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BIFURCATION AND HYPERBOLICITY FOR A NONLOCAL QUASILINEAR PARABOLIC PROBLEM

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Abstract. In this article, we study the scalar one-dimensional nonlocal quasilinear problem of the form

$$u_t = a(\|u_x\|^2)u_{xx} + \nu f(u),$$

with Dirichlet boundary conditions on the interval $[0, \pi]$, where $a : \mathbb{R}^+ \to [m, M] \subset (0, +\infty)$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous functions that satisfy suitable additional conditions. We give a complete characterization of the bifurcations and hyperbolicity for the corresponding equilibria. With respect to bifurcation, the existing result requires that the function $a(\cdot)$ be non-decreasing and shows that bifurcations are pitchfork supercritical bifurcations from zero. We extend these results to the case of a general smooth nonlocal diffusion function $a(\cdot)$ and show that bifurcations may be pitchfork or saddle-node, both subcritical or supercritical. Concerning hyperbolicity, we specifying necessary and sufficient conditions for its occurrence. We also explore some examples to exhibit the variety of possibilities, depending on the choice of the function $a(\cdot)$, that may occur as the parameter ν varies.

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1. Introduction

This paper is dedicated to the study of the stationary solutions of the following nonlocal quasilinear parabolic problem

$$\begin{cases} u_t = a(\|u_x\|^2)u_{xx} + \nu f(u), & x \in (0,\pi),\ t > 0, \\ u(0,t) = u(\pi,t) = 0, & t \geq 0, \\ u(\cdot,0) = u_0(\cdot) \in H_0^1(0,\pi), \end{cases} \tag{1.1}$$

where $\nu > 0$ is a parameter, $a : \mathbb{R}^+ \to [m, M] \subset (0, +\infty)$ is a continuously differentiable function, $f \in C^2(\mathbb{R})$, with

$$f(0) = 0,\ f'(0) = 1,\ f''(u)u < 0,\ \forall\, u \neq 0,\ \text{and} \tag{1.2}$$
$$\limsup_{|u|\to+\infty} \frac{f(u)}{u} < 0.$$

Here, $\|\cdot\|$ denotes the usual norm in $L^2(0,\pi)$.

We analyze the bifurcation and hyperbolicity of the stationary solutions of (1.1). This analysis has been carried out in [8, 3] for the particular case when a is increasing and f is odd (see also [2] for the study of the bifurcation when f is not necessarily odd). These two conditions considerably simplify the structure of the bifurcations. Indeed, in that case, bifurcations only occur from zero and they are all supercritical pitchfork bifurcations, just like the local case a =const. Here, we prove hyperbolicity and identify the bifurcations in the general case when a is not necessarily increasing and f is not necessarily odd. We will see that, in this general case, besides the supercritical pitchfork bifurcations, subcritical pitchfork bifurcations and saddle-node (subcritical and supercritical) bifurcations may occur.

The proof of hyperbolicity presented here is rather simple compared with the one in [3]. Nonetheless, an important part of the analysis is dependent on the analysis done in [3], where it was established a way to view the quasilinear nonlocal problem (1.1) as a semilinear nonlocal problem. We briefly recall this analysis to take advantage of it.

Consider the auxiliary semilinear nonlocal parabolic problem

$$\begin{cases} w_\tau = w_{xx} + \frac{\nu f(w)}{a(\|w_x\|^2)}, & x \in (0,\pi),\ \tau > 0, \\ w(0,\tau) = w(\pi,\tau) = 0, & \tau \geq 0, \\ w(\cdot,0) = u_0(\cdot) \in H_0^1(0,\pi). \end{cases} \tag{1.3}$$

Proceeding as in [3, 8], it can be shown that (1.3) is locally well-posed and the solutions are jointly continuous with respect to time and initial conditions.

Given a solution $w(x,\tau)$ of (1.3), changing the scale of time to

$$t = \int_0^\tau a(\|w_x(\cdot,\theta)\|^2)^{-1} d\theta,$$

we have that $u(x,t) = w(x,\tau)$ is the unique solution of (1.1). As a consequence, (1.1) is globally well-posed. If we define

$$S(t) : H_0^1(0,\pi) \to H_0^1(0,\pi)$$

by

$$S(t)u_0 = u(t,u_0), \quad t \geq 0,$$

where $u(\cdot,u_0) : \mathbb{R}^+ \to H_0^1(0,\pi)$ is the solution of (1.1), then $\{S(t) : t \geq 0\}$ is a semigroup that has a global attractor $\mathcal{A}$. We say that a continuous function $v : \mathbb{R} \mapsto H_0^1(0,\pi)$ is a global solution for the semigroup $\{S(t) : t \geq 0\}$ if it satisfies $v(t+s) = S(t)v(s)$ for all $t \geq 0$ and for all $s \in \mathbb{R}$. The global attractor can be characterized (see [7]) in terms of the global solutions in the following way

$$\mathcal{A} = \{v(0) : v : \mathbb{R} \to H_0^1(0,\pi) \text{ is a global bounded solution for } \{S(t) : t \geq 0\}\}.$$

In addition, the semigroup $\{S(t) : t \geq 0\}$ is gradient with Lyapunov function given by

$$V(u) = \frac{1}{2} \int_0^{\|u_x\|^2} a(s)ds - \nu \int_0^\pi \int_0^{u(x)} f(s)ds\, dx. \tag{1.4}$$

Denote by $\mathcal{E}$ the set of equilibria of (1.1), that is, the set of solutions of

$$\begin{cases} a(\|\varphi_x\|^2)\varphi_{xx} + \nu f(\varphi) = 0, \ x \in (0,\pi), \\ \varphi(0) = \varphi(\pi) = 0. \end{cases} \tag{1.5}$$

Then (see [7]), for each $u_0 \in H_0^1(0,\pi)$, $S(t)u_0 \overset{t\to+\infty}{\longrightarrow} \mathcal{E}$ and

$$\begin{aligned} \mathcal{A} &= W^u(\mathcal{E}) \\ &= \Big\{ u \in H_0^1(0,\pi) : \text{ there exists a global solution } \xi : \mathbb{R} \to H_0^1(0,\pi) \\ &\qquad \text{satisfying } \xi(0) = u \text{ and } \inf_{\varphi\in\mathcal{E}} \|\xi(t) - \varphi\|_{H_0^1(0,\pi)} \overset{t\to-\infty}{\longrightarrow} 0 \Big\}. \end{aligned}$$

In particular, if $\mathcal{E}$ is finite

$$\mathcal{A} = \bigcup_{\phi\in\mathcal{E}} W^u(\phi). \tag{1.6}$$

Additionally, for any $u_0 \in H_0^1(0,\pi)$ there is a $\phi \in \mathcal{E}$ such that

$$S(t)u_0 \overset{t\to+\infty}{\longrightarrow} \phi$$

and, for any bounded global solution $v : \mathbb{R} \to H_0^1(0,\pi)$ there are $\phi_-, \phi_+ \in \mathcal{E}$ such that

$$\phi_- \overset{t\to-\infty}{\longleftarrow} \xi(t) \overset{t\to+\infty}{\longrightarrow} \phi_+.$$

Let us recall the definitions of local stable and unstable manifolds and the notion of hyperbolicity (see [3]) which applies to (1.1).

Definition 1.1. *Given a neighborhood*

$$\mathcal{V}_\delta(\phi) = \{u \in H_0^1(0,\pi) : \|u - \phi\|_{H_0^1(0,\pi)} < \delta\}$$

of ϕ, the local stable and unstable sets of ϕ associated to $\mathcal{V}_\delta(\phi)$, are given by

$$\begin{aligned} W_{loc}^{s,\delta}(\phi) =&\{u \in H_0^1(0,\pi) : S(t)u \in \mathcal{V}_\delta \text{ for all } t \geq 0, \text{ and } S(t)u \overset{t\to+\infty}{\longrightarrow} \phi\}, \\ W_{loc}^{u,\delta}(\phi) =&\{u \in H_0^1(0,\pi) : \text{ there exists a global solution } v \text{ of } \{S(t)\colon t \geq 0\} \\ &\text{with } v(0) = u,\ v(t) \in \mathcal{V}_\delta \text{ for all } t \leq 0 \text{ and } v(t) \overset{t\to-\infty}{\longrightarrow} \phi\}. \end{aligned}$$

When ϕ is a maximal invariant set in a neighborhood of itself and

$$W_{loc}^{u,\delta}(\phi) = \{\phi\},$$

it is asymptotically stable; otherwise it is unstable. In this case, all solutions that remain in $\mathcal{V}_\delta(\phi)$ for all $t \geq 0$ ($t \leq 0$) must converge forwards (backwards) to ϕ. We refer to this property as topological hyperbolicity (see [1]).

