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Rate of convergence for reaction–diffusion equations with nonlinear Neumann boundary conditions and $\mathcal{C}^1$ variation of the domain

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Abstract. In this paper, we propose the compact convergence approach to deal with the continuity of attractors of some reaction–diffusion equations under smooth perturbations of the domain subject to nonlinear Neumann boundary conditions. We define a family of invertible linear operators to compare the dynamics of perturbed and unperturbed problems in the same phase space. All continuity arising from small smooth perturbations will be estimated by a rate of convergence given by the domain variation in a $\mathcal{C}^1$ topology.

1. Introduction

The nonlinear dynamics of reaction–diffusion equations under perturbations of the domain have been studied by several authors concerned with different types of domains. From pioneering to recent works, we can mention [4,5,21,29,30,32] and [18,23] where parabolic and elliptic equations have been considered, and theories to understand a huge class of perturbed problems are introduced. In this context, two interesting examples were extensively studied in [11,21], the so-called localized large diffusion and thin domain. For these problems, the works [1] and [8] have presented a rate of convergence to estimate the continuity of attractors as a positive parameter $\varepsilon \to 0$.

Indeed, a convergence rate theory for attractors has been developed (for instance in [1,12–15]), which enables us to estimate all convergences that appear when a fixed domain is smoothly perturbed and nonlinear Neumann boundary condition is considered. For example, it is possible to find a positive function $\tau(\varepsilon)$ that goes to zero as the parameter $\varepsilon \to 0$, to estimate the convergence of the resolvent operators and linear semigroup, the permanence of hyperbolic equilibrium points, the convergence of the nonlinear semigroup, the $\mathcal{C}^0$ convergence of unstable manifolds and the continuity of attractors.

The seminal paper [4] addresses many types of domain perturbations and their relations with the spectral behavior of the Laplace operator subject to homogeneous Neumann boundary conditions. The main difficulty to find a rate of convergence for this

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approach is due to the extended phase space $H^1(\Omega_\varepsilon \cap \Omega_0) \oplus H^1(\Omega_\varepsilon \backslash \Omega_0) \oplus H^1(\Omega_0 \backslash \Omega_\varepsilon)$ that does not allow to obtain estimates in the same space. The authors in [9,10,28,29] overcome this problem using the pull-back technique proposed by [23] in which, the perturbed nonlinear equation, is transferred to a fixed phase space. There they deal with nonlinear boundary conditions showing the continuity of the attractors but without estimates of convergence.

In this paper, we use the compact convergence approach introduced by Carvalho and Piscarev in [17], in a proper way, to estimate the convergence of the dynamics set by a reaction–diffusion equation under smooth perturbations of the domain. Our perspective allows us to advance and refine some existing results on the continuity of attractors for parabolic problems when a fixed domain undergoes smooth perturbation. Besides that, we show precisely how to estimate all convergence from the perturbed to the limiting problem when the perturbation parameter varies. In this way, we improve the results of the previous works [9,10,28,29] and [4] since we deal with nonlinear Neumann boundary conditions.

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a smooth $\mathcal{C}^1$ bounded domain and $h : \Omega \to \mathbb{R}^N$ be a diffeomorphism onto its image $\Omega_h := h(\Omega)$. Consider

$$\begin{cases} u_t - \Delta u = f(u), & \text{in } \Omega_h, \\ \frac{\partial u}{\partial \vec{n}_h} = \tilde{f}(u), & \text{on } \partial\Omega_h, \end{cases} \tag{1.1}$$

where $\Delta = \sum_{i=1}^N \partial^2/\partial y_i^2$ is the Laplacian differential operator in Ω_h, $\vec{n}_h$ is the outward unitary normal vector for the boundary $\partial\Omega_h$ and f, $\tilde{f}$ are smooth real functions defined in $\mathbb{R}$. It is well known that, under standard growth and dissipative conditions on f and $\tilde{f}$, problem (1.1) is globally well-posed in $H^1(\Omega_h)$. Also, the associated semigroup is gradient and possesses a global attractor $\mathcal{A}_h$ uniformly bonded in L^∞ (see, for instance, [6,13,20,27]).

We are interested here in finding estimates for the dynamics set by (1.1) as h approaches the inclusion $I_N : \Omega \to \mathbb{R}^N$ in the $\mathcal{C}^1$ topology. In fact, it is known by [9,29] that the perturbed problem (1.1) varies continuously concerning h under the condition that all the equilibria are hyperbolic. Thus, if we denote $\tau(h) = d_{C^1}(h, I_N)$, then $\tau(h) \to 0$ as $h \to I_N$ and the limiting problem of (1.1) is given by

$$\begin{cases} u_t - \Delta u = f(u), & \text{in } \Omega, \\ \frac{\partial u}{\partial \vec{n}} = \tilde{f}(u), & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\vec{n}$ is the outward unitary normal vector for the boundary $\partial\Omega$. The main result of this paper states that there exist constants $C > 0$ and $0 < \beta < 1$ independent of h and a linear invertible operator $E_h : H^1(\Omega) \to H^1(\Omega_h)$ such that the continuity of attractors can be estimated by

$$d_H(\mathcal{A}_h, E_h\mathcal{A}) \leq C\tau(h)^\beta \tag{1.3}$$

where d_H denotes the Hausdorff distance between closed sets in $H^1(\Omega_h)$.

In addition to the well-posedness of (1.2) and (1.1), we can assume that $f, \tilde{f} \in \mathcal{C}^2(\mathbb{R})$ are bounded with derivatives up to second order bounded. We also suppose all the equilibrium points of the limiting problem (1.2) are hyperbolic, and then, they compose a finite set $\mathcal{E} = \{u^{1,*}, \ldots, u^{p,*}\}$.

In the process to obtain (1.3), we prove the following results. Let $\lambda \geq 1$ and A_h be the linear operator $\lambda - \Delta$ in Ω_h with homogeneous Neumann boundary conditions. There are positive constants C, L, a, $0 < \theta < \frac{1}{2}$ and $\frac{1}{2} < s < 1$ independent of h such that:

(i) The rate of convergence of eigenvalues, spectral projections, and resolvent operators of A_h as $h \to I_N$ is given by $\tau(h)$. In particular,

$$\|A_h^{-1} E_h - E_h A_{I_N}^{-1}\|_{\mathcal{L}(H^{-s}(\Omega), H^1(\Omega_h))} \leq C\tau(h). \tag{1.4}$$

(ii) If u^* is an equilibrium point of (1.2), then there exists an equilibrium point u_h^* of (1.1) such that

$$\|u_h^* - E_h u^*\|_{H^1(\Omega_h)} \leq C\tau(h).$$

(iii) If $e^{-A_h t}$ is the linear semigroup generated by A_h and $T_h(\cdot)$ is the nonlinear semigroups generated by the solutions of (1.1) and (1.2) then

$$\|e^{-A_h t} E_h - E_h e^{-A_{I_N} t}\|_{\mathcal{L}(H^{-s}(\Omega), H^1(\Omega_h))} \leq C e^{-a(1-2\theta)t} \tau(h)^{2\theta} t^{-(\frac{1}{2}+\theta)}, \quad t > 0 \tag{1.5}$$

and

$$\|T_h(t) E_h u - E_h T_{I_N}(t) u\|_{H^1(\Omega_h)} \leq C e^{Lt} \tau(h)^{2\theta}, \quad u \in H^1(\Omega), \quad t > t_0. \tag{1.6}$$

(iv) The unstable manifolds of each equilibrium point are exponentially attracting, and the $\mathcal{C}^0$-convergence can be estimated by $C\tau(h)^{2\theta}$.

(v) The quantity $\eta = C\tau(h)^\beta$ measures how much the attractors $\mathcal{A}_h$ and $\mathcal{A}$ are not isometric.

It is worth noticing that the optimality of the estimates obtained in items (iii), (iv), (v), and (1.3) is an open question for a problem whose dynamics act in infinite-dimensional spaces. The optimal rate should be with $\beta = 1$ which is the rate of equilibria. We already know that for semiflows in finite-dimensional phase space, the estimates are sharp. Another class where the estimates are sharp is reaction–diffusion problems with large diffusion [14,31]. In this case, the limiting phase space is finite-dimensional. The paper [15] considers large diffusion only in a piece of the domain, but it is not enough to obtain the optimal rates once the dynamics act in infinite-dimensional phase space. It is still worth mentioning that the work [8] improves the estimates from [21] (but does not obtain the optimality) for a class of singular parabolic problems arising in thin domain problems.

The paper is organized as follows: In Sect. 2, we present the compact convergence approach together with the functional framework needed to get (1.3). As we can see in

[4], the spectral behavior of the linear part of (1.1) and (1.2) is essential to determine its nonlinear behavior. In Sect. 3 and 4, we developed the linear part of our problem, using the viewpoint of [17], aiming reaction–diffusion equations with nonlinear boundary conditions. We introduce the notions of E-convergence and admissibility for domains and operators for problems concerning domain perturbations. We prove several results related to the continuity of the resolvent operators and their perturbations by potentials getting precise estimates concerning $\tau(h)$, one of these results is (1.4). In Sect. 5, we show the permanence of equilibrium points. The key argument is to obtain estimates in nonlinear terms. We need to explore Sobolev immersion and trace theorems. The estimate (1.3) is proved in Sect. 6 where we also prove (1.5), (1.6) and the exponential attraction of the local unstable manifolds. Our results are used in Sect. 7 to show that the quantity $\eta = C\tau(h)^{\beta}$ measures how much the attractors are not isometric. In Sect. 8, we present a classical example to show that our technique works.

2. Functional setting

In this section, we establish the functional setting to deal with (1.1) and (1.2). Since Ω and Ω_h are smooth bounded domains, the appropriated phase space is the Sobolev spaces $H^s(\Omega)$ and $H^s(\Omega_h)$, $s > 0$, that can be defined as the fractional power space through the Laplace operator with homogeneous Newmann boundary conditions (see, for instance, [22,33]). In fact, we have from [33, Theorem 1.35 and Corollary 2.4] that $H^s(\Omega_h) = D((-\Delta_h + I_N)^{s/2})$, $0 \leq s \leq 1$ where Δ_h is the Laplacian operator with Neumann homogeneous boundary condition $\Delta_h : D(\Delta_h) \subset L^2(\Omega_h) \to L^2(\Omega_h)$ with

$$D(\Delta_h) = \left\{ u \in H^2(\Omega_h) : \frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \partial\Omega_h \right\} \quad \text{and} \quad \Delta_h u = \Delta u \ \text{ in } \ \Omega_h.$$

In the case $h = I_N$, we may just use $\Delta = \Delta_{I_N}$. The dual space of $H^s(\Omega)$ and $H^s(\Omega_h)$ are denoted by $H^{-s}(\Omega)$ and $H^{-s}(\Omega_h)$, and then, we also extend the scale for negative fractional exponent.

Lemma 2.1 gives us a way to consider the set of diffeomorphisms of Ω close to the inclusion as an appropriate set of parameters. We will define the operator E_h to compare the dynamics of (1.1) and (1.2) transferring the main concepts of [17] for our context.

Recall that we are denoting I_N the inclusion of Ω in $\mathbb{R}^N$ and $\tau(h) = d_{C^1}(h, I_N)$, where

$$\begin{aligned} d_{C^1}(h, I_N) = \sup_{x\in\Omega}\{h(x) - x\} + \sup_{\substack{|v|=1 \\ x\in\Omega}}\{h'(x)v - v\} \\ + \sup_{x\in\Omega_h}\{h^{-1}(x) - x\} + \sup_{\substack{|v|=1 \\ x\in\Omega}}\{(h')^{-1}(x)v - v\}. \end{aligned}$$

We consider $0 < \epsilon < 1$ and define the following set of diffeomorphisms ϵ-close to the inclusion

$$\begin{aligned}\mathrm{Diff}_\epsilon(\Omega) = &\{h \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R}^N) : h \text{ is a diffeomorphism onto its image } h(\Omega) = \Omega_h \\ &\text{satisfying } d_{C^1}(h, I_N) \leq \epsilon\}.\end{aligned}$$

The parameter ϵ measures how close h and I_N are. Eventually, we take ϵ sufficiently small to mean that h is sufficiently close to inclusion I_N.

Lemma 2.1. *Let $h \in \mathrm{Diff}_\epsilon(\Omega)$. If we denote $\tau(h) = d_{C^1}(h, I_N)$, then it is valid the following estimates for $x \in \Omega$*

$$\begin{aligned}|\det(h'(x))| &\leq 1 + \sum_{k=0}^{N-1} \binom{N}{k} \tau(h)^{N-k} \quad \textit{and} \\ |\det((h')^{-1}(x))| &\leq 1 + \sum_{k=0}^{N-1} \binom{N}{k} \tau(h)^{N-k}.\end{aligned}$$

Moreover, let $\Phi_i : U_i \subset \mathbb{R}^{N-1} \to \mathbb{R}^N$ be a family of $\mathcal{C}^1$ local parametrizations to $\partial\Omega$ such that $\partial\Omega \subset \cup \Phi_i(U_i)$. Then, if D_{ij} is the $(N-1)$-dimensional matrix obtaining by deleting the jth line of the matrix $D_i = h'\Phi_i'$ defined in U_i, there is a positive constant C_i, depending only on the parametrization Φ_i, such that it is valid the following estimate on $\partial\Omega$

$$\begin{aligned}&|\det(D_{ij})| \leq C_i \left(1 + \sum_{k=0}^{N-2} \binom{N-1}{k} \tau(h)^{N-1-k}\right) \\ &\textit{and} \quad |\det(D_{ij}^{-1})| \leq C_i \left(1 + \sum_{k=0}^{N-2} \binom{N-1}{k} \tau(h)^{N-1-k}\right),\end{aligned}$$

where D_{ij}^{-1} denotes the $(N-1)$-dimensional matrix obtaining by deleting the jth line of the matrix $D_i^{-1} = (h'\Phi_i')^{-1}$ defined in U_i.

Proof. The Hadamard's inequality says that for a matrix $A = [v_1 v_2 \ldots v_n]$, where $v_i = N$-vector, it is valid $|\det(A)| \leq \prod_{i=1}^n \|v_i\|_{\mathbb{R}^N}$, for the proof see [25]. Applying this inequality to Jacobian matrix of $h(x)$, $x \in \Omega$, we obtain

$$\begin{aligned}|\det(h'(x))| &\leq \prod_{i=1}^N \left\| \left(\frac{\partial h_1}{\partial x_i}, \frac{\partial h_2}{\partial x_i}, \ldots, \frac{\partial h_N}{\partial x_i}\right) \right\|_{\mathbb{R}^N} \leq d_{C^1}(h, 0)^N \\ &\leq (d_{C^1}(h, I_{\mathbb{R}^N}) + d_{C^1}(I_{\mathbb{R}^N}, 0))^N = (\tau(h) + 1)^N \\ &= 1 + \sum_{k=0}^{N-1} \binom{N}{k} \tau(h)^{N-k},\end{aligned}$$

where

$$d_{C^1}(h,0)=\sup_{x\in\Omega}h(x)+\sup_{|v|=1}h'(x)v+\sup_{x\in\Omega_h}h^{-1}(x)+\sup_{|v|=1}(h')^{-1}(x)v.$$

In the same way, we get the estimate for $(h')^{-1}$.

If $x\in\partial\Omega$, we have $x\in\Phi_i(U_i)$ for some i, and

$$\begin{aligned}|\det(D_{ij})| &\leq C_i\prod_{i=1}^{N-1}\left\|\left(\frac{\partial h_1}{\partial x_i},\ldots,\frac{\partial\hat{h}_j}{\partial x_i},\ldots,\frac{\partial h_N}{\partial x_i}\right)\right\|_{\mathbb{R}^{N-1}}\leq C_i d_{C^1}(h,0)^{N-1}\\ &\leq C_i(d_{C^1}(h,I_{\mathbb{R}^N})+d_{C^1}(I_{\mathbb{R}^N},0))^{N-1}=C_i(\tau(h)+1)^{N-1}\\ &=C_i\left(1+\sum_{k=0}^{N-2}\binom{N}{k}\tau(h)^{N-k-1}\right),\end{aligned}$$

where $C_i=d_{C^1}(\Phi_i,0)$.

