Modular invariance in conformal field theory

Yi-Zhi Huang

Department of Mathematics
Rutgers University
Piscataway, NJ 08854

March 14, 2014

Rutgers Lie Group / Quantum Math Seminar

Outline

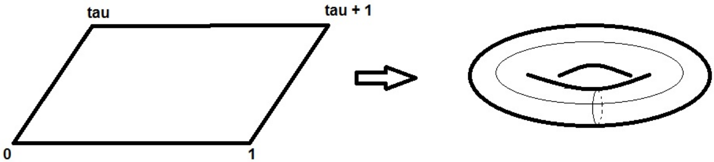
Outline

Outline

Outline

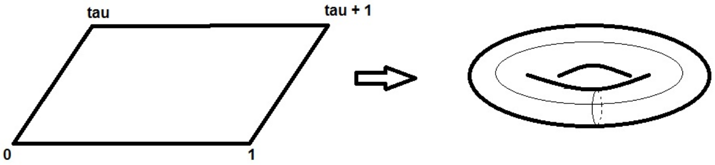
Tori as genus-one Riemann surfaces

- Tori can be constructed by identifying opposite sides of parallelograms in the complex plane. The complex structure of complex plane induces complex structures on tori.
- Any parallelogram is always conformally equivalent to a parallelogram with the four vertices 0, 1, τ and $\tau + 1$.



Tori as genus-one Riemann surfaces

- Tori can be constructed by identifying opposite sides of parallelograms in the complex plane. The complex structure of complex plane induces complex structures on tori.
- Any parallelogram is always conformally equivalent to a parallelogram with the four vertices 0, 1, τ and $\tau + 1$.



Genus-one Teichmüller space and moduli space

- Genus-one Teichmüller space: The torus constructed from a parallelogram contains additional information: The two circles corresponding to the four sides of the parallelogram. The space of such parallelograms is the genus-one Teichmüller space and is identified with the upper half plane (the space of τ).
- Genus-one moduli space: The genus-one moduli space is the space of all conformal equivalence classes of tori. It is identified with the upper half plane modulo the transformations on τ preserving the conformal structures. These transformations are the modular transformations.

Genus-one Teichmüller space and moduli space

- Genus-one Teichmüller space: The torus constructed from a parallelogram contains additional information: The two circles corresponding to the four sides of the parallelogram. The space of such parallelograms is the genus-one Teichmüller space and is identified with the upper half plane (the space of τ).
- Genus-one moduli space: The genus-one moduli space is the space of all conformal equivalence classes of tori. It is identified with the upper half plane modulo the transformations on τ preserving the conformal structures. These transformations are the modular transformations.

Modular invariance

- To construct a structure defined on the genus-one moduli space, a practical way is to construct first a structure on the genus-one Techmüller space and then to prove that the structure is invariant under the actions of the modular transformations, that is, to prove the modular invariance.
- Modular transformations are given by $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ where $\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \mathrm{SL}(2, \mathbb{Z})$. To prove modular invariance, we need to prove the invariance under the action of the group $\mathrm{SL}(2, \mathbb{Z})$.

Modular invariance

- To construct a structure defined on the genus-one moduli space, a practical way is to construct first a structure on the genus-one Techmüller space and then to prove that the structure is invariant under the actions of the modular transformations, that is, to prove the modular invariance.
- Modular transformations are given by $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ where $\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \mathrm{SL}(2, \mathbb{Z})$. To prove modular invariance, we need to prove the invariance under the action of the group $\mathrm{SL}(2, \mathbb{Z})$.

Modular invariance in conformal field theory

- The importance of modular invariance in conformal field theory and string theory was known for a long time. If one wants to construct conformal field theories on genus-one Riemann surfaces, one has to prove modular invariance.
- For WZNW models, minimal models, conformal field theories associated to lattices and the moonshine module, the modular invariance of the characters was obtained from explicit calculations.