Definition 1.2 (Strict Hyperbolicity, [3])**.** *An equilibrium ϕ of* (1.1) *is said to be* **hyperbolic** *if there are closed linear subspaces X_u and X_s of $H_0^1(0,\pi)$ with $H_0^1(0,\pi) = X_u \oplus X_s$ such that*

- *$\{\phi\}$ is topologically hyperbolic.*
- *The local stable and unstable sets are given as graphs of Lipschitz maps $\theta_u : X_u \to X_s$ and $\theta_s : X_s \to X_u$, with Lipschitz constants L_s, L_u in $(0,1)$ and such that $\theta_u(0) = \theta_s(0) = 0$, and there exists $\delta_0 > 0$ such that, given $0 < \delta < \delta_0$, there are $0 < \delta'' < \delta' < \delta$ such that*

$$\begin{aligned} &\{\phi + (x_u, \theta_u(x_u)) : x_u \in X_u, \|x_u\|_{H_0^1} < \delta''\} \\ &\subset W_{loc}^{u,\delta'}(\phi) \subset \{\phi + (x_u, \theta_u(x_u)) : x_u \in X_u, \|x_u\|_{H_0^1} < \delta\}, \\ &\{\phi + (\theta_s(x_s), x_s) : x_s \in X_s, \|x_s\|_{H_0^1} < \delta''\} \\ &\subset W_{loc}^{s,\delta'}(\phi) \subset \{\phi + (\theta_s(x_s), x_s) : x_s \in X_s, \|x_s\|_{H_0^1} < \delta\}. \end{aligned}$$

Proceeding as in [3], we will show strict hyperbolicity of equilibria for (1.1) in the following way. First, we note that the equilibria of (1.1) and (1.3) are the same. Then we consider the linearization around an equilibrium for the semilinear problem (1.3) and prove its hyperbolicity (showing that zero is not in the spectrum of the linearized self-adjoint nonlocal operator). Then we use the solution dependent change of time scale to conclude the hyperbolicity of equilibria for (1.1).

In [2, 8, 3], the authors proved the following:

Theorem 1.3. *Assume that the function $a(\cdot)$ is increasing and that f is odd. If*

$$a(0)N^2 < \nu \leq a(0)(N+1)^2,$$

then there are $2N+1$ equilibria of the equation (1.1);

$$\{0\} \cup \{\phi_j^{\pm} : j = 1, \ldots, N\},$$

where ϕ_j^+ and ϕ_j^- have $j-1$ zeros in $(0,\pi)$ and $\phi_j^-(x) = -\phi_j^+(x)$, for all $x \in [0,\pi]$, and $\phi_j^+(x) > 0$, for all $x \in (0, \frac{\pi}{j})$. The sequence of bifurcation given above satisfies:

Stability: If $\nu \leq a(0)$, 0 is the only equilibrium of (1.1) and it is stable. If $\nu > a(0)$, the positive equilibrium ϕ_1^+ and the negative equilibrium ϕ_1^- are stable and any other equilibrium is unstable.

Hyperbolicity: For all $\nu > 0$, the equilibria are hyperbolic with the exception of 0 in the cases $\nu = a(0)N^2$, for $N \in \mathbb{N}$.

Our main result in this paper is inspired by [4] and aims to show that the nonlocal diffusion brings many new interesting features to the bifurcation problem.

Consider the one-dimensional scalar local semilinear problem

$$\begin{cases} u_t = u_{xx} + \lambda f(u), & x \in (0,\pi),\ t > 0, \\ u(0,t) = u(\pi,t) = 0, & t \geq 0, \\ u(\cdot,0) = u_0(\cdot) \in H_0^1(0,\pi), \end{cases} \tag{1.7}$$

where $\lambda > 0$ is a parameter, $f \in C^2(\mathbb{R})$ satisfying (1.2).

Note that any equilibrium ψ of (1.1) is also an equilibrium of (1.7) with

$$\lambda = \frac{\nu}{a(\|\psi_x\|^2)}.$$

To proceed, we need to establish and recall a few properties of the equilibria of (1.7) and define an auxiliary function which precisely relates the equilibria of (1.1) and the equilibria of (1.7).

It is well-known (see [4]) that, for each

$$N^2 < \lambda \leqslant (N+1)^2, \quad N \in \mathbb{N},$$

problem (1.7) has exactly $2N+1$ stationary solutions,

$$\{0\} \cup \{\phi^+_{j,\lambda}, \phi^-_{j,\lambda} : j = 1, \dots, N\},$$

where $\phi^\pm_{j,\lambda}$ has $j-1$ zeros in $(0,\pi)$ and

$$\pm(\phi^\pm_{j,\lambda})'(0) > 0, \quad 1 \leqslant j \leqslant N.$$

The following result is proved in Section 2.

Theorem 1.4. *For $j \in \mathbb{N}$ and $\lambda \in (j^2, \infty)$, let $\phi^\pm_{j,\lambda}$ be the two equilibria of* (1.7)*. The functions*

$$(j^2, +\infty) \ni \lambda \mapsto \|(\phi^\pm_{j,\lambda})_x\|^2 \in (0, +\infty)$$

are C^1, strictly increasing and

$$0 \overset{\lambda \to j^2}{\longleftarrow} \|(\phi^\pm_{j,\lambda})_x\|^2 \overset{\lambda \to +\infty}{\longrightarrow} +\infty.$$

Definition 1.5. *For each $j \in \mathbb{N}$ and $r \geqslant 0$, let $\lambda^\pm_{j,r} \in [j^2, +\infty)$ be the unique λ such that $\|(\phi^\pm_{j,\lambda})_x\|^2 = r$. Let $c^\pm_j : [0, +\infty) \to (0, \frac{1}{j^2}]$ be the function defined by $c^\pm_j(r) = \frac{1}{\lambda^\pm_{j,r}}$, for each $r \geqslant 0$.*

The functions $c^\pm_j(\cdot)$ are strictly decreasing and continuously differentiable with

$$\lim_{r \to 0} c^\pm_j(r) = \frac{1}{j^2}.$$

With this functions, we rewrite

$$\begin{cases} (\phi^\pm_{j,\lambda})_{xx} + \lambda f(\phi^\pm_{j,\lambda}) = 0, \\ \phi^\pm_{j,\lambda}(0) = \phi^\pm_{j,\lambda}(\pi) = 0, \end{cases} \tag{1.8}$$

as the following ‘nonlocal’ problem

$$\begin{cases} \nu c^\pm_j(\|(\phi^\pm_{j,\lambda})_x\|^2)(\phi^\pm_{j,\lambda})_{xx} + \nu f(\phi^\pm_{j,\lambda}) = 0, \\ \phi^\pm_{j,\lambda}(0) = \phi^\pm_{j,\lambda}(\pi) = 0. \end{cases} \tag{1.9}$$

Theorem 1.6. *For each $j \in \mathbb{N}$, consider c^+_j and c^-_j as the functions defined above. For $\nu > 0$ and $r > 0$,* (1.1) *has an equilibrium ϕ, with $j-1$ zeros in the interval $(0,\pi)$, such that $\phi_x(0) > 0$ (resp. $\phi_x(0) < 0$) and $\|\phi_x\|^2 = r$ if and only if $\nu c^+_j(r) = a(r)$ (resp. $\nu c^-_j(r) = a(r)$). Furthermore, the equilibrium is hyperbolic if and only if, $a'(r) \neq \nu(c^+_j)'(r)$ (resp. $a'(r) \neq \nu(c^-_j)'(r)$).*

Remark 1.7. Note that, Theorem 1.6 characterizes all equilibria of (1.1). Also, it is only required for the function $a : \mathbb{R}^+ \to [m, M] \subset (0, +\infty)$ to be continuously differentiable, that is, $a(\cdot)$ is not necessarily increasing.

The study of the existence of equilibria requires only the continuity of $a(\cdot)$. The differentiability of $a(\cdot)$ is used to analyze the behavior near the equilibria.

Assuming only that f and a are continuous, $a(s) \geq m$, for all $s \in \mathbb{R}^+$ and that

$$\limsup_{|s|\to+\infty} \frac{f(s)}{s} = \beta < +\infty,$$

it has been proved in [2] that there exists a solution of (1.1) for each $u_0(\cdot) \in H_0^1(0, \pi)$, defined for all $t \geq 0$, and that (1.1) defines a multivalued semiflow. In addition, if a is either non-decreasing or bounded above and f satisfies some growth and dissipativity conditions, the authors show that the multivalued semiflow has a global attractor which is characterized as the unstable set of the equilibria. Under some additional assumptions, it is also proved in [2] that the set of equilibria has at least $2N + 1$ points if $\lambda > a(0)N^2$ and exactly $2N + 1$ if a is non-decreasing and

$$a(0)(N + 1)^2 \geqslant \lambda > a(0)N^2.$$

In this paper, our focus is on the bifurcation and hyperbolicity of equilibria assuming that a and f are smooth and that the function $a(\cdot)$ is not necessarily increasing.

