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THE EQUATIONAL DEFINABILITY OF TRUTH PREDICATES

In memory of Willem Blok

A b s t r a c t. Let $\mathcal{S}$ be a structural consequence relation, not assumed to be protoalgebraic. It is proved that the following conditions on $\mathcal{S}$ are equivalent, where 'algebra' means algebra in the signature of $\mathcal{S}$: (1) $\mathcal{S}$ is *truth-equational*, i.e., the truth predicate of the class of reduced matrix models of $\mathcal{S}$ is explicitly definable by some fixed set of unary *equations*. (2) The Leibniz operator Ω of $\mathcal{S}$ is *completely order reflecting* on all algebras, i.e., for any set of $\mathcal{S}$–filters $\mathcal{F} \cup \{G\}$ of an algebra, if $\bigcap \Omega[\mathcal{F}] \subseteq \Omega G$ then $\bigcap \mathcal{F} \subseteq G$. (3) The Leibniz operator is completely order reflecting on the *theories* of $\mathcal{S}$. (4) The Suszko operator of $\mathcal{S}$ is injective on all algebras.
It makes no difference to the meaning of (1) whether 'reduced' is interpreted as Leibniz-reduced or as Suszko-reduced. For the class of Suszko-reduced matrix models of $\mathcal{S}$, (4) $\Rightarrow$ (1) says that the *implicit* definability of the truth predicate entails its *equational* definability. Previously, this was known only for protoalgebraic systems. The corresponding assertion for the Leibniz-reduced models is shown to be false, i.e., global injectivity of the Leibniz operator does not entail truth-equationality.

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1 Introduction

By a *deductive system* we shall mean any structural consequence relation over an algebraic language. It is well known that every deductive system $\mathcal{S}$ possesses a *matrix semantics* consisting of 'reduced matrix models'. Here, a *matrix model* of $\mathcal{S}$ is a structure $\langle \mathbf{A}, F\rangle$ where

- $\mathbf{A}$ is an algebra in the signature of $\mathcal{S}$ and
- F is an $\mathcal{S}$*–filter* of $\mathbf{A}$, i.e., a subset closed under all derivable rules of $\mathcal{S}$ (and in particular containing all $\mathbf{A}$–instances of theorems of $\mathcal{S}$).

Such a structure is said to be (Leibniz-) *reduced* provided that

- no nontrivial congruence of $\mathbf{A}$ makes F a union of congruence classes.

The class of all reduced matrix models of $\mathcal{S}$ is denoted by $\mathsf{Mod}^*\mathcal{S}$, and every deductive system $\mathcal{S}$ is strongly sound and complete, in a natural sense, with respect to this class. The semantics $\mathsf{Mod}^*\mathcal{S}$ becomes more tangible when $\mathcal{S}$ is an *algebraizable logic* in the sense of [9] or [20], [26]. For example, the reduced matrix models of classical or intuitionistic propositional logic are just all pairs $\langle \mathbf{A}, \{\top\}\rangle$ where $\mathbf{A}$ is a Boolean or Heyting algebra, respectively, whose top element is $\top$. For the weaker 'substructural' logics $\mathcal{S}$, $\mathsf{Mod}^*\mathcal{S}$ consists of all pairs $\langle \mathbf{A}, \{a \in A : a \to a \leq a\}\rangle$ where $\mathbf{A}$ belongs to a fixed variety of residuated lattice-based algebras determined by $\mathcal{S}$.

As these examples suggest, every algebraizable logic $\mathcal{S}$ is completely interchangeable with an essentially *unique* class K of pure *algebras*, and the objects in K are exactly the algebra reducts of the reduced matrix models of $\mathcal{S}$. Moreover, each reduced matrix model is *determined* by its algebra reduct, i.e., when $\langle \mathbf{A}, F\rangle$ and $\langle \mathbf{A}, G\rangle$ are reduced matrix models of $\mathcal{S}$ then $F = G$. We express this by saying that *the truth predicate of* $\mathsf{Mod}^*\mathcal{S}$ *is implicitly definable* (when $\mathcal{S}$ is algebraizable).

Actually, the truth predicate of $\mathsf{Mod}^*\mathcal{S}$ is *equationally definable* whenever $\mathcal{S}$ is algebraizable. By this we mean that

(∗) there exists a set $\boldsymbol{\tau}$ of formal unary equations $\delta(x) \approx \varepsilon(x)$ such that for every $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^*\mathcal{S}$ and every $a \in A$, we have:
$$a \in F \;\text{ iff }\; [\,\delta^{\mathbf{A}}(a) = \varepsilon^{\mathbf{A}}(a) \text{ for all } \delta \approx \varepsilon \in \boldsymbol{\tau}\,].$$

In classical or intuitionistic logic this takes the form

$$a \in F \text{ iff } a = \top.$$

In substructural logics, e.g., linear or relevance logic, it becomes

$$a \in F \text{ iff } a = a \vee (a \to a).$$

We shall say that a deductive system $\mathcal{S}$ is *truth-equational* if $(*)$ is true. The exact relationship between this notion and algebraizability is as follows:

Algebraizable = *Truth-equational* + *Equivalential* (see [28], [20]).

The *equivalential* deductive systems are very well understood and we postpone further discussion of them until Section 11. The aim of this paper is to isolate the meaning of *truth-equationality* for *arbitrary* deductive systems. It is strictly intermediate between algebraizability and the much weaker demand of possessing an algebraic semantics. The truth-equational systems encompass the *weakly algebraizable logics* of [22], as well as the assertional logics of all pointed quasivarieties, and they must possess some *theorems* (as opposed to proper derivable rules). Further examples will be identified in Sections 12–14. But until now, no readily falsifiable characterization of truth-equationality was known.

In this paper we shall show that the truth-equational deductive systems are characterized intrinsically by an intelligible property of the 'Leibniz operator'. For a deductive system $\mathcal{S}$ and an algebra $\mathbf{A}$ of the same signature, the *Leibniz operator* of $\mathcal{S}$ associates with each $\mathcal{S}$–filter F of $\mathbf{A}$ the *largest* congruence θ of $\mathbf{A}$ for which F is a union of θ–classes. This congruence always exists; we denote it by $\Omega^{\mathbf{A}}F$ and call it the *Leibniz congruence* of F. The degree to which a deductive system admits algebraic treatment is known to correlate closely with transparent properties of its Leibniz operator: see [9], [20], [26].

The main result of this paper (Theorem 28) states that a deductive system $\mathcal{S}$ is truth-equational if and only if its Leibniz operator is *completely order reflecting* on the $\mathcal{S}$–filters of all algebras, i.e., whenever $\mathcal{F} \cup \{G\}$ is a set of $\mathcal{S}$–filters of an algebra $\mathbf{A}$ in the signature of $\mathcal{S}$, if $\bigcap_{F \in \mathcal{F}} \Omega^{\mathbf{A}}F \subseteq \Omega^{\mathbf{A}}G$ then $\bigcap \mathcal{F} \subseteq G$. This condition is readily falsifiable, as it makes no mention of any set of equations $\boldsymbol{\tau}$ or any special class of algebras. To verify that $\mathcal{S}$

is truth-equational, it proves sufficient to check that the Leibniz operator is completely order reflecting on the *theories* of $\mathcal{S}$, i.e., on the $\mathcal{S}$–filters of the absolutely free algebra generated by the variables of the language.

The *protoalgebraic logics* are the deductive systems whose Leibniz operators are order *preserving* on the filters of all algebras. Roughly speaking, they are the logics in which *implication* can be represented adequately by some family of connectives. Some significant truth-equational deductive systems fail to be protoalgebraic: see Section 13. Several authors have suggested that the classification of non-protoalgebraic logics may need to be based on properties of the 'Suszko operator', rather than the Leibniz operator.

The *Suszko operator* of $\mathcal{S}$ maps each $\mathcal{S}$–filter F of an algebra $\mathbf{A}$ to the intersection of the Leibniz congruences of all $\mathcal{S}$–filters *containing* F. If this intersection is the identity relation, the matrix $\langle \mathbf{A}, F \rangle$ is said to be *Suszko-reduced*. For protoalgebraic logics, the Leibniz and Suszko operators coincide. When $\mathcal{S}$ is not protoalgebraic, its *Suszko* operator still preserves order, and the role of $\mathsf{Mod}^*\mathcal{S}$ is to some extent taken over by a semantics $\mathsf{Mod}^{\mathrm{Su}}\mathcal{S}$, consisting of all Suszko-reduced matrix models. But the meaning of '$\mathcal{S}$ is truth-equational' is unaffected by this move, because $\mathsf{Mod}^{\mathrm{Su}}\mathcal{S}$ is known to be just the closure of $\mathsf{Mod}^*\mathcal{S}$ under subdirect products and isomorphisms [21], [25].

Theorem 28 also shows that a deductive system $\mathcal{S}$ is truth-equational if (and only if) its Suszko operator is *globally injective*, i.e., injective on the $\mathcal{S}$–filters of all algebras in the signature. It is *not* sufficient that this operator be injective on the *theories* of $\mathcal{S}$ (Example 1). The result can be rephrased as follows: if the *truth predicate* of $\mathsf{Mod}^{\mathrm{Su}}\mathcal{S}$ is *implicitly* definable then it is *equationally* definable, i.e., explicitly definable by a (possibly infinite) conjunction of equations. This conclusion is stronger than any general variant of Beth's Definability Theorem could have delivered. It completes a sequence of less general definability theorems for truth predicates, established by Blok and Pigozzi [9], by Herrmann [29], and by Czelakowski and Jansana [22]. Each of these earlier theorems extended the scope of the previous one, but all of them assumed protoalgebraicity and their proofs made definite use of this assumption.

All of this prompts the question: is it sufficient for truth-equationality that truth be implicitly definable in $\mathsf{Mod}^*\mathcal{S}$? Equivalently, does global injectivity of the *Leibniz* operator guarantee truth-equationality? For protoalgebraic systems, this was confirmed in [22]. We confirm it for certain classes of non-protoalgebraic logics also, e.g., Theorem 46. But in general, *a deductive system with a globally injective Leibniz operator need not be truth-equational*; it need not even possess an algebraic semantics. This is shown in Examples 2 and 3.

2 Algebras

Unless we say otherwise, $\boldsymbol{\mathcal{L}}$ shall denote a given but arbitrary algebraic language, i.e., a signature $\mathcal{L}$ together with an infinite set of variables, Var. For convenience, we fix one variable in Var and denote it by x. The symbols in the signature $\mathcal{L}$ are called the *basic operation symbols* or *connectives* of $\boldsymbol{\mathcal{L}}$, depending on the algebraic or logical context in which they are mentioned. Each has finite rank (alias 'arity'). The $\boldsymbol{\mathcal{L}}$*–constants* are, as usual, the symbols (if any) of $\mathcal{L}$ that have rank 0.

The models of the signature, i.e., the $\mathcal{L}$*–algebras*, shall be denoted by boldface capitals $\mathbf{A}, \mathbf{B}, \ldots$, and their respective universes by $A, B, \ldots$. As usual, the universe of an algebra is assumed to be non-empty. We use $\mathbf{Te}$ to symbolize the absolutely free $\mathcal{L}$–algebra generated freely by Var. The elements of Te are therefore just the $\boldsymbol{\mathcal{L}}$*–terms*; in logical contexts we sometimes call them the '$\boldsymbol{\mathcal{L}}$*–formulas*'. Endomorphisms of $\mathbf{Te}$ are called *substitutions*. Of course, these are determined (and often specified) by their restrictions to Var. We use $\mathbf{Te}(x)$ to stand for the subalgebra of $\mathbf{Te}$ generated by $\{x\}$.

The notation $\alpha = \alpha(x_1, \ldots, x_n) \in \mathrm{Te}$ shall mean that α is an $\boldsymbol{\mathcal{L}}$–term whose apparent variables are among the distinct variables $x_1, \ldots, x_n \in Var$; in this case we call α an *n–ary* term. Thus, every 0–ary term is an $\boldsymbol{\mathcal{L}}$–constant. The term function of an $\mathcal{L}$–algebra $\mathbf{A}$ induced by an $\boldsymbol{\mathcal{L}}$–term $\alpha \in \mathrm{Te}$ shall be denoted as usual by $\alpha^{\mathbf{A}}$. We omit the superscript when $\mathbf{A} = \mathbf{Te}$.

By a *unary polynomial function* of an $\mathcal{L}$–algebra $\mathbf{A}$ we shall mean a function $p : A \to A$ for which the following is true: there exist a nonnegative integer n, an $\boldsymbol{\mathcal{L}}$–term $\alpha = \alpha(x_1, \ldots, x_n) \in \mathrm{Te}$, and elements $b_2, \ldots, b_n$ of A such that for all $a \in A$, we have $p(a) = \alpha^{\mathbf{A}}(a, b_2, \ldots, b_n)$.

An expression of the form '$\vec{e} \in A$' shall always signify that $\vec{e}$ is some (finite or infinite) sequence of elements of the set A. The identity relation on a set A will be denoted by id_A. For any binary relation θ on A and any

$a \in A$, we use a/θ to denote the 'θ–class' $\{b \in A : \langle a, b\rangle \in \theta\}$.

When discussing classes of algebras, we make standard use of the class operator symbols $\mathrm{I}, \mathrm{H}, \mathrm{S}, \mathrm{P}, \mathrm{P_S}$ and $\mathrm{P_U}$. These stand, respectively, for the formation of isomorphic and homomorphic images, subalgebras, direct and subdirect products, and ultraproducts. In addition, the class operator $\mathrm{U} = \mathrm{U}_{Var}$ is defined by

$$\mathrm{U}(\mathsf{K}) = \{\mathbf{A} : \text{every subalgebra of } \mathbf{A} \text{ on } \leq |Var| \text{ generators belongs to } \mathsf{K}\}.$$

A class of $\mathcal{L}$–algebras is called an *ISP–class* if it is closed under I, S and P. An ISP–class is called a *UISP–class* if it closed under U; a *quasivariety* if it closed under $\mathrm{P_U}$; and a *variety* if it is closed under H. For any class K of $\mathcal{L}$–algebras,

$$\mathrm{ISP}(\mathsf{K}) \subseteq \mathrm{UISP}(\mathsf{K}) \subseteq \mathrm{ISPP_U}(\mathsf{K}) \subseteq \mathrm{HSP}(\mathsf{K})$$

and these are, respectively, the smallest ISP–class, the smallest UISP–class, the smallest quasivariety and the smallest variety containing K. We abbreviate $\mathrm{HSP}(\mathsf{K})$ as $\mathrm{V}(\mathsf{K})$, and $\mathrm{V}(\{\mathbf{A}\})$ as $\mathrm{V}(\mathbf{A})$, etc.

It is well known that varieties are exactly the model classes of sets of equations, and quasivarieties are the model classes of sets of finite quasi-identities. The operator U is significant because UISP–classes are precisely the model classes of sets of *infinitary* quasi-identities over Var (see for instance [6, Lec. 2] or [7]).

The congruence lattice of an $\mathcal{L}$–algebra $\mathbf{A}$ is denoted by $\mathbf{Con}\,\mathbf{A}$. For any class K of $\mathcal{L}$–algebras, a congruence θ of $\mathbf{A}$ will be called a K*–congruence* of $\mathbf{A}$ if $\mathbf{A}/\theta \in \mathsf{K}$. When K is closed under subdirect products and isomorphisms, the set of K–congruences of $\mathbf{A}$ is a complete lattice, and it will be denoted by $\mathbf{Con}_{\mathsf{K}}\,\mathbf{A}$. (Note that the empty set of K–congruences of $\mathbf{A}$ has the total congruence $A \times A$ as its intersection, and the trivial factor algebra belongs to $\mathrm{IP_S}(\emptyset)$, which is contained in $\mathrm{IP_S}(\mathsf{K}) = \mathsf{K}$.) The least K–congruence containing a subset B of $\mathbf{A}$ is denoted by $\Theta_{\mathsf{K}}^{\mathbf{A}} B$.

Suppose θ is a congruence of $\mathbf{A}$, and let $h : \mathbf{A} \to \mathbf{B} := \mathbf{A}/\theta$ be the canonical surjective homomorphism. We say that θ is *compatible with* a subset Y of A provided that Y is a union of congruence classes of θ, i.e., whenever $\langle a, b\rangle \in \theta$ and $a \in Y$ then $b \in Y$. This means just that $Y = h^{-1}[h[Y]]$. When it is clear that θ is compatible with Y, we may use the abbreviation

$$Y/\theta := \{y/\theta : y \in Y\}$$

in lieu of $h[Y]$, without risk of notational confusion. For in this case, if $y/\theta \in Y/\theta$ then $y \in Y$, by compatibility.

3 Deductive Systems

Unless we say otherwise, $\mathcal{S}$ shall denote a given but arbitrary *deductive system* over $\boldsymbol{\mathcal{L}}$, by which we mean a *structural* consequence relation in the sense of [9] or [51]. In other words, $\mathcal{S}$ is a relation from sets of $\boldsymbol{\mathcal{L}}$–terms to $\boldsymbol{\mathcal{L}}$–terms satisfying the three conditions set out below, for any set $\Gamma \cup \Phi \cup \{\alpha\}$ of $\boldsymbol{\mathcal{L}}$–terms. Here the expression $\Gamma \vdash_{\mathcal{S}} \alpha$ abbreviates that $\langle \Gamma, \alpha \rangle \in \mathcal{S}$.

- $\alpha \in \Gamma$ implies $\Gamma \vdash_{\mathcal{S}} \alpha$;
- $\Gamma \vdash_{\mathcal{S}} \alpha$ and $\Phi \vdash_{\mathcal{S}} \gamma$ for all $\gamma \in \Gamma$ implies $\Phi \vdash_{\mathcal{S}} \alpha$;
- (*structurality*) $\Gamma \vdash_{\mathcal{S}} \alpha$ implies $h[\Gamma] \vdash_{\mathcal{S}} h(\alpha)$ for every substitution h.

From these conditions, the next one follows:

- (*monotonicity*) $\Gamma \vdash_{\mathcal{S}} \alpha$ and $\Gamma \subseteq \Phi$ implies $\Phi \vdash_{\mathcal{S}} \alpha$.

We call $\mathcal{S}$ *finitary* if it also satisfies

- (*finitarity*) $\Gamma \vdash_{\mathcal{S}} \alpha$ implies $\Gamma' \vdash_{\mathcal{S}} \alpha$ for some finite $\Gamma' \subseteq \Gamma$.

The finitary deductive systems all arise as the deducibility relations of formal systems and are usually (but not always) specified in this way. A *formal system* over $\boldsymbol{\mathcal{L}}$ means any set $\mathcal{F}$ of expressions '$\Sigma \vdash \beta$' such that $\Sigma \cup \{\beta\}$ is a *finite* set of $\boldsymbol{\mathcal{L}}$–terms. These expressions are called the *postulated rules* of $\mathcal{F}$; the *axioms* of $\mathcal{F}$ are the terms β for which '$\emptyset \vdash \beta$' is a postulated rule of $\mathcal{F}$. The *deducibility relation* of $\mathcal{F}$ is the relation from (possibly infinite) sets of $\boldsymbol{\mathcal{L}}$–terms to single $\boldsymbol{\mathcal{L}}$–terms that contains a pair $\langle \Gamma, \alpha \rangle$ exactly when there is a proof of α from the 'premisses' in Γ that uses only instances of the axioms and postulated rules of $\mathcal{F}$. Here proofs are assumed to have finite length and the sense of 'proof from premisses' is as in classical logic. The deducibility relation of $\mathcal{F}$ is always the smallest finitary deductive system $\mathcal{S}$ containing $\mathcal{F}$ [33]; this system $\mathcal{S}$ is said to be *axiomatized* by $\mathcal{F}$. See [20], [26] for details, history and non-finitary analogies.

For (possibly non-finitary) deductive systems $\mathcal{S}$, we use the following terminology. We call $\Gamma \vdash \alpha$ a *rule*—or sometimes a *derivable rule*—of $\mathcal{S}$ provided that $\langle \Gamma, \alpha \rangle \in \mathcal{S}$. We call α a *theorem* of $\mathcal{S}$ if $\emptyset \vdash_{\mathcal{S}} \alpha$, and we signify this fact briefly by the expression $\vdash_{\mathcal{S}} \alpha$. We call $\mathcal{S}$ *consistent* if not all $\boldsymbol{\mathcal{L}}$–terms are theorems of $\mathcal{S}$. By structurality, this is equivalent to the requirement that $\nvdash_{\mathcal{S}} x$. We tend to abbreviate '$\Gamma \cup \{\beta_1, \ldots, \beta_n\} \vdash_{\mathcal{S}} \alpha$' as $\Gamma, \beta_1, \ldots, \beta_n \vdash_{\mathcal{S}} \alpha$, and '$\Gamma \vdash_{\mathcal{S}} \varphi$ for all $\varphi \in \Phi$' as $\Gamma \vdash_{\mathcal{S}} \Phi$.

By the *fragment* of $\mathcal{S}$ in an indicated subsignature of $\mathcal{L}$ we mean the set of all *rules* of $\mathcal{S}$ that use only connectives from this subsignature. This is clearly still a deductive system, and finitary if $\mathcal{S}$ was.

Given an $\mathcal{L}$–algebra $\mathbf{A}$, the *$\mathcal{S}$–filters* of $\mathbf{A}$ are the subsets F of A that are closed under the rules of $\mathcal{S}$ in the following sense: whenever $\Gamma \vdash_{\mathcal{S}} \alpha$ and $h : \mathbf{Te} \to \mathbf{A}$ is a homomorphism of $\mathcal{L}$–algebras,

$$\text{if} \quad h[\Gamma] \subseteq F \quad \text{then} \quad h(\alpha) \in F.$$

When $\mathcal{S}$ is axiomatized by a formal system $\mathcal{F}$, it is enough to check this closure property for the (axioms and) postulated rules $\Gamma \vdash \alpha$ of $\mathcal{F}$. We shall need to use the following straightforward and well known facts about filters.

Lemma 1. *Let $h : \mathbf{A} \to \mathbf{B}$ a homomorphism of $\boldsymbol{\mathcal{L}}$–algebras.*

(i) If G is an $\mathcal{S}$–filter of $\mathbf{B}$ then $h^{-1}[G]$ is an $\mathcal{S}$–filter of $\mathbf{A}$.

