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Instability of ground states for the NLS equation with potential on the star graph

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Abstract. We study the nonlinear Schrödinger equation with an arbitrary real potential $V(x) \in (L^1 + L^\infty)(\Gamma)$ on a star graph Γ. At the vertex an interaction occurs described by the generalized Kirchhoff condition with strength $-\gamma < 0$. We show the existence of ground states $\varphi_\omega(x)$ as minimizers of the action functional on the Nehari manifold under additional negativity and decay conditions on $V(x)$. Moreover, for $V(x) = -\frac{\beta}{x^\alpha}$, in the supercritical case, we prove that the standing waves $e^{i\omega t}\varphi_\omega(x)$ are orbitally unstable in $H^1(\Gamma)$ when ω is large enough. Analogous result holds for an arbitrary $\gamma \in \mathbb{R}$ when the standing waves have symmetric profile.

1. Introduction

We consider the following focusing nonlinear Schrödinger equation on an infinite star graph Γ:

$$\begin{cases} i\partial_t u(t,x) = -\Delta_\gamma u(t,x) + V(x)u(t,x) - |u(t,x)|^{p-1}u(t,x), & (t,x) \in \mathbb{R}\times\Gamma, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $\gamma > 0$, $p > 1$, $u(t,x) : \mathbb{R}\times\Gamma \to \mathbb{C}^N$, and Δ_γ is the Laplace operator with the generalized Kirchhoff condition at the vertex of Γ ($\cdot'$ stands for spatial derivative):

$$v_1(0) = \cdots = v_N(0), \quad \sum_{e=1}^{N} v_e'(0) = -\gamma v_1(0).$$

We assume that the potential $V(x) = (V_e(x))_{e=1}^N$ is real-valued and satisfies the *Assumptions* (see notation section):

1. *Self-adjointness assumption*: $V(x) \in L^1(\Gamma) + L^\infty(\Gamma)$.
2. *Weak continuity assumption*: $\lim\limits_{x\to\infty} V_e(x) = 0$.

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3. *Minimizing assumption*: $\int_{\mathbb{R}^+} V_e(x)|\phi(x)|^2 dx < 0$ for all $\phi(x) \in H^1(\mathbb{R}^+) \setminus \{0\}$.
4. *Virial identity assumption*: $xV'(x) \in L^1(\Gamma) + L^\infty(\Gamma)$.

Notice that *Assumption 3* essentially guarantees $(Vu, u)_2 < 0,\ u \in H^1(\Gamma) \setminus \{0\}$, and $V(x) \le 0$ a.e. on Γ (see Remark 1.2).

NLS equation (1.1) models wave propagation in thin waveguides (we refer the reader to [6,7,19,22] for the details). The study of stability properties of the multi-dimensional NLS with a linear potential

$$\begin{aligned}&i\partial_t u(t,x) = -\Delta u(t,x) + V(x)u(t,x) - |u(t,x)|^{p-1} u(t,x),\\ &(t,x) \in \mathbb{R} \times \mathbb{R}^n, \quad 1 + 4/n \le p < 1 + 4/(n-2),\end{aligned}$$

was initiated in [27]. More precisely, the authors proved orbital stability of $e^{i\omega t}\varphi_\omega(x)$ for ω sufficiently close to minus the smallest eigenvalue of the operator $-\Delta + V$ (under the assumptions $V(x) \in L^\infty(\mathbb{R}^n)$, $\lim\limits_{|x|\to\infty} V(x) = 0$). In [15], the stability results obtained by [27] were improved for $V(x)$ satisfying more general assumptions.

Recently in [25], the author studied strong instability (by blow-up) of the standing waves in the case of harmonic potential $V(x) = |x|^2$. In particular, he proved strong instability under certain concavity condition for the associated action functional (cf. Theorem 1.4 below). The same idea was applied in [13] to investigate strong instability for $V(x) = -\frac{\beta}{|x|^\alpha}$, $0 < \alpha < \min\{2, n\}$, $\beta > 0$. The reader is also referred to [24] for more information about NLS near soliton dynamics.

In the case $V(x) \equiv 0$, the well-posedness in $H^1(\Gamma)$, variational and stability/ instability properties of (1.1) have been extensively studied during the last decade. The well-posedness results were obtained in [2,18], whereas the existence, stability and variational properties of ground states were studied in [1–4,20]. Moreover, the regularity and strong instability results were elaborated in [18].

On the other hand, the NLS with potential on graphs is little studied. To our knowledge, the only results concerning the existence and stability of standing waves were obtained in [5,9,10]. In the subcritical ($1 < p < 5$) and critical ($p = 5$) case, orbitally stable standing waves $e^{i\omega t}\varphi_\omega(x)$ were constructed in [9,10] under specific conditions on $V(x)$. Subsequently, in [5] the orbital stability of $e^{i\omega t}\varphi_\omega(x)$ was studied in the supercritical case ($p > 5$). More precisely, it was shown (by solving a local energy minimization problem) that $e^{i\omega t}\varphi_\omega(x)$ is stable when the mass of $\varphi_\omega(x)$ is sufficiently small.

In this paper, we show the existence and orbital instability of the standing wave solutions to (1.1) relying on methods developed in [13,16]. Moreover, we state regularity of the solutions to the Cauchy problem for the initial data from the domain of the operator $-\Delta_\gamma + V(x)$. This result is used to show virial identity which is the key ingredient in the proof of the instability result.

1.1. Notation

We consider a graph Γ consisting of a central vertex v and N infinite half-lines attached to it. One may identify Γ with the disjoint union of the intervals $I_e = (0, \infty)$, $e = 1, \ldots, N$, augmented by the central vertex $v = 0$. Given a function $v : \Gamma \to \mathbb{C}^N$, $v = (v_e)_{e=1}^N$, where $v_e : (0, \infty) \to \mathbb{C}$ denotes the restriction of v to I_e. We denote by $v_e(0)$ and $v_e'(0)$ the limits of $v_e(x)$ and $v_e'(x)$ as $x \to 0^+$.

We say that a function v is continuous on Γ if every restriction v_e is continuous on I_e and $v_1(0) = \ldots = v_N(0)$. The space of continuous functions is denoted by $C(\Gamma)$.

