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Low frequency wave force spectrum influenced by wave-current interaction

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Abstract

The interaction between ocean current with sea waves plays an important role in the determination of the steady wave forces acting in a floating system and, in the limit when the velocity is small, an *exact* formula obtained [J Fluid Mech. 272 (1994) 147; J Fluid Mech. 313 (1996) 39], relates the steady wave forces with the standard drift forces in the seakeeping problems. This result is applied here, in conjunction to Newman's approximation, to express the first order influence of wave-current interaction in the second order low frequency wave force spectrum used in the study of the slow drift oscillation of a floating system.

Keywords: Wave force spectrum; Wave-current interaction; Steady wave forces

1. Introduction

The slow drift oscillation of a floating system is a resonant phenomenon being excited by the second order low frequency wave forces. These forces are, in general, expressed in terms of the so-called quadratic transfer function and are influenced, as the steady wave forces, by the presence of an ocean current. The complete determination of the quadratic transfer function is a complicated problem, even more when the interaction with the current is accounted for, and the purpose of this paper is to present a consistent approximation for the frequency spectrum of this force.

The obtained expression is just an extension of the so-called Newman's approximation [9] for the present case, where the force spectrum is described entirely now in terms of the second order steady wave force coefficients $\mathbf{D}_U(\omega,\beta)$ influenced by the current U; notice here that the tree components of the vector $\mathbf{D}_U(\omega,\beta)$ are the two force coefficients in surge and sway, the third component being the yaw moment, while (ω,β) are, respectively, the wave frequency and wave direction *before* the interaction with the current. The consistency of Newman's approximation has been analyzed in Ref. [1] and, as will be seen in the present work, it can be extended, with some minor modifications, to the present case, where the influence of the wave-current interaction is addressed.

On the other hand, as shown in Refs. [2,3], when terms of order U^2 are ignored the generalized force vector $\mathbf{D}_U(\omega,\beta)$ can be exactly expressed in terms of the standard ($U = 0$) generalized steady wave force $\mathbf{D}_0(\omega,\beta)$ by the formula

$$\mathbf{D}_U(\omega,\beta) = \left(1 - 4\frac{U\omega}{g}\cos\beta_{cw}\right)\mathbf{D}_0(\omega_e,\beta_1), \tag{1.1}$$

$$\omega_e = \omega\left(1 - \frac{U\omega}{g}\cos\beta_{cw}\right),$$

$$\beta_1 = \beta + 2\frac{U\omega}{g}\sin\beta_{cw},$$

where β_{cw} is the angle between the wave direction and the current (see Fig. 1) and (ω_e,β_1) are, respectively, the frequency of encounter and the refracted wave direction. Besides to be mathematically exact within the frame of the pertinent theory, where the flow is assumed potential and terms of order U^2 are ignored, two distinct experimental results seem to confirm the adequacy of Eq. (1.1) to describe the wave-current interaction effect. In fact, both Trassoudaine and Naciri [4], analyzing MARIN's decay test of a VLCC model, and Aranha et al. [5], analyzing the experimental results obtained at IPT for the equilibrium position of a FPSO model with turret, observed a very close agreement between the experiments and the theoretical results obtained from Eq. (1.1). Using Eq. (1.1) into the extension

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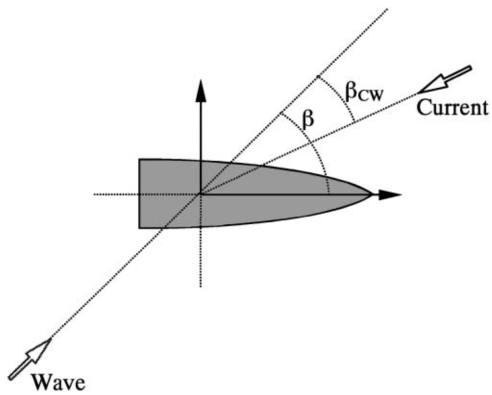

Fig. 1. Geometric definitions.

of Newman's approximation, it is possible now to express the low frequency force spectrum influenced by the wave-current interaction directly in terms of the standard drift force coefficients $\mathbf{D}_0(\omega,\beta)$. This is the main result of the present work.

In Section 2, some basic definitions and assumptions are introduced while in Section 3 the expression of the low frequency force spectrum is derived; in Section 4 a consistent approximation for this spectrum is obtained for a resonant response while in Section 5 some numerical results for a VLCC are presented, showing the importance of the wave-current interaction on the slow drift oscillation of a floating system. In Appendix A some results about the resonant response of a floating system and the kinematics of the wave-current interaction are reviewed; specifically, it is shown that the kinematic relations in Eq. (1.1) hold for a *particular history* of the ocean current and it is also indicated there how to extend to an arbitrary current field the main results of this work.

2. Basic definitions and assumptions

The environment is characterized by an ocean current with intensity U and an irregular wave being propagated in a direction that makes an angle β_{cw} with the current, see Fig. 1. The irregular wave is defined by its energy spectrum $S(\omega)$, where $S(\omega)$ is supposed to be the energy spectrum before the interaction with the ocean current, namely, the energy spectrum observed in the reference system where the medium is stationary. To make more precise this definition the attention is focused here on a *particular* current field, assumed to be *uniform* in the whole space. In the reference system moving with the current one observes then the body being displaced with the velocity $-\mathbf{U}$ while excited by a wave with spectrum $S(\omega)$. If ω_p is the peak frequency of $S(\omega)$, it is supposed here, consistent with the small velocity assumption behind Eq. (1.1), that

$$\tau = \frac{U\omega_p}{g} \ll 1. \tag{2.1}$$

Let ω_e be the *frequency of encounter*,[1] defined in Eq. (1.1), and $S_U(\omega_e)$ be the spectrum actually measured in the vicinity of the body; by conservation of the wave action the following relation can be derived (see Appendix A):

$$S(\omega)\,d\omega = S_U(\omega_e)\,d\omega_e \tag{2.2a}$$

and so, for $\tau \ll 1$, one obtains (see Ref. [6])

$$S(\omega) = S_U(\omega_e)\left(1 - 2\frac{U\omega_e}{g}\cos\beta_{cw}\right), \tag{2.2b}$$

$$\omega = \omega_e\left(1 + \frac{U\omega_e}{g}\cos\beta_{cw}\right).$$

The spectrum $S(\omega)$ is identified with the one existing before the interaction with the current while $S_U(\omega_e)$ is the one observed *after* the interaction. To make easier the derivation, in special, to use Eq. (1.1), the reference will be always the spectrum $S(\omega)$ although the final result can be expressed in terms of $S_U(\omega_e)$ with the help of Eqs. (2.2a) and (2.2b).

In a practical problem one is usually interested to estimate the dynamic behavior of a certain variable $V(t)$ dependent on the slow drift displacement of the body. The first task then is to define generically this variable and to estimate its low frequency spectrum $S_V(\Omega)$.