(ii) If h is surjective and its kernel $\ker h$ is compatible with an $\mathcal{S}$–filter F of $\mathbf{A}$ then $h[F]$ is an $\mathcal{S}$–filter of $\mathbf{B}$.[1]

The complete lattice of all $\mathcal{S}$–filters of an $\mathcal{L}$–algebra $\mathbf{A}$ (ordered by set inclusion) shall be denoted by $\mathbf{Fi}_{\mathcal{S}}\,\mathbf{A}$. The $\mathcal{S}$–filter of $\mathbf{A}$ generated by a subset Y of A shall be written as $\mathrm{Fg}^{\mathbf{A}}_{\mathcal{S}}\,Y$. The $\mathcal{S}$–filters of $\mathbf{Te}$ are usually called the *theories* of $\mathcal{S}$, or the *$\mathcal{S}$–theories*. We note that for any subset $\Gamma \cup \{\alpha\}$ of Te,

$$\Gamma \vdash_{\mathcal{S}} \alpha \quad \text{iff} \quad \alpha \in \mathrm{Fg}^{\mathbf{Te}}_{\mathcal{S}}\,\Gamma. \tag{1}$$

It is common (but not universal) practice to define a 'logic' as a deductive system in sense of this section.

4 Translations and Algebraic Semantics

An (*equational*) *$\boldsymbol{\mathcal{L}}$–translation* is a set $\boldsymbol{\tau}$ of pairs $\langle \delta, \varepsilon \rangle$ of unary $\boldsymbol{\mathcal{L}}$–terms in x. The set $\boldsymbol{\tau}$ is not assumed to be finite. The pairs in $\boldsymbol{\tau}$ may be thought

[1]Recall that $\ker h := \{\langle a, a' \rangle \in A \times A : h(a) = h(a')\}$.

of as formal equations $\delta(x) \approx \varepsilon(x)$, and we sometimes write $\boldsymbol{\tau}$ as $\boldsymbol{\tau}(x)$. Moreover, for the sake of readability, we sometimes write '$\boldsymbol{\tau}$ *is* $\delta(x) \approx \varepsilon(x)$' when we mean $\boldsymbol{\tau} = \{\langle \delta, \varepsilon \rangle\}$. The main purpose of a translation is to provide a deductive system with an 'algebraic semantics' (defined below), when this is possible.

For any $\boldsymbol{\mathcal{L}}$–translation $\boldsymbol{\tau}$ and any subset $Y \cup \{b\}$ of an $\mathcal{L}$–algebra $\mathbf{A}$, we employ the abbreviations

$$\boldsymbol{\tau}^{\mathbf{A}}[Y] = \{\langle \delta^{\mathbf{A}}(a), \varepsilon^{\mathbf{A}}(a) \rangle : \langle \delta, \varepsilon \rangle \in \boldsymbol{\tau} \text{ and } a \in Y\}; \quad \boldsymbol{\tau}^{\mathbf{A}}(b) = \boldsymbol{\tau}^{\mathbf{A}}[\{b\}].$$

We omit the superscript in $\boldsymbol{\tau}^{\mathbf{A}}$ when $\mathbf{A} = \mathbf{Te}$.

Definition 1. A class K of $\mathcal{L}$–algebras is called a *$\boldsymbol{\tau}$–algebraic semantics* for $\mathcal{S}$ provided that $\boldsymbol{\tau}$ is an $\boldsymbol{\mathcal{L}}$–translation and for any $\Gamma \cup \{\alpha\} \subseteq \mathrm{Te}$,

$$\Gamma \vdash_{\mathcal{S}} \alpha \quad \text{iff} \quad \boldsymbol{\tau}[\Gamma] \models_{\mathsf{K}} \boldsymbol{\tau}(\alpha). \tag{2}$$

The right hand side of (2) means, as usual, that for any homomorphism h from $\mathbf{Te}$ to any algebra $\mathbf{A} \in \mathsf{K}$, if $\delta^{\mathbf{A}}(h(\gamma)) = \varepsilon^{\mathbf{A}}(h(\gamma))$ for all $\gamma \in \Gamma$ and all $\langle \delta, \varepsilon \rangle \in \boldsymbol{\tau}$ then $\delta^{\mathbf{A}}(h(\alpha)) = \varepsilon^{\mathbf{A}}(h(\alpha))$ for all $\langle \delta, \varepsilon \rangle \in \boldsymbol{\tau}$.
We say that K is an *algebraic semantics* for $\mathcal{S}$ if there exists an $\boldsymbol{\mathcal{L}}$–translation $\boldsymbol{\tau}$ such that K is a $\boldsymbol{\tau}$–algebraic semantics for $\mathcal{S}$.

Now K and $\mathrm{UISP}(\mathsf{K})$ always satisfy the same infinitary quasi-identities over Var, so:

Lemma 2. *Suppose* K *is a class of* $\mathcal{L}$*–algebras with* $\mathsf{K} \subseteq \mathsf{M} \subseteq \mathrm{UISP}(\mathsf{K})$. *If one of* K, M, $\mathrm{UISP}(\mathsf{K})$ *is a* $\boldsymbol{\tau}$*–algebraic semantics for* $\mathcal{S}$ *then all of them are.*

The above definition of algebraic semantics originates in [9], where it was formulated for finitary systems. But in [9], translations were required to be *finite* sets. When K is a $\boldsymbol{\tau}$–algebraic semantics for $\mathcal{S}$ and $\boldsymbol{\tau}$ is finite, the quasivariety $\mathrm{ISPP_U}(\mathsf{K})$ will be a $\boldsymbol{\tau}$–algebraic semantics for $\mathcal{S}$ iff $\mathcal{S}$ is finitary.

For a single translation $\boldsymbol{\tau}$, several *different* UISP–classes may each be a $\boldsymbol{\tau}$–algebraic semantics for a fixed deductive system. For example, the variety of Boolean algebras is clearly an algebraic semantics for classical propositional logic with respect to the translation $\neg\neg x \approx \top$—but so is the variety of Heyting algebras, by a variant of Glivenko's Theorem (given explicitly in [13]).

At present there is no known *intrinsic* characterization of the deductive systems that possess an algebraic semantics. (For partial results see [9, Thm. 3.7] and [13].) For a characterization to be considered intrinsic, it should avoid making existential demands about extrinsic objects such as special classes of algebras that are not uniquely specified by the logic itself. And if we want the characterization to be *readily falsifiable*, it should exclude even existential demands like 'there exists a translation ...'.

We give below two consequences of the demand 'K is a $\boldsymbol{\tau}$–algebraic semantics for $\mathcal{S}$'. The first is independent of K; the second is independent of $\boldsymbol{\tau}$. Both will be required later.

Proposition 3. ([13, Thm. 2.16]) *If $\mathcal{S}$ has a $\boldsymbol{\tau}$–algebraic semantics then for every $\langle \delta, \varepsilon \rangle \in \boldsymbol{\tau}$ and every unary polynomial function p of* **Te**,

$$x,\, p(\delta(x)) \vdash_{\mathcal{S}} p(\varepsilon(x)), \quad \textit{and} \quad x,\, p(\varepsilon(x)) \vdash_{\mathcal{S}} p(\delta(x)).$$

The proof does not rely on the extra finiteness conditions assumed in [13].

Proposition 4. *Let K be an ISP–class that is an algebraic semantics for $\mathcal{S}$. Let $\mathbf{X}$ be an $\mathcal{L}$–algebra* (*not necessarily* in K) *that is K–free over a set Y of arbitrary cardinality.*[2]

Then the least K–congruence of $\mathbf{X}$ is compatible with every $\mathcal{S}$–filter of $\mathbf{X}$.

Proof. Recall that $\Theta^{\mathbf{X}}_{\mathsf{K}}\,\emptyset$ exists, because K is closed under subdirect products and isomorphisms. Let F be an $\mathcal{S}$–filter of $\mathbf{X}$ and let $\langle a, b\rangle \in \Theta^{\mathbf{X}}_{\mathsf{K}}\,\emptyset$. We need to show that $a \in F$ iff $b \in F$. Now $\langle a, b\rangle = \langle \alpha^{\mathbf{X}}(\vec{y}), \beta^{\mathbf{X}}(\vec{y})\rangle$ for some $\alpha, \beta \in \mathrm{Te}$, where $\vec{y}$ is some finite sequence of distinct elements of Y (see [15, Thm. II.10.3(c)] if necessary).

Suppose $\mathbf{A} \in \mathsf{K}$ and $\vec{a} \in A$, where the finite sequence $\vec{a}$ has the same length as $\vec{y}$, although its elements need not be distinct. The function assigning to each element in the sequence $\vec{y}$ the element in the corresponding position of $\vec{a}$ may be extended to a homomorphism $h : \mathbf{X} \to \mathbf{A}$, because $\mathbf{X}$ is K–free over Y. Now $\ker h$ is a K–congruence of $\mathbf{X}$, because

[2]The existence of $\mathbf{X}$ tacitly implies that $Y \neq \emptyset$ or $\mathcal{L}$ contains a constant symbol.

$\mathbf{X}/\ker h \cong h[\mathbf{X}] \in \mathrm{S}(\mathbf{A}) \subseteq \mathsf{K}$. So $\Theta^{\mathbf{X}}_{\mathsf{K}}\emptyset \subseteq \ker h$, whence $h(a) = h(b)$, i.e., $\alpha^{\mathbf{A}}(\vec{a}) = h(\alpha^{\mathbf{X}}(\vec{y})) = h(\beta^{\mathbf{X}}(\vec{y})) = \beta^{\mathbf{A}}(\vec{a})$. This shows that K satisfies $\alpha(\vec{x}) \approx \beta(\vec{x})$, where $\vec{x} \in Var$ is any sequence of distinct variables of the same length as $\vec{y}$. It follows that for any $\boldsymbol{\mathcal{L}}$–translation $\boldsymbol{\tau}$,

$$\boldsymbol{\tau}(\alpha(\vec{x})) \models_{\mathsf{K}} \boldsymbol{\tau}(\beta(\vec{x})), \text{ and } \boldsymbol{\tau}(\beta(\vec{x})) \models_{\mathsf{K}} \boldsymbol{\tau}(\alpha(\vec{x})).$$

Since K is a $\boldsymbol{\tau}$–algebraic semantics for $\mathcal{S}$ with respect to some translation $\boldsymbol{\tau}$, this tells us that $\alpha(\vec{x}) \vdash_{\mathcal{S}} \beta(\vec{x})$ and $\beta(\vec{x}) \vdash_{\mathcal{S}} \alpha(\vec{x})$. Consequently, $a = \alpha^{\mathbf{X}}(\vec{y}) \in F$ iff $b = \beta^{\mathbf{X}}(\vec{y}) \in F$. □

5 The Leibniz and Suszko Operators

Let $\mathbf{A}$ be any $\mathcal{L}$–algebra. For any subset F of A, there is a largest congruence of $\mathbf{A}$ that is compatible with F. When F is an $\mathcal{S}$–filter of $\mathbf{A}$, this congruence is denoted by $\Omega^{\mathbf{A}}F$. The *Leibniz operator of* $\mathcal{S}$ *on* $\mathbf{A}$, denoted by $\Omega^{\mathbf{A}}$, is the function *with domain* $\mathrm{Fi}_{\mathcal{S}}\,\mathbf{A}$ defined by $F \mapsto \Omega^{\mathbf{A}}F$. The functions $\Omega^{\mathbf{A}}$, taken over all $\mathcal{L}$–algebras $\mathbf{A}$, constitute the *Leibniz operator of* $\mathcal{S}$. Note that $\Omega^{\mathbf{A}}$ depends on $\mathcal{S}$ only insofar as $\mathcal{S}$ determines its domain by restriction. Thus, the *action* of the Leibniz operator is determined by the structure of the signature alone. But the extent to which a deductive system may be 'algebraized' is closely correlated with transparent properties of its Leibniz operator. This was one of the key discoveries of [9]. More recent accounts of this relationship usually describe a 'Leibniz hierarchy' of broadly algebraic conditions on deductive systems, each characterized intrinsically by a property of $\Omega^{\mathbf{Te}}$. (See [26, Sec. 3.4] and Section 11.)

The *injectivity* and the *isotonicity* of the Leibniz operator are two demands that turn out to carry significant information about a deductive system. The former demand makes sense only for deductive systems that possess *theorems* (as opposed to derivable proper rules), in view of the following simple fact:

Lemma 5. *If* $\mathcal{S}$ *has no theorem then its Leibniz operator is non-injective on every* $\mathcal{L}$*–algebra.*

Proof. Clearly, the system $\mathcal{S}$ has no theorem iff the empty set is an $\mathcal{S}$–filter of every $\mathcal{L}$–algebra. In this case, each $\mathcal{L}$–algebra $\mathbf{A}$ has distinct $\mathcal{S}$–filters $\emptyset$ and A with $\Omega^{\mathbf{A}}\emptyset = A \times A = \Omega^{\mathbf{A}}A$. □

Even when $\mathcal{S}$ has theorems, its Leibniz operator need not be *order preserving*, i.e., there may exist $\mathcal{S}$–filters F, G of an $\mathcal{L}$–algebra $\mathbf{A}$ with $F \subseteq G$ but $\Omega^{\mathbf{A}} F \not\subseteq \Omega^{\mathbf{A}} G$. This leads to the consideration of a second operator:

The *Suszko operator of* $\mathcal{S}$ *on* $\mathbf{A}$ is the function with domain $\mathrm{Fi}_{\mathcal{S}}\,\mathbf{A}$ that maps each $\mathcal{S}$–filter F of $\mathbf{A}$ to the congruence

$$\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}} F := \bigcap \{\Omega^{\mathbf{A}} G : F \subseteq G \in \mathrm{Fi}_{\mathcal{S}}\,\mathbf{A}\}$$

of $\mathbf{A}$. Note that $\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}} F \subseteq \Omega^{\mathbf{A}} F$. The functions $\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}}$, taken over all $\mathcal{L}$–algebras $\mathbf{A}$, together make up the *Suszko operator of* $\mathcal{S}$. By [21, Thm. 1.8], $\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}}$ is the 'largest' function H from $\mathrm{Fi}_{\mathcal{S}}\,\mathbf{A}$ into $\mathrm{Con}\,\mathbf{A}$ such that

(i) for every $\mathcal{S}$–filter F of $\mathbf{A}$, $H(F)$ is compatible with F, and

(ii) H is order preserving (with respect to set inclusion).

That is to say, a function $H : \mathrm{Fi}_{\mathcal{S}}\,\mathbf{A} \to \mathrm{Con}\,\mathbf{A}$ has these two properties iff $H(F) \subseteq \widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}} F$ for all $F \in \mathrm{Fi}_{\mathcal{S}}\,\mathbf{A}$. Because $\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}}$ is order preserving, we have

$$\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}} \bigcap \mathcal{F} \subseteq \bigcap_{F \in \mathcal{F}} \widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}} F \subseteq \bigcap_{F \in \mathcal{F}} \Omega^{\mathbf{A}} F \quad \text{for all } \mathcal{F} \subseteq \mathrm{Fi}_{\mathcal{S}}\,\mathbf{A}. \tag{3}$$

If F is an $\mathcal{S}$–filter of $\mathbf{A}$ then

$$F/\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}} F \in \mathrm{Fi}_{\mathcal{S}}\,(\mathbf{A}/\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}} F) \text{ and } F/\Omega^{\mathbf{A}} F \in \mathrm{Fi}_{\mathcal{S}}\,(\mathbf{A}/\Omega^{\mathbf{A}} F), \tag{4}$$

by Lemma 1(ii), because $\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}} F$ and $\Omega^{\mathbf{A}} F$ are both compatible with F.

For information on the Leibniz and Suszko operators (as well as motivation, applications and history) see [20], [26] and their references. In the case of the Suszko operator, see [21] also. All unproved statements about these operators that will be used here are either easy to prove or can be found in these sources.

Lemma 6. *Let F be an $\mathcal{S}$–filter of an $\mathcal{L}$–algebra $\mathbf{A}$ and let $a, b \in A$.*

(i) $\langle a, b\rangle \in \Omega^{\mathbf{A}} F$ iff the following is true: for every unary polynomial function p of $\mathbf{A}$, we have $p(a) \in F$ iff $p(b) \in F$.

(ii) $\langle a, b\rangle \in \widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}} F$ iff the following is true: for every unary polynomial function p of $\mathbf{A}$, the $\mathcal{S}$–filters of $\mathbf{A}$ generated by $F \cup \{p(a)\}$ and by $F \cup \{p(b)\}$ coincide.

Lemma 7. *For any homomorphism of $\mathcal{L}$–algebras, $h : \mathbf{A} \to \mathbf{B}$, and for any $\mathcal{S}$–filter G of $\mathbf{B}$, the following statements are true.*

(i) $h^{-1}[\Omega^{\mathbf{B}} G] \subseteq \Omega^{\mathbf{A}} h^{-1}[G]$.

(ii) If h is surjective then $h^{-1}[\Omega^{\mathbf{B}} G] = \Omega^{\mathbf{A}} h^{-1}[G]$.

The next result augments [13, Prop. 2.17].

Proposition 8. *For any $\boldsymbol{\mathcal{L}}$–translation $\boldsymbol{\tau}$, the following conditions are equivalent.*

(i) For all $\langle \delta, \varepsilon \rangle \in \boldsymbol{\tau}$ and for every unary polynomial function p of **Te**,

$$x,\, p(\delta(x)) \vdash_{\mathcal{S}} p(\varepsilon(x)), \quad \textit{and} \quad x,\, p(\varepsilon(x)) \vdash_{\mathcal{S}} p(\delta(x)).$$

(ii) For any $\mathcal{S}$–filter F of any $\mathcal{L}$–algebra $\mathbf{A}$, we have $\boldsymbol{\tau}^{\mathbf{A}}[F] \subseteq \widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}} F$.

(iii) $\boldsymbol{\tau}(x) \subseteq \widetilde{\Omega}^{\mathbf{Te}}_{\mathcal{S}}\, \mathrm{Fg}^{\mathbf{Te}}_{\mathcal{S}}\{x\}$.

Note. We assume no relationship between $\mathcal{S}$ and $\boldsymbol{\tau}$ here.

Proof. The equivalence of (i) and (iii) follows from (1), Lemma 6(ii) and the fact that for any $\boldsymbol{\mathcal{L}}$–term α, if $F = \mathrm{Fg}^{\mathbf{Te}}_{\mathcal{S}}\{x\}$ then $\mathrm{Fg}^{\mathbf{Te}}_{\mathcal{S}}\{x, \alpha\} = \mathrm{Fg}^{\mathbf{Te}}_{\mathcal{S}}(F \cup \{\alpha\})$. Obviously, (ii) ⇒ (iii).

(i) ⇒ (ii): Let $\mathbf{A}$ and F be as described, and let $a \in F$ and $\langle \delta, \varepsilon \rangle \in \boldsymbol{\tau}$. Consider a unary polynomial function p of $\mathbf{A}$, say $p(e) = \beta^{\mathbf{A}}(e, \vec{c})$ for all $e \in A$, where $\beta = \beta(x, \vec{y})$ is an $(n+1)$–ary term in Te whose distinct apparent variables are among $x, \vec{y} \in Var$, where $n \in \omega$, and $\vec{c} = c_1, \ldots, c_n \in A$. Then the function p' from Te to Te defined by $p'(\gamma) = \beta(\gamma, \vec{y})$ ($\gamma \in$ Te) is a unary polynomial function of **Te**. By (i),

$$x,\, p'(\delta(x)) \vdash_{\mathcal{S}} p'(\varepsilon(x)) \quad \text{and} \quad x,\, p'(\varepsilon(x)) \vdash_{\mathcal{S}} p'(\delta(x)).$$

Let h be a homomorphism from **Te** to $\mathbf{A}$ such that $h(x) = a$ and $h(y_i) = c_i$ for $i = 1, \ldots, n$. It follows from the displayed statement that any $\mathcal{S}$–filter of $\mathbf{A}$ containing $h(x) = a$ will contain $h(p'(\delta(x))) = p(\delta^{\mathbf{A}}(a))$ iff it contains $h(p'(\varepsilon(x))) = p(\varepsilon^{\mathbf{A}}(a))$. In particular, since $a \in F$, we have

$$\mathrm{Fg}^{\mathbf{A}}_{\mathcal{S}}(F \cup \{p(\delta^{\mathbf{A}}(a))\}) = \mathrm{Fg}^{\mathbf{A}}_{\mathcal{S}}(F \cup \{p(\varepsilon^{\mathbf{A}}(a))\}).$$

By Lemma 6(ii), $\langle \delta^{\mathbf{A}}(a), \varepsilon^{\mathbf{A}}(a) \rangle \in \widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}} F$. □

From Propositions 3 and 8, we deduce:

Corollary 9. *If $\mathcal{S}$ has a $\boldsymbol{\tau}$–algebraic semantics then for any $\mathcal{S}$–filter F of any $\mathcal{L}$–algebra $\mathbf{A}$, we have $\boldsymbol{\tau}^{\mathbf{A}}[F] \subseteq \widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}} F$.*

We shall say that the Leibniz (or the Suszko) operator of $\mathcal{S}$ is *globally injective* if it is injective on the $\mathcal{S}$–filters of *every* $\mathcal{L}$–algebra. The significance of this condition for the Suszko operator is partly revealed by the following result of Czelakowski.

Theorem 10. ([21, Thm. 7.8]) *The following conditions on $\mathcal{S}$ are equivalent.*

(i) The Suszko operator of $\mathcal{S}$ is globally injective.

(ii) For every $\mathcal{S}$–filter F of any $\mathcal{L}$–algebra $\mathbf{A}$, the least $\mathcal{S}$–filter of $\mathbf{A}/\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}} F$ is $F/\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}} F$.

Definition 2. Let $\mathbf{A}$ be an $\mathcal{L}$–algebra. We say that the Leibniz operator of $\mathcal{S}$ is *completely order reflecting on* (the $\mathcal{S}$–filters of) $\mathbf{A}$ provided that the following is true: whenever $\mathcal{F} \cup \{G\}$ is a set of $\mathcal{S}$–filters of $\mathbf{A}$,

$$\text{if} \quad \bigcap_{F \in \mathcal{F}} \Omega^{\mathbf{A}} F \subseteq \Omega^{\mathbf{A}} G \quad \text{then} \quad \bigcap \mathcal{F} \subseteq G.$$

If this is true for all $\mathcal{L}$–algebras, we say that the Leibniz operator of $\mathcal{S}$ is *globally completely order reflecting*. If it is true when $\mathbf{A} = \mathbf{Te}$, we say that the Leibniz operator is *completely order reflecting on $\mathcal{S}$–theories*.