The natural Hilbert space associated with the Laplace operator Δ_γ is $L^2(\Gamma)$, which is defined as $L^2(\Gamma) = \bigoplus_{e=1}^N L^2(\mathbb{R}^+)$, and is equipped with the norm

$$\|v\|_2^2 = \int_\Gamma |v|^2 \, dx = \sum_{e=1}^N \int_0^\infty |v_e(x)|^2 \, dx.$$

The inner product in $L^2(\Gamma)$ is denoted by $(\cdot, \cdot)_2$. The space $L^q(\Gamma)$ for $1 \le q \le \infty$ is defined analogously, and $\|\cdot\|_q$ stands for its norm. The Sobolev spaces $H^1(\Gamma)$ and $H^2(\Gamma)$ are defined as

$$H^1(\Gamma) = \left\{ v \in C(\Gamma) : \ v_e \in H^1(\mathbb{R}^+), \ \ e = 1, \ldots, N \right\},$$
$$H^2(\Gamma) = \left\{ v \in C(\Gamma) : \ v_e \in H^2(\mathbb{R}^+), \ \ e = 1, \ldots, N \right\}.$$

We consider the self-adjoint operator $H_{\gamma,V}$ on $L^2(\Gamma)$:

$$(H_{\gamma,V} v)_e = -(\Delta_\gamma v)_e + V_e v_e = -v_e'' + V_e v_e,$$
$$\mathrm{dom}(H_{\gamma,V}) = \left\{ v \in H^1(\Gamma) : \ -v_e'' + V_e v_e \in L^2(\mathbb{R}^+), \ \sum_{e=1}^N v_e'(0) = -\gamma v_1(0) \right\}. \tag{1.2}$$

When $\gamma = 0$, the condition at the vertex in (1.2) is usually referred as free or Kirchhoff boundary condition. For $\gamma \in \mathbb{R}$, the operator $H_{\gamma,V}$ has a precise interpretation as the self-adjoint operator on $L^2(\Gamma)$ uniquely associated with the closed semibounded quadratic form $F_{\gamma,V}$ defined on $H^1(\Gamma)$ by (see Lemma 4.10 in Appendix)

$$F_{\gamma,V}(v) = \|v'\|_2^2 - \gamma\, |v_1(0)|^2 + (Vv, v)_2$$
$$= \sum_{e=1}^N \int_0^\infty \left|v_e'(x)\right|^2 \, dx - \gamma\, |v_1(0)|^2 + \sum_{e=1}^N \int_0^\infty V_e(x)|v_e(x)|^2 \, dx. \tag{1.3}$$

Note that we can formally rewrite (1.1) as

$$i\partial_t u(t) = E'(u(t)),$$

where E is the energy functional defined by

$$E(u) = \frac{1}{2} F_{\gamma,V}(u) - \frac{1}{p+1} \|u\|_{p+1}^{p+1}.$$

The energy functional is well defined on $H^1(\Gamma)$ since the potential $V(x)$ belongs to $(L^1 + L^\infty)(\Gamma)$ (see Lemma 4.10 in Appendix).

1.2. Standing waves and instability results

By a standing wave of (1.1), we mean a solution of the form $e^{i\omega t}\varphi(x)$, where $\omega \in \mathbb{R}$ and φ is a solution of the stationary equation

$$H_{\gamma,V}\phi + \omega\phi - |\phi|^{p-1}\phi = 0. \tag{1.4}$$

We define two functionals on $H^1(\Gamma)$:

$$S_\omega(v) := \frac{1}{2} F_{\gamma,V}(v) + \frac{\omega}{2}\|v\|_2^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1} \quad \text{(action functional)},$$
$$I_\omega(v) := F_{\gamma,V}(v) + \omega\|v\|_2^2 - \|v\|_{p+1}^{p+1}.$$

Observe that (1.4) is equivalent to $S'_\omega(\phi) = 0$ (see [2, Theorem 4]) and $I_\omega(v) = \partial_\lambda S_\omega(\lambda v)\mid_{\lambda=1} = \langle S'_\omega(v), v\rangle$. Denote the set of non-trivial solutions to (1.4) by

$$\mathcal{B}_\omega = \left\{ v \in H^1(\Gamma)\backslash\{0\} \,:\, S'_\omega(v) = 0 \right\}.$$

A ground state for (1.4) is a function $\varphi \in \mathcal{B}_\omega$ that minimizes S_ω on $\mathcal{B}_\omega$, and the set of ground states is given by

$$\mathcal{G}_\omega = \left\{ \phi \in \mathcal{B}_\omega \,:\, S_\omega(\phi) \le S_\omega(v) \text{ for all } v \in \mathcal{B}_\omega \right\}.$$

We consider the minimization problem on the Nehari manifold

$$d_\omega = \inf\left\{ S_\omega(v) \,:\, v \in H^1(\Gamma)\backslash\{0\},\ I_\omega(v) = 0 \right\},$$

and the set of minimizers

$$\mathcal{M}_\omega = \left\{ \phi \in H^1(\Gamma)\backslash\{0\} \,:\, S_\omega(\phi) = d_\omega,\ \ I_\omega(\phi) = 0 \right\}.$$

We now state the first result, which provides the existence of the minimizer for d_ω when the strength $-\gamma$ is sufficiently strong. Denote (see Lemma 4.13)

$$-\omega_0 := \inf \sigma(H_{\gamma,V}) = \min \sigma_p(H_{\gamma,V}) < 0. \tag{1.5}$$

Proposition 1.1. *Let $p > 1$, $\omega > \omega_0$, and $V(x) = \overline{V(x)}$ satisfy Assumptions 1-3. Then there exists $\gamma^* > 0$ such that the set $\mathcal{G}_\omega$ is not empty for any $\gamma > \gamma^*$, in particular, $\mathcal{G}_\omega = \mathcal{M}_\omega$. If $\varphi_\omega \in \mathcal{G}_\omega$, then there exist $\theta \in \mathbb{R}$ and a positive function $\phi \in dom(H_{\gamma,V})$ such that $\varphi_\omega(x) = e^{i\theta}\phi(x)$.*

To be precise, γ^* is given in [2] by

$$\int_0^1 (1-t^2)^{\frac{2}{p-1}} \mathrm{d}t = \frac{N}{2} \int_{\frac{\gamma^*}{N\sqrt{\omega}}}^1 (1-t^2)^{\frac{2}{p-1}} \mathrm{d}t. \tag{1.6}$$

The condition $\gamma > \gamma^*$ guarantees that the action functional S_ω constrained to the Nehari manifold admits an absolute minimum when $V(x) \equiv 0$.

