Annals of Global Analysis and Geometry
https://doi.org/10.1007/s10455-022-09832-6

Timelike surfaces in the de Sitter space $\mathbb{S}^3_1(1) \subset \mathbb{R}^4_1$

M. P. Dussan[1] · A. P. Franco Filho[1] · M. Magid[2]

Received: 6 October 2019 / Accepted: 1 February 2022

Abstract
This paper studies timelike minimal surfaces in the De Sitter space $\mathbb{S}^3_1(1) \subset \mathbb{R}^4_1$ via a complex variable. Using complex analysis and stereographic projection of lightlike vectors in $\mathbb{C}\cup\{\infty\}$, we obtain a complex representation formula, together with some results about the existence of convenient isotropic coordinates. This allows us to construct timelike minimal surfaces in $\mathbb{S}^3_1(1)$ via local solutions of a certain PDE in a complex variable which arises when investigating our geometric conditions. Specifically, we find a new kind of complex functions which generalize the classes of holomorphic and anti-holomorphic functions, which we call quasi-holomorphic functions. We show that there is a correspondence between a timelike minimal surface in $\mathbb{S}^3_1(1)$ and a pair of quasi-holomorphic functions. In particular, when the two functions are holomorphic, we show that they are related by a Möbius transformation and then construct many families of minimal timelike surfaces in $\mathbb{S}^3_1(1)$ whose intrinsic Gauss map will also belong to the same class of surfaces. Several explicit examples are given.

Keywords Minimal surfaces · Timelike surfaces · Isotropic coordinates · de Sitter space · Holomorphic functions

Mathematics Subject Classification 53C42 · 53B30 · 30D60 · 34A26

1 Introduction

There have been many papers on timelike minimal surfaces in various ambient spaces. One of the first is Louise McNertney's thesis [11] in 1980, followed, in 1990, by the work of Van de Woestyne [13]. These papers work with either isotropic (null) coordinates or isothermal coordinates and examine various differential equations to analyze timelike minimal surfaces. Other techniques appear later. Beginning with the paper of Konderak [9] in 2005, we find the split-complex (para-complex) numbers used in place of the complex numbers to extend some results from positive definite surfaces to timelike minimal surfaces. This led, for example, to considering the Björling problem for timelike surfaces in different ambient spaces using the

✉ M. Magid
mmagid@wellesley.edu

1 Departamento de Matemática - IME, Universidade de São Paulo, São Paulo CEP:05508-090, Brazil

2 Wellesley College, Wellesley, USA

split-complex numbers by the authors in [1,5,6]. Following [1], other papers about timelike surfaces using split-complex numbers appear in the literature. These include: [2–4,8,12]. Other approaches to studying timelike minimal surface can be found in [7,10], using Loop groups or spinor theory, respectively. Although the use of the split-complex numbers allows many arguments to carry over to the timelike case, there are some difficulties, including that all split-meromorphic functions have singularities that consist of curves, not points, which restrict the kind of results one can obtain.

Our focus throughout this paper is to re-introduce complex variables and complex analysis into the study of timelike minimal surfaces in $\mathbb{R}^4_1$, via the light cone. In particular, we study timelike minimal surfaces in the de Sitter space $\mathbb{S}^3_1(1) \subset \mathbb{R}^4_1$. In order to do this, we first associate to two lightlike tangent vectors a convenient pair of complex functions (x, y) in such way that we can represent the surface by a complex representation formula involving the functions x and y. We identify two sets in $\mathbb{C}P^3$ with the set of timelike or spacelike oriented planes of $\mathbb{R}^4_1$, respectively, and using stereographic projection of lightlike vectors in $\mathbb{C} \cup \{\infty\}$, we establish our technique of constructing timelike minimal surfaces in $\mathbb{S}^3_1(1)$. This technique involves identifying the partial differential equations in a complex variable which describe the necessary conditions for our minimal surfaces. Investigating these conditions allows us to define a new kind of complex functions, which we call *quasi-holomorphic*, which are solutions to a PDE which generalizes the complex Cauchy-Riemann equations. We show that the set of the quasi-holomorphic functions contains the classes of holomorphic and anti-holomorphic functions. Moreover, we see that finding solutions of this complex PDE allows us to construct new examples of timelike minimal surfaces in $\mathbb{S}^3_1(1)$, and conversely, if we have a timelike minimal surface in $\mathbb{S}^3_1(1)$, then we can find two new solutions of the generalized complex Cauchy–Riemann-type equations, namely, the complex functions x, y which come from the stereographic projections of our chosen lightlike vectors.

In addition to the above, if we let x, y be holomorphic functions we show that the pairs of surfaces (M, f) and (M, ν), where ν represents the intrinsic Gauss map, are closely related. Indeed, if (M, f) is assumed, to be, for instance, a minimal non-totally geodesic isotropic surface in $\mathbb{S}^3_1(1)$ with Gauss map $\nu(w)$, then (M, ν) will also represent an isotropic minimal non-totally geodesic surface in $\mathbb{S}^3_1(1)$ with Gauss map $f(w)$. Moreover, the functions x and y are related by a Möbius function and the argument θ of the integration factor of the complex derivative f_w has to be a harmonic function on M. Finally we use our technique to construct explicit families of timelike minimal surfaces in $\mathbb{S}^3_1(1)$ with the associated families of (M, ν).

2 Preliminaries

We start by establishing some definitions, basic results and notation which we will use throughout the paper.

The Minkowski vector space $\mathbb{R}^4_1$ is the real vector space $\mathbb{R}^4$ endowed with the usual Euclidean topology and with the semi-Riemannian metric

$$\langle\ ,\ \rangle = -(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2.$$

It is oriented by $\partial_1 \wedge \partial_2 \wedge \partial_3 \wedge \partial_4$ and temporally by ∂_1, where $\{\partial_1, \partial_2, \partial_3, \partial_4\}$ is the canonical basis of $\mathbb{R}^4_1$.

Throughout this paper M will be an open connected and simply connected subset of the set of the complex numbers $\mathbb{C}$. We will denote by $\mathcal{H}(M)$ the set of holomorphic maps from $M \subset \mathbb{C}$ into $\mathbb{C}$. A complex map $f = P + iQ$ from M into $\mathbb{C}$ is an anti-holomorphic map if, and only if, its conjugate map $\overline{f} = P - iQ$ is a holomorphic map. The set of all anti-holomorphic maps will be denoted by $\overline{\mathcal{H}}(M)$. The set of all continuously differentiable maps from M into $\mathbb{C}$ will be denoted by $C^\infty(M, \mathbb{C})$, and we say that these maps are smooth maps from M into $\mathbb{C}$.

Let

$$\frac{\partial}{\partial w} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{w}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$$

be the differential operators defined over the set of all smooth maps from M into $\mathbb{C}$, where $w = u + iv \in M$. It follows from these expressions that a smooth map f from M into $\mathbb{C}$ is a holomorphic map if and only if $\frac{\partial f}{\partial \overline{w}}(w) = 0$ for all $w \in M$.

We will use often the notation $\frac{\partial f}{\partial w} = f_w$ and $\frac{\partial f}{\partial \overline{w}} = f_{\overline{w}}$.

Next we focus on the surfaces in the de Sitter space $\mathbb{S}^3_1(1)$.

A parametric surface of $\mathbb{R}^4_1$ is a function of two variables $f : M \longrightarrow \mathbb{R}^4_1$, where M is a connected open subset of $\mathbb{R}^2$, satisfying the following conditions:

(1) The function f is a homeomorphism from M onto $S = f(M)$, endowed with the subspace topology of $\mathbb{R}^4_1$.
(2) The function f is $C^\infty(M, \mathbb{R}^4_1)$.
(3) For each $w = (u, v) \in M$ the set $\{f_u(w), f_v(w)\}$ is a linearly independent set and the induced metric is given by

$$\mathrm{d}s^2(f) = E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2.$$

Here the functions $E(w)$, $F(w)$ and $G(w)$ are defined by

$$E(w) = \langle\frac{\partial f(w)}{\partial u}, \frac{\partial f(w)}{\partial u}\rangle, \qquad F(w) = \langle\frac{\partial f(w)}{\partial u}, \frac{\partial f(w)}{\partial v}\rangle \quad \text{and} \quad G(w) = \langle\frac{\partial f(w)}{\partial v}, \frac{\partial f(w)}{\partial v}\rangle.$$

Definition 2.1 A timelike surface in the sphere $\mathbb{S}^3_1(1)$ is a pair (M, f), where the function $f : M \to \mathbb{R}^4_1$ satisfies the conditions (1),(2) and (3) above, and for each $w \in M$, we have $\langle f(w), f(w)\rangle = 1$, with the metric tensor satisfying $EG - F^2 < 0$, i.e., it is a non-degenerate Lorentz metric. We call the local coordinates null or isotropic if the metric has the form: $\mathrm{d}s^2(f) = 2F\mathrm{d}u\mathrm{d}v$. It is always possible to find null coordinates locally.

In this paper we call a surface isotropic when we are using local null coordinates.