The condition just defined can be rephrased more economically using the Suszko operator. It is easy to show, using (3), that the Leibniz operator of $\mathcal{S}$ is completely order reflecting on $\mathbf{A}$ iff whenever F and G are $\mathcal{S}$–filters of $\mathbf{A}$,

$$\text{if} \quad \widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}} F \subseteq \Omega^{\mathbf{A}} G \quad \text{then} \quad F \subseteq G. \tag{5}$$

When the Leibniz operator of $\mathcal{S}$ is completely order reflecting on $\mathbf{A}$, then $\Omega^{\mathbf{A}}$ and $\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}}$ are both *order reflecting* functions in the standard sense, i.e., whenever F and G are $\mathcal{S}$–filters of $\mathbf{A}$ with $\Omega^{\mathbf{A}} F \subseteq \Omega^{\mathbf{A}} G$ or $\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}} F \subseteq \widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}} G$ then $F \subseteq G$ (use (5) in the latter case). Of course, every order reflecting function between ordered sets is *injective*. So when $\Omega^{\mathbf{A}}$ is completely order reflecting then both $\Omega^{\mathbf{A}}$ and $\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}}$ are injective on $\mathrm{Fi}_{\mathcal{S}}\, \mathbf{A}$. In the case of the *Suszko* operator, we establish a global converse of this statement:

Theorem 11. *The Suszko operator of $\mathcal{S}$ is globally injective iff the Leibniz operator of $\mathcal{S}$ is globally completely order reflecting.*

Proof. Assume that the Suszko operator of $\mathcal{S}$ is globally injective. Let $\mathbf{A}$ be an $\mathcal{L}$–algebra and let F and G be $\mathcal{S}$–filters of $\mathbf{A}$ with $\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}}F \subseteq \Omega^{\mathbf{A}}G$. It follows that the rule $a/\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}}F \mapsto a/\Omega^{\mathbf{A}}G$ ($a \in A$) describes a well-defined surjective homomorphism $h : \mathbf{A}/\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}}F \to \mathbf{A}/\Omega^{\mathbf{A}}G$. It also follows that $\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}}F$ is compatible with G, so we may safely use the abbreviation $G/\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}}F$ for $\{g/\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}}F : g \in G\}$. By (4), $G/\Omega^{\mathbf{A}}G$ is an $\mathcal{S}$–filter of $\mathbf{A}/\Omega^{\mathbf{A}}G$. So, by Lemma 1(i), $h^{-1}[G/\Omega^{\mathbf{A}}G]$ is an $\mathcal{S}$–filter of $\mathbf{A}/\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}}F$ and, clearly, $h^{-1}[G/\Omega^{\mathbf{A}}G] = G/\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}}F$. By Theorem 10, the least $\mathcal{S}$–filter of $\mathbf{A}/\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}}F$ is $F/\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}}F$, so $F/\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}}F \subseteq G/\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}}F$. Now let $a \in F$. By the inclusion just stated, we may choose $g \in G$ such that $\langle a, g\rangle \in \widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}}F$. Then $a \in G$, because $\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}}F$ is compatible with G. So $F \subseteq G$, as required. □

For the main result of this paper (Theorem 28), we shall require one more technical result about the Suszko operator:

Proposition 12. *Let k be the $\boldsymbol{\mathcal{L}}$–substitution sending all variables to x. Then*

$$k\,[\widetilde{\Omega}^{\mathbf{Te}}_{\mathcal{S}}\,\mathrm{Fg}^{\mathbf{Te}}_{\mathcal{S}}\{x\}] \;=\; (\mathrm{Te}(x) \times \mathrm{Te}(x)) \cap \widetilde{\Omega}^{\mathbf{Te}}_{\mathcal{S}}\,\mathrm{Fg}^{\mathbf{Te}}_{\mathcal{S}}\{x\}.$$

In particular, $k\,[\widetilde{\Omega}^{\mathbf{Te}}_{\mathcal{S}}\,\mathrm{Fg}^{\mathbf{Te}}_{\mathcal{S}}\{x\}]$ is a congruence of $\mathbf{Te}(x)$.

Proof. The expression on the right of the equality is evidently a subset of the one on the left, because elements of $\mathrm{Te}(x)$ are fixed points of k. Obviously, also, the expression on the left is contained in $\mathrm{Te}(x) \times \mathrm{Te}(x)$, so if we define $F = \mathrm{Fg}^{\mathbf{Te}}_{\mathcal{S}}\{x\}$, it remains only to prove that $k\,[\widetilde{\Omega}^{\mathbf{Te}}_{\mathcal{S}}F] \subseteq \widetilde{\Omega}^{\mathbf{Te}}_{\mathcal{S}}F$.[3]

To see that this is true, let $\langle\delta(x), \varepsilon(x)\rangle = \langle k(\alpha(x,\vec{y})), k(\beta(x,\vec{y}))\rangle$, where $\langle\alpha(x,\vec{y}), \beta(x,\vec{y})\rangle \in \widetilde{\Omega}^{\mathbf{Te}}_{\mathcal{S}}F$, and $\vec{y}$ is a finite sequence of distinct variables other than x. We must show that $\langle\delta(x), \varepsilon(x)\rangle \in \widetilde{\Omega}^{\mathbf{Te}}_{\mathcal{S}}F$. By Lemma 6(ii), for every unary polynomial function p of $\mathbf{Te}$,

$$\mathrm{Fg}^{\mathbf{Te}}_{\mathcal{S}}(F \cup \{p(\alpha(x,\vec{y}))\}) \;=\; \mathrm{Fg}^{\mathbf{Te}}_{\mathcal{S}}(F \cup \{p(\beta(x,\vec{y}))\}),$$
$$\text{i.e., } \mathrm{Fg}^{\mathbf{Te}}_{\mathcal{S}}\{x,\, p(\alpha(x,\vec{y}))\} \;=\; \mathrm{Fg}^{\mathbf{Te}}_{\mathcal{S}}\{x,\, p(\beta(x,\vec{y}))\}.$$

This means that for every $n \in \omega$, every $(n+1)$–ary $\boldsymbol{\mathcal{L}}$–term σ and every n–sequence $\vec{\gamma} = \gamma_1, \ldots, \gamma_n$ of (not necessarily distinct) $\boldsymbol{\mathcal{L}}$–terms,

$$\mathrm{Fg}^{\mathbf{Te}}_{\mathcal{S}}\{x,\, \sigma(\alpha(x,\vec{y}),\vec{\gamma})\} \;=\; \mathrm{Fg}^{\mathbf{Te}}_{\mathcal{S}}\{x,\, \sigma(\beta(x,\vec{y}),\vec{\gamma})\},$$

[3] This actually follows from a more general result, reported in [20, Remark. 1.5.6] and in [21, Prop. 1.5(8)], but these sources do not give a detailed proof.

i.e., both of the following are true (see (1)):

$$x,\, \sigma(\alpha(x,\vec{y}),\vec{\gamma}) \vdash_{\mathcal{S}} \sigma(\beta(x,\vec{y}),\vec{\gamma}), \quad \text{and} \quad x,\, \sigma(\beta(x,\vec{y}),\vec{\gamma}) \vdash_{\mathcal{S}} \sigma(\alpha(x,\vec{y}),\vec{\gamma}).$$

Consider the substitution that maps all variables in the sequence $\vec{y}$ to x and that leaves all other variables fixed. We apply this to all rules of the two forms displayed above. Invoking the structurality of $\mathcal{S}$, we obtain

$$x,\, \sigma(\delta(x),\vec{\mu}) \vdash_{\mathcal{S}} \sigma(\varepsilon(x),\vec{\mu}), \quad \text{and} \quad x,\, \sigma(\varepsilon(x),\vec{\mu}) \vdash_{\mathcal{S}} \sigma(\delta(x),\vec{\mu}) \qquad (6)$$

whenever n, σ are as previously described and $\vec{\mu}$ is an n–sequence of $\boldsymbol{\mathcal{L}}$–terms in which the variables of $\vec{y}$ do not occur. We use here the obvious fact that every term in such a sequence $\vec{\mu}$ belongs to the range of the substitution. (Recall that the domain of a substitution is, by definition, Te and not Var.)

Let $\vec{z}$ denote the sequence of all variables in Var other than x and $\vec{y}$, ordered in any way. Since Var is infinite and $\vec{y}$ is finite, there is a bijection from the range of $\vec{z}$ onto the union of the ranges of $\vec{y}$ and $\vec{z}$. Consider the substitution h that extends both this bijection and the identity function on the variables $x, \vec{y}$. Apply h to all possible cases of (6). By structurality, we conclude that (6) holds for all n, σ as previously described and for *all* n–sequences $\vec{\mu}$ of $\boldsymbol{\mathcal{L}}$–terms. Again, the point is that every $\boldsymbol{\mathcal{L}}$–term in such a sequence $\vec{\mu}$ is the image under h of an $\boldsymbol{\mathcal{L}}$–term in which the variables $\vec{y}$ do not occur. But this means that

$$x,\, p(\delta(x)) \vdash_{\mathcal{S}} p(\varepsilon(x)), \quad \text{and} \quad x,\, p(\varepsilon(x)) \vdash_{\mathcal{S}} p(\delta(x))$$

for all unary polynomial functions p of **Te**. So $\langle \delta(x), \varepsilon(x) \rangle \in \widetilde{\Omega}_{\mathcal{S}}^{\mathbf{Te}} F$, by Proposition 8. □

6 Matrix Semantics

Although a deductive system need not possess an algebraic semantics, it always possesses a 'matrix semantics', and this can always be chosen to consist of relatively simple objects called 'reduced matrices'. We give a brief account of matrix semantics here. For more thorough expositions see [51] and [20].

An $\mathcal{L}$–*matrix* is a pair $\langle \mathbf{A}, F \rangle$, where $\mathbf{A}$ is an $\mathcal{L}$–algebra and $F \subseteq A$. If in addition, F is an $\mathcal{S}$–filter of $\mathbf{A}$, we call $\langle \mathbf{A}, F \rangle$ a *matrix model* of $\mathcal{S}$. A class M of $\mathcal{L}$–matrices is called a *matrix semantics* for $\mathcal{S}$ if it satisfies the condition set out immediately below.

> For every set $\Gamma \cup \{\alpha\}$ of $\boldsymbol{\mathcal{L}}$–terms, we have $\Gamma \vdash_{\mathcal{S}} \alpha$ iff the following is true: for every $\langle \mathbf{A}, F\rangle \in \mathsf{M}$ and every homomorphism $h : \mathbf{Te} \to \mathbf{A}$, if $h[\Gamma] \subseteq F$ then $h(\alpha) \in F$.

Clearly, a matrix semantics for $\mathcal{S}$ must consist of matrix models of $\mathcal{S}$. When M is a matrix semantics for $\mathcal{S}$, we may call $\mathcal{S}$ the *consequence relation of* M.

We may consider an $\mathcal{L}$–matrix $\langle \mathbf{A}, F\rangle$ as a structure for the first order language without equality containing the operation symbols of $\mathcal{L}$ and one unary relation symbol. The unary predicate, which is realised as F, admits the intuitive interpretation 'it is true that' when $\langle \mathbf{A}, F\rangle$ is a matrix model of $\mathcal{S}$, and it is often called the *truth predicate*. This perspective on sentential logics originates with Bloom [14] and is fruitful for finitary systems; see [26, Sec. 3.1, 4.3] and its references. For *non-finitary* deductive systems, it is sometimes more convenient to think of the matrix models as models of infinitary languages, with or without equality, rather than first order ones. We shall return to this point in Section 9.

Recall that a *strict homomorphism* between similar structures is required to preserve all indicated operations and to preserve and reflect all indicated relations. So a *surjective* strict homomorphism $h : \langle \mathbf{A}, F\rangle \to \langle h[\mathbf{A}], h[F]\rangle$ between $\mathcal{L}$–matrices is essentially a surjective homomorphism of algebras $h : \mathbf{A} \to h[\mathbf{A}]$ such that $h^{-1}[h[F]] = F$. It is an *isomorphism* if, in addition, h is an embedding, i.e., h is injective on the elements of $\mathbf{A}$.

For any $\mathcal{S}$–filter F of an $\mathcal{L}$–algebra $\mathbf{A}$, we may characterize $\Omega^{\mathbf{A}} F$ as the largest congruence θ of $\mathbf{A}$ for which the canonical surjective homomorphism of algebras $h : \mathbf{A} \to \mathbf{A}/\theta$ defines a surjective *strict* homomorphism from $\langle \mathbf{A}, F\rangle$ to $\langle \mathbf{A}/\theta, h[F]\rangle$.

A matrix model $\langle \mathbf{A}, F\rangle$ of $\mathcal{S}$ is called *Suszko-reduced* if $\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}} F = \mathrm{id}_A$; it is called *Leibniz-reduced* if $\Omega^{\mathbf{A}} F = \mathrm{id}_A$.

It follows that $\langle \mathbf{A}, F\rangle$ is Leibniz-reduced iff every surjective strict homomorphism with domain $\langle \mathbf{A}, F\rangle$ is an isomorphism. Obviously, every Leibniz-reduced matrix model of $\mathcal{S}$ is Suszko-reduced. The converse is false. The next proposition is well known and easily verified.

Lemma 13. *The Suszko- [resp. Leibniz-] reduced matrix models of* $\mathcal{S}$ *are, up to isomorphism, just the* $\mathcal{L}$*–matrices of the form* $\langle \mathbf{A}/\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}} F,\ F/\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}} F\rangle$ *[resp.* $\langle \mathbf{A}/\Omega^{\mathbf{A}} F,\ F/\Omega^{\mathbf{A}} F\rangle$*], where* F *is an* $\mathcal{S}$*–filter of* $\mathbf{A}$.

By a Suszko- [resp. Leibniz-] reduced *formula matrix model* of $\mathcal{S}$, we mean an $\mathcal{L}$-matrix of the form

$$\langle \mathbf{Te}/\widetilde{\Omega}^{\mathbf{Te}}_{\mathcal{S}} F,\ F/\widetilde{\Omega}^{\mathbf{Te}}_{\mathcal{S}} F\rangle \quad \text{[resp. } \langle \mathbf{Te}/\Omega^{\mathbf{Te}} F,\ F/\Omega^{\mathbf{Te}} F\rangle],$$

where F is an $\mathcal{S}$–theory. The Suszko-reduced formula matrix models generalize the 'Lindenbaum-Tarski matrices' determined by the theories of familiar logics: see [26, Sec. 1]. We use each of the expressions listed on the left below to stand for the class of all objects of the kind listed on its right.

$\mathsf{Mod}^{\mathrm{Su}}\mathcal{S}$	Suszko-reduced matrix models of $\mathcal{S}$
$\mathsf{Mod}^{*}\mathcal{S}$	Leibniz-reduced matrix models of $\mathcal{S}$
$\mathsf{LMod}^{\mathrm{Su}}\mathcal{S}$	Suszko-reduced formula matrix models of $\mathcal{S}$
$\mathsf{LMod}^{*}\mathcal{S}$	Leibniz-reduced formula matrix models of $\mathcal{S}$

(The 'L' stands for Lindenbaum.) For each of the above classes M, the class of all algebra reducts $\mathbf{A}$ of matrices $\langle \mathbf{A}, F\rangle \in \mathsf{M}$ is denoted, respectively, by

$$\mathsf{Alg}\mathcal{S}, \quad \mathsf{Alg}^{*}\mathcal{S}, \quad \mathsf{LAlg}\mathcal{S}, \quad \mathsf{LAlg}^{*}\mathcal{S}.$$

Note that while $\mathsf{LMod}^{*}\mathcal{S} \subseteq \mathsf{Mod}^{*}\mathcal{S} \subseteq \mathsf{Mod}^{\mathrm{Su}}\mathcal{S}$, the class $\mathsf{LMod}^{*}\mathcal{S}$ need *not* be contained in the isomorphic closure of $\mathsf{LMod}^{\mathrm{Su}}\mathcal{S}$. This is witnessed by Example 1 below. The following result is also well known.

Proposition 14. *Each of* $\mathsf{Mod}^{\mathrm{Su}}\mathcal{S}$, $\mathsf{Mod}^{*}\mathcal{S}$, $\mathsf{LMod}^{\mathrm{Su}}\mathcal{S}$, $\mathsf{LMod}^{*}\mathcal{S}$ *is a matrix semantics for* $\mathcal{S}$.

Model-theoretic constructions on $\mathcal{L}$–matrices are defined just as for structures with a single unary relation. Thus, a *submatrix* of an $\mathcal{L}$–matrix $\langle \mathbf{B}, G\rangle$ is an $\mathcal{L}$–matrix of the form $\langle \mathbf{A}, G \cap A\rangle$, where $\mathbf{A}$ is a subalgebra of $\mathbf{B}$, and the *direct product* of a family of $\mathcal{L}$–matrices $\langle \mathbf{A}_\alpha, F_\alpha\rangle$ is defined as $\langle \Pi_\alpha\, \mathbf{A}_\alpha,\, \Pi_\alpha\, F_\alpha\rangle$. A submatrix $\langle \mathbf{A}, F\rangle$ of this direct product is called a *subdirect product* of the same family if $\mathbf{A}$ is a subdirect product of the family of algebras $\mathbf{A}_\alpha$, i.e., if $\pi_\beta[A] = A_\beta$ for each of the projection functions $\pi_\beta : \Pi_\alpha\, A_\alpha \to A_\beta$. The definition of ultraproducts is as expected (see, e.g., [20, p. 33]).

The class $\mathsf{Mod}^{\mathrm{Su}}\mathcal{S}$ is closed under subdirect products [21, Thm. 5.2]. A Suszko-reduced matrix model $\langle \mathbf{A}, F \rangle$ of $\mathcal{S}$ is naturally isomorphic to a subdirect product of Leibniz-reduced matrix models $\langle \mathbf{A}/\Omega^{\mathbf{A}}J,\ J/\Omega^{\mathbf{A}}J \rangle$ of $\mathcal{S}$, indexed by the $\mathcal{S}$–filters J of $\mathbf{A}$ containing F; this is essentially [25, Thm. 2.23]. Consequently:

Proposition 15. ([21], [25])

(i) $\mathsf{Mod}^{\mathrm{Su}}\mathcal{S}$ *is exactly the closure of* $\mathsf{Mod}^{*}\mathcal{S}$ *under subdirect products and isomorphisms.*

(ii) $\mathsf{LMod}^{\mathrm{Su}}\mathcal{S}$ *is contained in the closure of* $\mathsf{LMod}^{*}\mathcal{S}$ *under subdirect products and isomorphisms.*

(iii) $\mathsf{Alg}\mathcal{S} = \mathrm{IP_S}(\mathsf{Alg}^{*}\mathcal{S})$ *and* $\mathsf{LAlg}\mathcal{S} \subseteq \mathrm{IP_S}(\mathsf{LAlg}^{*}\mathcal{S})$.

Using Lemma 2, we deduce:

Corollary 16. $\mathsf{Alg}^{*}\mathcal{S}$ *is a* $\boldsymbol{\tau}$*–algebraic semantics for* $\mathcal{S}$ *iff* $\mathsf{Alg}\mathcal{S}$ *is.*

Recall that a class of structures for a fixed language is called *elementary* if it is axiomatized by a set of first order sentences. In general, $\mathsf{Mod}^{*}\mathcal{S}$ need not be an elementary class, even when $\mathcal{S}$ is finitary: see for instance [20, p. 50]. Rautenberg [41] has shown that for *finitary* systems $\mathcal{S}$, the class $\mathsf{Mod}^{*}\mathcal{S}$ is elementary iff it is closed under *ultraproducts*, and that this is true iff the Leibniz operator of $\mathcal{S}$ is *finitizable*. This last demand means that there is some *finite* set of $\boldsymbol{\mathcal{L}}$–terms Σ such that for any $\mathcal{L}$–algebra $\mathbf{A}$, the characterization of $\Omega^{\mathbf{A}}$ on $\mathrm{Fi}_{\mathcal{S}}\,\mathbf{A}$, given in Lemma 6(i), remains true even when we restrict the unary polynomials to those of the form $p(x) = \sigma^{\mathbf{A}}(x, \vec{c})$, $\sigma \in \Sigma$, $\vec{c} \in A$.

7 Definability of Truth

Definition 3. Let M be a class of $\mathcal{L}$–matrices.

(i) The assertion '*truth is implicitly definable in* M' shall mean that the matrices in M are uniquely determined by their algebra reducts, i.e., whenever $\langle \mathbf{A}, F \rangle$, $\langle \mathbf{A}, G \rangle \in \mathsf{M}$ then $F = G$.

(ii) The assertion '*truth is equationally definable in* M' shall mean that there exists an $\boldsymbol{\mathcal{L}}$–translation $\boldsymbol{\tau}$ such that for every $\langle \mathbf{A}, F \rangle \in \mathsf{M}$ and every $a \in A$, we have

$$a \in F \quad \text{iff} \quad [\, \delta^{\mathbf{A}}(a) = \varepsilon^{\mathbf{A}}(a) \ \text{ for all } \ \langle \delta, \varepsilon \rangle \in \boldsymbol{\tau} \,].$$

In this case, we say that $\boldsymbol{\tau}$ *defines truth in* M.

Here 'truth' really abbreviates '*the truth predicate of* M'. Obviously, over any class M of $\mathcal{L}$–matrices, the equational definability of truth entails its implicit definability. We aim to investigate the definability of truth in the various naturally occurring matrix semantics for deductive systems, such as $\mathsf{Mod}^{*}\mathcal{S}$ and $\mathsf{Mod}^{\mathrm{Su}}\mathcal{S}$. Lemma 13 and an easy compatibility argument yield:

Proposition 17. *Truth is implicitly definable in* $\mathsf{Mod}^{\mathrm{Su}}\mathcal{S}$ [*resp. in* $\mathsf{Mod}^{*}\mathcal{S}$] *iff the Suszko* [*resp. Leibniz*] *operator of* $\mathcal{S}$ *is globally injective.*

Thus, *global injectivity of the Suszko operator implies global injectivity of the Leibniz operator*, because $\mathsf{Mod}^{*}\mathcal{S} \subseteq \mathsf{Mod}^{\mathrm{Su}}\mathcal{S}$.

Beth's Definability Theorem (see [18, Thm. 2.2.22]) shows that over an *elementary* class M of $\mathcal{L}$–matrices, the implicit definability of the truth predicate entails its explicit *first order* definability. That is, it entails the existence of a formula Φ with just one free variable, belonging to the first order language with equality determined by $\mathcal{L}$ (which excludes the truth predicate), such that for all $\langle \mathbf{A}, F \rangle \in \mathsf{M}$ and all $a \in A$, we have $a \in F$ iff $\Phi[a]$ is true in $\mathbf{A}$. As we have mentioned, however, $\mathsf{Mod}^{*}\mathcal{S}$—and similarly $\mathsf{Mod}^{\mathrm{Su}}\mathcal{S}$—need not be an elementary class.