Remark 1.2. The proof of the last assertion of Proposition 1.1 essentially uses that $V(x) \le 0$ a.e. on Γ, which is a consequence of *Assumption 3*.

To show this, one observes that $\int_{\mathbb{R}^+} -V_e(x)\phi(x)\mathrm{d}x \ge 0$ for all nonnegative functions $\phi(x)$ from $C_c(\mathbb{R}^+)$ (the set of continuous functions with compact support). Indeed, let $\tilde{\phi}(x)$ be an extension onto $\mathbb{R}$ by zero of a nonnegative function $\phi(x) \in C_c(\mathbb{R}^+)$. Take $\{\phi_n(x)\} \subset C_c^\infty(\mathbb{R})$ such that $\phi_n \underset{n\to\infty}{\longrightarrow} \sqrt{\tilde{\phi}}$ uniformly, and $\mathrm{supp}\tilde{\phi}$, $\mathrm{supp}\phi_n \subset K \subset \mathbb{R}^+$, where K is a compact set. Then, $\phi_n^2 \underset{n\to\infty}{\longrightarrow} \tilde{\phi}$ uniformly, and, by the Dominated Convergence Theorem, we get

$$-\int_{\mathbb{R}^+} V_e(x)\phi_n^2(x)\mathrm{d}x \underset{n\to\infty}{\longrightarrow} -\int_{\mathbb{R}^+} V_e(x)\phi(x)\mathrm{d}x \ge 0.$$

Now, since $f(\phi) = -\int_{\mathbb{R}^+} V_e(x)\phi(x)\mathrm{d}x$ is a positive linear functional on $C_c(\mathbb{R}^+)$, then, by the Riesz–Markov–Kakutani representation theorem for positive linear functionals, we conclude the existence of a unique Radon measure μ on $\mathbb{R}^+$ such that $f(\phi) = \int_{\mathbb{R}^+} \phi(x) d\mu(x)$. On the other hand, $f(\phi) = \int_{\mathbb{R}^+} v(x)\phi(x) d\nu(x)$, where $\nu(A) = \int_A |V_e|\mathrm{d}x$ for A from the Borel σ-algebra on $\mathbb{R}^+$, and $v(x) = \begin{cases} \frac{V_e(x)}{|V_e(x)|}, & x \in \{x : V_e(x) \neq 0\} \\ 0, & \text{otherwise.} \end{cases}$ Finally, from the uniqueness stated in [12, Theorem 2.5.12] it follows that $\mu = \nu$ and $v = 1$ ν-a.e. on $\mathbb{R}^+$, hence $-V_e \ge 0$ ν-a.e. on $\mathbb{R}^+$. This implies $-V_e \ge 0$ Lebesgue-a.e. on $\mathbb{R}^+$ since the Lebesgue measure and the measure ν are mutually absolutely continuous on the set $\{x : V_e(x) \neq 0\}$.

The next step in the study of ground states for (1.4) is to investigate their stability properties. We define orbital stability as follows.

Definition 1.3. For $\varphi_\omega \in \mathcal{G}_\omega$, we set

$$N_\delta(\varphi_\omega) := \left\{ v \in H^1(\Gamma) : \inf_{\theta\in\mathbb{R}} \left\| v - e^{i\theta}\varphi_\omega \right\|_{H^1(\Gamma)} < \delta \right\}. \tag{1.7}$$

We say that a standing wave solution $e^{i\omega t}\varphi_\omega(x)$ of (1.1) is orbitally stable in $H^1(\Gamma)$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $u_0 \in N_\delta(\varphi_\omega)$, the solution $u(t)$ of (1.1) satisfies $u(t) \in N_\varepsilon(\varphi_\omega)$ for all $t \ge 0$. Otherwise, $e^{i\omega t}\varphi_\omega(x)$ is said to be orbitally unstable in $H^1(\Gamma)$.

Using the ideas developed in [13,16], we obtain a sufficient condition for the instability of standing waves when $p > 5$ (supercritical case). The main result of this paper is the following:

Theorem 1.4. *Assume that $p > 5$, $\gamma > \gamma^*$, $\omega > \omega_0$, and $V(x) = \overline{V(x)}$ satisfies Assumptions 1-4. If $\varphi_\omega(x) \in \mathcal{G}_\omega$ and $\partial_\lambda^2 E(\varphi_\omega^\lambda)\,|_{\lambda=1} < 0$, where $\varphi_\omega^\lambda(x) := \lambda^{1/2}\varphi_\omega(\lambda x)$ for $\lambda > 0$, then the standing wave solution $e^{i\omega t}\varphi_\omega(x)$ of (1.1) is orbitally unstable in $H^1(\Gamma)$.*

To prove Theorem 1.4, we use the variational characterization given in Proposition 1.1 and virial identity (2.4). Notice that the standing wave solution $e^{i\omega t}\varphi_\omega(x)$ of (1.1) with $\gamma > 0$ and $V(x) \equiv 0$ is unstable in $H^1(\Gamma)$ when $p > 5$ and ω is large enough (see [2, Remark 6.1] and also [18, Theorem 1.4]). Below we state that this also holds true for $\gamma > 0$ and slowly decaying potential $V(x) = \dfrac{-\beta}{x^\alpha}$, $0 < \alpha < 1$, $\beta > 0$ (i.e., $\partial_\lambda^2 E_\omega(\varphi_\omega^\lambda)\,|_{\lambda=1} < 0$ for sufficiently large ω). The choice of the potential is due to its "homogeneity" property, which is principal for the proof (see formula (4.7)).

Corollary 1.5. *Assume that $V(x) = \dfrac{-\beta}{x^\alpha}$, $\beta > 0$, $0 < \alpha < 1$, $\gamma > \gamma^*$, $p > 5$. If $\varphi_\omega(x) \in \mathcal{G}_\omega$, then there exists $\omega^* = \omega^*(\beta, \alpha, \gamma, p) \in (\omega_0, \infty)$ such that for any $\omega \in (\omega^*, \infty)$ the standing wave solution $e^{i\omega t}\varphi_\omega(x)$ of (1.1) is orbitally unstable in $H^1(\Gamma)$.*

As far as we know, these are the first results on instability of ground states for the NLS with potential on graphs. In Subsect. 4.3, we state the counterparts to Proposition 1.1, Theorem 1.4, Corollary 1.5 in the space $H^1_{\text{eq}}(\Gamma)$ of symmetric functions and arbitrary $\gamma \in \mathbb{R}$.