Observe that the diffusion coefficient $a(\|u_x\|^2)$ in (1.1) depends on the L^2 norm of the gradient of the solution. This means that, roughly speaking, if the function $a(\cdot)$ is increasing, then states with large gradients will have large diffusion coefficient and in some sense, the diffusion mechanism is more efficient in trying to smooth out the solution and definitely in stabilizing the system. Therefore, the dynamics, at least in terms of stability of positive equilibria, is expected to be similar to the classical case in which the diffusion does not depend on the state, see [3]. On the other hand, if the function $a(\cdot)$ is decreasing for some range of the parameter, it is possible that the system favors states with large gradients and it may destabilize the system. This is what actually may occur and, as we will see, we may have situations in which some non-sign-changing equilibria may become unstable, see Theorem 4.3 below.

This paper is organized as follows. In Section 2, we study fine properties of the solutions of the Chafee-Infante model (1.7). In Section 3, we explain how solutions of (1.1) can be retrieved from the solutions of (1.7). In Section 4, we give a full characterization of the bifurcations as a function of the parameter λ and of the function a. We also characterize the exact points where we may lose hyperbolicity of equilibria. Finally, in Section 5, we show some examples to exhibit the variety of behaviors one may identify for different functions a.

2. Properties of equilibria for the Chafee-Infante model

Problem (1.7) is known as the Chafee-Infante problem and is a very well-studied nonlinear dynamical system. In fact, we can say that it is the best understood example in the literature referring to the characterization of a non-trivial attractor of an infinite dimensional problem. Chafee and Infante started the description of the attractor of (1.7) in [5, 4] by showing that the problem admits only a finite number of equilibria which bifurcate from zero as the parameter $\lambda > 0$ increases. Also, these equilibria are all hyperbolic, with the exception of the zero equilibrium for $\lambda = N^2$, for $N \in \mathbb{N}$.

Remark 2.1. The dissipativity condition (1.2) can be relaxed (with very little changes) to include the possibility that inequality is not strict. We chose to keep the analysis as simple as possible.

Theorem 2.2. *For each $\lambda \in (N^2, +\infty)$, $N \in \mathbb{N}$, problem* (1.7) *admits exactly two equilibria $\phi^+_{N,\lambda}$ and $\phi^-_{N,\lambda}$ that vanish exactly $N-1$ times in the interval $(0, \pi)$ and such that*

$$(\phi^+_{N,\lambda})'(0) > 0 \quad \textit{and} \quad (\phi^-_{N,\lambda})'(0) < 0.$$

Hence, if $\lambda \in (N^2, (N+1)^2]$, then (1.7) *admits exactly the following $2N+1$ equilibria:*

$$\{0\} \cup \{\phi^+_{j,\lambda}, \phi^-_{j,\lambda} : j = 1, \dots, N\}.$$

For each $1 \leq j \leq N$, the linear operator

$$L^{\lambda,\pm}_j : H^2(0,\pi) \cap H^1_0(0,\pi) \subset L^2(0,\pi) \to L^2(0,\pi)$$

defined by

$$L^{\lambda,\pm}_j u = u_{xx} + \lambda f'(\phi^\pm_{j,\lambda})u, \quad u \in H^2(0,\pi) \cap H^1_0(0,\pi),$$

is a self-adjoint unbounded operator with compact resolvent. All eigenvalues of $L^{\lambda,\pm}_j$ are simple, zero is not an eigenvalue and exactly $j-1$ eigenvalues are positive, $j = 1, \dots, N$.

If j is a positive integer, many properties of the equilibrium $\phi^{\pm}_{j,\lambda}$ of (1.7) are proved using the properties of the time maps, which we briefly recall for later use.

Since, for $\lambda \in (j^2, +\infty)$, $\phi^{\pm}_{j,\lambda}$ are the solutions of (1.7) with $j-1$ zeros in the interval $(0,\pi)$ and

$$i(\phi^{i}_{j,\lambda})'(0) > 0, \quad i \in \{+,-\},$$

they are solutions of the initial value problem

$$\begin{aligned} &u_{xx} + \lambda f(u) = 0, \ x > 0, \\ &u(0) = 0, \ u'(0) = v_0, \end{aligned} \tag{2.1}$$

where $v_0 > 0$ is suitably chosen in such a way that $u(\pi) = 0$. For a given v_0, let

$$\lambda E = \frac{v_0^2}{2} \in [0, \min\{F(z^+), F(z^-)\}],$$

where

$$F(u) = \int_0^u f(s)ds,$$

z^+ (resp. z^-) is the positive (resp. negative) zero of f, and note that a solution of (2.1) must satisfy

$$\frac{u'(x)^2}{2} + \lambda F(u) = \lambda E.$$

Let $U^+(E) > 0$ and $U^-(E) < 0$ be defined as the unique numbers in $[0, z^+]$ and $[z^-, 0]$, respectively, with $F(U^{\pm}(E)) = E$. Then, if

$$\tau^i_\lambda(E) = i\left(\frac{2}{\lambda}\right)^{\frac{1}{2}} \int_0^{U^i(E)} (E - F(u))^{-\frac{1}{2}} du, \ i \in \{+,-\}, \tag{2.2}$$

we have, for j odd,

$$\begin{aligned} \mathcal{T}^+_\lambda(E) &= \frac{j+1}{2}\tau^+_\lambda(E) + \frac{j-1}{2}\tau^-_\lambda(E), \\ \mathcal{T}^-_\lambda(E) &= \frac{j+1}{2}\tau^-_\lambda(E) + \frac{j-1}{2}\tau^+_\lambda(E) \end{aligned}$$

or, for j even,

$$\mathcal{T}^{\pm}_\lambda(E) = \frac{j}{2}\tau^+_\lambda(E) + \frac{j}{2}\tau^-_\lambda(E).$$

The choices of E that gives us the solutions $\phi^+_{j,\lambda}$ are $\mathcal{T}^+_\lambda(E^+_j(\lambda)) = \pi$.

For completeness we give a simple proof that the equilibria of the Chafee-Infante equation (1.7) are all hyperbolic (see [9, Section 24F]) with the only exception being the equilibrium $\phi_0 \equiv 0$ and exactly when $\lambda = N^2$, N a positive integer. This shows, in particular, that bifurcations only occur from the ϕ_0.

We prove only the hyperbolicity of $\phi^+_{j,\lambda}$, the other case is similar. We consider the family $u(\cdot, E)$ of solutions of the problem

$$\begin{aligned} &u''(x) + \lambda f(u(x)) = 0, \\ &u(0,E) = 0,\ u'(0,E) = \sqrt{2\lambda E} \ \text{ and } \ u(\tau^+_\lambda(E)) = 0. \end{aligned} \tag{2.3}$$

Consequently, $\eta = (\phi^+_{j,\lambda})_x$ and $\psi = \frac{\partial u}{\partial E}(x,E)\big|_{E=E^+_j(\lambda)}$ are solutions of

$$v''(x) + \lambda f'(\phi^+_{j,\lambda})v(x) = 0 \tag{2.4}$$

with

$$\eta(0) \neq 0,\ \eta'(0) = 0\ \psi(0) = 0,\ \psi'(0) = \frac{\sqrt{\lambda}}{\sqrt{2E^+_j(\lambda)}} \neq 0.$$

This proves that η and ψ are linearly independent and any solution of (2.4) must be of the form

$$\omega = c_1\eta + c_2\psi, \quad \text{for } c_1, c_2 \in \mathbb{R}.$$

Let us show that if

$$\omega(0) = \omega(\mathcal{T}^+_\lambda(E^+_j(\lambda))) = 0,$$

then, necessarily, $w \equiv 0$. In fact, $\psi(0) = 0$, $\eta(0) \neq 0$ and $c_1\eta(0) + c_2\psi(0) = 0$ implies $c_1 = 0$. Now, since $u(\mathcal{T}^+_\lambda(E), E) = 0$ for all E, we have that

$$0 = \frac{\partial u}{\partial x}(\mathcal{T}^+_\lambda(E), E)(\mathcal{T}^+_\lambda(E))'(E) + \frac{\partial u}{\partial E}(\mathcal{T}^+_\lambda(E), E).$$

It is clear that

$$\frac{\partial u}{\partial x}(\mathcal{T}^+_\lambda(E), E) \neq 0$$

and since that $(\mathcal{T}^+_\lambda(E))'(E) \neq 0$ (see [4]), we have that

$$\psi(\mathcal{T}^+_\lambda(E^+_j(\lambda))) = \frac{\partial u}{\partial E}(\mathcal{T}^+_\lambda(E^+_j(\lambda), E^+_j(\lambda)) \neq 0.$$

Hence, we also have that $c_2 = 0$ and the only solution ω of (2.4) which satisfies $\omega(0) = \omega(\pi) = 0$ is $\omega \equiv 0$. This proves that 0 is not in the spectrum of the linearization around ϕ.

Now, we study the properties of the functions

$$(j^2, +\infty) \ni \lambda \mapsto \phi^\pm_{j,\lambda} \in H^1_0(0,\pi), \quad j = 1, 2, 3 \cdots,$$

proving Theorem 1.4.