In the same way, we get the estimate for D_{ij}^{-1}. □

Remark 2.2. Notice that if we define

$$\bar{\tau}(h):=\sum_{k=0}^{N-1}\binom{N}{k}\tau(h)^{N-k}$$

then $\bar{\tau}(h)\to 0$ as $h\to I_N$ and $\bar{\tau}(h)/\tau(h)\to\binom{N}{N-1}$ as $h\to I_N$. Hence, $\tau(h)$ and $\bar{\tau}(h)$ have the same order of convergence to zero as h converges to I_N. More precisely, we can find a constant C uniform in h such that $\bar{\tau}(h)/\tau(h)\leq C$, for h sufficiently close to I_N.

By Lemma 2.1, $\bar{\tau}(h)+1$ is an upper bound for $|\det(h'(x))|$ and $|\det((h')^{-1}(x))|$ for all $x\in\Omega\cup\partial\Omega$. If $x\in\partial\Omega$, C_i depends only on the fixed parametrization of the boundary. More precisely, since Ω is a $\mathcal{C}^1$ bounded domain its boundary $\partial\Omega$ is locally the graph of a $\mathcal{C}^1$ function. Therefore, if Φ_i is the parametrization of $\partial\Omega$ (as we have used in Lemma 2.1), then there is a $\mathcal{C}^1$ function $\varphi_i:U\to\mathbb{R}$, such that

(i) $\Phi_i(x')=(x',\varphi_i(x'))$, $x'\in U_i$.

(ii) $\varphi_i(U_i)\cap\Omega=\{x\in\Phi_i(U_i):x_N>\varphi_i(x')\}$, $x=(x',x_N)\in\mathbb{R}^N$.

(iii) $\|\nabla\varphi_i\|_{L^\infty(U_i)}\leq C_i$.

We can define a new parametrization $\varphi_i(\frac{1}{C_i}x')$ in order to obtain $\|\nabla\Phi_i\|_{L^\infty(U_i)}\leq 1$. Thus, we can take $C_i=1$ in Lemma 2.1.

We have $\{H^1(\Omega_h)\}_{h\in\mathrm{Diff}_\epsilon(\Omega)}$ is a family of Banach spaces indexed in the topological space $\mathrm{Diff}_\epsilon(\Omega)$ endowed with the $\mathcal{C}^1$ topology. It is worth noting that $I_N\in\mathrm{Diff}_\epsilon(\Omega)$ and the parameter ϵ is a upper bound to $\tau(h)$ independent of h. When we want to take h sufficiently close to I_N, we take ϵ sufficiently small.

We define the following family of linear operators

$$E_h:L^2(\Omega)\to L^2(\Omega_h),\quad E_hu=u\circ h^{-1},\quad u\in L^2(\Omega).$$

We have $E_h(H^1(\Omega)) = H^1(\Omega_h)$. It follows from the change of variables theorem that E_h is a continuous operator. Moreover, by Lemma 2.1, we have

$$\begin{aligned}\|E_h u\|^2_{H^1(\Omega_h)} &= \int_{\Omega_h} |\nabla(uh^{-1}(x))|^2\,\mathrm{d}x + \int_{\Omega_h} |(uh^{-1}(x))|^2\,\mathrm{d}x \\ &= \int_{\Omega} |\nabla(u(x))|^2|\det(h'(x))|\,\mathrm{d}x + \int_{\Omega} |(u(x))|^2|\det(h'(x))|\,\mathrm{d}x \\ &\le \|u\|^2_{H^1(\Omega)} + \|u\|^2_{H^1(\Omega)}\bar{\tau}(h).\end{aligned}$$

Hence,

$$\limsup_{h\to I_N} \|E_h u\|_{H^1(\Omega_h)} \le \|u\|_{H^1(\Omega)}. \tag{2.1}$$

In the same way, we obtain $\limsup_{h\to I_N} \|E_h u\|_{L^2(\Omega_h)} \le \|u\|_{L^2(\Omega)}$.

By the uniform boundedness principle, there is a constant $K > 0$, independent of h, such that, if we take ϵ sufficiently small, then the following uniform estimates are valid

$$\|E_h\|_{\mathcal{L}(H^1(\Omega),H^1(\Omega_h))}, \|E_h\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega_h))} \le K, \quad \text{for all} \;\; h \in \mathrm{Diff}_\epsilon(\Omega). \tag{2.2}$$

Thus,

$$\begin{aligned}\|u\|^2_{H^1(\Omega)} &= \int_{\Omega} |\nabla(u(x))|^2\,\mathrm{d}x + \int_{\Omega} |(u(x))|^2\,\mathrm{d}x \\ &= \int_{\Omega_h} |\nabla(uh^{-1}(x))|^2|\det((h')^{-1}(x))|\,\mathrm{d}x + \int_{\Omega_h} |(uh^{-1}(x))|^2|\det((h')^{-1}(x))|\,\mathrm{d}x \\ &\le \|E_h u\|^2_{H^1(\Omega_h)} + \|E_h u\|^2_{H^1(\Omega_h)}\bar{\tau}(h).\end{aligned}$$

Hence,

$$\|u\|_{H^1(\Omega)} \le \liminf_{h\to I_N} \|E_h u\|_{H^1(\Omega_h)}. \tag{2.3}$$

In the same way, we obtain $\|u\|_{L^2(\Omega)} \le \liminf_{h\to I_N} \|E_h u\|_{L^2(\Omega_h)}$.

The inequalities (2.1) and (2.3) imply

$$\|E_h u\|_{L^2(\Omega_h)} \to \|u\|_{L^2(\Omega)} \quad \text{as} \quad h \to I_N, \quad u \in L^2(\Omega), \quad \text{and} \tag{2.4}$$

$$\|E_h u\|_{H^1(\Omega_h)} \to \|u\|_{H^1(\Omega)} \quad \text{as} \quad h \to I_N, \quad u \in H^1(\Omega). \tag{2.5}$$

In order to connect the phase spaces, we also need to consider the inverse operator of E_h. It is defined as follows:

$$M_h : L^2(\Omega_h) \to L^2(\Omega), \quad M_h u_h = u_h \circ h, \quad u_h \in L^2(\Omega_h).$$

We have that M_h also acts in $H^1(\Omega_h)$. Similarly to E_h, we can prove that M_h is a continuous linear operator.

We are dealing with nonlinear boundary conditions; then, we need to extend E_h to an operator E_h^s acting in $H^s(\Omega)$. We have the following result.

Proposition 2.3. *Suppose $s \in (0,1)$ and $h \in \text{Diff}_\epsilon(\Omega)$ for $\epsilon \in [0,1]$. Then, $E_h^s : H^s(\Omega) \mapsto H^s(\Omega_h)$ given by $(E_h^s u)(y) = (u \circ h^{-1})(y)$ is well defined and satisfies*

$$\|E_h^s u\|_{H^s(\Omega_h)} \le C\|u\|_{H^s(\Omega)}$$

for some positive constant C independent of h. Moreover, E_h^s is an isomorphism with $(E_h^s)^{-1} = M_h^s$ where $M_h^s : H^s(\Omega_h) \mapsto H^s(\Omega)$ is given by $M_h^s v = v \circ h$ with

$$\|M_h^s v\|_{H^s(\Omega)} \le C\|v\|_{H^s(\Omega_h)}$$

for some $C > 0$ independent of h.

Proof. Let $B_R \subset \mathbb{R}^N$ be a ball of radius R such that $\Omega_h \subset B_R$ for all $h \in \text{Diff}_\epsilon(\Omega)$ and $\epsilon \in [0,1]$. From [23, Chapter 1], for any $h \in \text{Diff}_\epsilon(\Omega)$ with $\epsilon \in [0,1]$, there exists a diffeomorphism $H : \mathbb{R}^n \mapsto \mathbb{R}^N$ of class $\mathcal{C}^1$ such that its restriction to Ω is equal to h, $\left|\det(H'(x))\right|$ is strictly positive and uniformly bounded in $\mathbb{R}^N$ and $\text{Diff}_\epsilon(\Omega)$. Now, from [33, Section 11.4], $u \in H^s(\Omega)$, if and only if, there exists $U \in H^s(\mathbb{R}^N)$ with $U|_\Omega = u$ satisfying

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{i(y-x)\cdot\xi}(1+|\xi|^2)^{s/2}U(x)\mathrm{d}x\mathrm{d}\xi \in L^2(\mathbb{R}^N).$$

Hence, as $U \circ H^{-1}|_{\Omega_h} = u \circ h^{-1}$, we obtain $E_h^s u \in H^s(\Omega_h)$ whenever $U \circ H^{-1} \in H^s(\mathbb{R}^N)$. That is, whenever

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{i(y-x)\cdot\xi}(1+|\xi|^2)^{s/2}U(H^{-1}(x))\mathrm{d}x\mathrm{d}\xi\\ &\quad= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{i(y-H(x))\cdot\xi}(1+|\xi|^2)^{s/2}U(x)\left|\det(H'(x))\right|\mathrm{d}x\mathrm{d}\xi \in L^2(\mathbb{R}^N).\end{aligned} \tag{2.6}$$

Since $\left|\det(H'(x))\right|$ uniformly bounded in $\mathbb{R}^N$, we get that E_h^s is a well-defined operator from $H^s(\Omega)$ into $H^s(\Omega_h)$ with $\|E_h^s u\|_{H^s(\Omega_h)} \le C\|u\|_{H^s(\Omega)}$ for some $C > 0$ independent of $h \in \text{Diff}_\epsilon(\Omega)$. Notice that, a similar argument can be done to prove that M_h^s is also well defined with $\|M_h^s v\|_{H^s(\Omega)} \le C\|v\|_{H^s(\Omega_h)}$ for some $C > 0$ independent of h. Thus, in order to finish our proof, we just need to show that E_h^s is injective, but this follows from h being a diffeomorphisms, and $(u \circ h^{-1})(y) = (v \circ h^{-1})(y)$, for $y \in \Omega_h$, if and only if, $u(x) = v(x)$ for $x \in \Omega$. □

Remark 2.4. We also can obtain the uniform boundedness of E_h^s and M_h^s from [22, Theorem 1.4.4 and Exercise 5*] or from [33, Inequality 2.117]. From there, there is a positive constant C_s, independent of h, such that

$$\begin{aligned} \|E_h^s u\|_{H^s(\Omega_h)} &\le C_s \|E_h u\|_{H^1(\Omega_h)}^{1-s} \|E_h u\|_{L^2(\Omega_h)}^s \quad \forall u \in H^1(\Omega) \quad \text{and} \\ \|M_h^s v\|_{H^s(\Omega)} &\le C_s \|M_h v\|_{H^1(\Omega)}^{1-s} \|M_h v\|_{L^2(\Omega)}^s \quad \forall v \in H^1(\Omega_h). \end{aligned} \tag{2.7}$$

Notice that here we are using the fact that $H^s(\Omega_h) = D((-\Delta_h + I_N)^{s/2})$ for $0 \le s \le 1$.

As we have mentioned, we denote $H^{-s}(\Omega_h)$, $s > 0$, the dual space of $H^s(\Omega_h)$. We define $E_h^{-s} : H^{-s}(\Omega) \to H^{-s}(\Omega_h)$ by

$$\langle E_h^{-s} u, v \rangle = \langle u, M_h^s v \rangle, \quad \text{for all} \quad u \in H^{-s}(\Omega), \ v \in H^s(\Omega_h). \tag{2.8}$$

To obtain some properties of the operators E_h^{-s}, we need to impose $s \in (0, 1)$. With this restriction, we use an interpolation inequality to obtain the following result.

Corollary 2.5. *For $s \in (0, 1)$, the linear operators E_h^s and E_h^{-s} are uniformly bounded in h and*

$$\|E_h^s u\|_{H^s(\Omega_h)} \to \|u\|_{H^s(\Omega)} \quad as \quad h \to I_N, \quad u \in H^s(\Omega), \tag{2.9}$$
$$\|E_h^{-s} v\|_{H^{-s}(\Omega_h)} \to \|v\|_{H^{-s}(\Omega)} \quad as \quad h \to I_N, \quad v \in H^{-s}(\Omega). \tag{2.10}$$

Proof. The uniform boundedness follows from Proposition 2.3 and (2.8). Now, let us check (2.9). From [33, Section 11.4], for any $\delta > 0$, we have

$$\begin{aligned} \|E_h^s u\|_{H^s(\Omega_h)} &= \|E_h u\|_{L^2(\Omega_h)} + \left(\int\int_{\Omega_h \times \Omega_h} \frac{|(u \circ h^{-1})(x) - (u \circ h^{-1})(y)|^2}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}} \\ &= \|E_h u\|_{L^2(\Omega_h)} + \left(\int\int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|h(x) - h(y)|^{N+2s}} \left|\det(h'(x))\right| \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}} \\ &= \|E_h u\|_{L^2(\Omega_h)} + \left(\int\int_{(\Omega \times \Omega) \setminus D_\delta} \frac{|u(x) - u(y)|^2}{|h(x) - h(y)|^{N+2s}} \left|\det(h'(x))\right| \mathrm{d}x\mathrm{d}y \right. \\ &\quad \left. + \int\int_{D_\delta} \frac{|u(x) - u(y)|^2}{|h(x) - h(y)|^{N+2s}} \left|\det(h'(x))\right| \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}} \end{aligned}$$

with $D_\delta = \cup_{x \in \Omega} B_\delta(x)$ where $B_\delta(x) = \{(z, w) \in \Omega \times \Omega \ : \ |(z, w) - (x, x)| < \delta\}$. Since Ω is bounded, $\delta > 0$ is arbitrary, and $h \to I_N$ in $\mathrm{Diff}_\epsilon(\Omega)$, we obtain (2.9) from Proposition 2.3. Finally, since Ω is regular and bounded, we obtain (2.10) from (2.7) and

$$\|E_h^{-s} u\|_{H^{-s}(\Omega_h)} = \sup_{\substack{v \in H^s(\Omega_h) \\ \|v\|_{H^s(\Omega_h)=1}}} |\langle u, M_h^s v\rangle| \to \sup_{\substack{v \in H^s(\Omega) \\ \|v\|_{H^s(\Omega)=1}}} |\langle u, v\rangle| = \|u\|_{H^{-s}(\Omega)}$$

as $h \to I_N$. □

In what follows, we omit $-s$ in E_h^{-s}. We will denote $E_h : H_h^{-s}(\Omega) \to H^{-s}(\Omega_h)$, and the context will avoid confusion.

The boundedness and convergence properties of E_h and M_h enable us to use the functional framework proposed by [17] (see also [16,29]) taking $H^{-s}(\Omega_h)$ as the base space for a fixed s. More precisely, we are interested here in the abstract results from Section 3 of [17]. We will combine them with the techniques developed in [23] in order to show our results. In what follows, in this section, we adapt the main concepts and results from [17] for our context. We also recommend [2,7] to the interested reader.

Let $s \in (0, 1)$ be a fixed value and $Y_h \in \{L^2(\Omega_h), H^1(\Omega_h), H^s(\Omega_h), H^{-s}(\Omega_h)\}$. Then $\{Y_h\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ is a family of Banach spaces indexed in $\mathrm{Diff}_\epsilon(\Omega)$. When $h = I_N$ we just write Y.

Definition 2.6. We say that a family $\{g_h\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$, with $g_h \in Y_h$, E-converge to $g \in Y$ as $h \to I_N$ if $\|g_h - E_h g\|_{Y_h} \to 0$ as $h \to I_N$. In this case we denote $g_h \xrightarrow{\text{E}} g$.

Definition 2.7. We say that a sequence $\{g_{h_k}\}_{k\in\mathbb{N}}$, with $g_{h_k} \in Y_{h_k}$, is relatively compact if for each subsequence $\{g_{h_{k_l}}\}_{l\in\mathbb{N}}$ there is a subsequence $\{g_{h_{k_{l_j}}}\}_{j\in\mathbb{N}}$ and an element $g \in Y$ such that $g_{h_{k_{l_j}}} \xrightarrow{\text{E}} g$. The family $\{g_h\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$, with $g_h \in Y_h$, is relatively compact if any subsequence $\{g_{h_k}\}_{k\in\mathbb{N}}$ is relatively compact.

Definition 2.8. We say that a family $\{B_h : H^{-s}(\Omega_h) \to H^1(\Omega_h)\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ of bounded linear operators converges compactly to an operator $B : H^{-s}(\Omega) \to H^1(\Omega)$ as $h \to I_N$, which we denote $B_h \xrightarrow{\text{CC}} B$, if the following conditions are satisfied:

(i) B_h and B are compact operators.
(ii) $g_h \xrightarrow{\text{E}} g \Rightarrow B_h g_h \xrightarrow{\text{E}} Bg$.
(iii) Each family of the form $\{B_h g_h\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$, with $\|g_h\|_{H^1(\Omega_h)} = 1$, for all $h \in \mathrm{Diff}_\epsilon(\Omega)$, is relatively compact.