Modular invariance in conformal field theory

- The importance of modular invariance in conformal field theory and string theory was known for a long time. If one wants to construct conformal field theories on genus-one Riemann surfaces, one has to prove modular invariance.
- For WZNW models, minimal models, conformal field theories associated to lattices and the moonshine module, the modular invariance of the characters was obtained from explicit calculations.

The Verlinde formula

- In 1987, Verlinde obtained a conjectured formula which expresses fusion rules in a rational conformal field theory in terms of the action of the modular transformation $\tau \mapsto -1/\tau$ on the space of characters.
- In 1988, Moore and Seiberg formulated several natural conjectures on rational conformal field theories and derived from these conjectures the important Moore-Seiberg polynomial equations.The Verlinde formula was shown by Moore and Seiberg to be a consequence of these polynomial equations.

The Verlinde formula

- In 1987, Verlinde obtained a conjectured formula which expresses fusion rules in a rational conformal field theory in terms of the action of the modular transformation $\tau \mapsto -1/\tau$ on the space of characters.
- In 1988, Moore and Seiberg formulated several natural conjectures on rational conformal field theories and derived from these conjectures the important Moore-Seiberg polynomial equations.The Verlinde formula was shown by Moore and Seiberg to be a consequence of these polynomial equations.

Modular invariance conjecture of Moore and Seiberg

- Among the conjectures Moore and Seiberg formulated is a modular invariance conjecture for chiral vertex operators (or intertwining operators) in a rational conformal field theory. Let me copy the complete statement giving this conjecture in the paper of Moore and Seiberg:

The final equation is obtained from the two-point function on the torus. The conformal blocks for the two-point function of $\beta_1 \in \mathcal{H}_{j_1}$, $\beta_2 \in \mathcal{H}_{j_2}$ are given by

$$\mathrm{Tr}_i \left[q^{L_0 - \frac{c}{24}} \binom{i}{j_1 p}_{z_1} (\beta_1 \otimes \cdot) \binom{p}{j_2 i}_{z_2} (\beta_2 \otimes \cdot) \right] \cdot \cdot (dz_1)^{\Delta_{\beta_1}} (dz_2)^{\Delta_{\beta_2}} .$$

Modular invariance conjecture of Moore and Seiberg

- Among the conjectures Moore and Seiberg formulated is a modular invariance conjecture for chiral vertex operators (or intertwining operators) in a rational conformal field theory. Let me copy the complete statement giving this conjecture in the paper of Moore and Seiberg:

The final equation is obtained from the two-point function on the torus. The conformal blocks for the two-point function of $\beta_1 \in \mathcal{H}_{j_1}$, $\beta_2 \in \mathcal{H}_{j_2}$ are given by

$$\mathrm{Tr}_i \left[q^{L_0 - \frac{c}{24}} \binom{i}{j_1 p}_{z_1} (\beta_1 \otimes \cdot) \binom{p}{j_2 i}_{z_2} (\beta_2 \otimes \cdot) \right] \cdot \\ \cdot (dz_1)^{\Delta_{\beta_1}} (dz_2)^{\Delta_{\beta_2}}.$$

Modular invariance conjecture of Moore and Seiberg

- Among the conjectures Moore and Seiberg formulated is a modular invariance conjecture for chiral vertex operators (or intertwining operators) in a rational conformal field theory. Let me copy the complete statement giving this conjecture in the paper of Moore and Seiberg:

  The final equation is obtained from the two-point function on the torus. The conformal blocks for the two-point function of $\beta_1 \in \mathcal{H}_{j_1}$, $\beta_2 \in \mathcal{H}_{j_2}$ are given by

$$\mathrm{Tr}_i \left[q^{L_0 - \frac{c}{24}} \begin{pmatrix} i \\ j_1 p \end{pmatrix}_{z_1} (\beta_1 \otimes \cdot) \begin{pmatrix} p \\ j_2 i \end{pmatrix}_{z_2} (\beta_2 \otimes \cdot) \right] \cdot \cdot (dz_1)^{\Delta_{\beta_1}} (dz_2)^{\Delta_{\beta_2}}.$$