Proof of Theorem 1.4. Consider $i \in \{+,-\}$. To show that

$$(j^2,+\infty) \ni \lambda \mapsto \phi^i_{j,\lambda} \in H^1_0(0,\pi)$$

is continuously differentiable at a point $\lambda_0 \in (j^2,+\infty)$, we recall that, for each $\lambda \in (j^2,+\infty)$, we already know that $\phi^i_{j,\lambda}$ is hyperbolic. Hence, for λ near λ_0, $\phi^i_{j,\lambda} = \phi^i_{j,\lambda_0} + v$, where v is the only fixed point of the map

$$T^i_{j,\lambda} v := -\phi^i_{j,\lambda_0} - (L^{\lambda_0,i}_j)^{-1}\left(\lambda f(v+\phi^i_{j,\lambda_0}) - \lambda_0 f'(\phi^i_{j,\lambda_0})v - \lambda_0 f'(\phi^i_{j,\lambda_0})\phi^i_{j,\lambda_0}\right)$$

in a small neighborhood of zero in $H^1_0(0,\pi)$. Now, since

$$(j^2,+\infty) \ni \lambda \mapsto T^i_{j,\lambda} \in \mathcal{C}(H^1_0(0,\pi))$$

is continuously differentiable, we have that

$$(j^2,+\infty) \ni \lambda \mapsto \phi^i_{j,\lambda} \in H^1_0(0,\pi)$$

is continuously differentiable and the result follows.

The proof that

$$(j^2,+\infty) \ni \lambda \mapsto \|(\phi^i_{j,\lambda})_x\|^2 \in (0,+\infty)$$

is strictly increasing and that $\|(\phi^i_{j,\lambda})_x\|^2 \overset{\lambda\to+\infty}{\longrightarrow} +\infty$ follows from the results in [2, Lemma 5] and from the analysis done next.

It has been shown in [4] that the time maps $\tau^\pm_\lambda(\cdot)$, defined in (2.2), are strictly increasing functions. Clearly, for a fixed E, the functions $\lambda \mapsto \tau^\pm_\lambda(E)$ are strictly decreasing. Since $\mathcal{T}^i_\lambda(E^i_j(\lambda)) = \pi$, we must have that $iU^i(E^i_j(\lambda))$ is strictly increasing, $i \in \{+,-\}$.

It follows that

$$g(\lambda) := \int_0^{\tau^+_\lambda(E^\pm_j(\lambda))} ((\phi^\pm_{j,\lambda})_x)^2 dx = \sqrt{2\lambda}\int_0^{U^+(E^\pm_j(\lambda))} \sqrt{E^\pm_j(\lambda) - F(v)}dv$$

and

$$\int_0^{U^+(E^\pm_j(\lambda))} \sqrt{E^\pm_j(\lambda) - F(v)}dv$$

is a strictly increasing function of λ. Consequently, $g(\lambda) \overset{\lambda\to+\infty}{\longrightarrow} +\infty$ and we must have that $\|(\phi^i_{j,\lambda})_x\|^2 \overset{\lambda\to+\infty}{\longrightarrow} +\infty$, completing the proof. □

Let us consider an alternative simple direct proof of this theorem without using the differentiability results for fixed points. First, we show that $(j^2,+\infty) \ni \lambda \mapsto \phi^i_{j,\lambda} \in H^1_0(0,\pi)$ is continuous in $H^1_0(0,\pi)$ (or $C^1(0,\pi)$). For simplicity of notation, we will write ϕ_λ for $\phi^i_{j,\lambda}$.

Let us show that if $\lambda_n \to \lambda_0 \in (j^2, +\infty)$, we must have $\|\phi_{\lambda_n} - \phi_{\lambda_0}\|_{H_0^1(0,\pi)} \to 0$. Since

$$(\phi_\lambda)_{xx}(r) + \lambda f(\phi_\lambda) = 0 \tag{2.5}$$

and using the dissipativity condition in (1.2), we have that there is a constant $M > 0$ such that

$$\int_0^\pi ((\phi_\lambda)_x)^2 dx = \lambda \int_0^\pi f(\phi_\lambda)\phi_\lambda dx \le \lambda M.$$

Therefore, the function $(j^2, +\infty) \ni \lambda \mapsto \phi_\lambda \in H_0^1(0,\pi)$ is bounded in bounded subsets of $(j^2, +\infty)$ and, since $H_0^1(0,\pi) \hookrightarrow C([0,\pi])$, the function $(j^2, +\infty) \ni \lambda \mapsto \phi_\lambda \in C([0,\pi])$ is bounded in bounded subsets of $(j^2, +\infty)$. From the continuity of f, the same is true for $(j^2, +\infty) \ni \lambda \mapsto f \circ \phi_\lambda \in C([0,\pi])$ and, using (2.5), for $(j^2, +\infty) \ni \lambda \mapsto (\phi_\lambda)_{xx} \in C([0,\pi])$.

It follows from the compact embedding of $H^2(0,\pi)$ into $H_0^1(0,\pi)$ that there is a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ of $\{\lambda_n\}_{n\in\mathbb{N}}$ such that $\phi_{\lambda_{n_k}} \overset{k\to+\infty}{\longrightarrow} w$ in $H_0^1(0,\pi)$. Now, since

$$\int_0^\pi (\phi_{\lambda_{n_k}})_x v_x dx = \lambda_n \int_0^\pi f(\phi_{\lambda_{n_k}}) v dx,$$

for all $v \in H_0^1(0,\pi)$, passing to the limit as $k \to +\infty$, we have that

$$\int_0^\pi w_x v_x dx = \lambda_0 \int_0^\pi f(w) v dx,$$

and w is a weak solution of (2.5). Hence, since w also converges in the $C^1(0,\pi)$ norm, $w \equiv 0$ or $w = \phi_{\lambda_0}$. To see that $w \not\equiv 0$, we recall that

$$(j^2, +\infty) \ni \lambda \mapsto \int_0^\pi ((\phi_\lambda)_x)^2$$

is a strictly increasing function of λ. This shows the continuity of the function $(j^2, +\infty) \ni \lambda \mapsto \phi_\lambda \in H_0^1(0,\pi)$.

Let us now prove that

$$(j^2, +\infty) \ni \lambda \mapsto \phi_\lambda \in H_0^1(0,\pi) \quad \text{or} \quad C_0^1(0,\pi)$$

is continuously differentiable. Fix $\lambda \in (j^2, +\infty)$ and consider $\delta > 0$ such that $\lambda + h \in (j^2, +\infty)$, for all $h \in (-\delta, \delta)$. Denote $w(h) = \frac{\phi_{\lambda+h} - \phi_\lambda}{h}$.

Now,

$$\begin{aligned} w(h)_{xx} + f(\phi_{\lambda+h}) + \lambda\Big(\frac{f(\phi_{\lambda+h}) - f(\phi_\lambda)}{h}\Big) \\ = w(h)_{xx} + f(\phi_{\lambda+h}) + \lambda f'(\theta\phi_{\lambda+h} + (1-\theta)\phi_\lambda) w(h) = 0 \end{aligned}$$

and
$$\|w(h)_x\|^2 \leq C\|w(h)\|^2 + C.$$
Proceeding as before, we show that $\{w(h) : h \in (0,1]\}$ is uniformly bounded in $H_0^1(0,\pi)$ and so it is in $L^2(0,\pi)$ and $C(0,\pi)$.

Hence, using that $f \in C^2(\mathbb{R})$, $h \in (0,\delta]$, we must have that
$$\sup_{h\in(0,\delta]} \sup_{y\in[0,\pi]} |w(h)_{xx}(y)| < +\infty.$$
Therefore, the sequence $\{w(h) : h \in (0,1]\}$ is uniformly bounded in $H^2(0,\pi)$. Hence, we may assume that $w(h) \to \bar{w}$ in $H_0^1(0,\pi)$ (so as $C^1[0,\pi]$) as $h \to 0$.

Since, for all $v \in H_0^1(0,\pi)$, $h \in (0,\delta)$, we have
$$\int_0^\pi w(h)_x v_x = \int_0^\pi f(\phi_{\lambda+h})v + \lambda \int_0^\pi \lambda f'(\theta\phi_{\lambda+h} + (1-\theta)\phi_\lambda)w(h)v$$
we find, using the continuity of the function
$$(j^2, +\infty) \ni \lambda \mapsto \phi_\lambda \in H_0^1(0,\pi),$$
that
$$-\langle \bar{w}_x, v_x\rangle + \lambda \left\langle f'(\phi_\lambda)\bar{w}, v\right\rangle + \langle f(\phi_\lambda), v\rangle = 0, \tag{2.6}$$
for all $v \in H_0^1(0,\pi)$. That is, $\bar{w}$ is the only solution of
$$\begin{cases} u_{xx} + \lambda f'(\phi_\lambda)u + f(\phi_\lambda) = 0 \\ u(0) = u(\pi) = 0. \end{cases}$$
From this, we have the differentiability of the function
$$(j^2, +\infty) \ni \lambda \mapsto \phi_\lambda \in H_0^1(0,\pi).$$

Remark 2.3. The same reasoning can be used to show that $(j^2, +\infty) \ni \lambda \mapsto \phi_\lambda \in H_0^1(0,\pi)$ is twice continuously differentiable.