We assume that the lightlike vectors fields f_u and f_v are future directed, hence, $F(w) < 0$ for each $w \in M$. Moreover, we assume the surface is equipped with the Gauss map $\nu : M \longrightarrow \mathbb{S}^3_1(1)$, which is defined by the following conditions: for each $w \in M$,

(1) $\langle \nu(w), \nu(w)\rangle = 1$ and $\langle \nu(w), f(w)\rangle = 0$,
(2) $\langle \nu(w), f_u(w)\rangle = 0 = \langle \nu(w), f_v(w)\rangle$,
(3) the ordered set $\{f(w), f_u(w), f_v(w), \nu(w)\}$ is a positively oriented basis of $\mathbb{R}^4_1$.

We observe that, if we assume (M, f) with $f : M \to \mathbb{R}^4_1$ and the Gauss map $\nu(w)$ as defined above, it follows from conditions (1) and (2) of Definition 2.1 that $\nu_u(w), \nu_v(w) \in T_{f(w)}S$. For this reason we call the condition

$$(\forall w \in M) \;\; \{\nu_u(w), \nu_v(w)\} \subset T_{f(w)}S$$

the *spherical condition*. This means that the normal connection of this class of surface is flat.

From now on we will assume that the Gauss map $\nu(w)$ is not constant.

Next we will find the Gauss and Weingarten equations for an isotropic surface (M, f) of $\mathbb{S}^3_1(1)$ with Gauss map $\nu(w)$. Let

$$\mathcal{B}(w) = \{f(w), f_u(w), f_v(w), \nu(w)\}_{w \in M}$$

be the family of pointwise bases for $\mathbb{R}^4_1$ given by (3) of Definition 2.1.

Lemma 2.2 *Let (M, f) be an isotropic surface of $\mathbb{S}^3_1(1)$ equipped with the Gauss map $\nu(w)$. Since $\nu_u(w), \nu_v(w) \in span\{f_u(w), f_v(w)\}$, the structural equations for the surface are:*

$$\begin{cases} f_{uu} = \frac{F_u}{F} f_u + a\nu \\ f_{uv} = -Ff + b\nu \\ f_{vv} = \frac{F_v}{F} f_v + c\nu \end{cases} \text{(Gauss)}, \qquad \begin{cases} \nu_u = -\frac{b}{F} f_u - \frac{a}{F} f_v \\ \nu_v = -\frac{c}{F} f_u - \frac{b}{F} f_v \end{cases} \text{(Weingarten)}. \tag{1}$$

Moreover, the surface (M, f) is minimal if and only if $f_{uv}(w) = -F(w)f(w)$, which means that $b(w) = 0$ for each $w \in M$.

Proof We define $a = \langle f_{uu}, \nu\rangle$, $b = \langle f_{uv}, \nu\rangle$ and $c = \langle f_{vv}, \nu\rangle$. Once that is done, it is easy to verify the Gauss and Weingarten equations. For instance, since $\langle f_u, f\rangle = 0$ we have

$$\langle f_{uv}, f\rangle + \langle f_u, f_v\rangle = 0,$$

thereby obtaining the coefficient of f in the decomposition of f_{uv}. Finally, note that minimality means the trace of the shape operator is zero, or $b = 0$. □

Note that when the Gauss map $\nu(w) \in \mathbb{R}^4_1$ is a constant vector, the surface $f(M)$ is a totally geodesic surface, and so is a minimal surface in $\mathbb{S}^3_1(1)$. The timelike hyperplane $[\nu]^\perp$ contains $S = f(M)$ and the Gaussian curvature of S is $K(f)(w) = 1$, for all $w \in M$.

Corollary 2.3 *Let (M, f) be an isotropic surface of $\mathbb{S}^3_1(1)$ equipped with the non-constant Gauss map $\nu(w)$. Then the fundamental equations are given by*

$$K(f) = \frac{-1}{F}\left(\frac{F_u}{F}\right)_v = 1 - \frac{ac - b^2}{F^2} \quad \text{(Gauss)},$$

$$\frac{\partial b}{\partial u} - \frac{\partial a}{\partial v} = b\frac{F_u}{F} \quad \text{and} \quad \frac{\partial b}{\partial v} - \frac{\partial c}{\partial u} = b\frac{F_v}{F} \quad \text{(Codazzi)}.$$

Moreover if (M, f) is minimal, then $a(u, v) = a(u)$ and $c(u, v) = c(v)$, i.e., a and c are functions which depend only of u and v, respectively.

Proof The Gaussian curvature equation follows from $\langle (f_{uu})_v, f_v\rangle = \langle (f_{uv})_u, f_v\rangle$. Hence the Gauss equation follows immediately since

$$\left(\frac{F_u}{F}\right)_v - \frac{ac}{F} = -\langle f_u, f_v\rangle - \frac{b^2}{F}.$$

The Codazzi equations follow from $\langle (f_{uu})_v, \nu\rangle = \langle (f_{uv})_u, \nu\rangle$ and $\langle (f_{vv})_u, \nu\rangle = \langle (f_{uv})_v, \nu\rangle$. Indeed, $\langle (f_{uu})_v, \nu\rangle = a_v + b(F_u/F) = \langle (f_{uv})_u, \nu\rangle = b_u$ and $\langle (f_{vv})_u, \nu\rangle = b(F_v/F) + c_u = \langle (f_{uv})_v, \nu\rangle = b_v$. □

3 Sharing isotropic parameters

In this section we construct two isotropic surfaces which share isotropic parameters, and we show those are in correspondence through a dilatation and a translation.

Theorem 3.1 *Let (M, f) be an isotropic surface of $\mathbb{S}_1^3(1)$ equipped with a non-constant Gauss map $\nu(w)$. If the surface $f(M)$ is minimal then (M, ν) has the same isotropic parameters and is also minimal with*

$$\nu_u(w) = \frac{-a(w)}{F(w)} f_v(w) \quad \text{and} \quad \nu_v(w) = \frac{-c(w)}{F(w)} f_u(w).$$

Moreover, the Gaussian curvatures $K(f)$ of $f(M)$ and $K(\nu)$ of $\nu(S)$ are related by the equation:

$$F^2 K(f) + acK(\nu) = 0.$$

Hence, (M, f) is flat if and only if (M, ν) is flat.

Proof Since $f_u = (-F/c)\nu_v$ and $f_v = (-F/a)\nu_u$, we see that (M, ν) is isotropic and minimal. If we let $\langle \nu_u, \nu_v \rangle = \hat{F}$, so that the metric tensor of (M, ν) is

$$\mathrm{d}s^2(\nu) = 2\hat{F}\mathrm{d}u\mathrm{d}v,$$

then $\hat{F} = ac/F$. Now, from the Codazzi equations, we know $a = a(u)$ and $c = c(v)$. Then taking the v derivative of $(\log(\hat{F}))_u$ gives us

$$\left(\frac{\hat{F}_u}{\hat{F}}\right)_v = -\left(\frac{F_u}{F}\right)_v + \left(\frac{a_v}{a}\right)_u + \left(\frac{c_u}{c}\right)_v = -\left(\frac{F_u}{F}\right)_v.$$

Now the formula $F^2 K(f) + acK(\nu) = 0$ follows from the Gauss equation. □

3.1 The equation $\nu(w) = kf(w) + \mathbf{T}$

In this subsection we show that the Gauss map ν and the immersion f are directly related by a linear form.

Lemma 3.2 *Assume that the shape operator of the isotropic immersion f is diagonalized but never zero. Then there is a constant vector $\mathbf{T}$ so that $\nu(w) = kf(w) + \mathbf{T}$.*

Proof We are assuming that $a = 0 = c$ but $b \neq 0$ in the Weingarten equations. From the Codazzi equations we have

$$\frac{b_u}{b} = \frac{F_u}{F} \quad \text{and} \quad \frac{b_v}{b} = \frac{F_v}{F}.$$

It follows that $\frac{b(w)}{F(w)} = -k$ for some real number $k \neq 0$. Using the Weingarten equations again, we obtain $\nu_u = kf_u$ and $\nu_v = kf_v$, so $\nu(w) - kf(w) = \mathbf{T} \in \mathbb{R}_1^4 \setminus \{0\}$.

Note that $\mathbf{T}$ can not be 0, because we are assuming that $\{f(w), \nu(w)\}$ is a pointwise orthonormal basis of the normal bundle of (M, f). □

The following example shows that there exist non-minimal surfaces (M, f) and (M, ν) which share isotropic parameters.

Example 3.3 Let $X(w) = (X^1(w), X^2(w), X^3(w), 0)$ be an isotropic parametrization of an open subset of the sphere $\{X \in \mathbb{S}_1^3(1) | X^4 = 0\}$. Defining, for $\theta \in]0, \pi/2[$ and $w \in M = dom(X)$,

$$\nu(w) = \cos\theta\ \mathbf{e}_4 - \sin\theta\ X(w) \quad \text{and} \quad f(w) = \sin\theta\ \mathbf{e}_4 + \cos\theta\ X(w),$$

we have for, $k = -\tan\theta$ and $\mathbf{T} = \sec\theta\ \mathbf{e}_4$, a solution of the equation $\nu(w) = kf(w) + \mathbf{T}$.