Reserving the indices $(j,k) = 1,2,3$ for the degrees of freedom in the horizontal plane, let

$$\mathbf{d}_U(t) = \{d_{j,U}(t)\}, \tag{2.3a}$$

$$\mathbf{F}_U(t) = \{F_{k,U}(t)\}$$

be, respectively, the generalized slow drift displacement vector and the low frequency force in the horizontal plane, both influenced by the wave-current interaction, and

$$V(t) = \sum_j v_j d_{j,U}(t) \tag{2.3b}$$

with $\{v_j;\ j = 1,2,3\}$ being real numbers, be a generic low frequency variable of the floating system. The intention here is to obtain a consistent approximation for the frequency spectrum $S_V(\Omega)$ of $V(t)$ in terms of the standard drift forces coefficients $\mathbf{D}_0(\omega,\beta)$.

The dynamics of the floating system can be characterized by two parameters: by its natural frequency Ω_n in the horizontal plane and the factor ζ of the critical damping, see Appendix A. The following relations hold for these parameters:

$$\mu = \frac{\Omega_n}{\omega_p} \ll 1, \tag{2.4}$$

$$\zeta \ll 1.$$

The low frequency excitation has practical importance

[1] The suffix 'e' will be reserved here and in the following to identify frequencies after the interaction with the current.

when resonance is excited ($\Omega \approx \Omega_n$) and the dynamic amplification is large ($\zeta \ll 1$); those are the basic assumptions in the present work and since $\mu \ll 1$ one should be concerned with the low frequency force spectrum in a certain range Ω of frequencies where $\Omega \approx O(\mu\omega_p) \ll \omega_p$. It will be shown in Section 4 that the obtained approximation has an error factor of the form $[1 + O(\mu^2; \mu\zeta; \mu\tau; \tau^2)]$, see Eqs. (2.1) and (2.4).

3. The frequency spectrum $S_V(\Omega)$

As it is well known, a wave record with duration T_r can be represented by the Fourier series ($\Delta\omega = 2\pi/T_r$)

$$\eta(t) = \sum_{m=1}^{\infty} A_m \cos(\omega_m t - \varphi_m), \qquad \omega_m = m\Delta\omega, \tag{3.1a}$$

where the phases φ_m are random and the amplitudes are such that $E[A_m^2] = 2S(\omega_m)\Delta\omega$ in the limit $\Delta\omega \to 0$, with ω and $S(\omega)$ (or ω_m and $S(\omega_m)$) being, respectively, the frequencies and the wave energy spectrum of the free waves *before* the interaction with the ocean current.

Let $\{\mathbf{T}_{j,U}(\omega_1;\omega_2);\ j = 1,2,3\}$ be the quadratic transfer functions influenced by the wave-current interaction. These functions relate a pair of harmonic waves, with unit amplitudes and frequencies ($\omega_1;\omega_2$), with the second order wave forces at difference frequency $\omega_{1,e} - \omega_{2,e}$ *after* the wave-current interaction, with the *frequency of encounter* $\omega_{m,e}$ being defined by Eq. (1.1), see Appendix A. This transfer function depends, obviously, on the incidence angle β and it can be eventually measured in a wave tank by imposing, at the wavemaker, two harmonic waves with frequencies ($\omega_1;\omega_2$) and unit amplitudes while measuring the force in the floating body, being displaced with velocity $-\mathbf{U}$, at the difference frequency of encounter $\omega_{1,e} - \omega_{2,e}$.

If (*) stands for the complex conjugate of the term on the left, the components of the force vector (2.3a) can be written in the form

$$F_{j,U}(t) = \sum_q (\hat{F}_{j,U}(\Omega_q)\, e^{i\Omega_q t} + (^*)) \tag{3.1b}$$

with (notice, in Eq. (3.1c), that $(\omega_m + \hat{\Omega}_{q(m)})_e - \omega_{m,e} = \Omega_q$ when terms of order τ^2 are ignored)

$$\hat{F}_{j,U}(\Omega_q) = \sum_{m=1}^{\infty} A_m A_{m+q(m)}\, e^{i\varphi_{mq}}\, T_{j,U}(\omega_m; \omega_m + \hat{\Omega}_{q(m)}),\ \varphi_{mq} = \varphi_{m+q(m)} - \varphi_m,$$

$$\hat{\Omega}_{q(m)} = q(m)\Delta\omega = \Omega_q\left(1 + \frac{U}{g}(2\omega_m + \Omega_q)\cos\beta_{cw}\right). \tag{3.1c}$$

If $\mathbf{H}(\Omega)$ is the transfer matrix of the linearized dynamic system in the horizontal plane, see Appendix A, the generalized displacement can be written as

$$\mathbf{d}_U(t) = \sum_q \left(\mathbf{H}(\Omega_q)\{\hat{F}_{j,U}(\Omega_q)\}\, e^{i\Omega_q t} + (^*)\right) \tag{3.2a}$$

and introducing the row vector (see Eq. (2.2b))

$$\lfloor H_1(\Omega); H_2(\Omega); H_3(\Omega)\rfloor = \lfloor \nu_1; \nu_2; \nu_3 \rfloor \mathbf{H}(\Omega) \tag{3.2b}$$

the following expression is obtained for $V(t)$:

$$V(t) = \sum_{q=1}^{\infty}\left(\sum_j H_j(\Omega_q)\hat{F}_{j,U}(\Omega_q)\, e^{i\Omega_q t} + (^*)\right). \tag{3.2c}$$

Placing now Eq. (3.1c) into Eq. (3.2c), observing the randomness of φ_{mq} and using the relation $E[A_m^2] = 2S(\omega_m)\Delta\omega$, the frequency spectrum of $V(t)$ can be written in the form

$$S_V(\Omega) = \sum_j \sum_k (H_{jk}^{(R)}(\Omega)S_{U,R}^{jk}(\Omega) + H_{jk}^{(I)}(\Omega)S_{U,I}^{jk}(\Omega)), \tag{3.3a}$$

where

$$H_{jk}^{(R)}(\Omega) = \frac{1}{2}(H_j(\Omega)H_k^*(\Omega) + H_j^*(\Omega)H_k(\Omega)),$$
$$H_{jk}^{(I)}(\Omega) = -\frac{i}{2}(H_j(\Omega)H_k^*(\Omega) - H_j^*(\Omega)H_k(\Omega)) \tag{3.3b}$$

and (see Ref. [7])

$$S_{U,R}^{jk}(\Omega) = 8\int_0^{\infty} S(\omega)S(\omega + \hat{\Omega}(\omega))\frac{1}{2}[T_{j,U}(\omega; \omega + \hat{\Omega}(\omega))T_{k,U}^*(\omega; \omega + \hat{\Omega}(\omega)) + (^*)]\, d\omega,$$
$$S_{U,I}^{jk}(\Omega) = 8\int_0^{\infty} S(\omega)S(\omega + \hat{\Omega}(\omega))\frac{1}{2}[-iT_{j,U}(\omega; \omega + \hat{\Omega}(\omega))T_{k,U}^*(\omega; \omega + \hat{\Omega}(\omega)) + (^*)]\, d\omega, \tag{3.3c}$$

with the frequency $\hat{\Omega}(\omega)$ defined by the expression (see Eq. (3.1c))

$$\hat{\Omega}(\omega) = \Omega\left(1 + \frac{U}{g}(2\omega + \Omega)\cos\beta_{cw}\right). \tag{3.3d}$$

