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Regularization of a Mathematical Model of the Wheatley Heart Valve

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Abstract

This note considers the mathematical model published in the Journal of Biomechanical Engineering by McKee et al. [1]. The model presented there suffers from the fact that there is a line discontinuity in the first derivative producing what appears to be a kink in each of the leaflets. This note is concerned with regularizing the shape of the valve while holding to Wheatley's essential idea [2].

1 Background

The fifth author (D.J.W.) has a longstanding research experience in the field of heart valve replacement surgery([3, 4]) and holds a number of patents in this area ([2]); in particular, he has developed a novel design which has become known as the Wheatley valve. Existing artificial valves have not consistently demonstrated satisfactory durability or low thrombogenicity [5]. Before we can test the Wheatley valve for both durability and

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outgoing spiral flow (and thus lower thrombogenicity) it is useful to have a mathematical model of the Wheatley valve expressed in terms of simple mathematical functions. This note addresses this question and is essentially an addendum to the earlier paper [1].

2 Geometric description of the Wheatley heart valve

The original idea of the mathematical model arose from a realization that the valve could be constructed from a level set consisting of the arcs of three large intersecting circles and three smaller contiguous circles (see Fig. 2 of [1]). The arcs are coloured red in that figure and the contours (or level set) are the following arcs:

$$BP, \quad PQ, \quad B'P', \quad P'Q', \quad B''P'', \quad P''Q''. \tag{2.1}$$

This can be extended appropriately to three dimensions to produce a mathematical model of the Wheatley heart valve (see Fig. 6a of [1]).

3 Regularization of the mathematical model of the Wheatley heart valve

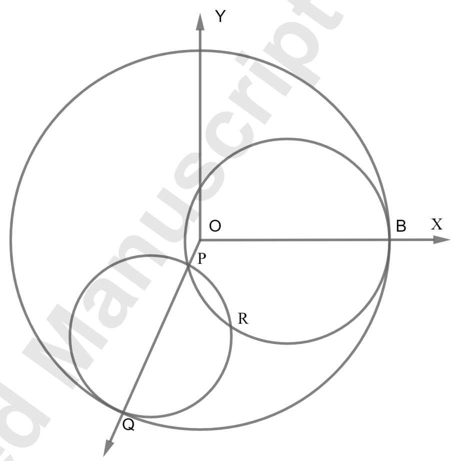

Figure 1: Two circles intersecting at P and R

Consider Fig. 2 of [1] and recall the equations of the large circle BPP' and the smaller circle PQ (see [1]):

$$\left(x - \frac{1-b^2}{2+b}\right)^2 + y^2 = \left(\frac{1+b+b^2}{2+b}\right)^2, \tag{3.2}$$

$$\left(x + \sqrt{3}y + (1+b)\right)^2 + \left(\sqrt{3}x - y\right)^2 = (1-b)^2, \tag{3.3}$$

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where $b = |OP|$. we note that the two circles pass through the point P, but do not meet at a tangent. Indeed, the circles intersect at two points P and R as we observe clearly from Fig. 1. We further note that the gradient of the tangent at P of the smaller circle is always $-1/\sqrt{3}$ and that the distance between the two intersecting points decreases with increasing b with the two points coinciding at $Q(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$ when $b = 1$.

One approach, and arguably close to the Wheatley philosophy, is to consider a third degree polynomial passing through the point P and satisfying both the function and its derivative at the points P and R (see Fig. 1). This is akin to attaching a paper sheet smoothly from P to R.

We recall that the coordinates of the point P are ([1])

$$(x_P, y_P) = \left(-\frac{1}{2}b, -\frac{\sqrt{3}}{2}b\right). \tag{3.4}$$

To determine the coordinates of the point R we need to solve the two equations for the intersecting circles, namely, (3.2), (3.3). These are found to be

$$(x_R, y_R) = \left(\frac{2b^2 - 7b + 2}{2(b^2 - 2b + 4)}, -\frac{\sqrt{3}(2b^2 - b + 2)}{2(b^2 - 2b + 4)}\right). \tag{3.5}$$

