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EQUIVALENTIAL LOGICS
(*AFTER 25 YEARS OF INVESTIGATIONS*)

A b s t r a c t. A short history of equivalential logics is sketched commencing with the pioneering work of Prucnal and Wroński [1974].

The notion of equivalential logic is due to Tadeusz Prucnal and Andrzej Wroński. The short note "An algebraic characterization of the notion of structural completeness" appeared in 1974 in Bulletin of the Section of Logic, Vol. 3, 30-33. The work dealt mainly with structural completeness, the property of deductive systems which was systematically investigated in Poland in the beginning of the 70ties. One may say that singling out the class of equivalential logics was a side-effect of the Prucnal and Wroński's investigations of structural completeness. The term "equivalential logic" is usually associated with the purely equivalential fragment of the classical or intuitionistic propositional logics. In the Prucnal - Wroński's paper the notion of an equivalential logic is understood much broader. The fragments mentioned above fall under this general notion of an equivalential logic.

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Logic is defined here in accordance with the Tarskian, purely consequential paradigm of logic which is dominant in the Polish logical tradition (see Tarski [1956]). Thus a logic (when restricted to sentential languages) is viewed as a pair

$$(S, C),$$

where S is an arbitrary sentential language and C is a consequence operation on S. Often the condition of structurality is added (see Łoś and Suszko [1958]):

$$eC(X) \subseteq C(eX),$$

for any substitution e in S. (Finitariness is not needed in our considerations.) As is customary, we often identify logics with their corresponding consequence operations.

The original definition due to Prucnal and Wroński states: a logic (S, C) is *equivalential* if there exists a set $E(x, y)$ of sentential formulas of S in two variables x and y such that the following conditions hold:

(i) $E(x, x) \subseteq C(\emptyset)$

(ii) $E(y, x) \subseteq C(E(x, y))$

(iii) $E(x, z) \subseteq C(E(x, y) \cup E(y, z))$

(iv) For any natural $n \geq 0$ and any n-ary connective F,

$$E(F(x_1, \ldots, x_n), F(y_1, \ldots, y_n)) \subseteq C(E(x_1, y_1) \cup \ldots \cup E(x_n, y_n))$$

(v) $y \in C(E(x, y) \cup \{x\})$.

Any set $E(x, y)$ with the above properties is called an *equivalence system* for the logic C. [In the above conditions $E(\alpha, \beta)$ stands for the set of formulas that results from $E(x, y)$ by the simultaneous substituting the formulas α and β for the variables x and y, respectively.] The consecutive conditions occurring in the above definition are referred to as to *reflexivity,* *symmetry, transitivity, congruentiality* and *detachment* (or the generalized Modus Ponens) for $E(x, y)$, respectively. $E(x, y)$ may be empty. It is easy to see that a logic C has the empty equivalence system if and only if $y \in C(x)$. The last condition means in turn that either C is inconsistent (i.e., $C(\emptyset) = S$) or C is almost inconsistent (i.e., $C(\emptyset) = \emptyset$ and $C(X) = S$ for any non-empty set X.)

If C is the consequence of classical or intuitionistic logic, then C is equivalential because the one-element set $E(x, y)$ containing the formula $x \leftrightarrow y$ only is an equivalence system. The connective $\leftrightarrow$ has the properties (i) - (v).

The above understanding of equivalence departs from the logical tradition, where equivalence is being understood as a binary *connective* with certain explicitly defined properties. The generalization of the notion of equivalence accomplished by Prucnal and Wroński turns out however to be very accurate and it has far-reaching consequences.

Wójcicki [1988] has noticed that the conditions (i) - (v) are not independent: (ii) and (iii) are consequences of the remaining conditions. Furthermore, he has singled out the important class of finitely equivalential logics. A logic C is *finitely equivalential* if it has a *finite* set $E(x, y)$ satisfying (i) - (v). Shortly, a logic C is finitely equivalential iff it is equivalential and has a finite equivalence system.

The very notion of an equivalential logic was not thoroughly examined by Prucnal and Wroński. No sooner than on the turn of the 70's and the 80's systematic investigations of this notion have been initiated in the then Section of Logic of the Polish Academy of Sciences. We mention here: Czelakowski's article [1981] "Equivalential logics", where the title notion was the subject of a detailed examination, a series of (published and unpublished) works of Czelakowski and Dziobiak (see e.g. Czelakowski [1985]) and, finally, Jacek Malinowski's monograph "Equivalence in intensional logics", which has essentially broadened our knowledge of this notion.