Again, if terms of order τ^2 are ignored, one can easily check that (see Eq. (1.1))

$$(\omega + \hat{\Omega}(\omega))_e - \omega_e = \Omega. \tag{3.3e}$$

4. Approximation for $S_V(\Omega)$ with error factor $[1 + O(\mu^2; \mu\zeta; \mu\tau; \tau^2)]$

The quadratic transfer function satisfies the conditions (see Ref. [7])

$$T_{j,U}(\omega_1; \omega_2) = T_{j,U}^*(\omega_2; \omega_1),$$
$$T_{j,U}(\omega; \omega) = D_{j,U}(\omega, \beta), \tag{4.1a}$$

where $D_{j,U}(\omega,\beta)$ is the second order steady wave force influenced by the wave-current interaction; given $T_{j,U}(\omega_1;\omega_2)$ the following function can also be defined (see Ref. [1]):

$$\partial T_{j,U}(\omega,\beta) = \frac{i}{2}\left[\frac{\partial T_{j,U}}{\partial\omega_1}(\omega;\omega) - \frac{\partial T_{j,U}}{\partial\omega_2}(\omega;\omega)\right]. \quad (4.1b)$$

Observing now that $\Omega/\omega \approx O(\mu)$ and expanding $T_{j,U}(\omega;\omega+\hat{\Omega}(\omega))$ in Taylor's series around $T_{j,U}(\omega+\hat{\Omega}(\omega)/2;\omega+\hat{\Omega}(\omega)/2) = D_{j,U}(\omega+\hat{\Omega}(\omega)/2,\beta)$ one obtains, with the help of Eq. (4.1b), that[2]

$$T_{j,U}(\omega;\omega+\hat{\Omega}(\omega)) = D_{j,U}(\omega+\hat{\Omega}(\omega)/2,\beta) + i\partial T_{j,U}(\omega,\beta)\hat{\Omega}(\omega) + O(\mu^2). \quad (4.1c)$$

The same expansion for $T_{j,U}(\omega+\hat{\Omega}(\omega);\omega)$ gives now

$$T_{j,U}(\omega+\hat{\Omega}(\omega);\omega) = D_{j,U}(\omega+\hat{\Omega}(\omega)/2,\beta) - i\partial T_{j,U}(\omega,\beta)\hat{\Omega}(\omega) + O(\mu^2)$$

and from the Hermitian property of the quadratic transfer function, namely, from the first relation in Eq. (4.1a), it follows that $\partial T_{j,U}(\omega,\beta)$ is *real*. Since $\Omega/\omega_p \cong O(\mu)$, it turns out that (see Eq. (4.1c))

$$\frac{1}{2}[T_{j,U}(\omega;\omega+\hat{\Omega})T^*_{k,U}(\omega;\omega+\hat{\Omega}) + (^*)] = D_{j,U}(\omega+\hat{\Omega}/2,\beta)D_{k,U}(\omega+\hat{\Omega}/2,\beta) + O(\mu^2),$$

$$\frac{1}{2}[-iT_{j,U}(\omega;\omega+\hat{\Omega})T^*_{k,U}(\omega;\omega+\hat{\Omega}) + (^*)] = \hat{\Omega}[\partial T_{j,U}(\omega,\beta)D_{k,U}(\omega,\beta) - D_{j,U}(\omega,\beta)\partial T_{k,U}(\omega,\beta)] + O(\mu^2)$$

and so, from Eq. (3.3c), the following approximations are obtained:

$$S^{jk}_{U,R}(\Omega) = 8\int_0^\infty S(\omega)S(\omega+\hat{\Omega}(\omega))D_{j,U}(\omega+\hat{\Omega}(\omega)/2,\beta)D_{k,U}(\omega+\hat{\Omega}(\omega)/2,\beta)\,d\omega + O(\mu^2),$$

$$S^{jk}_{U,I}(\Omega) = 8\int_0^\infty \hat{\Omega}(\omega)S^2(\omega)[\partial T_{j,U}(\omega,\beta)D_{k,U}(\omega,\beta) - D_{j,U}(\omega,\beta)\partial T_{k,U}(\omega,\beta)]\,d\omega + O(\mu^2). \quad (4.2a)$$

In a *resonant response* $H^{(I)}_{jk}(\Omega)$ is of order $\zeta \ll 1$ compared with $H^{(R)}_{jk}(\Omega)$, see Appendix A, and observing that $S^{jk}_{U,I}(\Omega) \approx O(\mu)$, since it is proportional to Ω, see Eq. (4.2a), the parcel $H^{(I)}_{jk}(\Omega)S^{jk}_{U,I}(\Omega)$ can be disregarded in Eq. (3.3a) with an error of the form $[1 + O(\mu^2;\zeta\mu)]$. On the other hand, one can easily check, from Eq. (4.2a), that

$$\frac{dS^{jk}_{U,R}}{d\Omega}(0) = 4\int_0^\infty \left(\frac{d\hat{\Omega}(\omega)}{d\Omega}\right)_{\Omega=0}\frac{d}{d\omega}\times(S^2(\omega)D_{j,U}(\omega,\beta)D_{k,U}(\omega,\beta))\,d\omega$$
$$= 4\int_0^\infty \left(1+2\frac{U\omega}{g}\cos\beta_{cw}\right)\frac{d}{d\omega}\times(S^2(\omega)D_{j,U}(\omega,\beta)D_{k,U}(\omega,\beta))\,d\omega$$
$$= -\frac{U}{g}\cos\beta_{cw}S^{jk}_{U,R}(0)$$

and so

$$S^{jk}_{U,R}(\Omega) = S^{jk}_{U,R}(0) + \frac{dS^{jk}_{U,R}}{d\Omega}(0)\Omega + O(\mu^2) = S^{jk}_{U,R}(0)[1 + O(\mu^2;\mu\tau)]. \quad (4.2b)$$

Then, with an error factor of the form $[1 + O(\mu^2;\mu\zeta;\mu\tau;\tau^2)]$, the spectrum $S_V(\Omega)$ can be approximated by

$$S_V(\Omega) \cong \sum_j\sum_k H^{(R)}_{jk}(\Omega)S^{jk}_{U,R}(0),$$
$$S^{jk}_{U,R}(0) = 8\int_0^\infty S^2(\omega)D_{j,U}(\omega,\beta)D_{k,U}(\omega,\beta)\,d\omega, \quad (4.3a)$$

where this last integral is given by (see Eq. (1.1))

$$S^{jk}_{U,R}(0) = 8\int_0^\infty \left(1-4\frac{U\omega}{g}\cos\beta_{cw}\right)^2\times S^2(\omega)D_{j,0}(\omega_e,\beta_1)D_{k,0}(\omega_e,\beta_1)\,d\omega, \quad (4.3b)$$

$$\omega_e = \omega\left(1-\frac{U\omega}{g}\cos\beta_{cw}\right), \qquad \beta_1 = \beta + 2\frac{U\omega}{g}\sin\beta_{cw}.$$

The frequency spectrum $S_V(\Omega)$ of the second order low frequency variable $V(t)$ can then be directly expressed in terms of the standard ($U=0$) second order steady drift force coefficients $D_{j,0}(\omega,\beta)$, determined from a linear frequency domain model. Expressions (4.3a) and (4.3b) are the main results of this work and, in Section 5, the practical importance of the wave-current interaction will be displayed by means of some numerical examples. First, however, it is necessary to extend the above result to a broader class of situations that may become important for application.