Theorem 3.4 *Let (M, f) be an isotropic surface of $\mathbb{S}_1^3(1)$ equipped with the non-constant Gauss map $\nu(w)$. If (M, f) and (M, ν) are isotropic solutions of the equation $\nu(w) = kf(w) + \mathbf{T}$, for a constant $k \neq 0$, then there exists a basis for $\mathbb{R}_1^4$, $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4\}$, for which $k = -\tan\theta$ and $\mathbf{T} = \sec\theta\ \mathbf{t}_4$ for some $\theta \in]0, \pi/2[$. In addition, there is the isotropic parametrization (M, X) of the open subset $\{X = X^1\mathbf{t}_1 + X^2\mathbf{t}_2 + X^3\mathbf{t}_3 | \langle X, X\rangle = 1\}$ for which the solution of $\nu(w) = kf(w) + \mathbf{T}$ is*

$$\nu(w) = \cos\theta\ \mathbf{t}_4 - \sin\theta\ X(w) \quad \textit{and} \quad f(w) = \sin\theta\ \mathbf{t}_4 + \cos\theta\ X(w).$$

The Gaussian curvatures are, respectively, $K(f) = sec^2(\theta)$ and $K(\nu) = \csc^2(\theta)$.

Proof We begin with the fact that $\mathbf{T} = \nu - kf$ is a constant vector. By computing $\langle \mathbf{T}, \mathbf{T}\rangle = 1 + k^2 > 1$, we see that there is $\theta \in]0, \pi/2[$ such that $1 + k^2 = \sec^2\theta$, and we can assume that $\tan\theta = -k$. We set $\mathbf{t}_4 = \mathbf{T}/\sec\theta$, a spacelike vector. Since

$$\frac{\nu - \cos\theta\ \mathbf{t}_4}{-\sin\theta} = \frac{f - \sin\theta\ \mathbf{t}_4}{\cos\theta},$$

we can set either side to be a function $X(w)$. By calculating $\langle kf + \mathbf{T}, kf + \mathbf{T}\rangle = 1$, we find that $\langle \mathbf{T}, f\rangle = \tan\theta$, so that $\langle f, \mathbf{t}_4\rangle = \sin\theta$. This shows that $X(w) \perp \mathbf{t}_4$ and we can choose orthonormal vectors $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$ so that $\{\mathbf{t}_1, \ldots, \mathbf{t}_4\}$ is an orthonormal basis of $\mathbb{R}_1^4$. □

Next we will give an example of a timelike minimal surface with non-null Gaussian curvature, together with a coordinate transformation which allows us to obtain an equivalent isotropic surface. This example is a type of Clifford torus for $\mathbb{S}_1^3(1)$. The (unique) coordinate transformation also forces (M, ν) to have isotropic parameters, by Theorem 3.1.

Example 3.5 Let

$$c_1(t) = (\sinh t, 0, 0, \cosh t) \quad \text{and} \quad c_2(s) = (0, \cos s, \sin s, 0)$$

be two curves in $\mathbb{S}_1^3(1)$. The first is a timelike curve and the second a spacelike curve. Taking the two-parameter map

$$X(x, y) = \cos x\ c_1(y) + \sin x\ c_2(y)$$

we have $X_x = -\sin x\ c_1(y) + \cos x\ c_2(y)$ and $X_y = \cos x\ c_1'(y) + \sin x\ c_2'(y)$. Thus, the metric tensor has $E(x, y) = 1$, $F(x, y) = 0$ and $G(x, y) = -\cos^2 x + \sin^2 x = -\cos 2x$. The unit normal is given by:

$$\nu(x, y) = \frac{1}{\sqrt{\cos 2x}}(\sin x\ c_1'(y) + \cos x\ c_2'(y)).$$

Since $X_{xx} = -X$, $X_{yy} = \cos x\ c_1(y) - \sin x\ c_2(y)$ and $X_{xy} = -\sin x\ c_1'(y) + \cos x\ c_2'(y)$, the second quadratic form $\Psi_{ij} = \langle D_{ij}X, X\rangle X + \langle D_{ij}X, \nu\rangle\nu = X_{ij}X + N_{ij}\nu$ or, in matrix form,

$$[\Psi_{ij}] = \begin{bmatrix} -1 & 0 \\ 0 & \cos 2x \end{bmatrix} X + \begin{bmatrix} 0 & 1/\sqrt{\cos 2x} \\ 1/\sqrt{\cos 2x} & 0 \end{bmatrix} \nu.$$

Therefore

$$[\Psi_i^j] = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} X + \begin{bmatrix} 0 & -1/\sqrt{\cos^3 2x} \\ 1/\sqrt{\cos 2x} & 0 \end{bmatrix} \nu,$$

$$H = trace(\Psi) = -X \quad \text{and} \quad K(f) = \det(X_i^j) + \det(N_i^j) = 1 + \sec^2 2x,$$

where H denotes the mean curvature vector of the immersion into $\mathbb{R}^4_1$.

Define the coordinate transformation $p = p(x)$, $q(y) = y$ and take $Y(p, q) = X(x(p), y(q))$. Then the metric coefficients for Y are given by

$$\overline{E}(p, q) = x'(p)^2, \quad \overline{F}(p, q) = 0 \quad \text{and} \quad \overline{G}(p, q) = -\cos(2x(p)).$$

Next let

$$\int \frac{\mathrm{d}x}{\sqrt{\cos 2x}} = p(x).$$

Finally, by setting $u = p + q$ and $v = p - q$ we obtain the equivalent surface (M, f), where $f(u, v) = Y(p + q, p - q)$ is equipped with isotropic parameters.

4 An integration problem

In this section we look for conditions which allows us to find an integral representation formula for the isotropic surfaces. We begin by identifying local representations for lightlike vectors L which are in the tangent spaces. Moreover, we identify a correspondence between the sets of oriented spacelike planes in $\mathbb{R}^4_1$ and the oriented timelike planes in $\mathbb{R}^4_1$.

If $L = (L^1, L^2, L^3, L^4)$ is a future directed lightlike vector with $L^1 > 0$, then there exists an unique vector $\mathbf{n} \in \mathbb{R}^3 = span\{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ such that

$$L = L^1(\mathbf{e}_1 + \mathbf{n}) \quad \text{where} \quad \mathbf{n} = (0, L^2/L^1, L^3/L^1, L^4/L^1).$$

Since $\langle L, L \rangle = 0$ we have $\langle \mathbf{n}, \mathbf{n} \rangle = 1$. Let the north pole be $(0, 0, 0, 1)$ and define the stereographic projection, st, by

$$st(L) = a + ib = \left(\frac{L^2/L^1 + iL^3/L^1}{1 - L^4/L^1} \right) = \frac{L^2 + iL^3}{L^1 - L^4} \in \mathbb{C} \cup \{\infty\},$$

where $st(L) = \infty$ if and only if $L = \mu(1, 0, 0, 1)$. Moreover, $st(L) = 0$ if and only if $L = \mu(1, 0, 0, -1)$, with $\mu > 0$.

Proposition 4.1 *For each isotropic plane* $span\{L_1, L_2\} \subset \mathbb{R}^4_1$ *there exists an unique pair* $(x, y) \in (\mathbb{C} \cup \{\infty\})^2$*, such that we can express, for* $\mu_1, \mu_2 > 0$,

$$\mu_1 L_1 = \tilde{L}_1 = (1 + x\overline{x}, x + \overline{x}, -i(x - \overline{x}), -1 + x\overline{x}), \quad \textit{with} \quad x = st(L_1),$$

$$\mu_2 L_2 = \tilde{L}_2 = (1 + y\overline{y}, y + \overline{y}, -i(y - \overline{y}), -1 + y\overline{y}), \quad \textit{with} \quad y = st(L_2).$$

Therefore $\langle \tilde{L}_1, \tilde{L}_2 \rangle = -2|x - y|^2$.

In addition, the map $\mathcal{F}$ *from the set of oriented isotropic planes to the square of the Riemann sphere* $(\mathbb{C} \cup \{\infty\})^2$ *given by*

$$\mathcal{F}(span\{L_1, L_2\}) = (st(L_1), st(L_2))$$

is one-to-one and onto the open subset $(\mathbb{C} \cup \{\infty\})^2 \setminus \{(x, x)|\ x \in \mathbb{C} \cup \{\infty\}\}$.

At this point, with a slight abuse of notation, we define

$$L(x) = (1 + x\overline{x}, x + \overline{x}, -i(x - \overline{x}), -1 + x\overline{x}).$$

Next we identify the orthogonal complement of the space spanned by L_1 and L_2, which we denote by $[L_1, L_2]^{\perp} = [W] \in \mathbb{C}P^3$. To do this, we proceed as follows.

Let $\langle ., .\rangle^{\mathbb{C}}$ be the natural extension of the Lorentz inner product to $\mathbb{C}^4$ and $\mathbb{R}^4_1 = T \oplus S$ be the direct sum of a timelike plane $T = span\{L_1, L_2\}$ and a spacelike plane $S = span\{X, Y\}$, where we assume that

(1) the lightlike vectors L_1 and L_2 are future directed,
(2) the ordered set $\{X, L_1, L_2, Y\}$ forms a positive basis of $\mathbb{R}^4_1$ obeying the relations:

$$\langle X, X\rangle = \langle Y, Y\rangle > 0, \quad \langle X, Y\rangle = 0 \ \text{ and } \ \langle X, L_i\rangle = 0 = \langle Y, L_i\rangle \ \text{ for } \ i = 1, 2.$$

Next we define the Grassmannians of the oriented spacelike planes and the oriented timelike planes of $\mathbb{R}^4_1$ within the complex projective space $\mathbb{C}P^3 = \mathbb{C}^4/_{\equiv}$.