Proposition 18.

(i) A translation defines truth in $\mathsf{Mod}^{*}\mathcal{S}$ *iff it defines truth in* $\mathsf{Mod}^{\mathrm{Su}}\mathcal{S}$.

(ii) If truth is equationally definable in $\mathsf{LMod}^{*}\mathcal{S}$ *then it is equationally definable, by the same translation, in* $\mathsf{LMod}^{\mathrm{Su}}\mathcal{S}$.

Proof. In (i), sufficiency is obvious, because $\mathsf{Mod}^{*}\mathcal{S} \subseteq \mathsf{Mod}^{\mathrm{Su}}\mathcal{S}$. Necessity in (i) and in (ii) follows because Suszko-reduced matrix (or formula matrix) models are subdirect products of Leibniz-reduced ones, by Proposition 15. The point is that subdirect products preserve infinitary special Horn sentences, such as

$$(\forall x)\,[\, Px \iff \textstyle\bigwedge_{\langle \delta,\, \varepsilon \rangle \,\in\, \boldsymbol{\tau}} \delta(x) \approx \varepsilon(x) \,]$$

(where P is any unary predicate symbol). □

The converse of Proposition 18(ii) is false, as Example 1 will witness. (This confirms that we can sometimes have $\mathsf{LMod}^*\mathcal{S} \nsubseteq \mathrm{I}(\mathsf{LMod}^{\mathrm{Su}}\mathcal{S})$.) Each of the conditions in Proposition 18(ii) has a syntactic characterization which makes no direct reference to any semantics. This is the content of the next result, which follows from (1), Lemma 6 and easy compatibility arguments.

Proposition 19. *Let* $\boldsymbol{\tau} = \{\langle \delta_i, \varepsilon_i \rangle : i \in I\}$ *be any* $\boldsymbol{\mathcal{L}}$*–translation.*

(i) $\boldsymbol{\tau}$ *defines truth in* $\mathsf{LMod}^{\mathrm{Su}}\mathcal{S}$ *iff the underivable rules* $\Gamma \nvdash_{\mathcal{S}} \alpha$ *of* $\mathcal{S}$ *are precisely those where* $\mathcal{S}$ *can 'refute' at least one of the equations* $\delta_i(\alpha) \approx \varepsilon_i(\alpha)$ *on the basis of* Γ, *in the sense that there is a unary polynomial function* p *of* **Te** *for which* $\Gamma, p(\delta_i(\alpha)) \nvdash_{\mathcal{S}} p(\varepsilon_i(\alpha))$ *or* $\Gamma, p(\varepsilon_i(\alpha)) \nvdash_{\mathcal{S}} p(\delta_i(\alpha))$.

(ii) $\boldsymbol{\tau}$ *defines truth in* $\mathsf{LMod}^*\mathcal{S}$ *iff the underivable rules* $\Gamma \nvdash_{\mathcal{S}} \alpha$ *of* $\mathcal{S}$ *are just those where* $\mathcal{S}$ *can 'strongly refute' at least one of the equations* $\delta_i(\alpha) \approx \varepsilon_i(\alpha)$ *on the basis of* Γ, *in the sense that there is a unary polynomial function* p *of* **Te** *for which exactly one of* $\Gamma \vdash_{\mathcal{S}} p(\delta_i(\alpha))$ *and* $\Gamma \vdash_{\mathcal{S}} p(\varepsilon_i(\alpha))$ *holds.*

The connection between algebraic semantics and the equational definability of truth is described in the next result, taken essentially from [9].

Proposition 20. *A class of algebras is an algebraic semantics for* $\mathcal{S}$ *iff it is the class of all algebra reducts of some matrix semantics for* $\mathcal{S}$ *in which truth is equationally definable.*

Any translation that witnesses one of these equivalent conditions witnesses the other also.

Proof. Let K be a $\boldsymbol{\tau}$–algebraic semantics for $\mathcal{S}$. For each $\mathcal{L}$–algebra $\mathbf{A}$, define

$$Eq^{\mathbf{A}}\boldsymbol{\tau} = \{a \in A : \delta^{\mathbf{A}}(a) = \varepsilon^{\mathbf{A}}(a) \text{ for all } \langle \delta, \varepsilon \rangle \in \boldsymbol{\tau}\}.$$

Then $\{\langle \mathbf{A}, Eq^{\mathbf{A}}\boldsymbol{\tau} \rangle : \mathbf{A} \in \mathsf{K}\}$ is clearly a matrix semantics for $\mathcal{S}$ in which truth is equationally definable (by $\boldsymbol{\tau}$). Conversely, if $\boldsymbol{\tau}$ defines truth in a matrix semantics M for $\mathcal{S}$, it follows from the definitions that

$$\{\mathbf{A} : \langle \mathbf{A}, F \rangle \in \mathsf{M} \text{ for some } F\}$$

is a $\boldsymbol{\tau}$–algebraic semantics for $\mathcal{S}$. □

Corollary 21. *If some translation $\boldsymbol{\tau}$ defines truth in any one of*

$$\mathsf{LMod}^{\mathrm{Su}}\mathcal{S},\ \ \mathsf{LMod}^*\mathcal{S},\ \ \mathsf{Mod}^*\mathcal{S}$$

then $\mathcal{S}$ has a $\boldsymbol{\tau}$–algebraic semantics. Indeed, the corresponding one of

$$\mathsf{LAlg}\mathcal{S},\ \ \mathsf{LAlg}^*\mathcal{S},\ \ \mathsf{Alg}^*\mathcal{S}\ \ (\textit{equivalently}\ \mathsf{Alg}\mathcal{S})$$

is then a $\boldsymbol{\tau}$–algebraic semantics for $\mathcal{S}$.

With regard to converses, the following should be noted. It may happen that $\mathsf{Alg}^*\mathcal{S}$ (say) is a $\boldsymbol{\tau}$–algebraic semantics for $\mathcal{S}$, forcing $\boldsymbol{\tau}$ to define truth in the matrix semantics $\{\langle \mathbf{A}, Eq^{\mathbf{A}}\boldsymbol{\tau}\rangle : \mathbf{A} \in \mathsf{Alg}^*\mathcal{S}\}$, but this class need not coincide with $\mathsf{Mod}^*\mathcal{S}$. The situation is illustrated by the next example, which shows that, in general:

(i) When truth is equationally definable in $\mathsf{LMod}^{\mathrm{Su}}\mathcal{S}$, it need not follow that truth is equationally—or even implicitly—definable in $\mathsf{LMod}^*\mathcal{S}$. (Contrast this with Proposition 18.)

(ii) When $\mathsf{Alg}^*\mathcal{S}$ is a $\boldsymbol{\tau}$–algebraic semantics for $\mathcal{S}$, it need not follow that $\boldsymbol{\tau}$ defines truth in $\mathsf{Mod}^*\mathcal{S}$, nor even that truth is implicitly definable in $\mathsf{Mod}^*\mathcal{S}$. (Contrast this with Corollary 21.)

(iii) The injectivity of the Suszko operator on just the *theories* of a deductive system does not force the existence of any theorems. (Contrast this with Lemma 5.)

Example 1. Let $\boldsymbol{\mathcal{L}}$ denote the algebraic language with one unary operation symbol $\Box$ and no other operation symbols. Here and in the sequel, we use the abbreviations $\Box^0 x = x$ and $\Box^{n+1} x = \Box(\Box^n x)$ for $n \in \omega$.

Let $\mathcal{S}$ be the finitary $\boldsymbol{\mathcal{L}}$–deductive system axiomatized by the single rule $x \vdash \Box x$. Let $\boldsymbol{\tau}$ be the translation $x \approx \Box x$.

For every $\mathcal{L}$–algebra $\mathbf{A}$, the $\mathcal{S}$–filters of $\mathbf{A}$ are evidently just the subsets of A that are closed under the term function $\Box^{\mathbf{A}}$ (including the empty set). Because $\mathcal{S}$ has no theorem, $\Omega^{\mathbf{Te}}$ is not injective on $\mathcal{S}$–theories (Lemma 5). So truth fails to be implicitly definable in $\mathsf{LMod}^*\mathcal{S}$—and therefore also in all larger semantics, such as $\mathsf{Mod}^*\mathcal{S}$ and $\mathsf{Mod}^{\mathrm{Su}}\mathcal{S}$. In particular, the Suszko operator of $\mathcal{S}$ is not globally injective (see Proposition 17).

Certainly, then, truth is not *equationally* definable in any of the three semantics just mentioned. We shall show, however, that in $\mathsf{LMod}^{\mathrm{Su}}\mathcal{S}$, truth *is* equationally definable (by $\boldsymbol{\tau}$) and therefore implicitly definable.

Consider an $\mathcal{L}$–algebra $\mathbf{A}$. For any $n \in \omega$, we shall use $\Box^n$ to abbreviate the term function of $\mathbf{A}$ defined by the $\boldsymbol{\mathcal{L}}$–term $\Box^n x$. Because $\Box$ is the sole connective of $\mathcal{L}$, it follows from Lemma 6(i) that for any $\mathcal{S}$–filter G of $\mathbf{A}$,

$$\Omega^{\mathbf{A}}G = \{\langle a,b\rangle \in A \times A : (\forall\, n \in \omega)\,(\Box^n a \in G \text{ iff } \Box^n b \in G)\} \tag{7}$$

and consequently, for all $a \in A$,

$$\langle a, \Box a\rangle \in \Omega^{\mathbf{A}}G \text{ iff } (\forall\, n \in \omega)\,(\Box^{n+1} a \in G \text{ implies } \Box^n a \in G). \tag{8}$$

Now the variety V of all $\mathcal{L}$–algebras is a $\boldsymbol{\tau}$–algebraic semantics for $\mathcal{S}$: soundness is obvious and completeness is proved in [13, p. 170]. Because a $\boldsymbol{\tau}$–algebraic semantics for $\mathcal{S}$ exists, it follows from Corollary 9 that for every $\mathcal{S}$–filter F of $\mathbf{A}$, we have $\boldsymbol{\tau}^{\mathbf{A}}[F] \subseteq \widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}}F$, i.e., $\langle a, \Box a\rangle \in \widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}}F$ for all $a \in F$.

Let F be an $\mathcal{S}$–theory and α an $\boldsymbol{\mathcal{L}}$–term. Suppose $\alpha \notin F$; we shall argue that $\langle \alpha, \Box\alpha\rangle \notin \widetilde{\Omega}^{\mathbf{Te}}_{\mathcal{S}}F$. We define

$$G := F \cup \{\Box^n \alpha : 0 < n \in \omega\}.$$

Since G is clearly closed under $\Box$, it is an $\mathcal{S}$–theory. Note that $\alpha \notin G$, but $\Box\alpha \in G$ so, by (8), $\langle \alpha, \Box\alpha\rangle \notin \Omega^{\mathbf{Te}}G$. It follows that

$$\langle \alpha, \Box\alpha\rangle \notin \bigcap\{\Omega^{\mathbf{Te}}M : F \subseteq M \in \mathrm{Fi}_{\mathcal{S}}\,\mathbf{Te}\} = \widetilde{\Omega}^{\mathbf{Te}}_{\mathcal{S}}F.$$

We have shown that for any $\alpha \in \mathrm{Te}$, we have $\alpha \in F$ iff $\langle \alpha, \Box\alpha\rangle \in \widetilde{\Omega}^{\mathbf{Te}}_{\mathcal{S}}F$. Now we apply Lemma 13, remembering that $\widetilde{\Omega}^{\mathbf{Te}}_{\mathcal{S}}F$ is always compatible with F. We conclude: for all $\langle \mathbf{B}, G\rangle \in \mathsf{LMod}^{\mathrm{Su}}\mathcal{S}$ and all $b \in B$, we have $b \in G$ iff $b = \Box^{\mathbf{B}} b$. Thus, $\boldsymbol{\tau}$ defines truth in $\mathsf{LMod}^{\mathrm{Su}}\mathcal{S}$. In particular, the Suszko operator of $\mathcal{S}$, although not globally injective, is injective on *$\mathcal{S}$–theories*.

Because $\boldsymbol{\tau}$ defines truth in $\mathsf{LMod}^{\mathrm{Su}}\mathcal{S}$, it follows from Corollary 21 that $\mathsf{LAlg}\mathcal{S}$ is another $\boldsymbol{\tau}$–algebraic semantics for $\mathcal{S}$. In fact this is true of *all* the classes of algebras mentioned in Corollary 21, despite the negative observations above. It is enough to verify this for $\mathsf{LAlg}^*\mathcal{S}$, because $\mathsf{LAlg}^*\mathcal{S} \subseteq \mathsf{Alg}^*\mathcal{S} \subseteq \mathsf{Alg}\mathcal{S} \subseteq \mathsf{V}$. And on the same grounds, for $\mathsf{LAlg}^*\mathcal{S}$, we need only verify completeness. We modify [13, p. 170]. Suppose $\Gamma \nvdash_{\mathcal{S}} \Box^n v$, where $v \in Var$ and $n \in \omega$. Then Γ cannot contain a term of the form $\Box^m v$ where $0 \leq m \leq n$, otherwise repeated use of the rule $x \vdash \Box x$ would yield a contradiction. The set $F := \Box^{n+1}[\mathrm{Te}]$ is clearly an $\mathcal{S}$–theory. Let $\mathbf{B} = \mathbf{Te}/\Omega^{\mathbf{Te}} F$. Using (7), we see that

$$\langle \mathbf{B},\, F/\Omega^{\mathbf{Te}} F\rangle \;=\; \langle\langle\{Var,\, \Box[Var],\, \ldots,\, \Box^n[Var],\, F\};\Box^{\mathbf{B}}\rangle,\, \{F\}\rangle.$$

Let $h : \mathbf{Te} \to \mathbf{B}$ be the homomorphism such that $h(v) = Var$ and $h(w) = F$ whenever $v \neq w \in Var$. Then $h(\gamma) = F = \Box^{\mathbf{B}} h(\gamma)$ for all $\gamma \in \Gamma$ but $h(\Box^n v) = \Box^n[Var] \neq F = \Box^{\mathbf{B}} h(\Box^n v)$. Since $\mathbf{B} \in \mathsf{LAlg}^*\mathcal{S}$, this completes the argument.

8 Testing for Equational Definability

By Example 1, the Suszko-reduced *formula* matrix models of $\mathcal{S}$ are not a reliable guide to whether truth is equationally definable in the full semantics $\mathsf{Mod}^{\mathrm{Su}}\mathcal{S}$ or, equivalently, in $\mathsf{Mod}^*\mathcal{S}$. In contrast, we shall show here that the *Leibniz*-reduced formula matrix models *are* a reliable guide.

Definition 4. For each $\mathcal{L}$–algebra $\mathbf{A}$, we define

$$\mathbf{A}\text{–}\mathsf{Mod}^*\mathcal{S} := \{\langle \mathbf{A}/\Omega^{\mathbf{A}} F,\, F/\Omega^{\mathbf{A}} F\rangle : F \in \mathrm{Fi}_{\mathcal{S}}\,\mathbf{A}\}.$$

Thus, $\mathsf{LMod}^*\mathcal{S} = \mathbf{Te}\text{–}\mathsf{Mod}^*\mathcal{S}$ and $\mathsf{Mod}^*\mathcal{S}$ is the isomorphic closure of the union of all $\mathbf{A}\text{–}\mathsf{Mod}^*\mathcal{S}$, where $\mathbf{A}$ ranges over all $\mathcal{L}$–algebras. Because $\Omega^{\mathbf{A}} F$ is always compatible with F, we have:

Proposition 22. *A translation* $\boldsymbol{\tau}$ *defines truth in* $\mathbf{A}\text{–}\mathsf{Mod}^*\mathcal{S}$ *iff*

$$(*)_{\mathbf{A}} \qquad (\forall F \in \mathrm{Fi}_{\mathcal{S}}\,\mathbf{A})\,(\forall a \in A)\,[\, a \in F \quad \textit{iff} \quad \boldsymbol{\tau}^{\mathbf{A}}(a) \subseteq \Omega^{\mathbf{A}} F\,].$$

Consequently, $\boldsymbol{\tau}$ *defines truth in* $\mathsf{Mod}^*\mathcal{S}$ *iff* $(*)_{\mathbf{A}}$ *holds for all* $\mathcal{L}$*–algebras* $\mathbf{A}$.

Sometimes, a deductive system $\mathcal{S}$ is *defined* in terms of an ISP–class K of algebras, which then serves as an algebraic semantics for $\mathcal{S}$ (see Section 12). In this case, it is convenient to have a test involving only the algebras in K which can establish whether a translation $\boldsymbol{\tau}$ defines truth in $\mathsf{Mod}^*\mathcal{S}$. We shall show that it is sufficient for $\boldsymbol{\tau}$ to define truth in $\bigcup_{\mathbf{A}\in\mathsf{K}} \mathbf{A}\text{–}\mathsf{Mod}^*\mathcal{S}$, provided that K is a $\boldsymbol{\tau}$–algebraic semantics for $\mathcal{S}$. We require two lemmas.

Lemma 23. *Let* $h : \mathbf{A} \to \mathbf{B}$ *be a surjective homomorphism of* $\boldsymbol{\mathcal{L}}$*–algebras. If a translation* $\boldsymbol{\tau}$ *defines truth in* $\mathbf{A}$–$\mathsf{Mod}^*\mathcal{S}$ *then* $\boldsymbol{\tau}$ *defines truth in* $\mathbf{B}$–$\mathsf{Mod}^*\mathcal{S}$*. The converse holds if* $\ker h$ *is compatible with all* $\mathcal{S}$*–filters of* $\mathbf{A}$.

Proof. Assume $(*)_{\mathbf{A}}$. Let G be an $\mathcal{S}$–filter of $\mathbf{B}$, so $h^{-1}[G]$ is an $\mathcal{S}$–filter of $\mathbf{A}$, by Lemma 1(i). Since h is surjective to $\mathbf{B}$, every element of B has the form $h(a)$ for some $a \in A$. Now for each $a \in A$, we have $h(a) \in G$ iff $a \in h^{-1}[G]$ iff $\boldsymbol{\tau}^{\mathbf{A}}(a) \subseteq \Omega^{\mathbf{A}} h^{-1}[G]$ (by $(*)_{\mathbf{A}}$) iff $\boldsymbol{\tau}^{\mathbf{A}}(a) \subseteq h^{-1}[\Omega^{\mathbf{B}} G]$ (by Lemma 7(ii)) iff $h[\boldsymbol{\tau}^{\mathbf{A}}(a)] \subseteq \Omega^{\mathbf{B}} G$ iff $\boldsymbol{\tau}^{\mathbf{B}}(h(a)) \subseteq \Omega^{\mathbf{B}} G$. This shows that $(*)_{\mathbf{B}}$ holds.

Conversely, suppose $\ker h$ is compatible with all $\mathcal{S}$–filters of $\mathbf{A}$ and that $(*)_{\mathbf{B}}$ holds. Let F be an $\mathcal{S}$–filter of $\mathbf{A}$. Since $\ker h$ is compatible with F, the set $h[F]$ is an $\mathcal{S}$–filter of $\mathbf{B}$ (Lemma 1(ii)) and $F = h^{-1}[h[F]]$. Consequently, for any $a \in A$, we have $\boldsymbol{\tau}^{\mathbf{A}}(a) \subseteq \Omega^{\mathbf{A}} F = \Omega^{\mathbf{A}} h^{-1}[h[F]]$ iff $\boldsymbol{\tau}^{\mathbf{A}}(a) \subseteq h^{-1}[\Omega^{\mathbf{B}} h[F]]$ (by Lemma 7(ii)) iff $h[\boldsymbol{\tau}^{\mathbf{A}}(a)] \subseteq \Omega^{\mathbf{B}} h[F]$ iff $\boldsymbol{\tau}^{\mathbf{B}}(h(a)) \subseteq \Omega^{\mathbf{B}} h[F]$ iff $h(a) \in h[F]$ (by $(*)_{\mathbf{B}}$) iff $a \in h^{-1}[h[F]] = F$. This proves $(*)_{\mathbf{A}}$. □

Lemma 24. *Let* $\mathbf{A}$ *be an* $\mathcal{L}$*–algebra and* $\boldsymbol{\tau}$ *a translation. Suppose* $\boldsymbol{\tau}$ *defines truth in* $\mathbf{B}$–$\mathsf{Mod}^*\mathcal{S}$ *for every finitely generated subalgebra* $\mathbf{B}$ *of* $\mathbf{A}$*. Then* $\boldsymbol{\tau}$ *defines truth in* $\mathbf{A}$–$\mathsf{Mod}^*\mathcal{S}$.

Proof. Let $F \in \mathrm{Fi}_{\mathcal{S}}\, \mathbf{A}$, and $a \in A$ with $\boldsymbol{\tau}^{\mathbf{A}}(a) \subseteq \Omega^{\mathbf{A}} F$. Let $\mathbf{B}$ be the subalgebra of $\mathbf{A}$ generated by a, so $\delta^{\mathbf{A}}(a), \varepsilon^{\mathbf{A}}(a) \in B$ for all $\langle \delta, \varepsilon \rangle \in \boldsymbol{\tau}$. By assumption, $(*)_{\mathbf{B}}$ holds. Let $G = F \cap B$ and note that G is an $\mathcal{S}$–filter of $\mathbf{B}$. Also, the congruence $(B \times B) \cap \Omega^{\mathbf{A}} F$ of $\mathbf{B}$ is clearly compatible with G, so it is contained in $\Omega^{\mathbf{B}} G$. Since $\boldsymbol{\tau}^{\mathbf{A}}(a) \subseteq (B \times B) \cap \Omega^{\mathbf{A}} F$, it follows that $\boldsymbol{\tau}^{\mathbf{B}}(a) \subseteq \Omega^{\mathbf{B}} G$. Then, by $(*)_{\mathbf{B}}$, $a \in G$, whence $a \in F$.

Conversely, suppose $a \in F$ and let $\langle \delta, \varepsilon \rangle \in \boldsymbol{\tau}$. We need to show that $\langle \delta^{\mathbf{A}}(a), \varepsilon^{\mathbf{A}}(a) \rangle \in \Omega^{\mathbf{A}} F$. We shall use the criterion of Lemma 6(i). Let p be a unary polynomial function of $\mathbf{A}$, so there exist $n \in \omega$, an $(n+1)$–ary $\boldsymbol{\mathcal{L}}$–term $\alpha \in \mathrm{Te}$ and a sequence $\vec{c}$ of n (not necessarily distinct) elements of A such that $p(e) = \alpha^{\mathbf{A}}(e, \vec{c})$ for all $e \in A$. Suppose that $p(\delta^{\mathbf{A}}(a)) \in F$. It is enough (by symmetry) to show that $p(\varepsilon^{\mathbf{A}}(a)) \in F$.

