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ON SECONDARY PROCESSES GENERATED BY A RANDOM POINT DISTRIBUTION OF POISSON TYPE

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Introduction

In the present paper we consider the mathematical model of secondary processes and give a general and rigorous method for solving special problems. We do not suppose that the basic "process" is a *time-process* but consider the problem more generally, i.e. we replace the time axis by an abstract space where the random points are distributed. The idea of our general method was suggested by a lecture of C. Ryll–Nardzewski, who gave an elegant solution of a telephone-problem.[1] We shall return to this problem and its solution in § 3 (Example 3.)

In § 1 we give a sufficient condition ensuring the Poisson character of a random point distribution. In § 2 the secondary process generated by a random point distribution of Poisson-type is considered and in § 3 some examples are given.

I express my sincere thank to P. Révész for his valuable remarks.

§. 1. The random point distribution of Poisson type

Let us consider an abstract space T in which a σ-algebra $\mathcal{S}_T$ is given (cf. [3], p. 28) with $\{t\} \in \mathcal{S}_T$ for $t \in T$. We suppose that in T a random point distribution is given, i.e. a random selection of a finite number of points of T. We suppose that if $A \in \mathcal{S}_T$, then the number of the random points which are inside of the set A is a random variable, and we denote it by $\xi(A)$.

Another equivalent conception is the following: we consider the set Ω of all the purely discontinuous measures defined on $\mathcal{S}_T$ which have a finite number of discontinuities and the magnitude of the discontinuities is equal to 1. We suppose that in the space Ω there is a σ-algebra $\mathcal{S}_\Omega$ on which a probability measure $\mathbf{P}$ is defined with $\mathbf{P}(\Omega) = 1$. Finally, if $\omega \in \Omega$ and $\xi(A) = \xi(\omega, A)$ denotes the number of discontinuities of the measure ω in the set $A \in \mathcal{S}_T$, then this function of ω is measurable with respect to $\mathcal{S}_\Omega$ for every fixed A.

[1]This lecture was held in Wroclaw at the Colloquium on Stochastic Processes in 1953.

We shall suppose that $\mathcal{S}_\Omega$ is constructed so that it contains those realizations which are determined by the conditions

$$\xi(A_1) = k_1, \ldots, \xi(A_n) = k_n, \tag{1}$$

where $A_1, \ldots, A_n$ are sets of $\mathcal{S}_\Omega$ and $k_1, \ldots, k_n$ non-negative integers, furthermore if $\mathcal{S}_0$ is the set of realizations satisfying a condition of the type (1), then $\mathcal{S}_\Omega$ is the smallest σ-algebra containing $\mathcal{S}_0$. The probabilities of the events of type (1) wholly determine the probability measure on $\mathcal{S}_\Omega$.

If the sets $A_1, \ldots, A_n$ in (1) are not disjoint, then there are disjoint sets of $\mathcal{S}_T$: $B_1, \ldots, B_r$ such that each A_k can be represented as

$$A_k = \sum_{l=1}^{l_k} B_{i_l}.$$

We suppose that the probability measure is constructed so that to disjoint sets there belong independent random variables $\xi(A)$. But in this case the probability distribution on $\mathcal{S}_\Omega$ is completely determined by the joint distributions of the variables

$$\xi(B_1), \ldots, \xi(B_r). \tag{2}$$

The notion of a random point distribution of Poisson type is formulated in

Definition 1. *A random point distribution is called of a Poisson type if there is a finite measure $\lambda(A)(A \in \mathcal{S}_T)$ such that*

$$\mathbf{P}(\xi(A) = k) = \frac{\lambda^k(A)}{k!} e^{-\lambda(A)} \qquad (a \in \mathcal{S}_T;\ k = 0, 1, 2, \ldots). \tag{3}$$

In the following we shall give sufficient conditions under which a random point distribution will be of Poisson type. First we introduce some other definitions.

Definition 2. *We say that the finite system of sets $\mathcal{B} = \{A_1, \ldots, A_r\}$ is a decomposition (or subdivision) of the set $A \in \mathcal{S}_T$ if $A_k \in \mathcal{S}_T$ $(k = 1, \ldots, r)$, $\sum_{k=1}^{r} A_k = A$ and $A_i A_k = 0$ for $i \neq k$.*

Definition 3. *Let $\mathcal{B}_1 = \left\{A_1^{(1)}, \ldots, A_{r_1}^{(1)}\right\}$, $\mathcal{B}_2 = \left\{A_1^{(2)}, \ldots A_{r_2}^{(2)}\right\}$ be two subdivisions of the set $A \in \mathcal{S}_T$. We say that $\mathcal{B}_1$ precedes $\mathcal{B}_2, \mathcal{B}_1 < \mathcal{B}_2$ if every $A_i^{(1)}$ can be decomposed with the aid of the sets $A_k^{(2)}$.*

Let us introduce the notation

$$P_k(A) = \mathbf{P}(\xi(A) = k) \qquad (k = 0, 1, 2, \ldots)$$

and formulate the notion of an *atomless random point distribution.*

Definition 4. *A random point distribution will be called atomless if for every $t \in T$ we have*

$$P_0(\{t\}) = 1.$$

This means that the points are well distributed.

For the space T we introduce the following condition:

a) *For every set $B \in \mathcal{S}_T$ there exists a sequence of subdivisions $\mathcal{B}_n$ with $\mathcal{B}_n < \mathcal{B}_{n+1}$ such that if $t_1 \in B$, $t_2 \in B$, $t_1 \neq t_2$, then for large n these are separated by the sets of $\mathcal{B}_n$, i.e. $t_1 \in A_i^{(n)}$, $T_2 \in A_k^{(n)}$ where $i \neq k$.*

THEOREM 1. [2] *Let us suppose that the space T has the property* a) *and the random point distribution satisfies the following conditions:*

b) *For every system $A_1, \ldots, A_r$ of disjoint sets of $\mathcal{S}_T$, the random variables $\xi(A_1), \ldots, \xi(A_r)$ are independent.*

c) *The random point distribution is atomless.*

Under these conditions we have

$$P_k(A) = \frac{\lambda^k(A)}{k!} e^{-\lambda(A)} \qquad (k = 0, 1, 2, \ldots), \tag{4}$$

where $\lambda(A)$ is a finite-valued, atomless measure defined on the σ-algebra $\mathcal{S}_T$.

PROOF. First we prove that for every $B \in \mathcal{S}_T$, $P_0(B) > 0$. In fact, if for a set $B \in \mathcal{S}_T$, $P_0(B) = 0$, then according to Condition a) there exists a point $t \in B$ with $P_0(\{t\}) = 0$ which contradicts Condition c). Let us introduce the notation

$$\lambda(B) = -\log P_0(B) \qquad (B \in \mathcal{S}_T). \tag{5}$$

We prove that $\lambda(B)$ is a bounded, atomless measure on $\mathcal{S}_T$.