If $\mu = a + ib \neq 0$ is a complex number and $Z = X + iY$ is the complex vector associated with the basis of the spacelike plane S, then $\mu Z = (aX - bY) + i(bX + aY)$ gives us another basis of S satisfying the condition (2) above. By definition

$$[Z] = [X + iY] = \{\mu Z|\mu \in \mathbb{C} \ \text{ and } \ \mu \neq 0\}$$

are the equivalence classes that define points of $\mathbb{C}P^3$. Now, taking the complex vector $T = L_1 + iL_2$ associated with a timelike plane, and a complex number $\mu = a + ib \neq 0$, we have the complex vector $A + iB = \mu T = (aL_1 - bL_2) + i(bL_1 + aL_2)$ satisfying

$$\langle A, A\rangle = -2ab\langle L_1, L_2\rangle = -\langle B, B\rangle \quad \text{and} \quad \langle A, B\rangle = (a^2 - b^2)\langle L_1, L_2\rangle.$$

Therefore, $\{A, B\}$ is also a basis of the timelike plane T, and the determinant of the matrix associated with that basis is $-\langle L_1, L_2\rangle^2|\mu|^2 < 0$. Then we have the following:

Definition 4.2 a) The set of equivalence classes $\{[Z] \in \mathbb{C}P^3|\langle Z, Z\rangle^{\mathbb{C}} = 0$ and $\langle Z, \overline{Z}\rangle^{\mathbb{C}} > 0\}$ is the complex quadric of $\mathbb{C}P^3$ of the set of oriented spacelike planes of $\mathbb{R}^4_1$. In this case, we denote the set by Q_{space}.

b) We represent every lightlike vector $L = (1, \mathbf{n})$ where $\mathbf{n}$ is a unit vector in $\mathbb{R}^3$. Then the set of equivalence classes $\{[Z] \in \mathbb{C}P^3 \mid Z = [L_1 + iL_2],$ where $\langle L_1, L_2\rangle \neq 0\}$, is the set of oriented timelike planes of $\mathbb{R}^4_1$. This set is denoted by Q_{time} and has complex dimension two.

Next we establish an important correspondence between Q_{space} and Q_{time}, using homogeneous coordinates for Q_{space}. In fact, we define the following complex vector:

a) for $x, y \in \mathbb{C}$, with $x \neq y$, let

$$W(x, y) = (1 + x\overline{y}, x + \overline{y}, -i(x - \overline{y}), -1 + x\overline{y}) \in \mathbb{C}^4. \tag{2}$$

b) For $x \in \mathbb{C}$ and $y = \infty$, or for $x = \infty$ and $y \in \mathbb{C}$, we set $W(x, \infty) = (x, 1, i, x)$ or $W(\infty, y) = (\overline{y}, 1, -i, \overline{y})$.

Proposition 4.3 *Given the isotropic plane* $span\{L_1, L_2\}$, *let* $W(x, y)$ *be the complex vector (2). Then* $\langle W(x, y), L_1\rangle^{\mathbb{C}} = 0$ *and* $\langle W(x, y), L_2\rangle^{\mathbb{C}} = 0$ *if, and only if* $x = st(L_1)$ *and* $y = st(L_2)$, *or* $y = st(L_1)$ *and* $x = st(L_2)$. *Moreover*

$$\langle W(x, y), W(x, y)\rangle^{\mathbb{C}} = 0 \quad \textit{and} \quad \langle W(x, y), \overline{W(x, y)}\rangle^{\mathbb{C}} = -\langle L(x), L(y)\rangle = 2|x - y|^2 > 0.$$

Hence, there exists a bijection $\mathcal{G} : Q_{time} \longrightarrow Q_{space}$ *defined by*

$$\mathcal{G}([L_1 + iL_2]) = [W(x, y)].$$

Now we construct the integral representation for vector fields along M.

Let M be a connected and simply connected open subset of $\mathbb{C}$ and let $w = u + iv \in M$ denote its points.

First let us recall that, given two smooth functions $A, B : M \to \mathbb{R}$, there exists another pair of smooth functions $a, b : M \to \mathbb{R}$ such that $\Gamma = aA\mathrm{d}u + bB\mathrm{d}v$ is a closed 1-form if and only if $a_v A - b_u B = -aA_v + bB_u$. Then, since M is simply connected, it follows that, if the form Γ is closed it is exact. This means there is a smooth function $\varphi : M \to \mathbb{R}$ such that $d\varphi = aA\mathrm{d}u + bB\mathrm{d}v$.

We will apply this last fact to vector fields along M. First suppose that

$$V(w) = (\varphi^1(w), \varphi^2(w), \varphi^3(w), \varphi^4(w))$$

is a smooth vector field along M such that $\{V_u(w), V_v(w)\}_{w\in M}$ is a set of lightlike vectors which is linearly independent. Then there exist complex functions x, y and real-valued functions α, β such that

$$V_u(w) = \alpha(w)L(x(w)) \quad \text{and} \quad V_v(w) = \beta(w)L(y(w)),$$

where $\langle L(x), L(y)\rangle = -2|x - y|^2 \neq 0$. So, if we take $L = (L^1, L^2, L^3, L^4)$, then the components of Γ are given by

$$\Gamma^i = \frac{\partial \varphi^i}{\partial u}\mathrm{d}u + \frac{\partial \varphi^i}{\partial v}\mathrm{d}v = \alpha L^i(x)\mathrm{d}u + \beta L^i(y)\mathrm{d}v.$$

In other words, we have an unique pair α and β for each coordinate 1-form $\Gamma^i = d\varphi^i$.

Now we assume that the vector 1-form $\Gamma = \alpha L(x)\mathrm{d}u + \beta L(y)\mathrm{d}v$ is defined over the ring $\mathcal{F}(M, \mathbb{R})$ of smooth functions from M into $\mathbb{R}$. Since we are assuming that M is a simply connected open subset of $\mathbb{C}$, we have:

Proposition 4.4 *The* 1*-form* $\Gamma = \alpha L(x)\mathrm{d}u + \beta L(y)\mathrm{d}v$ *is exact if, and only if it is closed. Then*

$$\mathrm{d}\Gamma = \left[-\left(\alpha_v L(x) + \alpha\frac{\partial L(x)}{\partial v}\right) + \left(\beta_u L(y) + \beta\frac{\partial L(y)}{\partial u}\right)\right]\mathrm{d}u \wedge \mathrm{d}v = 0 \tag{3}$$

is a necessary and sufficient condition for the existence of the vector field $V(w)$ *such that* $\mathrm{d}V = \Gamma$.

If equation (3) holds, then the vector field $V(w)$ *is given by:*

$$V(w) = V_0 + \int_0^w \alpha L(x)\mathrm{d}u + \beta L(y)\mathrm{d}v. \tag{4}$$

Moreover, from $\langle \mathrm{d}\Gamma(\partial u, \partial v), L(y)\rangle = 0$ *and* $\langle \mathrm{d}\Gamma(\partial u, \partial v), L(x)\rangle = 0$ *we obtain the following equations:*

$$\frac{1}{\alpha}\frac{\partial \alpha}{\partial v} = \frac{-\langle \partial_v L(x), L(y)\rangle}{\langle L(x), L(y)\rangle} \quad \text{and} \quad \frac{1}{\beta}\frac{\partial \beta}{\partial u} = \frac{-\langle \partial_u L(y), L(x)\rangle}{\langle L(x), L(y)\rangle}. \tag{5}$$

Equation (5) is a necessary condition, but it is not sufficient.

Proof Starting with $\langle \mathrm{d}\Gamma(\partial u, \partial v), L(y)\rangle = 0$ we have

$$\frac{\alpha_v}{\alpha} = -\frac{\langle (L(x))_v, L(y)\rangle}{\langle L(x), L(y)\rangle} = \frac{-x_v}{x-y} + \frac{-\overline{x}_v}{\overline{x}-\overline{y}}.$$

The same proof works for β, so that equations (5) become

$$\frac{\alpha_v}{\alpha} = \frac{-x_v}{x-y} + \frac{-\overline{x}_v}{\overline{x}-\overline{y}} \quad \text{and} \quad \frac{\beta_u}{\beta} = \frac{y_u}{x-y} + \frac{\overline{y}_u}{\overline{x}-\overline{y}}. \tag{6}$$

□

5 Constructing timelike minimal parametric surfaces in $\mathbb{S}_1^3(1)$

In this section we look carefully at the formulas for the minimal immersion f in terms μ, W, x and y defined below, as well as the real-valued functions α and β which satisfy $f_u(w) = \alpha L(x)$ and $f_v(w) = \beta L(y)$. This allows us to find the governing equations for our surface corresponding to three conditions: the existence of isotropic coordinates, the immersion of the surface in $\mathbb{S}_1^3(1)$ and the minimality of the immersion. With these equations we can construct our immersions.