The paper is organized as follows. In Sect. 2, we prove Proposition 2.2 that concerns local well-posedness in the energy domain. In Sect. 3, we provide the proof of Proposition 1.1. Section 4 is devoted to the proof of Theorem 1.4 and Corollary 1.5. In Appendix, we discuss some properties of the operator $H_{\gamma,V}$.

2. Local existence results and virial identity

We start with the proof of the following key lemma involving the estimate of H^1-norm of the unitary group generated by the self-adjoint operator $H_{\gamma,V}$.

Lemma 2.1. *Let $e^{-iH_{\gamma,V}t}$ be a unitary group generated by $H_{\gamma,V}$. Then, $e^{-iH_{\gamma,V}t}H^1(\Gamma) \subseteq H^1(\Gamma)$ and*

$$\|e^{-iH_{\gamma,V}t}v\|_{H^1(\Gamma)} \le C\|v\|_{H^1(\Gamma)}. \tag{2.1}$$

Proof. The idea of the proof was given in [10] (see formula (2.5)). However, some additional technical details seem useful.

Let $m > \omega_0$, where ω_0 is given by (1.5). Remark that $H^1(\Gamma) = \text{dom}\left(F_{\gamma,V}\right) = \text{dom}((H_{\gamma,V}+m)^{1/2})$ (see, for instance, [21, Chapter VI, Problem 2.25]). Since $e^{-iH_{\gamma,V}t}$ is bounded, we get for $v \in H^1(\Gamma)$

$$e^{-iH_{\gamma,V}t}(H_{\gamma,V}+m)^{1/2}v = (H_{\gamma,V}+m)^{1/2}e^{-iH_{\gamma,V}t}v.$$

Hence $e^{-iH_{\gamma,V}t}v \in H^1(\Gamma)$ and $e^{-iH_{\gamma,V}t}H^1(\Gamma) \subseteq H^1(\Gamma)$. Further, using L^2-unitarity of $e^{-iH_{\gamma,V}t}$, we obtain for $v \in H^1(\Gamma)$

$$\begin{aligned} F_{\gamma,V}(v) + m\|v\|_2^2 &= \left((H_{\gamma,V}+m)^{1/2}v, (H_{\gamma,V}+m)^{1/2}v\right)_2 \\ &= \left(e^{-iH_{\gamma,V}t}(H_{\gamma,V}+m)^{1/2}v, e^{-iH_{\gamma,V}t}(H_{\gamma,V}+m)^{1/2}v\right)_2 \\ &= \left((H_{\gamma,V}+m)^{1/2}e^{-iH_{\gamma,V}t}v, (H_{\gamma,V}+m)^{1/2}e^{-iH_{\gamma,V}t}v\right)_2 \\ &= F_{\gamma,V}(e^{-iH_{\gamma,V}t}v) + m\|e^{-iH_{\gamma,V}t}v\|_2^2. \end{aligned}$$

From the proof of Lemma 4.13-(ii), we get

$$\begin{aligned} &C_2\|e^{-iH_{\gamma,V}t}v\|_{H^1(\Gamma)}^2 \le F_{\gamma,V}(e^{-iH_{\gamma,V}t}v) + m\|e^{-iH_{\gamma,V}t}v\|_2^2 \\ &= F_{\gamma,V}(v) + m\|v\|_2^2 \le C_1\|v\|_{H^1(\Gamma)}^2, \end{aligned}$$

and (2.1) follows easily. □

The proposition below states the local well-posedness of (1.1).

Proposition 2.2. *For any* $u_0 \in H^1(\Gamma)$, *there exist* $T = T(u_0) > 0$ *and a unique solution* $u(t) \in C([0,T], H^1(\Gamma)) \cap C^1([0,T], (H^1(\Gamma))')$ *of problem* (1.1). *For each* $T_0 \in (0,T)$ *the mapping* $u_0 \in H^1(\Gamma) \mapsto u(t) \in C([0,T_0], H^1(\Gamma))$ *is continuous. Moreover, problem* (1.1) *has a maximal solution defined on an interval of the form* $[0, T_{H^1})$, *and the following "blow-up alternative" holds: either* $T_{H^1} = \infty$ *or* $T_{H^1} < \infty$ *and*

$$\lim_{t\to T_{H^1}} \|u(t)\|_{H^1(\Gamma)} = \infty.$$

Finally, the conservation of energy and charge holds: for $t \in [0, T_{H^1})$

$$E(u(t)) = \frac{1}{2}F_{\gamma,V}(u(t)) - \frac{1}{p+1}\|u(t)\|_{p+1}^{p+1} = E(u_0), \quad \|u(t)\|_2^2 = \|u_0\|_2^2. \tag{2.2}$$

Proof. A sketch of the proof was given in [10]. However, the rigorous proof (which serves for $p > 1$) might be obtained repeating the one of [11, Theorem 4.10.1]. In particular, one needs to use the fact that $g(u) = |u|^{p-1}u \in C^1(\mathbb{C},\mathbb{C})$ (i.e., $\mathrm{Im}(g)$ and $\mathrm{Re}(g)$ are C^1-functions of $\mathrm{Re}u$, $\mathrm{Im}u$) for $p > 1$ and apply inequality (2.1).

The proof of conservation laws (2.2) might be obtained involving regularization procedure analogous to the one introduced in the proof of [11, Theorem 3.3.5] and using the uniqueness of the solution (see [11, Theorem 3.3.9]). □

Remark 2.3. (i) For $p \geq 4$, the conservation laws follow easily from Proposition 2.4 below and continuous dependence on initial data.