The basic point is that the wave refraction by the ocean current is uniquely determined only if the field $\mathbf{U}(\mathbf{x},t)$ of the ocean current is known in a geographical scale. In the present context, however, the only information about the ocean current was supposed to be its value in the vicinity of the floating body and, in deriving the above result, a

[2] The function $\partial T_{j,U}(\omega,\beta)$ depends on the second order potential and it may become relevant in *shallow water*. As shown in Ref. [12], the influence of the second order potential can be determined from a trivial extension of Haskind's relation applied to the *steady* second order wave potential.

specific field $\mathbf{U}(\mathbf{x},t)$ had to be assumed: $\mathbf{U}(\mathbf{x},t) \cong \mathbf{U}(0,t) = -U(t)\mathbf{e}_1$, with $U(t) = 0$ for $t \le 0$ and $U(t) = U$ for $t \ge t_f$, where $\mathbf{x} = 0$ is the position of the floating system. This is the most natural way to define the ocean current from the given information and, clearly, the assumed field makes the problem identical to the one where the medium is stationary but the body is displaced with the velocity $U(t)\mathbf{e}_1$ (see Appendix A). In this case, then, the above result can be experimentally confirmed in laboratory by towing a model with the velocity $U\mathbf{e}_1$ while exposing it to a wave generated by the wavemaker, with spectrum $S(\omega)$; in particular, this experiment can be easily performed for following ($\beta_{cw} = 0°$) and head seas ($\beta_{cw} = 180°$), as it was done in Ref. [5] for the steady (harmonic) problem. Assuming, as it has been done so far, this particular current field one can express directly Eq. (4.3b) in terms of the actual spectrum $S_U(\omega_e)$; in fact, using Eqs. (2.2a) and (2.2b) into Eq. (4.3b) the following expression can be derived:

$$S^{jk}_{U,R}(0) = 8\int_0^\infty \left(1 - 5\frac{U\omega_e}{g}\cos\beta_{cw}\right)^2 \times S^2_U(\omega_e) D_{j,0}(\omega_e,\beta_1) D_{k,0}(\omega_e,\beta_1)\, d\omega_e, \qquad (4.3c)$$

$$\beta_1 = \beta + 2\frac{U\omega_e}{g}\sin\beta_{cw}.$$

The question remains, however, on how to deal with an arbitrary current field $\mathbf{U}(\mathbf{x},t)$ in the eventual situation where it is known in a geographical scale; as discussed by White and Fornberg [8] and in the references quoted there, the predictions obtained from the wave refraction theory seem to be reliable and may be used in the analysis of a floating system, at least in the few circumstances where a more detailed information about the spatial variation of the ocean current is known.

In this way, let again $S_U(\omega_e)$ be the *actual* wave spectrum in the vicinity of the floating body, this spectrum being obtained either from a direct measurement of the wave elevation or else theoretically, by the integration of the equations of the geometric optics using the actual field $\mathbf{U}(\mathbf{x},t)$. It is possible to argue then, as discussed in Appendix A, that the spectrum $S^{jk}_{U,R}(0)$ of the second order low frequency forces should be given by Eq. (4.3c). The basic argument is related with the ability that an observer placed at the floating body must have to predict the forces from the environment information he is able to measure in situ, namely, the spectrum $S_U(\omega_e)$ and the current $-U\mathbf{e}_1$, irrespective of the particular field $\mathbf{U}(\mathbf{x},t)$ existing in a geographical scale.

5. Numerical results

In the design analysis of a floating body one must study the behavior of the floating system subjected to different combinations of waves and currents. In this context, both the wave spectrum and the ocean current are defined independently and it seems natural, then, to identify the *design* wave spectrum as the one existing *before* the interaction with the current, since it is independent of it. As a consequence the design wave spectrum should be identified with $S(\omega)$ and the expression (4.3b) should be used.

In order to assess the importance of the wave-current interaction in the slow drift phenomenon one considers a VLCC, with length L, beam B and draft T, exposed to an irregular wave described by Pierson–Moskowitz spectrum with cut-off frequency[3] $2\omega_p$, namely

$$S(\omega) = \frac{5}{16}\frac{H_S^2}{\omega_p}\frac{1}{\varpi^5}\exp\left(-\frac{5}{4}\frac{1}{\varpi^4}\right), \qquad 0 \le \varpi = \frac{\omega}{\omega_p} \le 2, \qquad (5.1a)$$

$$\frac{\omega_p^2 H_S}{g} = 0.24$$

and to an ocean current with intensity U, incident in a direction that makes an angle β_{cw} with the wave direction, see Fig. 1.

As it is clear from Eqs. (4.3a) and (4.3b), the intensity of the low frequency second order wave forces can be gauged by the non-dimensional force coefficients

$$F_{jk,U}(\beta;\beta_{cw}) = \frac{\sqrt{\left|S^{jk}_{U,R}(0)\right|\omega_p}}{(1/8)\rho g H_S^2 l_{jk}},$$

$$[l_{jk}] = \begin{bmatrix} L & L & L\sqrt{L/2} \\ L & L & L\sqrt{L/2} \\ L\sqrt{L/2} & L\sqrt{L/2} & L^2/2 \end{bmatrix} \qquad (5.1b)$$

that is a function of both the incidence angle β and the wave-current angle β_{cw}. The importance of the wave-current interaction can be disclosed by comparing $F_{jk,U}(\beta;\beta_{cw})$ with $F_{jk,0}(\beta)$, the non-dimensional force coefficient in the standard problem, when $U = 0$ in Eq. (4.3b).

Figs. 2 and 3 present, for a VLCC with $L = 320$ m; $B = 54$ m; $T = 21$ m, the plot of $F_{jj,0}(\beta)$ and $F_{jj,U}(\beta;\beta_{cw})$ when the ship is exposed to the extreme environmental condition at *Campos Basin* ($H_S = 7.6$ m; $U = 1.8$ m/s) and at the *North Sea* ($H_S = 16$ m; $U = 1.5$ m/s). In these examples, the program WAMIT has been used to compute the steady force coefficients $\{D_{j,0}(\omega,\beta);\ j = 1,2,3\}$.

As a first observation about the obtained results it interesting to observe that the slow drift phenomenon is more important in Campos Basin, in spite of the fact that the significant wave height is then half the value as the one observed at the North Sea; this is caused by the invariance of the wave steepness assumed in Eq. (5.1a) and by the fact that the steady force coefficients are very small in the low frequency regime.

[3] Besides to be usual in the definition of an actual wave spectrum, a cut-off frequency as prescribed in Eq. (5.1a) is needed here since the essential assumption $U\omega/g \ll 1$ cannot be maintained if ω becomes very large.