The additiveness of λ is obvious. Let $B_1, B_2, \ldots$ be a non-increasing sequence of sets of $\mathcal{S}_T$ with $\prod_{k=1}^{\infty} B_k = 0$. Then $\mathbf{P}(\xi(B_n) \to 0) = 1$, hence $P_0(B_n) \to 1$ if $n \to \infty$ which implies the complete additiveness of λ. Since $\lambda(B) < \infty$ for every $B \in \mathcal{S}_T$, we conclude that the measure λ is bounded. Finally, according to Condition c), λ is atomless.

Let $\mathcal{B}^{(n)} = \left\{ B_1^{(n)}, \ldots, B_{k_n}^{(n)} \right\}$ be a sequence of subdivisions of the set $A \in \mathcal{S}_T$ such that

$$\lambda\left(B_k^{(n)}\right) \le \frac{\lambda(A)}{2^n} \qquad (k = 1, \ldots, k_n). \tag{6}$$

We may suppose that $\mathcal{B}^{(n)}$ has also the property described in Condition a). In fact, $\mathcal{B}^{(n)}$ can be chosen in such a way that $\mathcal{B}^{(n)} < \mathcal{B}^{(n+1)}$ and if $\mathcal{B}_0^{(n)}$ is a sequence of subdivisions satisfying c), then the superposition of $\mathcal{B}^{(n)}$ and $\mathcal{B}_0^{(n)}$ has the required property.

If we introduce the new random variables

$$\xi'\left(B_k^{(n)}\right) = \begin{cases} 1 & \text{if } \xi\left(B_k^{(n)}\right) = 1, \\ 0 & \text{otherwise,} \end{cases} \qquad (k = 1, \ldots, k_n;\ n = 1, 2, \ldots). \tag{7}$$

[2] Another proof for this theorem is given in [9].

then

$$\mathbf{P}\left(\lim_{n\to\infty}\sum_{k=1}^{k_n}\xi'\left(B_k^{(n)}\right)=\xi(A)\right)=1. \tag{8}$$

Clearly

$$\mathbf{M}\left(e^{i\xi'(B_k^{(n)})u}\right)=1-P_1\left(B_k^{(n)}\right)+P_1\left(B_k^{(n)}\right)e^{iu}=1+P_1\left(B_k^{(n)}\right)\left(e^{iu}-1\right). \tag{9}$$

According to (8)

$$\prod_{k=1}^{k_n}\left\{1+P_1\left(B_k^{(n)}\right)\left(e^{iu}-1\right)\right\}\Rightarrow\mathbf{M}\left(e^{i\xi(A)u}\right) \tag{10}$$

but $P_1\left(B_k^{(n)}\right)\le 1-P_0\left(B_k^{(n)}\right)\le-\log P_0\left(B_k^{(n)}\right)$ $(k=1,\ldots,k_n,\ n=1,2,\ldots)$, hence

$$\begin{aligned}&\left|\prod_{k=1}^{k_n}\left\{1+P_1\left(B_k^{(n)}\right)\left(e^{iu}-1\right)\right\}-\exp\sum_{k=1}^{k_n}P_1\left(B_k^{(n)}\right)\left(e^{iu}-1\right)\right|\\&\qquad\le 2\sum_{k=1}^{k_n}P_1^2\left(B_k^{(n)}\right)\le\frac{1}{2^{n-1}}\log\frac{1}{P_0(A)}\to 0,\end{aligned} \tag{11}$$

when $n\to\infty$. Thus

$$f(u,A)=\mathbf{M}\left(e^{i\xi(A)u}\right)=\lim_{n\to\infty}e^{\sum_{k=1}^{k_n}P_1(B_k^{(n)})(e^{iu}-1)} \tag{12}$$

which shows that $\xi(A)$ has a Poisson distribution. It follows from (12) that

$$\lim_{n\to\infty}\sum_{k=1}^{k_n}P_1\left(B_k^{(n)}\right)=-\log P_0(A)=\lambda(A). \tag{13}$$

Since $\lambda(A)(A\in\mathcal{S}_T)$ is a bounded, atomless measure, our theorem is completely proved.

□

Now, we introduce the notion of a σ-finite random point distribution of Poisson type.

Definition 5. *Let us suppose that in the space T there is a random selection of a countable number of points so that if $\xi(A)A\in\mathcal{S}_T$ is the number of random points lying in A, then the following conditions hold:*

1° *For every $A\in\mathcal{S}_T$, $\xi(A)$ is a random variable. We permit here random variables taking on the $+\infty$ with a positive probability.*

2° *If $A_1,A_2,\ldots$ is a sequence of disjoint sets of $\mathcal{S}_T$, then the random variables $\xi(A_1)$, $\xi(A_2),\ldots$ are independent.*

3° *There is a σ-finite measure λ defined on $\mathcal{S}_T$ such that*

$$\mathbf{P}(\xi(A) = k) = \frac{\lambda^k(A)}{k!} e^{-\lambda(A)} \qquad (k = 0, 1, 2, \ldots),$$

provided that $A \in \mathcal{S}_T$ and $\lambda(A) < \infty$.

In this case we say that this is a σ-finite random point distribution of Poisson type.

It follows from the above conditions that for every $A \in \mathcal{S}_T$ the probability $\mathbf{P}(\xi(A) = \infty)$ equals either 0 or 1. In fact, the set A can be represented as $A = \sum_{k=1}^{\infty} A_k$ where $A_k \in \mathcal{S}_T$ $(k = 1, 2, \ldots)$, $A_i A_k = 0$ for $i \neq k$ and $\lambda(A_k) < \infty$ $(k = 1, 2, \ldots)$. According to Condition 1° the random variables $\xi(A_1), \xi(A_2), \ldots$ are independent, hence the sum

$$\sum_{k=1}^{\infty} \xi(A_k)$$

is either finite or infinite with probability 1 (cf. [4], p. 60). But

$$\xi(A) = \sum_{k=1}^{\infty} \xi(A_k)$$

and the statement follows.

§ 2. The secondary process

In this section we suppose that in the space T there is a random point distribution of Poisson type with $\mathbf{M}(\xi(A)) = \lambda(A)(A \in \mathcal{S}_T)$ where $\xi(A)$ is the number of random points lying in the set A.

In many practical problems the case arises that every random point is a starting point of a further "happening". This happening may be e.g. the telephone conversation following the call or the random path of a microbe supposing that the microbes were originally distributed according to a Poisson law on the plane etc. In each case we may characterize the whole phenomenon by a finite sequence of data $(t_1, y_1), \ldots, (t_n, y_n)$, the elements $t_1, \ldots, t_n$ being a realization of a random point distribution in the space T and the elements $y_1, \ldots, y_n$ some characteristics of the corresponding happenings such as the durations of the conversations, the paths of the bacteria etc. Thus the sample space of the phenomenon will be the space of the random point distributions in the product space $Z = T \times Y$, where $Y = \{y\}$ is that space, the elements of which characterize the secondary happenings.

In many practical applications we deal with problems in which Y may not be considered as a Euclidean but a more general space. We consider the problem generally and regard Y as an abstract space.

We suppose that in the space Y there is a σ-algebra $\mathcal{S}_Y$ consisting of some subsets of Y and denote by $\mathcal{S}_Z$ the σ-ring $\mathcal{S}_T \times \mathcal{S}_Y$.