Having proved Theorem 1.4, we obtain that the functions $c_j^\pm : [0,+\infty) \to (0, \frac{1}{j^2}]$ given in Definition 1.5 are continuously differentiable, strictly decreasing functions with
$$\lim_{r\to 0} c_j^\pm(r) = \frac{1}{j^2}.$$
For simplicity of notation, we will write $c(\cdot)$ to denote one of the two functions $c_j^\pm(\cdot)$ and ϕ_λ to denote $\phi_{j,\lambda}^\pm$. Rewriting (1.9) as
$$\begin{cases} [\phi_\lambda]_{xx} + \dfrac{1}{c_j(\|[\phi_\lambda]_x\|^2)} f(\phi_\lambda) = 0, \\ \phi_\lambda(0) = \phi_\lambda(\pi) = 0. \end{cases} \tag{2.7}$$

Observe that (see Definition 1.5), for $r > 0$,

- $\|[\phi_{\lambda_r}]_x\|^2 = r$;
- $[\phi_{\lambda_r}]_{xx} + \frac{f(\phi_{\lambda_r})}{c(r)} = 0$.

Making $\psi(r) = \phi_{\lambda_r}$, differentiating with respect to r, and representing $\frac{d\psi(r)}{dr} = \dot{\psi}(r)$, we find

$$\dot{\psi}_{xx}(r) + \frac{f'(\psi(r))\dot{\psi}(r)}{c(\|(\psi(r))_x\|^2)} - \frac{f(\psi(r))c'(\|(\psi(r))_x\|^2)}{[c(\|(\psi(r))_x\|^2)]^2}\frac{d}{dr}\|(\psi(r))_x\|^2 = 0.$$

Now, since

$$\begin{aligned}\frac{d}{dr}\|(\psi(r))_x\|^2 &= 2\left\langle(\psi(r))_x, (\dot{\psi}(r))_x\right\rangle = -2\left\langle(\psi(r))_{xx}, \dot{\psi}(r)\right\rangle \\ &= \frac{2}{c(\|(\psi(r))_x\|^2)}\left\langle f(\psi(r)), \dot{\psi}(r)\right\rangle\end{aligned}$$

we may write

$$\dot{\psi}_{xx}(r) + \frac{f'(\psi(r))\dot{\psi}(r)}{c(\|(\psi(r))_x\|^2)} - \frac{2c'(\|(\psi(r))_x\|^2)}{c(\|(\psi(r))_x\|^2)^3}f(\psi(r))\left\langle f(\psi(r)), \dot{\psi}(r)\right\rangle = 0.$$

Now, if $L_c : D(L_c) \subset L^2(0,\pi) \to L^2(0,\pi)$ is given by

$$D(L_c) = H^2(0,\pi) \cap H^1_0(0,\pi)$$

and

$$L_c v = v'' + \frac{f'(\psi(r))}{c(\|(\psi(r))_x\|^2)}v - \frac{2c'(\|(\psi(r))_x\|^2)}{c(\|(\psi(r))_x\|^2)^3}f(\psi(r))\int_0^\pi f(\psi(r))v,$$

$v \in D(L_c)$, we have that $L_c\dot{\psi}(r) = 0$ and, since $\dot{\psi}(r) \in H^2(0,\pi) \cap H^1_0(0,\pi)$, it follows that $0 \in \sigma(L_c)$.

3. Identifying the equilibria of the nonlocal problem

Let us study the sequence of bifurcations for the nonlocal problem (1.1).

Theorem 3.1. *For each positive integer j, consider $c^+_j(\cdot)$ and $c^-_j(\cdot)$, the two maps defined above. For $\nu > 0$ and $r > 0$, (1.1) has an equilibrium ψ, with $j-1$ zeros in the interval $(0,\pi)$ such that $(\psi)_x(0) > 0$ (resp. $(\psi)_x(0) < 0$) and $\|\psi_x\|^2 = r$ if and only if $\nu c^+_j(r) = a(r)$ (resp. $\nu c^-_j(r) = a(r)$).*

Proof. If ψ is an equilibrium of (1.1), with $j-1$ zeros in the interval $(0,\pi)$ such that $\psi_x(0) > 0$ and $\|\psi_x\|^2 = r$, then

$$\psi = \phi^+_{j,\lambda^+_{j,r}} \quad \text{and} \quad \nu c^+_j(r) = a(r).$$

On the other hand, since $\phi^+_{j,\lambda^+_{j,r}}$ is a solution of (1.9) with $\|(\phi^+_{j,\lambda^+_{j,r}})_x\|^2 = r$ and since $\nu c^+_j(r) = a(r)$, $\psi = \phi^+_{j,\lambda^+_{j,r}}$ is an equilibrium of (1.1). A similar argument is used for c^-_j. □

Remark 3.2. For each positive integer k, if $\nu > k^2 a(0)$ there are at least $2k+1$ equilibria of the non-local problem (1.1). That is an immediate consequence of the fact that the functions

$$c^\pm_j : [0, +\infty) \to (0, \frac{1}{j^2}]$$

are continuous,

$$\nu c^\pm_j(0) = \frac{\nu}{j^2} > a(0), \quad c^\pm_j(r) \overset{r\to+\infty}{\longrightarrow} 0,$$

$1 \leq j \leq k$, and $a : [0, +\infty) \to [m, M]$ is continuous. In particular, if a is non-decreasing we have exactly $2k + 1$ equilibria of (1.1).

4. The hyperbolicity and Morse Index of equilibria

Consider the auxiliary initial boundary value problem (1.3) related to (1.1) by a solution dependent change of the time scale. As we have mentioned before, both problems have exactly the same equilibria and, as in [3], the spectral analysis of the self-adjoint operator associated to the linearization of (1.3) around an equilibrium ψ will determine its stability and hyperbolicity properties.

The linearization of (1.3) around an equilibrium ψ is given by the equation

$$v_t = Lv,$$

where $D(L) = H^2(0, \pi) \cap H^1_0(0, \pi)$ and

$$Lv = v'' + \frac{\nu f'(\psi)}{a(\|\psi_x\|^2)} v - \frac{2\nu^2 a'(\|\psi_x\|^2)}{a(\|\psi_x\|^2)^3} f(\psi) \int_0^\pi f(\psi) v, \ \ v \in D(L).$$

Given an equilibrium $\psi \neq 0$ of (1.3), let $r = \|\psi_x\|^2$ and let k be the positive integer such that ψ vanishes $k - 1$ times in the interval $(0, \pi)$. If $\psi_x(0) > 0$ and we consider

$$\lambda^+_{k,r} = (c^+_k(r))^{-1},$$

then $\psi = \phi^+_{k,\lambda^+_{k,r}}$. Similarly, if $\psi_x(0) < 0$ and we consider

$$\lambda^-_{k,r} = (c^-_k(r))^{-1},$$

then $\psi = \phi^-_{k,\lambda^-_{k,r}}$. For simplicity of notation, we will write $c(\cdot)$ instead of $c^\pm_k(\cdot)$, λ_r instead of $\lambda^\pm_{k,r}$ and ϕ_{λ_r} instead of $\phi^\pm_{k,\lambda^\pm_{k,r}}$ for the remainder of this section.

4.1. **Hyperbolicity.** This section is concerned with the characterization of hyperbolicity for the equilibria of (1.3) given by the theorem below.

Theorem 4.1. *With the notation above, the equilibrium ψ of* (1.3) *is not hyperbolic if, and only if,*

$$a'(\|\psi_x\|^2) = \nu c'(\|\psi_x\|^2).$$

Proof. $(\Leftarrow)$ Suppose initially that

$$a'(\|\psi_x\|^2) = \nu c'(\|\psi_x\|^2).$$

Let $r = \|\psi_x\|^2$. In the notation above, we have that $\psi = \phi_{\lambda_r}$.

Recall that, as we have seen at the end of Section 2, $v = \frac{d}{dr}\phi_{\lambda_r}$ satisfies

$$v_{xx} + \frac{f'(\phi_{\lambda_r})}{c(r)}v - \frac{2c'(r)}{c(r)^3}f(\phi_{\lambda_r})\int_0^\pi f(\phi_{\lambda_r})v = 0.$$

From Theorem 3.1, we have that $a(r) = \nu c(r)$. Since, $\|\psi_x\|^2 = r$, $\psi = \phi_{\lambda_r}$ and $a'(r) = \nu c'(r)$, we have

$$v_{xx} + \frac{\nu f'(\psi)}{a(\|\psi_x\|^2)}v - \frac{2\nu^2 a'(\|\psi_x\|^2)}{a(\|\psi_x\|^2)^3}f(\psi)\int_0^\pi f(\psi)v = 0.$$

Therefore, 0 is an eigenvalue of L, which implies that ψ is not a hyperbolic equilibrium.