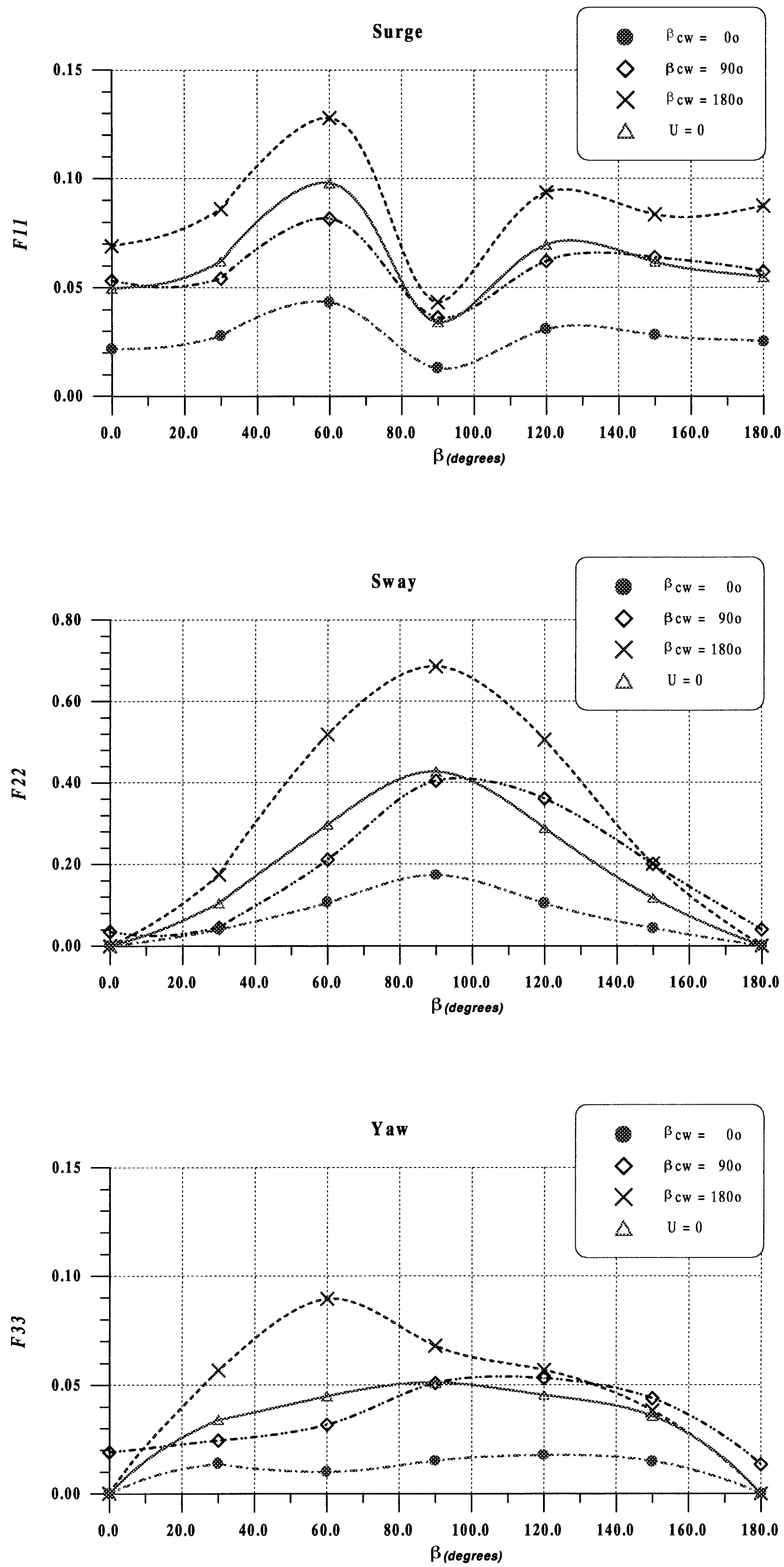

Fig. 2. Values of $F_{jj,0}(\beta)$ (–Δ–) and $\{F_{jj,U}(\beta;\beta_{cw});\ \beta_{cw} = 0°, 90°, 180°\}$. Campos Basin ($H_S = 7.6$ m; $U = 1.8$ m/s).