Let $\mathbf{B}$ be the subalgebra of $\mathbf{A}$ generated by a together with the range of $\vec{c}$. Then $\delta^{\mathbf{A}}(a)$, $\varepsilon^{\mathbf{A}}(a)$, $p(\delta^{\mathbf{A}}(a))$, $p(\varepsilon^{\mathbf{A}}(a)) \in B$. Also $G := F \cap B$ is an $\mathcal{S}$–filter of $\mathbf{B}$ with $a \in G$ and $p(\delta^{\mathbf{B}}(a)) = p(\delta^{\mathbf{A}}(a)) \in G$. Since, by assumption, $(*)_{\mathbf{B}}$ holds, $\boldsymbol{\tau}^{\mathbf{B}}(a) \subseteq \Omega^{\mathbf{B}} G$. In particular, $\langle \delta^{\mathbf{B}}(a), \varepsilon^{\mathbf{B}}(a) \rangle \in \Omega^{\mathbf{B}} G$. Now the restriction of p to $\mathbf{B}$ is a unary polynomial function of $\mathbf{B}$, by definition of $\mathbf{B}$. So from $p(\delta^{\mathbf{B}}(a)) \in G$ we may infer that $p(\varepsilon^{\mathbf{A}}(a)) = p(\varepsilon^{\mathbf{B}}(a)) \in G$. Thus, $p(\varepsilon^{\mathbf{A}}(a)) \in F$. □

Theorem 25. *Let $\boldsymbol{\tau}$ be an $\boldsymbol{\mathcal{L}}$–translation. Then the following conditions on $\mathcal{S}$ are equivalent:*

(i) $\boldsymbol{\tau}$ *defines truth in* $\mathsf{Mod}^*\mathcal{S}$.

(ii) $\boldsymbol{\tau}$ *defines truth in* $\mathsf{LMod}^*\mathcal{S}$.

(iii) $\mathcal{S}$ *has a* $\boldsymbol{\tau}$*–algebraic semantics and for any ISP–class* K *that is a* $\boldsymbol{\tau}$*–algebraic semantics for* $\mathcal{S}$, $\boldsymbol{\tau}$ *defines truth in* $\bigcup_{\mathbf{A}\in\mathsf{K}} \mathbf{A}$–$\mathsf{Mod}^*\mathcal{S}$.

Note. In (iii), the phrase 'for any' could be replaced by 'for some', without affecting the equivalence of the three conditions.

Proof. (i) ⇒ (ii) is trivial, as $\mathsf{LMod}^*\mathcal{S} \subseteq \mathsf{Mod}^*\mathcal{S}$.

(ii) ⇒ (iii): By (ii) and Corollary 21, $\mathcal{S}$ has a $\boldsymbol{\tau}$–algebraic semantics. So, by Lemma 2, there is an ISP–class (in fact a UISP–class) that is a $\boldsymbol{\tau}$–algebraic semantics for $\mathcal{S}$. Let K be any ISP–class with this property. Let $\mathbf{A} \in \mathsf{K}$. Since our set of variables, Var, is infinite and $\mathbf{Te}$ is K–free over Var, every countably generated algebra in K is a homomorphic image of $\mathbf{Te}$. Consequently, by (ii) and Lemma 23, $\boldsymbol{\tau}$ defines truth in $\mathbf{B}$–$\mathsf{Mod}^*\mathcal{S}$ for every countably generated algebra $\mathbf{B} \in \mathsf{K}$—in particular, for every finitely generated subalgebra $\mathbf{B}$ of $\mathbf{A}$. Using Lemma 24, we may infer that $\boldsymbol{\tau}$ defines truth in $\mathbf{A}$–$\mathsf{Mod}^*\mathcal{S}$. Since $\mathbf{A}$ was an arbitrary member of K, this establishes (iii).

(iii) ⇒ (i): Let K be an ISP–class that is a $\boldsymbol{\tau}$–algebraic semantics for $\mathcal{S}$, and assume that $\boldsymbol{\tau}$ defines truth in $\mathbf{B}$–$\mathsf{Mod}^*\mathcal{S}$ for all $\mathbf{B} \in \mathsf{K}$. Let $\mathbf{A}$ be any $\mathcal{L}$–algebra. Let $\mathbf{X}$ be the absolutely free $\mathcal{L}$–algebra generated by the elements of A. Then there is a surjective homomorphism $h : \mathbf{X} \to \mathbf{A}$. By the first assertion of Lemma 23, it is enough to show that $\boldsymbol{\tau}$ defines truth in $\mathbf{X}$–$\mathsf{Mod}^*\mathcal{S}$. Since $\mathbf{B} := \mathbf{X}/\Theta^{\mathbf{X}}_{\mathsf{K}}\emptyset \in \mathsf{K}$, we know that $\boldsymbol{\tau}$ defines truth in $\mathbf{B}$–$\mathsf{Mod}^*\mathcal{S}$. Now, by Proposition 4, $\Theta^{\mathbf{X}}_{\mathsf{K}}\emptyset$ is compatible with all $\mathcal{S}$–filters of $\mathbf{X}$. Thus, by the second assertion of Lemma 23, $\boldsymbol{\tau}$ defines truth in $\mathbf{X}$–$\mathsf{Mod}^*\mathcal{S}$.

The proof given here justifies the above Note as well. □

It follows that the demand '$\boldsymbol{\tau}$ defines truth in $\mathsf{Mod}^*\mathcal{S}$' is also characterized by the syntactic condition in Proposition 19(ii). The same applies to

'$\boldsymbol{\tau}$ defines truth in $\mathsf{Mod}^{\mathrm{Su}}\mathcal{S}$', by Proposition 18(i). Combining Theorem 25 with Corollaries 21 and 16, we get:

Corollary 26. *If an $\boldsymbol{\mathcal{L}}$–translation $\boldsymbol{\tau}$ defines truth in* $\mathsf{LMod}^*\mathcal{S}$ *then each of* $\mathsf{LAlg}^*\mathcal{S}$, $\mathsf{Alg}^*\mathcal{S}$ *and* $\mathsf{Alg}\mathcal{S}$ *is a $\boldsymbol{\tau}$–algebraic semantics for $\mathcal{S}$.*

9 Truth-Equational Systems

Definition 5. A deductive system $\mathcal{S}$ will be called *truth-equational* if truth is equationally definable in its class of Leibniz-reduced formula matrix models, $\mathsf{LMod}^*\mathcal{S}$. Any translation that defines truth in $\mathsf{LMod}^*\mathcal{S}$ is said to *witness* the truth-equationality of $\mathcal{S}$.

Note that when $\mathcal{S}$ is truth-equational then truth is also equationally definable (by the same translation) in the much larger matrix semantics $\mathsf{Mod}^{\mathrm{Su}}\mathcal{S}$, and therefore in $\mathsf{Mod}^*\mathcal{S}$ and in $\mathsf{LMod}^{\mathrm{Su}}\mathcal{S}$. This follows from Theorem 25 and Proposition 18(i) and it serves to justify the definition. Recall, in contrast, that the equational definability of truth in $\mathsf{LMod}^{\mathrm{Su}}\mathcal{S}$ does not imply its equational (or even implicit) definability in any of the other matrix semantics mentioned, by Example 1. We shall take no further interest in $\mathsf{LMod}^{\mathrm{Su}}\mathcal{S}$.

Truth-equationality has received attention in the literature, partly because it defines the difference between 'equivalential' and 'algebraizable' logics. Study of the condition has been confined almost entirely to systems that are 'protoalgebraic', a restriction that we do not impose here. (The adjectives in quotation marks will be defined in Section 11.)

Theorem 27. *If the Leibniz operator is completely order reflecting on the theories of $\mathcal{S}$ (see Definition 2) then $\mathcal{S}$ is truth-equational.*

Proof. Let $\Omega^{\mathbf{Te}}$ be completely order reflecting on $\mathrm{Fi}_{\mathcal{S}}\,\mathbf{Te}$. Let k be the $\boldsymbol{\mathcal{L}}$–substitution such that $k(y) = x$ for all variables $y \in Var$. Define

$$\boldsymbol{\tau} \;=\; \boldsymbol{\tau}(x) := \; k\,[\widetilde{\Omega}^{\mathbf{Te}}_{\mathcal{S}}\,\mathrm{Fg}^{\mathbf{Te}}_{\mathcal{S}}\{x\}].$$

This makes $\boldsymbol{\tau}$ an $\boldsymbol{\mathcal{L}}$–translation and we shall show that it defines truth in $\mathsf{LMod}^*\mathcal{S}$. By Proposition 22, it is enough to prove the condition $(*)_{\mathbf{Te}}$ for this $\boldsymbol{\tau}$. By Proposition 12, $\boldsymbol{\tau}(x) \subseteq \widetilde{\Omega}^{\mathbf{Te}}_{\mathcal{S}}\,\mathrm{Fg}^{\mathbf{Te}}_{\mathcal{S}}\{x\}$. By Proposition 8, therefore, we have $\boldsymbol{\tau}[F] \subseteq \widetilde{\Omega}^{\mathbf{Te}}_{\mathcal{S}}F \subseteq \Omega^{\mathbf{Te}}F$, whenever F is an $\mathcal{S}$–theory.

Conversely, let G be an $\mathcal{S}$–theory and suppose that $\boldsymbol{\tau}(\gamma) \subseteq \Omega^{\mathbf{Te}}G$. We must show that $\gamma \in G$. Let h be an $\boldsymbol{\mathcal{L}}$–substitution such that $h(x) = \gamma$, so

$$h[\boldsymbol{\tau}(x)] \;=\; \boldsymbol{\tau}(h(x)) \;=\; \boldsymbol{\tau}(\gamma) \subseteq \Omega^{\mathbf{Te}}G.$$

We therefore have $k\,[\widetilde{\Omega}^{\mathbf{Te}}_{\mathcal{S}}\,\mathrm{Fg}^{\mathbf{Te}}_{\mathcal{S}}\{x\}] \;=\; \boldsymbol{\tau}(x) \;\subseteq\; h^{-1}[\Omega^{\mathbf{Te}}G]$, i.e.,

$$\widetilde{\Omega}^{\mathbf{Te}}_{\mathcal{S}}\,\mathrm{Fg}^{\mathbf{Te}}_{\mathcal{S}}\{x\} \;\subseteq\; k^{-1}h^{-1}[\Omega^{\mathbf{Te}}G] \;=\; (hk)^{-1}[\Omega^{\mathbf{Te}}G] \;\subseteq\; \Omega^{\mathbf{Te}}(hk)^{-1}[G],$$

by Lemma 7(i). Since $\Omega^{\mathbf{Te}}$ is completely order reflecting on $\mathcal{S}$–theories, it follows that $\mathrm{Fg}^{\mathbf{Te}}_{\mathcal{S}}\{x\} \;\subseteq\; (hk)^{-1}[G]$ (see (5)) and, in particular, $x \in (hk)^{-1}[G]$, i.e., $\gamma = h(x) = hk(x) \in G$, as required. ☐

Putting together Theorems 11 and 27, we obtain some characterizations of truth-equational deductive systems that do not mention translations and that are in the spirit of the 'Leibniz hierarchy' (see [26]). The situation is summarized below in what may be considered the main theorem of this paper. Recall that $\mathcal{S}$ continues to denote an *arbitrary* deductive system over $\boldsymbol{\mathcal{L}}$.

Theorem 28. *The following conditions on $\mathcal{S}$ are equivalent.*

(i) $\mathcal{S}$ is truth-equational.

(ii) Truth is equationally definable in $\mathsf{Mod}^{\mathrm{Su}}\mathcal{S}$.

(iii) The Suszko operator of $\mathcal{S}$ is globally injective.

(iv) The Leibniz operator of $\mathcal{S}$ is globally completely order reflecting.

(v) The Leibniz operator is completely order reflecting on $\mathcal{S}$–theories.

(vi) For every $\mathcal{S}$–filter F of any $\mathcal{L}$–algebra $\mathbf{A}$, the least $\mathcal{S}$–filter of $\mathbf{A}/\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}}F$ is $F/\widetilde{\Omega}^{\mathbf{A}}_{\mathcal{S}}F$.

In this case $\mathcal{S}$ has a globally injective Leibniz operator, an algebraic semantics, and some theorems.

Proof. Note first that the equivalence of (iii) and (vi) is Czelakowski's result, Theorem 10. The implication (i) ⇒ (ii) was noted after Definition 5. (ii) ⇒ (iii) follows from Proposition 17, while (iii) ⇒ (iv) is just Theorem 11 and (iv) ⇒ (v) is trivial. (v) ⇒ (i) is Theorem 27. The last assertion is justified by the remark after Proposition 17, together with Corollary 21 and Lemma 5. □

Bearing Proposition 17 in mind, we see that (iii) ⇒ (ii) says:

Corollary 29. *When truth is implicitly definable in the class of all* Suszko-*reduced matrix models of a deductive system, it is equationally definable in the same class, i.e., it is explicitly definable by a (possibly infinite) conjunction of equations.*

This result is essentially a consequence of the *structurality* of deductive systems. Recall that even when $\mathsf{Mod}^{\mathrm{Su}}\mathcal{S}$ is an elementary class and (iii) holds, the Beth Definability Theorem predicts only the explicit first order definability of the truth predicate, not its equational definability. Previously, Corollary 29 was known only for protoalgebraic deductive systems; references will be given in Section 11.

The last statement of Theorem 28 prompts the question: are the truth-equational deductive systems *characterized* by the global injectivity of their *Leibniz* operators—possibly in conjunction with their possession of an algebraic semantics? This in turn prompts the question: *must* a deductive system with a globally injective Leibniz operator possess an algebraic semantics? The next two examples answer both of these questions in the negative.

Example 2 shows that *global injectivity of the Leibniz operator does not entail truth-equationality*, i.e., it does not entail global injectivity of the Suszko operator. Thus, the analogue of Corollary 29 for *Leibniz*-reduced matrix models is *false*. Moreover, the counter-example is a finitary system that has an elementary class of Leibniz-reduced matrix models and an algebraic semantics with respect to a finite translation.

Example 2. Let $\boldsymbol{\mathcal{L}}$ be a language with a constant symbol $\top$, two (distinct) unary connectives $\Box, \Diamond$ and no other connective. Let $\mathcal{S}$ be the finitary deductive system over $\boldsymbol{\mathcal{L}}$ axiomatized by the (axioms and) rules

(S1)	$\vdash \top$	(S4)	$\vdash \Diamond\Diamond x$
(S2)	$\vdash \Box\Box x$	(S5)	$\Box x \vdash \Diamond x$
(S3)	$\Box\top \vdash x$	(S6)	$\Diamond x \vdash \Box x.$

It is not difficult to show that $\mathcal{S}$ is *strongly finite* in the sense of Wójcicki [51]: it is the consequence relation of a single three-element matrix, viz.,

$$\langle\langle\{\bot, 0, \top\}; \Box, \Diamond, \top\rangle, \{\bot, \top\}\rangle, \tag{9}$$

where $\Box\bot = \bot = \Box 0$ and $\Box\top = 0$, and $\Diamond$ coincides with $\Box$. But we shall not rely on this fact in what follows.

The reader will wonder why the connective $\Diamond$ was included in the signature, as $\Diamond\alpha$ is interderivable with $\Box\alpha$, and $\Diamond$ has no feature that $\Box$ lacks. The reason is that, as we shall show, the presence of $\Diamond$ forces $\mathcal{S}$ to possess an algebraic semantics, whereas the $\Diamond$–free fragment of $\mathcal{S}$ has no algebraic semantics.

We shall say that an $\boldsymbol{\mathcal{L}}$–term is *deep* if it has the form $\alpha(\beta(\gamma))$ where $\alpha, \beta \in \{\Box, \Diamond\}$ and γ is any $\boldsymbol{\mathcal{L}}$–term. Let Dp be the set of all deep $\boldsymbol{\mathcal{L}}$–terms.

From $\vdash_{\mathcal{S}} \Box\Box x$ and (S5), we get $\vdash_{\mathcal{S}} \Diamond\Box x$. Similarly, $\vdash_{\mathcal{S}} \Diamond\Diamond x$ and (S6) give $\vdash_{\mathcal{S}} \Box\Diamond x$. Applying structurality, we see that every deep $\boldsymbol{\mathcal{L}}$–term is a theorem of $\mathcal{S}$. Evidently, $Dp \cup \{\top\}$ is an $\mathcal{S}$–theory, so the only theorem of $\mathcal{S}$ that is not deep is $\top$. In particular, x, $\Box x$ and $\Box\top$ are not $\mathcal{S}$–theorems.

Let $\mathbf{A}$ be an $\mathcal{L}$–algebra. We shall abuse notation by writing α for the term function $\alpha^{\mathbf{A}}$ of $\mathbf{A}$ induced by an $\boldsymbol{\mathcal{L}}$–term α. For each $a \in A$, every $\mathcal{S}$–filter of $\mathbf{A}$ contains both or neither of $\Box a, \Diamond a$, by (S5) and (S6). By (S3), the proper $\mathcal{S}$–filters of $\mathbf{A}$ all exclude $\Box\top$ and $\Diamond\top$ but contain $\{\top\} \cup Dp[\mathbf{A}]$, where $Dp[\mathbf{A}] := \{\sigma(a) : \sigma \in Dp \text{ and } a \in A\}$.

Let F be an $\mathcal{S}$–filter of $\mathbf{A}$. By Lemma 6(i) and the above observations,

$$\Omega^{\mathbf{A}} F = \{\langle a, b\rangle \in A \times A : (a \in F \text{ iff } b \in F) \text{ and } (\Box a \in F \text{ iff } \Box b \in F)\}. \tag{10}$$

This shows that the class of all Leibniz-reduced matrix models of $\mathcal{S}$ is elementary (see the remarks preceding Section 7).

Let $a \in F$. If $\Box a \in F$ then, by (10), $\langle a, \Box^2 a\rangle \in \Omega^{\mathbf{A}} F$. If $\Box a \notin F$ then $F \neq A$, whence $\Box\top \notin F$, so again by (10), $\langle a, \top\rangle \in \Omega^{\mathbf{A}} F$. Conversely, if $\langle a, \Box^2 a\rangle \in \Omega^{\mathbf{A}} F$ or $\langle a, \top\rangle \in \Omega^{\mathbf{A}} F$ then, since $\Box^2 a, \top \in F$, we have $a \in F$, by compatibility. We have shown that for any $a \in A$,

$$a \in F \text{ iff } [\,\langle a, \Box^2 a\rangle \in \Omega^{\mathbf{A}} F \text{ or } \langle a, \top\rangle \in \Omega^{\mathbf{A}} F\,]. \tag{11}$$

It follows immediately that $\Omega^{\mathbf{A}}$ is injective on the $\mathcal{S}$–filters of $\mathbf{A}$, so the Leibniz operator of $\mathcal{S}$ is globally injective, i.e., truth is implicitly definable in $\mathsf{Mod}^*\mathcal{S}$.

Notice that (11) could be rephrased as asserting that the truth predicate P of $\mathsf{Mod}^*\mathcal{S}$ is *explicitly* first order definable by the sentence

$$(\forall x)\,[\,Px \iff (x \approx \Box^2 x \ \text{ or } \ x \approx \top)\,], \tag{12}$$

in view of Lemma 13 and the definition of factor algebras.

To disprove truth-equationality, we use the subalgebra $\mathbf{B} = \mathbf{Te}(x)$ of $\mathbf{Te}$, so

$$B = \{\top,\, \Box\top,\, \Diamond\top,\, x,\, \Box x,\, \Diamond x\} \,\cup\, Dp[\mathbf{B}].$$

Consider the chain $F \subseteq G \subseteq H$ of proper $\mathcal{S}$–filters of $\mathbf{B}$, where

$$F = \{\top\} \cup Dp[\mathbf{B}]\,; \quad G = \{\top, x\} \cup Dp[\mathbf{B}]\,; \quad H = \{\top, x, \Box x, \Diamond x\} \cup Dp[\mathbf{B}].$$

We claim that the congruence $\theta := \Omega^{\mathbf{B}}G \cap \Omega^{\mathbf{B}}H$ is compatible with F. To see this, suppose $\langle \alpha, \beta\rangle \in \theta$ and $\alpha \in F$, hence $\alpha \in G$. Then $\beta \in G$, because $\langle \alpha, \beta\rangle \in \Omega^{\mathbf{B}}G$. We need to show that $\beta \in F$. Suppose $\beta \notin F$. Then $\beta = x$. Now $\langle \alpha, x\rangle \in \Omega^{\mathbf{B}}H$, whence $\langle \Box\alpha, \Box x\rangle \in \Omega^{\mathbf{B}}H$, and $\Box x \in H$ but $\Box\top \notin H$, so $\alpha \neq \top$. Since $\alpha \in F - \{\top\}$, it follows that $\Box\alpha \in G$, but $\Box x \notin G$. This contradicts the fact that $\langle \alpha, x\rangle \in \Omega^{\mathbf{B}}G$. We conclude that $\beta \in F$, vindicating the above claim. Consequently, $\Omega^{\mathbf{B}}G \cap \Omega^{\mathbf{B}}H \subseteq \Omega^{\mathbf{B}}F$, but $G \cap H = G \nsubseteq F$.

This shows that the Leibniz operator of $\mathcal{S}$ is not completely order reflecting on the $\mathcal{S}$–filters of $\mathbf{B}$ so, by Theorem 28, $\mathcal{S}$ is not truth-equational. In other words, the positive formula on the right of the biconditional in (12) cannot be replaced by any conjunction of unary equations in x.

We still need to justify the claim that $\mathcal{S}$ has an algebraic semantics. To do this we invoke the following result of Blok and Rebagliato.