As before we can extend M_h to $H^{-s}(\Omega_h)$ with $M_h(H^1(\Omega_h)) = H^1(\Omega)$ and $M_h(H^{-s}(\Omega_h)) = H^{-s}(\Omega)$. In what follows we use the same notation M_h for its restriction to $H^1(\Omega_h)$ and its extension to $H^{-s}(\Omega_h)$. With the above similar arguments for E_h, we see that M_h are bounded uniformly in h. Moreover, if $\{B_h : H^{-s}(\Omega_h) \to H^1(\Omega_h)\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ is a family of operators such that

$$\|B_h - E_h B M_h\|_{\mathcal{L}(H^{-s}(\Omega_h), H^1(\Omega_h))} \to 0 \quad \text{as} \quad h \to I_N \tag{2.11}$$

then

$$\|B_h E_h - E_h B\|_{\mathcal{L}(H^{-s}(\Omega), H^1(\Omega_h))} \to 0 \quad \text{as} \quad h \to I_N. \tag{2.12}$$

3. Rate of convergence for compact convergence of resolvent operators

In this section, we estimate the convergence of the resolvent operators $(\mu - \Delta)^{-1}$ in Ω_h. We use variational methods of elliptic equations to show that the function $h \to d_{C^1}(h, I_N)$ is a natural rate of convergence for the solutions of the elliptic parts of (1.1) and (1.2).

Our technique differs a little from that used in [1]. Here the boundary condition forces us to work in the Sobolev dual space $H^{-s}(\Omega)$, $s > 0$. First, we will consider particular elements of $H^{-s}(\Omega)$ having boundary traces, and then, we will consider more general functionals.

Let $0 < \epsilon < 1$. For each $h \in \mathrm{Diff}_\epsilon(\Omega)$, we recall that Δ_h denotes the Laplacian operator with homogeneous Neumann boundary condition: $\Delta_h : D(\Delta_h) \subset L^2(\Omega_h) \to L^2(\Omega_h)$ with

$$D(\Delta_h) = \left\{ u \in H^2(\Omega_h) : \frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \partial\Omega_h \right\} \quad \text{and} \quad \Delta_h u = \Delta u \ \text{ in } \Omega_h.$$

We omit the parameter I_N when considering the limiting problem $h = I_N$. In fact, we must say that $A_h = \lambda - \Delta_h$, $\lambda \geq 1$ is an operator of $H^1(\Omega_h)$ onto $H^{-s}(\Omega_h)$ whose realization in $L^2(\Omega_h)$ coincides with $\lambda - \Delta_h$, that is,

$$\langle A_h \phi, \psi \rangle_{-1,1} = \int_{\Omega_h} \nabla\phi \nabla\psi + \lambda \int_{\Omega_h} \phi\psi.$$

Now, let us take $s > \frac{1}{2}$ and $g_h \in H^{-s}(\Omega_h)$, assuming the following form: we suppose that there exist $g_{1,h} \in L^2(\Omega_h)$ and $g_{2,h} \in L^2(\partial\Omega_h)$, such that

$$\langle g_h, \phi \rangle_{-s,s} = \int_{\Omega_h} g_{1,h}\phi + \int_{\partial\Omega_h} g_{2,h}\phi, \quad \forall \phi \in H^s(\Omega_h). \tag{3.1}$$

Under these conditions, we have the following result:

Theorem 3.1. *For any $\lambda \geq 1$ and any family $\{g_h\}_{h \in \mathrm{Diff}_\epsilon(\Omega)}$ set by (3.1), the weak solution of*

$$\begin{cases} -\Delta_h u_h + \lambda u_h = g_{1,h}, & \text{in } \Omega_h \\ \frac{\partial u_h}{\partial \vec{n}} = g_{2,h} & \text{on } \partial\Omega_h, \end{cases} \tag{3.2}$$

satisfies

$$\|u_h\|_{H^1(\Omega_h)} \leq \|g_h\|_{H^{-s}(\Omega_h)} \tag{3.3}$$

and

$$\begin{aligned} \|u_h - E_h u\|_{H^1(\Omega_h)} \leq \|g_h - E_h g\|_{H^{-s}(\Omega_h)} + [\|g\|_{H^{-s}(\Omega)} \\ + \|\nabla u\|_{L^2(\Omega)} + \lambda \|u\|_{L^2(\Omega)}] C\tau(h), \end{aligned} \tag{3.4}$$

where $g = g_{I_N}$, C is a constant independent of h and $\tau(h) = d_{C^1}(h, I_N)$.

Proof. It is well known that Δ_h with Neumann boundary condition has infinity discrete spectrum set $\sigma(\Delta_h)$ contained in $(-\infty, 0]$. Thus, for any $\lambda > 0$, $\lambda - \Delta_h$ is an invertible operator. Since u_h is a weak solution of (3.2), we can write

$$\int_{\Omega_h} \nabla u_h \nabla \phi_h + \int_{\Omega_h} \lambda u_h \phi_h = \int_{\Omega_h} g_{1,h} \phi_h + \int_{\partial \Omega_h} g_{2,h} \phi_h, \quad \phi_h \in H^s(\Omega_h); \tag{3.5}$$

$$\int_{\Omega} \nabla u \nabla \phi + \int_{\Omega} \lambda u \phi = \int_{\Omega} g_1 \phi + \int_{\partial \Omega} g_2 \phi, \quad \phi \in H^s(\Omega). \tag{3.6}$$

Taking $\phi_h = u_h$ in (3.5), we obtain

$$\int_{\Omega_h} |\nabla u_h|^2 + \int_{\Omega_h} \lambda |u_h|^2 = \int_{\Omega_h} g_{1,h} u_h + \int_{\partial \Omega_h} g_{2,h} u_h.$$

Hence, by Holder's inequality, we get (3.3).

Now, taking $\phi_h = u_h - E_h u$ in (3.5), $\phi = M_h u_h - u$ in (3.6) and making the difference, we obtain

$$\begin{aligned} &\int_{\Omega_h} \nabla u_h (\nabla u_h - \nabla E_h u) - \int_{\Omega} \nabla u (\nabla M_h u_h - \nabla u) \\ &\quad + \int_{\Omega_h} \lambda u_h (u_h - E_h u) - \int_{\Omega} \lambda u (M_h u_h - u) \\ &\quad = \int_{\Omega_h} g_{1,h} (u_h - E_h u) - \int_{\Omega} g_1 (M_h u_h - u) \\ &\quad + \int_{\partial \Omega_h} g_{2,h} (u_h - E_h u) - \int_{\partial \Omega} g_2 (M_h u_h - u), \end{aligned}$$

and then

$$\begin{aligned} &\int_{\Omega_h} \nabla u_h (\nabla u_h - \nabla E_h u) + \int_{\Omega_h} \lambda u_h (u_h - E_h u) \\ &\quad = \int_{\Omega_h} g_{1,h} (u_h - E_h u) - \int_{\Omega} g_1 (M_h u_h - u) \\ &\quad + \int_{\Omega} \nabla u (\nabla M_h u_h - \nabla u) + \int_{\Omega} \lambda u (M_h u_h - u) \\ &\quad + \int_{\partial \Omega_h} g_{2,h} (u_h - E_h u) - \int_{\partial \Omega} g_2 (M_h u_h - u). \end{aligned}$$

We add appropriate terms to obtain

$$\begin{aligned}
&\int_{\Omega_h} \nabla u_h(\nabla u_h - \nabla E_h u) - \int_{\Omega_h} \nabla E_h u(\nabla u_h - \nabla E_h u) \\
&\quad + \int_{\Omega_h} \lambda u_h(u_h - E_h u) - \int_{\Omega_h} \lambda E_h u(u_h - E_h u) \\
&\quad = \int_{\Omega_h} g_{1,h}(u_h - E_h u) - \int_{\Omega} g_1(M_h u_h - u) \\
&\quad + \int_{\partial\Omega_h} g_{2,h}(u_h - E_h u) - \int_{\partial\Omega} g_2(M_h u_h - u) \\
&\quad + \int_{\Omega} \nabla u(\nabla M_h u_h - \nabla u) - \int_{\Omega_h} \nabla E_h u(\nabla u_h - \nabla E_h u) \\
&\quad + \int_{\Omega} \lambda u(M_h u_h - u) - \int_{\Omega_h} \lambda E_h u(u_h - E_h u) \\
&\quad := I_1 + I_2 + I_3,
\end{aligned}$$

where we denote the last three terms on the right-hand side, respectively, by I_1, I_2, and I_3.

Since $\lambda \geq 1$, we get

$$\|u_h - E_h u\|^2_{H^1(\Omega_h)} \leq \int_{\Omega_h} |\nabla u_h - \nabla E_h u|^2 + \int_{\Omega_h} \lambda |\nabla u_h - E_h u|^2 = I_1 + I_2 + I_3.$$

Next, we estimate I_1, I_2, and I_3. First, let us observe that

$$\begin{aligned}
-\int_{\Omega} g_1(M_h u_h - u) &= \int_{\Omega} g_1(u - M_h u_h) \\
&= \int_{\Omega_h} g_1(h^{-1})(u(h^{-1}) - M_h u_h(h^{-1}))|\det((h')^{-1})| \\
&= \int_{\Omega_h} E_h g_1(E_h u - u_h)|\det((h')^{-1})|
\end{aligned}$$

and

$$\begin{aligned}
-\int_{\partial\Omega} g_2(M_h u_h - u) &= \int_{\partial\Omega_h} g_2(h^{-1})(u(h^{-1}) - M_h u_h(h^{-1}))|\det((Dh)^{-1})| \\
&= \int_{\partial\Omega_h} E_h g_2(E_h u - u_h)|\det((Dh)^{-1})|
\end{aligned}$$

where $(Dh)^{-1}$ is the Jacobian matrix of $h^{-1} : \partial h(\Omega) \to \partial\Omega$ sets by a given coordinate on $\partial\Omega$. Thus,

$$I_1 = \int_{\Omega_h} (g_{1,h} - E_h g_1)(u_h - E_h u) + \int_{\partial\Omega_h} (g_{2,h} - E_h g_2)(u_h - E_h u)$$
$$+ \int_{\Omega_h} E_h g_1 (E_h u - u_h)(1 - |\det((h')^{-1})|) + \int_{\partial\Omega_h} E_h g_2 (E_h u - u_h)(1 - |\det((Dh)^{-1})|).$$

If we denote $\bar{\tau}(h) = \sum_{k=0}^{N-1} \binom{N}{k} \tau(h)^{N-k}$ then, by Lemma 2.1, we obtain

$$|I_1| \le \int_{\Omega_h} |(g_{1,h} - E_h g_1)(u_h - E_h u)| + \int_{\partial\Omega_h} |(g_{2,h} - E_h g_2)(u_h - E_h u)|$$
$$+ \int_{\Omega_h} |E_h g_1 (E_h u - u_h)| \bar{\tau}(h) + \int_{\partial\Omega_h} |E_h g_2 (E_h u - u_h)| \bar{\tau}(h),$$

where we have used $C_i = 1$ in Lemma 2.1 according to Remark 2.2.

For I_3, we have

$$\int_{\Omega} \lambda u (M_h u_h - u) = \int_{\Omega_h} \lambda u(h^{-1})(M_h u_h(h^{-1}) - u(h^{-1}))|\det((h')^{-1})|$$
$$= \int_{\Omega_h} \lambda E_h u (u_h - E_h u)|\det((h')^{-1})|,$$

which implies

$$I_3 = \int_{\Omega_h} \lambda E_h u (u_h - E_h u)(|\det((h')^{-1})| - 1).$$

Thus, by Lemma 2.1, we obtain

$$|I_3| \le \int_{\Omega_h} \lambda |E_h u (u_h - E_h u)| \bar{\tau}(h).$$

Finally, we have

$$I_2 = \int_{\Omega} \nabla u (\nabla M_h u_h - \nabla u) - \int_{\Omega_h} \nabla E_h u (\nabla u_h - \nabla E_h u)$$

In addition, if we denote $x = h^{-1}(y)$, then

$$\nabla u(x) = \nabla u(h^{-1}(y)) = E_h \nabla u(y)$$

and

$$\nabla E_h u(y) = \nabla u(h^{-1}(y)) = \nabla u(h^{-1}(y)) \cdot (h^{-1})'(y) = E_h \nabla u(y) \cdot (h^{-1})'(y)$$

thus,

$$\nabla E_h u = E_h \nabla u \cdot (h^{-1})'.$$

In the same way, we obtain

$$\nabla M_h u_h = (M_h \nabla u_h) h'$$

Therefore, we can write

$$\begin{aligned}
\int_\Omega \nabla u(\nabla M_h u_h - \nabla u) &= \int_{\Omega_h} \nabla u(h^{-1})(\nabla M_h u_h(h^{-1}) - \nabla u(h^{-1}))|\det((h')^{-1})| \\
&= \int_{\Omega_h} E_h \nabla u(E_h \nabla M_h u_h - E_h \nabla u)|\det((h')^{-1})| \\
&= \int_{\Omega_h} \nabla E_h u((h^{-1})')^{-1}(E_h M_h \nabla u_h E_h(h') \\
&\quad - \nabla E_h u((h^{-1})')^{-1}|\det((h')^{-1})| \\
&= \int_{\Omega_h} \nabla E_h u E_h(h')(\nabla u_h E_h(h') - \nabla E_h u E_h(h'))|\det((h')^{-1})|,
\end{aligned}$$

where we have used that

$$h^{-1}(h(x)) = x \Rightarrow (h^{-1})'(h(x))h'(x) = I \Rightarrow [(h^{-1})']^{-1}(y) = E_h(h'(y)),$$

where I denotes the identity in $\mathbb{R}^N$.

Thus,

$$\begin{aligned}
I_2 &= \int_{\Omega_h} \nabla E_h u E_h(h')(\nabla u_h - \nabla E_h u) E_h(h')|\det((h')^{-1})| - \int_{\Omega_h} \nabla E_h u(\nabla u_h - \nabla E_h u) \\
&= \int_{\Omega_h} \nabla E_h u(E_h(h') - I)(\nabla u_h - \nabla E_h u) E_h(h')|\det((h')^{-1})| \\
&\quad + \int_{\Omega_h} \nabla E_h u(\nabla u_h - \nabla E_h u)(E_h(h') - I)|\det((h')^{-1})| \\
&\quad + \int_{\Omega_h} \nabla E_h u(\nabla u_h - \nabla E_h u)|(\det((h')^{-1})| - 1)
\end{aligned}$$

Therefore,

$$|I_2| \leq \|\nabla E_h u\|_{L^2(\Omega_h)} \|\nabla u_h - \nabla E_h u\|_{L^2(\Omega_h)} \Big[\|E_h(h') - 1\|_{L^\infty(\Omega_h)} \|E_h(h')\|_{L^\infty(\Omega_h)} \sup_{x\in\Omega_h} \{|(\det((h')^{-1})|\} + \|E_h(h') - 1\|_{L^\infty(\Omega_h)} \sup_{x\in\Omega_h} \{|(\det((h')^{-1})|\} + \sup_{x\in\Omega_h} \{|(\det((h')^{-1})| - 1\} \Big].$$

But, by Lemma 2.1

$$\sup_{x\in\Omega_h} \{|(\det((h')^{-1})| - 1\} \leq \bar{\tau}(h)$$

and

$$\|E_h(h') - I\|_{L^\infty(\Omega_h)} = \sup_{\substack{|v|=1\\ y\in\Omega}} \|(h'h^{-1}(y) - I)v\| = \sup_{\substack{|v|=1\\ x\in\Omega}} \|(h'(x)) - I)v\| \leq \tau(h).$$

Hence, we take a constant C_1 independent of h such that

$$|I_2| \leq C_1 \|\nabla E_h u\|_{L^2(\Omega_h)} \|\nabla u_h - \nabla E_h u\|_{L^2(\Omega_h)} \tau(h)$$

Now, using the estimates for I_1, I_2 and I_3, we obtain

$$\begin{aligned} \|u_h - E_h u\|^2_{H^1(\Omega_h)} \leq & \int_{\Omega_h} |(g_{1,h} - E_h g_1)(u_h - E_h u)| + \int_{\Omega_h} |E_h g_1 (E_h u - u_h)| \bar{\tau}(h) \\ & + \int_{\partial\Omega_h} |(g_{2,h} - E_h g_2)(u_h - E_h u)| + \int_{\partial\Omega_h} |E_h g_2 (E_h u - u_h)| \bar{\tau}(h) \\ & + \int_{\Omega_h} \lambda |E_h u (u_h - E_h u)| \bar{\tau}(h) \\ & + C_1 \|\nabla E_h u\|_{L^2(\Omega_h)} \|\nabla u_h - \nabla E_h u\|_{L^2(\Omega_h)} \tau(h). \end{aligned}$$

By Holder's inequality, we obtain

$$\begin{aligned} \|u_h - E_h u\|^2_{H^1(\Omega_h)} \leq & \|g_h - E_h g\|_{H^{-s}(\Omega_h)} \|u_h - E_h u\|_{H^1(\Omega_h)} \\ & + \|E_h g\|_{H^{-s}(\Omega_h)} \|E_h u - u_h\|_{H^1(\Omega_h)} \bar{\tau}(h) \\ & + \lambda \|E_h u\|_{L^2(\Omega_h)} \|u_h - E_h u\|_{L^2(\Omega_h)} \bar{\tau}(h) \\ & + C_1 \|\nabla E_h u\|_{L^2(\Omega_h)} \|\nabla u_h - \nabla E_h u\|_{L^2(\Omega_h)} \tau(h) \end{aligned}$$

which implies (3.4) since E_h is bounded by K. □

We have the following result as an immediate consequence of Theorem 3.1.