We shall use the notation Ω for the sample space of the underlying random point distribution of Poisson type. The sample space of the secondary process arises then in such a way that we replace each $\omega \in \Omega$ by a set of point-systems in the space $T \times Y$.

Thus we form a new probability space Ω_1, the elements of which are the point distributions $(t_1, y_1), \dots, (t_n, y_n)$ where $(t_1, \dots, t_n) \in \Omega$ and $y_1, \dots, y_n$ are arbitrary elements of Y. For the probability measure on Ω_1 we keep the notation $\mathbf{P}$.

Let us define a class of sets $\mathcal{T}$, the elements of which are sets of Ω_1, as follows. If n is a non-negative integer and $D = A \times C (A \in \mathcal{S}_T, C \in \mathcal{S}_Y)$, then $\{(t_1, y_1), \dots, (t_n, y_n)\} \in \mathcal{T}$ provided that exactly k $(0 \leq k \leq n)$ of the points $(t_1, y_1), \dots, (t_n, y_n)$ fall in the set D. Now, $\mathcal{S}_{\Omega_1}$ is defined as the smallest σ-algebra containing the class of sets $\mathcal{T}$, or by symbols, $\mathcal{S}_\Omega = \mathcal{S}(\mathcal{T})$.

If $D \in \mathcal{S}_Z$, then the definition of Ω_1 implies that the number of random points being in the set D, which we shall denote in the sequel by $\eta(D) = \eta(\omega_1, D)$ $(\omega_1 \in \Omega_1)$, is a random variable. In other terms, $\eta(D)$ is measurable with regard to the σ-algebra $\mathcal{S}_{\Omega_1}$.

Consider the σ-algebra $\mathcal{S}_\Omega$. This is obviously isomorphic to a system of sets of $\mathcal{S}_{\Omega_1}$. From now on we shall not consider the underlying random point distribution separately from the whole phenomenon, hence it will not give rise to a misunderstanding if we denote this isomorphic σ-algebra also by $\mathcal{S}_\Omega$ and apply the notations ω, $\xi(A)(A \in \mathcal{S}_T)$ too. We consider the conditional probabilities

$$\begin{gathered} \mathbf{P}(\eta(D_1) = k_1, \dots, \eta(D_n) = k_n \mid \mathcal{S}_\Omega) \\ \times (D_i = A_i \times C_i,\ A_i \in \mathcal{S}_T,\ C_i \in \mathcal{S}_Y;\ i = 1, \dots, n) \end{gathered} \tag{14}$$

existing with probability 1 (cf. [2], Chapter I). If $\omega = (t_1, \dots, t_r)$ denotes the variable element (variable realization) of the probability space Ω, then another notation for (14) is the following:

$$\mathbf{P}(\eta(D_1) = k_1, \dots, \eta(D_n) = k_n \mid t_1, \dots, t_r). \tag{15}$$

We make an assumption expressing that the secondary happenings corresponding to different points of T are independent as follows:

α) If $D_i = A_i \times C_i$ where $A_i \in \mathcal{S}_T$, $C_i \in \mathcal{S}_Y$, $t_i \in A_i$ $(i = 1, \dots, n)$, $A_i A_k = 0$ for $i \neq k$, then

$$\mathbf{P}(\eta(D_1) = 1, \dots, \eta(D_n) = 1 \mid t_1, \dots, t_n) = \mu(C_1, t_1) \dots \mu(C_n, t_n), \tag{16}$$

where $\mu(C, t)(C \in \mathcal{S}_Y)$ denotes a probability measure defined on the σ-algebra $\mathcal{S}_Y$ while t is an element of T. In other terms, $\mu(C, t)(C \in \mathcal{S}_Y)$ is the probability distribution of the secondary happening if its starting point is t.

For every fixed $C \in \mathcal{S}_Y$ the function $\mu(C, t)(t \in T)$ is measurable with respect to the σ-ring $\mathcal{S}_T$. In fact, if we consider the set of realizations $\Omega^{(1)}$ of the sample space Ω which are composed only of one random point t, then by (16) we get

$$\mathbf{P}(\eta(D) = 1 \mid t) = \mu(C, t),$$

where $D = T \times C$, $C \in \mathcal{S}_Y$, $T \in \mathcal{S}_T$ and $t \in T$. Thus $\mu(C,t)$ as a function of $\omega = (t)$ is measurable relative to the σ-algebra $\mathcal{S}_\Omega$ given in the space Ω. Our assumption on $\mathcal{S}_\Omega$ implies that a set $\{\omega\} = \{(t)\}$ of $\Omega^{(1)}$ is measurable with respect to $\mathcal{S}_\Omega$ if and only if the set $\{t\}$ is measurable with respect to $\mathcal{S}_T$. But

$$\{\omega : \Omega^{(1)},\ a \leq \mathbf{P}(\eta(D) = 1 \mid \omega) \leq b\} \in \mathcal{S}_\Omega,$$

hence

$$\{t : a \leq \mu(C,t) \leq b\} \in \mathcal{S}_T$$

for every pair $a \leq b$.

Now we are in a position to formulate our main theorem.

THEOREM 2. *If for the random point distribution in the product space $T \times Y$ Condition α) holds, then it is of Poisson type, i.e. to disjoint sets $D_1, \ldots, D_n$ of $\mathcal{S}_Z$ there correspond independent random variables depending on Poisson distributions and if $D = A \times C(A \in \mathcal{S}_T,\ C \in \mathcal{S}_Y)$, then*

$$\mathbf{M}(\eta(D)) = \int_C \mu(A,t)\lambda(\mathrm{d}t). \tag{17}$$

If D' is an arbitrary set of $\mathcal{S}_Z$, then

$$\mathbf{M}(\eta(D')) = \nu^*(D') \tag{18}$$

where ν^* denotes the extended measure of $\mathbf{M}(\eta(D))$ which is defined on the rectangular sets $D = A \times C(A \in \mathcal{S}_T,\ C \in \mathcal{S}_Y)$.

Before turning to the proof of Theorem 2 we prove a

LEMMA. *Let us consider the subspace $\Omega^{(n)}$ of Ω determined by the condition $\xi(\omega, A) = n(A \in \mathcal{S}_T)$ with the corresponding conditional probability measure $\mathbf{P}\left(\cdot \mid \Omega^{(n)}\right)$. If $f(t)(t \in T)$ is a measurable complex-valued function and*

$$\varphi(\omega) = f(t_1) \ldots f(t_n) \ \ \textit{for} \ \ \omega = (t_1, \ldots, t_n) \in \Omega^{(n)}, \tag{19}$$

then

$$\int_{\Omega(n)} \varphi(\omega)\mathbf{P}\left(\mathrm{d}\omega \mid \Omega^{(n)}\right) = \left(\int_A f(t)\frac{\lambda(\mathrm{d}t)}{\lambda(A)}\right)^n. \tag{20}$$

PROOF OF THE LEMMA. We consider the product space $T^{(n)} = A \times \ldots \times A$. To every $\omega = (t_1, \ldots, t_n)$ we order the $n!$ points $(t_{i_1}, \ldots, t_{i_n})$ where $t_{i_j} \in A$ and $(i_1, \ldots, i_n)$ run over all permutations of the elements $1, \ldots, n$. In this case every point distribution $(t_1, \ldots, t_n)$ in the set A is represented by $n!$ points in the set $T^{(n)}$. This correspondence in one-to-one.