Now consider the polynomial

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3. \tag{3.6}$$

By requiring the polynomial (3.6) and it first derivative to be satisfied at the points (x_P, y_P) and (x_R, y_R) leads to the four equations

$$\begin{aligned}
a_0 + a_1 x_P + a_2 x_P^2 + a_3 x_P^3 - y_P &= 0, && (3.7)\\
a_0 + a_1 x_R + a_2 x_R^2 + a_3 x_R^3 - y_R &= 0, && (3.8)\\
a_1 + 2a_2 x_P + 3a_3 x_P^2 &= -1/\sqrt{3}, && (3.9)\\
a_1 + 2a_2 x_R + 3a_3 x_R^2 &= -(x_R - (-b^2 + 1)/(2 + b))/y_R, && (3.10)
\end{aligned}$$

where the gradient of the tangent to the larger circle at R is

$$-(x_R - (-b^2 + 1)/(2 + b))/y_R = \frac{\sqrt{3}(2b^4 - 2b^3 + 3b^2 - 8b - 4)}{6(2 + b)(b^2 - \frac{1}{2}b + 1)}.$$

Solving these we obtain

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$$
\begin{aligned}
a_0 &= -\left\{\frac{\sqrt{3}b(2b^8-5b^7+91b^6-352b^5+622b^4-608b^3+368b^2-32b-32)}{3(b^3-3b+2)(2b^5+5b^4-b^3-2b^2+4b-8)}\right\},\\
a_1 &= \left\{\frac{\sqrt{3}(2b^9-16b^8+151b^7-693b^6+1829b^5-3041b^4+2196b^3-1840b^2+352b+16)}{3(b^3-3b+2)(2b^5+5b^4-b^3-2b^2+4b-8)}\right\},\\
a_2 &= \left\{\frac{4\sqrt{3}(2b^8-20b^7+111b^6-385b^5+838b^4-1260b^3+1112b^2-656b+96)}{3(b^3-3b+2)(2b^5+5b^4-b^3-2b^2+4b-8)}\right\},\\
a_3 &= \left\{\frac{8\sqrt{3}(b^7-11b^6+48b^5-140b^4+256b^3-336b^2+256b-128)}{3(b^6+3b^5-3b^4-11b^3+6b^2+12b-8)(2b^2-b+2)}\right\}.
\end{aligned}
$$

In three dimensions, that is as b runs from 0 to 1, this "patch" is portrayed in the graph (see Fig. 2).

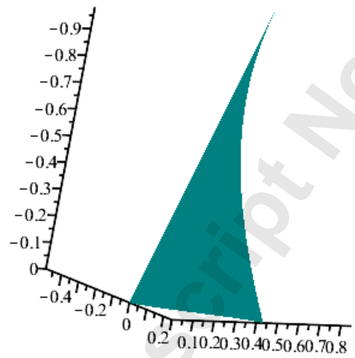

Figure 2: A graph of the 3D regularized surface joining the two "circles".

3.1 Starting and end points of arcs

We now need to put together this first leaflet. As we shall see the other leaflets can be constructed by rotations about $2\pi/3$ and $4\pi/3$, respectively. While in the original paper [1] the arcs stretched from B to P and from P to Q, they will now only run from B to R and from P to Q (see Fig. 1). Thus we need to determine the end point of the first arc; the starting point of the second arc, of course, remains unchanged ([1]).

Recall from [1] that the arc BP can be parameterized by

$$x - \frac{1-b^2}{2+b} = \left(\frac{1+b+b^2}{2+b}\right)\cos(t), y = \left(\frac{1+b+b^2}{2+b}\right)\sin(t).$$

At the point R, $x_R = \frac{2b^2-7b+2}{2(b^2-2b+4)}$ and so

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$$\cos(2\pi - t_R) = \cos(t_R) = \left\{\frac{b+2}{b^2+b+1}\right\}\left\{\frac{2b^2-7b+2}{2(b^2-2b+4)} + \frac{b^2-1}{2+b}\right\},$$

that is, the correct choice for t_R is

$$t_R = 2\pi - \arccos\left(\left\{\frac{b+2}{b^2+b+1}\right\}\left\{\frac{2b^2-7b+2}{2(b^2-2b+4)} + \frac{b^2-1}{2+b}\right\}\right). \tag{3.11}$$

Thus, in the notation of the paper [1],the correct choice is $t_B = 2\pi$ and t_R is given by (3.11).

For the remaining arc PQ, we have, as in [1], $t_P = 2\pi$ and $t_Q = \pi$.

With these starting and end points for the two arcs we can build the two pieces of the leaflet. This is displayed in the graph (see Fig. 3).

Note that there are multiple solutions to circular functions: care must be taken as to the starting point and the direction of travel around the different circles - clockwise for the larger circle and anticlockwise for the smaller circle.