Corollary 3.2. *There is a constant $C > 0$ independent of h such that*

$$\|A_h^{-1} E_h - E_h A^{-1}\|_{\mathcal{L}(H^{-s}(\Omega), H^1(\Omega_h))} \leq C\tau(h).$$

Remark 3.3. The Laplacian operator with Neumann boundary condition is not invertible; therefore, the number $\lambda \geq 1$ will be used to translate Δ in order to obtain $\lambda - \Delta$ an invertible sectorial operator in $L^2(\Omega)$ and then the analytic function $\mu \to \mu + (\lambda - \Delta)$, $\mu \in \rho(\lambda - \Delta)$ has $\sigma(\lambda - \Delta)$ as essential singularities, see [24].

Theorem 3.4. *For each $\lambda \geq 1$, we have*

$$(\lambda - \Delta_h)^{-1} \xrightarrow{\text{CC}} (\lambda - \Delta)^{-1}. \tag{3.7}$$

Moreover, there exists a constant $C > 0$ independent of h such that

$$\|(\lambda - \Delta_h)^{-1} g_h - E_h(\lambda - \Delta)^{-1} g\|_{H^1(\Omega_h)} \leq C(\|g_h - E_h g\|_{H^{-s}(\Omega_h)} + \tau(h)). \tag{3.8}$$

Proof. For all $h \in \mathrm{Diff}_\epsilon(\Omega)$, we have $(\lambda - \Delta_h)^{-1} : H^{-s}(\Omega_h) \to H^1(\Omega_h)$, well defined and since the inclusion $H^1(\Omega_h) \hookrightarrow L^2(\Omega_h)$ is compact, we obtain $(\lambda - \Delta_h)^{-1} : H^{-s}(\Omega_h) \to L^2(\Omega_h)$ a compact operator. Formally, $\lambda - \Delta_h$ is the realization of the bilinear form $a_h : H^1(\Omega_h) \times H^1(\Omega_h) \to \mathbb{R}$ defined by

$$a_h(u, v) = \int_{\Omega_h} \nabla u \nabla v + \lambda u v.$$

It is easy to see that a_h is coercive and continuous. Moreover, if we define $L_h(v) = \langle g_h, v\rangle_{-s,s}$ for $v \in H^1(\Omega_h) \subset H^s(\Omega_h)$ and $\{g_h\}_{h \in \mathrm{Diff}_\epsilon(\Omega)}$ a family with $g_h \in H^{-s}(\Omega_h)$ such that $g_h \xrightarrow{\text{E}} g$, we have

$$|L_h(v)| \leq \|g_h\|_{H^{-s}(\Omega_h)} \|v\|_{H^s(\Omega_h)} \leq \|g_h\|_{H^{-s}(\Omega_h)} \|v\|_{H^1(\Omega_h)}.$$

Thus, L_h is a continuous form and by Lax-Milgram theorem there exists a unique $u_h \in H^1(\Omega_h)$ such that $a_h(u_h, v) = L_h(v)$, for all $v \in H^1(\Omega_h)$.

Now, consider the problems

$$a_h(u_h, v_h) = \langle g_h, v_h\rangle_{-s,s}\,, \quad h \in \mathrm{Diff}_\epsilon(\Omega). \tag{3.9}$$

If we take $v_h = u_h - E_h u$ and $v = M_h u_h - u$ in (3.9) respectively to $h \neq I_N$ and $h = I_N$, we obtain

$$a_h(u_h, u_h - E_h u) - a(u, M_h u_h - u) = \langle g_h, u_h - E_h u\rangle_{-s,s} - \langle g, M_h u_h - u\rangle_{-s,s}$$

But,

$$\begin{aligned} \langle g_h, u_h - E_h u\rangle_{-s,s} - \langle g, M_h u_h - u\rangle_{-s,s} &= \langle g_h, u_h - E_h u\rangle_{-s,s} - \langle E_h g, u_h - E_h u\rangle_{-s,s} \\ &= \langle g_h - E_h g, u_h - E_h u\rangle_{-s,s} \end{aligned}$$

and then, u_h satisfies

$$\begin{aligned}\langle g_h - E_h g, u_h - E_h u\rangle_{-s,s} &= a_h(u_h, u_h - E_h u) - a(u, M_h u_h - u)\\ &= a_h(u_h - E_h u, u_h - E_h u) + a_h(E_h u, u_h - E_h u)\\ &\quad + a(M_h u_h - u, M_h u_h - u) - a(M_h u_h, M_h u_h - u).\end{aligned}$$

Now, we estimate

$$a_h(E_h u, u_h - E_h u) - a(M_h u_h, M_h u_h - u) = \int_{\Omega_h} \nabla(E_h u)\nabla(u_h - E_h u) + \lambda E_h u(u_h - E_h u)$$
$$- \int_{\Omega} \nabla(M_h u_h)\nabla(M_h u_h - u) + \lambda M_h u_h(M_h u_h - u).$$

But,

$$\begin{aligned}&\int_{\Omega_h} \nabla(E_h u)\nabla(u_h - E_h u) - \int_{\Omega} \nabla(M_h u_h)\nabla(M_h u_h - u)\\ &\quad= \int_{\Omega} \nabla u\nabla(M_h u_h - u)|\det(h')| - \int_{\Omega} \nabla(M_h u_h)\nabla(M_h u_h - u)\\ &\quad= -\int_{\Omega} \nabla(u - M_h u_h)^2 + \int_{\Omega} \nabla u\nabla(M_h u_h - u)(|\det(h')| - 1)\end{aligned}$$

and

$$\begin{aligned}&\int_{\Omega_h} \lambda E_h u(u_h - E_h u) - \int_{\Omega} \lambda M_h u_h(M_h u_h - u)\\ &\quad= \int_{\Omega} \lambda u(M_h u_h - u)|\det(h')| - \int_{\Omega} \lambda M_h u_h(M_h u_h - u)\\ &\quad= -\int_{\Omega} \lambda(u - M_h u_h)^2 + \int_{\Omega} \lambda u(M_h u_h - u)(|\det(h')| - 1).\end{aligned}$$

Thus,

$$\begin{aligned}&a_h(E_h u, u_h - E_h u) - a(M_h u_h, M_h u_h - u)\\ &\quad= -a(u - M_h u_h, u - M_h u_h) + \int_{\Omega} \nabla u\nabla(M_h u_h - u)(|\det(h')| - 1)\\ &\quad\quad + \int_{\Omega} \lambda u(M_h u_h - u)(|\det(h')| - 1),\end{aligned}$$

which implies

$$
\begin{aligned}
&a_h(u_h, u_h - E_h u) - a(u, M_h u_h - u) \\
&\qquad = a_h(u_h - E_h u, u_h - E_h u) + \int_\Omega \nabla u \nabla (M_h u_h - u)(|\det(h')| - 1) \\
&\qquad\quad + \int_\Omega \lambda u (M_h u_h - u)(|\det(h')| - 1).
\end{aligned}
$$

But $|\det(h')| - 1 \to 0$ as $h \to I_N$ uniformly in Ω, thus

$$\int_\Omega \nabla u \nabla (M_h u_h - u)(|\det(h')| - 1) \to 0 \quad \text{as } h \to I_N$$

and

$$\int_\Omega \lambda u (M_h u_h - u)(|\det(h')| - 1) \to 0 \quad \text{as } h \to I_N.$$

Moreover, $g_h \xrightarrow{E} g$, and then, $\langle g_h - E_h g, u_h - E_h u\rangle \to 0$, and $a_h(u_h - E_h u, u_h - E_h u) \to 0$ as $h \to I_N$ (here we have also used that $u_h - E_h u$ is uniformly bounded in $H^1(\Omega_h)$ with respect to h).

Finally, $a_h(u_h - E_h u, u_h - E_h u) \to 0$ as $h \to I_N$ implies $u_h \xrightarrow{E} u$ since

$$a_h(u_h - E_h u, u_h - E_h u) \geq \|u_h - E_h u\|^2_{H^1(\Omega_h)},$$

which proves (3.7).

Now, we obtain the estimates (3.8).

$$
\begin{aligned}
\langle g_h - E_h g, u_h - E_h u\rangle = a_h(u_h - E_h u, u_h - E_h u) \\
+ \int_\Omega \nabla u \nabla (M_h u_h - u)(|\det(h')| - 1) \\
+ \int_\Omega \lambda u (M_h u_h - u)(|\det(h')| - 1)
\end{aligned}
$$

which implies

$$
\begin{aligned}
\|u_h - E_h u\|^2_{H^1(\Omega_h)} \leq \|g_h - E_h g\|_{H^{-s}(\Omega_h)} \|u_h - E_h u\|_{H^1(\Omega_h)} \\
+ \sup_{x \in \Omega_h} \{|(\det((h')^{-1})| - 1\} \|\nabla u\|_{L^2(\Omega)} \\
\times \left(\|\nabla (M_h u_h - u)\|_{L^2(\Omega)} + \lambda \|M_h u_h - u\|_{L^2(\Omega)} \right)
\end{aligned}
$$

But,

$$\int_\Omega (M_h u_h - u)^2 = \int_{\Omega_h} (u_h - E_h u)^2 |\det(h')^{-1}|$$

and

$$\int_{\Omega} \nabla(M_h u_h - u)^2 = \int_{\Omega_h} \nabla(u_h - E_h u)^2 E_h(h')|\det(h^{-1})'|.$$

Hence,

$$\begin{aligned}
\|u_h - E_h u\|^2_{H^1(\Omega_h)} &\leq \|g_h - E_h g\|_{H^{-s}(\Omega_h)} \|u_h - E_h u\|_{H^1(\Omega_h)} \\
&\quad + \sup_{x\in\Omega_h} \{|(\det((h')^{-1})| - 1\} \|\nabla u\|_{L^2(\Omega)} \sup_{x\in\Omega_h} \{|(\det((h')^{-1})|\}^{\frac{1}{2}} \cdot \\
&\quad \cdot \Big(\|u_h - E_h u\|_{L^2(\Omega)} \|E_h(h')\|_{L^\infty(\Omega_h)} + \lambda \|u_h - E_h u\|_{L^2(\Omega_h)} \Big) \\
&\leq \Big(\|g_h - E_h g\|^2_{H^{-s}(\Omega_h)} + \{|(\det((h')^{-1})| - 1\} \|\nabla u\|_{L^2(\Omega)} \sup_{x\in\Omega_h} \{|(\det((h')^{-1})|\}^{\frac{1}{2}} \Big) \cdot \\
&\quad \cdot \|u_h - E_h u\|_{H^1(\Omega_h)} \bar{C},
\end{aligned}$$

where $\bar{C} = \max\{\sup_{x\in\Omega_h} \{|(\det((h')^{-1})|\}, \lambda\}$.

By Lemma 2.1, the result follows. □

Remark 3.5. Notice that in the proof of Theorem 3.4 we need to consider a abstract family $g_h \in H^{-s}(\Omega_h)$ which may not have the form $g_{1,h} + g_{2,h}$, where $g_{1,h} \in L^2(\Omega_h)$ and $g_{2,h} \in L^2(\partial\Omega_h)$ as in Theorem 3.1. In fact, not every function in $H^{-s}(\Omega_h)$ can be written this way but, this decomposition works well when we are interested in estimates as (3.3) and (3.4).

As a consequence of Theorem 3.4, we have the following corollaries.

Corollary 3.6. *For each $\lambda \geq 1$, there exists a constant C independent of h such that*

$$\|(\lambda - \Delta_h)^{-1} - E_h(\lambda - \Delta)^{-1} M_h\|_{\mathcal{L}(H^{-s}(\Omega_h), H^1(\Omega_h))} \leq C\tau(h) \tag{3.10}$$

and

$$\|(\lambda - \Delta_h)^{-1} E_h - E_h(\lambda - \Delta)^{-1}\|_{\mathcal{L}(H^{-s}(\Omega), H^1(\Omega_h)} \leq C\tau(h). \tag{3.11}$$

Proof. Since (2.11) implies (2.12), we just have to prove (3.10). The result follows from Theorem 3.4. □

Corollary 3.7. *Let $\lambda \geq 1$. For each $\mu \in \rho(-\Delta + \lambda)$, there exists $\epsilon = \epsilon(\mu)$ such that, $\mu \in \rho(-\Delta_h + \lambda)$ for all $h \in \mathrm{Diff}_\epsilon(\Omega)$ and*

$$(\mu + (-\Delta_h + \lambda))^{-1} \xrightarrow{\text{CC}} (\mu + (-\Delta + \lambda))^{-1}.$$

Moreover, there exists a constant $C = C(\mu)$ independent of h such that

$$\|(\mu + (-\Delta_h + \lambda))^{-1} - E_h(\mu + (-\Delta + \lambda))^{-1} M_h\|_{\mathcal{L}(H^{-s}(\Omega_h), H^1(\Omega_h))} \leq C\tau(h)$$

and

$$\|(\mu + (-\Delta_h + \lambda))^{-1} E_h - E_h(\mu + (-\Delta + \lambda))^{-1}\|_{\mathcal{L}(H^{-s}(\Omega), H^1(\Omega_h)} C\tau(h).$$

Proof. The result follows from Theorem 3.4. □

Remark 3.8. It is interesting to compare the results of this section with those of Section 2 in [28] and Section 3 in [29]. Notice that we have not fixed the domain, that is, the parameter h and the domain Ω_h vary simultaneously and our estimates are uniform concerning h and Ω_h. All sectorial inequalities to estimate the resolvent operators in [28,29] are here naturally absorbed in inequalities (3.10) and (3.11).

4. Rate of convergence for resolvent operator perturbations

The attractors $\mathcal{A}_h$ are characterized by the union of unstable manifolds of each equilibrium point. In this way, understanding the local behavior of the equilibrium set is essential to obtain the continuity of attractors. In order to describe the unstable manifold, we take a linearization around each equilibrium point. This type of argument involves making perturbations of the resolvent operators by the derivative of the vector field. In this section, we study the resolvent perturbations by potentials establishing some results that will be used in the next sections.

Definition 4.1. We say that a family of potentials $\{V_h : H^1(\Omega_h) \to H^{-s}(\Omega_h)\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ is admissible if $\sup_{h\in \mathrm{Diff}_\epsilon(\Omega)} \|V_h\|_{\mathcal{L}(H^1(\Omega_h), H^{-s}(\Omega_h))} < \infty$ and V_h E-converges to V in $H^{-s}(\Omega_h)$ as $h \to I_N$, that is, for any family $\{g_h\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ with $g_h \in H^1(\Omega_h)$ such that $g_h \xrightarrow{\text{E}} g$, we have $\|V_h g_h - E_h V g\|_{H^{-s}(\Omega_h)} \to 0$ as $h \to I_N$.