(ii) For $1 < p < 5$, problem (1.1) is globally well-posed in $H^1(\Gamma)$. To see that one might repeat the proof of [11, Theorem 3.4.1], where condition (3.4.1) follows from

$$\begin{aligned} &\|u\|_{p+1}^{p+1} - (Vu,u)_2 + \gamma|u_1(0)|^2 \le C\|u'\|_2^{\frac{p-1}{2}}\|u\|_2^{\frac{p+3}{2}} + 2\varepsilon\|u'\|_2^2 + C_1\|u\|_2^2 \\ &\le 3\varepsilon\|u'\|_2^2 + C_2\|u\|_2^{\frac{2(p+3)}{5-p}} + C_1\|u\|_2^2 \le 3\varepsilon\|u\|_{H^1(\Gamma)}^2 + C(\|u_0\|_2). \end{aligned}$$

The above estimate is induced by the conservation of charge, estimate (4.19), the Gagliardo–Nirenberg inequality (see (2.1) in [10]) and the Young inequality $ab \leq \delta a^q + C_\delta b^{q'}$, $\frac{1}{q} + \frac{1}{q'} = 1$, $q, q' > 1$, $a, b \geq 0$. Observe that the key point is that $q = \frac{4}{p-1} > 1$ for $1 < p < 5$.

Now, let $m \geq 1+2\omega_0$. Introduce the norm $\|v\|_{H_{\gamma,V}} := \|(H_{\gamma,V}+m)v\|_2$ that endows $\mathrm{dom}(H_{\gamma,V})$ with the structure of a Hilbert space. We denote $D_{H_{\gamma,V}} = (\mathrm{dom}(H_{\gamma,V}), \|\cdot\|_{H_{\gamma,V}})$.

Proposition 2.4. *Let $p \geq 4$ and $u_0 \in dom(H_{\gamma,V})$. Then, there exists $T > 0$ such that problem (1.1) has a unique solution $u(t) \in C([0,T], D_{H_{\gamma,V}}) \cap C^1([0,T], L^2(\Gamma))$. Moreover, problem (1.1) has a maximal solution defined on an interval of the form $[0, T_{H_{\gamma,V}})$, and the following "blow-up alternative" holds: either $T_{H_{\gamma,V}} = \infty$ or $T_{H_{\gamma,V}} < \infty$ and*

$$\lim_{t \to T_{H_{\gamma,V}}} \|u(t)\|_{H_{\gamma,V}} = \infty.$$

Proof. The proof repeats the one of [18, Theorem 2.3] observing that $\mathrm{dom}(H_{\gamma,V}) \subset H^1(\Gamma) = \mathrm{dom}((H_{\gamma,V}+m)^{1/2})$ and, by $m \geq 1+2\omega_0$,

$$\|u\|_\infty \leq C_1\|u\|_{H^1(\Gamma)} \leq C_2\|(H_{\gamma,V}+m)^{1/2}u\|_2 \leq C_2\|(H_{\gamma,V}+m)u\|_2.$$

□

Remark 2.5. Notice that due to estimate (4.19), Propositions 2.2 and 2.4 hold for any $\gamma \in \mathbb{R}$ and $V(x) \in (L^1 + L^\infty)(\Gamma)$.

Set

$$P(v) = \|v'\|_2^2 - \frac{1}{2}\int_\Gamma xV'(x)|v(x)|^2\,\mathrm{d}x - \frac{\gamma}{2}|v_1(0)|^2 - \frac{p-1}{2(p+1)}\|v\|_{p+1}^{p+1}, \quad v \in H^1(\Gamma).$$

Proposition 2.6. *Let $\Sigma(\Gamma) = \{v \in H^1(\Gamma) : xv \in L^2(\Gamma)\}$. Assume that $u_0 \in \Sigma(\Gamma)$, and $u(t)$ is the corresponding maximal solution to (1.1). Then, $u(t) \in C([0, T_{H^1}), \Sigma(\Gamma))$, and the function*

$$f(t) := \int_\Gamma x^2|u(t)|^2\mathrm{d}x = \|xu(t)\|_2^2$$

belongs to $C^2[0, T_{H^1})$. Moreover,

$$f'(t) = 4Im\int_\Gamma x\overline{u}\partial_x u\,\mathrm{d}x, \qquad and \tag{2.3}$$

$$f''(t) = 8P(u(t)), \quad t \in [0, T_{H^1}). \quad (virial\ identity) \tag{2.4}$$

Proof. The proof is similar to the one of [11, Proposition 6.5.1]. We provide the details since the virial identity is the key ingredient in the instability analysis. Firstly, we show (2.3), secondly we prove (2.4) for $u_0 \in \mathrm{dom}(H_{\gamma,V})$, and then, we conclude that (2.4) holds for $u_0 \in H^1(\Gamma)$ using continuous dependence on the initial data.

Step 1. Let $\varepsilon > 0$, define $f_\varepsilon(t) = \|e^{-\varepsilon x^2} x u(t)\|_2^2$, for $t \in [0,T]$, $T \in (0, T_{H^1})$. Then, observing that $e^{-2\varepsilon x^2} x^2 u(t) \in H^1(\Gamma)$ and taking $(H^1)' - H^1$ duality product of equation (1.1) with $i e^{-2\varepsilon x^2} x^2 u(t)$, we get

$$\begin{aligned} f_\varepsilon'(t) &= 2\mathrm{Im} \int_\Gamma \left(\partial_x u\, \partial_x (e^{-2\varepsilon x^2} x^2 \overline{u}) - e^{-2\varepsilon x^2} x^2 |u|^{p+1} \right) \mathrm{d}x \\ &= 4\mathrm{Im} \int_\Gamma \left\{ e^{-\varepsilon x^2} (1 - 2\varepsilon x^2) \right\} \overline{u} x e^{-\varepsilon x^2} \partial_x u \, \mathrm{d}x. \end{aligned} \tag{2.5}$$

Remark that $|e^{-\varepsilon x^2}(1-2\varepsilon x^2)| \le 2$ for any x. From (2.5), by the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} |f_\varepsilon'(t)| &\le 4 \left| \int_\Gamma \left\{ e^{-\varepsilon x^2}(1-2\varepsilon x^2) \right\} \overline{u} x e^{-\varepsilon x^2} \partial_x u \, \mathrm{d}x \right| \le 8 \int_\Gamma |e^{-\varepsilon x^2} x u \partial_x u| \, \mathrm{d}x \\ &\le 8 \sum_{j=1}^N \|\partial_x u_j\|_2 \|e^{-\varepsilon x^2} x u_j\|_2 \le C \|u\|_{H^1(\Gamma)} \sqrt{f_\varepsilon(t)}. \end{aligned} \tag{2.6}$$

