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CONCIRCULAR VECTOR FIELDS AND PSEUDO-KAEHLER MANIFOLDS

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Abstract. A vector field on a pseudo-Riemannian manifold N is called concircular if it satisfies $\nabla_X v = \mu X$ for any vector X tangent to N, where ∇ is the Levi-Civita connection of N. A concircular vector field satisfying $\nabla_X v = \mu X$ is called a non-trivial concircular vector field if the function μ is non-constant. A concircular vector field v is called a concurrent vector field if the function μ is a non-zero constant. In this article we prove that every pseudo-Kaehler manifold of complex dimension > 1 does not admit a non-trivial concircular vector field. We also prove that this result is false whenever the pseudo-Kaehler manifold is of complex dimension one. In the last section we provide some remarks on pseudo-Kaehler manifolds which admit a concurrent vector field.

1. Introduction

A. Fialkow introduced in [8] the notion of *concircular vector fields* on a Riemannian manifold N as vector fields which satisfy

$$\nabla_X v = \mu X \tag{1.1}$$

for vectors X tangent to N, where ∇ denotes the Levi-Civita connection of N and μ is a non-trivial function on N. A concircular vector field satisfying (1.1) is called non-trivial if the function μ is non-constant. Concircular vector fields can also be defined on pseudo-Riemannian manifolds exactly in the same way.

A concircular vector field v is called a concurrent vector field if the function μ in (1.1) is equal to one. However, for simplicity we call a concircular vector field v a *concurrent vector field* if the function μ in (1.1) is a nonzero constant throughout this article.

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Concircular vector fields also known as geodesic fields in literature since integral curves of such vector fields are geodesics. Concircular vector fields appeared in the study of concircular mappings, i.e., conformal mappings preserving geodesic circles [8, 13]. Concircular vector fields also play an important role in the theory of projective and conformal transformations. Such vector fields have interesting applications in general relativity, e.g. trajectories of time-like concircular fields in the de Sitter model determine the world lines of receding or colliding galaxies satisfying the Weyl hypothesis [12]. Furthermore, it was proved by the author in [3] that a Lorentzian manifold is a generalized Robertson-Walker spacetime if and only if it admits a time-like concircular vector field. For some further results related to concircular vector fields, see [4, 6–8, 11–13] for instance.

A pseudo-Riemannian metric g on a complex manifold (M, J) is called *pseudo-Hermitian* if the metric g and the complex structure J on M are compatible, i.e.,

$$g(JX, JY) = g(X, Y), \quad X, Y \in T_pM, \quad p \in M. \tag{1.2}$$

A *pseudo-Hermitian manifold*, by definition, is a complex manifold equipped with a pseudo-Hermitian metric. A pseudo-Hermitian manifold is called a *pseudo-Kaehler manifold* if its complex structure J is parallel with respect to its Levi-Civita connection ∇, i.e., $\nabla J = 0$. Notice that the real index of a pseudo-Hermitian metric is always even, say $2s$, due to (1.2). The integer s is called the *complex index.*

The main result of this article is the following.

Theorem 1.1. *We have:*

(a) *every pseudo-Kaehler manifold M^n with $n = \dim_{\mathbf{C}} M^n > 1$ does not admit a non-trivial concircular vector field;*
(b) *the result is false for pseudo-Kaehler manifolds M^n with $n = 1$.*

In the last section we provide some remarks on pseudo-Kaehler manifolds which admit a concurrent vector field.

2. Proof of Theorem 1.1

For general references on pseudo-Riemannian and pseudo-Kaehler manifolds, we refer to books [1, 2, 5, 9, 10]. Throughout this article, we shall follow the notations given in the book [2] closely.

Assume that M^n is a complex n-dimensional pseudo-Kaehler manifold which admits a non-zero concircular vector field v satisfying condition (1.1).

Let R denote the Riemann curvature tensor of M^n which is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{2.1}$$

for vector fields X, Y, Z tangent to M^n. It is well-known that the Riemann curvature tensor R satisfies

$$R(X,Y) = -R(Y,X), \tag{2.2}$$

$$R(X,Y)JZ = J(R(X,Y)Z), \tag{2.3}$$

$$R(JX,JY)Z = R(X,Y)Z, \tag{2.4}$$

$$g(R(X,Y)Z,W) = g(R(Z,W)X,Y). \tag{2.5}$$

It follows from (1.1) and (2.1) that the curvature tensor R satisfies

$$\begin{aligned} R(X,v)v &= \nabla_X\nabla_v v - \nabla_v\nabla_X v - \nabla_{[X,v]}v \\ &= \nabla_X(\mu v) - \nabla_v(\mu X) - \mu\nabla_X v + \mu\nabla_v X \\ &= (X\mu)v - (v\mu)X, \end{aligned} \tag{2.6}$$