Wójcicki's book "Theory of logical calculi" (Wójcicki [1988]) is a competent and complete source of information on the state of the theory to the late 80-ties. It contains almost all more important results obtained up to 1988.

Malinowski's investigations were very helpful in establishing the exact topography of the modal equivalential logics. It is interesting to know that the logics (= consequence operations) determined by weak normal modal systems, such as Kripke's system **K** or Feys-von Wright's system **T**, are finitary, equivalential but not finitely equivalential. This means that they possess only infinite equivalence systems. E.g. for the system **K**, we define

$$\mathbf{K}^{\rightarrow}$$

to be the consequence operation (in the modal language $(S, \wedge, \vee, \neg, \Box)$ determined by the system **K** in the following way: **K** is a set of logical axioms of $\mathbf{K}^{\rightarrow}$ and the rule Modus Ponens for the material implication $\rightarrow$ is its only primitive rule of inference. **K** is equivalential and the infinite set

$$\{\Box^n(x \leftrightarrow y) : n = 0, 1, 2, \ldots\}$$

forms an equivalence system for $\mathbf{K}^{\rightarrow}$. Malinowski has proved that $\mathbf{K}^{\rightarrow}$ is not finitely equivalential.

In light of the investigations carried out by Malinowski, almost all more important intensional deductive systems are known to be equivalential e.g. temporal logics, dynamic logics, multi-modal logics, ortho-modular logics. At the same time only some of them exhibit the property of being finitely equivalential. This fact is in itself worth mentioning because it shows how good intuition Prucnal and Wroński had when they defined equivalence as a set of binary terms allowing infinite sets.

To continue the story, a few words must be said about the hierarchy of deductive systems. Before discussing the hierarchy, we first formulate the thesis we adhere to:

> *the class of equivalential logics is the most important class in the hierarchy of deductive systems.*

The above thesis should be assessed in purely pragmatic terms - the plausible arguments we present here merely show that all the known deductive systems which in the literature are qualified as non-trivial ones are necessarily equivalential.

In the middle of the 80-ties Blok and Pigozzi carried out a research concerning the abstract treatment of the Tarski- Lindenbaum method. It is a commonly known fact that if T is a theory (in a sentential or any higher order language), then the relation $\sim_T$ between formulas defined by the stipulation:

$$\varphi \sim_T \psi \text{ iff } T \vdash \varphi \leftrightarrow \psi$$

(i.e., $\varphi \leftrightarrow \psi$ is classically provable on the basis of T) is an equivalence relation on the set of formulas. Moreover, $\sim_T$ is a congruence on the formula algebra, i.e., it is compatible with the operations determined by the connectives of the language. The resulting quotient algebra is a Boolean algebra and it is called the *Lindenbaum-Tarski* algebra of the theory T.

The above remarks (with a suitably modified conclusion) can be repeated without major changes for many other deductive systems, different from classical logic. E.g. if the logic in the language (with $\rightarrow, \wedge, \vee, \neg$) is the intuitionistic system, the above procedure assigns a Heyting algebra to each theory.

The question arises - what is the scope of the Lindenbaum—Tarski method (LTM, for short)? As is well-known, LTM is the simplest and most important tool linking logic with algebra: the Lindenbaum—Tarski algebras of classical logic are Boolean algebras, the Lindenbaum—Tarski algebras of intuitionistic logic are Heyting algebras. (The underlying sentential languages of classical and intuitionistic logics are assumed to have an arbitrary infinite stock of variables.) Two elements are inherent to LTM: firstly, LTM assigns the congruence $\sim_T$ on the formula algebra to each closed theory T of a given logic, and secondly, the congruence $\sim_T$ is the largest among all congruences Φ that have the compatibility property which means that for any formulas φ, ψ:

(*) $\varphi \equiv \psi \pmod{\Phi}$ and $\varphi \in T$ imply $\psi \in T$.

