ARTICLE TEMPLATE

Bayesian Analysis of the Inverse Generalized Gamma Distribution Using Objective Priors

Pedro L. Ramos[a*], Alex L. Mota[b], Paulo H. Ferreira[a,c], Eduardo Ramos[a], Vera L.D. Tomazella[b], and Francisco Louzada[a]

[a]Institute of Mathematical and Computer Sciences, University of São Paulo, São Carlos, SP, Brazil
[b]Department of Statistics, Federal University of São Carlos, São Carlos, SP, Brazil
[c]Department of Statistics, Federal University of Bahia, Salvador, BA, Brazil

ARTICLE HISTORY
Compiled September 28, 2020

ABSTRACT
The inverse generalized gamma (IGG) distribution can be particularly useful for modeling reliability (survival) data with an upside-down bathtub hazard rate function. The mathematical properties and estimation methods are not known in the literature. In this paper, we provide Bayesian inferences for the IGG distribution parameters using non-informative priors, namely, the Jeffreys prior and the reference prior. Extensive numerical simulations are conducted to investigate the performance of the proposed estimation method when compared with the classical inference. Finally, the potentiality of the IGG model is analyzed by employing real environmental data.

KEYWORDS
Environmental data; Inverse generalized gamma distribution; Jeffreys prior; Maximum likelihood estimation; Reference prior.

1. Introduction

In recent years, several new probability distributions have been proposed in the literature for describing real problems in many applied sciences. In this context, the inverse distributions, also called inverted or reciprocal distributions, have been widely used to a broad range of situations [5, 30, 44, 47, 56, 64]. Often, some inverse distributions appear in Bayesian applications as prior or posterior distributions [20, 27]. Louzada et al. [44] argued that the study of inverse distributions has provided a better comprehension of standard distributions and contributed to adding more flexibility for fitting data.

In particular, the inverse generalized gamma (IGG) distribution was presented by Hoq and Ali [28] in a life testing context and can be seen as the inverse of the generalized gamma (GG) distribution introduced by Stacy [69]. However, its mathematical properties and inferential procedures have not received attention so far. The IGG distribution includes several submodels as particular cases, such as the inverse exponen-

*Corresponding author. Email: pedrolramos@usp.br

tial, inverse log-normal, inverse Weibull, inverse gamma, inverse half-normal, inverse Nakagami-m, inverse Rayleigh, inverse Maxwell-Boltzmann and inverse chi-squared distributions. Generalizations of the IGG distribution were also proposed by Kalla et al. [34] and, more recently, by Mead [47].

In this paper, we consider a Bayesian approach to obtain the parameter estimates of the IGG model. Although a common approach is to consider proper priors with high variance, Bernardo [9] argued that using simple proper priors, presumed to be non-informative, often hides unwarranted critical assumptions, which may easily dominate, or even invalidate the analysis. To overcome this problem, we consider objective priors, where the data provide the dominant information. These objective priors are obtained by formal rules and are usually improper [35], which may lead to improper posteriors. Here, we consider two classes of objective priors, the Jeffreys prior [31] and the reference priors [6]. For the Jeffreys prior, we show that the obtained posterior is improper. Further, four different reference priors are obtained and we show that only one returned a proper posterior density. The obtained reference posterior distribution has essential properties, such as one-to-one invariance, consistent marginalization, and consistent sampling properties. An extensive numerical simulation study is carried out to investigate and compare the proposed estimation methods' performance. The IGG distribution can be particularly useful for modeling reliability (survival) data with an upside-down bathtub hazard rate function. Additionally, we also present several mathematical properties of the IGG distribution, such as the r-th moment, r-th central moment, mean residual life function, harmonic mean, Shannon and Rényi entropies, among others. To illustrate the potentiality of the IGG distribution, we apply our proposed methodology to describe the average flows of water (in cubic meters per second, m^3/s) from July till November (1972-2014) in the Piracicaba River, Brazil.

The remainder of this paper is organized as follows. Section 2 revises the IGG distribution and some of its basic properties, including the behavior of the density and hazard rate functions. Section 3 derive several other important properties of the IGG distribution. Section 4 outlines some special cases (submodels) of the IGG distribution. Section 5 presents the inferential procedures based on MLEs and Bayes estimators considering objective priors for the IGG distribution parameters. Section 6 discusses the results of a simulation study aimed at investigating and comparing the performance of the proposed estimators. Section 7 illustrates the usefulness of the IGG distribution, as well as the relevance of our proposed methodology, through a real environmental data set. Finally, Section 8 presents some concluding remarks.

2. The IGG distribution

A positive random variable T has an IGG distribution, denoted by $\mathrm{IGG}(\phi, \lambda, \alpha)$, if its probability density function (PDF) is given by

$$f(t|\phi,\lambda,\alpha) = \frac{\alpha}{\Gamma(\phi)}\lambda^{\alpha\phi}t^{-\alpha\phi-1}\exp\left\{-\left(\frac{\lambda}{t}\right)^{\alpha}\right\}, \quad t>0, \tag{1}$$

where $\Gamma(\phi) = \int_0^\infty e^{-t}t^{\phi-1}dt$ is the gamma function, $\phi > 0$ and $\alpha > 0$ are the shape parameters, and $\lambda > 0$ is the scale parameter.

The proposition below relates to the IGG distribution with the GG distribution.

Proposition 2.1. *Let* $T \sim \mathrm{IGG}(\phi, \lambda, \alpha)$*, then* $X = 1/T \sim \mathrm{GG}(\phi, \lambda, \alpha)$*.*

Proof. Define the transformation $X = g(T) = \dfrac{1}{T}$, then the resulting PDF is

$$f_X(x) = f_T\left(g^{-1}(x)\right)\left|\frac{d}{dx}g^{-1}(x)\right| = \frac{\alpha}{\Gamma(\phi)}\lambda^{\alpha\phi}x^{\alpha\phi+1}\exp\left\{-\lambda^{\alpha}x^{\alpha}\right\}\frac{1}{x^2}$$
$$= \frac{\alpha}{\Gamma(\phi)}\lambda^{\alpha\phi}x^{\alpha\phi-1}\exp\left\{-(\lambda x)^{\alpha}\right\}.$$

□

It is worth mentioning that the GG distribution includes as special cases several well-known distributions, such as the exponential ($\alpha = \phi = 1$), gamma ($\alpha = 1$), Weibull ($\phi = 1$), Nakagami-m ($\alpha = 2$ and $\lambda = \sqrt{\phi/\Omega}$), half-normal ($\alpha = 2, \phi = 1/2$ and $\lambda = 1/\sqrt{2\sigma^2}$), Rayleigh ($\alpha = 2, \phi = 1$ and $\lambda = 1/\sqrt{2\sigma^2}$), Maxwell-Boltzmann ($\alpha = 2, \phi = 3/2$) and chi-squared ($\alpha = 1, \phi = \nu/2$ and $\lambda = 1/2$) distributions. The log-normal distribution is also obtained as a limiting distribution when $\phi \to \infty$. Different estimation procedures for the GG distribution and its related models, considering both classical and Bayesian approaches, can be found in [25, 55, 57, 59, 68].

Figure 1 gives examples of the PDF shapes of the IGG distribution (1) for different values of ϕ, λ and α.

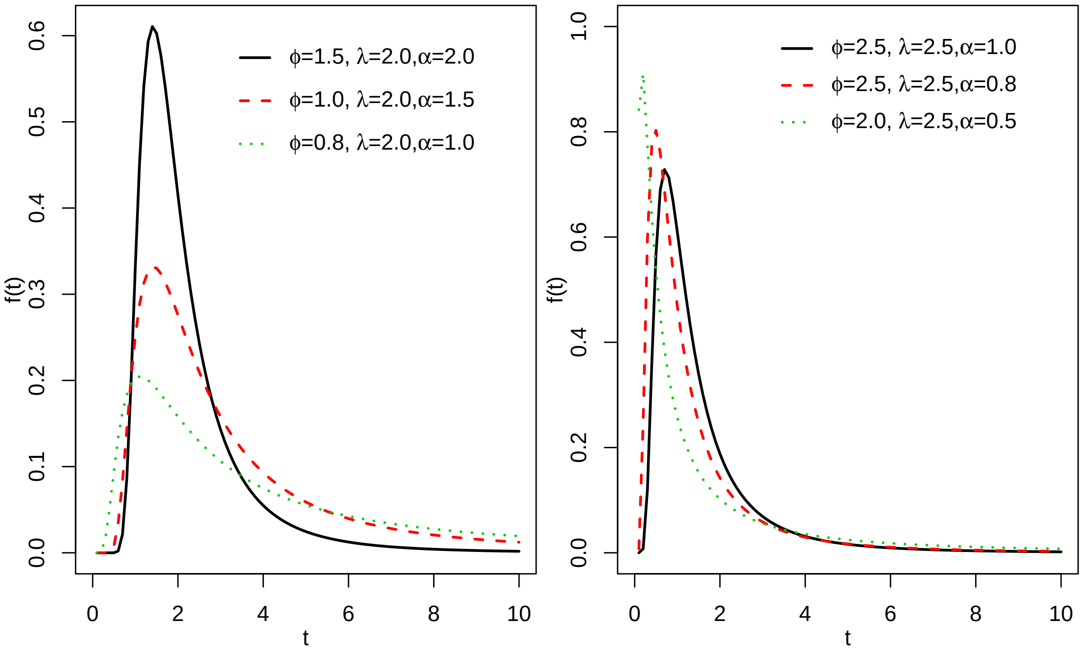

Figure 1. The PDF shapes of the IGG distribution considering different values of ϕ, λ and α.

We observe a unimodal behavior for the PDF as well as a positive asymmetry. The cumulative distribution function (CDF) of the IGG distribution is given by

$$F(t|\phi,\lambda,\alpha) = \frac{1}{\Gamma(\phi)}\Gamma\left(\phi, \left(\frac{\lambda}{t}\right)^{\alpha}\right),$$

where $\Gamma(a,b) = \int_b^{\infty} y^{a-1}e^{-y}dy$ is the upper incomplete gamma function.

The corresponding survival (or reliability) function, which represents the probability of an observation not failing until time t, is

$$S(t|\phi,\lambda,\alpha) = 1 - F(t|\phi,\lambda,\alpha) = \frac{1}{\Gamma(\phi)}\gamma\left(\phi, \left(\frac{\lambda}{t}\right)^{\alpha}\right),$$

where $\gamma(a,b) = \int_0^b y^{a-1}e^{-y}dy$ is the lower incomplete gamma function.

The hazard rate function plays an essential role in reliability theory. It is quite useful for describing the lifetime distribution of engineered systems or components. It describes how the instantaneous failure rate changes over time.

The hazard rate function of the IGG distribution is given by

$$h(t|\phi,\lambda,\alpha) = \frac{f(t|\phi,\lambda,\alpha)}{S(t|\phi,\lambda,\alpha)} = \frac{\alpha\lambda^{\alpha\phi}}{\gamma\left(\phi, \left(\frac{\lambda}{t}\right)^{\alpha}\right)} t^{-\alpha\phi-1}\exp\left\{-\left(\frac{\lambda}{t}\right)^{\alpha}\right\}.$$

Figure 3 shows different shapes for the hazard rate function of the IGG distribution considering distinct values of ϕ, λ and α. It can be noted that the hazard rate function has unimodal shapes regardless of the parameter values.

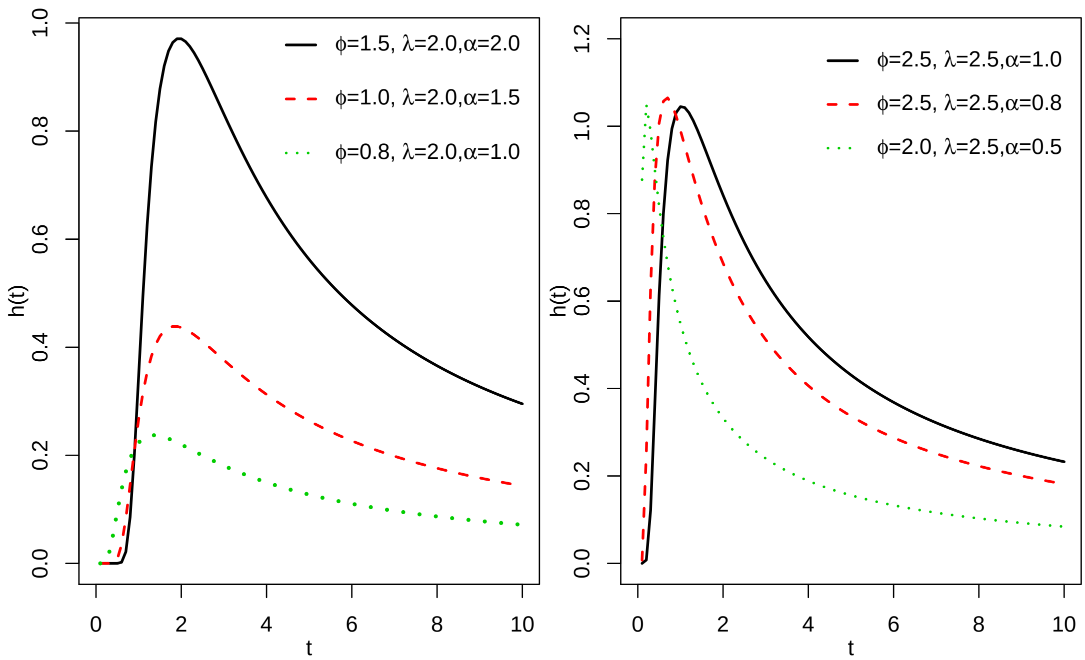

Figure 2. The hazard rate function shapes of the IGG distribution considering different values of ϕ, λ and α.

3. Some properties of the IGG distribution

In this section, we present some mathematical properties of the IGG distribution.