Let us introduce a terminology: a set $\{(t_1, \ldots, t_n)\}$ of $T^{(n)}$ will be called symmetric if for every permutation $(i_1, \ldots, i_n)$ of the numbers $1, \ldots, n$ it is identical with $\{(t_{i_1}, \ldots, t_{i_n})\}$. In this sense to every set $\{(t_1, \ldots, t_n)\}$ of $\Omega^{(n)}$ there corresponds a symmetric set in $T^{(n)}$.

A complex-valued function $g(t_1, \dots, t_n)$ will be called symmetric if for every permutation $(i_1, \dots, i_n)$ of the numbers $1, \dots, n$ we have

$$g(t_1, \dots, t_n) = g(t_{i_1}, \dots, t_{i_n}).$$

In the mapping $\Omega^{(n)} \to T^{(n)}$ we have constructed a new probability space instead of $\Omega^{(n)}$. Let $\mathcal{J}^{(n)}$ denote the product σ-algebra in $T^{(n)}$ which is the smallest σ-algebra containing the sets of the type $A_1 \times \dots \times A_n$ where A_i is a measurable subset of A.

If a set $\{(t_1, \dots, t_n)\}$ was measurable in $\Omega^{(n)}$, its image in $T^{(n)}$ is measurable with respect to $\mathcal{J}^{(n)}$. Conversely, every symmetric set belonging to $\mathcal{J}^{(n)}$ is an image of a measurable set of $\Omega^{(n)}$. In fact, $\mathcal{S}_\Omega$ is generated by the conditions $\xi(A_1) = k_1, \dots, \xi(A_n) = k_n$ where $A_i A_k = 0$ for $i \neq k$. But this can be formulated with the language applied for $T^{(n)}$ so that the random element (i.e. the $n!$ points) $\sum_{(i_1,\dots,i_n)} (t_{i_1}, \dots, t_{i_n})$ belongs to a measurable (with respect to $\mathcal{J}^{(n)}$) symmetric set of $T^{(n)}$. On the other hand, the event that $\sum_{(i_1,\dots,i_n)} (t_{i_1}, \dots, t_{i_n})$ belongs to a measurable (with respect to $(\mathcal{J}^{(n)})$ and symmetric set, which is a finite sum of rectangular measurable sets of $T^{(n)}$, can be expressed so that $\xi(A_1) = k_1, \dots, \xi(A_n) = k_n$ with some $A_1, \dots, A_n$ and $k_1, \dots, k_n$.

Let $\mathcal{J}_1^{(n)}$ denote the σ-algebra of the symmetric sets of $\mathcal{J}^{(n)}$. If a symmetric function $g(t_1, \dots, t_n)$ is measurable in $\Omega^{(n)}$, then, according to what has been said above, the function $g(t_1, \dots, t_n)$ is measurable with respect to $\mathcal{J}_1^{(n)}$.

Let us calculate the probability

$$\mathbf{P}\left(\xi(A_1) = k_1, \dots, \xi(A_r) = k_r \mid \Omega^{(n)}\right)$$

for $A_i \in A\ \mathcal{S}_T$ $(i = 1, \dots, n)$, $A_i A_k = 0$ if $i \not\equiv k$ and $A = \sum_{i=1}^r A_i$. A simple argument shows that $(n = k_1 + \dots + k_r)$

$$\mathbf{P}\left(\xi(A_1) = k_1, \dots, \xi(A_r) = k \mid \Omega^{(n)}\right) = \frac{n!}{k_1! \dots k_r!} \left(\frac{\lambda(A_1)}{\lambda(A)}\right)^{k_1} \dots \left(\frac{\lambda(A_r)}{\lambda(A)}\right)^{k_r}. \tag{21}$$

This means that if $A' \subseteq T^{(n)}$ is the image of the set determined by $\xi(A_1) = k_1, \dots, \xi(A_r) = k_r$, then

$$\mathbf{P}\left(\sum_{(i_1,\dots,i_n)} (t_{i_1}, \dots, t_{i_n}) \in A'\right) = \frac{n!}{k_1! \dots k_r!} \left(\frac{\lambda(A_1)}{\lambda(A)}\right)^{k_1} \dots \left(\frac{\lambda(A_r)}{\lambda(A)}\right)^{k_r} \tag{22}$$

which is equal to

$$\lambda^{(n)}(A'), \tag{23}$$

where $\lambda^{(n)}$ denotes the product measure

$$\lambda^{(n)} = \frac{\lambda}{\lambda(A)} \times \dots \times \frac{\lambda}{\lambda(A)} \tag{24}$$

defined on $\mathcal{J}^{(n)}$.

The sets of the type A' generate the σ-algebra $\mathcal{J}_1^{(n)}$, hence if $B \in \mathcal{J}_1^{(n)}$, then the random element of $T^{(n)}$ belongs to B with the probability $\lambda^{(n)}(B)/\lambda^n(A)$.

This implies at once that

$$\int_{\Omega^{(n)}} \varphi(\omega)\mathbf{P}\left(\mathrm{d}\omega \mid \Omega^{(n)}\right) = \int_{T^{(n)}} f(t_1)\dots f(t_n)\,\mathrm{d}\lambda^{(n)} = \left(\frac{1}{\lambda(A)}\int_A f(t)\,\mathrm{d}\lambda\right)^n \tag{25}$$

what was to be proved. □

Proof of Theorem 2. In the first step of the proof we show that if

$$D_k = A_k \times C_k \qquad (A_k \in \mathcal{S}_T,\ C_k \in \mathcal{S}_Y;\ k = 1,\dots,n)$$

and $D_iD_k = 0$ for $i \neq k$, then the random variables $\eta(D_1),\dots,\eta(D_n)$ are independent, $\eta(D_k)$ has a Poisson-distribution with the parameter

$$\int_{A_k} \mu(C_k,t)\lambda(\mathrm{d}t) \tag{26}$$

for $k = 1,\dots,n$.

We reduce this problem to another one which is more convenient for our method. The sets $A_1,\dots,A_n$ and $C_1,\dots,C_n$ can be represented as sums of disjoint groups of sets $A'_1,\dots A'_m$ and $C'_1,\dots,C'_r$ belonging to $\mathcal{S}_T$ and $\mathcal{S}_Y$, respectively. Thus it suffices to prove that the random variables $\eta(D'_{jl})$ $(D'_{jl} = A'_j \times C'_l;\ j = 1,\dots,m;\ l = 1,\dots,r)$ are independent and depend on Poisson distributions with

$$\mathbf{M}(\eta(D'_{jl})) = \int_{A_j} \mu(C'_l,t)\lambda(\mathrm{d}t). \tag{27}$$

Let $\Omega^{(N_1,\dots,N_m)}$ denote the set of those ω's for which $\xi(A'_1) = N_1,\dots,\xi(A'_r) = N_r$ and consider the conditional probability

$$\begin{aligned}&\mathbf{P}\left(\eta(D'_{jl}) = k_{jl},\ j = 1,\dots,m,\ l = 1,\dots,r \mid \Omega^{(N_1,\dots,N_m)}\right)\\ &= \int_{\Omega^{(N_1,\dots,N_m)}} \mathbf{P}\left(\eta(D'_{jl}) = k_{jl},\ j = 1,\dots,m,\ l = 1,\dots,r \mid \omega\right)\mathbf{P}\left(\mathrm{d}\omega \mid \Omega^{(N_1,\dots,N_m)}\right),\end{aligned} \tag{28}$$

where the integration is taken with respect to the conditional probability measure

$$\mathbf{P}\left(E \mid \Omega^{(N_1,\dots,N_m)}\right)\left(E \in \Omega^{(N_1,\dots,N_m)}\mathcal{S}_\Omega\right)$$

and $N_j \geq \sum_{l=1}^r k_{jl}$. (If for at least one $j, N_j < \sum_{l=1}^r k_{jl}$, then (28) is equal to 0.)