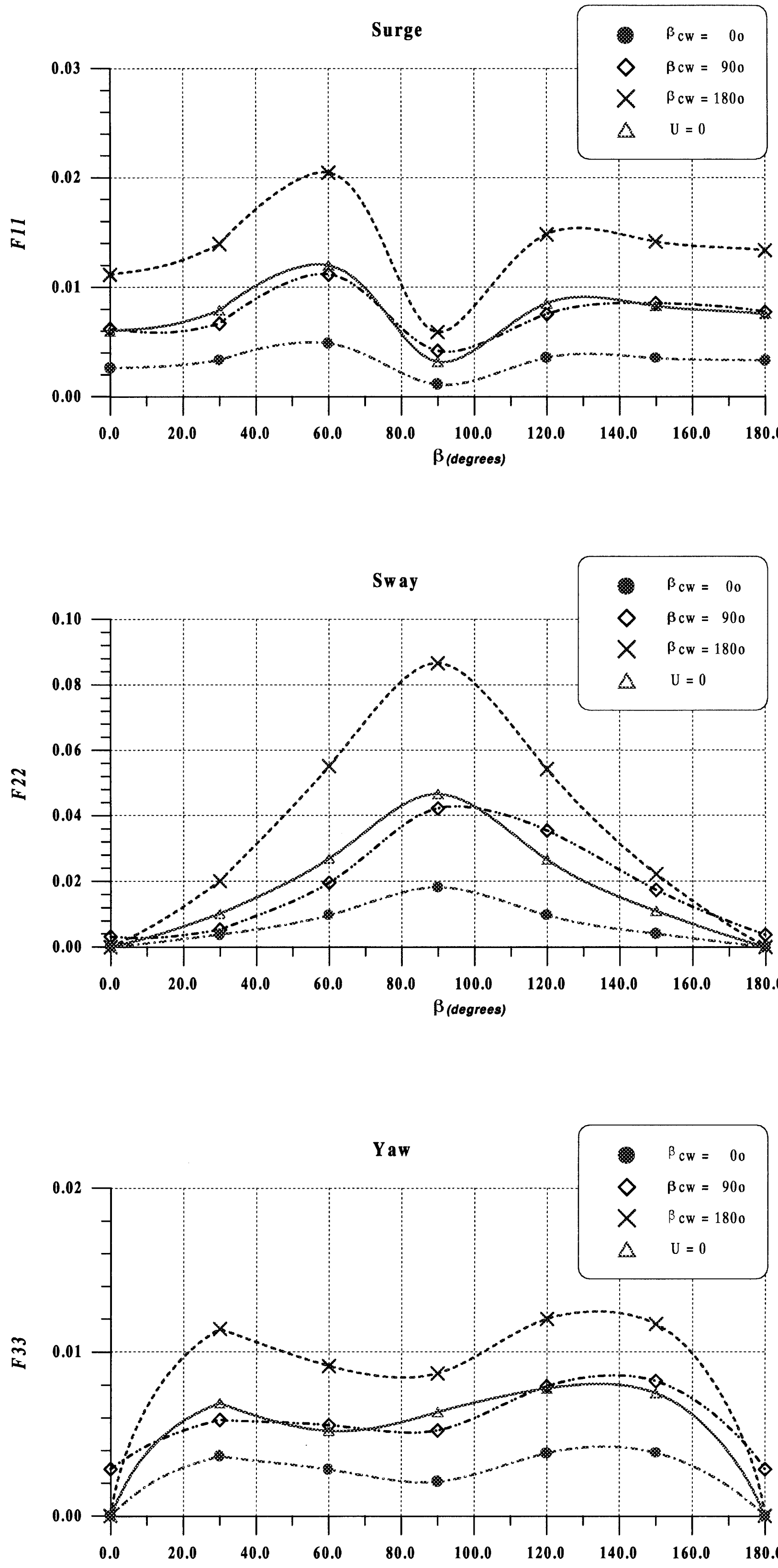

Fig. 3. Values of $F_{jj,0}(\beta)$ (–Δ–) and $\{F_{jj,U}(\beta;\beta_{cw});\ \beta_{cw} = 0°, 90°, 180°\}$. North Sea ($H_S = 16$ m; $U = 1.5$ m/s).

As it is shown in Figs. 2 and 3, the wave-current interaction may have an important influence in the slow drift phenomenon, being able to change the response by a factor of order 2 when $\beta_{cw} = 0°; 180°$. Similar conclusions were also found in the determination of the steady equilibrium position of a FPSO with a turret, where the importance of the wave-current interaction was then confirmed experimentally, see Ref. [5]. It is certainly important to check experimentally the main results derived in this work, even more in the situations where such interaction is apparently so relevant; in the other hand, the proposed leading order approximation in the small parameters τ, ζ, μ can be very easily computed in a routine analysis of a floating system, see Eqs. (4.3a) and (4.3b).

Appendix A

A.1. Resonant response

In this section, some basic results about the resonant response of a floating system in the horizontal plane are reviewed, aiming to clarify some arguments used in the main text and also to provide some results that are thought to be useful. When the equations of the motion of the floating system in the horizontal plane are linearized around a given *stable* equilibrium position, the *transfer matrix* can be written as

$$\mathbf{H}(\Omega) = (-\Omega^2\mathbf{M} + \mathrm{i}\Omega\mathbf{B}(\sigma) + \mathbf{R})^{-1}, \tag{A1.1a}$$

where $\mathbf{M}$ and $\mathbf{R}$ are, respectively, the inertia and the restoring matrices and $\mathbf{B}(\sigma)$ is the damping matrix, in general a function of the standard deviation σ of the response. The matrix $\mathbf{M}$ is symmetric positive definite and the matrix $\mathbf{R}$, being due to the restoring forces caused by the mooring lines,[4] may be assumed to satisfy also these properties; the natural frequencies and normal modes $\{(\Omega_j; \mathbf{q}_j);\ j = 1,2,3\}$ are then solutions of the eigenvalue problem

$$(-\Omega^2\mathbf{M} + \mathbf{R})\mathbf{q} = 0, \tag{A1.1b}$$

with $\Omega_n = \mathrm{Max}\{\Omega_j;\ j = 1,2,3\}$ in Eq. (2.3a) and (2.3b). Let $\mathbf{T} = [\mathbf{q}_1; \mathbf{q}_2; \mathbf{q}_3]$ be the matrix of transformation to modal coordinates, assumed normalized by the conditions

$$\mathbf{T}^{t}\mathbf{M}\mathbf{T} = \mathbf{I}, \tag{A1.1c}$$

$$\mathbf{T}^{t}\mathbf{R}\mathbf{T} = \left[\Omega_j^2\delta_{jk}\right],$$

$$\hat{\mathbf{B}}(\sigma) = [\hat{b}_{jk}(\sigma)] = \mathbf{T}^{t}\mathbf{B}(\sigma)\mathbf{T}$$

with $\hat{\mathbf{B}}(\sigma)$ being the damping matrix in the modal coordinates. The following damping coefficients can then be introduced (see Eqs. (2.3a) and (2.3b)):

$$\zeta_j(\sigma) = \frac{\hat{b}_{jj}(\sigma)}{2\Omega_j}, \tag{A1.2a}$$

$$\zeta = \underset{j=1,2,3}{\mathrm{Max}}\{\zeta_j(\sigma)\}.$$

Assuming that $\zeta \ll 1$, with $\hat{b}_{jk}(\sigma)/\hat{b}_{jj}(\sigma) \leq \mathrm{O}(1)$, the resonant response in the modal coordinates can be obtained, with an error of order $(1 + \mathrm{O}(\zeta))$, from the simplified transfer matrix

$$\hat{\mathbf{H}}_{\mathrm{S}}(\Omega) = [\hat{H}_{jk}(\Omega)], \tag{A1.2b}$$

$$\hat{H}_{jk}(\Omega) = \frac{\delta_{jk}}{(\Omega_j^2 - \Omega^2) + 2\mathrm{i}\zeta_j(\sigma)\Omega_j\Omega}.$$

Returning to the original physical variables $\mathbf{d}_{\mathrm{U}}(t)$, see Eq. (2.2a), it turns out that the resonant response can be obtained, with the same error of order $(1 + \mathrm{O}(\zeta))$, from the approximated transfer matrix defined by

$$\mathbf{H}(\Omega) \cong \mathbf{T}\hat{\mathbf{H}}_{\mathrm{S}}(\Omega)\mathbf{T}^{t}. \tag{A1.2c}$$

Introducing the auxiliary matrices

$$\mathbf{I}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{I}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{I}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{A1.3a}$$

and the row vectors (see Eq. (3.2b))

$$\lfloor h_{1m}; h_{2m}; h_{3m}\rfloor = \lfloor v_1; v_2; v_3\rfloor(\mathbf{T}\mathbf{I}_m\mathbf{T}^{t}). \tag{A1.3b}$$

It follows, from Eq. (3.2b) and Eq. (A1.2d), that

$$H_j(\Omega) \cong \sum_m \frac{h_{jm}}{(\Omega_m^2 - \Omega^2) + 2\mathrm{i}\zeta_m(\sigma)\Omega_m\Omega}(1 + \mathrm{O}(\zeta)). \tag{A1.3c}$$

In a resonant response, where $|\Omega - \Omega_j| \approx \mathrm{O}(\zeta\Omega_j)$ for $j = 1,2,3$, the leading order behavior of the products that appear in Eq. (3.3b) is then given by

$$H_j(\Omega)H_k^{*}(\Omega) \cong \sum_m \frac{h_{jm}h_{km}}{(\Omega_m^2 - \Omega^2)^2 + 4\zeta_m^2(\sigma)\Omega_m^2\Omega^2}(1 + \mathrm{O}(\zeta)). \tag{A1.4a}$$

Observing that h_{jm} are real numbers, since v_j are (see Eq. (A1.3b)), Eq. (A1.4a) implies that $H_{jk}^{(I)}$ is of order ζ compared to $H_{jk}^{(R)}$, as assumed in this work.