3.1. *Quantile function*

The quantile function plays an essential role in statistical analysis. Generally, a probability distribution can be specified either in terms of the distribution function or by the quantile function [51].

For a random variable with IGG distribution, the p-th quantile, t_p, is given by the solution of

$$\frac{\Gamma\left(\phi, \left(\frac{\lambda}{t_p}\right)^{\alpha}\right)}{\Gamma(\phi)} = p, \quad \text{for } p \in (0, 1). \tag{2}$$

Although the quantile function does not have a closed mathematical expression, we can use the `uniroot` function of the `R` software to find the desired quantiles; see [11, 71].

3.2. *Moments*

Many essential features and properties of a distribution can be obtained through its moments, such as mean, variance, kurtosis, and skewness. In this subsection, essential moment functions, such as the characteristic function, r-th moment, r-th central moment, are presented.

Theorem 3.1. *If $T \sim IGG(\phi, \lambda, \alpha)$, then the r-th power, logarithmic and negative moments are given, respectively, by*

$$E\left[T^r\right] = \frac{\lambda^r}{\Gamma(\phi)} \Gamma\left(\phi - \frac{r}{\alpha}\right), \quad \textit{for} \quad \frac{r}{\alpha} < \phi, \tag{3}$$

$$E\left[\log(T^r)\right] = r\left(\log(\lambda) - \frac{\psi(\phi)}{\alpha}\right), \quad \textit{and} \tag{4}$$

$$E\left[T^{-r}\right] = \frac{\Gamma\left(\phi + \frac{r}{\alpha}\right)}{\lambda^r \Gamma(\phi)}, \tag{5}$$

where $\psi(k) = \dfrac{\partial}{\partial k} \log\left(\Gamma(k)\right)$ *is the digamma function.*

Proof. We have

$$\begin{aligned} E\left[T^r\right] &= \int_0^{\infty} t^r f(t|\phi, \lambda, \alpha) dt = \frac{\alpha}{\Gamma(\phi)} \int_0^{\infty} t^r \left(\frac{\lambda}{t}\right)^{\alpha\phi} \frac{1}{t} \exp\left\{-\left(\frac{\lambda}{t}\right)^{\alpha}\right\} dt. \\ &= \frac{\lambda^r}{\Gamma(\phi)} \int_0^{\infty} z^{(\phi - r/\alpha) - 1} \exp\{-z\} dz = \frac{\lambda^r}{\Gamma(\phi)} \Gamma\left(\phi - \frac{r}{\alpha}\right), \quad \text{for} \quad \frac{r}{\alpha} < \phi. \end{aligned}$$

Similarly, we can prove (4) and (5). □

Corollary 3.2. *If $T \sim IGG(\phi, \lambda, \alpha)$, then the mean and variance are given, respec-*

tively, by

$$\mu = E[T] = \frac{\lambda}{\Gamma(\phi)}\Gamma\left(\phi - \frac{1}{\alpha}\right), \quad \textit{for} \quad \frac{1}{\alpha} < \phi, \tag{6}$$

$$\sigma^2 = Var[T] = \frac{\lambda^2}{\Gamma(\phi)}\left[\Gamma\left(\phi - \frac{2}{\alpha}\right) - \frac{\Gamma^2\left(\phi - \frac{1}{\alpha}\right)}{\Gamma(\phi)}\right], \quad \textit{for} \quad \frac{2}{\alpha} < \phi. \tag{7}$$

Corollary 3.3. *For the random variable T with IGG distribution, the r-th central moment is given by*

$$M_r = E[(T-\mu)^r] = \sum_{i=0}^{r}\binom{r}{i}\left[-\frac{\lambda}{\Gamma(\phi)}\Gamma\left(\phi - \frac{1}{\alpha}\right)\right]^{r-i}\frac{\lambda^i}{\Gamma(\phi)}\Gamma\left(\phi - \frac{i}{\alpha}\right),$$

for $\frac{r}{\alpha} < \phi$, where μ is the mean given in (6).

From Corollary 3.3, we can obtain the skewness and kurtosis measures of the IGG distribution by computing, respectively, $\gamma_1 = \frac{M_3}{(\sigma^2)^{3/2}}$ and $\kappa = \frac{M_4}{(\sigma^2)^2}$, where σ^2 is the variance given in (7).

Proposition 3.4. *The characteristic function (CF) $\Psi_T(s)$ of the IGG distribution is given by*

$$\Psi_T(s) = E\left[e^{isT}\right] = \frac{1}{\Gamma(\phi)}\sum_{r=0}^{\infty}\frac{(is\lambda)^r\Gamma\left(\phi - \frac{r}{\alpha}\right)}{r!}, \quad \textit{for} \quad \frac{r}{\alpha} < \phi \textit{ and } s \in \mathbb{R}.$$

As can be seen in the proposition above, the CF of the IGG distribution cannot be computed analytically. However, it is important to emphasize that the CF always exists since the random variable $e^{isT} = \cos(sT) + i\sin(sT)$ is limited, for all $s \in \mathbb{R}$; see [10].

3.3. *Mean residual life function*

The mean residual life (MRL) function has been widely used in survival and reliability analysis, and represents the expected additional lifetime given that a component has survived or not failed until time t. The MRL function is computed by

$$r(t|\boldsymbol{\theta}) = \frac{1}{S(t|\boldsymbol{\theta})}\int_t^{\infty} y f(y|\boldsymbol{\theta})dy - t,$$

where $\boldsymbol{\theta}$ denotes the parameter vector and $S(.)$ is the survival (or reliability) function.

Proposition 3.5. *The MRL function of the random variable T with IGG distribution* *is given by*

$$r(t|\phi,\lambda,\alpha) = \frac{\lambda\gamma\left(\phi - \frac{1}{\alpha}, \left(\frac{\lambda}{t}\right)^{\alpha}\right)}{\gamma\left(\phi, \left(\frac{\lambda}{t}\right)^{\alpha}\right)} - t,$$

where $\gamma(a,b)$ is the lower incomplete gamma function.

Figure 3 presents examples of the shapes of the MRL function for different parameter values. Since the hazard rate function has unimodal shapes, the MRL function has bathtub shapes [24].

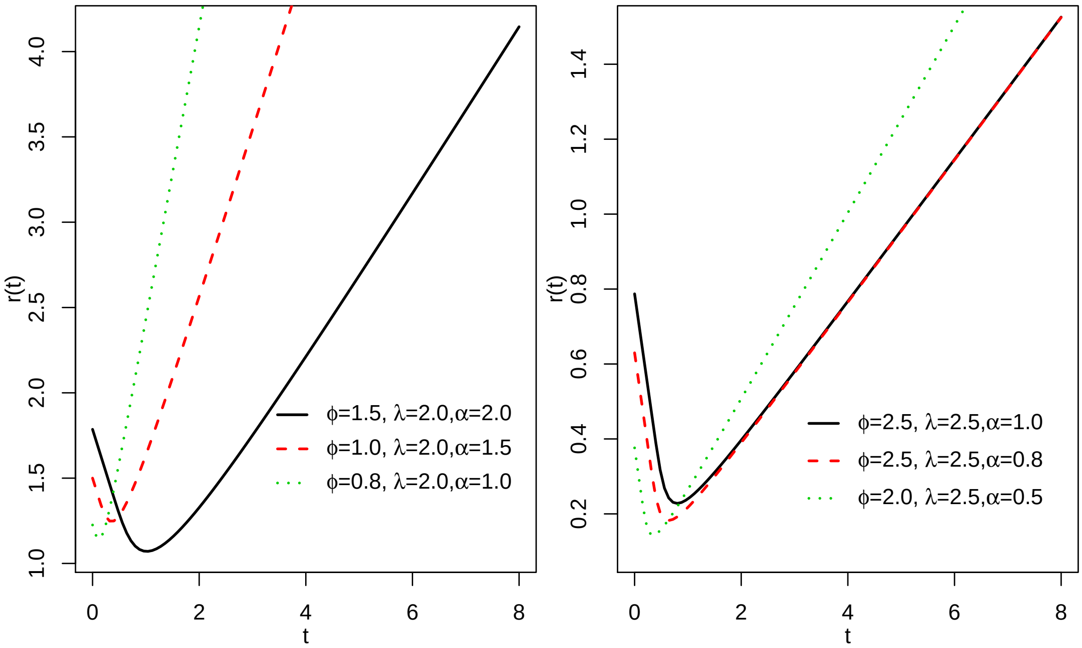

Figure 3. The MRL function shapes for the IGG distribution considering different values of ϕ, λ and α.

3.4. *Entropy*

Entropy plays a significant role in information theory as a measure of the uncertainty associated with a random variable. Shannon's entropy [63] is one of the most important and widely used metrics in information theory. For a PDF f defined on the real line, it is given by

$$H_{\mathrm{S}}(f) = -\int_{-\infty}^{\infty} \log\left(f(t)\right) f(t)dt = E\left[-\log(f(T))\right].$$

Proposition 3.6. *Let $T \sim IGG(\phi,\lambda,\alpha)$, then the Shannon's entropy is given by*

$$H_{\mathrm{S}}(f) = \log(\Gamma(\phi)) + \log(\lambda) - \log(\alpha) - (\alpha\phi + 1)\frac{\psi(\phi)}{\alpha} + \phi.$$

Proof. In fact,

$$\begin{aligned} H_{\mathrm{S}}(f) &= -\int_0^{\infty} \log\left(\frac{\alpha}{\Gamma(\phi)}\lambda^{\alpha\phi}t^{-\alpha\phi-1}\exp\left\{-\left(\frac{\lambda}{t}\right)^{\alpha}\right\}\right) f(t|\phi,\lambda,\alpha)dt \\ &= -\log\left(\frac{\alpha\lambda^{\alpha\phi}}{\Gamma(\phi)}\right) + (\alpha\phi+1)E\left[\log(T)\right] + \lambda^{\alpha}E\left[T^{-\alpha}\right] \\ &= \log(\Gamma(\phi)) + \log(\lambda) - \log(\alpha) - (\alpha\phi+1)\frac{\psi(\phi)}{\alpha} + \phi. \end{aligned}$$

□

A popular generalization of the Shannon's entropy is the Rényi's entropy [60]. If T has PDF f, the Rényi's entropy of order ρ, where $\rho \geq 0$ and $\rho \neq 1$, is defined as

$$H_{\mathrm{R}}(\rho) = \frac{1}{1-\rho}\log\left\{\int_{-\infty}^{\infty}\left[f(t)\right]^{\rho}dt\right\}.$$

Some applications of the the Rényi's entropy can be seen in [4, 61, 70]. The Shannon's entropy appears as a special case of the Rényi's entropy by taking the limit of it as $\rho \to 1$.

Proposition 3.7. *A random variable T with IGG(ϕ,λ,α) distribution has the Rényi's entropy given by*

$$H_{\mathrm{R}}(\rho) = \frac{(\rho-1)\left[\log(\alpha) - \log(\lambda) - \frac{\log(\rho)}{\alpha}\right] + \log\left(\Gamma\left(\rho\phi + \frac{\rho-1}{\alpha}\right)\right) - \rho\left[\log\left(\Gamma(\phi)\right) + \phi\log(\rho)\right]}{1-\rho}.$$

Proof. Note that

$$H_{\mathrm{R}}(\rho) = \frac{1}{1-\rho}\log\left\{\left(\frac{\alpha}{\Gamma(\phi)}\right)^{\rho}\int_0^{\infty}\lambda^{\alpha\phi\rho}t^{-\rho(\alpha\phi+1)}\exp\left\{-\rho\left(\frac{\lambda}{t}\right)^{\alpha}\right\}dt\right\}.$$

Now, using the transformation $z = \rho\left(\frac{\lambda}{t}\right)^{\alpha}$, we have $dz = -\frac{\alpha\rho}{t}\left(\frac{\lambda}{t}\right)^{\alpha}dt$. Thus,

$$\begin{aligned} H_{\mathrm{R}}(\rho) &= \frac{1}{1-\rho}\log\left\{\frac{\alpha^{\rho-1}}{\left[\Gamma(\phi)\right]^{\rho}\lambda^{\rho-1}\rho^{\rho\phi+\frac{\rho-1}{\alpha}}}\int_0^{\infty} z^{\rho\phi+\frac{\rho-1}{\alpha}-1}e^{-z}dz\right\} \\ &= \frac{1}{1-\rho}\log\left\{\left(\frac{\alpha}{\lambda}\right)^{\rho-1}\frac{\Gamma\left(\rho\phi+\frac{\rho-1}{\alpha}\right)}{\left[\Gamma(\phi)\right]^{\rho}\rho^{\rho\phi+\frac{\rho-1}{\alpha}}}\right\}, \end{aligned}$$

and with some algebraic manipulations, we get the result. □

3.5. *Harmonic mean*

The harmonic mean is a measure of central tendency, and in many applications, it can be a natural choice for describing real problems. Some recent applications can be seen in [26, 40, 53].

The harmonic mean of a random variable T is defined as the reciprocal of the expected value of the reciprocal of the random variable T, i.e.,

$$H_m = \left(E\left[\frac{1}{T}\right]\right)^{-1} = \left(\int_{-\infty}^{\infty} \frac{1}{t} f(t)dt\right)^{-1}.$$

Proposition 3.8. *The harmonic mean of the random variable T with IGG distribution is given by*

$$H_m = \frac{\lambda\Gamma(\phi)}{\Gamma\left(\phi + \frac{1}{\alpha}\right)}.$$

4. Particular cases

In this section, we present several inverse models that arise as particular cases of the IGG distribution. These models have been studied and applied in various contexts.