Let us take $W(x, y)$ given by equation (2), where

$$x(w) = st(f_u(w)) \qquad \text{and} \qquad y(w) = st(f_v(w)), \tag{7}$$

and (M, f) is an isotropic surface of $\mathbb{S}_1^3(1)$ equipped with the non-constant Gauss map $\nu(w)$. Then one can find a map $\mu(x, y) \in \mathbb{C}$ such that

$$f(w) = \frac{\mu\, W(x, y) + \overline{\mu}\, \overline{W(x, y)}}{2} \quad \text{and} \quad |\mu|^2 \langle W(x, y), \overline{W(x, y)}\rangle^{\mathbb{C}} = 2. \tag{8}$$

Next we look for complex partial differential equations which relate the functions $\mu(w)$, $x(w)$ and $y(w)$ for (M, f), where $f(w)$ is given by (8), and such that its Gauss map $\nu(w)$ has the form:

$$\nu(w) = \frac{\mu\, W(x, y) - \overline{\mu}\, \overline{W(x, y)}}{2i}. \tag{9}$$

In addition we ask that $\{\nu_u(w), \nu_v(w)\} \subset T_{f(w)}S$ for all $w \in M$. We seek those partial differential equations whose solution will guarantee that (M, f) is a parametric surface in $\mathbb{S}_1^3(1)$ with Gauss map $\nu(w)$. This means we are looking for the spherical condition for equation (8).

Lemma 5.1 (Spherical condition) *Let $f(w)$ be the map given by (8) with $x, y, \mu \in \mathcal{F}(M, \mathbb{C})$ and $W(x(w), y(w))$ as in (2). Let $\nu(w)$ be the map given by (9). Then (M, f) is a parametric surface of a scaled $\mathbb{S}_1^3(1)$ equipped with Gauss map $\nu(w)$ if, and only if,*

$$\frac{\mu_w}{\mu} = -\frac{\langle W_w, \overline{W}\rangle^{\mathbb{C}}}{\langle W, \overline{W}\rangle^{\mathbb{C}}} \quad \text{and} \quad \frac{\mu_{\overline{w}}}{\mu} = -\frac{\langle W_{\overline{w}}, \overline{W}\rangle^{\mathbb{C}}}{\langle W, \overline{W}\rangle^{\mathbb{C}}}. \tag{10}$$

Proof From equation (8) we have $\mu\bar{\mu}\langle W, \bar{W}\rangle^{\mathbb{C}} = 2$, so that

$$\frac{\mu_w}{\mu} + \frac{\langle W_w, \overline{W}\rangle^{\mathbb{C}}}{\langle W, \overline{W}\rangle^{\mathbb{C}}} + \frac{\overline{\mu}_w}{\overline{\mu}} + \frac{\langle \overline{W}_w, W\rangle^{\mathbb{C}}}{\langle W, \overline{W}\rangle^{\mathbb{C}}} = 0.$$

Since $\nu(w)$ is the Gauss map, it follows that, for all $w \in M$, $\{\nu_u(w), \nu_v(w)\} \subset T_{f(w)}S$. As we saw above, $\langle f_w, W\rangle^{\mathbb{C}} = 0 = \langle f_w, \overline{W}\rangle^{\mathbb{C}}$. Thus,

$$\mu_w\langle W, \overline{W}\rangle = -\mu\langle W_w, \overline{W}\rangle \quad \text{and} \quad \overline{\mu}_w\langle W, \overline{W}\rangle = -\overline{\mu}\langle \overline{W}_w, W\rangle.$$

These yield equations (10).

Now if equations (10) are satisfied, then $\mu\bar{\mu}\langle W, \bar{W}\rangle^{\mathbb{C}} = c > 0$, hence $\langle f, f\rangle$ is a positive constant. Since $2f_w = (\mu W)_w + (\overline{\mu}\overline{W})_w$, from (10) we find

$$\langle f_w, W\rangle^{\mathbb{C}} = \frac{1}{2}[\overline{\mu}_w\langle \overline{W}, W\rangle + \overline{\mu}\langle \overline{W}_w, W\rangle] = 0,$$
$$\langle f_w, \overline{W}\rangle^{\mathbb{C}} = \frac{1}{2}[\mu_w\langle \overline{W}, W\rangle + \mu\langle W_w, \overline{W}\rangle] = 0.$$

Therefore, for all $w \in M$, $\{\nu_u(w), \nu_v(w)\} \subset T_{f(w)}S$. Thus we see that (M, f) is a parametric surface of a scaled $\mathbb{S}^3_1(1)$ with Gauss map $\nu(w)$. □

Next we look for the conditions which imply that we can choose the parametric coordinates to be isotropic at every point of M.

Lemma 5.2 (Isotropic condition) *Let (M, f) and ν be the maps given, respectively, by (8) and (9) for which equations (10) hold. Then the pair (M, f) is a parametric isotropic surface of $\mathbb{S}^3_1(1)$ with Gauss map ν if and only if the following equations*

$$\begin{cases} \mathfrak{Im}\left(\mu\langle W_w, L(y)\rangle^{\mathbb{C}} + \overline{\mu}\langle \overline{W}_w, L(y)\rangle^{\mathbb{C}}\right) = 0, \\ \mathfrak{Re}\left(\mu\langle W_w, L(x)\rangle^{\mathbb{C}} + \overline{\mu}\langle \overline{W}_w, L(x)\rangle^{\mathbb{C}}\right) = 0 \end{cases} \tag{11}$$

are satisfied.

Proof By hypothesis, we are taking $W(x, y)$ with $x = st(f_u(w))$ and $y = st(f_v(w))$. Hence $f_u(w) = \alpha L(x)$ and $f_v(w) = \beta L(y)$ for α, β real-valued functions. Since $\langle f_w, L(y)\rangle$ is real-valued, and $2\langle f_w, L(y)\rangle = \mu\langle W_w, L(y)\rangle + \overline{\mu}\langle \overline{W}_w, L(y)\rangle$, it follows that $\mathfrak{Im}\left(\mu\langle W_w, L(y)\rangle + \overline{\mu}\langle \overline{W}_w, L(y)\rangle\right) = 0$. In a similar way, since $\langle f_w, L(x)\rangle$ is imaginary-valued, the second equation of (11) holds.

We now show sufficiency. The map $f(w)$ is given, and equation (9) tells us that $\nu(w)$ is its Gauss map, so that we have a timelike surface in $\mathbb{S}^3_1(1)$. Proposition 4.1 shows that a pointwise isotropic basis for the tangent bundle $T_{f(w)}S$ is given by $\{L(x(w)), L(y(w))\}_{w\in M}$. Next we need to show that f_u and f_v are isotropic. In fact, since $f_u = f_w + f_{\overline{w}} = AL(x) + BL(y)$ and $f_v = -i(CL(x) - DL(y))$, for A, B, C, D complex valued functions, the first and second equations in (11) imply, respectively, that $C(w) = 0$ and $B(w) = 0$ for all $w \in M$. □

Now we show that the pairs (M, f) and (M, ν) constructed above are closely related. In fact if (M, f) is assumed to be a minimal non-totally geodesic isotropic surface in $\mathbb{S}^3_1(1)$ with Gauss map $\nu(w)$, then (M, ν) will also represent an isotropic minimal surface in $\mathbb{S}^3_1(1)$ which is non-totally geodesic with Gauss map $f(w)$, and conversely.

Theorem 5.3 *Let (M, f) be a minimal parametric isotropic surface given by (8) equipped with Gauss map given by (9). Then (M, ν) is also a minimal non-totally geodesic isotropic surface in $\mathbb{S}^3_1(1)$ with Gauss map $f(w)$. Moreover, the isotropic condition for (M, ν) is given by the equations*

$$\begin{cases} \mathfrak{Im}\left(\mu\langle W_w, L(y)\rangle^{\mathbb{C}} - \overline{\mu}\langle \overline{W}_w, L(y)\rangle^{\mathbb{C}}\right) = 0, \\ \mathfrak{Re}\left(\mu\langle W_w, L(x)\rangle^{\mathbb{C}} - \overline{\mu}\langle \overline{W}_w, L(x)\rangle^{\mathbb{C}}\right) = 0. \end{cases} \tag{12}$$

Proof Since (M, f) is minimal, by the Weingarten equations we have $\nu_u = \frac{-a}{F} f_v$ and $\nu_v = \frac{-c}{F} f_u$. Hence

$$\nu_w = \frac{1}{2}(\frac{-a\beta}{F} L(y) + i\frac{c\alpha}{F} L(x)),$$

where $f_u(w) = \alpha L(x)$, $f_v(w) = \beta L(y)$, since, by hypothesis, x and y are chosen so that $x = st(f_u(w))$ and $y = st(f_v(w))$.

We see easily that $span\{f_u, f_v\} \subset T_{\nu(w)} S$. So (M, ν) is an isotropic surface in $\mathbb{S}^3_1(1)$ with Gauss map given by (M, f), which is also minimal non-totally geodesic.