for any vector field X tangent to M^n. Thus after taking the inner product of (2.6) with v we find

$$(X\mu)g(v,v) = (v\mu)g(X,v), \quad \text{for all } X \in TM^n. \tag{2.7}$$

In particular, (2.7) implies the following

$$v\mu = 0, \quad \text{whenever } g(v,v) = 0, \tag{2.8}$$

$$X\mu = 0, \quad \text{whenever } g(v,v) \neq 0 \text{ and } g(X,v) = 0. \tag{2.9}$$

Let X be a vector satisfying $g(X,v) = 0$. By taking the inner product of (2.6) with X, we find

$$g(R(X,v)v,X) = -(v\mu)g(X,X), \quad \text{whenever } g(X,v) = 0. \tag{2.10}$$

Similarly, by applying (1.1), (2.1) and (2.3), we also have

$$\begin{aligned} R(Y,Jv)Jv &= J(R(Y,Jv)v) \\ &= J\{\nabla_Y\nabla_{Jv}v - \nabla_{Jv}\nabla_Y v - \nabla_{[Y,Jv]}v\} \\ &= J\{\nabla_Y(\mu Jv) - \nabla_{Jv}(\mu Y) - \mu\nabla_Y(Jv) + \mu\nabla_{Jv}Y\} \\ &= -(Y\mu)v - ((Jv)\mu)JY \end{aligned} \tag{2.11}$$

for any vector field Y tangent to M^n. Therefore, by combining (2.11) with (2.7), we obtain

$$g(R(Y,Jv)Jv,Y) = 0 \tag{2.12}$$

for any tangent vector Y satisfying $g(Y,v) = 0$.

Next, by applying (2.2), (2.4), (2.5) and (2.12) we have

$$\begin{aligned} 0 &= -g(R(Y,Jv)Jv,Y) = g(R(JY,v)Jv,Y) \\ &= g(R(Jv,Y)JY,v) = -g(R(v,JY)JY,v) \\ &= g(R(JY,v)v,JY) \end{aligned} \tag{2.13}$$

for any Y satisfying $g(Y,v) = 0$.

Now, by combining (2.10) and (2.13) we get

$$(v\mu)g(X,X)=0 \tag{2.14}$$

for any tangent vector X satisfying $g(X,v)=g(JX,v)=0$.

Proof of Theorem 1.1. *Statement* (a). Let p be an arbitrary fixed point in M^n.

Case (a.i)*: $v(p)$ is either space-like or time-like.* In this case, there exists an orthonormal basis $e_1,\ldots,e_{2n}$ of T_pM^n such that

$$e_2=Je_1,\quad g(e_i,e_j)=\epsilon_i\delta_{ij},\quad v(p)=ce_1, \tag{2.15}$$

where $c\neq 0$ and $\epsilon_i=\pm 1$. It follows from (2.14) and (2.15) that $v\mu=0$ whenever $n=\dim_{\mathbf{C}} M^n>1$.

On the other hand, we find from (2.9) that $e_2\mu=\cdots=e_{2n}\mu=0$. Hence we have

$$U\mu=0,\quad \text{for all } U\in T_pM^n.$$

Case (a.ii): *$v(p)$ is light-like.* In this case, it follows from (1.1), (2.1), and $\nabla J=0$ that

$$R(X,Jv)v=(X\mu)Jv-((Jv)\mu)X, \tag{2.16}$$

for any vector X tangent to M^n. Thus, after taking the inner product of (2.16) with v, we find

$$0=((Jv)\mu)g(X,v).$$

Since $v(p)$ is a light-like vector, there exists another light-like vector u at p such that $g(u,v)=-1$. Therefore we also have

$$(Jv)\mu=0. \tag{2.17}$$

Now, by combining (2.16) and (2.17) we obtain

$$R(X,Jv)v=(X\mu)Jv. \tag{2.18}$$

Similarly, we also find from (1.1), (2.1) and (2.8) that

$$R(JX,v)v=(JX\mu)v. \tag{2.19}$$

Since we have $R(X,Jv)=-R(JX,v)$ from (2.4), we may obtain from (2.18) and (2.19) that

$$(X\mu)Jv+(JX\mu)v=0. \tag{2.20}$$

Because v and Jv are linearly independent vector, (2.20) implies that $X\mu=JX\mu=0$. Therefore we also have $U\mu=0$ for any vector $U\in T_pM^n$. Because p can be chosen to any any arbitrary point with $v(p)\neq 0$, this shows that μ is a constant function. Consequently, M^n admits no non-trivial concircular vector fields whenever $n>1$. This proves statement (a) of the theorem.