4.1. *Inverse exponential*

Keller et al. [38] introduced the inverse exponential (IE) distribution as an alternative to the exponential distribution for modeling real phenomena with constant failure rate assumption, and memory loss property is not adequate. The IE distribution has an inverted bathtub (unimodal) hazard rate function, and it has been discussed as a lifetime model by Lin et al. [41].

The IE distribution is obtained from the IGG distribution (1) when $\alpha = \phi = 1$ and its PDF is given by

$$f(t|\lambda) = \frac{\lambda}{t^2} \exp\left\{-\frac{\lambda}{t}\right\},$$

for all $t > 0$, where $\lambda > 0$ is the scale parameter.

The IE distribution has no finite moments, and its inability to correctly model datasets that are highly skewed or have fat tails have been noticed in Abouammoh and Alshingiti [2], where the generalized inverted exponential distribution was introduced. The inference for the parameters was conducted under classical and Bayesian viewpoints. In the classical approach, Lin et al. [41] derived the MLE and showed that it is unbiased for λ. Moreover, they estimated and built $100(1 - \vartheta)\%$ confidence interval for the reliability function. Under a Bayesian perspective, Prakash [52] studied the properties of Bayes estimators of the parameter, reliability function, and hazard rate function, using symmetric and asymmetric loss functions. Singh et al. [66] proposed Bayes estimators of the reliability function and parameter of the IE distribution using informative and non-informative priors under general entropy loss function for complete, type I and type II censored samples.

4.2. *Inverse gamma*

The inverse gamma (IG) distribution, also called the inverted gamma distribution, is the reciprocal of a random variable distributed according to the gamma distribution.

In the Bayesian context, it arises as to the marginal posterior distribution for the unknown variance of the normal distribution when a non-informative prior is used. This distribution also appears as an analytically tractable conjugate prior when an informative prior is required [27]. Besides, it is a conjugate prior for an exponential likelihood function [13]. As well as the gamma distribution, the IG distribution belongs to the exponential family, and it has positive support, becoming useful to model several real problems [23, 29, 41, 62]. Some advantages between the gamma and IG distributions can be found in [42].

The IG distribution is obtained from the IGG distribution when $\alpha = 1$, which leads to the PDF given by

$$f(t|\phi, \lambda) = \frac{\lambda^{\phi}}{\Gamma(\phi)} t^{-\phi-1} \exp\left\{-\frac{\lambda}{t}\right\},$$

for all $t > 0$, where $\lambda > 0$ and $\phi > 0$.

The IG distribution has finite m-th moments for $\phi > m$. In particular, if $\phi > 2$ it has mean and variance [13]. In survival analysis context, Glen [23] studied the properties of the IG distribution and showed that it always has an upside-down bathtub shaped hazard rate function. Hence, the IG distribution can be used to model transistors, metals subjected to alternating stress levels, insulation degradation, mechanical devices subjected to wear, and bearings [36].

Different methods to estimate the IG distribution parameters can be found in the literature. For example, Llera and Beckmann [42] introduced five different inferential procedures based on the method of moments (MM), maximum likelihood (ML), and Bayesian estimation. Abid and Al-Hassany [1] studied and compared the ML, MM, percentile, least squares (LS), and weighted LS estimators. Iranmanesh et al. [29] obtained the MLE and the approximate MLE of the reliability function. They also discussed the Bayes estimator of the reliability function under the assumption of independent gamma prior, squared error loss, and Linex error loss functions.

4.3. *Inverse Weibull*

The inverse Weibull (IW) distribution plays an important role in statistical analysis because of its versatility in modeling various real-life phenomena [12, 15, 55]. The IW distribution appears as a particular case of the IGG distribution when $\phi = 1$. Its PDF is given by

$$f(t|\alpha, \lambda) = \alpha\lambda^{\alpha} t^{-\alpha-1} \exp\left\{-\left(\frac{\lambda}{t}\right)^{\alpha}\right\}, \tag{8}$$

for all $t > 0$, and the quantities $\alpha > 0$ and $\lambda > 0$ are the shape and scale parameters, respectively. The PDF (8) can be unimodal or decreasing, depending on the choice of the shape parameter; while the hazard rate function is always unimodal. In this respect, the behavior of the IW and log-normal distributions is quite similar. Erto [17] argued that the most distinctive applicative feature of the IW model is its heavy right-tail, as well as the upside-down bathtub shaped hazard rate function.

The IW distribution is widely known and referred to by different names, like complementary Weibull [15], reciprocal Weibull [45, 49], and Fréchet-type distribution [33, 39]. It was firstly derived for describing the degradation phenomena of mechanical components. Keller et al. [37] used the IW distribution for reliability analysis of commercial vehicle engines. Erto [16] showed that several generative mechanisms lead to the IW distribution, such as deterioration, stress-strength, and shocks mechanisms. Ramos et al. [55] revisited many different estimation methods under both classical and Bayesian approaches and outlined that the reference posteriors should be used to perform inference on the parameters of this distribution.

4.4. *Inverse Nakagami-m*

The inverse Nakagami-m (INK) distribution was proposed by Louzada et al. [44] and can be obtained as a particular case of the IGG distribution when $\alpha = 2$ and $\lambda = \sqrt{\frac{\phi}{\Omega}}$. Its PDF is given by

$$f(t|\phi,\Omega) = \frac{2}{\Gamma(\phi)}\left(\frac{\phi}{\Omega}\right)^{\phi} t^{-2\phi-1}\exp\left\{-\frac{\phi}{\Omega t^2}\right\},$$

for all $t > 0$, where $\phi > 0$ and $\Omega > 0$ are the shape and scale parameters, respectively.

Louzada et al. [44] showed that the hazard rate function of the INK distribution has a unimodal shape for all $\phi > 0$ and $\Omega > 0$. Thus, it can be quite useful, for example, for describing devices that are subjected to high stress, providing a high failure rate after a short repair time.

4.5. *Inverse Rayleigh*

The inverse Rayleigh (IR) distribution is a one-parameter continuous probability distribution on the positive real line. It was introduced by Trayer [72] in the context of reliability theory. After that, in this same context, Voda [73] studied its statistical properties, in particular, the ML estimation, confidence intervals, and hypothesis tests. The author mentioned that an IR distribution could approximate the distribution of lifetimes of several types of experimental units.

The IR distribution appears as a particular case of the IGG distribution when $\phi = 1$ and $\alpha = 2$, and its PDF is given by

$$f(t|\lambda) = \frac{2\lambda^2}{t^3}\exp\left\{-\left(\frac{\lambda}{t}\right)^2\right\},$$

for all $t > 0$, where $\lambda > 0$ is the scale parameter. The variance and higher order moments do not exist for this distribution. Gharraph [22] provided closed-form expressions for the mean, harmonic mean, geometric mean, mode and median of this distribution. Mukherjee and Saran [50] showed that the hazard rate function of a single-parameter IR distribution is increasing for $t < 1.0694543\sqrt{\lambda}$ and decreasing for $t > 1.0694543\sqrt{\lambda}$.

Different inferential procedures for estimating the parameter of the IR distribution have been proposed in the literature. Gharraph [22] obtained estimators of the unknown parameter using different methods of estimation. A comparison of these estimators was discussed numerically in terms of their bias and root mean square error. Soliman et al. [67] discussed the problems of classical and Bayesian estimation based on lower record values. In the classical approach, they derived an MLE, and, in the Bayesian approach, the Bayes estimator was found using informative prior under squared error and zero-one loss functions. Furthermore, they obtained and discussed the Bayesian prediction interval of future record values. Recently, Dey [14] found Bayes estimators of the reliability function under symmetric (squared error loss) and asymmetric linear exponential loss functions using a non-informative prior from Bayes-Laplace (or vague prior).

4.6. *Inverse Maxwell-Boltzmann*

The inverse Maxwell-Boltzmann (IMB) distribution is a one-parameter continuous probability distribution of the reciprocal of a random variable distributed according to the Maxwell-Boltzmann distribution. Singh and Srivastava [65] introduced it, and it has applications in reliability and life testing.

The IMB distribution is a particular case of the IGG distribution when $\alpha = 2$ and $\phi = 3/2$, with PDF

$$f(t|\lambda) = \frac{4\lambda^3}{\sqrt{\pi}} t^{-4} \exp\left\{ -\left(\frac{\lambda}{t}\right)^2 \right\},$$

for all $t > 0$ and scale parameter $\lambda > 0$. Here, we use $\Gamma(3/2) = \sqrt{\pi}/2$. Singh and Srivastava [65] obtained the mean, variance, harmonic mean, and mode of the IMB distribution. They also showed that the hazard rate function is unimodal, therefore, the IMB distribution has the potential to model items with a higher chance of failing as they age for some time, but after survival to a specific age, the probability of failure decreases as time increases.

The estimation of parameter λ has been discussed under classical and Bayesian approaches. Singh and Srivastava [65] obtained estimators of the unknown parameter using the methods of moments and the ML-based on uncensored data and type II censored data. Loganathan et al. [43] derived the Bayes estimator using Jeffreys non-informative prior to the weighted quadratic loss function. Since the risk function is constant, the Bayes estimator is also a minimax estimator. Moreover, simulation studies showed that the Bayes estimator had performed better than the MLE.

4.7. *Inverse chi-squared*

The inverse chi-squared (ICS) distribution, also called the inverted chi-square distribution, is defined as the reciprocal of a random variable distributed according to the chi-squared distribution. In Bayesian inference, it is the conjugate prior distribution for a normal distribution with unknown variance [20, 27]. The ICS distribution is derived from the IGG distribution when $\alpha = 1$, $\phi = \nu/2$ and $\lambda = 1/2$. Its PDF is given by

$$f(t|\nu) = \frac{1}{\Gamma(\nu/2)2^{\nu/2}} t^{-\nu/2-1} \exp\left\{ -\frac{1}{2t} \right\}, \tag{9}$$

for all $t > 0$ and degrees of freedom parameter $\nu > 0$. The PDF (9) is unimodal, positively skewed and has a long right-tail. The power moments, mean, variance, skewness and kurtosis of the ICS distribution can be found in [3].

4.8. *Inverse half-normal*

The inverse half-normal (IHN) distribution is another particular case of the IGG distribution when $\alpha = 2$, $\phi = 1/2$ and $\lambda = 1/\sqrt{2\sigma^2}$. Its PDF is given by

$$f(t|\sigma^2) = \frac{\sqrt{2}}{\sqrt{\pi\sigma^2}} t^{-2} \exp\left\{-\frac{1}{2\sigma^2 t^2}\right\},$$

for all $t > 0$ and scale parameter $\sigma > 0$. Unfortunately, the IHN distribution has not received attention so far. Its mathematical properties, estimation procedures and applications need still be discussed.

5. Inferential procedures

In this section, we discuss the classical (via MLEs) and Bayesian (considering objective priors) estimation procedures for the IGG distribution parameters.

5.1. *Maximum likelihood estimation*

Let $T_1, T_2, \ldots, T_n$ be a random sample of size n from an IGG(ϕ, λ, α) population. Then, the likelihood function of (1) is given by

$$L(\phi, \lambda, \alpha|\boldsymbol{t}) = \frac{\alpha^n}{[\Gamma(\phi)]^n} \lambda^{n\alpha\phi} \left(\prod_{i=1}^{n} t_i^{-\alpha\phi-1}\right) \exp\left\{-\lambda^\alpha \sum_{i=1}^{n} t_i^{-\alpha}\right\}. \tag{10}$$

The natural logarithm of the likelihood function (10), the so-called log-likelihood function, is given by

$$\ell(\phi, \lambda, \alpha|\boldsymbol{t}) = n\log(\alpha) - n\log\left(\Gamma(\phi)\right) + n\alpha\phi\log(\lambda) - (\alpha\phi + 1)\sum_{i=1}^{n}\log\left(t_i\right) - \lambda^\alpha \sum_{i=1}^{n} t_i^{-\alpha}.$$

From the partial derivatives: $\frac{\partial}{\partial\phi}\ell(\phi, \lambda, \alpha|\boldsymbol{t}) = 0$, $\frac{\partial}{\partial\lambda}\ell(\phi, \lambda, \alpha|\boldsymbol{t}) = 0$ and $\frac{\partial}{\partial\alpha}\ell(\phi, \lambda, \alpha|\boldsymbol{t}) = 0$, one gets the following nonlinear equations, respectively:

$$n\hat{\alpha}\log(\hat{\lambda}) - \hat{\alpha}\sum_{i=1}^{n}\log(t_i) = n\psi(\hat{\phi}), \tag{11}$$

$$n\hat{\phi} = \hat{\lambda}^{\hat{\alpha}} \sum_{i=1}^{n} {t_i}^{-\hat{\alpha}} \quad \text{and} \tag{12}$$

$$\frac{n}{\hat{\alpha}} + n\hat{\phi}\log(\hat{\lambda}) - \hat{\phi}\sum_{i=1}^{n}\log(t_i) = \hat{\lambda}^{\hat{\alpha}} \sum_{i=1}^{n} {t_i}^{-\hat{\alpha}} \log\left(\frac{\hat{\lambda}}{t_i}\right). \tag{13}$$

The solution of the equations (11)-(13) yields the MLEs. After some algebraic manipulations, we have

$$\hat{\lambda} = \left(\frac{n\hat{\phi}}{\sum_{i=1}^{n} t_i^{-\hat{\alpha}}}\right)^{\frac{1}{\hat{\alpha}}}, \tag{14}$$

$$\hat{\phi} = \frac{\sum_{i=1}^{n} t_i^{-\hat{\alpha}}}{n\sum_{i=1}^{n} t_i^{-\hat{\alpha}}\log(t_i^{-\hat{\alpha}}) - \sum_{i=1}^{n} t_i^{-\hat{\alpha}}\sum_{i=1}^{n}\log(t_i^{-\hat{\alpha}})}$$

and the MLE of α is obtained by solving the nonlinear equation

$$h(\hat{\alpha}) = n\log\left(n\hat{\phi}\right) - n\log\left(\sum_{i=1}^{n} t_i^{-\hat{\alpha}}\right) - \hat{\alpha}\sum_{i=1}^{n}\log(t_i) - n\psi(\hat{\phi}) = 0.$$

Although only one nonlinear equation has to be solved, usually there are different local maxima, which lead to different estimates than expected. On the other hand, under mild conditions, the MLEs are asymptotically normally distributed with a joint trivariate normal distribution given by

$$(\hat{\phi}, \hat{\lambda}, \hat{\alpha}) \sim \mathrm{N}_3\left((\phi, \lambda, \alpha), I^{-1}(\phi, \lambda, \alpha)\right) \quad \text{for} \quad n \to \infty,$$

where $I(\phi, \lambda, \alpha)$ is the Fisher information matrix (see Hager and Bain [25] for a detailed discussion) given by

$$I(\phi, \lambda, \alpha) = n\begin{bmatrix} \psi^{'}(\phi) & \dfrac{\alpha}{\lambda} & -\dfrac{\psi(\phi)}{\alpha} \\ \dfrac{\alpha}{\lambda} & \dfrac{\phi\alpha^2}{\lambda^2} & -\dfrac{1+\phi\psi(\phi)}{\lambda} \\ -\dfrac{\psi(\phi)}{\alpha} & -\dfrac{1+\phi\psi(\phi)}{\lambda} & \dfrac{1+2\psi(\phi)+\phi\psi^{'}(\phi)+\phi[\psi(\phi)]^2}{\alpha^2} \end{bmatrix} \tag{15}$$

and $\psi'(k) = \frac{\partial}{\partial k}\psi(k)$ is the trigamma function.