From this result it follows, with an error factor of the form $[1 + \mathrm{O}(\mu^2; \mu\zeta; \mu\tau; \tau^2)]$, that $S_{\mathrm{V}}(\Omega)$ can be approximated by Eq. (4.3a); if this expression is further integrated in Ω and

[4] There may be also a parcel of $\mathbf{R}$ due to the variation of the steady forces with the dynamic yaw angle. In this case, one can write $\mathbf{R} = \mathbf{R}_S + \mathbf{R}_A$ with $\mathbf{R}_{\mathrm{S}}$ symmetric positive definite and $\mathbf{R}_{\mathrm{A}}$ anti-symmetric and of the same order ζ of the damping matrix. The same results can be obtained then with $\mathbf{R}_{\mathrm{S}}$ in place of $\mathbf{R}$ in Eqs. (A1.1b) and (A1.1c).

Eq. (A1.4a) is used, the standard deviation σ_V of the variable $V(t)$ can be approximated, now with an error factor of the form $[1 + O(\zeta)]$, by

$$\gamma_m = \sum_j \sum_k S^{jk}_{U,R}(0) h_{jm} h_{km}, \qquad \sigma_V^2 = \sum_m \frac{\pi \gamma_m}{4\Omega_m \zeta_m(\sigma)}. \tag{1}$$

Using Eq. (A1.4b) for the displacements whose standard deviations enter in the definition of $\mathbf{B}(\sigma)$ one can determine the damping factors $\{\zeta_m(\sigma);\ m = 1,2,3\}$ by iteration.

A.2. Wave-current interaction: kinematic relations and spectrum

The intention here is to recover some basic kinematic results used in the main text and to discuss also the relation between the wave spectrum *before* and *after* the interaction with the current. The focus of this discussion is specially concerned with the assessment of the performance of a floating production system, pursued either by a theoretical means or else experimentally in the existing facilities. The purpose is to try to make a little more objective and explicit some implicit assumptions made in these studies.

Let $\sigma(K)$ be the *intrinsic frequency*, namely, the wave frequency with respect to the medium; for example, $\sigma(K) = (gK)^{1/2}$ for a deep water gravity waves. If the medium is moving with a *slowly varying* velocity $\mathbf{U}(\mathbf{x},t)$, from a Galilean transformation of velocities one can easily check that the *dispersion relation* is then defined by the function ($\mathbf{K} = K_1\mathbf{e}_1 + K_2\mathbf{e}_2$)

$$W(\mathbf{K},\mathbf{x},t) = \sigma(K) + \mathbf{U}(\mathbf{x},t)\mathbf{K}, \qquad K = \sqrt{K_1^2 + K_2^2}. \tag{A2.1a}$$

Let now $\mathbf{x}(t)$ be the *wave ray*, tangent to the group velocity vector; as shown in Ref. [10], the wave ray $\mathbf{x}(t)$ and the values of $\mathbf{K}(t)$ and $\omega(t)$ along it can be determined by the solution of the following system (kinematic equations of *geometric optics*):

$$\frac{dx_j}{dt} = \frac{\partial W}{\partial K_j}, \qquad \frac{dK_j}{dt} = -\frac{\partial W}{\partial x_j}, \qquad \frac{d\omega}{dt} = \frac{\partial W}{\partial t}. \tag{A2.1b}$$

The *Hamiltonian structure* of Eq. (A2.1b) has a considerable interest in itself: as it is known, by using de Broglie's wave-particle relations $(\mathbf{H};\mathbf{p};E) = \hbar(W;\mathbf{K};\omega)$ and the expression $E = \mathbf{p}^2/2m + V(\mathbf{x},t)$ for the energy of a particle in the potential field $V(\mathbf{x},t)$, one obtains the dispersion relation $\hbar\omega = \hbar^2\mathbf{K}^2/2m + V(\mathbf{x},t)$ of Schrödinger's wave equation; in this context, Hamilton's equations for the particle is just the geometric optics limit of Schrödinger's wave equation. If now $C_g(K) = d\sigma/dK$ is the intrinsic group velocity, placing Eq. (A2.1a) into Eq. (A2.1b) one obtains ($j = 1,2$)

$$\frac{dx_j}{dt} = C_g(K)\frac{K_j}{K} + U_j(\mathbf{x},t), \qquad \frac{dK_j}{dt} = -K_j\frac{\partial U_j}{\partial x_j}, \qquad \frac{d\omega}{dt} = \sum_j K_j \frac{\partial U_j}{\partial t}. \tag{A2.1c}$$

To integrate Eq. (A2.1c) it is necessary to define the *initial state* and also the current $\mathbf{U}(\mathbf{x},t)$, more precisely, the *history* of how was the evolution of the ocean current, as observed in the floating system, until the design value $-U\mathbf{e}_1$ was reached. It seems natural, in this context, to postulate an initial state without any current and where a wave with frequency ω and wave number K, being propagated in the direction β_{cw}, is detected by an observer placed in the floating system. The wave number vector and frequency are in this case defined by

$$\mathbf{K} = K(\cos\beta_{cw}\,\mathbf{e}_1 + \sin\beta_{cw}\,\mathbf{e}_2), \qquad \omega = \sigma(K) = \sqrt{gK}. \tag{A2.2a}$$

Suppose further that at time $t = 0$ the presence of an ocean current in the direction $-\mathbf{e}_1$ starts to be felt. Since the ocean current is supposed to vary weakly in the space, and this is an underlying assumption in the geometric optics approximation, the observer in the floating system feels only the variation of the current in time and, for him, the ocean current is given[5] by $\mathbf{U}(\mathbf{x},t) \cong \mathbf{U}(0,t) = -U(t)\mathbf{e}_1$, with $U(t) \equiv 0$ for $t \le 0$ and $U(t) \equiv U$ for $t \ge t_f$. As it is clear from (A2.1c), in this case all wave variables change only in time and, in particular, the wave numbers $\{K_j;\ j = 1,2\}$ remain constant along all the transition.

From the last equation in Eqs. (A2.1c) and (A2.2a) it follows, however, that the frequency changes from ω, at $t = 0$, to ω_e, at $t = t_f$, with (see Eq. (1.1))

$$\omega_e = \omega\left(1 - \frac{U\omega}{g}\cos\beta_{cw}\right). \tag{A2.2b}$$

The observer at the floating body also detects a change in the wave direction (wave refraction), the new direction $\beta_{cw,1}$ being given by

$$\tan\beta_{cw,1} = \frac{dx_2}{dx_1} = \frac{C_g(K)\sin\beta_{cw}}{C_g(K)\cos\beta_{cw} - U},$$

or, when terms of order U^2 are ignored and Eq. (A2.2a) is used, by (see Eq. (1.1))

$$\beta_{cw,1} = \beta_{cw} + 2\frac{U\omega}{g}\sin\beta_{cw}. \tag{A2.2c}$$

[5] The case of an arbitrary $\mathbf{U}(\mathbf{x},t)$, with $\mathbf{U}(0,t) \to -U\mathbf{e}_1$ when $t \to \infty$, is briefly addressed in the discussion that follows Eq. (A2.3b); notice that $\mathbf{x} = 0$ is assumed to be the position of the floating body.