Statement (b). Assume that M^n is a pseudo-Kaehler manifold with $n=1$. Then it follows from (1.2) that M^1 is either space-like or time-like.

Case (b.i): M^1 is space-like. Let us consider the complex projective line $CP^1(4)$ equipped with the Fubini-Study metric of constant Gauss curvature 4. Let $z = x + \mathrm{i}y$ be a local complex coordinate on $CP^1(4)$ so that the metric tensor of $CP^1(4)$ is given by

$$g = \frac{dzd\bar{z}}{(1+z\bar{z})^2}. \tag{2.21}$$

It is easy to verify that the Levi-Civita connection of $CP^1(4)$ satisfies

$$\begin{aligned} \nabla_{\partial/\partial x}\frac{\partial}{\partial x} &= -\frac{2x}{1+z\bar{z}}\frac{\partial}{\partial x} + \frac{2y}{1+z\bar{z}}\frac{\partial}{\partial y}, \\ \nabla_{\partial/\partial x}\frac{\partial}{\partial y} &= -\frac{2y}{1+z\bar{z}}\frac{\partial}{\partial x} - \frac{2x}{1+z\bar{z}}\frac{\partial}{\partial y}, \\ \nabla_{\partial/\partial y}\frac{\partial}{\partial y} &= \frac{2x}{1+z\bar{z}}\frac{\partial}{\partial x} - \frac{2y}{1+z\bar{z}}\frac{\partial}{\partial y}. \end{aligned} \tag{2.22}$$

Let us consider the function φ defined by

$$\varphi = \frac{1-z\bar{z}}{1+z\bar{z}}.$$

By applying (2.21) and (2.22). It is straight-forward to verify that the gradient vector of φ, denoted by grad φ, is given by

$$\text{grad } \varphi = -4\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right).$$

It is direct to show that grad φ satisfies

$$\nabla_X \text{ grad } \varphi = -4\varphi X$$

for any X tangent to $CP^1(4)$. Therefore grad φ is a non-trivial concircular vector field defined on $CP^1(4)$.

Case (b.ii): M^1 is time-like. Let $\overline{CP}^1(4)$ denote the unit disc $D = \{z \in \mathbf{C} : z\bar{z} < 1\}$ equipped with the following time-like metric:

$$g = \frac{-dzd\bar{z}}{(1-z\bar{z})^2}. \tag{2.23}$$

It is easy to verify that the Levi-Civita connection of $\overline{CP}^1(4)$ satisfies

$$\begin{aligned} \nabla_{\partial/\partial x}\frac{\partial}{\partial x} &= \frac{2x}{1-z\bar{z}}\frac{\partial}{\partial x} - \frac{2y}{1-z\bar{z}}\frac{\partial}{\partial y}, \\ \nabla_{\partial/\partial x}\frac{\partial}{\partial y} &= \frac{2y}{1-z\bar{z}}\frac{\partial}{\partial x} + \frac{2x}{1-z\bar{z}}\frac{\partial}{\partial y}, \\ \nabla_{\partial/\partial y}\frac{\partial}{\partial y} &= -\frac{2x}{1-z\bar{z}}\frac{\partial}{\partial x} + \frac{2y}{1-z\bar{z}}\frac{\partial}{\partial y}. \end{aligned} \tag{2.24}$$

It follows from (2.24) that $\overline{CP}^1(4)$ has constant Gauss curvature 4 as well.

Now, let us consider the function

$$\psi = \frac{1+z\bar{z}}{1-z\bar{z}}$$

defined on $\overline{CP}^1(4)$. By applying (2.23) and (2.24), we find

$$\begin{aligned} &\text{grad } \psi = -4\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \\ &\nabla_X \text{grad } \psi = \mu X, \end{aligned}$$

with $\mu = -4\psi$. Therefore grad ψ is a non-trivial concircular vector field on $\overline{CP}^1(4)$. This proves statement (b) of the theorem.

□

3. Some Remarks on Concurrent Vector Fields

Recall that a vector field v on a pseudo-Riemannian manifold is called concurrent if it satisfies

$$\nabla_X v = cX \tag{3.1}$$

for any tangent vector X, where c is a non-zero constant.