Modular invariance conjecture of Moore and Seiberg

- By stating that the conformal blocks for the two-point function are given by the traces above, Moore and Seiberg in fact assumed the modular invaraince of the space spanned by these traces. This was a powerful conjecture when Moore and Seiberg wrote their paper. Note that even for WZNW and minimal models, this was a conjecture, not a theorem. It took me 15 years to eventually prove this conjecture, including, in particular, the conjectures in the case of WZNW and minimal models.
- The main purpose of this talk is to survey the results that led to my proof of this conjecture in 2003 and the progress towards the formulation and solution of the modular invariance conjecture in logarithmc conformal field theory.

Modular invariance conjecture of Moore and Seiberg

- By stating that the conformal blocks for the two-point function are given by the traces above, Moore and Seiberg in fact assumed the modular invaraince of the space spanned by these traces. This was a powerful conjecture when Moore and Seiberg wrote their paper. Note that even for WZNW and minimal models, this was a conjecture, not a theorem. It took me 15 years to eventually prove this conjecture, including, in particular, the conjectures in the case of WZNW and minimal models.
- The main purpose of this talk is to survey the results that led to my proof of this conjecture in 2003 and the progress towards the formulation and solution of the modular invariance conjecture in logarithmc conformal field theory.

Zhu's theorem

- In 1990, in his important Ph.D. thesis work, Zhu proved a partial result on the modular invariance conjecture of Moore and Seiberg.
- Let V be a vertex operator algebra of central charge c satisfying the following conditions:
  1. For $n < 0$, $V_{(n)} = 0$.
  2. Every $\mathbb{N}$-gradable weak V-module is completely reducible.
  3. V is C_2-cofinite.

  Then up to equivalence, there are only finitely many irreducible V-modules. Let $W_1, \ldots, W_m$ be the complete list of irreducible V-modules up to equivalence and $Y_1, \ldots, Y_n$ be their vertex operator maps, respectively. Zhu proved the following result:

Zhu's theorem

- In 1990, in his important Ph.D. thesis work, Zhu proved a partial result on the modular invariance conjecture of Moore and Seiberg.
- Let V be a vertex operator algebra of central charge c satisfying the following conditions:
  1. For $n < 0$, $V_{(n)} = 0$.
  2. Every $\mathbb{N}$-gradable weak V-module is completely reducible.
  3. V is C_2-cofinite.

  Then up to equivalence, there are only finitely many irreducible V-modules. Let $W_1, \ldots, W_m$ be the complete list of irreducible V-modules up to equivalence and $Y_1, \ldots, Y_n$ be their vertex operator maps, respectively. Zhu proved the following result:

Zhu's theorem

Theorem (Zhu, 1990)

For $v_1, \ldots, v_n \in V$, the series

$$\mathrm{Tr}_{W_i} Y_i(e^{2\pi i z_1 L(0)} v_1, e^{2\pi i z_1}) \cdots Y_i(e^{2\pi i z_n L(0)} v_n, e^{2\pi i z_n}) e^{2\pi i \tau (L(0) - \frac{c}{24})}$$

for $i = 1, \ldots, m$ are absolutely convergent on $1 > |e^{2\pi i z_1}| > \cdots > |e^{2\pi i z_n}| > |e^{2\pi i \tau}| > 0$ and can be analytically extended to meromorphic functions $S_{W_i}((v_1, z_1), \ldots, (v_n, z_n), \tau)$ on the region $z_i \neq z_j + \alpha\tau + \beta$ for $i \neq j$ and $\alpha, \beta \in \mathbb{Z}$. Moreover, the vector space spanned by $S_{W_i}(((dz_1)^{L(0)} v_1, z_1), \ldots, ((dz_n)^{L(0)} v_n, z_n), \tau)$ is invariant under the modular transformation given by

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \; z_1 \mapsto \frac{z_1}{c\tau + d}, \ldots, z_n \mapsto \frac{z_n}{c\tau + d}.$$