If $\omega = (T_{11},\dots,t_{1N_1},\ t_{21},\dots,t_{2N_2},\dots,t_{m1},\dots,t_{mN_m})$ where $t_{js} \in A'_j$ $(s = 1,\dots,N_j;$ $j = 1,\dots,m)$, then the characteristic function of the integrand in (28) equals

$$\sum_{j=1}^{m}\sum_{l=1}^{r} e^{iu_{jl}k_{jl}}\mathbf{P}(\eta(D_{jl}) = k_{jl},\ j = 1,\dots,m,\ l = 1,\dots,r \mid \omega)$$

$$= \prod_{j=1}^{m}\prod_{s=1}^{N_j}\left\{1 - \sum_{l=1}^{r}\mu(C_l', t_{js}) + \sum_{l=1}^{r}\mu(C_l', t_{js})e^{iu_{jl}}\right\} \tag{29}$$

$$= \prod_{j=1}^{m}\prod_{s=1}^{N_j}\left\{1 + \sum_{l=1}^{r}\mu(C_l', t_{js})(e^{iu_{jl}} - 1)\right\}.$$

Using (29), the characteristic function of the conditional probabilities in (28) equals

$$\sum_{j=1}^{m}\sum_{l=1}^{r} e^{iu_{jl}k_{jl}}\mathbf{P}\left(\eta(D_{jl}') = k_{jl},\ j = 1,\dots,m,\ l = 1,\dots,r \mid \Omega^{(N_1,\dots,N_m)}\right)$$

$$= \int_{\Omega^{(N_1,\dots,N_m)}} \sum_{j=1}^{m}\sum_{l=1}^{r} e^{iu_{jl}k_{jl}}\mathbf{P}\left(\eta(D_{jl}') = k_{jl},\ j = 1,\dots,m,\right.$$

$$\left. l = 1,\dots,r \mid \omega\right)\mathbf{P}\left(\mathrm{d}\omega \mid \Omega^{(N_1,\dots,N_m)}\right) \tag{30}$$

$$= \int_{\Omega^{(N_1,\dots,N_m)}} \sum_{j=1}^{N}\sum_{s=1}^{N}\left\{1 + \sum_{l=1}^{r}\mu(C_l', t_{js})(e^{iu_{jl}} - 1\right\}\mathbf{P}\left(\mathrm{d}\omega \mid \Omega^{(N_1,\dots,N_m)}\right).$$

Since

$$\mathbf{P}\left(\Omega^{(N_1,\dots,N_m)}\right) = \mathbf{P}\left(\Omega^{(N_1)}\dots\Omega^{(N_m)}\right) = \mathbf{P}\left(\Omega^{(N_1)}\right)\mathbf{P}\left(\Omega^{(N_m)}\right), \tag{31}$$

(30) equals furthermore

$$\prod_{j=1}^{m}\int_{\Omega^{(N_j)}}\prod_{s=1}^{N_j}\left\{1 + \sum_{l=1}^{r}\mu(C_l', t_{is})(e^{iu_{jl}} - 1)\right\}\mathbf{P}\left(\mathrm{d}\omega \mid \Omega^{(N_j)}\right). \tag{32}$$

Now, let us apply the Lemma for the factors in (32). For a fixed j the function

$$\prod_{s=1}^{N_j}\left\{1 + \sum_{l=1}^{r}\mu(C_l', t_{js})(e^{iu_{jl}} - 1)\right\}$$

satisfies the condition of the Lemma, hence

$$\int_{\Omega^{(N_j)}}\prod_{s=1}^{N_j}\left\{1 + \sum_{l=1}^{r}\mu(C_l', t_{js})(e^{iu_{jl}} - 1)\right\}\mathbf{P}\left(\mathrm{d}\omega \mid \Omega^{(N_j)}\right)$$

$$= \left(\int_{A_j'}\left\{1 + \sum_{l=1}^{r}\mu(C_l', t)(e^{iu_{jl}} - 1)\right\}\frac{\lambda(\mathrm{d}t)}{\lambda(A_j')}\right)^{N_j} \tag{33}$$

$$= \left(1 + \sum_{l=1}^{r}\frac{1}{\lambda(A_j')}\int_{A_j'}\mu(C_l', t)\lambda(\mathrm{d}t)(e^{iu_{jl}} - 1)\right)^{N_j} \qquad (j = 1,\dots,m).$$

Introducing the notation

$$(34)\qquad b_j = \sum_{l=1}^{r} \frac{1}{\lambda(A'_j)} \int_{A'_j} \mu(C'_l, t)\lambda(\mathrm{d}t)(e^{iu_{jl}} - 1) \qquad (j = 1, \ldots, m),$$

by (30), (32) and (33) we conclude

$$\sum_{j=1}^{m}\sum_{l=1}^{r} e^{iu_{jl}k_{jl}} \mathbf{P}(\eta(D'_{jl}) = k_{jl},\ j = 1, \ldots, m,\ l = 1, \ldots, r)$$
$$= \sum_{N_1,\ldots,N_m} \sum_{j=1}^{m}\sum_{l=1}^{r} e^{iu_{jl}k_{jl}} \mathbf{P}\left(\eta(D'_{jl}) = k_{jl},\ j = 1, \ldots, m,\ l = 1, \ldots, r \mid \Omega^{(N_1,\ldots,N_m)}\right).$$

(35)

$$\prod_{s=1}^{m} \frac{\lambda^{N_s}(A'_s)}{N_s!} e^{-\lambda(A'_s)} = \sum_{N_1,\ldots,N_m} \prod_{j=1}^{m} (1 + b_j)^{N_j} \frac{\lambda^{N_j}(A'_j)}{N_j!} e^{-\lambda(A'_j)}$$
$$= \sum_{N_1,\ldots,N_m} \prod_{j=1}^{m} \frac{(\lambda(A'_j) + \lambda(A'_j)b_j)}{N_j!} e^{-\lambda(A'_j)},$$

where $N_1, \ldots, N_m$ run independently of each other through the non-negative integers. Thus the joint characteristic function of the random variables $\eta(D'_{jl})$ $(j = 1, \ldots, m;\ l = 1, \ldots, r)$ has the form

$$(36)\qquad \prod_{j=1}^{m}\sum_{N_j=0}^{\infty} \frac{(\lambda(A'_j) + \lambda(A'_j)b_j)^{N_j}}{N_j!} e^{-\lambda(A'_j)} = \prod_{j=1}^{m} e^{\lambda(A'_j)+\lambda(A'_j)b_j} e^{-\lambda(A'_j)} = \prod_{j=1}^{m} e^{\lambda(A'_j)b_j}.$$

Taking into account (34), we get finally

$$(37)\qquad \mathbf{M}\left(e^{i\sum_{j=1}^{m}\sum_{l=1}^{r} u_{jl}\eta(D'_{jl})}\right) = \prod_{j=1}^{m}\prod_{l=1}^{r} \exp\left\{\int_{A'_j} \mu(C'_l, t)\lambda(\mathrm{d}t)\right\}(e^{iu_{jl}} - 1)$$

and this completes the first step of the proof.