Theorem 30. ([13, Thm. 3.1]) *Let* M *be a matrix semantics for a finitary deductive system* $\mathcal{S}$ *over* $\boldsymbol{\mathcal{L}}$*. Suppose that* $\boldsymbol{\mathcal{L}}$ *contains an* n*–ary connective* δ*, where* $1 \leq n < \omega$*, and a unary term* $\varepsilon(x)$ *whose main symbol (if any) is different from* δ*. If for every* $\langle \mathbf{A}, F\rangle \in \mathsf{M}$ *and every* $a \in F$*, we have*

$$\delta^{\mathbf{A}}(a, \ldots, a) \,=\, \varepsilon^{\mathbf{A}}(a)$$

then $\mathcal{S}$ *has a* $\boldsymbol{\tau}$*–algebraic semantics, where* $\boldsymbol{\tau}$ *is* $\delta(x, \ldots, x) \approx \varepsilon(x)$*.*

Returning to our example, we recall that $\mathsf{Mod}^*\mathcal{S}$ is a matrix semantics for $\mathcal{S}$, by Proposition 14. Note that for any $\mathcal{S}$–filter F of an $\mathcal{L}$–algebra

A and any $a \in A$, we have $\langle \Diamond^{\mathbf{A}} a, \Box^{\mathbf{A}} a \rangle \in \Omega^{\mathbf{A}} F$, by (10), (S5) and (S6). Thus, the algebra $\mathbf{A}/\Omega^{\mathbf{A}} F$ satisfies $\Diamond x \approx \Box x$. Then by Lemma 13, $\mathsf{Alg}^* \mathcal{S}$ satisfies $\Diamond x \approx \Box x$. Also, $\Diamond$ and $\Box$ are distinct unary connectives of $\mathcal{S}$, so Theorem 30 shows that $\mathcal{S}$ has an algebraic semantics with respect to the translation $\Diamond x \approx \Box x$.[4]

We shall now show that a deductive system with a globally injective Leibniz operator need not have an algebraic semantics (even if it is finitary, with an elementary class of Leibniz-reduced matrix models).

Example 3. Let $\mathcal{S}'$ be the finitary deductive system got from the system $\mathcal{S}$ of Example 2 by deleting $\Diamond$ from the signature and (S4)–(S6) from the axiomatization. It is quite easy to show that $\mathcal{S}'$ is the consequence relation of the $\Diamond$–free reduct of the matrix displayed in (9), but again we shall not rely on this knowledge. Let $\boldsymbol{\mathcal{L}}'$ be the language of $\mathcal{S}'$. Following Example 2 but ignoring $\Diamond$, we can establish the condition (10) and deduce that $\mathcal{S}'$ has a globally injective Leibniz operator. We claim that $\mathcal{S}'$ has no algebraic semantics.

Suppose, on the contrary, that K is a $\boldsymbol{\tau}$–algebraic semantics for $\mathcal{S}'$, where $\boldsymbol{\tau} = \{\langle \delta_i, \varepsilon_i \rangle : i \in I\}$ is an $\boldsymbol{\mathcal{L}}'$–translation. There are only four kinds of value that each $\langle \delta_i(x), \varepsilon_i(x) \rangle$ may have, viz.

$$
\begin{array}{lrcl}
\text{(i)} & \langle \delta_i(x), \varepsilon_i(x) \rangle & = & \langle \Box^{n_i} \top, \, \Box^{m_i} \top \rangle \\
\text{(ii)} & \langle \delta_i(x), \varepsilon_i(x) \rangle & = & \langle \Box^{n_i} x, \Box^{m_i} x \rangle \\
\text{(iii)} & \langle \delta_i(x), \varepsilon_i(x) \rangle & = & \langle \Box^{n_i} x, \Box^{m_i} \top \rangle \\
\text{(iv)} & \langle \delta_i(x), \varepsilon_i(x) \rangle & = & \langle \Box^{n_i} \top, \Box^{m_i} x \rangle,
\end{array}
$$

for suitable integers $n_i, m_i \in \omega$ in each case.

We claim that K satisfies $\delta_i(\Box \top) \approx \varepsilon_i(\Box \top)$ for all $i \in I$. Then, because K is a $\boldsymbol{\tau}$–algebraic semantics for $\mathcal{S}'$, it will follow that $\vdash_{\mathcal{S}'} \Box \top$, a contradiction. Thus we will conclude that $\mathcal{S}'$ has no algebraic semantics. So it remains only to verify the above claim.

Since $\vdash_{\mathcal{S}'} \top$ and $\vdash_{\mathcal{S}'} \Box^2 x$ and K is a $\boldsymbol{\tau}$–algebraic semantics for $\mathcal{S}'$, we

[4] If we apply the proof in [13] of Theorem 30 directly to the singleton matrix semantics for $\mathcal{S}$ displayed in (9), we get a singleton $\{\langle \Diamond x, \Box x \rangle\}$–algebraic semantics $\{\mathbf{A}\}$ for $\mathcal{S}$, where $\mathbf{A}$ is a four-element algebra.

have

$$\models_{\mathsf{K}} \ \delta_i(\top) \approx \varepsilon_i(\top) \ \text{ and } \tag{13}$$

$$\models_{\mathsf{K}} \ \delta_i(\Box^2 x) \approx \varepsilon_i(\Box^2 x) \tag{14}$$

for all $i \in I$.

Let $i \in I$. In Case (i) above, (13) says that K satisfies $\Box^{n_i}\top \approx \Box^{m_i}\top$, which is precisely the same equation as $\delta_i(\Box\top) \approx \varepsilon_i(\Box\top)$.

In Case (ii), (13) says that K satisfies $\Box^{n_i}\top \approx \Box^{m_i}\top$. Therefore K satisfies

$$\Box^{n_i}(\Box\top) \ \approx \ \Box(\Box^{n_i}\top) \ \approx \ \Box(\Box^{m_i}\top) \ \approx \ \Box^{m_i}(\Box\top),$$

i.e., K satisfies $\delta_i(\Box\top) \approx \varepsilon_i(\Box\top)$.

In Case (iii), (13) and (14) say that K satisfies

$$\Box^{n_i}\top \approx \Box^{m_i}\top \tag{15}$$

$$\Box^{n_i+2}x \approx \Box^{m_i}\top. \tag{16}$$

Using (15) and (16), we see that K satisfies

$$\Box^{n_i}(\Box\top) \ \approx \ \Box(\Box^{n_i}\top) \ \approx \ \Box(\Box^{m_i}\top) \ \approx \ \Box(\Box^{n_i+2}x) \ \approx \ \Box^{n_i+2}(\Box x) \ \approx \ \Box^{m_i}\top,$$

the last equality being a substitution instance of (16). In other words, K satisfies $\delta_i(\Box\top) \approx \varepsilon_i(\Box\top)$. The same conclusion follows in Case (iv), by a symmetric argument, and this completes the proof.

In view of Example 2, it is natural to ask: when the Leibniz operator of $\mathcal{S}$ is globally injective, but $\mathsf{Mod}^*\mathcal{S}$ is not necessarily elementary, is truth explicitly definable in some 'weaker-than-equational' sense in $\mathsf{Mod}^*\mathcal{S}$?

To answer this, let us define $\kappa = \kappa(\boldsymbol{\mathcal{L}}) = \max\{|\mathcal{L}|, |Var|\}$. In general, $\mathsf{Mod}^*\mathcal{S}$ is merely an '$L_{(2^\kappa)^+\,|Var|^+}$ *class*' in the sense of infinitary model theory. That is, it is the class of all models of a 'sentence' of the infinitary language $L_{(2^\kappa)^+\,|Var|^+}(\mathcal{L} \cup \{P\})$, where P is the unary truth predicate. The sentences of this language are defined recursively like those of the first order language with equality induced by $\mathcal{L} \cup \{P\}$, except that conjunctions of up to 2^κ formulas and quantification over up to $|Var|$ variables may occur.[5] No reasonable analogue of Beth's Definability Theorem holds for

[5]It is understood that the infinitary language has $(2^\kappa)^+ + |Var|^+$ variables of its own.

$L_{(2^\kappa)^+\,|Var|^+}$ classes, even if we assume that Var is denumerable (see [23, Counterex. 2.4.5]).

Now let us define $\lambda = \lambda(\boldsymbol{\mathcal{L}}) = \max\{|\mathcal{L}|, \omega\}$. When $\mathcal{S}$ is finitary, $|Var|$ may as well be assumed denumerable; then $\mathsf{Mod}^*\mathcal{S}$ becomes an $L_{\lambda^+\,\omega}$ class. The proof makes use of Proposition 14 and Lemma 6(i). In this case, by Malitz' extension of Beth's Theorem ([34] or [23, pp. 132–3, 149]), if the truth predicate of $\mathsf{Mod}^*\mathcal{S}$ is implicitly definable then it is explicitly definable by some $L_{(2^\lambda)^+\,\lambda^+}$ formula Φ in one free variable. Note that a conjunction of unary $\mathcal{L}$–equations in x is logically equivalent to an $L_{\lambda^+\,\omega}$ formula (because $\lambda = |\,\mathrm{Te}\,|$).

When $\mathcal{S}$ is finitary, if $|\mathcal{L}|$ and $|Var|$ are countable then a variant of Beth's Theorem due to López-Escobar ([32] or [30, Thm. 5, p. 21]) gives a better conclusion than Malitz' Theorem. In this case, if truth is implicitly definable in $\mathsf{Mod}^*\mathcal{S}$ then it is explicitly defined by an $L_{\omega_1\omega}$ formula, i.e., by a formula Φ in one free variable that may involve denumerable conjunctions but in which only finite quantification occurs. Herrmann [29] has drawn attention to this fact. The point of Example 2 is that, even under these favourable conditions, Φ need not amount to a conjunction of equations.

The following corollary of Theorem 28 indicates how we should strengthen the demand that a system have an algebraic semantics and an injective Leibniz operator, in order to ensure that the system be truth-equational.

Corollary 31. *The following conditions on* $\mathcal{S}$ *are equivalent*:

(i) $\mathcal{S}$ *is truth-equational.*

(ii) $\mathcal{S}$ *has an algebraic semantics* K *such that on every* $\mathcal{L}$*–algebra* $\mathbf{A}$*, the Leibniz operator of* $\mathcal{S}$ *defines a function from* $\mathrm{Fi}_{\mathcal{S}}\,\mathbf{A}$ *into* $\mathrm{Con}_{\,\mathrm{UISP(K)}}\,\mathbf{A}$*, and this function has a left inverse that commutes with arbitrary meets.*

(iii) Some homomorphism from the complete meet semilattice reduct of $\mathbf{Con}_{\,\mathrm{UISP(LAlg^*\mathcal{S})}}\,\mathbf{Te}$ *to that of* $\mathbf{Fi}_{\mathcal{S}}\,\mathbf{Te}$ *serves as a left inverse for the Leibniz operator on* $\mathcal{S}$*–theories.*

Proof. (i) ⇒ (ii): Set $\mathsf{K} = \mathsf{Alg}^*\mathcal{S}$ and let $\boldsymbol{\tau}$ witness the truth-equationality of $\mathcal{S}$. For each $\mathcal{L}$–algebra $\mathbf{A}$ and each $\theta \in \mathrm{Con}_{\,\mathrm{UISP(K)}}\,\mathbf{A}$, define $\Lambda\theta = \{a \in A : \boldsymbol{\tau}^{\mathbf{A}}(a) \subseteq \theta\}$. By Corollary 21, K is a $\boldsymbol{\tau}$–algebraic semantics for $\mathcal{S}$, and from this it follows easily that $\Lambda\theta$ is an $\mathcal{S}$–filter of $\mathbf{A}$. Obviously

Λ commutes with arbitrary intersections of UISP(K)–congruences. Note that for any $\mathcal{S}$–filter F of $\mathbf{A}$, the congruence $\Omega^{\mathbf{A}}F$ is a K–congruence, by Lemma 13, and therefore a UISP(K)–congruence. Finally, Λ is a left inverse for $\Omega^{\mathbf{A}}$, because $\boldsymbol{\tau}$ defines truth in $\mathsf{Mod}^*\mathcal{S}$.

(ii) $\Rightarrow$ (iii): By (ii) and Lemma 13, $\mathsf{LAlg}^*\mathcal{S} \subseteq$ UISP(K). Then setting $\mathbf{A} = \mathbf{Te}$ in (ii), we get (iii).

(iii) $\Rightarrow$ (i): Let Λ denote a homomorphism as described in (iii). We show that $\Omega^{\mathbf{Te}}$ is completely order reflecting. Suppose $\bigcap \Omega^{\mathbf{Te}}[\mathcal{F}] \subseteq \Omega^{\mathbf{Te}}G$, where $\mathcal{F} \cup \{G\} \subseteq \mathrm{Fi}_{\mathcal{S}}\,\mathbf{Te}$. Recall that whenever K is a class of $\mathcal{L}$–algebras that is closed under subdirect products, then the K–congruences of *any* $\mathcal{L}$–algebra are closed under arbitrary intersections. In particular, $\bigcap \Omega^{\mathbf{Te}}[\mathcal{F}]$ is a UISP($\mathsf{LAlg}^*\mathcal{S}$)–congruence of $\mathbf{Te}$. Now

$$\bigcap \mathcal{F} \;=\; \bigcap \Lambda\Omega^{\mathbf{Te}}[\mathcal{F}] \;=\; \Lambda \bigcap \Omega^{\mathbf{Te}}[\mathcal{F}] \subseteq \Lambda\Omega^{\mathbf{Te}}G = G,$$

as required. By Theorem 28, therefore, $\mathcal{S}$ is truth-equational. □

The following problem is open. (Partial solutions are discussed in Sections 11 and 14 below.)

Problem 1. *If the Leibniz operator of $\mathcal{S}$ is injective on the* theories *of $\mathcal{S}$, must it be globally injective*?

Equivalently, if truth is implicitly definable in $\mathsf{LMod}^*\mathcal{S}$, *must it be implicitly definable in* $\mathsf{Mod}^*\mathcal{S}$?

10 Finite Translations

It seems at present that in all interesting applications of the theory of algebraic semantics, the translations used are finite, or can be chosen finite. In fact they are almost always singletons. This is rather at odds with the extravagant choice of $\boldsymbol{\tau}$ that we made when proving Theorem 27. Some remarks about the existence of adequate finite translations for truth-equational systems are therefore in order (see Proposition 39 also).

Recall that $\mathcal{S}$ is a fixed (but arbitrary) deductive system over $\boldsymbol{\mathcal{L}}$. We define

$$\boldsymbol{\tau}_\infty \;=\; (\mathrm{Te}(x) \times \mathrm{Te}(x)) \cap \widetilde{\Omega}^{\mathbf{Te}}_{\mathcal{S}}\, \mathrm{Fg}^{\mathbf{Te}}_{\mathcal{S}}\{x\} \;=\; k\,[\widetilde{\Omega}^{\mathbf{Te}}_{\mathcal{S}}\, \mathrm{Fg}^{\mathbf{Te}}_{\mathcal{S}}\{x\}],$$

where k is the substitution sending all variables to x. The equality of the second and third expressions displayed above was demonstrated in Proposition 12.

Proposition 32. *If $\mathcal{S}$ is truth-equational then $\boldsymbol{\tau}_\infty$ is the largest $\boldsymbol{\mathcal{L}}$–translation $\boldsymbol{\tau}$ satisfying either one of the following conditions.*

(i) $\boldsymbol{\tau}$ witnesses the truth-equationality of $\mathcal{S}$.

(ii) $\mathcal{S}$ has a $\boldsymbol{\tau}$–algebraic semantics.

Proof. Let $\mathcal{S}$ be truth-equational. We know that (i) implies (ii). By Theorem 28 and the proof of Theorem 27, $\boldsymbol{\tau}_\infty$ witnesses the truth-equationality of $\mathcal{S}$. Also, whenever $\mathcal{S}$ has a $\boldsymbol{\tau}$–algebraic semantics, where $\boldsymbol{\tau}$ is an $\boldsymbol{\mathcal{L}}$–translation, we must have $\boldsymbol{\tau} \subseteq \boldsymbol{\tau}_\infty$, by Corollary 9. □

Now assume that $\mathcal{S}$ is truth-equational and let $\mathsf{K} := \mathrm{UISP}(\mathsf{LAlg}^*\mathcal{S})$. By Corollary 21 and Lemma 2, K is a $\boldsymbol{\tau}$–algebraic semantics for $\mathcal{S}$ whenever $\boldsymbol{\tau}$ witnesses the truth-equationality of $\mathcal{S}$. In particular K is a $\boldsymbol{\tau}_\infty$–algebraic semantics for $\mathcal{S}$. Note that K contains $\mathsf{LAlg}\mathcal{S}$, by Proposition 15(iii). By the Homomorphism Theorem, $\mathbf{Te}(x)/\boldsymbol{\tau}_\infty$ can be embedded into

$$\mathbf{Te}/\widetilde{\Omega}_{\mathcal{S}}^{\mathbf{Te}}\,\mathrm{Fg}_{\mathcal{S}}^{\mathbf{Te}}\{x\} \in \mathsf{K},$$

so $\boldsymbol{\tau}_\infty$ is a K–congruence of $\mathbf{Te}(x)$. Using the definition of $\mathsf{LAlg}^*\mathcal{S}$, we can easily see that for any $\boldsymbol{\mathcal{L}}$–translation $\boldsymbol{\tau}$, the following conditions are equivalent (assuming still that $\mathcal{S}$ is truth-equational):

(i) $\boldsymbol{\tau}$ witnesses the truth-equationality of $\mathcal{S}$.

(ii) $\boldsymbol{\tau} \subseteq \boldsymbol{\tau}_\infty$ and $\boldsymbol{\tau}(x) \models_{\mathsf{LAlg}^*\mathcal{S}} \boldsymbol{\tau}_\infty(x)$.

(iii) $\boldsymbol{\tau}_\infty = \Theta^{\mathbf{Te}(x)}_{\mathrm{UISP}(\mathsf{LAlg}^*\mathcal{S})}\,\boldsymbol{\tau}$.

Consequently, we have:

Proposition 33. *The following conditions on $\mathcal{S}$ are equivalent.*

(i) Some finite *translation witnesses the truth-equationality of $\mathcal{S}$.*

(ii) $\mathcal{S}$ is truth-equational and the $\mathrm{UISP}(\mathsf{LAlg}^*\mathcal{S})$*–congruence $\boldsymbol{\tau}_\infty$ of $\mathbf{Te}(x)$ is finitely generated in $\mathbf{Con}_{\,\mathrm{UISP}(\mathsf{LAlg}^*\mathcal{S})}\,\mathbf{Te}(x)$.*

Theorem 34. *Suppose $\mathcal{S}$ is a finitary deductive system for which* $\mathsf{Mod}^*\mathcal{S}$ *is closed under ultraproducts (and is therefore an elementary class).*[6]

If $\mathcal{S}$ is truth-equational then some finite *translation $\boldsymbol{\tau}$ witnesses its truth-equationality, and $\mathcal{S}$ has a $\boldsymbol{\tau}$–algebraic semantics that is a quasivariety.*

Proof. Let $\mathcal{S}$ be truth-equational, so $\boldsymbol{\tau}_\infty$ witnesses its truth-equationality and $\mathcal{S}$ has a $\boldsymbol{\tau}_\infty$–algebraic semantics. Corollary 9 and Lemma 13 show that whenever $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^*\mathcal{S}$ and $a \in F$ then $\delta^{\mathbf{A}}(a) = \varepsilon^{\mathbf{A}}(a)$ for all $\langle \delta, \varepsilon\rangle \in \boldsymbol{\tau}_\infty$.

Suppose the first claim in the theorem is false. Then for every finite subset $\boldsymbol{\tau}$ of $\boldsymbol{\tau}_\infty$, there is a Leibniz-reduced matrix model $\langle \mathbf{B}, G_{\boldsymbol{\tau}}\rangle = \langle \mathbf{B}_{\boldsymbol{\tau}}, G_{\boldsymbol{\tau}}\rangle$ of $\mathcal{S}$ and an element $b_{\boldsymbol{\tau}} \in B$ such that $\delta^{\mathbf{B}}(b_{\boldsymbol{\tau}}) = \varepsilon^{\mathbf{B}}(b_{\boldsymbol{\tau}})$ for all $\langle \delta, \varepsilon\rangle \in \boldsymbol{\tau}$, but $b_{\boldsymbol{\tau}} \notin G_{\boldsymbol{\tau}}$.

Let Λ be the set of all finite subsets of $\boldsymbol{\tau}_\infty$ and $\langle \mathbf{C}, H\rangle$ the direct product of all the matrices $\langle \mathbf{B}_{\boldsymbol{\tau}}, G_{\boldsymbol{\tau}}\rangle$, $\boldsymbol{\tau} \in \Lambda$. The sets of the form $\{\boldsymbol{\tau} \in \Lambda : \langle \delta, \varepsilon\rangle \in \boldsymbol{\tau}\}$, ranging over all $\langle \delta, \varepsilon\rangle \in \boldsymbol{\tau}_\infty$, constitute a family $\mathcal{F}$ with the finite intersection property, so $\mathcal{F}$ is contained in an ultrafilter $\mathcal{U}$ over Λ. The ultraproduct $\langle \mathbf{D}, J\rangle = \langle \mathbf{C}, H\rangle/\mathcal{U}$ also belongs to $\mathsf{Mod}^*\mathcal{S}$, by assumption.

Set $d = c/\mathcal{U}$, where $c(\boldsymbol{\tau}) = b_{\boldsymbol{\tau}}$ for each $\boldsymbol{\tau} \in \Lambda$. Then, by the construction of $\langle \mathbf{D}, J\rangle$ and standard properties of ultraproducts, we have $d \notin J$.

On the other hand, for each $\langle \delta, \varepsilon\rangle \in \boldsymbol{\tau}_\infty$, the indices $\boldsymbol{\tau} \in \Lambda$ for which $\delta^{\mathbf{B}}(b_{\boldsymbol{\tau}}) = \varepsilon^{\mathbf{B}}(b_{\boldsymbol{\tau}})$ form a superset of a member of $\mathcal{U}$, viz. $\{\boldsymbol{\tau} \in \Lambda : \langle \delta, \varepsilon\rangle \in \boldsymbol{\tau}\}$, so $\delta^{\mathbf{D}}(d) = \varepsilon^{\mathbf{D}}(d)$. Since $\boldsymbol{\tau}_\infty$ defines truth in $\mathsf{Mod}^*\mathcal{S}$ and $\langle \mathbf{D}, J\rangle \in \mathsf{Mod}^*\mathcal{S}$, it follows that $d \in J$, a contradiction.

So some finite $\boldsymbol{\tau}$ must witness the truth-equationality of $\mathcal{S}$, whence $\mathcal{S}$ has a $\boldsymbol{\tau}$–algebraic semantics, K say. Since $\mathcal{S}$ is finitary and $\boldsymbol{\tau}$ is finite, the quasivariety $\mathrm{ISPP}_{\mathrm{U}}(\mathsf{K})$ is also a $\boldsymbol{\tau}$–algebraic semantics for $\mathcal{S}$. $\square$

11 Protoalgebraic Systems

The class of 'protoalgebraic' deductive systems was introduced in [8] and has been studied extensively. It has a number of natural and appealing characterizations, of which the following may as well serve here as a definition:

[6] See the remarks preceding Section 7.