From (2.6), one implies

$$\int_0^t \frac{f_\varepsilon'(s)}{\sqrt{f_\varepsilon(s)}} \mathrm{d}s \le C \int_0^t \|u(s)\|_{H^1(\Gamma)} \mathrm{d}s,$$

and therefore,

$$\sqrt{f_\varepsilon(t)} \le \|x u_0\|_2 + \frac{C}{2} \int_0^t \|u(s)\|_{H^1(\Gamma)} \mathrm{d}s, \;\; t \in [0,T].$$

Letting $\varepsilon \downarrow 0$ and applying Fatou's lemma, we get that $xu(t) \in L^2(\Gamma)$ and $f(t)$ is bounded in $[0,T]$. Observe that from (2.5) one induces

$$f_\varepsilon(t) = f_\varepsilon(0) + 4\mathrm{Im} \int_0^t \int_\Gamma \left\{ e^{-\varepsilon x^2}(1-2\varepsilon x^2) \right\} \overline{u} x e^{-\varepsilon x^2} \partial_x u \, \mathrm{d}x \, \mathrm{d}s. \tag{2.7}$$

We have the following estimates for any positive x and ε:

$$\begin{aligned} & e^{-2\varepsilon x^2} x^2 |u(t)|^2 \le x^2 |u(t)|^2, \\ & e^{-2\varepsilon x^2} x^2 |u_0|^2 \le x^2 |u_0|^2, \\ & |e^{-\varepsilon x^2}(1-2\varepsilon x^2) \overline{u} x e^{-\varepsilon x^2} \partial_x u| \le 2 |\partial_x u| |x u|. \end{aligned} \tag{2.8}$$

Having pointwise convergence, and using (2.8), by the Dominated Convergence Theorem, we get from (2.7)

$$f(t) = \|x u(t)\|_2^2 = \|x u_0\|_2^2 + 4\mathrm{Im} \int_0^t \int_\Gamma x \overline{u} \partial_x u \, \mathrm{d}x \, \mathrm{d}s.$$

Since $u(t)$ is strong H^1-solution, $f(t)$ is C^1-function, and (2.3) holds for any $t \in [0, T_{H^1})$.

Using continuity of $\|xu(t)\|_2$ and the inclusion $u(t) \in C([0, T_{H^1}), H^1(\Gamma))$, by the Brezis–Lieb lemma [8], we get for $t_0, t_n \in [0, T_{H^1})$

$$\lim_{t_n \to t_0} \|xu(t_n) - xu(t_0)\|_2^2 = \lim_{t_n \to t_0} \|xu(t_n)\|_2^2 - \|xu(t_0)\|_2^2 = 0,$$

hence $u(t) \in C([0, T_{H^1}), \Sigma(\Gamma))$.

Step 2. Let $u_0 \in \operatorname{dom}(H_{\gamma,V})$. By Proposition 2.4, the solution $u(t)$ to the corresponding Cauchy problem belongs to $C([0, T_{H_{\gamma,V}}), D_{H_{\gamma,V}}) \cap C^1([0, T_{H_{\gamma,V}}), L^2(\Gamma))$.

Let $\varepsilon > 0$ and $\theta_\varepsilon(x) = e^{-\varepsilon x^2}$. Define

$$h_\varepsilon(t) = \operatorname{Im} \int_\Gamma \theta_\varepsilon x \overline{u} \partial_x u \, \mathrm{d}x \quad \text{for } t \in [0, T], \; T \in (0, T_{H_{\gamma,V}}). \tag{2.9}$$

First, let us show that

$$h'_\varepsilon(t) = -\operatorname{Im} \int_\Gamma \partial_t u \left\{ 2\theta_\varepsilon x \overline{\partial_x u} + (\theta_\varepsilon + x\theta'_\varepsilon)\overline{u} \right\} \mathrm{d}x \tag{2.10}$$

or equivalently

$$h_\varepsilon(t) = h_\varepsilon(0) - \operatorname{Im} \int_0^t \int_\Gamma \partial_s u \left\{ 2\theta_\varepsilon x \overline{\partial_x u} + (\theta_\varepsilon + x\theta'_\varepsilon)\overline{u} \right\} \mathrm{d}x \, \mathrm{d}s. \tag{2.11}$$

Let us prove that identity (2.11) holds for $u(t) \in C([0, T], H^1(\Gamma)) \cap C^1([0, T], L^2(\Gamma))$. Note that by density argument it is sufficient to show (2.11) for $u(t) \in C^1([0, T], H^1(\Gamma)) \cap C^1([0, T], L^2(\Gamma))$. From (2.9), it follows

$$h'_\varepsilon(t) = -\operatorname{Im} \int_\Gamma \left\{ \theta_\varepsilon x \partial_t u \overline{\partial_x u} + \theta_\varepsilon x u \overline{\partial^2_{xt} u} \right\} \mathrm{d}x. \tag{2.12}$$

Note that

$$\theta_\varepsilon x u \overline{\partial^2_{xt} u} = \theta_\varepsilon x u \overline{\partial^2_{tx} u} = \partial_x \left(\theta_\varepsilon x u \overline{\partial_t u} \right) - \theta_\varepsilon u \overline{\partial_t u} - \theta_\varepsilon x \partial_x u \overline{\partial_t u} - x\theta'_\varepsilon u \overline{\partial_t u},$$

which induces

$$\int_\Gamma \theta_\varepsilon x u \overline{\partial^2_{xt} u} \, \mathrm{d}x = - \int_\Gamma \overline{\partial_t u} \left\{ \theta_\varepsilon (u + x\partial_x u) + x\theta'_\varepsilon u \right\} \mathrm{d}x.$$

Therefore, from (2.12) we get

$$h'_\varepsilon(t) = -\operatorname{Im} \int_\Gamma \left\{ \theta_\varepsilon x \partial_t u \overline{\partial_x u} + \partial_t u \left(\theta_\varepsilon (\overline{u} + x\overline{\partial_x u}) + x\theta'_\varepsilon \overline{u} \right) \right\} \mathrm{d}x.$$

Consequently, we obtain (2.11) for $u(t) \in C^1([0, T], H^1(\Gamma)) \cap C^1([0, T], L^2(\Gamma))$, and hence, for $u(t) \in C([0, T], H^1(\Gamma)) \cap C^1([0, T], L^2(\Gamma))$ which implies (2.10).