Now, let us consider the random variables $\eta(D)(D \in \mathcal{S}_z)$. We have proved that $\eta(D)$ is a completely additive stochastic set function on the ring of sets the elements of which are finite sums of sets belonging to $\mathcal{T}$. (Cf. [8], p. 215–216). Let $\mathcal{R}(\mathcal{T}_0)$ denote this ring and $\mathcal{S}(\mathcal{R}(\mathcal{T}_0))$ the smallest σ-ring containing $\mathcal{R}(\mathcal{T}_0)$. Clearly $\mathcal{S}_Z = \mathcal{S}(\mathcal{R}(\mathcal{T}_0))$. Since $\eta(D) \geq 0$ for $D \in \mathcal{R}(\mathcal{T}_0)$, by Theorem 3.14 of [8] there is one and only one completely additive stochastic set function $\eta^*(D)(D \in \mathcal{S}_Z)$ for which

$$\mathbf{P}(\eta^*(D) = \eta(D)) = 1 \text{ if } D \in \mathcal{R}(\mathcal{T}_0).$$

We prove that $\mathbf{P}(\eta(D) = \eta^*(D)) = 1$ for $D \in \mathcal{S}_Z$. Let $\mathbf{M}$ denote the set of those D's for which $\mathbf{P}(\eta(D) = \eta^*(D)) = 1$ and let $D_1, D_2, \ldots$ be a monotone sequence of sets of $\mathfrak{M}$ with $\lim_{n\to\infty} D_n = D$. η^* is a completely additive stochastic set function on $\mathcal{S}_Z$, hence

$$\mathbf{P}\left(\lim_{n\to\infty} \eta^*(D_n) = \eta^*(D)\right) = 1.$$

On the other hand, $\eta(D)$ denotes the number of random points lying in the set D, hence

$$\mathbf{P}\left(\lim_{n\to\infty}\eta(D_n)=\eta(D)\right)=1.$$

Thus

$$\mathbf{P}(\eta^*(D)=\eta(D))=1.$$

This means that $\mathfrak{M}$ is a monotone class of sets (cf. [3] , p. 27). Since $\mathcal{R}(\mathcal{T}_0)\subseteq\mathfrak{M}$, by Theorem B of [3], p. 27, $\mathfrak{M}=\mathcal{S}(\mathcal{R}(\mathcal{T}_0))$, what was to be proved. Thus $\eta(D)(D\in\mathcal{S}_z)$ is a completely additive stochastic set function.

Let us introduce the notation

$$\nu(D)=\int_A \mu(C,t)\lambda(\mathrm{d}t)\qquad (A\in\mathcal{S}_T,\ C\in\mathcal{S}_Y). \tag{38}$$

We have shown that if $D'\in\mathcal{R}(\mathcal{T}_0)$, then $\eta(D')$ has a Poisson distribution and

$$\mathbf{M}(\eta(D'))=\nu(D').$$

Let $D_1,D_2,\dots$ be a sequence of disjoint sets of $\mathcal{R}(\mathcal{T}_0)$ with $D=\sum_{k=1}^{\infty}D_k\in\mathcal{R}(\mathcal{T}_0)$. Since $\eta(D_k)\geq 0$ $(k=1,2,\dots)$ and

$$\mathbf{P}\left(\eta(D)=\sum_{k=1}^{\infty}\eta(D_k)\right)=1,$$

it follows that

$$\nu(D)=\mathbf{M}(\eta(D))=\sum_{k=1}^{\infty}\mathbf{M}(\eta(D_k))=\sum_{k=1}^{\infty}\nu(D_k).$$

Thus $\nu(D)$ is a measure on $\mathcal{R}(\mathcal{T}_0)$. If ν^* denotes the extended measure of ν from $\mathcal{R}(\mathcal{T}_0)$ to $\mathcal{S}_Z$, then

$$\mathbf{M}(\eta(D))=\nu^*(D)\text{ for }D\in\mathcal{S}_Z. \tag{39}$$

It is sufficient to show that $\mathbf{M}(\eta(D))$ is a measure on $\mathcal{S}_Z$. But if $D_1,D_2,\dots$ is a sequence of disjoint sets of $\mathcal{S}_Z$, then

$$\mathbf{P}\left(\eta(D)=\sum_{k=1}^{\infty}\eta(D_k)\right)=1\qquad\left(D=\sum_{k=1}^{\infty}D_k\right),$$

hence ($\mathbf{M}(\eta(D))<\infty$, $\eta(D_k)\geq 0$, $k=1,2,\dots$)

$$\mathbf{M}(\eta(D))=\sum_{k=1}^{\infty}\mathbf{M}(\eta(D_k)).$$

this completes the proof of Theorem 2. □

Remark. If the probability distributions $\mu(C,t)(C \in \mathcal{S}_Y,\ t \in T)$ do not depend on t, i.e. if the secondary happenings have the same distributions independently of their starting points, then the random variable $\eta(D)(D \in \mathcal{S}_Z)$ has a Poisson distribution with the parameter $\nu^*(D)$ where ν^* is the product measure

$$\nu^* = \mu \times \lambda. \tag{40}$$

In many of the practical applications occurs this simple case.

The case when the random point distribution in T is σ-finite, can be treated quite analogously. In this case Ω and Ω_1 are the ensembles $\{(t_1, t_2, \ldots)\}$ and $\{((t_1, y_1)(t_2, y_2), \ldots)\}$, respectively. According to Condition 3° in Definition 5 for every $B \in \mathcal{S}_T$ there is a sequence of disjoint sets $B_1, B_2, \ldots$ $(B_k \in \mathcal{S}_T,\ k = 1, 2, \ldots)$ such that $\sum_{k=1}^{\infty} B_k = B$ and $\lambda(B_k) < \infty$ $(k = 1, 2, \ldots)$. Thus every problem can be solved first inside of the sets $B_1, B_2, \ldots$ and the formulate the general assertion. Hence we obtain

Theorem 3. *Let us suppose that in the space T there is a σ-finite random point distribution of Poisson type with*

$$\mathbf{M}(\xi(A)) = \lambda(A) \qquad (A \in \mathcal{S}_T)$$

where $\xi(A)$ denotes the number of random points being in the set A.