Generalizations of Zhu's theorem

- Adapting Zhu's method, Dong, Li and Mason in 1997 generalized Zhu's theorem to a corresponding partial result on the modular invariance in orbifold rational conformal field theory associated with cyclic groups.
- Adapting Zhu's method, Miyamoto in 2000 generalized Zhu's theorem to a modular invariance result involving one intertwining operator and an arbitrary number of vertex operators associated to irreducible modules.
- Unfortunately, the method developed by Zhu cannot be used or adapted to prove the (full) modular invariance conjecture of Moore and Seiberg. This is the reason why for almost 13 years after Zhu's theorem was proved, there was not much progress towards the proof of this full modular invariance conjecture.

Generalizations of Zhu's theorem

- Adapting Zhu's method, Dong, Li and Mason in 1997 generalized Zhu's theorem to a corresponding partial result on the modular invariance in orbifold rational conformal field theory associated with cyclic groups.
- Adapting Zhu's method, Miyamoto in 2000 generalized Zhu's theorem to a modular invariance result involving one intertwining operator and an arbitrary number of vertex operators associated to irreducible modules.
- Unfortunately, the method developed by Zhu cannot be used or adapted to prove the (full) modular invariance conjecture of Moore and Seiberg. This is the reason why for almost 13 years after Zhu's theorem was proved, there was not much progress towards the proof of this full modular invariance conjecture.

Generalizations of Zhu's theorem

- Adapting Zhu's method, Dong, Li and Mason in 1997 generalized Zhu's theorem to a corresponding partial result on the modular invariance in orbifold rational conformal field theory associated with cyclic groups.
- Adapting Zhu's method, Miyamoto in 2000 generalized Zhu's theorem to a modular invariance result involving one intertwining operator and an arbitrary number of vertex operators associated to irreducible modules.
- Unfortunately, the method developed by Zhu cannot be used or adapted to prove the (full) modular invariance conjecture of Moore and Seiberg. This is the reason why for almost 13 years after Zhu's theorem was proved, there was not much progress towards the proof of this full modular invariance conjecture.

The modular invariance theorem

- In 2003, I proved a precisely formulated modular invariance conjecture of Moore and Seiberg. This modular invariance conjecture is now a theorem. In particular, the modular invariance conjectures for WZNW and minimal models are now theorems.
- Let V be a vertex operator algebra of central charge c satisfying the following conditions:
  1. For $n < 0$, $V_{(n)} = 0$, $V_{(0)} = \mathbb{C}\mathbf{1}$ and V' as a V-module is equivalent to V.
  2. Every lower bounded generalized V-module is completely reducible.
  3. V is C_2-cofinite.

  Let $W_1, \ldots, W_m$ be the complete list of irreducible V-modules up to equivalence.

The modular invariance theorem

- In 2003, I proved a precisely formulated modular invariance conjecture of Moore and Seiberg. This modular invariance conjecture is now a theorem. In particular, the modular invariance conjectures for WZNW and minimal models are now theorems.
- Let V be a vertex operator algebra of central charge c satisfying the following conditions:
  1. For $n < 0$, $V_{(n)} = 0$, $V_{(0)} = \mathbb{C}\mathbf{1}$ and V' as a V-module is equivalent to V.
  2. Every lower bounded generalized V-module is completely reducible.
  3. V is C_2-cofinite.

  Let $W_1, \dots, W_m$ be the complete list of irreducible V-modules up to equivalence.

