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Technical Note

On the second-order slow drift force spectrum

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In this paper, it is shown that the slow drift force spectrum of a floating body, obtained from the exact quadratic transfer function, is flat in the low-frequency range of interest and can be written in the form $S_F(\mu) = S_F(0) + O(\mu^2)$, where $S_F(0)$ can be computed from the known drift force coefficient in harmonic waves and the wave energy spectrum. It is also shown here that a special and normally used form of Newman's approximation for the exact quadratic transfer function has an error of the form $[1 + O(\mu^2)]$ at low frequencies. Copyright © 1996 Elsevier Science Ltd.

1 INTRODUCTION

Low-frequency wave excitation on a floating body can be described by the so-called quadratic transfer function $T(\Omega_1; \Omega_2)$, namely the force that appears at the 'difference frequency' $\Delta\Omega = \Omega_2 - \Omega_1$, in the second-order interaction between two harmonic waves with unit amplitude and frequencies Ω_1 and Ω_2, respectively.

The numerical computation of $T(\Omega_1; \Omega_2)$ is difficult, since one needs to evaluate not only the quadratic interaction of the linear potential, but also to compute the second-order potential at the difference frequency $\Delta\Omega$. Observing that practical interest is focused on small values of $\Delta\Omega$, Newman[1] proposed the approximation $T(\Omega_1, \Omega_2) \cong T(\Omega_1 + \alpha\Delta\Omega; \Omega_1 + \alpha\Delta\Omega) \equiv D(\Omega_1 + \alpha\Delta\Omega)$, with $0 \leq \alpha \leq 1$ and $D(\ldots)$ being the drift force coefficient in harmonic waves.

If $T(\Omega_1; \Omega_2)$ is known, one can compute the low-frequency force spectrum $S_F(\Delta\Omega)$ and so the pertinent parameters of the response. In particular, if the low-frequency damping is small in the horizontal x-motion, its RMS value is given by the expression

$$\sigma_x = \sqrt{\frac{\pi}{4\zeta} \frac{S_F(\Omega_n)\Omega_n}{R^2}} \tag{1a}$$

where Ω_n is the small natural frequency, ζ is the percentage of the critical damping and R the restoring coefficient. In the context of this paper, the important point in the above expression is to make clear that the function $S_F(\ldots)$ needs to be computed only for small values of its argument.

The purpose of this paper is to show that $dS_F/d\Omega$ is zero at $\Omega = 0$ and so $S_F(\Omega)$ is 'flat' in the region of interest. If Ω_0 is the typical wave frequency, for example the sea spectrum peak frequency, one can introduce the variables:

$$\omega = \Omega/\Omega_0$$

$$\mu_n = \Omega_n/\Omega_0$$

$$\mu = \Delta\Omega/\Omega_0 \tag{1b}$$

Interest is centered in determining $S_F(\ldots)$ at frequencies $\mu \cong \mu_n \ll 1$, since usually $\mu_n \cong 0{\cdot}1$ or smaller. It is shown here that in this range of frequency one has

$$S_F(\mu) = S_F(0) + O(\mu^2), \qquad \mu \ll 1, \tag{2a}$$

with

$$S_F(0) = 8\int_0^\infty S^2(\omega)D^2(\omega)\,d\omega \tag{2b}$$

In the above integral, $S(\omega)$ is the wave energy spectrum and $D(\omega)$ is the drift force coefficient, namely, the drift force for a harmonic wave with unit amplitude.

The result expressed in eqn (2a) is general, it does not depend on any hydrodynamic approximation besides the usual ones (potential flow corrected to second-order in the wave amplitude) and it is valid for all six generalized low-frequency forces acting on the floating body. As will be seen next, this result depends only on the formal structure of the quadratic transfer function $T(\omega_1; \omega_2)$ and the definition of $S_F(\omega)$.

2 DEMONSTRATION OF EQN (2a)

Let $\Phi_j(\mathbf{x}, t) = \phi_j(\mathbf{x})e^{-i\omega_j t}$, $j = 1, 2$, be the total linear potential related to the radiation–diffraction problem of a harmonic wave with frequency ω_j and unit amplitude. The coupling between $\Phi_1(\mathbf{x}, t)$ and $\Phi_2^*(\mathbf{x}, t)$, where (*) stands for the complex conjugate, produces, at the second order, a force of the form $T(\omega_1; \omega_2)e^{i(\omega_2-\omega_1)t}$; the coupling between $\Phi_2(\mathbf{x}, t)$ and $\Phi_1^*(\mathbf{x}, t)$ produces the force $T(\omega_2; \omega_1)e^{i(\omega_1-\omega_2)t}$. One then obtains the known relation

$$T(\omega_1; \omega_2) = T^*(\omega_2; \omega_1) \tag{3}$$

The result expressed in eqn (2a) depends solely on this Hermitian property of the quadratic transfer function. In fact, if one writes

$$T(\omega_1; \omega_2) = P(\omega_1; \omega_2) + iQ(\omega_1; \omega_2) \tag{4a}$$

it follows, from eqn (3), that

$$P(\omega; \omega+\mu) = P(\omega+\mu; \omega)$$
$$Q(\omega; \omega+\mu) = -Q(\omega+\mu; \omega) \tag{4b}$$