$(\Rightarrow)$ Assume that we find a $0 \neq u \in H^2(0,\pi) \cap H^1_0(0,\pi)$ satisfying

$$u_{xx} + \frac{\nu f'(\psi)}{a(\|\psi_x\|^2)}u - \frac{2\nu^2 \alpha a'(\|\psi_x\|^2)}{a(\|\psi_x\|^2)^3}f(\psi) = 0$$

for

$$\alpha = \int_0^\pi f(\psi(s))u(s)ds.$$

Now, since $a(r) = \nu c(r)$ and $\phi_{\lambda_r} = \psi$, $v = \frac{d}{dr}\phi_{\lambda_r}$ satisfies

$$v_{xx} + \frac{\nu f'(\psi)}{a(\|\psi_x\|^2)}v - \frac{2\nu^3 \beta c'(\|\psi_x\|^2)}{a(\|\psi_x\|^2)^3}f(\psi) = 0,$$

for

$$\beta = \int_0^\pi f(\psi(s))v(s)ds.$$

Consequently, $w = \beta\nu c'(r)u - \alpha a'(r)v$ is the solution of

$$\begin{cases} w_{xx} + \dfrac{\nu f'(\phi_{\lambda_r})}{a(\|\phi_{\lambda_r}\|^2)} w = 0 \text{ or } w_{xx} + \lambda_r f'(\phi_{\lambda_r}) w = 0, \\ w(0) = w(\pi) = 0, \end{cases} \tag{4.1}$$

which means $w \equiv 0$. Thus,

$$\beta\nu c'(r)u - \alpha a'(r)v = 0$$

and, by multiplying both sides of equality by $f(\phi)$ and integrating from 0 to π, we find

$$\alpha\beta\nu c'(r) = \alpha\beta a'(r).$$

Clearly, $\alpha\beta \neq 0$. Otherwise, either u or v should be a solution of (4.1), that is, either $u = 0$ or $v = 0$, which would be a contradiction.

Therefore, we conclude that $a'(r) = \nu c'(r)$. □

4.2. **Morse Index.** Now, we analyze what happens to the dimension of the unstable manifolds for the equilibria of (1.3) as they bifurcate.

For $\varepsilon \in \mathbb{R}$, define the operator

$$L_\varepsilon : H^2(0,\pi) \cap H_0^1(0,\pi) \subset L^2(0,\pi) \to L^2(0,\pi)$$

by

$$L_\varepsilon u(x) = u'' + p(x)u + \varepsilon q(x) \int_0^\pi q(s)u(s)ds,$$

where $p, q : [0,\pi] \to \mathbb{R}$ are continuous functions with $q \not\equiv 0$.

When $\varepsilon = 0$, $L_0 u = u'' + p(x)u$ is a Sturm-Liouville operator. Hence, L_0 is a self-adjoint with compact resolvent and its spectrum consists of a decreasing sequence of simple eigenvalues, that is,

$$\sigma(L_0) = \{\gamma_j : j = 1, 2, 3 \cdots\}$$

with, $\gamma_j > \gamma_{j+1}$ and $\gamma_j \longrightarrow -\infty$ as $j \to +\infty$.

Note that, for all $\varepsilon \in \mathbb{R}$, we can decompose L_ε as a sum of two operators:

$$L_\varepsilon u = L_0 u + \varepsilon B u,$$

where

$$Bu = q(x) \int_0^\pi q(s)u(s)ds,$$

for all $u \in H^2(0,\pi) \cap H_0^1(0,\pi)$, is a bounded operator with rank one. It is easy to see that L_ε is also self-adjoint with compact resolvent.

Then, we write $\{\mu_j(\varepsilon) : j = 1, 2, 3, \cdots\}$ to represent the eigenvalues of L_ε, ordered in such a way that, for $j = 1, 2, 3, \cdots$, the function $\mathbb{R} \ni \varepsilon \mapsto \mu_j(\varepsilon) \in \mathbb{R}$ satisfies $\mu_j(0) = \gamma_j$.

Throughout this paper, we will use, in an essential way, Theorems 3.4 and 4.5 of [6], which are summarized in the next result.

Theorem 4.2. *Let L_ε and $\{\mu_j(\varepsilon) : j = 1, 2, 3, \cdots\}$ be as above. The following holds:*

- *i) For all $j = 1, 2, 3, \cdots$, the function $\mathbb{R} \ni \varepsilon \mapsto \mu_j(\varepsilon) \in \mathbb{R}$ is non-decreasing.*
- *ii) If for some $j = 1, 2, 3 \cdots$, and $\varepsilon \in \mathbb{R}$, $\mu_j(\varepsilon) \notin \{\gamma_k : k = 1, 2, 3, \cdots\}$, then $\mu_j(\varepsilon)$ is a simple eigenvalue of L_ε.*

We wish to determine the Morse Index of the equilibria of (1.3) by looking carefully to the points where the graphs of the functions $a(\cdot)$ and $\nu c(\cdot)$ intercept, that is, depending on how these curves intersect, we will be able to determine the Morse Index of the equilibria. Recall that the function $c(\cdot)$ is in fact $c_j^+(\cdot)$ or $c_j^-(\cdot)$ which is associated to an equilibrium ϕ^+_{j,λ_r^+} or ϕ^-_{j,λ_r^-} which change sign $j-1$ times in the interval $(0, \pi)$. The intersection of the graphs of $a(\cdot)$ and $\nu c_j^\pm(\cdot)$ necessarily gives rise to an equilibrium that changes sign $j-1$ times for (1.1). Hence, as ν increases, if the first intersection between the graphs of $a(\cdot)$ and $\nu c_j^\pm(\cdot)$ happens with a value of $r \neq 0$ and before the intersection with $r = 0$, we must have at least one saddle-node bifurcation that precedes the pitchfork bifurcation from zero (see, for instance, Example 5.3).

This is the main result of this section:

Theorem 4.3. *Suppose that ψ is an equilibrium of (1.3) with $k-1$ zeros in $(0, \pi)$ for some positive integer k. Let $r = \|\psi_x\|^2$ and λ_r such that $\psi = \phi^+_{k,\lambda_r}$ (resp. $\psi = \phi^-_{k,\lambda_r}$). If we denote $c_k^\pm(\cdot)$ by $c(\cdot)$, then*

(i) If

$$a'(\|\psi_x\|^2) > \nu c'(\|\psi_x\|^2),$$

then ψ is hyperbolic and its Morse index is $k-1$.

(ii) If

$$a'(\|\psi_x\|^2) < \nu c'(\|\psi_x\|^2),$$

then ψ is hyperbolic and its Morse index is k.

Proof. The hyperbolicity follows from Theorem 4.1. Define the operator

$$L_\varepsilon v = v'' + \frac{\nu f'(\psi)}{a(\|\psi'\|^2)} v + \varepsilon f(\psi) \int_0^\pi f(\psi) v,$$

$v \in D(L_\varepsilon) = H^2(0,\pi) \cap H_0^1(0,\pi)$, for each $\varepsilon > 0$.

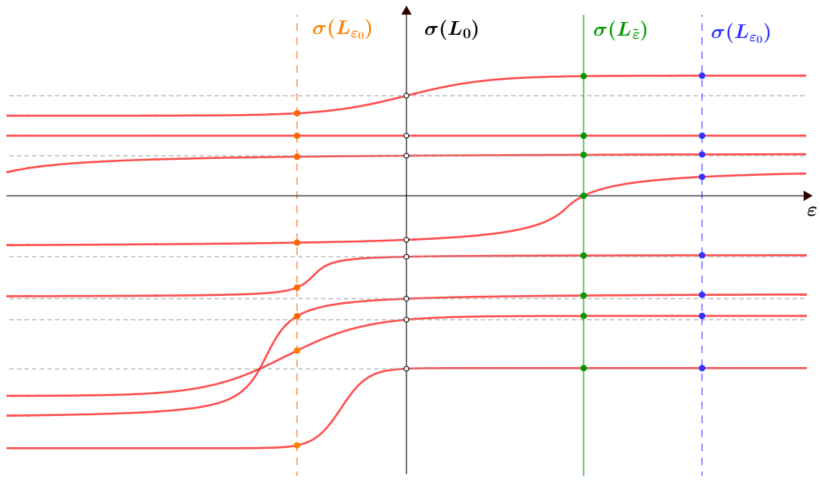


FIGURE 1. Spectrum of L_ϵ

Note that L_0 is the linearization of (1.7) at ψ for the parameter

$$\nu_0 = \frac{\nu}{a(\|\psi_x\|^2)}.$$

The spectrum of L_0 is given by an unbounded ordered sequence $\{\mu_j(0)\}_{j\in\mathbb{N}}$ of simple eigenvalues, that is,

$$\mu_1(0) > \mu_2(0) > \cdots > \mu_{k-1}(0) > \mu_k(0) > \mu_{k+1}(0) > \dots$$

Since $0 \notin \sigma(L_0)$, we may have that $0 > \mu_j(0)$, for all $j = 1, 2, 3 \cdots$, or there is a positive integer k such that $\mu_{k-1}(0) > 0 > \mu_k(0)$.