Proposition 3.1. *Let M^n be a pseudo-Kaehler manifold. Then we have the following.*

(a) *For any concurrent vector field v on M^n, Jv is never a concurrent vector field on M^n.*
(b) *The complex distribution $\mathcal{D} := \text{Span}\,\{v, Jv\}$ is always integrable.*
(c) *If $\mathcal{D}$ is non-degenerate, then the leaves of $\mathcal{D}$ are flat totally geodesic surfaces. Moreover, M^n is foliated by totally geodesic flat holomorphic curves.*

Proof. For a concurrent vector field v on M^n, we have

$$\nabla_X Jv = J\nabla_X v = cJX, \quad X \in TM^n, \tag{3.2}$$

which implies that Jv is not a concurrent vector field. This proves statement (a).

From (3.1) and (3.2), we find

$$\nabla_{Jv} v = \nabla_v Jv = cJv. \tag{3.3}$$

Thus we have

$$[v, Jv] = \nabla_v Jv - \nabla_{Jv} v = 0,$$

which implies that the distribution $\mathcal{D}$ is an integrable distribution. Moreover, it follows from (3.1) and (3.2) that

$$\nabla_v v = -\nabla_{Jv} Jv = cv. \tag{3.4}$$

Hence the leaves of $\mathcal{D}$ are totally geodesic surfaces in M^n. Also, it is easy to verify from (3.3) and (3.4) that the leaves are flat surfaces. Therefore, we also have statements (b) and (c). □

Let $e_1, \ldots, e_{2n}$ be an orthonormal frame on the pseudo-Kaehler manifold M^n, the Ricci tensor Ric of M^n is defined by

$$\mathrm{Ric}(X,Y) = \sum_{i=1}^{2n} \epsilon_i g(R(e_i,X)Y,e_i), \tag{3.5}$$

where $g(e_i,e_j) = \epsilon_i \delta_{ij}$. Since v is a concurrent vector field, it follows from (2.6), (2.11) and (3.1) that

$$R(X,v)v = R(X,Jv)Jv = 0. \tag{3.6}$$

Therefore, we find from (3.5) and (3.6) that

$$\mathrm{Ric}(v,v) = \mathrm{Ric}(Jv,Jv) = 0. \tag{3.7}$$

Consequently, we have the following.

Proposition 3.2. *Every pseudo-Kaehler manifold with positive (or negative) Ricci curvature admits no concurrent vector fields.*

An immediate consequence this proposition is the following.

Corollary 3.1. *If an Einstein pseudo-Kaehler manifold admits a concurrent vector field, then it is Ricci-flat.*

Remark 3.1. The complex pseudo-Euclidean n-space $\mathbf{C}^n_s$ with the pseudo-Kaehlerian metric

$$g = -\sum_{i=1}^{s} dz_i d\bar{x}_i + \sum_{j=s+1}^{n} dz_j d\bar{z}_j,$$

is a Ricci-flat pseudo-Kaehler manifold which admits a concurrent vector field; namely, the position vector field of $\mathbf{C}^n_s$.

Let N^m be a Riemannian m-manifold with $m > 2$. Denote by $K(\pi)$ the sectional curvature of a plane section $\pi \subset T_pN^m$, $p \in N^m$. For any orthonormal basis $e_1, \ldots, e_m$ of T_pN^m, the scalar curvature τ at p is defined by

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j).$$

Let L be a r-subspace of T_pN^m with $r \geq 2$ and $\{e_1, \ldots, e_r\}$ an orthonormal basis of L. The scalar curvature $\tau(L)$ of L is defined by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r.$$

For given integer $k \geq 1$, we denote by $\mathcal{S}(m,k)$ the finite set consisting of k-tuples $(n_1, \ldots, n_k)$ of integers satisfying $2 \leq n_1, \cdots, n_k < m$ and $\sum_{j=1}^{k} i \leq m$.

For each k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(m,k)$, the author introduced in the early 1990s the invariant $\delta(n_1, \ldots, n_k)$ defined by

$$\delta(n_1, \ldots, n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \cdots + \tau(L_k)\}, \quad p \in N^m,$$

where $L_1, \ldots, L_k$ run over all k mutually orthogonal subspaces of T_pN^m such that $\dim L_j = n_j$, $j = 1, \ldots, k$ (cf. [2, page 253] for details).

Another immediate consequence of (3.6) is the following.

Corollary 3.2. *If a Kaehler manifold M^n admits a concurrent vector field, then we have*

$$\delta(2n-1) \geq 0.$$

Proof. This corollary follows from (3.7) and the fact that

$$\delta(2n-1) = \max_{u \in T_1M^n} \mathrm{Ric}(u,u),$$

where T_1M^n is the unit tangent bundle of M^n. □

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