Let $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3) = (\phi, \lambda, \alpha)$ be the IGG distribution parameter vector. When the sample size is large, one can construct approximate confidence intervals for the individual parameters, with $100(1-\vartheta)\%$ confidence coefficient, through the marginal distributions

$$\hat{\theta}_w \sim \mathrm{N}\left(\theta_w, I_{ww}^{-1}(\boldsymbol{\theta})\right) \quad \text{for} \quad n \to \infty, \quad w = 1, 2, 3,$$

where $I_{ww}(\boldsymbol{\theta})$ denotes the (w, w)-th element of the Fisher information matrix (15).

5.2. *Bayesian approach*

In this subsection, we discuss the use of objective priors to obtain the posterior densities. The primary motivation lies in the fact that objective Bayesian analysis allows us to make inferences without expert opinion. In this way, the dominant information in the posterior distribution is provided by the data.

The first prior considered here is the Jeffreys [32] prior. For the IGG distribution, the Jeffreys prior is obtained by taking the square root from the determinant of the Fisher information matrix (15), i.e.,

$$\pi_{\mathrm{J}}(\phi,\lambda,\alpha) \propto \frac{\sqrt{\phi^2\left[\psi'(\phi)\right]^2-\psi'(\phi)-1}}{\lambda}. \tag{16}$$

The joint posterior distribution for ϕ, λ and α, using the Jeffreys prior (16), is

$$\pi_{\mathrm{J}}(\phi,\lambda,\alpha|\boldsymbol{t}) = \frac{\alpha^n\sqrt{\phi^2\left[\psi'(\phi)\right]^2-\psi'(\phi)-1}}{c_1(\boldsymbol{t})\left[\Gamma(\phi)\right]^n}\lambda^{n\alpha\phi-1}\left\{\prod_{i=1}^n t_i^{-\alpha\phi-1}\right\}\exp\left\{-\lambda^\alpha\sum_{i=1}^n t_i^{-\alpha}\right\}, \tag{17}$$

where

$$c_1(\boldsymbol{t}) = \int\limits_{\mathcal{A}} \frac{\alpha^n\sqrt{\phi^2\left[\psi'(\phi)\right]^2-\psi'(\phi)-1}}{\left[\Gamma(\phi)\right]^n}\lambda^{n\alpha\phi-1}\left\{\prod_{i=1}^n t_i^{-\alpha\phi-1}\right\}\exp\left\{-\lambda^\alpha\sum_{i=1}^n t_i^{-\alpha}\right\}d\boldsymbol{\theta}$$

and $\mathcal{A}=\{(0,\infty)\times(0,\infty)\times(0,\infty)\}$ is the parameter space for $\boldsymbol{\theta}$. Hereafter, we assume the same definition given above for $\mathcal{A}$.

Notice that, since the prior (16) is improper, the derived posterior distribution (17) may be improper. The following proposition shows that the posterior distribution obtained using the Jeffreys prior is improper and should not be used.

Proposition 5.1. *The posterior distribution (17) is not a proper PDF, i.e.,* $c(\boldsymbol{t})=\infty$.

Proof. See Appendix 9.1. □

Another essential objective prior, the so-called reference prior, was introduced by Bernardo [8], with further developments by [6, 7, 9]. The reference prior has essential properties, such as invariance under one-to-one transformation and consistency under marginalization. There are different approaches to obtain reference priors. The following proposition will be used to obtain them.

Proposition 5.2. *[Bernardo [9], page 40, Theorem 14]. Consider the vector of ordered parameters of interest* $\boldsymbol{\theta}=(\theta_1,\ldots,\theta_d)'$*, and let* $\boldsymbol{t}=(t_1,\ldots,t_n)'$ *be a random sample of size* n *from a statistical model* $f(\cdot|\boldsymbol{\theta})$*. Also, let* $\mathcal{P}$ *be the class of all continuous priors with support* $\mathcal{A}$*. If the posterior distribution of* $\boldsymbol{\theta}$ *is asymptotically normal with dispersion matrix* $V(\hat{\boldsymbol{\theta}}_n)/n$*, where* $\hat{\boldsymbol{\theta}}_n$ *is a consistent estimator of* $\boldsymbol{\theta}$*, then* $H(\boldsymbol{\theta})=V^{-1}(\boldsymbol{\theta})$*,* V_j *is the upper* $j\times j$ *submatrix of* V*,* $H_j=V_j$ *and* $h_{j,j}(\boldsymbol{\theta})$ *is the lower right element of* H_j*, for* $j=1,\ldots,d$*. Finally, if the parameter space of* θ_j *does not depend on* $\boldsymbol{\theta}_{-j}=(\theta_1,\ldots,\theta_{j-1},\theta_{j+1},\ldots,\theta_d)'$ *and* $h_{j,j}(\boldsymbol{\theta})$*, for* $j=1,\ldots,d$*, are factorized in the form*

$$h_{j,j}^{\frac{1}{2}}(\boldsymbol{\theta}) = f_j(\theta_j)g_j(\boldsymbol{\theta}_{-j}),$$

then the reference prior for the ordered parameter vector $\boldsymbol{\theta}$ *is given by*

$$\pi_{\mathrm{R}}(\boldsymbol{\theta}) = \pi_{\mathrm{R}}(\boldsymbol{\theta}|\mathcal{P}) = \pi(\theta_j|\theta_1,\ldots,\theta_{j-1}) \times \cdots \times \pi(\theta_2|\theta_1)\pi(\theta_1),$$

where $\pi(\theta_j|\theta_1,\ldots,\theta_{j-1}) = f_j(\theta_j)$*, and there is no need for compact approximations, even if the conditional priors are improper. That is to say, the* $\boldsymbol{\theta}$*-reference prior is obtained from* $\pi_{\mathrm{R}}(\boldsymbol{\theta}) = \prod_{j=1}^{d} f_j(\theta_j)$*.*

Theorem 5.3. *Let* $\boldsymbol{\theta} = (\phi,\lambda,\alpha)$ *be the vector of ordered parameters. Then, the* $\boldsymbol{\theta}$*-reference prior is given by*

$$\pi_{\mathrm{R1}}(\phi,\lambda,\alpha) \propto \frac{1}{\alpha\lambda}\sqrt{\frac{\phi^2\left[\psi'(\phi)\right]^2 - \psi'(\phi) - 1}{\phi + \phi^2\psi'(\phi) - 1}}. \tag{18}$$

Proof. See Appendix 9.2. □

The joint posterior distribution for ϕ, λ and α, using the reference prior distribution (18), is given by

$$\pi_{\mathrm{R1}}(\phi,\lambda,\alpha|\boldsymbol{t}) = \frac{\pi(\phi)}{d(\boldsymbol{t})}\frac{\alpha^{n-1}}{[\Gamma(\phi)]^n}\lambda^{n\alpha\phi-1}\left\{\prod_{i=1}^{n} t_i^{-\alpha\phi-1}\right\}\exp\left\{-\lambda^{\alpha}\sum_{i=1}^{n} t_i^{-\alpha}\right\}, \tag{19}$$

where

$$d(\boldsymbol{t}) = \int\limits_{\mathcal{A}} \frac{\alpha^{n-1}\pi(\phi)}{[\Gamma(\phi)]^n}\lambda^{n\alpha\phi-1}\left\{\prod_{i=1}^{n} t_i^{-\alpha\phi-1}\right\}\exp\left\{-\lambda^{\alpha}\sum_{i=1}^{n} t_i^{-\alpha}\right\} d\boldsymbol{\theta},$$

with $\pi(\phi) = \sqrt{\frac{\phi^2[\psi'(\phi)]^2 - \psi'(\phi) - 1}{\phi + \phi^2\psi'(\phi) - 1}}$.

Theorem 5.4. *The posterior distribution (19) is a proper PDF, i.e.,* $b(\boldsymbol{t}) < \infty$*.*

Proof. See Appendix 9.3.

□

It is worth mentioning that the (ϕ,λ,α)-reference prior is the same as the (ϕ,α,λ)-reference prior. Additionally, different order for the parameters can lead to different reference priors. Here, we have only presented the prior that led to a proper posterior, but in Appendix 9.4 we present other reference priors and the proof that they lead to improper posteriors.

The marginal reference posterior distribution for ϕ and α is given by

$$\pi_{\mathrm{R1}}(\phi,\alpha|\boldsymbol{t}) \propto \alpha^{n-2}\frac{\pi(\phi)\Gamma(n\phi)}{[\Gamma(\phi)]^n}\left(\frac{\sqrt[n]{\prod_{i=1}^{n} t_i^{-\alpha}}}{\sum_{i=1}^{n} t_i^{-\alpha}}\right)^{n\phi}.$$

The conditional reference posterior distributions for ϕ, λ and α are given, respectively, by

$$\pi_{\mathrm{R1}}(\phi|\alpha,\boldsymbol{t}) \propto \frac{\pi(\phi)\Gamma(n\phi)}{[\Gamma(\phi)]^n}\left(\frac{\sqrt[n]{\prod_{i=1}^{n} t_i^{-\alpha}}}{\sum_{i=1}^{n} t_i^{-\alpha}}\right)^{n\phi}, \tag{20}$$

$$\pi_{\mathrm{R1}}(\lambda|\phi,\alpha,\boldsymbol{t}) \sim \mathrm{GG}\left(n\phi, \left(\sum_{i=1}^{n} t_i^{-\alpha}\right)^{\frac{1}{\alpha}}, \alpha\right) \quad \text{and} \tag{21}$$

$$\pi_{\mathrm{R1}}(\alpha|\phi,\boldsymbol{t}) \propto \alpha^{n-2}\left(\frac{\sqrt[n]{\prod_{i=1}^{n} t_i^{-\alpha}}}{\sum_{i=1}^{n} t_i^{-\alpha}}\right)^{n\phi}. \tag{22}$$

The conditional posterior distributions (20)-(22) are useful during the use of the Markov chain Monte Carlo (MCMC) methods (see Gamerman and Lopes [19]). Particularly, the Metropolis-Hastings (MH) algorithm can be applied, since the conditional distributions (20) and (22) do not have closed-form expressions. From the MH algorithm, we will obtain MCMC samples from the marginal reference posterior densities used to compute the posterior median and the 95% highest posterior density intervals (HPDIs).

6. Simulation study

We performed a Monte Carlo (MC) simulation study to evaluate and compare the performance of the classical and Bayesian approaches for estimating the parameters of the IGG distribution. To that end, we used two criteria: the mean relative estimate (MRE) and mean square error (MSE), which are defined, respectively, as follows:

$$\mathrm{MRE}\left(\hat{\theta}_w\right) = \frac{1}{N}\sum_{j=1}^{N}\frac{\hat{\theta}_{w,j}}{\theta_w} \quad \text{and} \quad \mathrm{MSE}\left(\hat{\theta}_w\right) = \frac{1}{N}\sum_{j=1}^{N}(\hat{\theta}_{w,j}-\theta_w)^2,$$

for $w = 1,2,3$, where $N = 1,000$ is the number of MC realizations. The coverage probabilities (CPs) of the asymptotic normal 95% confidence intervals and Bayesian 95% HPDIs, were also computed. By this procedure, the best estimators will provide both MRE closer to one and MSE closer to zero. Besides, the relative frequencies of intervals containing the true parameter values should be close to 0.95.

In order to obtain the ML estimates, the Newton-Raphson iteration method was applied. In this case, the starting values chosen to initialize the algorithm were the same values used to generate the samples. All calculations and simulations were done using the R software [71].