Suppose furthermore that for every $B \in \mathcal{S}_T$ with $\lambda(B) < \infty$, the corresponding random point distribution (i.e. when T is replaced by B) satisfies the conditions of Theorem 2.

In this case the random point distribution given in the space $Z = T \times Y$ is σ-finite and if Poisson-type. Finally, if $\eta(D)$ is the number of random points being in the set $D \in \mathcal{S}_Z$, then

$$\mathbf{M}(\eta(D)) = \nu^*(D), \tag{41}$$

where

$$\nu^*(D') = \int_A \mu(C,t)\lambda(\mathrm{d}t) \quad \textit{for} \quad D' = A \times C(A \in \mathcal{S}_T,\ C \in \mathcal{S}_Y). \tag{42}$$

The proof can be carried out simply by reducing it to Theorem 2 and will be omitted.

§ 3. Examples

1. A telephone-problem. Let us suppose that the calls arrive at the telephone centre according to a Poisson process and denote ξ_t the number of calls arrived during the time interval $(0, t)$. We restrict ourselves to a finite time interval $0 \le t \le t_0$ where $t_0 > 0$. Suppose that each call is followed by a conversation the duration of which is also a random variable. If a conversation began at time t, then the probability distribution of its duration will be denoted by $F(x, t)$.

In this case the space Y is the half real axis $[0, \infty)$ and if $J = [a, b)$ is an interval where $0 \le a < b < \infty$, then

$$\mu(J, t) = F(b, t) - F(a, t). \tag{43}$$

The product space $T \times Y$ is the following stripe of the plane

$$\{(t, y) \,:\, 0 \leq t \leq t_0,\ 0 \leq y\}.$$

We can imagine the situation so that at every point of the calls we draw a vertical line and measure on it the duration of the corresponding conversation. If $\xi(A)$ is the number of calls arriving in the set $A \subseteq [0, t_0]$ where A is a Borel-set, then we suppose $\xi(A)$ to be a completely additive stochastic set function defined on the Borel-sets of the interval $[0, t_0]$. Thus if $\mu(C, t)$ is defined on the Borel-sets of the axis $[0, \infty)$, then in the plane to every Borel set $D \subseteq T \times Y$ there corresponds a random variable $\eta(D)$.

According to Theorem 2, $\eta(D)(D \in \mathcal{S}_Z)$ has a Poisson distribution, and if $D = J \times I$ where

$$J = [a, b),\ 0 \leq a < b < t_0,\ I = [c, d),\ 0 \leq c < d < \infty,$$

then

$$\mathbf{M}(\eta(D)) = \int_I \mu(J, t)\lambda(\mathrm{d}t) = \int_I (F(b, t) - F(a, t))\lambda(\mathrm{d}t).$$

Let ζ_t denote the number of conversations going on at time t $(t \leq t_0)$. Then clearly $\zeta_t = \eta(D)$, where D denotes the closed set straffed in Figure 1. (This was the idea of C. Ryll–Nardzewski). Thus by Theorem 2, $\zeta_t = \eta(D)$ has a Poisson distribution with

$$\mathbf{M}(\zeta_t) = \int_{[0,t]} (1 - F(t - \tau, \tau))\lambda(\mathrm{d}\tau).$$

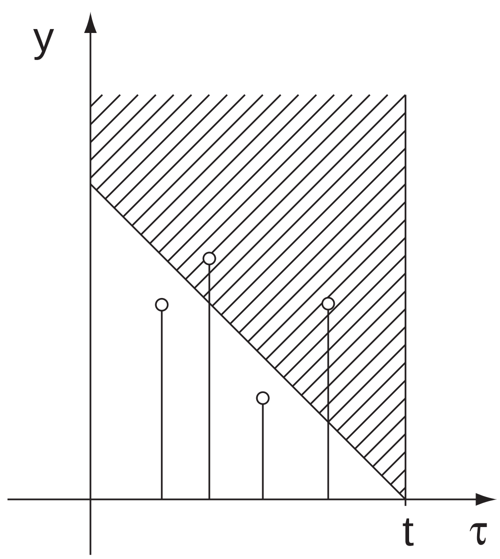

Figure 1:

If $F(x, t)$ is independent of t, then

$$\mathbf{M}(\zeta_t) = \int_{[0,t]} (1 - F(t - \tau))\lambda(\mathrm{d}\tau). \tag{44}$$

In particular if $F(x,t) = 1 - e^{-0x}$, then

$$\mathbf{M}(\zeta_t) = e^{-ct} \int_{[0,t]} e^{c\tau} \lambda(\mathrm{d}\tau). \tag{45}$$

Finally, if ξ_t is a homogeneous process with $\mathbf{M}(\xi_t) = \lambda t$, then

$$\mathbf{M}(\zeta_t) = \lambda e^{-ct} \int_0^t e^{c\tau}\,\mathrm{d}\tau = \frac{\lambda}{c}(1 - e^{-ct}). \tag{46}$$

The case when $t > t_0$, can be treated similarly.

If ζ_t is defined for all $t(-\infty < t < \infty)$, then we can apply Theorem 3 and in the homogeneous case (46) becomes λ/c.

With the aid of our model it is possible to calculate the correlation coefficients $R(\zeta_t, \zeta_s)$ and solve other problems, but we will not enter into the details.

Formulae (46) and (44) (the letter with a continuous $\lambda(t)$) were published in [10] and [11], respectively. Formula (46) can be derived from the results of [5] and [7] where the problem is solved with other assumptions.

Example 2. In the phototechnique an interesting problem is to calculate the transparentness of a film darked by a finite number of emulsion spheres distributed at random in the film.

Following B. Picinbono we make for the problem a plane model and imagine that the emulsions are circles the centres of which are distributed according to a random point distribution of Poisson type.

For simplicity we suppose the film to be infinite large so that it covers the whole plane $\{(x_1, x_2)\} = \{(t)\}$. Each emulsion radius is supposed to be a random variable having a probability distribution $F(x,t) = F(x; x_1, x_2)$, provided that the centre is at the point $t = (x_1, x_2)$.

In this case $T = \{t = (x_1, x_2) : -\infty < x_1 < \infty,\ -\infty < x_2 < \infty\}$, $Y = \{y : 0 \le y\}$ and $T \times Y$ is the upper half of the three-dimensional Euclidean space $\{(x_1, x_2, y)\}$. $\mathcal{S}_T$ is the system of Borel sets of T (including the sets with infinite measure) $\xi(A)(A \in \mathcal{S}_T)$ is the number of centres being in the set A. $\xi(A)(A \in \mathcal{S}_T)$ is supposed to be a completely additive stochastic set function, the random variables $\xi(A)$ of which have Poisson distributions with

$$\mathbf{M}(\xi(A)) = \lambda(A) \qquad (A \in \mathcal{S}_T).$$

In other terms, the emulsion centres form a σ-finite random point distribution of Poisson type. $\mathcal{S}_Y$ is the system of Borel sets of Y and $\mathcal{S}_Z$ is the system of Borel sets of $Z = T \times Y$.