Let $\{V_h\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ be a family of potentials. We denote $\{\lambda_h^n\}_{n=1}^\infty$ the set of eigenvalues, ordered and counting multiplicity, of the operator $-\Delta_h + V_h$ with Neumann boundary condition in Ω_h and by $\{\phi_h^n\}_{n=1}^\infty$ a corresponding associated family of eigenfunctions. If $\lambda_h^n \to \lambda^n$ as $h \to I_N$, we can define the spectral projection $P_h^n : H^{-s}(\Omega_h) \to H^1(\Omega_h)$ by

$$P_h^n = \frac{1}{2\pi i} \int_{\Gamma_n} (\mu + (-\Delta_h + V_h))^{-1}\, \mathrm{d}\mu, \tag{4.1}$$

where Γ_n is a curve in $\rho(-\Delta + V)$ involving $\{\lambda^1, \dots, \lambda^n\}$.

Definition 4.2. Let $\{V_h\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ be a family of potentials. We say that the spectra of $-\Delta_h + V_h$ behaves continuously as $h \to I_N$ when $\lambda_h^n \to \lambda^n$ and $P_h^n \xrightarrow{CC} P^n$ as $h \to I_N$. We say that the spectra of $-\Delta_h$ behave continuously as $h \to I_N$ when the spectra of $-\Delta_h + V_h$ behave continuously as $h \to I_N$ for any family of admissible potentials $\{V_h\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$.

Definition 4.3. We say that a family of domains $\{\Omega_h \subset \mathbb{R}^N\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ is admissible if it satisfies the following conditions

(i) For any $K \subset\subset \Omega$, there exists $\epsilon = \epsilon(K)$ such that, $K \subset\subset \Omega_h$ for each $h \in \mathrm{Diff}_\epsilon(\Omega)$.

(ii) The spectra of $-\Delta_h$ behave continuously as $h \to I_N$.

The main result of this section states as follows.

Theorem 4.4. *The family of domains* $\{\Omega_h\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ *associated with* (1.1) *is admissible. In particular, the spectra of* $-\Delta_h$ *behave continuously as* $h \to I_N$.

To proof Theorem 4.4, we need some auxiliary results

Proposition 4.5. *For any* $\lambda \geq 1$ *and any family of admissible potentials* $\{V_h\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$, *it is valid*

$$(\lambda - \Delta_h)^{-1} V_h \xrightarrow{\mathrm{CC}} (\lambda - \Delta)^{-1} V.$$

Proof. Let $\{g_h\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ be a family with $g_h \in H^1(\Omega_h)$ such that $g_h \xrightarrow{\mathrm{E}} g$. Define $u_h = (\lambda - \Delta_h)^{-1} V_h g_h$ and $u = (\lambda - \Delta)^{-1} V g$, then, for all $h \in \mathrm{Diff}_\epsilon(\Omega)$, we have $\langle(-\Delta_h + \lambda)u_h, \phi\rangle = \langle V_h g_h, \phi\rangle$ for all $\phi \in H^1(\Omega_h)$. The result follows as in the proof of Theorem 3.4 since $\| V_h g_h - E_h V g\|_{H^{-s}(\Omega_h)} \to 0$ as $h \to I_N$. □

Corollary 4.6. *Assume* $0 \in \rho(-\Delta + V)$. *Then, there exists* ϵ *sufficiently small such that,* $0 \in \rho(-\Delta_h + V_h)$ *for all* $h \in \mathrm{Diff}_\epsilon(\Omega)$ *and*

$$(-\Delta_h + V_h)^{-1} \xrightarrow{\mathrm{CC}} (-\Delta + V)^{-1}.$$

Proof. We denote $A_h = \lambda - \Delta_h$, for $h \in \mathrm{Diff}_\epsilon(\Omega)$ and $\lambda \geq 1$. Since $0 \in \rho(-\Delta + V)$ we can write

$$(-\Delta + V)^{-1} = (I + A^{-1}(V - \lambda))^{-1} A^{-1}.$$

By Proposition 4.5, we have $A_h^{-1} V_h \xrightarrow{\mathrm{CC}} A^{-1} V$ and it is easy to see that $A_h^{-1}(V_h - \lambda) \xrightarrow{\mathrm{CC}} A^{-1}(V - \lambda)$.

Claim. The operator $[I + A_h^{-1}(V_h - \lambda)]^{-1}$ is bounded, where I denotes the identity in $\mathbb{R}^N$.

This statement is equivalent to the existence of $C > 0$ independent of h such that

$$\|[I + A_h^{-1}(V_h - \lambda)]u_h\|_{H^1(\Omega_h)} \geq \frac{1}{C}, \quad \text{for all} \quad u_h \in H^1(\Omega_h), \quad \|u_h\|_{H^1(\Omega_h)} = 1.$$

If it is not true, then there is a sequence $\{u_{h_n}\}_n$, $u_{h_n} \in H^1(\Omega_{h_n})$, $\|u_h\|_{H^1(\Omega_{h_n})} = 1$ and $h_n \to I_N$ such that $\|[I + A_h^{-1}(V_h - \lambda)]u_h\|_{H^1(\Omega_h)} \to 0$. But, (taking subsequence) $\{A_h^{-1}(\lambda + V_h)u_{h_n}\}_n$ E-converges to some $u \in H^1(\Omega)$, $\|u\|_{H^1(\Omega)} = 1$ which implies $u_{h_n} + A_h^{-1}(V_h - \lambda)u_{h_n} \xrightarrow{\mathrm{E}} 0$ and $u_{h_n} \xrightarrow{\mathrm{E}} -u$. Therefore, $[I + A^{-1}(V - \lambda)]u = 0$ is an absurd since $I + A^{-1}(V - \lambda)$ is invertible.

Now, we can write

$$I = (-\Delta_h + V_h)(I + A_h^{-1}(V_h - \lambda))^{-1} A_h^{-1}.$$

Since $[I + A_h^{-1}(V_h - \lambda)]^{-1}$ is bounded, we obtain $(-\Delta_h + V_h)$ invertible.

Now, let $\{g_h\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ be a family with $g_h \in H^1(\Omega_h)$ such that $g_h \xrightarrow{\mathrm{E}} g$. Define $u_h = (-\Delta_h + V_h)^{-1} V_h g_h$ and $u = (-\Delta + V)^{-1} V g$, then, for all $h \in \mathrm{Diff}_\epsilon(\Omega)$, we have $\langle(-\Delta_h + V_h)u_h, \phi\rangle = \langle V_h g_h, \phi\rangle$, for all $\phi \in H^1(\Omega_h)$. The result follows as in the proof of Theorem 3.4. □

Remark 4.7. It is worth comparing Corollary 4.6 with Proposition 2.3 of [4]. Here the compact convergence approach implies spectral convergence. In [4], the authors have used the spectral convergence to conclude the resolvent operator convergence.

Corollary 4.8. *For each $\mu \in \rho(-\Delta + V)$, there exists $\epsilon = \epsilon(\mu)$ such that, $\mu \in \rho(-\Delta_h + V_h)$ for all $h \in \mathrm{Diff}_\epsilon(\Omega)$ and*

$$(\mu + (-\Delta_h + V_h))^{-1} \xrightarrow{\mathrm{CC}} (\mu + (-\Delta + V))^{-1}. \tag{4.2}$$

Moreover, if $\|V_h - E_h V\|_{H^{-s}(\Omega_h)} \leq \tau(h)$, then there exists a constant $C = C(\mu)$ independent of h such that

$$\|(\mu + (-\Delta_h + V_h))^{-1} - E_h(\mu + (-\Delta + V))^{-1} M_h\|_{\mathcal{L}(H^{-s}(\Omega_h), H^1(\Omega_h))} \leq C\tau(h)$$

and

$$\|(\mu + (-\Delta_h + V_h))^{-1} E_h - E_h(\mu + (-\Delta + V))^{-1}\|_{\mathcal{L}(H^{-s}(\Omega), H^1(\Omega_h)} \leq C\tau(h).$$

Proof. Similar to the proof of Corollary 4.6. □

Next let us show Theorem 4.4.

Proof of Theorem 4.4. Since h is close to inclusion I_N, the condition (i) in Definition 4.3 is immediate. Now, we claim that $\lambda_h^n \to \lambda^n$ as $h \to I_N$. If this does not occurs then there exist $\delta > 0$ and a sequence $h_k \to I_N$ such that

$$\int_{|\mu-\lambda^n|=\delta} (\mu - \lambda^n)^l (\mu + (-\Delta_{h_k} + V_{h_k}))^{-1}\, \mathrm{d}\mu = 0, \quad k, l \in \mathbb{N}.$$

But, by (4.2), we have

$$\int_{|\mu-\lambda^n|=\delta} (\mu - \lambda^n)^l (\mu + (-\Delta + V))^{-1}\, \mathrm{d}\mu = 0, \quad l \in \mathbb{N},$$

which is an absurd since the eigenvalue λ^n is not a removable singularity of the resolvent map $\mu \to (\mu + (-\Delta_I + V))^{-1}$, $\mu \in \rho(-\Delta_I + V)$.

Since the spectral projection is given by (4.1), the compact convergence $P_h^n \xrightarrow{CC} P^n$ follows from the fact that $(-\Delta_h + V_h)^{-1}$ is compact and satisfies (4.2). □

Corollary 4.9. *There exists a constant $C > 0$ independent of h such that it is valid the following estimates*

(i) $|\lambda_h^n - \lambda^n| \leq C\tau(h)$;
(ii) $\|P_h^n - E_h P^n M_h\|_{\mathcal{L}(H^{-s}(\Omega_h), H^1(\Omega_h))} \leq C\tau(h)$;
(iii) $\|P_h^n E_h - E_h P^n\|_{\mathcal{L}(H^{-s}(\Omega), H^1(\Omega_h))} \leq C\tau(h)$.

Proof. The proof of (i) is the same as in [14, Corollary 3.8] or [13, Corollary 14.11]. By (4.1), we have

$$P_h^n - P^n = \frac{1}{2\pi i} \int_{\Gamma_n} (\mu + (-\Delta_h + V_h))^{-1} - (\mu + (-\Delta + V))^{-1} \, \mathrm{d}\mu,$$

the estimates (ii) and (iii) follow from Corollary 4.8. Here, the constant C depends on n but independent of h. □

5. Rate of convergence for permanence of equilibrium points

We are assuming here the set $\mathcal{E}$ of equilibrium points of (1.2) is composed of hyperbolic points. As consequence, we will see that the set $\mathcal{E}_h$ of equilibrium points of (1.1) is also composed of hyperbolic points as h is sufficiently close to I_N. Moreover, we estimate the convergence of elements of $\mathcal{E}_h$ to elements of $\mathcal{E}$ when $h \to I_N$.

Let $\lambda \geq 1$. For $h \in \mathrm{Diff}_\epsilon(\Omega)$, denote $A_h = \lambda - \Delta_h$ and define, for each $h \in \mathrm{Diff}_\epsilon(\Omega)$ the nonlinear operator $F_h : H^1(\Omega_h) \to H^{-s}(\Omega)$ by

$$F_h(u)\phi = \int_{\Omega_h} f(u)\phi + \int_{\Omega_h} \lambda u\phi + \int_{\partial\Omega_h} \tilde{f}(u)\phi, \quad u \in H^1(\Omega_h), \;\; \phi \in H^s(\Omega_h),$$

where, by convenience, we omit the trace operator. It is well known that if f and $\tilde{f}$ are $\mathcal{C}^2$ bounded functions with derivatives up to second order bounded and if $\frac{1}{2} < s < 1$, then F_h is a well-defined Nemytskii function which is Fréchet continuously differentiable, see for instance [29]. Hence, throughout the remainder of the text, we fix $\frac{1}{2} < s < 1$.

Lemma 5.1. *There are positive constants $L_{f,\tilde{f}}$ and C_0 such that*

$$\|F_h(u) - E_h F(v)\|_{H^{-s}(\Omega_h)} \leq L_{f,\tilde{f}} \|u - E_h v\|_{H^1(\Omega_h)} + C_0\tau(h),$$
$$u \in H^1(\Omega_h), \;\; v \in H^1(\Omega), \tag{5.1}$$

and

$$\|F_h(u) - F_h(v)\|_{H^{-s}(\Omega_h)} \leq L_{f,\tilde{f}} \|u - v\|_{H^1(\Omega_h)}, \quad u, v \in H^1(\Omega_h). \tag{5.2}$$

Proof. Let L_f and $L_{\tilde{f}}$ the Lipschitz constants of f and $\tilde{f}$. Then, for $\phi \in H^1(\Omega_h)$, we have

$$|F_h(u)\phi - E_h F(v)\phi| = |\int_{\Omega_h} f(u)\phi + \int_{\partial\Omega_h} \tilde{f}(u)\phi - \int_{\Omega} f(v)M_h\phi - \int_{\partial\Omega} \tilde{f}(v)M_h\phi|.$$

Also,

$$
\begin{aligned}
|\int_{\Omega_h} f(u)\phi - \int_{\Omega} f(v)\phi| &\le \int_{\Omega_h} |f(u) - f(E_h v)||\phi| \\
&\quad + \int_{\Omega_h} |f(E_h v)||\phi| \left(1 - |\det((h')^{-1})|\right) \\
&\le L_f \|u - E_h v\|_{L^2(\Omega_h)} \|\phi\|_{L^2(\Omega_h)} + \sup|f| \bar{\tau}(h) \|\phi\|_{L^1(\Omega_h)}.
\end{aligned}
$$

In the same way, we get

$$
\begin{aligned}
|\int_{\partial\Omega_h} \tilde{f}(u)\phi - \int_{\partial\Omega} \tilde{f}(v) M_h \phi| &\le L_f \|u - E_h v\|_{L^2(\partial\Omega_h)} \|\phi\|_{L^2(\partial\Omega_h)} \\
&\quad + \sup|f| \bar{\tau}(h) \|\phi\|_{L^1(\partial\Omega_h)}
\end{aligned}
$$

which proves (5.1). Inequality (5.2) is left to the interested reader. □

The next result shows how to extend the derivative F_h' to a family of potentials indexed in $\mathrm{Diff}_\epsilon(\Omega)$. This extension is fundamental in the next section to characterize the local behavior of the nonlinear semigroup in a neighborhood of its equilibrium points. Recall that we are denoting $F = F_h|_{h=I_N}$.

Lemma 5.2. *Let $\{v_h\}_{h \in \mathrm{Diff}_\epsilon(\Omega)}$ be a family with $v_h \in H^1(\Omega_h)$ and $v_h \xrightarrow{\text{E}} v$. Then,*

(i) The family $\{F_h'(v_h) : H^1(\Omega_h) \to H^{-s}(\Omega_h)\}_{h \in \mathrm{Diff}_\epsilon(\Omega)}$ is admissible.

(ii) If $0 \notin \sigma(A - F'(v))$, then $A_h^{-1} F_h'(v_h) \xrightarrow{CC} A^{-1} F'(v)$, where $A_h = \lambda - \Delta_h$, $h \in \mathit{Diff}_\epsilon(\Omega)$.

Proof. (i) Since f, $\tilde{f}$ and its derivatives up to second order are bounded, we have

$$
\sup_{h \in \mathrm{Diff}_\epsilon(\Omega)} \|F_h'(v_h)\|_{\mathcal{L}(H^1(\Omega_h), H^{-s}(\Omega_h))} < \infty.
$$

Since F_h is $\mathcal{C}^2$ we can perform a Frechét version of the mean value theorem to F_h' to obtain a constant C independent of h such that,

$$
\|F_h'(v_h) - E_h F'(v)\|_{\mathcal{L}(H^1(\Omega_h), H^{-s}(\Omega_h))} \le C \|v_h - E_h v\|_{H^1(\Omega_h)}.
$$

Hence, $\|F_h'(v_h) - E_h F'(v)\|_{\mathcal{L}(H^1(\Omega_h), H^{-s}(\Omega_h))} \to 0$ whenever $v_h \xrightarrow{E} v$.

(ii) Since $\{F_h'(v_h)\}_{h \in \mathrm{Diff}_\epsilon(\Omega)}$ is admissible, the result follows from Proposition (4.5). □

The solutions to the elliptic problem

$$
A_h u_h - F_h u_h = 0 \quad \text{with } u_h \in H^1(\Omega_h) \tag{5.3}
$$

are the equilibrium points of (1.1) ($h \neq I_N$) and (1.2) ($h = I_N$). We denote $\mathcal{E}_h$ the set of all solutions of (5.3). Recall that we are assuming that $\mathcal{E}$ is composed of p hyperbolic equilibrium points, that is, $0 \notin \sigma(A - F_h'(u^*))$ for all $u^* \in \mathcal{E}$.