Definition 6. A deductive system $\mathcal{S}$ is called a *protoalgebraic logic* if there is a set $\mathbf{\Delta} = \mathbf{\Delta}(x, y)$ of binary $\boldsymbol{\mathcal{L}}$–terms $\Delta(x, y)$ such that

$$\vdash_{\mathcal{S}} \mathbf{\Delta}(x, x), \quad \text{i.e.,} \quad \vdash_{\mathcal{S}} \Delta(x, x) \text{ for all } \Delta(x, y) \in \mathbf{\Delta}(x, y); \quad (17)$$

$$\{x\} \cup \mathbf{\Delta}(x, y) \vdash_{\mathcal{S}} y. \quad (18)$$

Consequently, whenever a deductive system $\mathcal{S}$ has a connective $\rightarrow$ for which $x \rightarrow x$ is a theorem and modus ponens is a derivable rule, then $\mathcal{S}$ is protoalgebraic. The 'almost inconsistent' deductive systems, i.e., those for which $x \vdash y$ is a derivable rule, are evidently protoalgebraic (with $\mathbf{\Delta} = \emptyset$). Except for these systems, all protoalgebraic logics must clearly possess theorems, and their languages must give rise to some strictly *binary* terms (not merely projections and unary terms with fictitious extra variables). The following alternative characterizations of protoalgebraic logics are well known. For the proofs and origins of these and other properties mentioned below, see [10], [20], [26].

Theorem 35. *The following conditions on* $\mathcal{S}$ *are equivalent*:

(i) $\mathcal{S}$ *is protoalgebraic.*

(ii) The Leibniz operator of $\mathcal{S}$ *is globally order preserving, i.e., whenever* F, G *are* $\mathcal{S}$*–filters of an* $\mathcal{L}$*–algebra* $\mathbf{A}$, *if* $F \subseteq G$ *then* $\Omega^{\mathbf{A}} F \subseteq \Omega^{\mathbf{A}} G$.

(iii) The Leibniz operator is order preserving on $\mathcal{S}$*–theories.*

(iv) The Leibniz and Suszko operators of $\mathcal{S}$ *coincide.*

(v) $\mathsf{Mod}^{*}\mathcal{S}$ *is closed under subdirect products, i.e.,* $\mathsf{Mod}^{\mathrm{Su}}\mathcal{S} = \mathsf{Mod}^{*}\mathcal{S}$.

(vi) Whenever $\Gamma, \alpha \nvdash_{\mathcal{S}} \beta$ *or* $\Gamma, \beta \nvdash_{\mathcal{S}} \alpha$, *there is a unary polynomial function* p *of* $\mathbf{Te}$ *such that exactly one of* $\Gamma \vdash_{\mathcal{S}} p(\alpha)$ *and* $\Gamma \vdash_{\mathcal{S}} p(\beta)$ *is true.*

Condition (vi) is usually phrased contrapositively: over any theory, the 'indiscernibility' of formulas implies their interderivability. The name 'protoalgebraic' derives from a further characterization of the same class of systems, which is a filter-theoretic analogue of the Correspondence Theorem of universal algebra: see [20, Thm. 1.1.8]. In a protoalgebraic logic, the Leibniz (i.e., the Suszko) operator *commutes* with arbitrary intersections of $\mathcal{S}$–filters, so it could be described as 'completely' order preserving.

With regard to Problem 1 of Section 9: Czelakowski and Jansana have shown that when $\mathcal{S}$ is protoalgebraic, if its Leibniz operator is injective on *$\mathcal{S}$–theories* then it *is* globally injective: see [22, Thm. 3.6]. So, when $\mathcal{S}$ is protoalgebraic, we may use the expression '*$\mathcal{S}$ has an injective Leibniz operator*', without fear of ambiguity. We can now explain a further result of Czelakowski and Jansana in a more general way:

Theorem 36. ([22, Thm. 3.8]) *The following conditions on a protoalgebraic system $\mathcal{S}$ are equivalent*:

(i) $\mathcal{S}$ has an injective Leibniz operator.

(ii) $\mathcal{S}$ is truth-equational.

Proof. We already know that (ii) implies (i), regardless of protoalgebraicity. Conversely, since $\mathcal{S}$ *is* protoalgebraic, its Leibniz and Suszko operators coincide on all algebras. If in addition, this common operator is globally injective, then $\mathcal{S}$ is truth-equational, by Theorem 28. □

A protoalgebraic deductive system $\mathcal{S}$ is called a *weakly algebraizable logic* if it satisfies the equivalent conditions of this theorem. This terminology is justified by further characterizations of the notion in [22], which we shall not discuss here. The above result is itself a generalization of narrower theorems published earlier by Herrmann [28], [29] (for the 'equivalential logics' defined below) and by Blok and Pigozzi [9] (for the finitary 'finitely equivalential logics'). The following definition has its origins in [40].

Definition 7. A deductive system $\mathcal{S}$ is said to be *equivalential* if there is a set $\boldsymbol{\Delta} = \boldsymbol{\Delta}(x, y)$ of binary $\boldsymbol{\mathcal{L}}$–terms $\Delta(x, y)$ such that conditions (17) and (18) hold and, moreover,

$$\boldsymbol{\Delta}(x, y) \vdash_{\mathcal{S}} \boldsymbol{\Delta}(y, x) \tag{19}$$

$$\boldsymbol{\Delta}(x, y) \cup \boldsymbol{\Delta}(y, z) \vdash_{\mathcal{S}} \boldsymbol{\Delta}(x, z) \tag{20}$$

$$\textstyle\bigcup_{i=1}^{n} \boldsymbol{\Delta}(x_i, y_i) \vdash_{\mathcal{S}} \boldsymbol{\Delta}(\alpha(x_1, \ldots, x_n),\, \alpha(y_1, \ldots, y_n)) \tag{21}$$

whenever α is an n–ary basic operation symbol of $\boldsymbol{\mathcal{L}}$, where $0 < n \in \omega$. We call $\boldsymbol{\Delta}$ a set of *equivalence terms* for $\mathcal{S}$ when this is true. We call $\mathcal{S}$ *finitely equivalential* if it has a finite set of equivalence terms.

Of course, every equivalential deductive system is protoalgebraic. It is known that $\mathcal{S}$ is equivalential iff $\mathsf{Mod}^*\mathcal{S}$ is closed under submatrices and direct products; it is finitely equivalential iff for every algebra $\mathbf{A}$, the function $\Omega^{\mathbf{A}}$ commutes with the unions of those directed families of $\mathcal{S}$–filters for which the union is again an $\mathcal{S}$–filter: see for instance [20]. In a finitary and finitely equivalential deductive system $\mathcal{S}$, the class $\mathsf{Mod}^*\mathcal{S}$ is elementary: see [19] or [20]. So, when a system of this kind is truth-equational, some *finite* translation must witness this fact, by Theorem 34. (This was pointed out by Herrmann in [28, Remark 3.5].)

Suppose that $\mathbf{\Delta}$ is a set of equivalence terms for an equivalential deductive system $\mathcal{S}$. It is well known that in this case, for every $\mathcal{L}$–algebra $\mathbf{A}$ and every $\mathcal{S}$–filter F of $\mathbf{A}$,

$$\Omega^{\mathbf{A}}F = \{\langle a,b\rangle \in A \times A : \mathbf{\Delta}^{\mathbf{A}}(a,b) \subseteq F\}\,; \tag{22}$$

see for instance [20, Thm. 3.1.2]. For our purposes this observation combines usefully with Theorem 30 to give the next result, which is implicit in [13].

Corollary 37. *Suppose that a finitary deductive system $\mathcal{S}$ has binary connectives $\to$ and $\wedge$ such that $\{x \to y,\, y \to x\}$ is a set of equivalence terms for $\mathcal{S}$ and $x \to (x \wedge x)$ and $(x \wedge x) \to x$ are theorems of $\mathcal{S}$. Then $\mathcal{S}$ has a $\boldsymbol{\tau}$–algebraic semantics, where $\boldsymbol{\tau}$ is $x \wedge x \approx x$.*

Proof. Recall that $\mathsf{Mod}^*\mathcal{S}$ is a matrix semantics for $\mathcal{S}$, by Proposition 14. Let F be an $\mathcal{S}$–filter of an $\mathcal{L}$–algebra $\mathbf{A}$, and let $a \in A$. Since $\vdash_{\mathcal{S}} (x \wedge x) \to x$ and $\vdash_{\mathcal{S}} x \to (x \wedge x)$, we have $(a \wedge a) \to a,\ a \to (a \wedge a) \in F$. Because $\{x \to y,\, y \to x\}$ is a set of equivalence terms for $\mathcal{S}$, it follows from (22) that $\langle a \wedge a,\, a\rangle \in \Omega^{\mathbf{A}}F$. Thus $\mathbf{A}/\Omega^{\mathbf{A}}F$ satisfies $x \wedge x \approx x$. By Lemma 13, $\mathsf{Alg}^*\mathcal{S}$ satisfies $x \wedge x \approx x$. Choosing $\wedge$ for δ and setting $\varepsilon(x) = x$, we deduce from Theorem 30 that $\mathcal{S}$ has a $\{\langle x \wedge x,\, x\rangle\}$–algebraic semantics. □

A finitary and finitely equivalential deductive system can fail to possess an algebraic semantics, in which case it cannot be truth-equational. This is shown by the next example. A protoalgebraic logic with no algebraic semantics was already exhibited in [13], but that example is not equivalential.

Example 4. We denote by **P–W** the deducibility relation of the formal system from relevance logic that is known variously as **P–W** or $\mathbf{T}_{\to}$–**W** or

BB′I. This finitary system has a binary implication connective $\rightarrow$ and no other connective; it is axiomatized by

(B)	$\vdash (x \rightarrow y) \rightarrow ((z \rightarrow x) \rightarrow (z \rightarrow y))$	(*prefixing*)
(B)′	$\vdash (x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z))$	(*suffixing*)
(I)	$\vdash x \rightarrow x$	(*identity*)
(MP)	$x,\, x \rightarrow y \vdash y$	(*modus ponens*)

Obviously, **P–W** is consistent, as it is contained in the implication fragment of classical logic. It is straightforward to show that $\{x \rightarrow y,\, y \rightarrow x\}$ is a set of equivalence terms for **P–W**, so **P–W** is finitely equivalential.

The '**P–W** Problem' was the conjecture, formulated in [1], that there do not exist syntactically *distinct* implicational terms α and β such that both $\alpha \rightarrow \beta$ and $\beta \rightarrow \alpha$ are theorems of **P–W**. The correctness of this conjecture was proved by semantic methods in [35]; see [31] for a syntactic proof.

Proposition 38. **P–W** *has no algebraic semantics.*

Proof. Suppose K is a $\boldsymbol{\tau}$–algebraic semantics for **P–W**, where $\boldsymbol{\tau}$ is some $\{\rightarrow\}$–translation. Let $\langle \delta, \varepsilon \rangle \in \boldsymbol{\tau}$. As a special case of Proposition 3,

$$x,\, \delta(x) \rightarrow \delta(x) \vdash_{\mathbf{P-W}} \varepsilon(x) \rightarrow \delta(x)\,;$$
$$x,\, \varepsilon(x) \rightarrow \varepsilon(x) \vdash_{\mathbf{P-W}} \delta(x) \rightarrow \varepsilon(x)\,.$$

Structurality permits us to substitute $x \rightarrow x$ for x and, invoking (I), we see that $\varepsilon(x \rightarrow x) \rightarrow \delta(x \rightarrow x)$ and $\delta(x \rightarrow x) \rightarrow \varepsilon(x \rightarrow x)$ are both theorems of **P–W**. It follows from the solution to the **P–W** Problem that $\delta(x \rightarrow x)$ and $\varepsilon(x \rightarrow x)$ are identical terms. It is then easy to show (by induction if necessary) that $\delta(x) = \varepsilon(x)$, whence of course K satisfies $\delta(x) \approx \varepsilon(x)$. Since we chose $\langle \delta, \varepsilon \rangle \in \boldsymbol{\tau}$ arbitrarily, and since K is a $\boldsymbol{\tau}$–algebraic semantics for **P–W**, it follows that x is a theorem of **P–W**, contradicting consistency. Thus, **P–W** has no algebraic semantics. In particular, **P–W** is not truth-equational. □

Roughly speaking, a deductive system $\mathcal{S}$ is called 'algebraizable' if it is essentially the equational consequence relation $\models_{\mathsf{K}}$ of some class K of similar algebras. When this is true, $\mathsf{Alg}\mathcal{S} = \mathsf{Alg}^{*}\mathcal{S}$ is a UISP–class and it is the *only* UISP–class K that is interchangeable with $\mathcal{S}$ in the intended sense (see [6] or [7]); it is called the *equivalent algebraic semantics* of $\mathcal{S}$. For our purposes, however, the [*finitely*] *algebraizable* deductive systems may be defined as the [finitely] equivalential deductive systems that are truth-equational. (These definitions are equivalent to the ones given in [20] and [26].) When $\mathcal{S}$ is finitary, it is *finitely* algebraizable iff $\mathsf{Alg}^*\mathcal{S}$ is a quasivariety.

It follows from Theorem 36 that an equivalential deductive system is algebraizable iff it has an injective Leibniz operator. This result appears already in Herrmann's paper [29]. The implication from left to right yields a frequently used strategy for *disproving* the algebraizability of an equivalential logic and, as Herrmann observes in [28], it instantiates Padoa's method of proving non-definability (see [37], [45]). All of these results about algebraizability generalize earlier findings of [9]; different proofs can be found in [22] and in [20]. The term 'algebraizable logic' comes from [9], where the theory of finitary finitely algebraizable deductive systems was developed.

Even when a finitary finitely equivalential deductive system has an algebraic semantics, it may fail to be truth-equational (i.e., algebraizable). The known counter-examples include the deducibility relation of Anderson and Belnap's formal system **E** for 'Entailment' (axiomatized in [1]) and the pure implication fragments of linear and relevance logic. The existence of algebraic semantics for these systems was proved in [13] and, in the case of **E**, Corollary 37 provides the explanation. Non-algebraizability was proved in [9].

12 Assertional Logics

Definition 8. (Pointed Assertional Logics) Let K be a class of $\mathcal{L}$–algebras and c an $\boldsymbol{\mathcal{L}}$–constant, or at least an $\boldsymbol{\mathcal{L}}$–term that is constant over K. The c–*assertional logic* $\mathcal{S}(\mathsf{K},\mathsf{c})$ of K is the deductive system over $\boldsymbol{\mathcal{L}}$ defined by

$$\Gamma \vdash_{\mathcal{S}(\mathsf{K},\,\mathsf{c})} \alpha \ \text{ iff } \ \{\gamma \approx \mathsf{c} : \gamma \in \Gamma\} \models_{\mathsf{K}} \alpha \approx \mathsf{c}.$$

This definition is a variant of one introduced by Pigozzi [39]. Note that $\mathcal{S}(\mathsf{K},\mathsf{c})$ is indeed a deductive system and that K is a $\boldsymbol{\tau}$–algebraic semantics for $\mathcal{S}(\mathsf{K},\mathsf{c})$, where $\boldsymbol{\tau}$ is the 'pointed' translation $x \approx \mathsf{c}$. It is easy to show directly that $\mathcal{S}(\mathsf{K},\mathsf{c})$ is truth-equational (with $\boldsymbol{\tau}$ as witness) but this will also follow from a more general result, Corollary 40, given below.

The pointed assertional logics obviously include classical and intuitionistic propositional logic and the normal modal logics. They also encompass those 'substructural logics' whose theorems include the 'weakening' axiom

(K) $\vdash x \to (y \to x)$,

in particular, **BCK**–logic. (See [36], [42] for background on substructural logics.) All of these examples are algebraizable systems, so the *equivalent* algebraic semantics $\mathsf{Alg}^*\mathcal{S}$ is the natural choice for K. In each case, the constant c is equationally definable as $x \to x$ over K.

A number of significant deductive systems—such as the principal systems of linear and relevance logic—cannot be represented as pointed assertional logics $\mathcal{S}(\mathsf{K}, \mathsf{c})$, regardless of how we choose K and c. In fact this is true of any deductive system $\mathcal{S}$ with a binary connective $\to$ such that

$$\vdash_{\mathcal{S}} x \to x \quad \text{and} \quad \nvdash_{\mathcal{S}} (x \to x) \to (y \to y). \tag{23}$$

For if $\vdash_{\mathcal{S}(\mathsf{K},\mathsf{c})} x \to x$ then the term $x \to x$ is constantly c over K, whence $(x \to x) \to (y \to y)$ is also constantly c over K, making it a theorem of $\mathcal{S}(\mathsf{K}, \mathsf{c})$. Nevertheless, the following generalization of pointed assertional logics encompasses many systems that satisfy (23).

Definition 9. For any class K of $\mathcal{L}$–algebras and any $\boldsymbol{\mathcal{L}}$–translation $\boldsymbol{\tau}$, we define a deductive system $\mathcal{S}(\mathsf{K}, \boldsymbol{\tau})$ over $\boldsymbol{\mathcal{L}}$, as follows:

$$\Gamma \vdash_{\mathcal{S}(\mathsf{K},\boldsymbol{\tau})} \alpha \;\text{ iff }\; \boldsymbol{\tau}[\Gamma] \models_{\mathsf{K}} \boldsymbol{\tau}(\alpha).$$

Again, note that for any such K and $\boldsymbol{\tau}$, the relation $\mathcal{S}(\mathsf{K}, \boldsymbol{\tau})$ is indeed a deductive system and K is a $\boldsymbol{\tau}$–algebraic semantics for $\mathcal{S}(\mathsf{K}, \boldsymbol{\tau})$. Conversely, *every* deductive system that has an algebraic semantics arises in this way. In particular, Corollary 21 and Lemma 2 show that:

Fact. *Every truth-equational deductive system has the form $\mathcal{S}(\mathsf{K}, \boldsymbol{\tau})$ for some UISP–class K and some translation $\boldsymbol{\tau}$.*

The notion of a truth-equational deductive system can therefore be reduced in principle to a condition on a class of algebras K and a translation—and we can safely test for it in the ISP–closure of K alone, i.e., it is enough to prove $(*)_{\mathbf{A}}$ for all $\mathbf{A} \in \mathrm{ISP}(\mathsf{K})$, by Theorem 25. The next result goes a little way toward explaining why the applications almost invariably involve *singleton* translations.

Proposition 39. *Let* K *be a* $\boldsymbol{\tau}$*–algebraic semantics for* $\mathcal{S}$*. Suppose* $\boldsymbol{\tau}$ *contains a pair* $\langle \delta, \varepsilon \rangle$ *such that* $\delta(x) \vdash_{\mathcal{S}} x$ *and* $\vdash_{\mathcal{S}} \varepsilon(x)$.

Then $\mathcal{S} = \mathcal{S}(\mathsf{K}, \{\langle \delta, \varepsilon \rangle\})$*. Also,* $\mathcal{S}$ *is truth-equational, with* $\delta(x) \approx \varepsilon(x)$ *as witnessing translation.*

Proof. Let $\boldsymbol{\tau} = \{\langle \delta_i, \varepsilon_i \rangle : i \in I\}$. By assumption, $\mathcal{S} = \mathcal{S}(\mathsf{K}, \boldsymbol{\tau})$,

$$\models_{\mathsf{K}} \ \delta_i(\varepsilon(x)) \approx \varepsilon_i(\varepsilon(x)) \ \text{ and} \tag{24}$$

$$\{\delta_j(\delta(x)) \approx \varepsilon_j(\delta(x)) : j \in I\} \ \models_{\mathsf{K}} \ \delta_i(x) \approx \varepsilon_i(x) \tag{25}$$

for all $i \in I$. Using (24), we obtain

$$\delta(x) \approx \varepsilon(x) \ \models_{\mathsf{K}} \ \delta_j(\delta(x)) \approx \delta_j(\varepsilon(x)) \approx \varepsilon_j(\varepsilon(x)) \approx \varepsilon_j(\delta(x))$$

for all $j \in I$. This, together with (25), yields

$$\delta(x) \approx \varepsilon(x) \ \models_{\mathsf{K}} \ \delta_i(x) \approx \varepsilon_i(x)$$

for all $i \in I$. It follows that $\mathcal{S} = \mathcal{S}(\mathsf{K}, \{\langle \delta, \varepsilon \rangle\})$.

Let F be an $\mathcal{S}$–filter of an $\mathcal{L}$–algebra $\mathbf{A}$. If $a \in F$ then $\langle \delta^{\mathbf{A}}(a), \varepsilon^{\mathbf{A}}(a) \rangle \in \Omega^{\mathbf{A}} F$, by Corollary 9. Conversely, suppose $\langle \delta^{\mathbf{A}}(a), \varepsilon^{\mathbf{A}}(a) \rangle \in \Omega^{\mathbf{A}} F$. Now $\varepsilon^{\mathbf{A}}(a) \in F$, because $\vdash_{\mathcal{S}} \varepsilon(x)$, and $\Omega^{\mathbf{A}} F$ is compatible with F, so $\delta^{\mathbf{A}}(a) \in F$. Then $a \in F$, because $\delta(x) \vdash_{\mathcal{S}} x$. □

The hypotheses of this proposition, when phrased as conditions on K and $\boldsymbol{\tau}$, are rather complicated. The following specialization of the result is simpler and still widely applicable.

Corollary 40. *Let* K *be a class of* $\mathcal{L}$*–algebras with a unary term* ε *such that* K *satisfies* $\varepsilon(x) \approx \varepsilon(\varepsilon(x))$*. Then the deductive system* $\mathcal{S}(\mathsf{K}, \{\langle x, \varepsilon(x) \rangle\})$ *is truth-equational, with* $x \approx \varepsilon(x)$ *as witnessing translation.*

Proof. In Proposition 39, set $\delta(x) = x$. Let $\boldsymbol{\tau}$ be $x \approx \varepsilon(x)$ and $\mathcal{S} = \mathcal{S}(\mathsf{K}, \boldsymbol{\tau})$, and observe that $\vdash_{\mathcal{S}} \varepsilon(x)$, because K satisfies $\varepsilon(x) \approx \varepsilon(\varepsilon(x))$. □

Note that when $\mathcal{S}$ is as in the above proof, we have:

$$\Gamma \vdash_{\mathcal{S}} \alpha \ \text{ iff } \ \{\gamma \approx \varepsilon(\gamma) : \gamma \in \Gamma\} \models_{\mathsf{K}} \alpha \approx \varepsilon(\alpha).$$

For convenience, we make the following definition.