The modular invariance theorem

Theorem (H, 2003)

For elements $w_1, \ldots, w_n$ of irreducible V-modules, the series

$$\mathrm{Tr}_{W_i} \mathcal{Y}_1((e^{2\pi i z_1} dz_1)^{L(0)} w_1, e^{2\pi i z_1}) \cdot \\ \cdots \mathcal{Y}_i((e^{2\pi i z_n} dz_n)^{L(0)} w_n, e^{2\pi i z_n}) e^{2\pi i \tau (L(0) - \frac{c}{24})}$$

for $i = 1, \ldots, m$ and for intertwining operators $\mathcal{Y}_1, \ldots, \mathcal{Y}_n$ of suitable types are absolutely convergent and can be analytically extended to multivalued analytic forms on the region $z_i \neq z_j + \alpha\tau + \beta$ for $i \neq j$ and $\alpha, \beta \in \mathbb{Z}$. Moreover, the vector space spanned by these multivalued analytic forms is invariant under the modular transformations given by

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \; z_1 \mapsto \frac{z_1}{c\tau + d}, \ldots, z_n \mapsto \frac{z_n}{c\tau + d}.$$

The modular invariance theorem

- Why does Zhu's method not work for the full modular invariance conjecture? Zhu's method uses the commutator formula for vertex operators to reduce the construction of genus-one n-point functions to the construction of genus-one one-point functions. Since there is no commutator formula for general intertwining operators, one cannot use Zhu's method to construct genus-one n-point functions from general intertwining operators.
- The commutator formula and locality (weak associativity) for vertex operators are very useful. The commutator formula allows us to apply Lie-theoretic methods to vertex operator algebras and modules. The locality simplifies the formulations and proofs of many results. Unfortunately, they do not hold for general intertwining operators.

The modular invariance theorem

- Why does Zhu's method not work for the full modular invariance conjecture? Zhu's method uses the commutator formula for vertex operators to reduce the construction of genus-one n-point functions to the construction of genus-one one-point functions. Since there is no commutator formula for general intertwining operators, one cannot use Zhu's method to construct genus-one n-point functions from general intertwining operators.
- The commutator formula and locality (weak associativity) for vertex operators are very useful. The commutator formula allows us to apply Lie-theoretic methods to vertex operator algebras and modules. The locality simplifies the formulations and proofs of many results. Unfortunately, they do not hold for general intertwining operators.

The modular invariance theorem

The new method developed by me consists of the following steps:

1. Establish a general theory of intertwining operators, including, in particular, the genus-zero operator product expansion for intertwining operators (or associativity for intertwining operators) which is in fact also a conjecture stated in the paper of Moore and Seiberg. This theory uses the $P(z)$-tensor product bifunctor whose existence is in fact another conjecture given in the paper of Moore and Seiberg and proved by Lepowsky and me.

The modular invariance theorem

2. Show that the q-traces of products of intertwining operators satisfy systems of modular invariant differential equations with regular singular points and show that the q-traces of products of intertwining operators are absolutely convergent and have genus-one operator product expansions (or genus-one associativity for intertwining operators).
3. Reduce the proof of the modular invariance for genus-one n-point functions to that for genus-one one-point functions using the genus-one operator product expansion.

The modular invariance theorem

- The modular invariance theorem above is for chiral rational conformal field theories. In 2005, using the results obtained by me on chiral rational conformal field theories, including the modular invariance theorem, Kong and I constructed genus-zero full rational conformal field theories. It was a surprise that the construction of genus-zero full conformal field theories needs the modular invariance theorem, which is a result on genus-one chiral conformal field theories. See more discussions on this later.
- In 2006, Kong and I proved the modular invariance of the genus-zero full rational conformal field theories constructed in 2005.

The modular invariance theorem

- The modular invariance theorem above is for chiral rational conformal field theories. In 2005, using the results obtained by me on chiral rational conformal field theories, including the modular invariance theorem, Kong and I constructed genus-zero full rational conformal field theories. It was a surprise that the construction of genus-zero full conformal field theories needs the modular invariance theorem, which is a result on genus-one chiral conformal field theories. See more discussions on this later.
- In 2006, Kong and I proved the modular invariance of the genus-zero full rational conformal field theories constructed in 2005.

Miyamoto's work in the nonsemisimple case

- In 2002, Miyamoto claimed to have generalized Zhu's theorem to the nonsemisimple case in which $\mathbb{N}$-gradable weak V-modules might not be completely reducible.
- In this case, the usual q-traces of products of vertex operators for V-modules are not modular invariant. It is necessary to introduce what Miyamoto called pseudo-traces. The main new ingredient in Miyamoto's work is the introduction and study of pseudo-traces.