Let $\zeta_t = \zeta(x_1, x_2)$ denote the number of emulsion circles covering their point t. This equals the number of random points lying in the reversed straight cone the apex of which is at $t = (x_1, x_2)$ and has an angle 90° (Fig. 2).

Thus if D denotes this closed cone, then $\zeta_t = \eta(D)$ and according to Theorem 3

$$\mathbf{M}(\zeta_t) = \mathbf{M}(\eta(D)) = \int_0^\infty \int_{(\mu - x_1)^2 + (v - x_2)^2 \le x^2} (1 - F(x, u - x_1, v - x_2))\lambda(\mathrm{d}u, \mathrm{d}v)\,\mathrm{d}x. \tag{47}$$

If $F(x, x_1, x_2)$ is independent of (x_1, x_2) and has a density function $f(x)$, then (47) reduces to

$$\mathbf{M}(\zeta_t) = \int_0^\infty f(x)\lambda(A_x)\,\mathrm{d}x, \tag{48}$$

where $A_x = \{(u, v) \,:\, (u - x_1^2) + (v - x_2)^2 \leq x^2\}$.

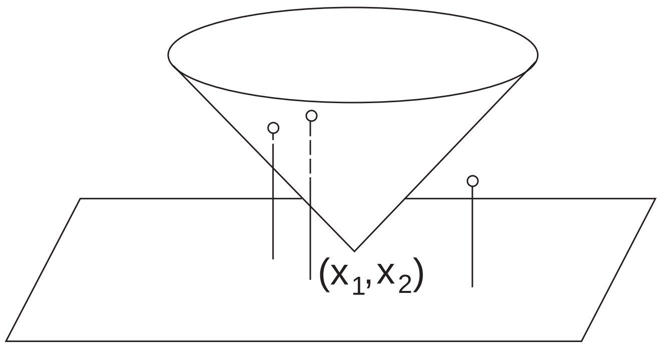

Figure 2:

If we have on the plane a homogeneous random point distribution, then on the whole plane there is an infinite number of random points with probability 1. In this case

$$\mathbf{M}(\xi(A)) = \lambda|A| \qquad (A \in \mathcal{S}),$$

where $|A|$ is the Lebesgue-measure of the set A. Hence (48) will be equal to

$$\mathbf{M}(\zeta_t) = \lambda\pi \int_0^\infty x^2 f(x)\,\mathrm{d}x. \tag{49}$$

In the practically interesting cases we suppose that the integral in (49) is finite. Formula (49) was published by B. Picinbono [6].

It is possible to solve also the problem for the case of a finite film and calculate other quantities so as the correlation coefficients $R(\zeta_{t_1}, \zeta_{t_2})$ etc.

Particular interesting has the random variable

$$\zeta_t' = \begin{cases} 1 & \text{if} \quad \zeta_t = 0, \\ 0 & \text{if} \quad \zeta_t \neq 0. \end{cases}$$

If we consider the case of a finite film and a homogeneous random point distribution, then the value of the stochastic integral

$$\chi = \int_0^{M_1}\int_0^{M_2} \zeta'_{x_1,x_2}\,\mathrm{d}x_1\,\mathrm{d}x_2 = \lim_{\substack{\max \Delta x_1^{(k)} \to 0 \\ \max \Delta x_2^{(l)} \to 0}} \sum_k \sum_l \zeta_{x_1^{(k)}, x_2^{(l)}} \Delta x_1^{(k)} \Delta x_2^{(l)}$$

is the transparence of the film covering the rectangle $\{(x_1, x_2) \,:\, 0 \le x_1 \le M_1,\, 0 \le x_2 \le M_2\}$. Since $0 \le \xi'_{x_1,x_2} \le 1$, we have

$$M(\chi) = \int_0^{M_1} \int_0^{M_2} M(\zeta'_{x_1,x_2})\,\mathrm{d}x_1\,\mathrm{d}x_2 = \int_0^{M_1} \int_0^{M_2} P(\zeta_{x_1,x_2} = 0)\,\mathrm{d}x_1\,\mathrm{d}x_2.$$

An analogous formula can be derived for the case of an infinite film.

Example 3. Let us consider a σ-finite random point distribution of Poisson type in the n-dimensional Euclidean space T. Let $\mathcal{S}_T$ be the system of Borel sets of T and $\xi(A)(A \in \mathcal{S}_T)$ the number of random points being in the set A with $\mathbf{M}(\xi(A)) = \lambda(A)$.

In many practical problems occurs the situation that the random points distributed in the space T wander in the time, and the curves belonging to different random points are independent of each other.

In our model the phenomenon can be described as follows. Let $Y = \{\varphi_1(x), \dots, \varphi_n(x)\}$ (x denotes the time parameter) be a function vector space, where the functions φ_k are defined on the half time axis $0 \le x < \infty$ and denote $\mathcal{S}_Y$ a σ-algebra of sets of Y. We suppose that every set $\{(\varphi_1(x), \dots, \varphi_n(x)) \,:\, a_k \le \varphi_k(x_0) \le b_k \; k = 1, \dots, n\}$ is an element of $\mathcal{S}_Y$ where $[a_k, b_k]$ are arbitrary intervals and $0 \le x_0 < \infty$. Denote $\mu(C, t)(C \in \mathcal{S}_Y,\ t \in T)$ a system of measures depending on the parameter $t = (x_1, \dots, x_n)$. $\mu(C, t)$ is the probability distribution of the path of a random point if its original place was at t.

Suppose that our condition α) holds in the product space $Z = T \times Y$. We can then solve different problems. The first is the following: what is the probability that at time x there are exactly k points in a set $A \in \mathcal{S}_T$.

This event can completely be characterized so that

$$\eta(D) = k,$$

where $D \le \mathcal{S}_Z$ is the set of those elements $(x_1, \dots, x_n,\ \varphi_1(x), \dots, \varphi_n(x))$ of $T \times Y$ for which $(\varphi_1(x) - x_1, \dots, \varphi_n(x) - x_n) \in A$. Now, the solution is that $\eta(D)$ has a Poisson distribution with the parameter

$$\mathbf{M}(\eta(D)) = \int_T \mu(C_x - t, t)\lambda(\mathrm{d}t), \tag{50}$$

where $C_x - t$ denotes the set of those elements $y = (\varphi_1(x), \dots, \varphi_n(x))$ for which $(\varphi_1(x) - x_1, \dots, \varphi_n(x) - x_n) \in A$.

If $\xi(A)$ is a homogeneous point distribution with $\mathbf{M}(\xi(A)) = \lambda|A|$ where $|A|$ is the Lebesgue-measure of the set A, $\mu(C, t)$ is independent of t for every $C \in \mathcal{S}_Y$ and $\xi_X(A)$ is the number of the particles in the set A at time x, then

$$\mathbf{M}(\xi_x(A)) = \lambda \int_T \mu(C_x - t)\,\mathrm{d}t. \tag{51}$$

The integral in (51) is an n-dimensional Lebesgue-integral. If $n = 1$, then we obtain as a special case the result of Doob (cf. [2] pp.).

With the aid of our method it is possible also to solve and generalize the problems considered by L. Takács [12] , [13]. We can thus obtain similar results but as the practically interesting formulae are given in [12] and [13], we do not enter into the details.

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