In order to see that equations (12) hold, we proceed as follows. First, one has

$$\nu_w = \frac{1}{i}\left((\mu W)_w - f_w\right) = \frac{1}{i}\left(-(\overline{\mu}\overline{W})_w + f_w\right).$$

Since $\langle \nu_w, L(x)\rangle$ is real-valued, then $\langle (\mu W)_w - (\overline{\mu}\overline{W})_w, L(x)\rangle$ is pure imaginary. This corresponds to $\mathfrak{Re}\left(\mu\langle W_w, L(x)\rangle - \overline{\mu}\langle \overline{W}_w, L(x)\rangle\right) = 0$, the second equation of (12). Similarly, using the fact that $\langle \nu_w, L(y)\rangle$ is pure imaginary, one gets the first equation of (12). □

In order to finish this section, we introduce a new complex basis which will make many of our computations simpler. Define the set of complex vectors

$$c_1 = (1, 0, 0, -1), \quad c_2 = (0, 1, -i, 0), \quad c_3 = (0, 1, i, 0), \quad c_4 = (1, 0, 0, 1).$$

Each of these vectors is null with respect to the bilinear form $\langle , \rangle^{\mathbb{C}}$, and the matrix of $\langle c_i, c_j\rangle^{\mathbb{C}} = C_{ij}$ is given by

$$C_{ij} = \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}.$$

Using this new basis, we see that the expressions for $L(x)$ and $W(x, y)$ become

$$L(x) = c_1 + xc_2 + \overline{x}c_3 + x\overline{x}c_4, \qquad W(x, y) = c_1 + xc_2 + \overline{y}c_3 + x\overline{y}c_4, \tag{13}$$

and one easily checks that $W(y, x) = \overline{W(x, y)}$. Moreover, if $x = x(w)$ and $y = y(w)$, then

$$W_w = x_w(c_2 + \overline{y}c_4) + \overline{y}_w(c_3 + xc_4) \quad \text{and} \quad \langle W_w, L(x)\rangle^{\mathbb{C}} = 2(\overline{x} - \overline{y})x_w.$$

In addition, the spherical condition (10) given by Lemma 5.1 is equivalent to

$$|\mu| = \frac{1}{|x - y|}, \qquad \frac{\mu_w}{\mu} = \frac{-x_w}{x - y} + \frac{\overline{y}_w}{\overline{x} - \overline{y}} \quad \text{and} \quad \frac{\mu_{\overline{w}}}{\mu} = \frac{-x_{\overline{w}}}{x - y} + \frac{\overline{y}_{\overline{w}}}{\overline{x} - \overline{y}}. \tag{14}$$

Furthermore, the isotropic condition (11) given by Lemma 5.2, with the orientation given by $\{L(x(w)), L(y(w))\}$, is equivalent to

$$\frac{\overline{\mu} y_v}{x - y} + \frac{\mu\, \overline{y}_v}{\overline{x} - \overline{y}} = 0 \quad \text{and} \quad \frac{\mu x_u}{x - y} + \frac{\overline{\mu}\, \overline{x}_u}{\overline{x} - \overline{y}} = 0. \tag{15}$$

5.1 Formulas for the mean curvature of timelike parametric surfaces in $\mathbb{S}^3_1(1)$

In this subsection we look for expressions for the mean curvature of timelike surfaces in $\mathbb{S}^3_1(1)$. We will use these expressions in the next section in order to define a new kind of complex function, which will generalize holomorphic functions.

Recall we are assuming that (M, f) is an isotropic surface in $\mathbb{S}^3_1(1) \subset \mathbb{R}^4_1$. Thus, there exist two smooth functions $\alpha, \beta : M \to \mathbb{R}$ and two smooth functions $x, y : M \to \mathbb{C}$ such that

$$f_u(w) = \alpha(w)L(x(w)) \quad \text{and} \quad f_v(w) = \beta(w)L(y(w)),$$

with $F = \langle f_u, f_v\rangle = -2\alpha\beta|x-y|^2$. Moreover, there exists a smooth complex function $\mu : M \to \mathbb{C}$ such that $f(w)$, $W(x, y)$ are given by formulas (8) and (2), and the intrinsic Gauss map is the function $\nu(w)$ given by (9). We also have the fixed reference frame $\mathcal{B} = \{f(w), L(x(w)), L(y(w)), \nu(w)\}$.

The mean curvature of this surface is the trace of $A_\nu = \langle H_f, \nu\rangle = \langle \frac{f_{uv}}{F}, \nu\rangle$, where $F = \langle f_u, f_v\rangle$ and H_f is the mean curvature vector. We simplify our notation by writing $\langle \frac{f_{uv}}{F}, \nu\rangle = \frac{\Phi(w)}{F}$, where $\Phi = \langle f_{uv}, \nu\rangle$.

Next we look more closely at Φ. In fact, since $\Phi = \langle f_{uv}, \nu\rangle = \langle(\alpha L(x))_v, \nu\rangle = -\alpha\langle L(x), \nu_v\rangle$ we have

$$\Phi = -\alpha\langle L(x), \nu_v\rangle = -\alpha\langle L(x), (\mu W_v - \overline{\mu}\overline{W}_v)/2i\rangle.$$

Using (13) we find $\langle L(x), W_v\rangle = 2(\overline{x}-\overline{y})x_v$ and $\langle L(x), \overline{W}_v\rangle = 2(x-y)\overline{x}_v$. Thus,

$$\Phi = -2\frac{\alpha}{2i}\left(\mu(\overline{x}-\overline{y})x_v - \overline{\mu}(x-y)\overline{x}_v\right) = -2\alpha\mathfrak{Im}\left(\mu(\overline{x}-\overline{y})x_v\right). \tag{16}$$

Again, since $\Phi = \langle f_{uv}, \nu\rangle = \langle(\beta L(y))_u, \nu\rangle$, we have

$$\Phi = -\beta\langle L(y), \nu_u\rangle = -\beta\langle L(y), (\mu W_u - \overline{\mu}\overline{W}_u)/2i\rangle = 2\beta\mathfrak{Im}\left(\mu(x-y)\overline{y}_u\right).$$

Altogether then one has

$$\Phi = 2\beta\mathfrak{Im}\left(\mu(x-y)\overline{y}_u\right) \quad \text{and} \quad \Phi = -2\alpha\mathfrak{Im}(\mu(\overline{x}-\overline{y})x_v). \tag{17}$$

We codify this in the next lemma.

Lemma 5.4 *Let (M, f) be an isotropic parametric surface of the de Sitter space $\mathbb{S}^3_1(1)$. Then, with the notation above,*

$$\alpha\mathfrak{Im}\left(\mu(\overline{x}-\overline{y})x_v\right) + \beta\mathfrak{Im}\left(\mu(x-y)\overline{y}_u\right) = 0.$$

Now we continue looking at the formulas for F, and for the functions α and β.

Lemma 5.5 *Let (M, f) be an isotropic parametric surface of the de Sitter space $\mathbb{S}^3_1(1)$. Assume that $\frac{\Phi(w)}{F}$ is the mean curvature of $S = f(M)$. Then*

$$F = -2\alpha\beta|x-y|^2 = 2\alpha\mathfrak{Re}\left(\mu(\overline{x}-\overline{y})x_v\right) = -2\beta\mathfrak{Re}\left(\mu(x-y)\overline{y}_u\right), \tag{18}$$

and therefore:

$$\alpha = \mathfrak{Re}\left(\mu\frac{\overline{y}_u}{\overline{x}-\overline{y}}\right) \quad \text{and} \quad \beta = -\mathfrak{Re}\left(\mu\frac{x_v}{x-y}\right). \tag{19}$$

In particular, if $\Phi = 0$, then the real-valued functions α and β become

$$\alpha = \mu\frac{\overline{y}_u}{\overline{x}-\overline{y}} \quad \text{and} \quad \beta = -\mu\frac{x_v}{x-y}. \tag{20}$$

Proof Since $\langle f_{uv}, f\rangle = -\langle f_u, f_v\rangle = 2\alpha\beta|x-y|^2$, we obtain

$$-F = \frac{1}{2}\left(\langle f_{uv}, \mu W(x,y)\rangle + \langle f_{uv}, \overline{\mu} W(y,x)\rangle\right) = \alpha\left(\frac{\mu}{2}\langle L_v(x), W(x,y)\rangle + \frac{\overline{\mu}}{2}\langle L_v(x), \overline{W}\rangle\right).$$

Then equation (18) follows from $F = 2\alpha\mathfrak{Re}\,(\mu x_v(\overline{x}-\overline{y}))$. In similar way, one gets $F = -2\beta\mathfrak{Re}\left(\mu(x-y)\overline{y}_u\right)$. The equation (19) follows by substitution, and (20) follows from (17). □

6 When Φ = 0 and a new class of functions

In this section we investigate isotropic surfaces in $\mathbb{S}^3_1(1)$ with $\Phi = 0$. We then define a new class of complex functions which we will call *quasi-holomorphic* and which appear naturally when we consider an isotropic surface (M, f) in $\mathbb{S}^3_1(1)$, with f given by (8) and $\Phi = 0$. We will also see that the set of quasi-holomorphic functions contains as subsets both holomorphic and anti-holomorphic functions. When we restrict ourselves to the case where x, y are holomorphic functions and $\Phi = 0$, we find our results are simplified and that the argument of the function μ has to be harmonic. After this, we construct several explicit examples.

We continue under the same conditions as in Sect. 5. Our first result establishes equations (22), which we will use frequently in the rest of the paper.