Theorem 5.3. *For ϵ sufficiently small, $\mathcal{E}_h$ is a finite set with constant cardinality p, that is, $\mathcal{E}_h = \{u_h^{1,*}, \dots, u_h^{p,*}\}$ for all $h \in \mathrm{Diff}_\epsilon(\Omega)$. Moreover, $\mathcal{E}_h$ behaves continuously as $h \to I_N$ with*

$$\max_{1\le k\le p} \|u_h^{k,*} - E_h u^{k,*}\|_{H^1(\Omega_h)} \le C\tau(h) \tag{5.4}$$

for some constant C independent of h.

Proof. Section 4.1 in [4] inspires the proof. Let $u^* \in \mathcal{E}$ and define the operator $\Theta_h : H^1(\Omega_h) \to H^1(\Omega_h)$ by

$$\Theta_h(u_h) = (A_h - F_h'(E_h u^*))^{-1}(F_h(u_h) - F_h'(E_h u^*)u_h).$$

We have u^* is hyperbolic, and it is easy to see that $E_h u^* \xrightarrow{E} u^*$. Thus by Lemma 5.2 $\{F_h'(E_h u^*) : H^1(\Omega_h) \to H^{-s}(\Omega_h)\}_{h\in\mathrm{Diff}_\epsilon(\Omega)}$ is a admissible family and by Corollary 4.6 we have $0 \in \rho(A_h - F'(E_h(u^*)))$ thus, Θ_h is well defined. Notice that a fixed point of Θ_h is equivalent to a solution of (5.3). Arguing as [4, Proposition 4.1], we first show that Θ_h is a strict contraction in a closed ball centered in $E_h u^*$, which proves the existence of a unique equilibrium point u_h^* close to $E_h u^*$.

For this, let us take v and w in a ball of radius $\delta > 0$ centered at $E_h u^*$ in $H^1(\Omega_h)$. We have

$$\begin{aligned}\|\Theta_h(v) - \Theta_h(w)\|_{H^1(\Omega_h)} \le &\left\| \left(A_h - F_h'(E_h u^*)\right)^{-1} \right\|_{\mathcal{L}(H^{-s}(\Omega_h), H^1(\Omega_h))} \\ &\times \|F_h(v) - F_h(w) - F_h'(E_h u^*)(v-w)\|_{H^{-s}(\Omega_h)}.\end{aligned}$$

Also, for some ξ and $\tilde{\xi}$ between v and w, we have from the mean value theorem that

$$\begin{aligned}&(F_h(v) - F_h(w) - F_h'(E_h u^*)(v-w))\phi \\ &= \int_{\Omega_h} \left(f'(\xi) - f'(E_h u^*)\right)(v-w)\phi + \int_{\partial\Omega_h} \left(\tilde{f}'(\tilde{\xi}) - \tilde{f}'(E_h u^*)\right)(v-w)\phi \\ &\le \int_{\Omega_h} \theta_h |v-w||\phi| + \int_{\partial\Omega_h} \tilde{\theta}_h |v-w||\phi|\end{aligned}$$

where

$$\begin{aligned}\theta_h(x) &= 2\sup|f''| \min\{1, |v(x) - E_h u^*(x)| + |w(x) - E_h u^*(x)|\} \\ &\ge |f'(\xi(x)) - f'(E_h u^*(x))| \quad \text{and} \\ \tilde{\theta}_h(x) &= 2\sup|\tilde{f}''| \min\{1, |v(x) - E_h u^*(x)| + |w(x) - E_h u^*(x)|\} \\ &\ge |\tilde{f}'(\tilde{\xi}(x)) - \tilde{f}'(E_h u^*(x))|.\end{aligned}$$

Now, due to $\|\theta_h\|_{L^\infty(\Omega_h)} \le 1$ and $\|\theta_h\|_{L^2(\Omega_h)} \le \|v - E_h u^*\|_{L^2(\Omega_h)} + \|w - E_h u^*\|_{L^2(\Omega_h)} \le 2\delta$ we have $\|\theta_h\|_{L^p(\Omega_h)} \le 2\delta^{2/p}$ for all $p \in [2,\infty)$. Similarly, we can get $\|\tilde{\theta}_h\|_{L^p(\partial\Omega_h)} \le 2\delta^{2/p}$ for all $p \in [2,\infty)$.

Thus,

$$\begin{aligned}
&|(F_h(v) - F_h(w) - F_h'(E_h u^*)(v - w))\phi| \\
&\quad \le \|\theta_h(v - w)\|_{L^2(\Omega_h)}\|\phi\|_{L^2(\Omega_h)} + \|\tilde{\theta}_h(v - w)\|_{L^2(\partial\Omega_h)}\|\phi\|_{L^2(\partial\Omega_h)} \\
&\quad \le \|\theta_h\|_{L^N(\Omega_h)}\|v - w\|_{L^{\frac{2N}{N-2}}(\Omega_h)}\|\phi\|_{L^2(\Omega_h)} \\
&\qquad + \|\tilde{\theta}_h\|_{L^{2(N-1)}(\partial\Omega_h)}\|v - w\|_{L^{\frac{2(N-1)}{N-2}}(\partial\Omega_h)}\|\phi\|_{L^2(\partial\Omega_h)} \\
&\quad \le 2\min\{\delta^{2/N}, \delta^{1/(N-1)}\}\|\phi\|_{H^s(\Omega_h)}\|v - w\|_{H^1(\Omega_h)}.
\end{aligned}$$

Then, for δ small enough, it follows from Lemma 5.2 and Corollary 4.6 that Θ_h is a contraction near to $E_h u^*$. Hence, there exists a unique equilibrium solution u_h^* to (5.3) close to $E_h u^*$.

It only remains to prove the estimate (5.4). We have u^* and u_h^* given by

$$u^* = (A + V)^{-1}[F(u^*) + V u^*] \quad \text{and} \quad u_h^* = (A_h + V_h)^{-1}[F_h(u_h^*) + V_h u_h^*]$$

where $V = -F'(u^*)$ and $V_h = -F_h'(E_h u^*)$. Thus,

$$\begin{aligned}
\|u_h^* - E_h u^*\|_{H^1(\Omega_h)} &\le \|(A_h + V_h)^{-1} - E_h(A + V)^{-1}M_h[F_h(u_h^*) + V_h u_h^*]\|_{H^1(\Omega_h)} \\
&\quad + \|E_h(A + V)^{-1}[M_h(F_h(u_h^*) + V_h u_h^*) - F(u^*) \\
&\quad - V u^*]\|_{H^1(\Omega_h)}.
\end{aligned} \tag{5.5}$$

By Corollary 4.8, we have

$$\|(A_h + V_h)^{-1} - E_h(A + V)^{-1}M_h[F_h(u_h^*) + V_h u_h^*]\|_{H^1(\Omega_h)} \le \bar{C}\tau(h), \tag{5.6}$$

for some constant $\bar{C}$ independent of h.

<u>Claim.</u> For all $\eta > 0$, there is ϵ sufficiently small, and positive constants C_0 and C_1, independent of η and h, such that

$$\|M_h(F_h(u_h^*) + V_h u_h^*) - F(u^*) - V u^*\|_{H^1(\Omega)} \le \eta C_0\|u_h^* - E_h u^*\|_{H^1(\Omega_h)} + C_1\tau(h) \tag{5.7}$$

for all $h \in \mathrm{Diff}_\epsilon(\Omega)$. In fact,

$$\begin{aligned}
&M_h(F_h(u_h^*) + V_h u_h^*) - F(u^*) - V u^* \\
&\quad = M_h(F_h(u_h^*) - F_h'(E_h u^*)u_h^*) - F(u^*) + F'(u^*)u^* \\
&\quad = M_h[F_h(u_h^*) - F_h'(E_h u^*)u_h^* - E_h F(u^*) + E_h F'(u^*)u^*].
\end{aligned}$$

But, for $\phi \in H^s(\Omega_h)$,

$$(F_h(u_h^*) - E_h F(u^*))\phi = \int_{\Omega_h} f(u_h^*)\phi + \lambda \int_{\Omega_h} u_h^*\phi + \int_{\partial\Omega_h} \tilde{f}(u_h^*)\phi \\ - \int_{\Omega} f(u^*)M_h\phi - \lambda \int_{\Omega} u^* M_h\phi - \int_{\partial\Omega} \tilde{f}(u^*)M_h\phi$$

and

$$(F_h'(E_h u^*)u_h^* - E_h F'(u^*)u^*)\phi = \int_{\Omega_h} f'(E_h u^*)u_h^*\phi + \lambda \int_{\Omega_h} u_h^*\phi \\ + \int_{\partial\Omega_h} \tilde{f}'(E_h u^*)u_h^*\phi \\ - \int_{\Omega} f'(u^*)u^* M_h\phi - \lambda \int_{\Omega_h} u^* M_h\phi \\ - \int_{\partial\Omega} \tilde{f}'(u^*)u^* M_h\phi.$$

Now, for w_h and $\tilde{w}_h$ between u_h^* and $E_h u^*$, we have

$$\int_{\Omega_h} f(u_h^*)\phi - \int_{\Omega} f(u^*)M_h\phi = \int_{\Omega_h} (f(u_h^*) - f(E_h u^*))\phi \\ + \int_{\Omega_h} f(E_h u^*)\phi(1 - |\det((h')^{-1})|) \\ = \int_{\Omega_h} f'(w_h)(u_h^* - E_h u^*)\phi \\ + \int_{\Omega_h} f(E_h u^*)\phi(1 - |\det((h')^{-1})|)$$

and

$$\int_{\partial\Omega_h} \tilde{f}(u_h^*)\phi - \int_{\partial\Omega} \tilde{f}(u^*)M_h\phi = \int_{\partial\Omega_h} (\tilde{f}(u_h^*) - \tilde{f}(E_h u^*))\phi \\ + \int_{\partial\Omega_h} \tilde{f}(E_h u^*)\phi(1 - |\det((Dh)^{-1})|) \\ = \int_{\partial\Omega_h} \tilde{f}'(\tilde{w}_h)(u_h^* - E_h u^*)\phi \\ + \int_{\partial\Omega_h} \tilde{f}(E_h u^*)\phi(1 - |\det((Dh)^{-1})|)$$

where $(Dh)^{-1}$ is the Jacobian matrix of $h^{-1} : \partial h(\Omega) \to \partial\Omega$ sets by a given parametrization of $\partial\Omega$.

We also have for $\phi \in H^s(\Omega_h)$ that

$$\int\limits_{\Omega_h} f'(E_h u^*)u_h^*\phi - \int\limits_{\Omega} f'(u^*)u^* M_h\phi = \int\limits_{\Omega_h} f'(E_h u^*)(u_h^* - E_h u^*)\phi + \int\limits_{\Omega_h} f'(E_h u^*)E_h u^*\phi(1 - |\det((h')^{-1})|)$$

and

$$\int\limits_{\partial\Omega_h} \tilde{f}'(E_h u^*)u_h^*\phi - \int\limits_{\partial\Omega} \tilde{f}'(u^*)u^* M_h\phi = \int\limits_{\partial\Omega_h} \tilde{f}'(E_h u^*)(u_h^* - E_h u^*)\phi + \int\limits_{\partial\Omega_h} \tilde{f}'(E_h u^*)E_h u^*\phi(1 - |\det((Dh)^{-1})|).$$

Consequently,

$$\begin{aligned}
&(F_h(u_h^*) - F_h'(E_h u^*)u_h^* - E_h F(u^*) + E_h F'(u^*)u^*)\phi \\
&= \int\limits_{\Omega_h} f'(w_h)(u_h^* - E_h u^*)\phi - \int\limits_{\Omega_h} f'(E_h u^*)(u_h^* - E_h u^*)\phi \\
&\quad + \int\limits_{\Omega_h} f(E_h u^*)\phi(1 - |\det((h')^{-1})|) - \int\limits_{\Omega_h} f'(E_h u^*)E_h u^*\phi(1 - |\det((h')^{-1})|) \\
&\quad + \int\limits_{\partial\Omega_h} \tilde{f}'(\tilde{w}_h)(u_h^* - E_h u^*)\phi - \int\limits_{\partial\Omega_h} \tilde{f}'(E_h u^*)(u_h^* - E_h u^*)\phi \\
&\quad + \int\limits_{\partial\Omega_h} \tilde{f}(E_h u^*)\phi(1 - |\det((Dh)^{-1})|) \\
&\quad - \int\limits_{\partial\Omega_h} \tilde{f}'(E_h u^*)E_h u^*\phi(1 - |\det((Dh)^{-1})|) \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Now, let us estimate I_1. We have

$$\begin{aligned}
I_1 &= \int\limits_{\Omega_h} f'(w_h)(u_h^* - E_h u^*)\phi - \int\limits_{\Omega_h} f'(E_h u^*)(u_h^* - E_h u^*)\phi \\
&= \int\limits_{\Omega_h} (f'(w_h) - f'(E_h u^*))(u_h^* - E_h u^*)\phi.
\end{aligned}$$

Since $|f'(w_h) - f'(E_h u^*)| \leq \kappa_h(x)$ where $\kappa_h(x) = \sup |f''| \min\{1, w_h - E_h u^*\}$, we have

$$|\int_{\Omega_h} f'(w_h)(u_h^* - E_h u^*)\phi - \int_{\Omega_h} f'(E_h u_h^*)(u_h^* - E_h u^*)\phi| \leq \int_{\Omega_h} \kappa_h(x)|u_h^* - E_h u^*||\phi|.$$

But, $\|\kappa_h(x)\|_{L^\infty(\Omega_h)} \leq 1$ and $\|\kappa_h(x)\|_{L^2(\Omega_h)} \leq \|u_h^* - E_h u^*\|_{L^2(\Omega_h)}$. Hence,

$$\|\kappa_h(x)\|_{L^p(\Omega_h)} \leq C_p \|u_h^* - E_h u^*\|_{L^2(\Omega_h)}^{\frac{2}{p}}, \quad \text{for all } p \in [2, \infty).$$

Next, it follows from [2, Proposition 4.2] that, if $N > 2$, then $L^{\frac{2N}{N-2}}(\Omega_h) \hookrightarrow H^1(\Omega_h)$ uniformly in h (the case $N = 2$ is simpler). Thus,

$$\begin{aligned}
&|\int_{\Omega_h} f'(w_h)(u_h^* - E_h u^*)\phi - \int_{\Omega_h} f'(E_h u_h^*)(u_h^* - E_h u^*)\phi| \\
&\quad \leq \|\kappa_h\|_{L^N(\Omega_h)} \|u_h^* - E_h u^*\|_{L^{\frac{2N}{N-2}}(\Omega_h)} \|\phi\|_{L^2(\Omega_h)} \\
&\quad \leq C_N \|\kappa_h\|_{L^N(\Omega_h)} \|u_h^* - E_h u^*\|_{H^1(\Omega_h)} \|\phi\|_{H^s(\Omega_h)} \\
&\quad \leq C_N \delta^{\frac{2}{N}} \|u_h^* - E_h u^*\|_{H^1(\Omega_h)} \|\phi\|_{H^s(\Omega_h)}
\end{aligned}$$

where $\delta > 0$ is such that $\|u_h^* - E_h u^*\|_{H^1(\Omega_h)} < \delta$ and C_N is a constant independent of h.

Hence,

$$\begin{aligned}
I_1 + I_2 &\leq C_N \delta^{\frac{2}{N}} \|u_h^* - E_h u^*\|_{H^1(\Omega_h)} \|\phi\|_{H^s(\Omega_h)} \\
&\quad + \int_{\Omega_h} |f(E_h u^*)\phi(1 - |\det((h')^{-1})|) \\
&\quad + \int_{\Omega_h} |f'(E_h u^*) E_h u^* \phi (1 - |\det((h')^{-1})|) \\
&\leq C_N \delta^{\frac{2}{N}} \|u_h^* - E_h u^*\|_{H^1(\Omega_h)} \|\phi\|_{H^s(\Omega_h)} \\
&\quad + \max\{\sup|f|, \sup|f'|\} \|\phi\|_{H^s(\Omega_h)} (1 + \|E_h u^*\|_{L^2(\Omega_h)}) \bar{\tau}(h).
\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}
I_3 + I_4 &\leq C_{N-1} \delta^{\frac{1}{N-1}} \|u_h^* - E_h u^*\|_{H^1(\Omega_h)} \|\phi\|_{H^s(\Omega_h)} \\
&\quad + \max\{\sup|\tilde{f}|, \sup|\tilde{f}'|\} \|\phi\|_{H^s(\Omega_h)} (1 + \|E_h u^*\|_{L^2(\partial\Omega_h)}) \bar{\tau}(h).
\end{aligned}$$

Thus, we can conclude (5.7) setting $\eta = \delta^{\frac{2}{N}}$ since $N \geq 2$.