The normalizing constants for the marginal reference posterior densities require two-dimensional integration. Hence, the MCMC methods were used to obtain the posterior median estimates. Since the conditional reference posterior distributions of ϕ and α are not easily identified, the MH algorithm was applied to sample from the

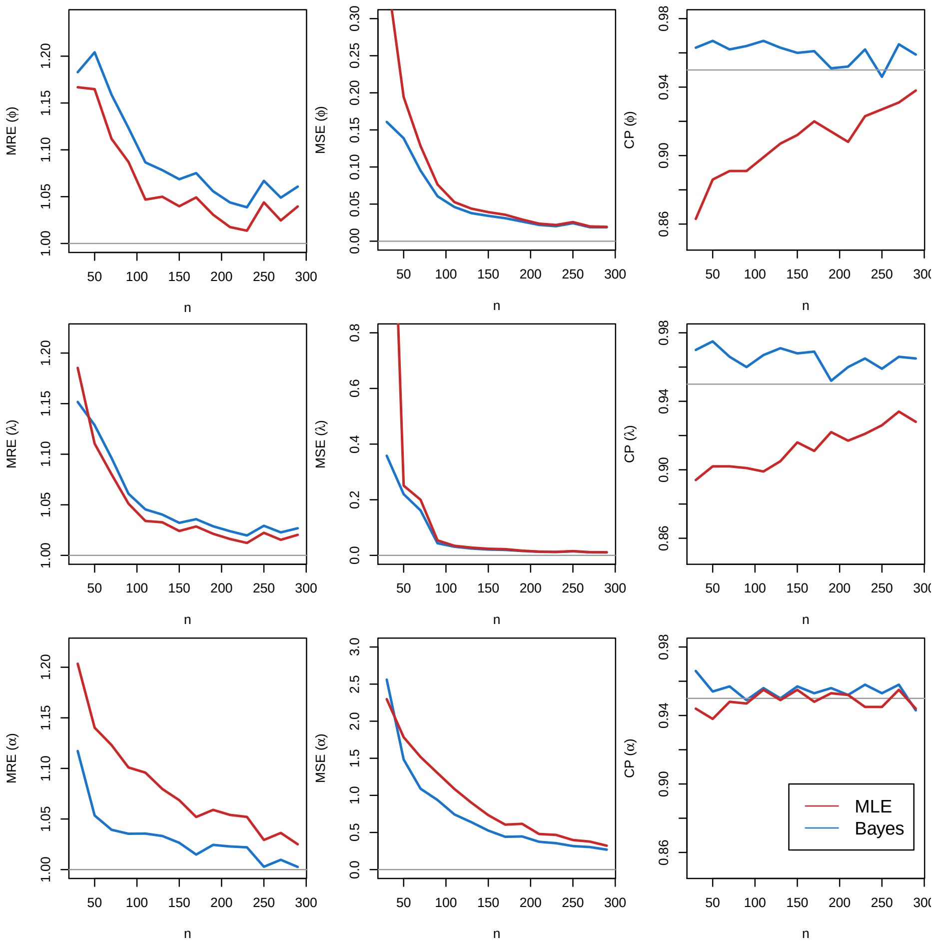

Figure 4. MREs, MSEs and CPs related to the ML and Bayesian estimates of $\phi = 0.5$, $\lambda = 1.0$ and $\alpha = 3$, for $N = 1,000$ simulated samples and $n = \{40, 60, \ldots, 300\}$.

marginals distributions and used to compute the posterior quantities of interest. For each simulated data set, 15,500 iterations were performed using MCMC methods. The initial 1,000 iterations were discarded as burn-in, and a thin of 30 was used to reduce the autocorrelation of successive realizations. The Geweke's diagnostic criterion [21] was applied to assess convergence of the obtained chains under a 95% confidence level. These samples were used to calculate the posterior median estimates of ϕ, λ and α, and the 95% HPDIs.

The parameter values selected to perform the simulations were: $\boldsymbol{\theta} = (0.5, 1.0, 3)$, with $n = \{40, 60, \ldots, 300\}$. The seed used to generate the pseudo-random samples in the R software was 2019.

Figure 4 presents the MRE, MSE, and CP of the estimates obtained through the MLEs and Bayes estimators, for 1,000 simulated samples under different values of n. As can be seen from this figure, the Bayes estimators returned better estimates than the corresponding MLEs in terms of MSE values for all parameters, mainly for small and moderate sample sizes. In terms of MRE, the MLEs returned better results for ϕ and similar results for λ, while the Bayes approach was superior for obtaining estimates for α. On the other hand, Figure 4 shows that the asymptotic normal confidence intervals did not have desired coverage rates, while the HPDIs provided good CPs that are close to the nominal level (95%), even for small sample sizes. For these reasons, the Bayes estimators (posterior medians) should be considered for improved estimation of the IGG distribution parameters.

7. Application

In this section, we apply the IGG distribution to five real data sets related to the average flows of water (in m^3/s), from July to November, in the Piracicaba River, Brazil. The data sets (see Appendix 9.5) were obtained from the Department of Water Resources and Power agency manager of water resources of the State of São Paulo, including a period from 1972 to 2014. The control of the water flow is often essential for safety reasons. Water flow rates either above or below the desired limit can affect both the population and the ecosystems. Thus, estimating well, the flow of water is essential for the definition of environmental planning and public policies.

Table 1 shows the Bayes estimates (posterior median) and 95% HPDIs for the parameters ϕ, λ and α of the IGG distribution.

Table 1. Bayes estimates and 95% HPDIs for ϕ, λ and α, considering the data sets related to the average flows of water (in m^3/s) during July-November 1972-2014 in the Piracicaba river, Brazil.

Month	Parameter	Estimate	95% HPDI
July	ϕ	0.8622	(0.0825 ; 4.6452)
	λ	9.3877	(6.6941 ; 17.0599)
	α	3.7612	(0.5957 ; 7.9839)
August	ϕ	1.3637	(0.2112 ; 7.5886)
	λ	9.7080	(6.8989 ; 14.7985)
	α	3.6240	(0.8358 ; 6.9295)
September	ϕ	0.4741	(0.0550 ; 1.7201)
	λ	7.1545	(5.7549 ; 9.7028)
	α	5.3741	(0.9313 ; 13.7751)
October	ϕ	1.3505	(0.1960 ; 10.0741)
	λ	11.9997	(7.2400 ; 23.2962)
	α	2.3073	(0.3250 ; 4.5499)
November	ϕ	1.2023	(0.0984 ; 8.1300)
	λ	16.0169	(7.6476 ; 27.5436)
	α	1.9791	(0.4592 ; 5.9490)

The results obtained using the IGG distribution are compared to the corresponding ones achieved with the use of the GG [69], exponentiated Weibull (EW) [48], Marshall-Olkin Weibull (MOW) [46], and extended Poisson-Weibull (EPW) [54] distributions. We consider the most common model selection/discrimination criteria, namely the BIC (Bayesian or Schwarz Information Criteria), AIC (Akaike Information Criteria) and AICc (Corrected AIC), which are calculated, respectively, by $\text{BIC} = -2\ell(\hat{\boldsymbol{\theta}}|\boldsymbol{t}) + d\log(n)$, $\text{AIC} = -2\ell(\hat{\boldsymbol{\theta}}|\boldsymbol{t}) + 2d$ and $\text{AICc} = \text{AIC} + 2\,d\,(d+1)/(n-d-1)$, where d is the number of model parameters and $\hat{\boldsymbol{\theta}}$ is the Bayes estimate of $\boldsymbol{\theta}$. Given a set of candidate models for the data at hand, the preferred model is the one that provides the minimum values of these criteria.

Table 2 presents the BIC, AIC, and AICc values for different probability distributions. The goodness-of-fit can also be checked through the over a plot of the survival functions adjusted by the proposed theoretical models onto the empirical survival function (Kaplan-Meier estimate), as shown in Figure 5. Comparing the empirical survival function with the adjusted distributions, we observe that the IGG distribution better fits these data than its competitor models. These results are confirmed by the BIC, AIC, and AICc values since the IGG distribution has the minimum values for all proposed data sets.

Table 2. The AIC, AICc and BIC values for different probability distributions, considering the data sets related to the average flows of water (in m^3/s) during July-November 1972-2014 in the Piracicaba river, Brazil.

Month	Criterion	IGG	GG	EW	MOW	EPW
	AIC	**221.134**	236.692	232.009	252.830	246.044
July	AICc	**221.819**	237.378	232.695	253.516	246.730
	BIC	**226.124**	241.683	237.000	257.821	251.035
	AIC	**195.969**	206.131	200.925	220.267	222.772
August	AICc	**196.617**	206.779	201.574	220.916	223.420
	BIC	**201.109**	211.272	206.066	225.408	227.912
	AIC	**219.681**	243.388	235.706	256.432	254.323
September	AICc	**220.367**	244.074	236.392	257.118	254.323
	BIC	**224.672**	248.379	240.697	261.423	258.628
	AIC	**249.894**	257.992	255.494	269.773	262.800
October	AICc	**250.560**	258.659	256.161	270.439	263.467
	BIC	**254.960**	263.059	260.561	274.839	267.867
	AIC	**276.520**	280.949	278.278	287.283	282.818
November	AICc	**277.225**	281.655	278.984	287.989	283.524
	BIC	**281.432**	285.861	283.190	292.196	287.731

Therefore, our proposed methodology can be applied successfully to analyze the average flows of water during July-November in the Piracicaba river, using the IGG distribution with the Bayesian approach.

8. Concluding remarks

In this paper, we derived and discussed many important mathematical properties of the IGG distribution, which allows its application in a wide range of practical problems. Further, we revised several inverted models that arise as particular cases of this critical distribution.

The inferential methods for the parameters were discussed, considering both classical and Bayesian approaches. In the classical approach, we obtained the estimators for two parameters in closed-form expressions, while the estimate of α can be achieved using unidimensional optimization methods. The Fisher information matrix was presented, which allows us to construct asymptotic confidence intervals. Under the Bayesian approach, we considered an objective Bayesian analysis where the dominant information is provided by the data. We derived five objective priors, the Jeffreys prior and four reference priors. Since the obtained priors are improper, we proved that only one reference prior leads to a proper posterior density. The obtained reference posterior distribution has essential properties, such as one-to-one invariance, consistent marginalization, and consistent sampling properties.

An extensive Monte Carlo simulation study showed that the Bayes estimators, using the absolute loss function (posterior median), performed better than the corresponding MLEs and, therefore, should be considered to obtain improved estimates for the

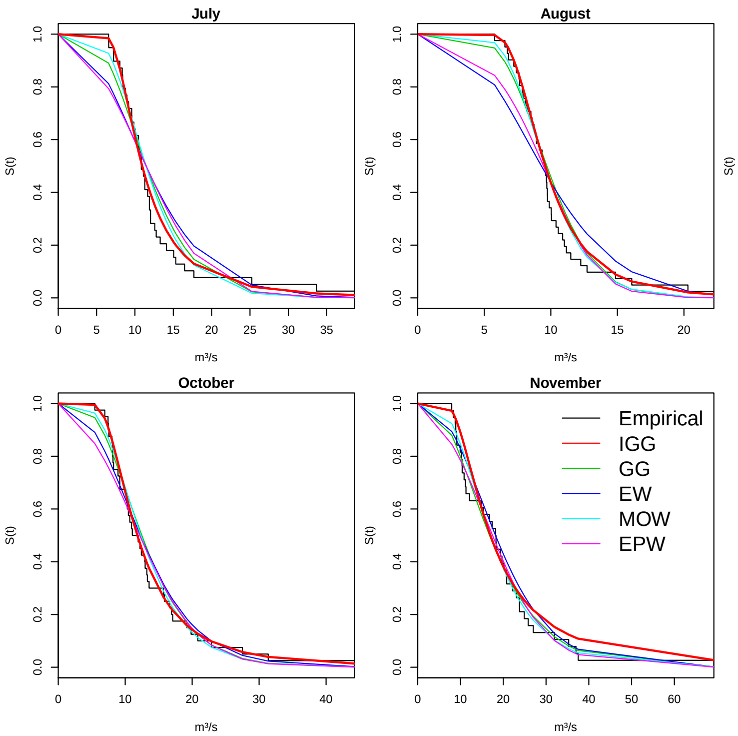

Figure 5. Fitted survival functions superimposed to the empirical survival function (Kaplan-Meier estimate), considering the data sets related to the average flows of water (in m^3/s) during July-November 1972-2014 in the Piracicaba river, Brazil.

parameters of the IGG distribution. Moreover, we observed that the HPDIs returned more accurate intervals for the parameters of interest. Finally, our proposed methodology was used in an application considering five real data sets related to the average flows of water in the Piracicaba river, Brazil, from July to November. The results showed that the IGG distribution returned a better fit in comparison with other critical three-parameter distributions.

There is a large number of possible extensions of this current work. For example, the presence of covariates and long-term survivals is widespread in practice. Moreover, a regression model for censored and uncensored data could be useful. Hence, our approach should be investigated further in these contexts. The IGG distribution is also promising to be used in studies involving degradation and accelerated life test data.

Disclosure statement

No potential conflict of interest was reported by the authors.

Acknowledgements

The research was partially supported by CNPq, FAPESP, and CAPES of Brazil. Pedro L. Ramos acknowledges support from São Paulo State Research Foundation (FAPESP Proc. 2017/25971-0). Eduardo Ramos acknowledges financial support from São Paulo

State Research Foundation (FAPESP Proc. 2019/27636-9). Francisco Louzada is supported by the Brazilian agencies CNPq (grant number 301976/2017-1) and FAPESP (grant number 2013/07375-0).

References

[1] S.H. Abid and S. Al-Hassany, *On the inverted gamma distribution*, Int. J. Syst. Sci. Appl. Math 1 (2016), pp. 16–22.

[2] A. Abouammoh and A.M. Alshingiti, *Reliability estimation of generalized inverted exponential distribution*, Journal of Statistical Computation and Simulation 79 (2009), pp. 1301–1315.

[3] K.C. Ayienda, *Gamma and Related Distributions*, BoD–Books on Demand, 2014.

[4] R.G. Baraniuk, P. Flandrin, A.J. Janssen, and O.J. Michel, *Measuring time-frequency information content using the rényi entropies*, IEEE Transactions on Information theory 47 (2001), pp. 1391–1409.

[5] O.E. Barndorff-Nielsen, *Normal inverse gaussian distributions and stochastic volatility modelling*, Scandinavian Journal of statistics 24 (1997), pp. 1–13.