Theorem 6.1 *If (M, f) is an isotropic parametric surface of $\mathbb{S}^3_1(1)$ with mean curvature vector H_f, then*

$$\langle H_f, \nu\rangle = \frac{1}{\beta}\mathfrak{Im}\left(\mu\frac{x_v}{x-y}\right) = \frac{1}{\alpha}\mathfrak{Im}\left(\overline{\mu}\frac{y_u}{x-y}\right). \tag{21}$$

Moreover, if $\langle H_f, \nu\rangle = 0$, so that $\Phi = 0$, then

$$x_{uv} = \frac{2x_u x_v}{x-y} \quad \text{and} \quad y_{uv} = \frac{-2y_u y_v}{x-y}. \tag{22}$$

Proof First note that (21) follows from (17) and (18), using the fact that $\mathfrak{Im}\gamma = -\mathfrak{Im}(\overline{\gamma})$. Next we derive equations (22). Taking the logarithmic derivative of equation (20) for the function β we obtain

$$\frac{\beta_u}{\beta} = \frac{\mu_u}{\mu} + \frac{x_{uv}}{x_v} - \frac{x_u - y_u}{x-y}.$$

From second part of equation (6) for β and from the version of equation (10) for the variable u, namely

$$\frac{\mu_u}{\mu} = \frac{-x_u}{x-y} + \frac{\overline{y}_u}{\overline{x}-\overline{y}},$$

we get the first equation of (22). The second equation follows in a similar way. □

Now we are able to construct explicit examples of minimal isotropic surfaces in $\mathbb{S}^3_1(1)$.

Example 6.2 We observe that a solution of the system (22) is given by the real-valued functions $x = v$ and $y = u$. We then define the parametric surface

$$f(u,v) = \frac{W(v,u) + W(u,v)}{2(u-v)} \quad \text{for} \quad M = \{(u,v) \in \mathbb{C} |\ u > v\}.$$

Since in this case $\mu = 1/(u - v)$, the spherical condition (14) and isotropic conditions (15) are satisfied trivially. Furthermore, because the third coordinate $f^3(u, v) = 0$, the subset $f(M)$ is an open subset of the sphere

$$\{(t, x, 0, z) \in \mathbb{R}^4_1 | -t^2 + x^2 + z^2 = 1\}.$$

Thus, this surface is a totally geodesic open submanifold of the 2-dimensional de Sitter space form, away from the set $u = v$. In consequence, it is minimal in $\mathbb{S}^3_1(1)$.

Example 6.3 For each $w = u + iv \in \mathbb{C}$ let

$$\mu(u, v) = \frac{\sqrt{2}(1+i)}{4} e^{(v-u)},$$

and x and y the solution of the system (22) given by

$$x(u, v) = e^{(u-v)+i(v+u)} \quad \text{and} \quad y(u, v) = -e^{(u-v)+i(v+u)}.$$

We see that we have an isotropic surface in $\mathbb{S}^3_1(1)$ and the shape operator, with respect to the flat null coordinates $\{u, v\}$, is given by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. In fact, the functions $f(w) = \frac{\mu W + \overline{\mu} \overline{W}}{2}$ and $\nu(w) = \frac{\mu W - \overline{\mu} \overline{W}}{2i}$ take the form

$$f(u, v) = \frac{\sqrt{2}}{2}(\sinh(v-u), -\sin(u+v), \cos(u+v), -\cosh(v-u)),$$
$$\nu(u, v) = \frac{\sqrt{2}}{2}(\sinh(v-u), \sin(u+v), -\cos(u+v), -\cosh(v-u)).$$

With these formulas, we find $\langle f, f\rangle = 1 = \langle \nu, \nu\rangle$, $\langle f, \nu\rangle = 0 = \langle f_u, \nu\rangle = \langle f_v, \nu\rangle$ and f_u, f_v are lightlike vectors with $\langle f_u, f_v\rangle = F = 1$. Our basis of $\mathbb{R}^4_1$ is given by $\{f, f_u, f_v, \nu\}$. Moreover $\langle f_{uv}, \nu\rangle = 0$, which implies that the surface is minimal, so $\Phi = 0$. Hence, using formulas (20), the real-valued functions α and β take the form

$$\alpha = -\frac{\sqrt{2}}{4} e^{v-u} = -\beta.$$

Now it is easy to see that the spherical and isotropic conditions (14) and (15) are satisfied.

Finally, we note that, from Theorem 5.3, the pair (M, ν) also represents a timelike minimal surface in $\mathbb{S}^3_1(1)$ with Gauss map given by $f(w)$, whose isotropic conditions are given by formula (15).

Our explicit construction of isotropic surfaces in $\mathbb{S}^3_1(1)$ with $\Phi = 0$ allows us to define a new set of complex functions which contains, as a subset, the holomorphic functions. In fact,

Theorem 6.4 *Let (M, f) be an isotropic parametric surface in $\mathbb{S}^3_1(1)$ such that $\Phi = 0$ and*

$$f(w) = \frac{\mu(w) W(x(w), y(w)) + \overline{\mu(w)} W(y(w), x(w))}{2},$$

with $f_u = \alpha L(x)$ and $f_v = \beta L(y)$. Then the functions $x, y : M \to \mathbb{C}$ belong to a class of complex functions $Z(w) = \varphi(w) + i\psi(w)$ such that

$$\frac{\partial Z}{\partial v} = i\sigma(w) \frac{\partial Z}{\partial u}, \quad \text{where} \quad \sigma : M \longrightarrow \mathbb{R} \quad \text{with} \quad \sigma(w) \neq 0 \;\; (\forall w \in M). \tag{23}$$

In addition, φ and ψ satisfy the following Cauchy–Riemann-type equations:

$$\begin{cases} \varphi_u = \frac{1}{\sigma}\ \psi_v \\ \varphi_v = -\sigma\psi_u. \end{cases} \tag{24}$$

Proof Assuming that $\Phi = 0$, we obtain, from the second equation of (17),

$$\frac{\mu x_v}{x-y} = \frac{\overline{\mu}\ \overline{x}_v}{\overline{x}-\overline{y}}.$$

Combining this last equation with the second equation of (15), we find that $x_u\ \overline{x}_v + \overline{x}_u\ x_v = 0$. Writing $x = a + ib$ in the last equation, we see that $a_u\, a_v + b_u\, b_v = 0$, which means that the set of $\mathbb{R}^2$-vectors $\{(b_v, -a_v), (a_u, b_u)\}$ is a linearly dependent set. So, pointwise, there exists a real-valued function $\sigma = \sigma(u, v)$ such that

$$x_v(u, v) = i\sigma(u, v)\ x_u(u, v) \quad \text{for} \quad (u, v) \in M.$$

An analogous computation shows that the function $y = y(w)$ satisfies $y_u(u, v) = i\xi(u, v)\ y_v(u, v)$ for some real-valued function $\xi = \xi(u, v)$ defined over M. Thus we have derived equation (23) for x and y. Moreover, equations (24) are then satisfied.

In light of Theorem 6.4 we define a new class of functions, as follows.

Definition 6.5 A complex function $Z : M \longrightarrow \mathbb{C}$ is called quasi-holomorphic if, and only if, there exists a real-valued function $\sigma : M \longrightarrow \mathbb{R}$ such that

$$\frac{\partial Z}{\partial v} = i\sigma\frac{\partial Z}{\partial u}.$$

We denote this set of functions by $\mathcal{O}(M)$. Observe that $\sigma = 1$ implies that Z is a holomorphic function on M, which means that $Z' = Z_u$ and $Z' = -iZ_v$.

As mentioned above, we see that $\mathcal{O}(M)$ contains the following subsets:

Proposition 6.6 *The class of holomorphic and anti-holomorphic functions, $\mathcal{H}(M)$, $\overline{\mathcal{H}(M)}$, are contained in $\mathcal{O}(M)$. Moreover $\mathcal{O}(M)$ is closed under conjugation, i.e., $\mathcal{O}(M) = \overline{\mathcal{O}(M)}$.*

Example 6.7 Let $Z(w) = \varphi(u, v) + i\psi(u, v)$ be a holomorphic function and $a(u)$, $b(v)$ two real-valued functions. Then the function

$$\Psi = \varphi\,(a(u), b(v)) + i\psi\,(a(u), b(v)) \in \mathcal{O}(M).$$

Indeed, since $\Psi_u = a'(u)Z_u$, $\Psi_v = b'(v)Z_v$ and $Z \in \mathcal{H}(M)$, it follows from $Z_v = iZ_u = iZ'$, that $\Psi_v = i\sigma\Psi_u$, with $\sigma = \frac{b'(v)}{a'(u)}$.

For instance, if we take $Z(w) = w^2$, $a(u) = u$ and $b(v) = v^2$, then $\Psi(u, v) = u^2 - v^4 + 2iuv^2 \in \mathcal{O}(M)$ with $\sigma(u, v) = 2v$. Here then $\Psi_v = 2vi(2u + 2iv^2) = 2vi\Psi_u$.

6.1 When $\Phi = 0$ and x, y are holomorphic functions satisfying the system (22)

In this last subsection we focus on isotropic surfaces with $\Phi = 0$ for which x, y are holomorphic functions that satisfy the system (22). We show that, in this case, the functions x and y are related by a Möbius transformation and that the argument θ of function μ has to be a harmonic function in M. In particular, we give explicit formulas for x and y when (M, f) is a minimal isotropic surface in $\mathbb{S}^3_1(1)$ with f being given by (8). We also explicitly construct families of timelike surfaces in $\mathbb{S}^3_1(1)$ with $\Phi = 0$. This example will be a generalization of Example 6.3.

In what follows, we use the symbols x' or y' to mean the complex derivative x_w or y_w, respectively. We will assume $x \neq y$.