Since $u(t) \in C([0, T_{H_{\gamma,V}}), D_{H_{\gamma,V}})$, from (2.10) we get

$$h'_\varepsilon(t) = \mathrm{Re} \int_\Gamma (H_{\gamma,V} u - |u|^{p-1} u) \left\{ 2\theta_\varepsilon x \overline{\partial_x u} + (x\theta_\varepsilon)' \overline{u} \right\} \mathrm{d}x. \tag{2.13}$$

Below we will consider separately linear and nonlinear part of identity (2.13). Integrating by parts, we obtain

$$\begin{aligned} &- \mathrm{Re} \int_\Gamma \Delta_\gamma u \left\{ 2\theta_\varepsilon x \overline{\partial_x u} + (x\theta_\varepsilon)' \overline{u} \right\} \mathrm{d}x = -\gamma |u_1(0)|^2 \\ &\quad + 2 \int_\Gamma x\theta'_\varepsilon |\partial_x u|^2 \mathrm{d}x + \int_\Gamma (2\theta'_\varepsilon + x\theta''_\varepsilon) \mathrm{Re}(\overline{u} \partial_x u) \mathrm{d}x + 2 \int_\Gamma \theta_\varepsilon |\partial_x u|^2 \mathrm{d}x. \end{aligned} \tag{2.14}$$

Noting that

$$\mathrm{Re} \left(V(x) u \left\{ 2\theta_\varepsilon x \overline{\partial_x u} + (x\theta_\varepsilon)' \overline{u} \right\} \right) = \partial_x \left(x V(x) \theta_\varepsilon |u|^2 \right) - x V'(x) \theta_\varepsilon |u|^2,$$

we get

$$\mathrm{Re} \int_\Gamma V(x) u \left\{ 2\theta_\varepsilon x \overline{\partial_x u} + (x\theta_\varepsilon)' \overline{u} \right\} \mathrm{d}x = - \int_\Gamma x V'(x) \theta_\varepsilon |u|^2 \mathrm{d}x. \tag{2.15}$$

Moreover,

$$\begin{aligned} &\mathrm{Re} \int_\Gamma -|u|^{p-1} u \left\{ 2\theta_\varepsilon x \overline{\partial_x u} + (x\theta_\varepsilon)' \overline{u} \right\} \mathrm{d}x \\ &= - \int_\Gamma |u|^{p+1} \theta_\varepsilon \, \mathrm{d}x - \int_\Gamma |u|^{p+1} x\theta'_\varepsilon \, \mathrm{d}x - \int_\Gamma (|u|^2)^{\frac{p-1}{2}} \partial_x (|u|^2) x\theta_\varepsilon \, \mathrm{d}x \\ &= - \frac{p-1}{p+1} \int_\Gamma |u|^{p+1} \theta_\varepsilon \, \mathrm{d}x - \frac{p-1}{p+1} \int_\Gamma |u|^{p+1} x\theta'_\varepsilon \, \mathrm{d}x. \end{aligned} \tag{2.16}$$

Finally, from (2.13)-(2.16) we get

$$\begin{aligned} h'_\varepsilon(t) &= \left[2 \int_\Gamma \theta_\varepsilon |\partial_x u|^2 \mathrm{d}x - \int_\Gamma x V'(x) \theta_\varepsilon |u|^2 \mathrm{d}x - \gamma |u_1(0)|^2 - \frac{p-1}{p+1} \int_\Gamma |u|^{p+1} \theta_\varepsilon \mathrm{d}x \right] \\ &\quad + \left[2 \int_\Gamma x\theta'_\varepsilon |\partial_x u|^2 \mathrm{d}x + \int_\Gamma (2\theta'_\varepsilon + x\theta''_\varepsilon) \mathrm{Re}(\overline{u} \partial_x u) \, \mathrm{d}x \right] - \frac{p-1}{p+1} \int_\Gamma |u|^{p+1} x\theta'_\varepsilon \, \mathrm{d}x. \end{aligned}$$

Since $\theta_\varepsilon,\ \theta'_\varepsilon,\ x\theta'_\varepsilon,\ x\theta''_\varepsilon$ are bounded with respect to x and ε, and

$$\theta_\varepsilon \to 1,\ \theta'_\varepsilon \to 0,\ x\theta'_\varepsilon \to 0,\ x\theta''_\varepsilon \to 0 \text{ pointwise as } \varepsilon \downarrow 0,$$

by the Dominated Convergence Theorem, we have

$$\lim_{\varepsilon \downarrow 0} h'_\varepsilon(t) = 2\|\partial_x u\|_2^2 - \int_\Gamma x V'(x) |u|^2 \mathrm{d}x - \gamma |u_1(0)|^2 - \frac{p-1}{p+1} \|u\|_{p+1}^{p+1} =: g(t).$$

Moreover, again by the Dominated Convergence Theorem,

$$\lim_{\varepsilon \downarrow 0} h_\varepsilon(t) = \mathrm{Im} \int_\Gamma x \overline{u} \partial_x u \, \mathrm{d}x =: h(t).$$

Using continuity of $g(t)$ and the fact that the operator $A = \dfrac{\mathrm{d}}{\mathrm{d}t}$ in the space $C[0, T]$ with $\mathrm{dom}(A) = C^1[0, T]$ is closed, we arrive at $h'(t) = g(t)$, $t \in [0, T]$, i.e.,

$$h'(t) = 2\|\partial_x u\|_2^2 - \int_\Gamma xV'(x)|u|^2 \mathrm{d}x - \gamma|u_1(0)|^2 - \frac{p-1}{p+1}\|u\|_{p+1}^{p+1},$$

and $h(t)$ is C^1 function. Finally, (2.4) holds for $u_0 \in \mathrm{dom}(H_{\gamma,V})$.

Step 3. To conclude the proof, consider $\{u_0^n\}_{n\in\mathbb{N}} \subset \mathrm{dom}(H_{\gamma,V})$ such that $u_0^n \to u_0$ in $H^1(\Gamma)$ and $xu_0^n \to xu_0$ in $L^2(\Gamma)$ as $n \to \infty$. Let $u^n(t)$ be the maximal solutions of the corresponding Cauchy problem associated with (1.1). From (2.3) and (2.4), we obtain

$$\|xu^n(t)\|_2^2 = \|xu_0^n\|_2^2 + 4t\mathrm{Im}\int_\Gamma x\overline{u_0^n}\partial_x u_0^n \,\mathrm{d}x + \int_0^t\int_0^s 8P(u^n(y))\mathrm{d}y\,\mathrm{d}s.$$

Using continuous dependence and repeating the arguments from [11, Corollary 6.5.3], we obtain as $n \to \infty$

$$\|xu(t)\|_2^2 = \|xu_0\|_2^2 + 4t\mathrm{Im}\int_\Gamma x\overline{u_0}\partial_x u_0 \,\mathrm{d}x + \int_0^t\int_0^s 8P(u(y))\mathrm{d}y\,\mathrm{d}s,$$

that is, (2.4) holds for $u_0 \in H^1(\Gamma)$. □

3. Existence of ground states

In this section, we prove Proposition 1.1. We begin with two technical lemmas. Throughout this section, we assume that $\omega > \omega_0$.