Definition 10. (Normal Assertional Logics) Let ε be a unary $\mathcal{L}$–term, let $\boldsymbol{\tau}$ be $x \approx \varepsilon(x)$ and let K be a class of $\mathcal{L}$–algebras. If K satisfies $\varepsilon(x) \approx \varepsilon(\varepsilon(x))$, we say that the deductive system $\mathcal{S}(\mathsf{K}, \boldsymbol{\tau})$ is a *normal assertional logic*, and we call $\boldsymbol{\tau}$ a *normal translation* for K.

Thus, the normal assertional logics are all truth-equational. Note that they include all pointed assertional logics. The assumption of normality simplifies the algebraic analysis and makes for a relatively smooth theory of logical *filter generation*, worked out in [12], which we shall not discuss here.

Example 5. (Substructural Logics) The principal formal systems **LL** and **R** for linear and relevance logic (respectively) axiomatize normal assertional logics (see [47] and [1] for the axioms). These systems are finitely algebraizable: see [9], [27]. So, as in the case of **BCK**–logic, the equivalent algebraic semantics $\mathsf{Alg}^*\mathcal{S}$ is the natural candidate for the class K in the normal representation. For **LL** and **R**, this class K is a variety of enriched residuated lattice-ordered commutative semigroups. The residuation operator interprets the implication connective $\rightarrow$ of the logic, and

$$\mathsf{Mod}^*\mathcal{S} \;=\; \{\langle \mathbf{A}, F\rangle : \mathbf{A} \in \mathsf{K} \text{ and } F = \{a \in A : a \rightarrow a \leq a\}\}$$

(see, e.g. [48, Sec. 6]). The normal translations $\boldsymbol{\tau}$ therefore encode the inequality $x \rightarrow x \leq x$ in equational form. For instance, we can take $\boldsymbol{\tau}$ to be

$$x \;\approx\; x \vee (x \rightarrow x).$$

When dealing with fragments of these systems that include implication and conjunction but not disjunction, we may instead use

$$x \;\approx\; (x \wedge (x \rightarrow x)) \rightarrow x$$

for $\boldsymbol{\tau}$. It can be shown that these two translations are indeed normal for K. All of these remarks apply equally to many neighbouring substructural logics that lack the weakening axiom (K), such as **LR** and **RW**.

In the case of linear logic, the enriched semigroups in K are in fact monoids; their identity elements interpret a constant t of the signature (denoted by **1** in [47]). Also, the inequality $x \rightarrow x \leq x$ is equivalent over K to $\mathsf{t} \leq x$, so either of the simpler normal translations

$$x \;\approx\; x \vee \mathsf{t}\,; \qquad x \;\approx\; (x \wedge \mathsf{t}) \rightarrow x$$

may be used instead of the previous two.

Example 6. (Mingle) In certain extensions of **LL**, **R** and their relatives, the lattice-like connectives can be dispensed with in the normal representation, because the order $\leq$ can be defined equationally over K without using them. This applies in particular to extensions in which the 'mingle' axiom

(M) $\quad \vdash x \to (x \to x)$

is provable. These may be represented as normal assertional logics using the translation $x \approx x \to x$, because K already satisfies $x \leq x \to x$. Even the pure implication fragment of the system is then a normal assertional logic. The 'semi-relevant' logic **R**–*mingle* (**RM**) [24], [1], [9] is an example of this kind, and so are the alternative relevance logics $\mathbf{RMI}_{\min}$ and **RMI** [3], [4] which cannot accommodate the constant t.

If a class of algebras K with a unary term ε satisfies $\varepsilon^n(x) \approx \varepsilon^{n+1}(x)$ for some positive integer n, then clearly $\mathcal{S}(\mathsf{K}, \{\langle x, \varepsilon^n(x)\rangle\})$ is a normal assertional logic, and therefore truth-equational. But the following slightly stronger result is also true, and will be useful in Section 14. It generalizes Corollary 40 in a direction different from Proposition 39.

Theorem 41. *Let ε be a unary $\mathcal{L}$–term, let $\boldsymbol{\tau}$ be $x \approx \varepsilon(x)$ and let* K *be a class of $\mathcal{L}$–algebras.*

If K *satisfies $\varepsilon^n(x) \approx \varepsilon^{n+1}(x)$ for some nonnegative integer n, then the deductive sytem $\mathcal{S}(\mathsf{K}, \boldsymbol{\tau})$ is truth-equational, with witnessing translation $\boldsymbol{\tau}$.*

Proof. Let $\mathcal{S} = \mathcal{S}(\mathsf{K}, \boldsymbol{\tau})$. If $n = 0$ then $\vdash_{\mathcal{S}} x$, so for every $\mathcal{L}$–algebra $\mathbf{A}$, the only $\mathcal{S}$–filter of $\mathbf{A}$ is A. In this case, the result is trivially true. We may therefore assume that $n > 0$.

Let F be an $\mathcal{S}$–filter of an $\mathcal{L}$–algebra $\mathbf{A}$. We omit the superscript $\mathbf{A}$ from term functions of $\mathbf{A}$ of the form ε^m, $m \in \omega$. Since K is a $\boldsymbol{\tau}$–algebraic semantics for $\mathcal{S}$, it follows from Corollary 9 that $\boldsymbol{\tau}^{\mathbf{A}}[F] \subseteq \Omega^{\mathbf{A}} F$. Conversely, let $a \in A$ with $\langle a, \varepsilon(a)\rangle \in \Omega^{\mathbf{A}} F$. Then for all $m \in \omega$, since $\Omega^{\mathbf{A}} F$ is a congruence of $\mathbf{A}$, we have $\langle \varepsilon^m(a), \varepsilon^{m+1}(a)\rangle \in \Omega^{\mathbf{A}} F$. Also $\vdash_{\mathcal{S}} \varepsilon^n(x)$, by definition of $\mathcal{S}$, since K satisfies $\varepsilon^n(x) \approx \varepsilon(\varepsilon^n(x))$. Consequently, $\varepsilon^n(a) \in F$. Now because $\Omega^{\mathbf{A}} F$ is compatible with F, it follows that $\varepsilon^{n-1}(a) \in F$. If $n = 1$, this tells us that $a \in F$. If not, we repeat the argument a further $n - 1$ times, eventually getting $a \in F$, as required. $\square$

13 Non-Protoalgebraic Systems

The abstract study of non-protoalgebraic systems is a recent development. Czelakowski lists a number of significant examples in [21], and others can be found in papers cited in [26, p. 35] and in [25]. Several of these logics lack theorems, so they cannot be truth-equational. But non-protoalgebraic systems *with* theorems are to be found, for instance, among the so-called 'subintuitionistic logics' (see [16] and its references). These systems have been a focus of recent interest, and some of them are truth-equational.[7]

We have noted that every pointed assertional logic is truth-equational. The examples of this kind discussed thusfar were all protoalgebraic logics. We mention two non-protoalgebraic examples.

Example 7. Let $\mathbf{IPL}^*$ denote the *implication-less fragment of intuitionistic propositional logic* (**IPL**). The signature of $\mathbf{IPL}^*$ is $\{\wedge, \vee, \neg, \bot, \top\}$. If K denotes the variety of pseudocomplemented distributive lattices then, by [9, Thm. 2.6], $\mathbf{IPL}^*$ coincides with $\mathcal{S}(\mathsf{K}, \top)$ and is therefore truth-equational. It is not protoalgebraic; this was demonstrated in [9, Thm. 5.13]. Note that the 'material conditional' $x \supset y := (\neg x) \vee y$ of $\mathbf{IPL}^*$ satisfies modus ponens, but for $\supset$, the 'reflexivity' demand (17) of Definition 6 is just the law of the excluded middle, which of course fails in **IPL**.

Example 8. *Visser's Propositional Logic* **VPL** is a subintuitionistic logic, defined in [50] and motivated by considerations of constructivity. It can be embedded into the 'basic modal logic' **K4** by a well known translation that also embeds **IPL** into **S4**, and classical propositional logic into **S5**. Unlike $\mathbf{IPL}^*$, the system **VPL** has a reflexive implication $\to$, but it lacks modus ponens. It is not protoalgebraic [44]. It is the $\top$–assertional logic of a variety—called the variety B of 'Basic algebras' in [2], [16] and [17]—so it is truth-equational. In B, the constant $\top$ may be defined by $x \to x$.[8]

Obviously, any number of distinct non-protoalgebraic pointed assertional logics can be produced using languages that contain no connective

[7] A logic is called *subintuitionistic* if it has the language of intuitionistic propositional logic (**IPL**) and can be defined semantically in terms of Kripke frames, where the connectives satisfy the truth conditions used to define **IPL** but the binary relation of a frame is not required to satisfy all of the postulates from the intuitionistic case.

[8] **VPL** is called the 'Basic Propositional Calculus' **BPC** in [2].

of binary or higher rank, and classes that include at least one nontrivial algebra.

It is useful to have some nontrivial criteria of a purely algebraic kind for the failure of protoalgebraicity. The following result, proved in [12], identifies a class of non-pointed normal assertional logics that cannot be protoalgebraic.

Proposition 42. *Let* V *be a variety of* $\mathcal{L}$*–algebras and* $\boldsymbol{\tau}$ *an* $\boldsymbol{\mathcal{L}}$*–translation. Suppose* V *contains a three-element algebra* $\mathbf{A}$*, with universe* $A = \{-1, 0, 1\}$*, say, such that the following statements are true.*

(i) Whenever α *is a basic operation symbol in* $\mathcal{L}$ *of rank* $n > 0$*, and* $\alpha^{\mathbf{A}}(a_1, a_2, \ldots, a_n) = 0$ *then* $a_i = 0$ *for all* i*.*

(ii) For each $\langle \delta, \varepsilon \rangle \in \boldsymbol{\tau}$*, the term functions* $\delta^{\mathbf{A}}$ *and* $\varepsilon^{\mathbf{A}}$ *are, respectively, the identity function and the absolute value function on* A*.*

Then $\mathcal{S}(\mathsf{V}, \boldsymbol{\tau})$ *is not protoalgebraic.*

Note that under the assumptions of Proposition 42, $\mathcal{S}(\mathrm{V}(\mathbf{A}), \boldsymbol{\tau})$ is a normal assertional logic and is therefore truth-equational.

A simple example of an algebra $\mathbf{A}$ as described in Proposition 42 is a pointed join semilattice $\langle \{-1, 0, 1\}; \vee, 0 \rangle$ in which -1 and 0 are incomparable. The term $x \vee 0$ defines the absolute value function. A more interesting example is:

Example 9. Let Z_3 be the universe of the three-element idempotent commutative totally ordered monoid on the set $\{-1, 0, 1\}$, where 0 is the identity element, $-1 < 0 < 1$ and $-1 \cdot 1 = -1$. Let $\neg$ denote the usual additive inverse operation on Z_3, and define $x \rightarrow y = \neg(x \cdot \neg y)$. Then $x \rightarrow x$ defines the absolute value function on Z_3, so we shall abbreviate $x \rightarrow x$ as $|\, x \,|$. The algebra $\mathbf{Z}_3 = \langle Z_3; \rightarrow, \neg \rangle$ is called the *Sobociński algebra* in [12], in recognition of a detailed logical study of it in [43]. In $\mathbf{Z}_3$, the element 0 is irreducible in the sense of Proposition 42(i). Now let $\boldsymbol{\tau}$ be $x \approx |\, x \,|$ and define $\mathcal{S} = \mathcal{S}(\mathrm{V}(\mathbf{Z}_3), \boldsymbol{\tau})$.

By Proposition 42, $\mathcal{S}$ is not protoalgebraic; this was proved directly in [11]. But since $\mathbf{Z}_3$ satisfies $|\, x \,| \approx ||\, x \,||$, $\mathcal{S}$ is a normal assertional logic, and is therefore truth-equational. The *theorems* of $\mathcal{S}$ are known to be exactly the theorems of the implication-negation fragment of **RM** [38], but when it comes to rules, $\mathcal{S}$ is much weaker than this fragment, e.g., $\mathcal{S}$ lacks modus ponens. (On the other hand, if we replace $\mathrm{V}(\mathbf{Z}_3)$ by $\mathrm{ISP}(\mathbf{Z}_3)$ in the definition, the resulting system has some rules not derivable in **RM**, e.g. $\neg(x \to \neg\, y) \vdash x$ [5].)

Observe that $|\,x\,|$ is a theorem of $\mathcal{S}$, but $|\,x\,| \to |\,y\,|$ is not (e.g., in $\mathbf{Z}_3$, we have $|\,1\,| \to |\,0\,| = -1 \neq |-1\,|$). Thus, $\mathcal{S}$ cannot be represented as a *pointed* assertional logic—see the remarks preceding Definition 9.

14 Mono-unary Systems

A deductive system is called *mono-unary* if its signature consists of just one connective and this connective has rank 1. (Constants are not permitted.) A mono-unary deductive system has no genuinely binary terms, so it cannot be protoalgebraic, unless it contains the rule $x \vdash y$. Because mono-unary systems are relatively easy to analyze, they are a natural laboratory for testing conjectures about deductive systems that have already been confirmed in the case of protoalgebraic logics. Problem 1 of Section 9 meets this description, and we shall settle the mono-unary case of this problem affirmatively here. In fact, we shall prove a stronger theorem: if the Leibniz operator is injective on the *theories* of a mono-unary deductive system $\mathcal{S}$ then $\mathcal{S}$ is truth-equational.

From now on, when discussing an unspecified mono-unary system $\mathcal{S}$, we shall always denote its sole connective by ε. In this case, obviously, every $\boldsymbol{\mathcal{L}}$–term has the form $\varepsilon^m(v)$ for some $v \in Var$ and some $m \in \omega$. We shall need the following result of Blok and Rebagliato, which was formulated in [13] with additional finiteness assumptions not invoked in the proof:

Theorem 43. ([13, Thm. 2.20]) *A mono-unary deductive system* $\mathcal{S}$ *has an algebraic semantics iff* $x \vdash_{\mathcal{S}} \varepsilon(x)$*. In this case* $\mathcal{S}$ *has a* $\boldsymbol{\tau}$*–algebraic semantics, where* $\boldsymbol{\tau}$ *is* $x \approx \varepsilon(x)$*.*

Blok and Rebagliato point out that, by Theorem 43, the pure negation fragments of classical and intuitionistic propositional logic have no algebraic semantics (regardless of the choice of translation), whereas they each have a translation $\boldsymbol{\tau}$ such that the conclusion of Proposition 3 holds, viz. $x \approx \neg\neg\, x$ and $\neg\, x \approx \neg\neg\neg\, x$, respectively. Thus, we cannot upgrade Proposition 3 to a characterization of the deductive systems having an algebraic semantics.

Example 1 shows that a mono-unary deductive system with an algebraic semantics need not be truth-equational. In contrast:

Corollary 44. *If a mono-unary deductive system has an algebraic semantics and at least one theorem then it is truth-equational, with witnessing translation* $x \approx \varepsilon(x)$.

Proof. Let $\mathcal{S}$ satisfy the hypotheses of the corollary. By structurality, $\mathcal{S}$ must have a theorem of the form $\varepsilon^n(x)$ for some $n \in \omega$. By Theorem 43, $\mathcal{S}$ has a $\boldsymbol{\tau}$–algebraic semantics, K say, where $\boldsymbol{\tau}$ is $x \approx \varepsilon(x)$. Now $\mathcal{S} = \mathcal{S}(\mathsf{K}, \boldsymbol{\tau})$, and the theoremhood of $\varepsilon^n(x)$ in $\mathcal{S}$ forces K to satisfy $\varepsilon^n(x) \approx \varepsilon^{n+1}(x)$. Then by Theorem 41, $\mathcal{S}$ is truth-equational, with $\boldsymbol{\tau}$ as witness. □

Incidentally, it is easy to show that a mono-unary deductive system with at least one theorem must be *finitary*, so we could have chosen the class K in the above proof to be a *quasivariety*. The next result should be contrasted with Example 3, where the signature was 'almost' mono-unary but had a constant symbol as well.

Theorem 45. *If the Leibniz operator of a mono-unary deductive system* $\mathcal{S}$ *is injective on the theories of* $\mathcal{S}$ *then* $\mathcal{S}$ *has an algebraic semantics.*

Proof. Let $\Omega^{\mathbf{Te}}$ be injective on $\mathcal{S}$–theories, so $\mathcal{S}$ must have at least one theorem, by Lemma 5. As in the previous proof, structurality ensures that $\mathcal{S}$ has a theorem of the form $\varepsilon^n(x)$ for some $n \in \omega$. Let n be the least nonnegative integer such that $\vdash_{\mathcal{S}} \varepsilon^n(x)$.

Suppose that $\mathcal{S}$ has no algebraic semantics. Then $x \nvdash_{\mathcal{S}} \varepsilon(x)$, by Theorem 43. So $n \geq 2$ and, by structurality, $Var \nvdash_{\mathcal{S}} \varepsilon(x)$. Using structurality again, we see that $\vdash_{\mathcal{S}} \varepsilon^m(v)$ for all $v \in Var$ and all integers $m \geq n$, so every $\mathcal{S}$–theory contains $\varepsilon^n[\mathrm{Te}]$.

Let F be the $\mathcal{S}$–theory generated by Var. Let k be the least nonnegative integer such that $\varepsilon^k[\mathrm{Te}] \subseteq F$. By the previous remarks, k exists and $k \leq n$. Also $k > 1$, because $Var \nvdash_{\mathcal{S}} \varepsilon(x)$. Moreover, $\varepsilon^{k-1}[Var] \cap F = \emptyset$, by the minimality of k (using structurality and monotonicity).

Let G be the $\mathcal{S}$–theory generated by $\varepsilon^k[\mathrm{Te}]$, so $G \subseteq F$. We claim that $x \notin G$. To see this, suppose $x \in G$, i.e., $\varepsilon^k[\mathrm{Te}] \vdash_{\mathcal{S}} x$. Substituting $\varepsilon^{n-k}(x)$ for all variables and using structurality, we get $\varepsilon^n[\mathrm{Te}(x)] \vdash_{\mathcal{S}} \varepsilon^{n-k}(x)$. It follows that $\vdash_{\mathcal{S}} \varepsilon^{n-k}(x)$ (because $\varepsilon^n[\mathrm{Te}(x)]$ consists of $\mathcal{S}$–theorems). But this contradicts the minimality of n, so $x \notin G$. Since $x \in F$, we have $F \neq G$ and so $\Omega^{\mathbf{Te}}F \neq \Omega^{\mathbf{Te}}G$, by assumption.

Now for any $\mathcal{S}$–theory H, we have

$$\Omega^{\mathbf{Te}} H = \{\langle \alpha, \beta \rangle \in \mathrm{Te} \times \mathrm{Te} : (\forall i \in \omega)\, (\varepsilon^i(\alpha) \in H \text{ iff } \varepsilon^i(\beta) \in H)\},$$

by Lemma 6(i), because ε is the sole connective of $\mathcal{S}$. But since $\varepsilon^n[\mathrm{Te}] \subseteq H$, this becomes

$$\Omega^{\mathbf{Te}} H = \{\langle \alpha, \beta \rangle \in \mathrm{Te} \times \mathrm{Te} : (\forall i < n)\, (\varepsilon^i(\alpha) \in H \text{ iff } \varepsilon^i(\beta) \in H)\}. \tag{26}$$

Let θ be the equivalence relation on Te corresponding to the partition

$$\{Var,\, \varepsilon[Var],\, \varepsilon^2[Var],\, \dots ,\, \varepsilon^{k-1}[Var],\, \varepsilon^k[\mathrm{Te}]\}.$$

Clearly, θ is a congruence of **Te**. Also, θ is compatible with F and with G; this follows easily from the definitions, using structurality. So

$$\theta \subseteq \Omega^{\mathbf{Te}} F \text{ and } \theta \subseteq \Omega^{\mathbf{Te}} G. \tag{27}$$

We intend to show now that $\Omega^{\mathbf{Te}} F = \Omega^{\mathbf{Te}} G$, contradicting the already established fact that these congruences are distinct. This contradiction will allow us to conclude that $\mathcal{S}$ must have an algebraic semantics after all, completing the proof. In view of (27), it is enough to prove the following

Claim. $\Omega^{\mathbf{Te}} F \subseteq \theta$ and $\Omega^{\mathbf{Te}} G \subseteq \theta$.

To prove this claim, consider any two $\boldsymbol{\mathcal{L}}$–terms that lie in different θ–classes. These must have the form $\varepsilon^l(v)$ and $\varepsilon^r(w)$ where $v, w \in Var$ and, necessarily, l and r are distinct nonnegative integers at least one of which is less than k. We must show that $\varepsilon^l(v)$ and $\varepsilon^r(w)$ are separated both by $\Omega^{\mathbf{Te}} F$ and by $\Omega^{\mathbf{Te}} G$. By symmetry, we may assume that $l < r$ and that $l < k$.

By (26), it is enough to find a nonnegative integer p such that $\varepsilon^p(\varepsilon^l(v)) \notin F$ and $\varepsilon^p(\varepsilon^r(w)) \in G$ (whence $\varepsilon^p(\varepsilon^l(v)) \notin G$ and $\varepsilon^p(\varepsilon^r(w)) \in F$, because $G \subseteq F$). The integer $p = k - 1 - l$ has these properties. Indeed, for this p, the term $\varepsilon^p(\varepsilon^l(v))$ belongs to $\varepsilon^{k-1}[Var]$, which is disjoint from F, but because $r > l$, the term $\varepsilon^p(\varepsilon^r(w))$ belongs to $\varepsilon^k[\mathrm{Te}]$, which is contained in G. □

From the previous three results, Theorem 28 and Lemma 5, we deduce:

Theorem 46. *For a mono-unary deductive system $\mathcal{S}$, the following conditions are equivalent*:

(i) $\mathcal{S}$ *is truth-equational.*

(ii) The Leibniz operator of $\mathcal{S}$ *is globally injective.*

(iii) The Leibniz operator is injective on the theories *of* $\mathcal{S}$.

(iv) $\mathcal{S}$ *has an algebraic semantics and some theorems.*

(v) $x \vdash_{\mathcal{S}} \varepsilon(x)$ *and* $\vdash_{\mathcal{S}} \varepsilon^n(x)$ *for some* $n \in \omega$.

In this case the translation $x \approx \varepsilon(x)$ *witnesses* (i) *and* (iv).

We note that a *multi*-unary system with theorems and an algebraic semantics need *not* be truth-equational. Indeed, it follows from Example 2 that the $\top$–free fragment of the system discussed there has a $\{\langle \Diamond x, \Box x\rangle\}$–algebraic semantics. But the Leibniz operator of this fragment is not injective on its theories: the least theory and the theory generated by Var can easily be shown to have the same Leibniz congruence.

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