[6] J.O. Berger and J.M. Bernardo, *Estimating a product of means: Bayesian analysis with reference priors*, Journal of the American Statistical Association 84 (1989), pp. 200–207.

[7] J.O. Berger and J.M. Bernardo, *Ordered group reference priors with application to the multinomial problem*, Biometrika 79 (1992), pp. 25–37.

[8] J.M. Bernardo, *Reference posterior distributions for bayesian inference*, Journal of the Royal Statistical Society. Series B (Methodological) (1979), pp. 113–147.

[9] J.M. Bernardo, *Reference analysis*, Handbook of statistics 25 (2005), pp. 17–90.

[10] P. Billingsley, *Probability and measure*, John Wiley & Sons, 2008.

[11] R.P. Brent, *Algorithms for minimization without derivatives, chap. 4* (1973).

[12] R. Calabria and G. Pulcini, *On the maximum likelihood and least-squares estimation in the inverse weibull distributions*, Statistica Applicata 2 (1990), pp. 53–66.

[13] J.D. Cook, *Inverse gamma distribution*, online: http://www. johndcook. com/inverse gamma. pdf, Tech. Rep (2008).

[14] S. Dey, *Bayesian estimation of the parameter and reliability function of an inverse rayleigh distribution*, Malaysian Journal of Mathematical Sciences 6 (2012), pp. 113–124.

[15] A. Drapella, *The complementary weibull distribution: unknown or just forgotten?*, Quality and Reliability Engineering International 9 (1993), pp. 383–385.

[16] P. Erto, *Genesi, proprietaed identificazione del modello di sopravvivenza weibull inverso*, Statistica Applicata 1 (1989), pp. 117–128.

[17] P. Erto, *The inverse weibull survival distribution and its proper application*, arXiv preprint arXiv:1305.6909 (2013).

[18] G.B. Folland, *Real analysis: modern techniques and their applications*, 2nd ed., Wiley, New York, 1999.

[19] D. Gamerman and H.F. Lopes, *Markov chain Monte Carlo: stochastic simulation for Bayesian inference*, CRC Press, 2006.

[20] A. Gelman, J.B. Carlin, H.S. Stern, D.B. Dunson, A. Vehtari, and D.B. Rubin, *Bayesian data analysis*, Chapman and Hall/CRC, 2013.

[21] J. Geweke, *et al.*, *Evaluating the accuracy of sampling-based approaches to the calculation of posterior moments*, Vol. 196, Federal Reserve Bank of Minneapolis, Research Department Minneapolis, MN, USA, 1991.

[22] M. Gharraph, *Comparison of estimators of location measures of an inverse rayleigh distribution*, The Egyptian Statistical Journal 37 (1993), pp. 295–309.

[23] A.G. Glen, *On the inverse gamma as a survival distribution*, Journal of Quality technology 43 (2011), pp. 158–166.

[24] A. H. Olcay, *Mean residual life function for certain types of non-monotonic ageing*, Com-

munications in statistics. Stochastic models 11 (1995), pp. 219–225.

[25] H.W. Hager and L.J. Bain, *Inferential procedures for the generalized gamma distribution*, Journal of the American Statistical Association 65 (1970), pp. 1601–1609.

[26] M.O. Hasna and M.S. Alouini, *Harmonic mean and end-to-end performance of transmission systems with relays*, IEEE Transactions on communications 52 (2004), pp. 130–135.

[27] P.D. Hoff, *A first course in Bayesian statistical methods*, Vol. 580, Springer, 2009.

[28] A. Hoq and M. Ali, *Templetion, estimation of parameters of a generalized life testing models*, Journal of Statistical Research 9 (1974), pp. 67–79.

[29] A. Iranmanesh, K.F. Vajargah, and M. Hasanzadeh, *On the estimation of stress strength reliability parameter of inverted gamma distribution*, Mathematical Sciences 12 (2018), pp. 71–77.

[30] K. Jain, N. Singla, and S.K. Sharma, *The generalized inverse generalized weibull distribution and its properties*, Journal of Probability 2014 (2014).

[31] H. Jeffreys, *An invariant form for the prior probability in estimation problems*, in *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, Vol. 186. The Royal Society, 1946, pp. 453–461.

[32] H. Jeffreys, *An invariant form for the prior probability in estimation problems*, in *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, Vol. 186. The Royal Society, 1946, pp. 453–461.

[33] N. Johnson and S. Kotz, *Balakrishnan (1994), continuous univariate distributions*, New York: VViley .

[34] S.L. Kalla and B. Al-Saqabi, *Further results on a unified form of gamma-type distributions*, Fractional Calculus and Applied Analysis 4 (2001), pp. 91–100.

[35] R.E. Kass and L. Wasserman, *The selection of prior distributions by formal rules*, Journal of the American Statistical Association 91 (1996), pp. 1343–1370.

[36] D. Kececioglu, *Reliability engineering handbook. vol. 2. nj: Ptr prentice-hall*, Inc: Englewood Cliffs (1991).

[37] A. Keller, M. Goblin, and N. Farnworth, *Reliability analysis of commercial vehicle engines*, Reliability Engineering 10 (1985), pp. 15–25.

[38] A. Keller, A. Kamath, and U. Perera, *Reliability analysis of cnc machine tools*, Reliability engineering 3 (1982), pp. 449–473.

[39] S. Kotz and S. Nadarajah, *Extreme value distributions: theory and applications*, World Scientific, 2000.

[40] J.F. Limbrunner, R.M. Vogel, and L.C. Brown, *Estimation of harmonic mean of a lognormal variable*, Journal of hydrologic engineering 5 (2000), pp. 59–66.

[41] C. Lin, B. Duran, and T. Lewis, *Inverted gamma as a life distribution*, Microelectronics Reliability 29 (1989), pp. 619–626.

[42] A. Llera and C. Beckmann, *Estimating an inverse gamma distribution*, arXiv preprint arXiv:1605.01019 (2016).

[43] A. Loganathan, S. Chelvi, and M. Uma, *Bayes estimation of parameter in inverse maxwell distribution under weighted quadratic loss function*, International Journal of Scientific Research in Mathematical and Statistical Sciences 4 (2017), pp. 13–16.

[44] F. Louzada, P.L. Ramos, and D. Nascimento, *The inverse nakagami-m distribution: A novel approach in reliability*, IEEE Transactions on Reliability (2018), pp. 1–13.

[45] C.J. Lu and W.O. Meeker, *Using degradation measures to estimate a time-to-failure distribution*, Technometrics 35 (1993), pp. 161–174.

[46] A.W. Marshall and I. Olkin, *A new method for adding a parameter to a family of distributions with application to the exponential and weibull families*, Biometrika 84 (1997), pp. 641–652.

[47] M. Mead, *Generalized inverse gamma distribution and its application in reliability*, Communications in Statistics-Theory and Methods 44 (2015), pp. 1426–1435.

[48] G.S. Mudholkar, D.K. Srivastava, and M. Freimer, *The exponentiated weibull family: A reanalysis of the busmotor-failure data*, Technometrics 37 (1995), pp. 436–445.

[49] G.S. Mudholkar and G.D. Kollia, *Generalized weibull family: a structural analysis*, Com-

munications in statistics-theory and methods 23 (1994), pp. 1149–1171.

[50] S. Mukherjee and L. Saran, *Bivariate inverse rayleigh distributions in reliability studies*, Journal of the Indian Statistical Association 22 (1984), pp. 23–31.

[51] N.U. Nair, P. Sankaran, and N. Balakrishnan, *Quantile-based reliability analysis*, Springer, 2013.

[52] G. Prakash, *Inverted exponential distribution under a bayesian viewpoint*, Journal of Modern Applied Statistical Methods 11 (2012), p. 16.

[53] A.E. Raftery, M.A. Newton, J.M. Satagopan, and P.N. Krivitsky, *Estimating the integrated likelihood via posterior simulation using the harmonic mean identity* (2006).

[54] P.L. Ramos, D.K. Dey, F. Louzada, and V.H. Lachos, *An extended poisson family of life distribution: A unified approach in competitive and complementary risks*, Journal of Applied Statistics 47 (2020), pp. 306–322.

[55] P.L. Ramos, F. Louzada, E. Ramos, and S. Dey, *The fréchet distribution: Estimation and application-an overview*, Journal of Statistics and Management Systems 23 (2020), pp. 549–578.

[56] P.L. Ramos, F. Louzada, T.K. Shimizu, and A.O. Luiz, *The inverse weighted lindley distribution: Properties, estimation and an application on a failure time data*, Communications in Statistics-Theory and Methods 48 (2019), pp. 2372–2389.

[57] P.L. Ramos, D.C. Nascimento, P.H. Ferreira, K.T. Weber, T.E. Santos, and F. Louzada, *Modeling traumatic brain injury lifetime data: Improved estimators for the generalized gamma distribution under small samples*, PloS one 14 (2019), p. e0221332.

[58] P.L. Ramos, F.A. Rodrigues, E. Ramos, D.K. Dey, and F. Louzada, *Power laws distributions in objective priors*, arXiv preprint arXiv:2005.07674 (2020).

[59] P.L. Ramos and F. Louzada, *Bayesian reference analysis for the generalized gamma distribution*, IEEE Communications Letters 22 (2018), pp. 1950–1953.

[60] A. Rényi, *On measures of entropy and information*, In: Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics 1 (1961), pp. 547–561.

[61] D. Rocchini, L. Delucchi, G. Bacaro, P. Cavallini, H. Feilhauer, G.M. Foody, K.S. He, H. Nagendra, C. Porta, C. Ricotta, *et al.*, *Calculating landscape diversity with information-theory based indices: A grass gis solution*, Ecological informatics 17 (2013), pp. 82–93.

[62] X. Shang and H. Song, *Radar detection based on compound-gaussian model with inverse gamma texture*, IET radar, sonar & navigation 5 (2011), pp. 315–321.

[63] C.E. Shannon, *A mathematical theory of communication*, Bell system technical journal 27 (1948), pp. 379–423.

[64] V.K. Sharma, S.K. Singh, U. Singh, and F. Merovci, *The generalized inverse lindley distribution: A new inverse statistical model for the study of upside-down bathtub data*, Communications in Statistics-Theory and Methods 45 (2016), pp. 5709–5729.

[65] K.L. Singh and R. Srivastava, *Inverse maxwell distribution as a survival model, genesis and parameter estimation*, Research Journal of Mathematical and Statistical Sciences 2 (2014), pp. 23–28.

[66] S.K. Singh, U. Singh, and D. Kumar, *Bayes estimators of the reliability function and parameter of inverted exponential distribution using informative and non-informative priors*, Journal of Statistical computation and simulation 83 (2013), pp. 2258–2269.

[67] A. Soliman, E.A. Amin, and A.A. Abd-El Aziz, *Estimation and prediction from inverse rayleigh distribution based on lower record values*, Applied Mathematical Sciences 4 (2010), pp. 3057–3066.

[68] E.W. Stacy and G.A. Mihram, *Parameter estimation for a generalized gamma distribution*, Technometrics 7 (1965), pp. 349–358.

[69] E.W. Stacy, *et al.*, *A generalization of the gamma distribution*, The Annals of mathematical statistics 33 (1962), pp. 1187–1192.

[70] J.M. Stéphan, G. Misguich, and V. Pasquier, *Phase transition in the rényi-shannon entropy of luttinger liquids*, Physical Review B 84 (2011), p. 195128.

[71] R.C. Team, *A language and environment for statistical computing. vienna, austria: R*

foundation for statistical computing; 2012, URL: https://www.R-project.org (2019).
[72] V.N. Trayer, Doklady Acad, Nauk, Belorus, USSR (1964).
[73] V.G. Voda, *On the inverse rayleigh distributed random variable*, Reports of Statistical Application Research 19 (1972), pp. 13–21.

9. Appendix

9.1. *Proof of Proposition 5.1*

Since $\alpha^n\sqrt{\phi^2\left[\psi'(\phi)\right]^2-\psi'(\phi)-1}\,[\Gamma(\phi)]^{-n}\lambda^{n\alpha\phi-1}\prod_{i=1}^n t_i^{-\alpha\phi-1}\exp\left\{-\lambda^\alpha\sum_{i=1}^n t_i^{-\alpha}\right\}\geq 0$, by Tonelli's theorem (see Folland [18]) and from the asymptotic relations proved in [58] we have

$$
\begin{aligned}
c_1(\boldsymbol{t}) &= \int\limits_{\mathcal{A}} \alpha^n \frac{\sqrt{\phi^2\left[\psi'(\phi)\right]^2-\psi'(\phi)-1}}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n t_i^{-\alpha\phi-1}\right\}\lambda^{n\alpha\phi-1}\exp\left\{-\lambda^\alpha\sum_{i=1}^n t_i^{-\alpha}\right\}d\boldsymbol{\theta}\\
&= \int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty \alpha^n \frac{\sqrt{\phi^2\left[\psi'(\phi)\right]^2-\psi'(\phi)-1}}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n t_i^{-\alpha\phi-1}\right\}\lambda^{n\alpha\phi-1}\exp\left\{-\lambda^\alpha\sum_{i=1}^n t_i^{-\alpha}\right\}d\lambda\,d\phi\,d\alpha\\
&= \int\limits_0^\infty\int\limits_0^\infty \alpha^{n-1}\sqrt{\phi^2\left[\psi'(\phi)\right]^2-\psi'(\phi)-1}\,\frac{\Gamma(n\phi)}{[\Gamma(\phi)]^n}\frac{\left(\prod_{i=1}^n t_i\right)^{-\alpha\phi-1}}{\left(\sum_{i=1}^n t_i^{-\alpha}\right)^{n\phi}}d\phi\,d\alpha\\
&\geq \int\limits_0^\infty\int\limits_0^1 \alpha^{n-1}\sqrt{\phi^2\left[\psi'(\phi)\right]^2-\psi'(\phi)-1}\,\frac{\Gamma(n\phi)}{[\Gamma(\phi)]^n}\frac{\left(\prod_{i=1}^n t_i\right)^{-\alpha\phi-1}}{\left(\sum_{i=1}^n t_i^{-\alpha}\right)^{n\phi}}d\phi\,d\alpha\\
&\propto \int\limits_0^\infty\int\limits_0^1 \alpha^{n-1}\times\phi^0\times\phi^{n-1}\left(\frac{\sqrt[n]{\prod_{i=1}^n t_i^{-\alpha}}}{\sum_{i=1}^n t_i^{-\alpha}}\right)^{n\phi}d\phi\,d\alpha\\
&\propto \int\limits_1^\infty \alpha^{n-1}\int\limits_0^1 \phi^{n-1}e^{-n\,\mathrm{q}(\alpha)\phi}\,d\phi\,d\alpha\\
&= \int\limits_1^\infty \alpha^{n-1}\frac{\gamma(n,n\,\mathrm{q}(\alpha))}{(n\,\mathrm{q}(\alpha))^n}\,d\alpha \propto \int\limits_1^\infty \alpha^{-1}\,d\alpha=\infty,
\end{aligned}
$$

where $\mathrm{q}(\alpha)=\log\left(\dfrac{\sum_{i=1}^n t_i^{-\alpha}}{\sqrt[n]{\prod_{i=1}^n t_i^{-\alpha}}}\right)$.