This observation was the starting point in formulating the so-called *abstract* treatment of the Lindenbaum-Tarski method. Blok and Pigozzi have introduced the Leibniz operator to metalogic, usually denoted by Ω. By definition, if T is a set of sentential formulas (we restrict ourselves to sentential languages here), then

$$\Omega T$$

is the largest congruence Φ on the formula algebra that is compatible with T, i.e., ΩT is the greatest congruence Φ for which the implication (*) holds. The function which assigns the congruence ΩT to each set T is called the *Leibniz operator* and the congruence ΩT is called the *Leibniz congruence of the theory* T.

Two remarks are appropriate here. Firstly - the very notion of the Leibniz operator (but not its name) has been known for years. We find it in Suszko's and Wójcicki's papers (see Wójcicki [1988]) and even in Łoś [1949]. The congruence ΩT bears there the name "the largest strict congruence of the matrix (S, T)". However, the term "Leibniz congruence" has recently become widespread in the literature. Secondly, the definition of ΩT is independent of whatever deductive system admitted in the language of T. The congruence ΩT depends entirely on the structural (i.e. grammatical) properties of the language. ΩT is therefore often referred to as the *synonymy relation* relative to T. This is due to the fact that for any sentential language S and any set $T \subseteq S$ the following equivalence holds:

> $\varphi \equiv \psi \pmod{\Omega T}$ iff, for every formula $\chi \in S$ and any variable x occurring in χ, $\chi(x/\varphi) \in T$ iff $\chi(x/\psi) \in T$.

[$\chi(x/\varphi)$ is the result of the uniform substituting every occurrence of x in the formula χ by φ.] In other words, $\varphi \equiv \psi \pmod{\Omega T}$ iff φ and ψ are interchangeable relative to T in every context represented by χ.

The process of relating the operator Ω to a given deductive system C is effected by way of restricting the domain of the operator Ω to the family $Th(C)$ of closed theories of C and then investigating the run of Ω on $Th(C)$.

It may happen that the operator Ω is

(a) *monotonic* on $Th(C)$, i.e., $T_1 \subseteq T_2$ implies $\Omega T_1 \subseteq \Omega T_2$, for any $T_1, T_2 \in Th(C)$.

(b) *injective* on $Th(C)$, i.e., $T_1 = T_2$ implies $\Omega T_1 = T_2$, for any $T_1, T_2 \in Th(C)$.

The specification of properties of Ω, similar to the above ones, give us a hierarchy of deductive systems called the *hierarchy of protoalgebraic logics*. We shall describe it briefly. In the above hierarchy, which is depicted in the figure below, moving upwards one reaches smaller and smaller classes of deductive systems. Protoalgebraic logics constitute the largest class of the hierarchy (the bottom of the hierarchy) while Fregean protoalgebraic systems form the smallest class (the top of the hierarchy).

The classes placed in the right-hand side of the figure are the most important ones because almost all non-trivial systems studied in the literature belong to the classes exhibited there. The left-hand side comprises certain special and rather weak deductive systems as e.g. quantum logics determined by non- orthomodular ortholattices.

Here are the definitions of particular classes of the hierarchy. The letter C represents here an arbitrary deductive system.

1. C is *protoalgebraic* if Ω is monotonic on $Th(C)$,

2. C is *equivalential* (in the operator sense) if C is protoalgebraic and Ω commutes with pre-images of substitutions, i.e., for any substitution e in the language of C and any theory $T \in Th(C)$,

$$e^{-1}\Omega T = \Omega e^{-1} T,$$

where $e^{-1}\Omega T := \{\langle \alpha, \beta \rangle \in S \times S : e\alpha \equiv e\beta \ (\bmod \ \Omega T)\}$,

3. C is *algebraizable* if C is equivalential in the operator sense and Ω is injective on $Th(C)$,

4. C is *regularly algebraizable* if C is equivalential in the operator sense and Ω glues together the theorems of any theory T. The last condition is formally expressed by the formula

   (i) $\quad x \equiv y \ (\bmod \ \Omega C(x,y))$,

   where x and y are arbitrary but fixed distinct variables.

5. C is a *Fregean protoalgebraic logic* if Ω satisfies the condition:

   (ii) $\quad \alpha \equiv \beta \ (\bmod \ \Omega T)$ iff $C(T,\alpha) = C(T,\beta)$,

   for any theory $T \in Th(C)$ and any formulas α, β.