Theorem 6.8 *Let $x(w)$ and $y(w)$ be two holomorphic functions from M into $\mathbb{C}$, such that $x - y \neq 0$ and $x'y' \neq 0$. Since $x_u = x'$ and $x_v = ix'$, and the same is true for y, the system (22) for these functions becomes*

$$x'' = \frac{2x'^2}{x-y} \quad \text{and} \quad y'' = \frac{-2y'^2}{x-y}. \tag{25}$$

Then, there exists a Möbius transformation

$$M_c(z) = \frac{z}{cz-1} \quad \text{where} \quad c \in \overline{\mathbb{C}},$$

such that $y'(w) = M_c(x'(w))$ for each $w \in M$. Conversely, if $x(w)$ and $y(w)$ are related by $M_c(z)$ and $x(w)$ is a solution of the first equation in (25), then $y(w)$ is solution of second one.

Proof Since

$$\frac{x''}{x'^2} + \frac{y''}{y'^2} = \left(\frac{-1}{x'}\right)' + \left(\frac{-1}{y'}\right)' = 0 \iff \left(\frac{1}{x'}\right) + \left(\frac{1}{y'}\right) = c \in \mathbb{C},$$

we obtain the family of relations $y' = M_c(x')$. Conversely, if we assume that $y' = M_c(x')$ and x satisfies (25), then it follows that $y''/(y')^2 = -2/(x-y)$. □

Corollary 6.9 *Let $x(w)$, $y(w)$ be holomorphic functions from M into $\mathbb{C}$, such that $x - y \neq 0$, $x'y' \neq 0$ and which satisfy equations (25). If $c = 0$ in the Möbius transformation, then $y' = -x'$, and if $c = \infty$ then $y' = 0$.*

Example 6.10 If $c = 0$, so that $x' + y' = 0$, then $x + y = 2a$ for some $a \in \mathbb{C}$. Letting $x - y = 2z(w)$ it follows that $x = z(w) + a$. Then equations (25) become $\frac{z''}{z'} = \frac{z'}{z}$. This implies $\frac{z'}{z} = k$, for some complex number k. Finally, we find that the solution of the system (25) is

$$x = a + e^{kw+b} \quad \text{and} \quad y = a - e^{kw+b}$$

for complex numbers a, b and $k \neq 0$.

Next we obtain information about the argument of the integration factor μ. Since $|\mu| = \frac{1}{|x-y|}$, its polar form is

$$\mu(w) = \frac{e^{i\theta(w)}}{|x(w) - y(w)|}.$$

Lemma 6.11 *For holomorphic functions x, y from M into $\mathbb{C}$, the spherical condition (14) for the polar form of μ is*

$$\theta_w = \frac{i}{2}\,\frac{x' + y'}{x - y}. \tag{26}$$

Therefore, the real-valued function θ is harmonic in M.

Proof As $x_w = x'$, $x_{\overline{w}} = 0$ and $\log\mu = i\theta - \frac{1}{2}\log(x-y) - \frac{1}{2}\log(\overline{x}-\overline{y})$, then $\frac{\mu_w}{\mu} = i\theta_w - \frac{1}{2}\frac{x'-y'}{x-y}$. Since the same equations hold for y, we obtain from equations (14) that

$$\frac{-x'}{x-y} = i\theta_w - \frac{1}{2}\frac{x'}{x-y} + \frac{1}{2}\frac{y'}{x-y},$$

which implies equation (26). □

For the next lemma we recall that we are assuming that f_u and f_v are multiples of $L(x)$ and $L(y)$, respectively.

Lemma 6.12 *For holomorphic functions x, y from M into $\mathbb{C}$, the isotropic conditions (15) correspond to the equations*

$$\mathfrak{Re}\left(e^{i\theta}\frac{x'}{x-y}\right) = 0 \quad \text{and} \quad \mathfrak{Re}\left(e^{-i\theta}\frac{iy'}{x-y}\right) = 0. \tag{27}$$

Proof It follows from equations (15) that, since $|x-y|$ is real, $x_u = x'$ and $y_v = iy'$. Indeed, equations (15) say that $\frac{e^{i\theta}\,x'}{x-y}$ and $\frac{ie^{-i\theta}\,y'}{x-y}$ are imaginary-valued functions. □

The next corollary says that the function θ contains quite a bit of information about the holomorphic functions x', y' and $x-y$.

Corollary 6.13 *For x, y holomorphic functions from M into $\mathbb{C}$, the equations (27) mean that*

$$arg\left(\frac{x'}{x-y}\right) = -\theta \pm \frac{\pi}{2} + 2k\pi \quad \text{and} \quad arg\left(\frac{y'}{x-y}\right) = \theta \pm \pi + 2k\pi, \quad \textit{for} \quad k \in \mathbb{Z}.$$

Theorem 6.14 *(Necessity) Assume that (M, f) is a minimal isotropic surface into $\mathbb{S}^3_1(1)$, such that θ is a non-constant real-valued harmonic function, with f given by equation (8), and x and y holomorphic functions. Then there exists constants $k, c \in \mathbb{C}\setminus\{0\}$ such that*

$$x(w) = \frac{1}{c}\int_{w_0}^{w}(1+ke^{\psi(\xi)})\mathrm{d}\xi \quad \text{and} \quad y(w) = \frac{1}{ck}\int_{w_0}^{w}(k+e^{-\psi(\xi)})\mathrm{d}\xi, \tag{28}$$

where ψ is the harmonic function given by

$$\psi(w) = \theta(w_0) - 4i\int_{w_0}^{w}\theta_w(\xi)\mathrm{d}\xi. \tag{29}$$

Proof By Theorem 6.8 we see that there exists a Möbius transformation M_c such that $y' = M_c(x')$. This implies $y'(cx'-1) = x'$. Hence $x'+y' = cx'y'$ for $c \neq 0$.

Now, from equations (25) and (26) it follows that

$$\frac{x''}{x'} - \frac{y''}{y'} = 2\frac{x'+y'}{x-y} = -4i\theta_w =: \psi_w.$$

Then we have the system

$$x'+y' = cx'y' \quad \text{and} \quad \frac{x'}{y'} = ke^{\psi},$$

since the logarithmic derivative x'/y' equals ψ_w. From these two equations we see that $cx' = 1+ke^{\psi}$ and $kcy' = k+e^{-\psi}$. Therefore we get the expressions in (28). Moreover, since θ is a harmonic function, it follows immediately that ψ is also a harmonic function. □

In our last example we construct a family of isotropic surfaces in $\mathbb{S}^3_1(1)$ by varying the real parameter r.

Example 6.15 For complex numbers c and $k \neq 0$ and taking $0 \neq r \in \mathbb{R}$, we define, for each $w = u + iv \in \mathbb{C}$ the following functions:

$$\mu(u, v) = \frac{e^{i\theta}}{2|k|} e^{-\mathfrak{Re}(aw)},$$
$$x(u, v) = c + ke^{(1+i)r(u+iv)},$$
$$y(u, v) = c - ke^{(1+i)r(u+iv)},$$

where $a = r(1 + i)$ and θ is one of $\{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}$.

We can derive this by assuming that x, y have the form above and that $\mu(u, v) = \frac{\sqrt{2}(1+i)}{4|k|} e^{r(v-u)}$. We begin by taking x, y holomorphic functions such that $x + y = 2c$ and $x - y = 2z = 2ke^{aw}$, where $c, a \in \mathbb{C}$, and $k \in \mathbb{C} - \{0\}$. We see that $z'/z = a$.

Now we look for the function μ satisfying the spherical and isotropic equations (14) and (15), to obtain an isotropic immersion in $\mathbb{S}^3_1(1) \subset \mathbb{R}^4_1$.

From equation (14), since $x_w = x' = -y'$, we find

$$\frac{\mu_w}{\mu} = \frac{-x_w}{x - y} = \frac{\overline{\mu}_w}{\overline{\mu}} = \frac{y_w}{x - y} = \frac{-a}{2}.$$

Now, since we must have $|\mu| = \frac{1}{|2ke^{aw}|}$ with $0 \neq k \in \mathbb{C}$, we set

$$\mu(u, v) = \frac{e^{i\theta}}{2|k|} e^{-\mathfrak{Re}(aw)}.$$

Moreover, since $\frac{\mu_w}{\mu} = \frac{-a}{2}$ we have $\theta_w = 0$, which implies that $\theta \in \mathbb{C}$.

Since $x_w = x_u = x'$, the second equation of (15) implies that $\mathfrak{Re}(\mu \frac{a}{2}) = \mathfrak{Re}(\mu \frac{x_u}{x-y}) = 0$, hence $\mu \frac{a}{2}$ is imaginary. In the same way, the first equation of (15) says that $\overline{\mu} i \frac{a}{2}$ is imaginary. Taking $e^{i\theta} = p$, we obtain

$$pa = -\overline{p}\,\overline{a} \qquad \text{and} \qquad \overline{p}a = p\overline{a}. \tag{30}$$

The latter implies that $(\frac{a}{\overline{a}})^2 = -1 = (\frac{p}{\overline{p}})^2$. Then, from $p^2 = -(\overline{p})^2$, we have $p = b(1 \pm i)$ for some real number b. Similarly, we find that $a = r(1 \pm i)$ for some real number $r \neq 0$. Finally, since $|p| = 1$ and a, p have to satisfy equation (30), we choose from a set of four possible solutions for $p^4 = -1$, as seen above.

Acknowledgements The first author's research was supported by Projeto Temático Fapesp No. 2016/23746-6. São Paulo. Brazil.

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