9.2. *Proof of Theorem 5.3*

Considering some algebraic manipulations in (15), we have that

$$
h_{1,1}^{\frac{1}{2}}(\boldsymbol{\theta})=\sqrt{\frac{\phi^2\left[\psi'(\phi)\right]^2-\psi'(\phi)-1}{\phi+\phi^2\psi'(\phi)-1}}=f_1(\phi)g_1(\lambda)g_1(\alpha),
$$

where $f_1(\phi)=\sqrt{\frac{\phi^2[\psi'(\phi)]^2-\psi'(\phi)-1}{\phi+\phi^2\psi'(\phi)-1}}$, $g_1(\lambda)=1$ and $g_1(\alpha)=1$;

$$h_{2,2}^{\frac{1}{2}}(\boldsymbol{\theta})=\frac{\alpha}{\lambda}\sqrt{\frac{\phi+\phi^2\psi(\phi)-1}{1+2\psi(\phi)+\phi\psi'(\phi)+\phi\left[\psi(\phi)\right]^2}}=g_2(\alpha)f_2(\lambda)g_2(\phi),$$

where $g_2(\alpha)=\alpha,\ f_2(\lambda)=\frac{1}{\lambda}$ and $g_2(\phi)=\sqrt{\frac{\phi+\phi^2\psi(\phi)-1}{1+2\psi(\phi)+\phi\psi'(\phi)+\phi[\psi(\phi)]^2}}$; and finally,

$$h_{3,3}^{\frac{1}{2}}(\boldsymbol{\theta})=\frac{\sqrt{1+2\psi(\phi)+\phi\psi'(\phi)+\phi\left[\psi(\phi)\right]^2}}{\alpha}=f_3(\alpha)g_3(\lambda)g_3(\phi),$$

where $g_3(\phi)=\sqrt{1+2\psi(\phi)+\phi\psi'(\phi)+\phi\left[\psi(\phi)\right]^2}$, $g_3(\lambda)=1$ and $f_3(\alpha)=\frac{1}{\alpha}$.

Using Proposition 5.2 and assuming the ordered parameters (ϕ,λ,α), the conditional reference priors are

$$\pi(\alpha|\lambda,\phi)\propto f_3(\alpha)=\frac{1}{\alpha},\quad \pi(\lambda|\phi)\propto f_2(\lambda)=\frac{1}{\lambda},\quad \pi(\phi)\propto f_1(\phi)=\sqrt{\frac{\phi^2\left[\psi'(\phi)\right]^2-\psi'(\phi)-1}{\phi+\phi^2\psi'(\phi)-1}}.$$

Hence, the reference prior is

$$\pi_{\mathrm{R1}}(\phi,\lambda,\alpha)\propto\pi(\alpha|\lambda,\phi)\pi(\lambda|\phi)\pi(\phi)=\frac{1}{\alpha\lambda}\sqrt{\frac{\phi^2\left[\psi'(\phi)\right]^2-\psi'(\phi)-1}{\phi+\phi^2\psi'(\phi)-1}}.$$

9.3. *Proof of Theorem 5.4*

Again, $\alpha^{n-1}\pi(\phi)[\Gamma(\phi)]^{-n}\lambda^{n\alpha\phi-1}\prod_{i=1}^{n}t_i^{-\alpha\phi-1}\exp\left\{-\lambda^{\alpha}\sum_{i=1}^{n}t_i^{-\alpha}\right\}\ \geq\ 0$. Then, by Tonelli's theorem and the asymptotic relations proved in [58] we have

$$\begin{aligned}d(\boldsymbol{t})&=\int_0^\infty\int_0^\infty\int_0^\infty\frac{\alpha^{n-1}\pi(\phi)}{[\Gamma(\phi)]^n}\lambda^{n\alpha\phi-1}\left\{\prod_{i=1}^{n}t_i^{-\alpha\phi-1}\right\}\exp\left\{-\lambda^{\alpha}\sum_{i=1}^{n}t_i^{-\alpha}\right\}d\lambda\,d\phi\,d\alpha\\&=\int_0^\infty\int_0^\infty\frac{\alpha^{n-2}\pi(\phi)}{[\Gamma(\phi)]^n}\left\{\prod_{i=1}^{n}t_i^{-\alpha\phi-1}\right\}\frac{\Gamma(n\phi)}{\left(\sum_{i=1}^{n}t_i^{-\alpha}\right)^{n\phi}}d\phi\,d\alpha=s_1+s_2+s_3+s_4,\end{aligned}$$

where

$$\begin{aligned}s_1&=\int_0^1\int_0^1\frac{\alpha^{n-2}\pi(\phi)}{[\Gamma(\phi)]^n}\left\{\prod_{i=1}^{n}t_i^{-\alpha\phi-1}\right\}\frac{\Gamma(n\phi)}{\left(\sum_{i=1}^{n}t_i^{-\alpha}\right)^{n\phi}}d\phi\,d\alpha\\&\propto\int_0^1\int_0^1\alpha^{n-2}\times1\times\phi^{n-\frac{3}{2}}\left(\frac{\sqrt[n]{\prod_{i=1}^{n}t_i^{-\alpha}}}{\sum_{i=1}^{n}t_i^{-\alpha}}\right)^{n\phi}d\phi\,d\alpha=\int_0^1\alpha^{n-2}\int_0^1\phi^{n-\frac{3}{2}}e^{-n\phi\,\mathrm{q}(\alpha)}d\phi\,d\alpha\\&=\int_0^1\alpha^{n-2}\frac{\gamma(n-\frac{1}{2},n\,\mathrm{q}(\alpha))}{[n\,\mathrm{q}(\alpha)]^{n-\frac{1}{2}}}d\alpha\propto\int_0^1\alpha^{n-2}\frac{1}{1^{n-\frac{1}{2}}}\times1\,d\alpha=\int_0^1\alpha^{n-2}d\alpha<\infty;\end{aligned}$$

$$s_2 = \int_1^\infty \int_0^1 \frac{\alpha^{n-2}\pi(\phi)}{[\Gamma(\phi)]^n} \left\{ \prod_{i=1}^n t_i^{-\alpha\phi-1} \right\} \frac{\Gamma(n\phi)}{\left(\sum_{i=1}^n t_i^{-\alpha}\right)^{n\phi}} d\phi\, d\alpha$$

$$\propto \int_1^\infty \int_0^1 \alpha^{n-2} \times 1 \times \phi^{n-\frac{3}{2}} \left(\frac{\sqrt[n]{\prod_{i=1}^n t_i^{-\alpha}}}{\sum_{i=1}^n t_i^{-\alpha}} \right)^{n\phi} d\phi\, d\alpha = \int_1^\infty \alpha^{n-2} \int_0^1 \phi^{n-\frac{3}{2}} e^{-n\phi\, \mathrm{q}(\alpha)} d\phi\, d\alpha$$

$$= \int_1^\infty \alpha^{n-2} \frac{\gamma(n-\frac{1}{2}, n\, \mathrm{q}(\alpha))}{[n\, \mathrm{q}(\alpha)]^{n-\frac{1}{2}}} d\alpha \propto \int_1^\infty \alpha^{n-2} \frac{1}{\alpha^{n-\frac{1}{2}}} \times 1\, d\alpha = \int_1^\infty \alpha^{-\frac{3}{2}} d\alpha < \infty;$$

$$s_3 = \int_0^1 \int_1^\infty \frac{\alpha^{n-2}\pi(\phi)}{[\Gamma(\phi)]^n} \left\{ \prod_{i=1}^n t_i^{-\alpha\phi-1} \right\} \frac{\Gamma(n\phi)}{\left(\sum_{i=1}^n t_i^{-\alpha}\right)^{n\phi}} d\phi\, d\alpha$$

$$\propto \int_0^1 \int_1^\infty \alpha^{n-2} \times \frac{1}{\phi^{\frac{3}{2}}} \times n^{n\phi} \phi^{\frac{n-1}{2}} \left(\frac{\sqrt[n]{\prod_{i=1}^n t_i^{-\alpha}}}{\sum_{i=1}^n t_i^{-\alpha}} \right)^{n\phi} d\phi\, d\alpha = \int_0^1 \alpha^{n-2} \int_1^\infty \phi^{\frac{n-2}{2}-1} e^{-n\phi\, \mathrm{p}(\alpha)} d\phi\, d\alpha$$

$$= \int_0^1 a^{n-2} \frac{\Gamma\left(\frac{n-2}{2}, n\, \mathrm{p}(\alpha)\right)}{[n\, \mathrm{p}(\alpha)]^{\frac{n-2}{2}}} d\alpha \propto \int_0^1 \alpha^{n-2} \frac{1}{(\alpha^2)^{\frac{n-2}{2}}} \times 1\, d\alpha = \int_0^1 \alpha^0 d\alpha < \infty;$$

and

$$s_4 = \int_1^\infty \int_1^\infty \frac{\alpha^{n-2}\pi(\phi)}{[\Gamma(\phi)]^n} \left\{ \prod_{i=1}^n t_i^{-\alpha\phi-1} \right\} \frac{\Gamma(n\phi)}{\left(\sum_{i=1}^n t_i^{-\alpha}\right)^{n\phi}} d\phi\, d\alpha$$

$$\propto \int_0^1 \int_1^\infty \alpha^{n-2} \times \frac{1}{\phi^{\frac{3}{2}}} \times n^{n\phi} \phi^{\frac{n-1}{2}} \left(\frac{\sqrt[n]{\prod_{i=1}^n t_i^{-\alpha}}}{\sum_{i=1}^n t_i^{-\alpha}} \right)^{n\phi} d\phi\, d\alpha$$

$$= \int_0^1 \alpha^{n-2} \int_1^\infty \phi^{\frac{n-2}{2}-1} e^{-n\phi\, \mathrm{p}(\alpha)} d\phi\, d\alpha$$

$$= \int_1^\infty a^{n-2} \frac{\Gamma\left(\frac{n-2}{2}, n\, \mathrm{p}(\alpha)\right)}{[n\, \mathrm{p}(\alpha)]^{\frac{n-2}{2}}} d\alpha \propto \int_1^\infty \alpha^{n-2} \frac{1}{\alpha^{\frac{n-2}{2}}} \times \alpha^{\frac{n-2}{2}-1} e^{-n\log\left(\frac{t_m}{\sqrt[n]{\prod_{i=1}^n t_i}}\right)\alpha} d\alpha$$

$$= \int_1^\infty \alpha^{n-3} e^{-L\alpha} d\alpha < \infty,$$

where $\mathrm{p}(\alpha) = \log\left(\frac{\frac{1}{n}\sum_{i=1}^n t_i^{-\alpha}}{\sqrt[n]{\prod_{i=1}^n t_i^{-\alpha}}} \right)$, $L = n\log\left(\frac{t_m}{\sqrt[n]{\prod_{i=1}^n t_i}} \right) > 0$, and $t_m = \max\{t_1, \ldots, t_n\}$. Therefore, we have $d(\boldsymbol{t}) = s_1 + s_2 + s_3 + s_4 < \infty$.

9.4. *Reference priors*

Here, we present other reference priors considering different ordered parameters. The derivations are similar to the ones shown in Appendix 9.2.