Evidently (ii) yields the monotonicity of Ω (and hence protoalgebraicity). It is also clear that (ii) entails (i). Therefore every Fregean protoalgebraic logic is regularly algebraizable.

In turn, every regularly algebraizable logic is algebraizable. Indeed, (i) implies that

(iii) $\alpha \equiv \beta \ (\bmod\ \Omega T)$ for any formulas $\alpha, \beta \in T$,

where T is any theory in $Th(C)$. It is easy to see that (iii) implies the injectivity of Ω on $Th(C)$. Therefore, for any regularly algebraizable logic C, the operator Ω is injective on $Th(C)$. From this and the equivalentiality of C, the algebraizability of C follows.

Let us add, for the sake of completeness, that a logic C is *weakly algebraizable* if C is protoalgebraic and Ω is injective on $Th(C)$. A deductive system C is *regularly weakly algebraizable* if it is protoalgebraic and it satisfies (i). Since (i) implies the injectivity of Ω on $Th(C)$, every regularly weakly algebraizable logic is weakly algebraizable.

The definitions of the classes forming the above hierarchy are all uniformly formulated in terms of the properties of the Leibniz operator. Some of the above definitions may seem to be not intuitive and formulated *ad hoc*. We must however remember that the process of discovering the consecutive classes from the hierarchy continued for years. The initial, original definitions of some of the above classes were not even formulated in terms of Ω. The development of the theory has shown that all these definitions are equivalent to the above "operator" ones. This fact has accounted for establishing the operator paradigm in the classification of deductive systems. E.g. the above operator definition of an equivalential logic is equivalent, as expected, to the original definition given by Prucnal and Wroński. For if C is equivalential in the Prucnal-Wroński's sense and $E(x, y)$ is an equivalence system for C, then the Leibniz congruence ΩT for any theory $T \in Th(C)$ has a simple characterization in terms of $E(x, y)$:

(iv) $\alpha \equiv \beta \ (\bmod\ \Omega T)$ iff $E(\alpha, \beta) \subseteq T$,

for any $\alpha, \beta \in S$.

It follows from (iv) that Ω is monotonic and commutes with pre-images of substitutions on $Th(C)$. To show the second property it suffices to notice that the following conditions are equivalent:

$$\alpha \equiv \beta \ (\bmod e^{-1}\Omega T),$$

$$e\alpha \equiv e\beta \ (\bmod \Omega T),$$

$$E(e\alpha, e\beta) \subseteq T,$$

$$eE(\alpha, \beta) \subseteq T,$$

$$E(\alpha, \beta) \subseteq e^{-1}T,$$

$$\alpha \equiv \beta \ (\bmod \Omega e^{-1}T),$$

for any substitution e and any $\alpha, \beta \in S$. Consequently, C is equivalential in the sense of the above operator definition. The proof of the reverse implication is much harder - if C is equivalential in the sense of Ω then C possesses an equivalence system $E(x, y)$, i.e., C is equivalential in the sense of Prucnal and Wroński.

Notes. 1. The classes from the above hierarchy are also characterized in semantic terms via their model classes. With each deductive system C the class $\mathbf{Mod}^*(C)$ of reduced matrix models of C is uniquely associated. Much of the theory of deductive systems is devoted to structural properties of the class $\mathbf{Mod}^*(C)$ for logics C varying over definite levels in the protoalgebraic logics hierarchy. E.g. C is protoalgebraic iff the class $\mathbf{Mod}^*(C)$ is closed under the formation of subdirect products of logical matrices. C is equivalential iff $\mathbf{Mod}^*(C)$ is closed under the formation of submatrices and direct products of matrices.

2. The finitely equivalential logics are also simply characterized in terms of the Leibniz operator. Let C be a deductive system. We say that Ω is *continuous* on $Th(C)$ if

$$\Omega \bigcup_{i \in I} T_i = \bigcup_{i \in I} \Omega T_i$$

for any (upper) directed family T_i $(i \in I)$ of closed theories of C such that the theory $\bigcup_{i \in I} T_i$ is closed. (The proviso that the union $\bigcup_{i \in I} T_i$ be closed for any directed system $T_i (i \in I)$ of closed theories automatically holds if it is additionally assumed that the logic C is finitary.) The following theorem is due to Herrmann: C is finitely equivalential iff is continuous on $Th(C)$.