Miyamoto's work in the nonsemisimple case

- In 2002, Miyamoto claimed to have generalized Zhu's theorem to the nonsemisimple case in which $\mathbb{N}$-gradable weak V-modules might not be completely reducible.
- In this case, the usual q-traces of products of vertex operators for V-modules are not modular invariant. It is necessary to introduce what Miyamoto called pseudo-traces. The main new ingredient in Miyamoto's work is the introduction and study of pseudo-traces.

Miyamoto's work in the nonsemisimple case

- Arike and Nagatomo found a mistake in Miyamoto's paper. There is in fact a counterexample to a statement in a proof in Miyamoto's paper. For the conclusion of Miyamoto's main theorem to hold, one needs the additional assumption that all irreducible V-modules are infinite dimensional. Because of this additional assumption, Miyamoto's theorem is weaker than what he conjectured.
- What Miyamoto claimed originally is a generalization of Zhu's theorem in the nonsemisimple case. Just as Zhu's theorem, it would still be a partial result on the modular invaraince in this case. Miyamoto did not have a conjecture for the full modular invariance in this case.

Miyamoto's work in the nonsemisimple case

- Arike and Nagatomo found a mistake in Miyamoto's paper. There is in fact a counterexample to a statement in a proof in Miyamoto's paper. For the conclusion of Miyamoto's main theorem to hold, one needs the additional assumption that all irreducible V-modules are infinite dimensional. Because of this additional assupmption, Miyamoto's theorem is weaker than what he conjectured.
- What Miyamoto claimed originally is a generalization of Zhu's theorem in the nonsemisimple case. Just as Zhu's theorem, it would still be a partial result on the modular invaraince in this case. Miyamoto did not have a conjecture for the full modular invariance in this case.

Logarithmic modular invariance conjecture

- A few years ago, I made a (full) modular invariance conjecture in the nonsemisimple case (or in the case of logarithmic conformal field theory): Analytic extensions of suitably generalized q-pseudo-traces of products of logarithmic intertwining operators span a modular invariant vector space.
- To even formulate this conjecture precisely, we need to generalize Miyamoto's pseudo-traces to the case of logarithmic intertwining operators and study their properties.

Logarithmic modular invariance conjecture

- A few years ago, I made a (full) modular invariance conjecture in the nonsemisimple case (or in the case of logarithmic conformal field theory): Analytic extensions of suitably generalized q-pseudo-traces of products of logarithmic intertwining operators span a modular invariant vector space.
- To even formulate this conjecture precisely, we need to generalize Miyamoto's pseudo-traces to the case of logarithmic intertwining operators and study their properties.

Logarithmic modular invariance conjecture

- In his thesis work, Fiordalisi has successfully obtained such generalizations and has made substantial progress towards the proof of this conjecture. In fact, there is only one conjectured lemma that still needs to be proved.
- In the study of this logarithmic modular invariance, a result by Yang and me in 2010 on associative algebras and logarithmic intertwining operators plays a crucial role.

Logarithmic modular invariance conjecture

- In his thesis work, Fiordalisi has successfully obtained such generalizations and has made substantial progress towards the proof of this conjecture. In fact, there is only one conjectured lemma that still needs to be proved.
- In the study of this logarithmic modular invariance, a result by Yang and me in 2010 on associative algebras and logarithmic intertwining operators plays a crucial role.

Verlinde conjecture and Verlinde formula

- In 2004, the Verlinde conjecture and Verlinde formula for rational conformal field theories were finally proved by me using the modular invariance theorem together with some other results.
- The Verlinde formula for WZNW was proved in 1992 and 1993 by Faltings (for most classes of simple Lie algebras) and Teleman. But even in this case, the conceptual proof of the Verlinde formula that confirms the idea of Moore and Seiberg was first given as a special case of the theorem above.