Finally, we have

$$\begin{aligned} &\|E_h(A+V)^{-1}[M_h(F_h(u_h^*)+V_hu_h^*)-F(u^*)-Vu^*]\|_{H^1(\Omega_h)} \\ &\quad\leq \|E_h(A+V)^{-1}M_h\|_{\mathcal{L}(H^{-s}(\Omega_h),H^1(\Omega))}\left(\eta C_0\|u_h^*-E_hu^*\|_{H^1(\Omega_h)}+C_1\tau(h)\right). \end{aligned} \tag{5.8}$$

We can choose η sufficiently small such that $\eta C_0\|E_h(A+V)^{-1}M_h\|_{\mathcal{L}(H^{-s}(\Omega_h),H^1(\Omega))} \leq \frac{1}{2}$. Hence, due to (5.5) and (5.6), we obtain

$$\begin{aligned} \|u_h^*-E_hu^*\|_{H^1(\Omega_h)} &\leq \frac{1}{2}\|u_h^*-E_hu^*\|_{H^1(\Omega_h)} \\ &\quad+\left(\bar{C}+C_1\|E_h(A+V)^{-1}M_h\|_{\mathcal{L}(H^{-s}(\Omega_h),H^1(\Omega))}\right)\tau(h) \end{aligned}$$

which proves the theorem. □

We finish this section by stating an important well-known estimate for the linear semigroup generated by A_h. For instance, we have

$$e^{-A_ht}=\frac{1}{2\pi i}\int_\Gamma(\mu+A_h)^{-1}e^{\mu t}\,\mathrm{d}\mu, \tag{5.9}$$

where Γ is a curve delimiting an appropriated sector in $\rho(-A_h)$ independent of $h \in \text{Diff}_\epsilon(\Omega)$. It follows from [13, Section 6.4] that, if $\lambda_n < a < \lambda_{n+1}$, then there exists a constant $\bar{M}$ independent of h such that

$$\|e^{-A_ht}\phi\|_{H^1(\Omega_h)} \leq \bar{M}e^{-at}t^{-\frac{1+s}{2}}\|\phi\|_{H^{-s}(\Omega_h)}, \quad t \geq 0. \tag{5.10}$$

6. Rate of convergence for continuity of attractors

In this section, we obtain the exponential attraction of the attractors $\mathcal{A}_h$, $h \in \text{Diff}_\epsilon(\Omega)$. This property together with the continuity of the nonlinear semigroups generated by solutions of (1.1) and (1.2) will imply the continuity of attractors in a way that the modulus of continuity of semigroups will define the rate of convergence of attractors as $h \to I_N$. It is worth mentioning that our definitions and estimates are made such that the uniform condition in the parameter h needs to be checked at each step. Notice that, different from [12], our dynamics act in different phase spaces. Our adaptations allow us to use the Theorems 1.1 and Proposition 1.1 of [12].

Recall that the Hausdorff distance between closed sets $A, B \subset H^1(\Omega_h)$ is defined by

$$d_H(A,B)=\sup_{u\in A} dist(u,B)+\sup_{v\in B} dist(v,A),$$

where $dist(u,B)=\inf_{v\in B}\|u-v\|_{H^1(\Omega_h)}$.

Definition 6.1. We say that a family $\{\mathcal{A}_h\}_{h\in\text{Diff}(\Omega)}$ is continuous at I_N if

$$d_H(\mathcal{A}_h,E_h\mathcal{A}) \to 0 \quad \text{as} \quad h \to I_N.$$

The nonlinear semigroup $T_h(\cdot)$ given by solutions of (1.1) and (1.2) satisfy the variation of constant formula

$$T_h(t)u = e^{-A_h t}u + \int_0^t e^{-A_h(t-s)} F_h(T_h(s)u)\,ds, \quad u \in H^1(\Omega_h),\ h \in \mathrm{Diff}_\epsilon(\Omega), \tag{6.1}$$

where $e^{-A_h t}$ is the linear analytic semigroup with infinitesimal generator $A_h = \lambda - \Delta_h$ which is a sectorial operator. Let $\{\mathcal{A}_h\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ be its family of attractors.

Definition 6.2. We say that a family $\{\mathcal{A}_h\}_{h\in \mathrm{Diff}(\Omega)}$ is uniformly bounded at I_N if there exist $r > 0$ independent of h, such that $\|u_h\|_{L^\infty(\Omega_h)} \leq r$, for all $u_h \in \mathcal{A}_h$, $h \in \mathrm{Diff}_\epsilon(\Omega)$.

Proposition 6.3. *The family of attractors $\{\mathcal{A}_h\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ of (6.1) is uniformly bounded at I_N.*

Proof. The well-posedness of (1.1) and (1.2) that we are assuming requires growth and dissipativeness conditions which implies the uniform boundedness of $\{\mathcal{A}_h\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ in $H^1(\Omega_h)$ and $L^\infty(\Omega_h)$, see Theorem 4.5 in [6]. Since E_h is uniformly bounded in h the result follows. It is important to note that the upper bound for the attractors may depend on Ω but it is independent of h. □

Definition 6.4. We say that a family of nonlinear semigroups $\{T_h(\cdot)\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ having global attractors $\{\mathcal{A}_h\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ has a κ-modulus of continuity at I_N if there exits a continuous function $\kappa : \mathrm{Diff}_\epsilon(\Omega) \to [0,\infty)$ with $\kappa(I_N) = 0$ such that

$$\|T_h(t)u - E_h T(t) M_h u\|_{H^1(\Omega_h)} \leq C e^{Lt}\kappa(h), \quad u \in \mathcal{A}_h,\ t > t_0,$$

where C, L and t_0 are positive constants independent of h.

Theorem 6.5. *The family of nonlinear semigroups $\{T_h(\cdot)\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ satisfying (6.1) has a κ-modulus of continuity at I_N. In addition, there exist positive constants L, a, C_1, C and $\theta \in (0, \frac{1}{2})$ independent of h, such that*

$$\|e^{-A_h t} - E_h e^{At} M_h\|_{\mathcal{L}(H^{-s}(\Omega_h), H^1(\Omega_h))} \leq C_1 e^{-a(1-2\theta)t}\tau(h)^{2\theta} t^{-(\frac{1+s}{2}+\theta)}, \quad t > 0 \tag{6.2}$$

and

$$\|T_h(t)u - E_h T(t) M_h u\|_{H^1(\Omega_h)} \leq C\tau(h)^{2\theta} e^{Lt} t^{-(\frac{1+s}{2}+\theta)}, \quad u \in H^1(\Omega_h),\ t > 0. \tag{6.3}$$

Proof. It follows from (5.10) and (2.2) that we can find positive constants M_1 and a independent of h, such that

$$\begin{aligned}\|e^{-A_h t} - E_h e^{At} M_h\|_{\mathcal{L}(H^{-s}(\Omega_h), H^1(\Omega_h))} &\leq \|e^{-A_h t}\|_{\mathcal{L}(H^{-s}(\Omega_h), H^1(\Omega_h))} \\ &\quad + \|E_h e^{-At}\|_{\mathcal{L}(H^{-s}(\Omega_h), H^1(\Omega_h))} \\ &\leq M_1 e^{-at} t^{-\frac{1+s}{2}}.\end{aligned}$$

On the other hand, by (5.9) and Corollary 4.8, we obtain a constant M_2 independent of h, such that

$$\begin{aligned}\|e^{-A_h t} - E_h e^{At} M_h\|_{\mathcal{L}(H^{-s}(\Omega_h), H^1(\Omega_h))} &\leq \int_\Gamma \|(\mu + A_h)^{-1} \\ &\quad - E_h(\mu + A_h)^{-1} M_h\|_{\mathcal{L}(H^{-s}(\Omega_h), H^1(\Omega_h))} |e^{\mu t}| \, |\mathrm{d}\mu| \\ &\leq M_2 \tau(h) t^{-1}\end{aligned}$$

where the term t^{-1} is due to the unbounded curve Γ involving the spectra of $-\Delta + \lambda$.

Following [1], we take $\theta \in (0, \frac{1}{2})$ and interpolate the above inequalities with exponents $1 - 2\theta$ and 2θ, to obtain a constant C_1 independent of h such that (6.2) is valid.

Now, let $u \in \mathcal{A}_h$ and $t > 0$. By (6.1), we have

$$\begin{aligned}&\|T_h(t)u - E_h T(t) M_h u\|_{H^1(\Omega_h)} \\ &\quad \leq \|e^{-A_h t} - E_h e^{At} M_h\|_{\mathcal{L}(H^{-s}(\Omega_h), H^1(\Omega_h))} \|u\|_{H^1(\Omega_h)} \\ &\quad + \int_0^t \|e^{-A_h(t-s)}[F_h(T_h(s)u) - E_h F(T(s) M_h u)]\|_{H^1(\Omega_h)} \, \mathrm{d}s \\ &\quad + \int_0^t \|[e^{-A_h(t-s)} - E_h e^{A(t-s)} M_h] F(T(s) M_h u)\|_{H^1(\Omega_h)} \, \mathrm{d}s.\end{aligned}$$

By (5.1), we can find positive constants $L_{f,\tilde{f}}$ and C_0 independent of h such that

$$\begin{aligned}&\|F_h(T_h(s)u) - E_h F(T(s) M_h u)\|_{H^{-s}(\Omega_h)} \\ &\quad \leq L_{f\tilde{f}} \|T_h(s)u - E_h T(s) M_h u\|_{H^1(\Omega_h)} + C_0 \tau(h)\end{aligned} \tag{6.4}$$

and since F_h is uniformly bounded in h, by (6.2), (6.4), (5.10) and Proposition 6.3 we can find a constant $r > 0$ independent of h such that

$$\begin{aligned}\|T_h(t)u - E_h T(t) M_h u\|_{H^1(\Omega_h)} &\leq C_1 e^{-a(1-2\theta)} \tau(h)^{2\theta} t^{-(\frac{1+s}{2} + \theta)} r \\ &\quad + \bar{M} L_{f,\tilde{f}} \int_0^t e^{-a(t-s)} (t-s)^{-\frac{1+s}{2}} \\ &\quad \|T_h(s)) - E_h T(s) M_h u\|_{H^1(\Omega_h)} \, \mathrm{d}s\end{aligned}$$

$$+ C_0 \bar{M} L_{f,\tilde{f}} \tau(h) \int_0^t e^{-a(t-s)} (t-s)^{-\frac{1+s}{2}} \, \mathrm{d}s$$

$$+ r C_1 \tau(h)^{2\theta} \int_0^t (t-s)^{-\frac{1+s}{2}+\theta} e^{-a(1-2\theta)(t-s)} \, \mathrm{d}s.$$

But, since $s \in (\frac{1}{2}, 1)$ and $\theta \in (0, \frac{1}{2})$, we have

$$C_\theta := \int_0^t e^{-a(t-s)} (t-s)^{-\frac{1+s}{2}} \, \mathrm{d}s + \int_0^t (t-s)^{-\frac{1+s}{2}+\theta} e^{-a(1-2\theta)(t-s)} \, \mathrm{d}s$$

$$\leq \frac{1}{a^{1-\frac{(1+s)}{2}}} \bar{\Gamma}\Big(\frac{1}{2} - \frac{s}{2}\Big) + \frac{1}{a^{\frac{1}{2}-(\frac{s}{2}-\theta)} (1-2\theta)^{\frac{1}{2}-(\frac{s}{2}-\theta)}} \bar{\Gamma}\Big(\frac{1}{2} - \Big(\frac{s}{2} - \theta\Big)\Big) < \infty,$$

where $\bar{\Gamma}(\cdot)$ denotes the gamma function.

Thus, if we take $C_3 = 2C_2$ with $C_2 = \max\{C_0 \bar{M} L_{f,\tilde{f}}, r C_1\}$, we have

$$\|T_h(t)u - E_h T(t) M_h u\|_{H^1(\Omega_h)} \leq r C_1 e^{-a(1-2\theta)} \tau(h)^{2\theta} t^{-(\frac{1+s}{2}+\theta)} + C_3 C_\theta \tau(h)^{2\theta}$$

$$+ \bar{M} L_{f,\tilde{f}} \int_0^t e^{-a(t-s)} (t-s)^{-\frac{1+s}{2}}$$

$$\|T_h(s)u - E_h T(s) M_h u\|_{H^1(\Omega_h)} \, \mathrm{d}s.$$

Now, we can take $\delta = \delta(\theta) > 0$ such that $1 \leq t^{-(\frac{1+s}{2}+\theta)} e^{a\delta t}$. Thus, since $e^{-a(1-2\theta)} \leq 1$ and $e^{a\delta t} \geq 1$, we have

$$\|T_h(t)u - E_h T(t) M_h u\|_{H^1(\Omega_h)} \leq (C_1 r + C_3 C_\theta) \tau(h)^{2\theta} t^{-(\frac{1+s}{2}+\theta)} e^{a\delta t}$$

$$+ \bar{M} L_{f,\tilde{f}} \int_0^t (t-s)^{-\frac{1+s}{2}} \|(T_h(s)u) - E_h T(s) M_h u\|_{H^1(\Omega_h)} \, \mathrm{d}s$$

If we denote $\phi(t) = \|T_h(t)u - E_h T(t) M_h u\|_{H^1(\Omega_h)} e^{-a\delta t}$, we have

$$\phi(t) \leq (C_1 r + C_3 C_\theta) \tau(h)^{2\theta} t^{-(\frac{1+s}{2}+\theta)} + \bar{M} L_{f,\tilde{f}} \int_0^t (t-s)^{-\frac{1+s}{2}} \phi(s) \, \mathrm{d}s$$

where we have used $e^{-a\delta t} \leq e^{-a\delta s}$ for $s \leq t$.

By singular Gronwall inequality, we find positive constants C and L independent of h such that (6.3) is valid. The result follows taking $t_0 = 1$ and $\kappa(h) = \tau(h)^{2\theta}$. □

Definition 6.6. We say that a family $\{\mathcal{A}_h\}_{h \in \mathrm{Diff}_\epsilon(\Omega)}$ is eventually uniformly exponentially attracting if there exists $\epsilon \in (0, 1), \delta > 0, t_0 > 0, C > 0$ and $\gamma > 0$ independents of h such that

$$dist_H(T_h(t)\mathcal{O}_\delta(\mathcal{A}_h), \mathcal{A}_h) \leq C e^{-\gamma t}, \quad t \geq t_0, \; h \in \mathrm{Diff}_\epsilon(\Omega),$$

where $\mathcal{O}_\delta(\mathcal{A}_h) = \{v \in H^1(\Omega) : dist(v, \mathcal{A}_h) < \delta\}$.

The main requirement to obtain the continuity of attractors with a rate of convergence is that $\mathcal{A}_h$ uniformly attracts a δ neighborhood of itself. Notice that the parameter δ is the same for all $h \in \mathrm{Diff}_\epsilon(\Omega)$. A beautiful theorem to guarantee the exponential attraction of $\{\mathcal{A}_h\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ is [12, Theorem 1.1].