Firstly, let us consider that the ordered parameters are given by (α,ϕ,λ). Then, the (α,ϕ,λ)-reference prior is

$$\pi_{\mathrm{R2}}(\phi,\lambda,\alpha)\propto\frac{1}{\alpha}\sqrt{\frac{\phi\psi'(\phi)-1}{\phi}}. \tag{23}$$

The joint posterior distribution for ϕ, λ and α, using the reference prior distribution (23), is given by

$$\pi_{\mathrm{R2}}(\phi,\lambda,\alpha|\boldsymbol{t})=\frac{1}{c_2(\boldsymbol{t})}\sqrt{\frac{\phi\psi'(\phi)-1}{\phi}}\frac{\alpha^{n-1}}{\Gamma(\phi)^n}\lambda^{n\alpha\phi-1}\left\{\prod_{i=1}^{n}t_i^{-\alpha\phi-1}\right\}\exp\left\{-\lambda^{\alpha}\sum_{i=1}^{n}t_i^{-\alpha}\right\}, \tag{24}$$

where

$$c_2(\boldsymbol{t})=\int\limits_{\mathcal{A}}\sqrt{\frac{\phi\psi'(\phi)-1}{\phi}}\frac{\alpha^{n-1}}{\Gamma(\phi)^n}\lambda^{n\alpha\phi-1}\left\{\prod_{i=1}^{n}t_i^{-\alpha\phi-1}\right\}\exp\left\{-\lambda^{\alpha}\sum_{i=1}^{n}t_i^{-\alpha}\right\}d\boldsymbol{\theta}.$$

Since $\alpha^{n-1}\phi^{-\frac{1}{2}}\sqrt{\phi\psi'(\phi)-1}\,\Gamma(\phi)^{-n}\lambda^{n\alpha\phi-1}\prod_{i=1}^{n}t_i^{-\alpha\phi-1}\exp\left\{-\lambda^{\alpha}\sum_{i=1}^{n}t_i^{-\alpha}\right\}\geq 0$, by Tonelli's theorem and the asymptotic relations proved in [58] we have

$$\begin{aligned}
c_2(\boldsymbol{t})&=\int\limits_{\mathcal{A}}\alpha^{n-1}\frac{\phi^{-\frac{1}{2}}\sqrt{\phi\psi'(\phi)-1}}{\Gamma(\phi)^n}\left\{\prod_{i=1}^{n}t_i^{-\alpha\phi-1}\right\}\lambda^{n\alpha\phi-1}\exp\left\{-\lambda^{\alpha}\sum_{i=1}^{n}t_i^{-\alpha}\right\}d\boldsymbol{\theta}\\
&=\int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty\alpha^{n-1}\frac{\phi^{-\frac{1}{2}}\sqrt{\phi\psi'(\phi)-1}}{\Gamma(\phi)^n}\left\{\prod_{i=1}^{n}t_i^{-\alpha\phi-1}\right\}\lambda^{n\alpha\phi-1}\exp\left\{-\lambda^{\alpha}\sum_{i=1}^{n}t_i^{-\alpha}\right\}d\lambda\,d\phi\,d\alpha\\
&=\int\limits_0^\infty\int\limits_0^\infty\alpha^{n-2}\phi^{-\frac{1}{2}}\sqrt{\phi\psi'(\phi)-1}\,\frac{\Gamma(n\phi)}{[\Gamma(\phi)]^n}\frac{\left(\prod_{i=1}^{n}t_i\right)^{-\alpha\phi-1}}{\left(\sum_{i=1}^{n}t_i^{-\alpha}\right)^{n\phi}}d\phi\,d\alpha\\
&\geq\int\limits_0^\infty\int\limits_0^1\alpha^{n-2}\phi^{-\frac{1}{2}}\sqrt{\phi\psi'(\phi)-1}\,\frac{\Gamma(n\phi)}{[\Gamma(\phi)]^n}\frac{\left(\prod_{i=1}^{n}t_i\right)^{-\alpha\phi-1}}{\left(\sum_{i=1}^{n}t_i^{-\alpha}\right)^{n\phi}}d\phi\,d\alpha\\
&\propto\int\limits_0^\infty\int\limits_0^1\alpha^{n-2}\times\phi^{-1}\times\phi^{n-1}\left(\frac{\sqrt[n]{\prod_{i=1}^{n}t_i^{-\alpha}}}{\sum_{i=1}^{n}t_i^{-\alpha}}\right)^{n\phi}d\phi\,d\alpha\\
&\propto\int\limits_1^\infty\alpha^{n-2}\int\limits_0^1\phi^{n-2}e^{-n\,\mathrm{q}(\alpha)\phi}\,d\phi\,d\alpha\\
&=\int\limits_1^\infty\alpha^{n-2}\frac{\gamma(n-1,n\,\mathrm{q}(\alpha))}{(n\,\mathrm{q}(\alpha))^{n-1}}\,d\alpha\propto\int\limits_1^\infty\alpha^{-1}\,d\alpha=\infty
\end{aligned}$$

Hence, the posterior distribution (24) is improper.

Now, the (α, λ, ϕ)-reference prior, as well as the (λ, α, ϕ)-reference prior, have the same form and are given by

$$\pi_{\mathrm{R3}}(\phi, \lambda, \alpha) \propto \frac{\sqrt{\psi'(\phi)}}{\alpha}. \tag{25}$$

The joint posterior distribution for ϕ, λ and α, using the reference prior distribution (25), is given by

$$\pi_{\mathrm{R3}}(\phi, \lambda, \alpha|\boldsymbol{t}) = \frac{1}{c_3(\boldsymbol{t})}\alpha^{n-1}\frac{\sqrt{\psi'(\phi)}}{\Gamma(\phi)^n}\lambda^{n\alpha\phi-1}\left\{\prod_{i=1}^{n} t_i^{-\alpha\phi-1}\right\}\exp\left\{-\lambda^{\alpha}\sum_{i=1}^{n} t_i^{-\alpha}\right\}, \tag{26}$$

where

$$c_3(\boldsymbol{t}) = \int\limits_{\mathcal{A}} \alpha^{n-1}\frac{\sqrt{\psi'(\phi)}}{\Gamma(\phi)^n}\lambda^{n\alpha\phi-1}\left\{\prod_{i=1}^{n} t_i^{-\alpha\phi-1}\right\}\exp\left\{-\lambda^{\alpha}\sum_{i=1}^{n} t_i^{-\alpha}\right\} d\boldsymbol{\theta}.$$

Since $\alpha^{n-1}\frac{\sqrt{\psi'(\phi)}}{\Gamma(\phi)^n}\lambda^{n\alpha\phi-1}\prod_{i=1}^{n} t_i^{-\alpha\phi-1}\exp\left\{-\lambda^{\alpha}\sum_{i=1}^{n} t_i^{-\alpha}\right\} \geq 0$, by Tonelli's theorem and from the asymptotic relations proved in [58], we have, just as in the case of $c_1(\boldsymbol{t})$, that

$$\begin{aligned}
c_3(\boldsymbol{t}) &= \int\limits_{\mathcal{A}} \alpha^{n-1}\frac{\sqrt{\psi'(\phi)}}{\Gamma(\phi)^n}\left\{\prod_{i=1}^{n} t_i^{-\alpha\phi-1}\right\}\lambda^{n\alpha\phi-1}\exp\left\{-\lambda^{\alpha}\sum_{i=1}^{n} t_i^{-\alpha}\right\} d\boldsymbol{\theta} \\
&= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \alpha^{n-1}\frac{\sqrt{\psi'(\phi)}}{\Gamma(\phi)^n}\left\{\prod_{i=1}^{n} t_i^{-\alpha\phi-1}\right\}\lambda^{n\alpha\phi-1}\exp\left\{-\lambda^{\alpha}\sum_{i=1}^{n} t_i^{-\alpha}\right\} d\lambda\, d\phi\, d\alpha \\
&\geq \int_0^{\infty}\int_0^{1} \alpha^{n-2}\sqrt{\psi'(\phi)}\,\frac{\Gamma(n\phi)}{[\Gamma(\phi)]^n}\,\frac{\left(\prod_{i=1}^{n} t_i\right)^{-\alpha\phi-1}}{\left(\sum_{i=1}^{n} t_i^{-\alpha}\right)^{n\phi}} d\phi\, d\alpha \\
&\propto \int_0^{\infty}\int_0^{1} \alpha^{n-2}\times\phi^{-1}\times\phi^{n-1}\left(\frac{\sqrt[n]{\prod_{i=1}^{n} t_i^{-\alpha}}}{\sum_{i=1}^{n} t_i^{-\alpha}}\right)^{n\phi} d\phi\, d\alpha = \infty.
\end{aligned}$$

Therefore, the posterior distribution (26) is improper.

Finally, the last reference prior has the ordered parameters given by (λ, ϕ, α). Then, the (λ, ϕ, α)-reference prior is

$$\pi_{\mathrm{R4}}(\phi, \lambda, \alpha) \propto \sqrt{\psi'(\phi) - \frac{\psi(\phi)^2}{2\psi(\phi) + \phi\psi'(\phi) + \phi\psi(\phi^2) + 1}} = \pi_4(\phi)\cdot \tag{27}$$

The joint posterior distribution for ϕ, λ and α, using the reference prior distribution (27), is given by

$$\pi_{\mathrm{R4}}(\phi, \lambda, \alpha|\boldsymbol{t}) = \frac{1}{c_4(\boldsymbol{t})}\alpha^{n}\frac{\pi_4(\phi)}{\Gamma(\phi)^n}\lambda^{n\alpha\phi-1}\left\{\prod_{i=1}^{n} t_i^{-\alpha\phi-1}\right\}\exp\left\{-\lambda^{\alpha}\sum_{i=1}^{n} t_i^{-\alpha}\right\}, \tag{28}$$

where

$$c_4(\boldsymbol{t}) = \int\limits_{\mathcal{A}} \alpha^n \frac{\pi_4(\phi)}{\Gamma(\phi)^n} \lambda^{n\alpha\phi-1} \left\{ \prod_{i=1}^{n} t_i^{-\alpha\phi-1} \right\} \exp\left\{ -\lambda^{\alpha} \sum_{i=1}^{n} t_i^{-\alpha} \right\} d\boldsymbol{\theta}.$$

Since $\alpha^n \pi_4(\phi)\,\Gamma(\phi)^{-n} \lambda^{n\alpha\phi-1} \prod_{i=1}^{n} t_i^{-\alpha\phi-1} \exp\left\{-\lambda^{\alpha} \sum_{i=1}^{n} t_i^{-\alpha}\right\} \geq 0$, by Tonelli's theorem and from the asymptotic relations proved in [58] we have, just as in the case of $c_1(\boldsymbol{t})$, that

$$\begin{aligned}
c_4(\boldsymbol{t}) &= \int\limits_{\mathcal{A}} \alpha^n \frac{\pi_4(\phi)}{\Gamma(\phi)^n} \left\{ \prod_{i=1}^{n} t_i^{-\alpha\phi-1} \right\} \lambda^{n\alpha\phi-1} \exp\left\{ -\lambda^{\alpha} \sum_{i=1}^{n} t_i^{-\alpha} \right\} d\boldsymbol{\theta} \\
&= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \alpha^n \frac{\pi_4(\phi)}{\Gamma(\phi)^n} \left\{ \prod_{i=1}^{n} t_i^{-\alpha\phi-1} \right\} \lambda^{n\alpha\phi-1} \exp\left\{ -\lambda^{\alpha} \sum_{i=1}^{n} t_i^{-\alpha} \right\} d\lambda\, d\phi\, d\alpha \\
&\geq \int_0^{\infty}\int_0^{1} \alpha^{n-1} \pi_4(\phi) \frac{\Gamma(n\phi)}{[\Gamma(\phi)]^n} \frac{\left(\prod_{i=1}^{n} t_i\right)^{-\alpha\phi-1}}{\left(\sum_{i=1}^{n} t_i^{-\alpha}\right)^{n\phi}} d\phi\, d\alpha \\
&\propto \int_0^{\infty}\int_0^{1} \alpha^{n-1} \times \phi^{-1} \times \phi^{n-1} \left(\frac{\sqrt[n]{\prod_{i=1}^{n} t_i^{-\alpha}}}{\sum_{i=1}^{n} t_i^{-\alpha}} \right)^{n\phi} d\phi\, d\alpha \\
&\geq \int_0^{\infty}\int_0^{1} \alpha^{n-2} \times \phi^{-1} \times \phi^{n-1} \left(\frac{\sqrt[n]{\prod_{i=1}^{n} t_i^{-\alpha}}}{\sum_{i=1}^{n} t_i^{-\alpha}} \right)^{n\phi} d\phi\, d\alpha = \infty.
\end{aligned}$$

Thus, the posterior distribution (28) is improper.

9.5. *Data sets*

July										
10.78	12.03	10.03	38.61	9.16	11.88	10.52	12.03	8.96	16.47	33.66
8.40	10.46	8.74	11.66	11.29	12.59	8.04	14.10	13.29	9.86	8.34
15.33	10.83	12.77	10.83	11.29	7.21	8.45	6.58	15.05	9.60	7.19
25.24	9.58	11.88	11.10	17.69	6.56	-	-	-	-	-
August										
10.03	8.92	8.50	22.25	7.91	7.85	8.86	9.43	7.61	16.08	20.30
9.88	9.13	14.86	9.70	8.21	9.74	7.66	11.17	8.59	9.34	6.73
10.05	9.62	9.67	9.75	9.57	7.42	11.02	6.55	8.73	7.22	6.82
10.89	11.50	12.72	8.09	12.25	10.39	10.56	5.78	-	-	-
September										
10.40	7.77	6.91	27.34	12.48	8.80	13.33	10.77	6.32	7.94	60.30
9.36	9.00	9.04	11.74	6.52	9.42	11.49	11.32	18.90	5.96	8.73
20.39	10.46	10.95	15.14	16.52	9.62	9.17	6.36	6.36	7.35	7.24
8.15	8.45	15.27	8.83	11.34	9.53	-	-	-	-	-
October										
12.41	10.11	10.31	27.50	8.93	7.56	15.70	11.94	19.20	44.23	31.36
8.26	7.44	8.11	13.24	22.88	6.96	7.47	15.87	10.69	17.10	13.31
12.98	20.85	9.20	16.94	10.47	10.20	13.57	8.11	8.18	12.93	8.03
11.06	10.94	12.21	9.19	19.84	16.63	5.47	-	-	-	-
November										
27.01	19.69	8.34	23.79	23.08	18.30	23.80	20.16	10.84	37.00	19.39
31.95	17.39	11.28	12.11	18.33	25.82	9.92	10.36	37.51	10.39	35.30
14.96	22.18	69.24	9.03	8.76	24.90	8.90	15.37	18.21	11.15	9.14
16.75	20.71	10.25	20.80	7.95	-	-	-	-	-	-