If $D(\omega)$ is the drift force for a harmonic wave with unit amplitude then, from the definition of the quadratic transfer function and from eqn (4b), one obtains

$$P(\omega;\omega) = D(\omega)$$
$$Q(\omega;\omega) = 0 \tag{4c}$$

If the first expression in eqn (4b) is derived with respect to μ and μ is made equal to zero afterwards, one has

$$\frac{\partial P}{\partial \omega_1}(\omega; \omega) = \frac{\partial P}{\partial \omega_2}(\omega; \omega) \tag{4d}$$

Deriving now the first relation in eqn (4c) with respect to ω one gets

$$\frac{\partial D}{\partial \omega}(\omega) = \frac{\partial P}{\partial \omega_1}(\omega; \omega) + \frac{\partial P}{\partial \omega_2}(\omega; \omega) = 2\frac{\partial P}{\partial \omega_2}(\omega; \omega)$$

and so

$$\frac{\partial P}{\partial \omega_2}(\omega; \omega) = \frac{1}{2}\frac{dD}{d\omega}(\omega) \tag{4e}$$

From eqns (4a), (4c) and (4e) one obtains the result

$$\frac{d}{d\mu}|T(\omega; \omega+\mu)|^2\big|_{\mu=0} = D(\omega)\frac{dD}{d\omega}(\omega) \tag{5}$$

Following Kim and Yue,[2] the low-frequency spectrum $S_F(\mu)$ is defined by the convolution-like integral

$$S_F(\mu) = 8\int_0^\infty S(\omega)S(\omega+\mu)|T(\omega; \omega+\mu)|^2\,d\omega \tag{6}$$

If the above expression is derived with respect to μ and μ is taken equal to zero afterwards, one gets, with the help of eqns (4c) and (5)

$$\frac{dS_F}{d\mu}(0) = 8\int_0^\infty \left[S(\omega)\frac{dS}{d\omega}(\omega)D^2(\omega) + S^2(\omega)D(\omega)\frac{dD}{d\omega}(\omega)\right]d\omega$$
$$= 4\int_0^\infty \frac{d}{d\omega}[S^2(\omega)D^2(\omega)]\,d\omega = 0$$

since $S(0) = S(\infty) = 0$. Equation (2a) follows directly from this equality and from eqns (4c) and (6).

From eqns (4c) and (4e) it follows also that

$$P^2(\omega; \omega+\mu) = D^2(\omega) + \left[D(\omega)\frac{dD}{d\omega}(\omega)\right]\mu + O(\mu^2)$$
$$Q^2(\omega; \omega+\mu) = O(\mu^2)$$

Placing these relations in the expression that defines the quadratic transfer function one obtains

$$|T(\omega; \omega+\mu)|^2 = D^2(\omega) + \left(2D(\omega)\frac{dD}{d\omega}(\omega)\right)\frac{\mu}{2} + O(\mu^2)$$

and so

$$|T(\omega; \omega+\mu)|^2 = D^2\left(\omega+\frac{\mu}{2}\right) + O(\mu^2), \quad \mu \ll 1 \tag{7}$$

This is one of the possible ways to express Newman's approximation, corresponding to taking $|T(\omega; \omega+\mu)| = D(\omega+\mu/2)$; other approximations, for example $|T(\omega; \omega+\mu)| = D(\omega)$, have an error of order μ, and so are less precise.

3 CONCLUSIONS

It has been shown in this paper that the exact second-order low-frequency force spectrum is flat in the frequency range of interest and its value can be well approximated by a single number, defined in eqn (2b); this value depends only on the sea spectrum $S(\omega)$ and on the drift force coefficient $D(\omega)$ in harmonic waves. Apart from its numerical simplicity, this result may have some importance from a more practical point of view, since it shows that it is hopeless to try to detune the mooring line system from an eventual frequency where the excitation would be higher. Once the force spectrum is flat, the excitation is the same across the entire frequency range of practical interest.

It has been shown also that within the family of Newman's approximation there is one instance which is more precise, with an error of order $O(\mu^2)$, the remaining ones have an larger error, of $O(\mu)$.

In the specialized literature, one can find several numerical computations of the low-frequency force spectrum using, however, different sorts of 'approximations' to estimate the influence of the second-order potential. Most of them do not satisfy the exact relation $dS_F/d\mu=0$ at $\mu=0$ and some of them show a sharp disagreement with the force spectrum computed with Newman's approximation $|T(\omega;\ \omega+\mu)| \cong D(\omega+\mu/2)$. Apparently, the quadratic transfer function used in this numerical work does not satisfy the Hermitian property [eqn (3)] since, otherwise, the related force spectrum would also be 'flat' and coincident, in the small μ range, with the force spectrum computed from Newman's approximation.

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