To obtain the variation of the wave spectrum one should observe, first, that the intrinsic frequency $\sigma(K)$ remains constant, since the wave number K does, and so conservation of the wave action implies here in energy conservation. If now $S_{U(t)}(\omega_e(t))$ is the wave energy spectrum at the time t, with $S(\omega)$ and $S_U(\omega_e)$ being, respectively, the spectrum *before* $(t \leq 0)$ and *after* $(t \geq t_f)$ the interaction with the current, let $E(t) = S_{U(t)}(\omega_e(t))\,d\omega_e(t)$ be the energy around a certain frequency $\omega_e(t)$; the equation for the energy conservation is given by (see Eq. (A2.1c))

$$\frac{\partial E}{\partial t} + \sum_j \frac{\partial}{\partial x_j}(c_{g,j}E) = 0, \qquad c_{g,j}(t) = \frac{dx_j}{dt}. \tag{A2.3a}$$

Since in the present case the wave variables do not change in space, the above equation implies in $dE/dt = 0$ or $E(t)$ is a constant[6] and then

$$S_U(\omega_e)\,d\omega_e = S(\omega)\,d\omega.$$

From this equality and Eq. (A2.2b) it follows, disregarding again terms of order U^2, that

$$S(\omega) = S_U(\omega_e)\left(1 - 2\frac{U\omega_e}{g}\cos\beta_{cw}\right), \tag{A2.3b}$$

$$\omega = \omega_e\left(1 + \frac{U\omega_e}{g}\cos\beta_{cw}\right).$$

Notice that Eqs. (A2.2b), (A2.2c) and (A2.3b) coincide exactly with the relations that would be observed at the floating body if the medium is stationary but the body is advancing with velocity $U(t)\mathbf{e}_1$, see, for example Ref. [6].

Placing now Eq. (A2.3b) into Eq. (4.3b) one obtains Eq. (4.3c) and this expression is useful to extend the main result of this work to an *arbitrary* current field $\mathbf{U}(\mathbf{x},t)$. In fact, if for a time long the same ocean current $-U\mathbf{e}_1$ and wave spectrum $S_U(\omega_e)$ are observed in the vicinity of the floating body then, irrespective of the previous history of both, the response should always be given by Eq. (4.3c), that expresses the low frequency force spectrum directly in terms of the in situ measured quantities. This can be raised to a status of a postulate, where one affirms the ability that a floating body observer must have to determine the response from the environmental measurements he is able to make. Another way to place this argument, in perhaps a less enlightened and more direct perspective, is the following: let $S_U(\omega_e)$ be the *actual* wave spectrum in the vicinity of the floating body when $t \to \infty$, this spectrum being obtained either by a direct measurement of the wave elevation or else by a theoretical calculation, namely, by integrating Eq. (A2.1c), with the prescribed $\mathbf{U}(\mathbf{x},t)$ and initial conditions, in conjunction with the equation for the conservation of the wave action. If now $(\omega;S(\omega))$ are determined from the actual values of $(\omega_e;S_U(\omega_e))$ by means of Eq. (A2.3b) then, necessarily, $(\omega;S(\omega))$ are the frequency and wave spectrum that would exist before the interaction *if the current were given by* $\mathbf{U}(\mathbf{x},t) = -U(t)\mathbf{e}_1$, as assumed in the main text; but then the derived results Eqs. (4.3a) and (4.3b) continue to be valid if the values of $(\omega;S(\omega))$, obtained from Eq. (A2.3b), are used in them, and so Eq. (4.3c) follows trivially when terms of order U^2 are ignored. In other words: the $(\omega;S(\omega))$ obtained from Eq. (A2.3b) are the ones observed in the reference frame that moves with the current, where the medium is stationary and the body advances with velocity $U\mathbf{e}_1$; all results derived in this work were obtained in this reference system.

In the same way that Eqs. (4.3a) and (4.3b) can be verified in a wave tank by towing a model with velocity $U\mathbf{e}_1$, expression (4.3c) can be verified in those large wave tanks provided with devices to generate current. As it known, the obtained current fields are not homogeneous in space (sometimes, they are not even in time) and so they refract the waves generated by the wavemaker; then, by measuring the actual $S_U(\omega_e)$ in the wave tank together with the response of the floating body model, one could check experimentally Eq. (4.3c). However, some word of caution is needed here. First, it must be granted that the spatial variation of the current in the wave tank is weak compared with the main dimensions of the floating body model since, otherwise, the whole set of measurements has very little meaning; second, as shown by White [11], the vorticity associated with the depth dependence of the current field imposes a phase shift that should not be ignored; third, and perhaps more important for practical application, the obtained responses at these facilities cannot be extrapolated directly to reality, since they are influenced by the particular wave refraction pattern of the tank. It is necessary to correct the distortions caused by it, using then a prescribed $S(\omega)$, together with Eq. (A2.3b) and the measured $S_U(\omega_e)$, to control the wavemaker in a closed loop.

[6] Eq. (A2.3a) is usually integrated by the method of the characteristic: along the wave ray $\mathbf{x}(t)$ this equation reduces to $(dE/dt)_{\mathbf{x}(t)} = -E \operatorname{div} c_g$. But here $c_g = c_g(t)$ and so $\operatorname{div} c_g = 0$ leading to $dE/dt = 0$.

References

[1] Aranha JAP, Fernandes AC. On the second order slow drift force spectrum. Appl Ocean Res 1996;17(5):311–3.

[2] Aranha JAP. A formula for wave damping in the drift of a floating body. J Fluid Mech 1994;272:147–55.

[3] Aranha JAP. Second order horizontal steady forces and moment on a floating body with small forward speed. J Fluid Mech 1996;313:39–54.

[4] Trassoudaine D, Naciri M. A comparison of a heuristic wave drift damping formula with experimental results. Appl Ocean Res 1999;21(2):93–97.

[5] Aranha JAP, Silva S, Martins MR, Leite AJP. An experimental verification of the wave drift damping formula. 14th International Workshop on Water Waves and Floating Bodies, Ann Harbour, Michigan. 1999.

[6] Beck RF, Cummins WE, Dalzell JF, Mandel P, Webster WC. Principles of naval architecture, vol. III, Motions in waves. Jersey City, NJ: The Society of Naval Architects and Marine Engineers, 1989.

[7] Kim KH, Yue DKP. The complete second-order diffraction solutions

for na axisymmetric body. Part 2: Bichromatic incident waves and body motions. J Fluid Mech 1990;211:557–93.

[8] White SB, Fornberg B. On the chance of freak waves at sea. J Fluid Mech 1998;355:113–38.

[9] Newman JN. Second-order slowly varying forces on vessels in irregular waves. Proceedings of the International Symposium on Dynamics of Marine Vehicles, London. 1974.

[10] Whitham GB. Linear and nonlinear waves. New York: Wiley, 1974.

[11] White SB. Wave action on currents with vorticity. J Fluid Mech 1999;386:329–44.

[12] Aranha JAP, Pesce CP. The effect of the second order potential in the slow drift oscillation of a floating structure in irregular waves. J Ship Res 1986;30(2):103–22.