For

$$\tilde{\varepsilon} = -\frac{2c'(\|\psi_x\|^2)}{c(\|\psi_x\|^2)^3} = -\frac{2\nu^3 c'(\|\psi_x\|^2)}{a(\|\psi_x\|^2)^3},$$

we have $0 \in \sigma(L_{\tilde{\varepsilon}})$ and 0 is a simple eigenvalue, by Theorem 4.2. Using the same reasoning applied in the proof of the second part of Theorem 4.1, we can show that if $0 \in \sigma(L_\varepsilon)$, then $\varepsilon = \tilde{\varepsilon}$.

Using Theorem 4.2, part *i*), we deduce that $\mu_j(\varepsilon) > 0$, $j = 1, \cdots, k-1$, for all $\varepsilon \geq 0$. Since $0 \in \sigma(L_\varepsilon)$ if and only if $\varepsilon = \tilde{\varepsilon} > 0$, we must have that $\mu_j(\varepsilon) > 0$, $j = 1, \cdots, k-1$, for all $\varepsilon < 0$. That means at least $k-1$ eigenvalues are positive, for all $\varepsilon \in \mathbb{R}$.

Also, since $\mu_k(0) > \mu_j(0)$, for all $j > k$, $\mu_k(\cdot)$ is increasing and L_0 does not have an eigenvalue in the interval $(\mu_k(0), 0]$ we have that $\mu_k(\tilde{\varepsilon}) = 0$. Otherwise $\mu_j(\varepsilon) = \mu_k(\varepsilon) \in (\mu_k(0), 0]$, for some $j > k$ and $\varepsilon \in (0, \tilde{\varepsilon}]$, which is not possible by Theorem 4.2, part *ii*). Since $0 \notin \sigma(L_0)$, $\mu_j(\varepsilon) < 0$ for all $\varepsilon \in \mathbb{R}$ and $j > k$.

As a consequence, the number of positive eigenvalues of L_ε is $k-1$ if $\varepsilon < \tilde{\varepsilon}$ and k if $\varepsilon > \tilde{\varepsilon}$ (see Figure 1).

Let

$$\varepsilon_0 = -\frac{2\nu^2 a'(\|\psi_x\|^2)}{a(\|\psi_x\|^2)^3}.$$

Then, if

$$a'(\|\psi_x\|^2) > \nu c'(\|\psi_x\|^2),$$

we have that $\varepsilon_0 < \tilde{\varepsilon}$ and L_{ε_0} has exactly $k-1$ positive eigenvalues and $0 \notin \sigma(L_{\varepsilon_0})$. On the other hand, if $a'(\|\psi_x\|^2) < \nu c'(\|\psi_x\|^2)$, we have $\varepsilon_0 > \tilde{\varepsilon}$ and L_{ε_0} has exactly k positive eigenvalues and $0 \notin \sigma(L_{\varepsilon_0})$. □

5. Analyzing the attractor for a few examples

Denote by $c^L_{j,\pm}(\cdot)$, $L > 0$, $j \in \mathbb{N}$, the function $c^\pm_j(\cdot)$ related to the equilibria that have $j-1$ zeros in $(0, \pi)$ of the problem

$$\begin{cases} u_{xx} + \lambda f(u) = 0, x \in (0, L), \\ u(0) = u(L) = 0, \end{cases} \tag{5.1}$$

for $\lambda > 0$ a parameter.

Recall that the following holds.

Lemma 5.1. *Consider $f \in C^2(\mathbb{R})$ satisfying (1.2). If $j \in \mathbb{N}$, $j \geq 2$, and ϕ_j is an equilibrium of (1.7) with $j-1$ zeros in $(0, \pi)$, then ϕ_{2j} is $\frac{\pi}{j}$ periodic. In addition, if f is odd, then*

$$\phi_j(\frac{\pi}{j} + x) = -\phi_j(\frac{\pi}{j} - x), \quad \textit{for } x \in [0, \frac{\pi}{j}].$$

Proposition 5.2. *If $f \in C^2(\mathbb{R})$ and satisfies (1.2), then*

(i) $c^L_{j,\pm}(r) = \left(\frac{L}{\pi}\right)^2 c^\pi_{j,\pm}(\frac{Lr}{\pi})$, *for all* $r \in \mathbb{R}^+$, $j = 1, 2, 3 \cdots$.

(ii) *For all* $r \in \mathbb{R}^+$, $c^\pi_{2j,\pm}(r) = \frac{1}{j^2} c^\pi_{2,\pm}(\frac{r}{j^2})$.

If we also assume that f is odd, then

(iii) $c^L_{j,+}(\cdot) = c^L_{j,-}(\cdot)$ *and* $c^\pi_{j,\pm}(r) = c^{\frac{\pi}{j}}_{1,\pm}(\frac{r}{j})$, *for all* $r \in \mathbb{R}^+$ *and* $j \in \mathbb{N}$.

(iv) $c^\pi_{j,\pm}(r) = \frac{1}{j^2} c^\pi_{1,\pm}(\frac{r}{j^2})$, *for all* $r \in \mathbb{R}^+$ *and* $j \in \mathbb{N}$.

Proof. The proof follows by a simple change of variables.

(i) Let $r \in \mathbb{R}^+$. In what follows, we fix one of the symbols $+$ or $-$ and omit it in the notation. If $c_j^L(r) = \frac{1}{\lambda_r}$, then there is a $\phi \in C^2(0, L)$, with $\|\phi_x\|^2 = r$, such that $\phi \neq 0$ in $(0, \pi)$ and satisfies (5.1) with λ replaced by λ_r.

For $x \in [0, \pi]$, define $\psi(x) = \phi(\frac{Lx}{\pi})$. Then ψ satisfies

$$\psi_{xx}(s) = \Big(\frac{L}{\pi}\Big)^2 \phi_{xx}\Big(\frac{Ls}{\pi}\Big) = -\Big(\frac{L}{\pi}\Big)^2 \lambda_r f\Big(\phi\Big(\frac{Ls}{\pi}\Big)\Big).$$

In other words, ψ is a solution of (5.1) with L replaced by π and λ replaced by $\left(\frac{L}{\pi}\right)^2 \lambda_r$. Also,

$$\begin{aligned}\|\psi_x\|^2 &= \int_0^\pi (\psi_x(s))^2 ds = \int_0^\pi \Big(\frac{L}{\pi}\Big)^2 \Big(\phi_x\Big(\frac{Ls}{\pi}\Big)\Big)^2 ds \\ &= \frac{L}{\pi}\int_0^L (\phi_x(u))^2 du = \frac{Lr}{\pi}.\end{aligned}$$

Hence, by definition of c_j^π, we conclude that

$$c_j^\pi\Big(\frac{Lr}{\pi}\Big) = \Big(\frac{\pi}{L}\Big)^2 \frac{1}{\lambda_r}.$$

Therefore,

$$c_j^L(r) = \frac{1}{\lambda_r} = \Big(\frac{L}{\pi}\Big)^2 c_j^\pi(\frac{Lr}{\pi}).$$

Since $r \in \mathbb{R}^+$ is arbitrary, the result follows.

(ii) Once again, we fix one of the symbols $+$ or $-$ and omit it in the notation. Let $r > 0$ and $j \in \mathbb{N}$, $j \geq 2$. By the definition, $c_{2j}^\pi(r) = \frac{1}{\lambda_r}$ implies that there is a ϕ, with $2j-1$ zeros in $(0, \pi)$, an equilibrium of (1.7) when $\lambda = \lambda_r$ and satisfying $\|\phi_x\|^2 = r$.

By Lemma 5.1, we have

$$r = \int_0^\pi (\phi_x(s))^2 ds = j \int_0^{\frac{\pi}{j}} (\phi_x(s))^2 ds.$$

Hence, $\psi = \phi\big|_{[0,\frac{\pi}{j}]}$ is the solution of (5.1) that changes sing one time for $L = \frac{\pi}{j}$ and $\|\psi_x\|^2_{L^2(0,\frac{\pi}{2})} = \frac{r}{j}$. Therefore,

$$c_2^{\frac{\pi}{j}}(\frac{r}{j}) = \frac{1}{\lambda_r} = c_{2j}^\pi(r).$$

By the previous item, the desired result follows.

(iii) Assume that f is odd. In this case, if ϕ is an equilibrium (5.1), then $-\phi$ is an equilibrium (5.1), for the same parameter $\lambda > 0$ and having the same norm.