The fundamentals of the theory of algebraizable logics were created by Blok and Pigozzi [1989]. The class of algebraizable logics is the intersection of the classes of equivalential logics and weakly algebraizable logics. As we remarked earlier, their basic goal was to define adequately the scope of the Lindenbaum—Tarski method in metalogic. In their approach, when relativized to finitary systems, the essence of LTM lies in the equivalence between the process of deducibilty of individual formulas from sets of formulas and the process of generating (relative) congruences in certain algebras associated with the given deductive system. This equivalence can be given a strict, purely technical meaning (see also the forthcoming Czelakowski's monograph "Protoalgebraic Logics" for more information). A (*finitary*) logic is algebraizable (in the sense of Blok and Pigozzi) when the above equivalence holds for it. The algebras associated with a given logic C algebraizable in Blok-Pigozzi's sense form a quasivariety. Since this quasivariety is uniquely determined by C, it is called the *equivalent algebraic semantics for C*. The notion of algebraizable logic admitted in this paper is much broader than the one considered by Blok and Pigozzi. (E.g. the finitariness of deductive systems is not assumed here.)

The origin of Fregean logics goes back to the reflection on the Fregean principle in the logical sense and the critique of this principle contained in Suszko's writings. The main feature of non-Fregean logic is the distinction it makes between the meaning and the truth-value of a sentence. (In the sequel we interchangeably use the terms "meaning", "denotation" and "semantic reference".) In the logical systems defined by Suszko the distinction between the reference and truth-value is embodied in a new binary connective called *identity*. This connective is denoted by $\equiv$. Let S be any sentential language which among its connectives has the identity and the equivalence connectives $\equiv$ and $\leftrightarrow$ (and possibly some other connectives) and let C be any logic in S. The connectives $\equiv$ and $\leftrightarrow$ are assumed to have their "usual" meanings in C, i.e., the conditions

$$\alpha \leftrightarrow \beta \in C(T) \ \text{ iff } \quad C(T, \alpha) = C(T, \beta)$$

and

$$\alpha \equiv \beta \in C(T) \ \text{ iff } \ C(T, \varphi(x/\alpha)) = C(T, \varphi(x/\beta))$$

hold for any theory $T \subseteq S$, any formulas α, β, φ and for any variable x. We say, following Suszko, that the *Fregean Principle holds for* C (equivalently, C satisfies the Fregean Principle) if the formula

(FP) $$(x \equiv y) \equiv (x \leftrightarrow y)$$

is the thesis of C. The above formula states that the fact that two sentences have identical meanings is identical with the fact that these two sentences are equivalent. In other words, the above formula reduces the identity of meanings of two sentences to the property of bearing the same truth-value by them. (FP) is referred to as the Fregean Principle (in the logical sense). It is not difficult to prove that the Fregean Principle holds for C iff, for any theory $T \in Th(C)$ and any two formulas α, β, the following equivalence holds:

(v) $$\alpha \equiv \beta \ (\bmod \ \Omega T) \ \text{ iff } \ C(T, \alpha) = C(T, \beta).$$

This observation is the key to the general definition of a Fregean logic formulated at point 5. This general definition entirely abstracts from the grammatical structure of the language and from the supply of connectives the language may possess. (S may not involve the connectives $\equiv$ and $\leftrightarrow$ at all.) We thus see that the general notion of a Fregean logic is a conservative extension of the Fregean logic in the sense of Suszko. (The latter is restricted to the languages which contain $\equiv$ and $\leftrightarrow$.)

Contemporary investigations concerning classification of deductive systems are mainly focused on subclasses of the class of equivalential logics.

The most important intensional logics are equivalential; they are *not* even algebraizable and therefore located not very high in the hierarchy of deductive systems we have outlined here. We may say that the notion (and the name) of an equivalential logic belongs to the arsenal of contemporary logic. It is still an object of extensive research - in Poland, Germany, Russia, U.S.A or in Spain - and a source of deep results. The "career" it has made is amazing - from a short note published in the Bulletin of the Section of Logic many years ago, where it was mentioned first time, it reached the columns of all more important logical journals.

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