Verlinde conjecture and Verlinde formula

- In 2004, the Verlinde conjecture and Verlinde formula for rational conformal field theories were finally proved by me using the modular invariance theorem together with some other results.
- The Verlinde formula for WZNW was proved in 1992 and 1993 by Faltings (for most classes of simple Lie algebras) and Teleman. But even in this case, the conceptual proof of the Verlinde formula that confirms the idea of Moore and Seiberg was first given as a special case of the theorem above.

Rigidity and modularity

- In 2005, I proved the rigidity and modularity of the braided tensor category of modules for a vertex operator algebra satisfying the three conditions in the modular invariance theorem. The proof uses a strong version of the Verlinde formula and thus logically uses the modular invariance theorem.
- It is surprising that the proof of the rigidity needs the modular invariance theorem. This fact in fact shows why the works of Tsuchiya-Ueno-Yamada, Beilinson-Feigin-Mazur and Bakalov-Kirillov did not give a proof of the rigidity.

Rigidity and modularity

- In 2005, I proved the rigidity and modularity of the braided tensor category of modules for a vertex operator algebra satisfying the three conditions in the modular invariance theorem. The proof uses a strong version of the Verlinde formula and thus logically uses the modular invariance theorem.
- It is surprising that the proof of the rigidity needs the modular invariance theorem. This fact in fact shows why the works of Tsuchiya-Ueno-Yamada, Beilinson-Feigin-Mazur and Bakalov-Kirillov did not give a proof of the rigidity.

Rigidity and modularity

- In the case of WZNW models, Finkelberg's tensor-category-equivalence theorem together with Kazhdan-Lusztig's rigidity theorem for negative levels had been thought to prove the rigidity for almost all (but not all) cases. But it turned out that Finkelberg's paper had a gap and it required either Teleman's or my proof of the Verlinde formula or, alternatively, my proof of the rigidity, to fill the gap and prove the equivalence theorem (again, for almost al,l but not all, cases). Note that, as is mentioned above, my proof of the Verlinde formula or my proof of the rigidity needs the modular invariance theorem.

Full conformal field theories

- As is mentioned before, in 2005, Kong and I constructed genus-zero full rational conformal field theories. It is surprising that the modular invariance theorem, a genus-one property, is needed in this construction of genus-zero full conformal field theories.
- Let me explain how this construction needs the modular invariance theorem. Our construction of full rational conformal field theories amounts to a construction of a nondegenerate bilinear form on the space of intertwining operators. The hard part is the proof of the nondegeneracy of the bilinear form. In a suitable basis, this bilinear form is given by a matirx formed by certain special entries of the fusing matrix.

Full conformal field theories

- As is mentioned before, in 2005, Kong and I constructed genus-zero full rational conformal field theories. It is surprising that the modular invariance theorem, a genus-one property, is needed in this construction of genus-zero full conformal field theories.
- Let me explain how this construction needs the modular invariance theorem. Our construction of full rational conformal field theories amounts to a construction of a nondegenerate bilinear form on the space of intertwining operators. The hard part is the proof of the nondegeneracy of the bilinear form. In a suitable basis, this bilinear form is given by a matirx formed by certain special entries of the fusing matrix.

Full conformal field theories

- The nondegeneracy of the bilinear form is equivalent to the invertibility of the matrix. But calculations show that this matrix is invertible if and only if one special entry of the fusing matrix is nonzero. A strong version of the Verlinde conjecture is needed to prove that this special entry is nonzero. Since this strong version of the Verlinde conjecture is proved using, among other things, the modular invariance theorem, the proof of the nondegeneracy of the bilinear form needs the modular invariance theorem.

Logarithmic tensor category theory

- In 2009, I stated a conjecture on the rigidity of the braided tensor categories in the logarithmic case constructed by Lepowsky, Zhang and me. It is expected that the proof of the logarithmic modular invariance conjecture will be one of the most important steps in the future proof of this conjecture.