Fix $j \in \mathbb{N}$ and $r \in \mathbb{R}^+$. If $c_j^\pi(r) = \frac{1}{\lambda_r}$, then there is $\phi \in C^2(0,\pi)$ with $j-1$ zeros in $(0,\pi)$, with $\|\phi_x\|^2 = r$, and satisfying (1.7). Since f is odd, ϕ has a lot of symmetries and

$$r = \int_0^\pi (\phi_x(s))^2 ds = j\int_0^{\frac{\pi}{j}} (\phi_x(s))^2 ds.$$

Consider $\psi = \phi\big|_{[0,\frac{\pi}{j}]}$. Then, we have $\psi > 0$ in $(0,\frac{\pi}{j})$, $\|\psi_x\|^2_{L^2(0,\frac{\pi}{j})} = \frac{r}{j}$, and ψ satisfies (5.1), for $L = \frac{\pi}{j}$ and $\lambda = \lambda_r$. Hence, by the definition of $c_1^{\frac{\pi}{j}}$, we find

$$c_1^{\frac{\pi}{j}}(\frac{r}{j}) = \frac{1}{\lambda_r} = c_j^\pi(r).$$

(iv) It follows from the previous items. $\square$

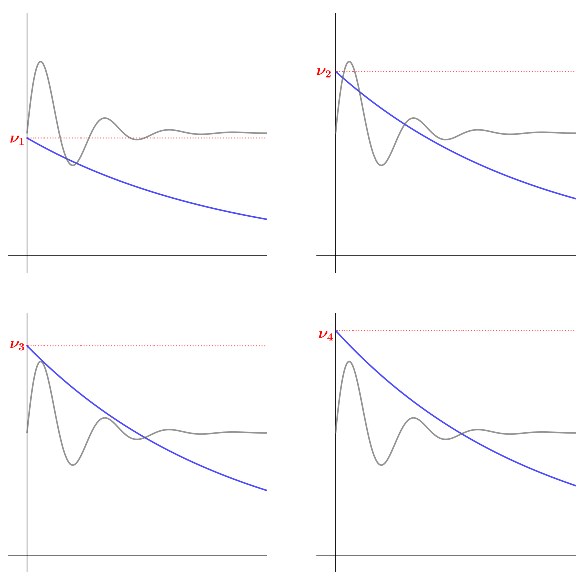


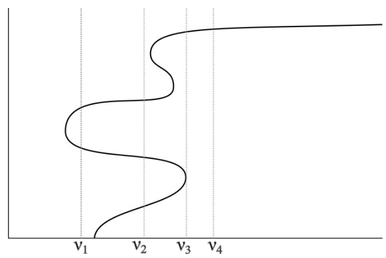
FIGURE 2. Graphs of a_1 (in gray) and $\nu c_1^{\pm}$ (in blue) for different choices of ν

The result from Proposition 5.2 provides a very good understanding of the bifurcations of equilibria for (1.1) with particular emphasis to the case of suitably large $j \in \mathbb{N}$. We remark that, if f is odd, for large values of j, the functions $j^2 c_j^{\pm}$ are very slowly decreasing.

Next, we exhibit a few pictorial examples of possible bifurcations that will happen depending on our choice of the functions a and f.

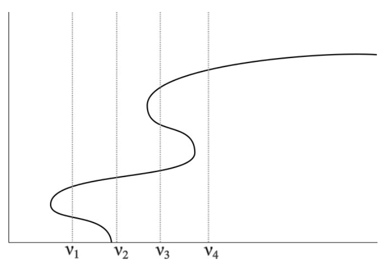
Example 5.3. Consider in this example the function $a = a_1$ as in Figure 2:

In that case, the bifurcation from zero is a supercritical pitchfork bifurcation and four other saddle-node bifurcations occur, two subcritical and two supercritical. The bifurcation curve looks like this:



Example 5.4. Consider in this example the function $a = a_2$, with graph pictured in gray, in Figure 3:

In that case, the bifurcation from zero is a subcritical pitchfork bifurcation and three other saddle-node bifurcations occur, two supercritical and one subcritical. The bifurcation curve looks like this:



Example 5.5. Consider the function given by $a = a_3$, with graph pictured in gray, as in Figure 5.

The first bifurcation from zero is a supercritical pitchfork bifurcation and the second bifurcation from zero is a subcritical pitchfork bifurcation.

Between the two bifurcations from zero occurs saddle-node bifurcations.

Suppose that $\nu_3 \in (\nu_1, \nu_5)$ is the moment for which the saddle-node bifurcation of the equilibria that change sign one time in $(0, \pi)$ appears. In this

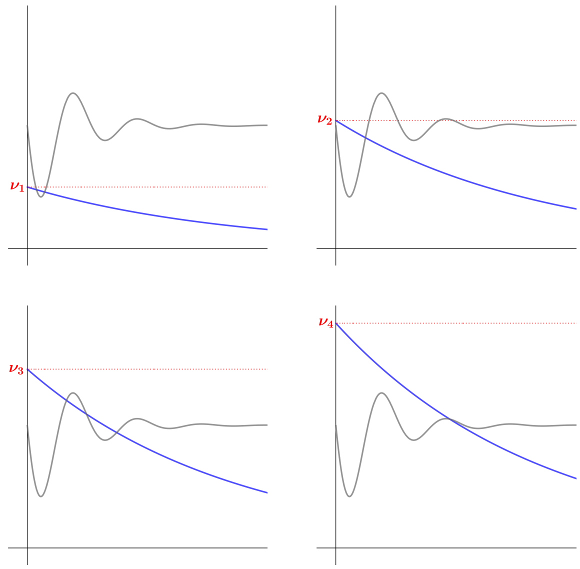
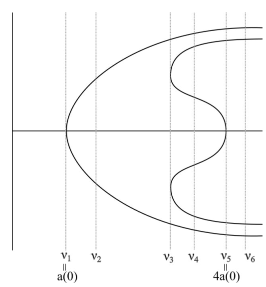


FIGURE 3. Graphs of a_2 and $\nu c_1^{\pm}$ (in blue) for different choices of ν



case, if f is odd, a pictorial representation of the global attractor is given in Figure 4.

For $\nu \in (\nu_3, \nu_5)$, it is also expected that the two more unstable equilibria collapses at 0 as ν approaches $4a(0)$.

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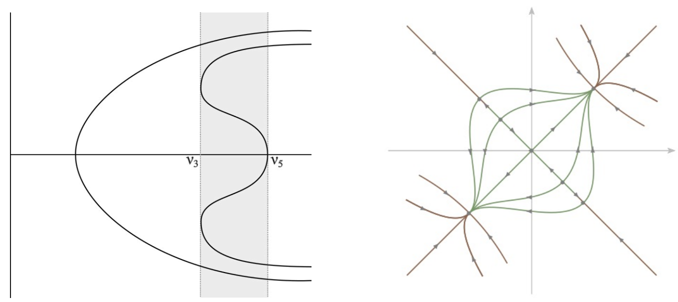

Figure 4. Expected structure of the attractor, when $\nu \in (\nu_3, \nu_5)$.

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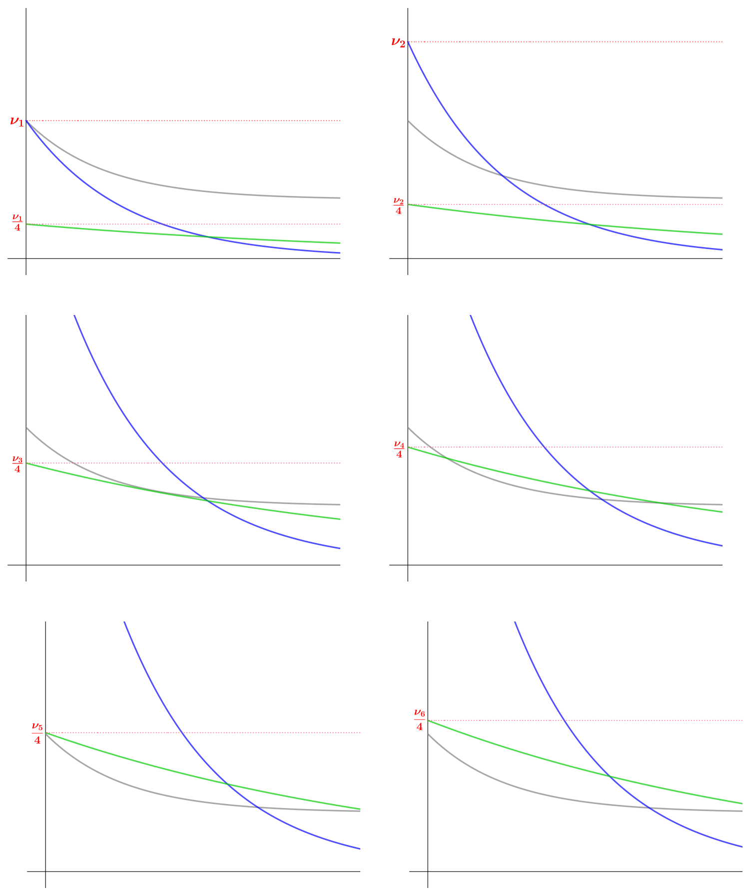

FIGURE 5. Graphs of a_3 (in gray), $\nu c_1^{\pm}$ (in blue) and $\nu c_2^{\pm}$ (in green) for different choices of ν

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