,\frac{u(t,w)-u(t,s)}{w-s}$$

$$\mathcal{F}' : \quad u(z,w) \to (1+z)(1-t)\,\frac{u(z,w)-u(t,w)}{z-t} + i(1-s)(1+w)\,\frac{u(t,w)-u(t,s)}{w-s}$$

$$\mathcal{F}'' : \quad u(z,w) \to (1-z)(1-t)\,\frac{u(z,w)-u(t,w)}{z-t} + i(1-s)(1-w)\,\frac{u(t,w)-u(t,s)}{w-s}$$

From these formulas we deduce easily that the corresponding convolution products indeed preserve discrete - analyticity (i.e., the property of being a solution of (3.1)). They can also be used to advantage in giving short proofs of the commutativity and associativity of these products.

b) For a general partial difference equation with constant coefficients $\mathcal{P}u \equiv 0$, in Z^2, Duffin and Rohrer [1] introduced a 'product' which can be shown, by a straightforward but a little lengthy calculation, to be induced by

$$\mathcal{F}(u(z,w)) = ts\left\{\frac{u(t,s)-u(t,w)}{s-w}\left[\frac{P(z,w)-P(t,w)}{z-t}\right] - \frac{u(z,w)-u(t,w)}{z-t}\left[\frac{P(t,s)-P(t,w)}{s-w}\right]\right\}$$

$$= \frac{ts}{(s-w)(z-t)}[u(t,s)[P(z,w)-P(t,w)] - u(t,w)[P(z,w)-P(t,s)] - u(z,w)[P(t,s)-P(t,w)]],$$

where $P(z,w)$ is the symbol of $\mathcal{P}$. $\mathcal{F}$ is seen to satisfy the hypothesis of the theorem, thus furnishing a short proof to the fact that if $\mathcal{P}f \equiv 0$ and $\mathcal{P}g \equiv 0$ then $\mathcal{P}(f*g) \equiv 0$, (see Duffin and Rohrer [1], pp. 691-693, for the original proof).

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