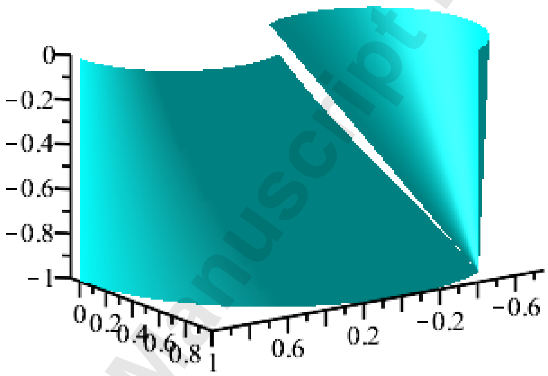

Figure 3: A graph of the 3D regularized surface with the missing ”patch”.

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The next step is to combine the three surfaces to produce the regularized surface of the first leaflet. This is displayed in Fig. 4.

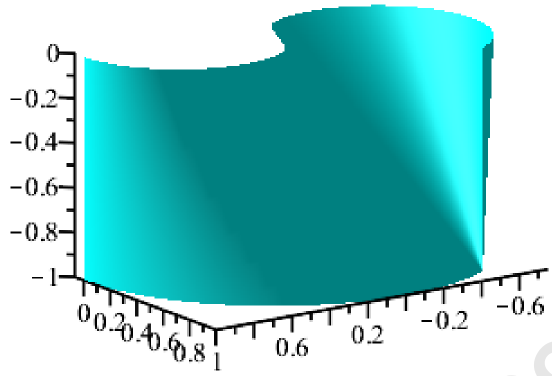

Figure 4: A graph of the first regularized leaflet

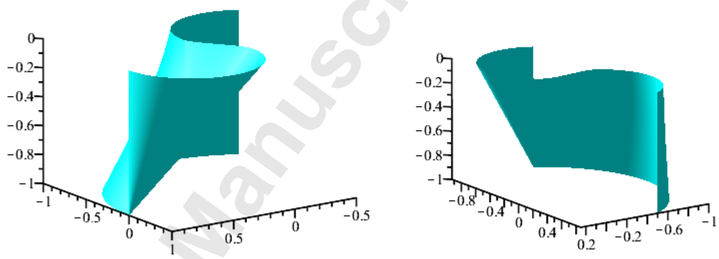

Figure 5: Graphs of the two remaining leaflets

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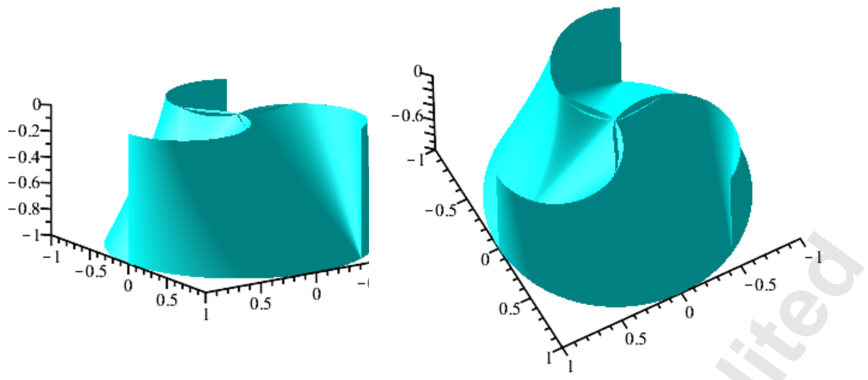

Figure 6: A graph of the regularized Wheatley heart valve from two view points

Returning to Fig. 2 of the earlier paper [1] we see that the large and small circles have a certain symmetry: rotation of BPP' and $P''Q''$ about the z-axis (the z-axis points vertically downwards from the paper) by an amount $2/3\pi$ and $4/3\pi$, produces the circles $B''P''P$, $P'Q'$ and $B'P'P''$, PQ, respectively. This creates the other two leaflets which are displayed in Fig. 5.

Finally, combining the three leaflets together we obtain the regularized form of the Wheatley valve (see Fig. 6). Comparing Fig. 2 of [1] with Fig. 6 it is clear that the discontinuity or "kink" has been smoothed out.

4 Concluding Remarks

The original mathematical model [1] suffered from the fact that it had a line discontinuity. This note has been concerned with removing this discontinuity. This was achieved by noticing that the two sets of circles in Fig. 2 of [1] intersect at two distinct points P and R and, as $b \to 1$, these two point coalesce at the point Q: this allowed a polynomial to be fitted between these two point, thus smoothing out the line discontinuity.

The importance of these mathematical models in the context of a novel heart valve has been shown in empiric development of prototype valves where the absence of a mathematical model has been sorely missed. This note and its predecessor [1] should therefore provide a major help to future heart valve researchers and other cardiovascular devices developers in the medical device industry.

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