Recall that the unstable manifold of $u_h^* \in \mathcal{E}_h$ for the semigroup $T_h(\cdot)$ generated by solutions of (1.1) is the set

$$W^u(u_h^*) = \{u \in H^1(\Omega_h) : \exists \text{ global solution } \xi_h : \mathbb{R} \to H^1(\Omega_h) \text{ such that,}\\ \xi_h(0) = u \text{ and } \|\xi_h(t) - u_h^*\|_{H^1(\Omega_h)} \to 0 \text{ as } t \to -\infty\}.$$

Given $\delta > 0$, the local unstable manifold of u_h^* for $T_h(\cdot)$ is defined as

$$W^u_{\mathrm{loc}}(u_h^*) = \{u \in H^1(\Omega_h) : \exists \text{ global solution } \xi_h : \mathbb{R} \to H^1(\Omega_h) \text{ such that,}\\ \xi_h(0) = u,\ \|\xi_h(t) - u_h^*\|_{H^1(\Omega_h)} < \delta,\ t \le 0 \text{ and}\\ \|\xi_h(t) - u_h^*\|_{H^1(\Omega_h)} \to 0 \text{ as } t \to -\infty\}.$$

Definition 6.7. We say that a family of local unstable manifolds $\{W^u_{\mathrm{loc}}(u_h^*)\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ is pointwise exponentially attracting if there exist positive constants C, γ and δ such that, for each $h \in \mathrm{Diff}_\epsilon(\Omega)$,

$$dist(T_h(t)u, W^u_{\mathrm{loc}}(u_h^*)) \le Ce^{-\gamma t}$$

whenever $\|u - u_h^*\|_{H^1(\Omega_h)} < \delta, t \ge 0$ and $\{T_h(s)u : s \in [0, t]\} \subset \{v \in H^1(\Omega_h) : \|v - u_h^*\|_{H^1(\Omega_h)} < \delta\}$. We say that $\mathcal{E}_h$ has uniformly pointwise exponentially attracting local unstable manifolds if, for each $u_h^* \in \mathcal{E}_h$, the family $\{W^u_{\mathrm{loc}}(u_h^*)\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ is pointwise exponentially attracting with the same parameters C, γ, δ independent of h.

Theorem 6.8. *The set of equilibrium points $\mathcal{E}_h$ of (5.3) has uniformly pointwise exponentially attracting local unstable manifolds. In addition, the C^0-convergence of the local unstable manifold can be estimate by $C\tau(h)^{2\theta}$, for $\theta \in (0, \frac{1}{2})$ and $C > 0$ constants independent of h.*

Proof. The construction of the unstable manifold as a graph of a Lipschitz function is a well-known result present in several papers (we refer [1,17]). Thus, we can state that there exists a Lipschitz function $s_h^* : P_h^n H^1(\Omega_h) \to (I - P_h^n)H^1(\Omega_h)$ such that the unstable manifold of $u_h^* \in \mathcal{E}_h$ is given as graph of s_h^*, that is,

$$W^u_{\mathrm{loc}}(u_h^*) = \{(v, z) \in H^1(\Omega_h) : z = s_h^*(v),\ \ v \in P_h^n H^1(\Omega_h)\}.$$

We can proceed as [1] being careful with the H^{-s} dual spaces to obtain the following estimate

$$\sup_{v\in P_h^n H^1(\Omega_h)} \|s_h^*(v) - E_h s^*(M_h v)\|_{H^1(\Omega_h)} \le C\tau(h)^{2\theta},$$

where $C > 0$ and $\theta \in (0, \frac{1}{2})$ are constants independents of h. Moreover, we can use the projection P_h^n to decompose the equation (5.3) in order to obtain that the family $\{W^u(u_h^*)\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ is pointwise exponentially attracting with the same parameters C, γ, δ independent of h. □

Given all we have obtained so far, it remains to show the following property to complete all assumptions of [12, Theorem 1.1].

Definition 6.9. We say that a family of nonlinear semigroups $\{T_h(\cdot)\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ having attractors $\{\mathcal{A}_h\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ is exponentially Lipschitz continuous relatively to its family of attractors if there exist constants $C > 0$, $L > 0$ independent of h and $\epsilon \in (0, 1)$ such that

$$\|T_h(t)u - T_h(t)v\|_{H^1(\Omega_h)} \le Ce^{Lt}\|u - v\|_{H^1(\Omega_h)}, \quad u, v \in \mathcal{A}_h.$$

Proposition 6.10. *The family of nonlinear semigroups $\{T_h(\cdot)\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ satisfying (6.1) is exponentially Lipschitz continuous relatively to its family of attractors.*

Proof. Let $u \in \mathcal{A}_h$ and $v \in \mathcal{A}$. By (6.1), (5.10) and (5.2), we can write

$$\begin{aligned}\|T_h(t)u - T_h(t)v\|_{H^1(\Omega_h)} &\le \bar{M}e^{-at}\|u - v\|_{H^1(\Omega_h)}t^{-\frac{1}{2}} \\ &+ \bar{M}L_{f,\tilde{f}}\int_0^t e^{-a(t-s)}(t-s)^{-\frac{1+s}{2}}\|T_h(s)u - T_h(s)v\|_{H^1(\Omega_h)}\,\mathrm{d}s\end{aligned}$$

The result follows from Gronwall inequality as in the proof of Theorem 6.5. □

Proposition 6.11. *The family of attractors $\{\mathcal{A}_h\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ of (6.1) is eventually uniformly exponentially attracting.*

Proof. We can see in [27] that $T_h(\cdot)$ is a gradient semigroup. The existence of $\mathcal{A}_h$ and Theorem 6.5 implies that the family $\{T_h(\cdot)\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ is asymptotically compact and continuous at $h = I_N$. Theorem 5.3 states the continuity of $\mathcal{E}_h \to \mathcal{E}$ as $h \to I_N$. Proposition 6.10 ensures that $\{T_h(\cdot)\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ is exponentially Lipschitz continuous relatively to $\{\mathcal{A}_h\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$, and Theorem 6.8 provides that the family of local unstable manifolds $\{W^u_{\mathrm{loc}}(u_h^*)\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ is pointwise exponentially attracting for all $u_h^* \in \mathcal{E}_h$. These are all assumptions of [12, Theorem 1.1] which implies $\{\mathcal{A}_h\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ uniformly exponentially attracting. □

Finally, we can state, in our context, the [12, Proposition 1.1].

Theorem 6.12. *If a family of nonlinear semigroups $\{T_h(\cdot)\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ with attractors $\{\mathcal{A}_h\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ has a κ-modulus of continuity at I_N and $\{\mathcal{A}_h\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ is eventually uniformly exponentially attracting, then for $\epsilon \in (0, 1)$ sufficiently small*

$$d_h(\mathcal{A}_h, E_h\mathcal{A}) \le \bar{C}\kappa(h)^{\frac{\gamma}{\gamma+L}}, \quad h \in \mathrm{Diff}_\epsilon(\Omega),$$

where $\bar{C}$ is a constant independent of h, γ is the uniform constant given by exponential attraction of $\{\mathcal{A}_h\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ and L is the uniform Lipschitz constant of $\{T_h(\cdot)\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$.

Now, we have all the conditions to show the main result of this paper.

Theorem 6.13. *The family of attractors* $\{\mathcal{A}_h\}_{h\in \mathrm{Diff}_\epsilon(\Omega)}$ *is continuous at* I_N *and this continuity can be estimated by*

$$d_h(\mathcal{A}_h, E_h\mathcal{A}) \leq C\tau(h)^\beta, \quad h \in \mathrm{Diff}_\epsilon(\Omega), \tag{6.5}$$

for constants $C > 0$ *and* $0 < \beta < 1$ *independent of* h.

Proof. By Proposition 6.5, $\{T_h(\cdot)\}_{h\in\mathrm{Diff}_\epsilon(\Omega)}$ has $\kappa(h) = \tau(h)^{2\theta}$ as modulus of continuity at $h = I_N$. Proposition 6.11 ensures that $\{\mathcal{A}_h\}_{h\in\mathrm{Diff}_\epsilon(\Omega)}$ is eventually uniformly exponentially attracting. Thus, by Theorem 6.12, we can take $\epsilon \in (0, 1)$ sufficiently small such that

$$d_H(\mathcal{A}_h, E_h\mathcal{A}) \leq \bar{C}\kappa(h)^{\frac{\gamma}{\gamma+L}} = \bar{C}\tau(h)^\beta,$$

where $\beta = \frac{2\theta\gamma}{\gamma+L}$. □

Remark 6.14. Finally, we notice that the choice of $H^1(\Omega_h)$ as the phase space to obtain the estimate (6.5) has no advantage over $H^1(\Omega)$. Since (2.11) implies (2.12), we can remake all the results of the previous sections to obtain

$$d_H^\Omega(M_h\mathcal{A}_h, \mathcal{A}) \leq C\tau(h)^\beta,$$

where d_H^Ω denotes the Hausdorff distance in $H^1(\Omega)$.

7. Rate of convergence of attractors in the Gromov–Hausdorff distance

The continuity of attractors gives information about how the shape of attractors approaches each other as $h \to I_N$. It does not give information on the internal structure of the attractors. The works in this direction are of high importance and involve more delicate questions related to the structural stability of the problem. We do not intend to address these questions here, but we can use the previous results to quantify how much the attractors $\mathcal{A}_h$ and $\mathcal{A}$ are no longer isometric.

From [26], we take the following definition.

Definition 7.1. An η−isometry ($\eta > 0$) is a map $i_h : \mathcal{A}_h \to \mathcal{A}$ (not necessarily continuous) satisfying

$$\big|\|i_h(u) - i_h(v)\|_{H^1(\Omega)} - \|u - v\|_{H^1(\Omega_h)}\big| \leq \eta, \quad u, v \in \mathcal{A}_h \tag{7.1}$$

and $d_H(i_h(\mathcal{A}_h), \mathcal{A}) \leq \eta$. The Gromov–Hausdorff distance between $\mathcal{A}_h$ and $\mathcal{A}$ is defined by

$$d_{\mathrm{GH}} = \inf\{\eta : \exists\ i_h : \mathcal{A}_h \to \mathcal{A} \text{ and } j_h : \mathcal{A} \to \mathcal{A}_h\ \eta-\text{isometries}\}.$$

Remark 7.2. Notice that an isometry ($\eta = 0$) is a map that preserves distance, and then, it is continuous. On the other side, condition (7.1) does not imply i_h continuous. The distance d_{GH} originated from the work [19]. It quantifies how much the attractors $\mathcal{A}_h$ and $\mathcal{A}$ are not isometric.

Recently, [26] have shown that there exists a $\eta-$isometry ($\eta > 0$) between $\mathcal{A}_h$ and $\mathcal{A}$ for η sufficiently small. In the next result, we show that we can take η of the order $\tau(h)^\beta$, $0 < \beta < 1$.

Theorem 7.3. *The Gromov–Hausdorff distance of the attractors can be estimated by*

$$d_{\mathrm{GH}}(\mathcal{A}_h, \mathcal{A}_0) \leq C \min\{\tau(h)^\beta, \tau(h)^{\frac{1}{2}}\}. \tag{7.2}$$

for constants $C > 0$, $0 < \beta < 1$ independent of h.

Proof. For all $u, v \in \mathcal{A}_h$, we have

$$\begin{aligned}\int_\Omega |M_h u - M_h v|^2 &= \int_{\Omega_h} |M_h u h^{-1} - M_h v h^{-1}|^2 |\det((h')^{-1})| \\ &\leq \int_{\Omega_h} |u - v|^2 + \int_{\Omega_h} |u - v|^2 \bar{\tau}(h)\end{aligned}$$

and

$$\begin{aligned}\int_\Omega |\nabla M_h u - \nabla M_h v|^2 &= \int_{\Omega_h} |\nabla M_h u h^{-1} - \nabla M_h v h^{-1}|^2 |\det((h')^{-1})| \\ &\leq \int_{\Omega_h} |\nabla u - \nabla v|^2 + \int_{\Omega_h} |\nabla u - \nabla v|^2 \bar{\tau}(h).\end{aligned}$$

Thus,

$$\|M_h(u) - M_h(v)\|_{H^1(\Omega)} \leq \sqrt{\|u - v\|^2_{H^1(\Omega_h)} + (\|u - v\|_{H^1(\Omega_h)} \sqrt{\bar{\tau}(h)})^2}$$

which implies

$$\|M_h(u) - M_h(v)\|_{H^1(\Omega)} - \|u - v\|_{H^1(\Omega_h)} \leq \|u - v\|_{H^1(\Omega_h)} \bar{\tau}(h)^{\frac{1}{2}}.$$

In the same way, one can obtain that

$$\|u - v\|_{H^1(\Omega_h)} - \|M_h(u) - M_h(v)\|_{H^1(\Omega)} \leq \|u - v\|_{H^1(\Omega_h)} \bar{\tau}(h)^{\frac{1}{2}}.$$

Since the attractors are uniformly bounded, we have $M_h : \mathcal{A}_h \to H^1(\Omega)$ is an $r\bar{\tau}(h)^{\frac{1}{2}}$-isometry for some $r > 0$ independent of h. In the same way, we can prove that $E_h : \mathcal{A} \to H^1(\Omega_h)$ is an $r\bar{\tau}(h)^{\frac{1}{2}}$-isometry.

Now, we can argue as in [26] to take, for each h, two maps $i_h : \mathcal{A}_h \to \mathcal{A}$ and $j_h : \mathcal{A} \to \mathcal{A}_h$ such that, by Theorem 6.13 and Remark 6.14, we have

$$\|i_h(u) - M_h(u_h)\|_{H^1(\Omega)} \leq C\bar{\tau}(h)^\beta \text{ and } \|j_h(u) - E_h(u)\|_{H^1(\Omega_h)} \leq C\bar{\tau}(h)^\beta.$$

Hence, i_h and j_h are both $C \min\{\bar{\tau}(h)^\beta, \bar{\tau}(h)^{\frac{1}{2}}\}$-isometries. Since d_{GH} is the infimum on the η-isometries, we obtain (7.2). □

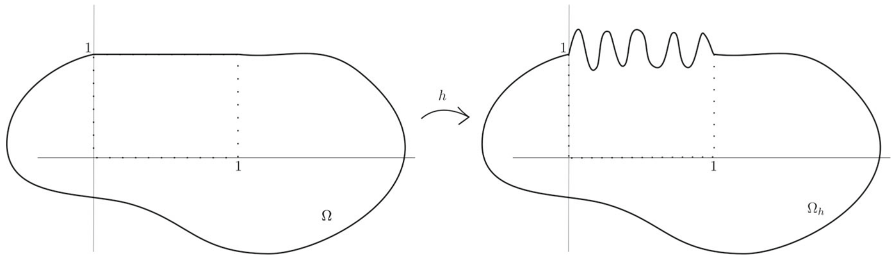

Figure 1. A local oscillating perturbation of the boundary of a domain Ω

8. Example: oscillating perturbation of a piece of the boundary

Let $\Omega \subset \mathbb{R}^2$ be a smooth C^2 domain such that $R_1 = [0,1] \times [0,1] \subset \Omega$ and $\partial\Omega \cap R_1 = \{(x,1) \in \mathbb{R}^2 \ : \ x \in (0,1)\}$, see Fig. 1. We define

$$h_\epsilon(x,y) = \begin{cases} (x,y), & (x,y) \in \Omega \setminus int(R_1), \\ (x, y + \epsilon y \sin(\frac{x}{\epsilon^\alpha})), & (x,y) \in int(R_1), \end{cases} \tag{8.1}$$

where $0 < \alpha < 1$ is fixed and $\epsilon \in (0,1)$ is a parameter.

We have that h_ε is a diffeomorphism from Ω into its image Ω_h. If $(x,y) \in \Omega \setminus int(R_1)$ then $\det(h'_\epsilon) = 1$ and if $(x,y) \in int(R_1)$, then

$$h'_\epsilon(x,y) = \begin{bmatrix} 1 & 0 \\ y\epsilon^{1-\alpha}\cos\left(\frac{x}{\epsilon^\alpha}\right) & 1 + \epsilon\sin\left(\frac{x}{\epsilon^\alpha}\right) \end{bmatrix},$$

which implies $|\det(h'_\epsilon(x,y))| = \left|1 + \epsilon\sin\left(\frac{x}{\epsilon^\alpha}\right)\right|$. It is easy to see $\tau(h) = d_{C^1}(h, I_2) \leq C\epsilon^{1-\alpha}$. Hence,

$$d_H(\mathcal{A}_h, E_h\mathcal{A}) \leq C\epsilon^\beta$$

for some $0 < \beta < 1$.

Remark 8.1. It is worth mentioning that the case $\alpha = 1$ has been addressed in [3]. In this case, the problem presents a nonuniform Lipschitz deformation and the limiting problem is different. Hence, to obtain the rate of convergence $\tau(h)$, a differential framework is essential (as we can see in Lemma 2.1). Thus, dealing with Lipschitz (not differentiable) perturbation of the domain is an interesting open question of the viewpoint of the rate of convergence of attractors for parabolic equations that we intend to address in a future work.

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Declarations

Conflict of interest The authors disclose that there is no financial interest directly or indirectly related to the work.

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