Lemma 3.1. *If $I_\omega(v) < 0$, then*

$$d_\omega < \frac{p-1}{2(p+1)}\|v\|_{p+1}^{p+1}, \quad \textit{and} \;\; d_\omega < \frac{p-1}{2(p+1)}\left(F_{\gamma,V}(v) + \omega\|v\|_2^2\right).$$

Moreover,

$$\begin{aligned} d_\omega &= \inf\left\{\frac{p-1}{2(p+1)}\|v\|_{p+1}^{p+1} : v \in H^1(\Gamma)\backslash\{0\},\ I_\omega(v) \le 0\right\} \\ &= \inf\left\{\frac{p-1}{2(p+1)}\left(F_{\gamma,V}(v) + \omega\|v\|_2^2\right) : v \in H^1(\Gamma)\backslash\{0\},\ I_\omega(v) \le 0\right\}. \end{aligned} \tag{3.1}$$

Proof. Noting that

$$S_\omega(v) = \frac{1}{2}I_\omega(v) + \frac{p-1}{2(p+1)}\|v\|_{p+1}^{p+1}, \quad v \in H^1(\Gamma), \tag{3.2}$$

we get

$$d_\omega = \inf\left\{\frac{p-1}{2(p+1)}\|v\|_{p+1}^{p+1} : v \in H^1(\Gamma)\backslash\{0\},\ I_\omega(v) = 0\right\}.$$

Set

$$d_\omega^* := \inf\left\{\frac{p-1}{2(p+1)}\|v\|_{p+1}^{p+1} \ : \ v \in H^1(\Gamma)\backslash\{0\},\ I_\omega(v) \le 0\right\}.$$

It is clear that $d_\omega^* \le d_\omega$. Let $v \in H^1(\Gamma)\backslash\{0\}$ and $I_\omega(v) < 0$. Put

$$\lambda_1 := \left(\frac{F_{\gamma,V}(v) + \omega\|v\|_2^2}{\|v\|_{p+1}^{p+1}}\right)^{\frac{1}{p-1}}.$$

Then, since $I_\omega(\lambda v) = \lambda^2\left(F_{\gamma,V}(v) + \omega\|v\|_2^2\right) - \lambda^{p+1}\|v\|_{p+1}^{p+1} =: f(\lambda)$, we obtain $I_\omega(\lambda_1 v) = 0$ and $0 < \lambda_1 < 1$ (one needs to remark that $f(1) < 0$, $f(0) = 0$, and $f'(\lambda) > 0$ for small positive λ). Hence, we have

$$d_\omega \le \frac{p-1}{2(p+1)}\|\lambda_1 v\|_{p+1}^{p+1} = \frac{p-1}{2(p+1)}\lambda_1^{p+1}\|v\|_{p+1}^{p+1} < \frac{p-1}{2(p+1)}\|v\|_{p+1}^{p+1}.$$

Thus, we obtain $d_\omega \le d_\omega^*$. Similarly, we can show $d_\omega < \frac{p-1}{2(p+1)}\left(F_{\gamma,V}(v) + \omega\|v\|_2^2\right)$ and the second part of (3.1) since we can rewrite

$$d_\omega = \inf\left\{\frac{p-1}{2(p+1)}\left(F_{\gamma,V}(v) + \omega\|v\|_2^2\right) \ : \ u \in H^1(\Gamma)\backslash\{0\},\ I_\omega(v) = 0\right\}.$$

□

To get the existence of the minimizers of d_ω, one has at a certain point to compare the action S_ω for $\gamma > 0$ with the action S_ω^0 of the nonpotential case ($V(x) \equiv 0, \gamma > 0$). Set

$$\begin{aligned}
S_\omega^0(v) &= \frac{1}{2}\|v'\|_2^2 + \frac{\omega}{2}\|v\|_2^2 - \frac{\gamma}{2}|v_1(0)|^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1},\\
I_\omega^0(v) &= \|v'\|_2^2 + \omega\|v\|_2^2 - \gamma|v_1(0)|^2 - \|v\|_{p+1}^{p+1},\\
d_\omega^0 &= \inf\left\{S_\omega^0(v) \ : \ v \in H^1(\Gamma)\backslash\{0\},\ \ I_\omega^0(v) = 0\right\}\\
&= \inf\left\{\frac{p-1}{2(p+1)}\|v\|_{p+1}^{p+1} \ : \ v \in H^1(\Gamma)\backslash\{0\},\ \ I_\omega^0(v) = 0\right\},
\end{aligned}$$

and

$$\mathcal{M}_\omega^0 := \left\{\phi \in H^1(\Gamma)\backslash\{0\} \ : \ I_\omega^0(\phi) = 0,\ S_\omega^0(\phi) = d_\omega^0\right\}.$$

It is known that for $\gamma > \gamma^*$, where γ^* is defined by (1.6), the set $\mathcal{M}_\omega^0$ is not empty (see [2]). Throughout this section, we assume $\gamma > \gamma^*$.

Lemma 3.2. $d_\omega^0 > d_\omega > 0$.

Proof. First, we show that $d_\omega > 0$. Let $v \in H^1(\Gamma)\backslash\{0\}$ satisfy $I_\omega(v) = 0$. Then,

$$\|v\|_{p+1}^{p+1} = F_{\gamma,V}(v) + \omega \|v\|_2^2 .$$

Since $\omega > \omega_0$, by the Sobolev embedding and Lemma 4.13-(ii), we have

$$\|v\|_{p+1}^2 \leq C_1 \|v\|_{H^1(\Gamma)}^2 \leq C_2 \left(F_{\gamma,V}(v) + \omega \|v\|_2^2\right) = C_2 \|v\|_{p+1}^{p+1} .$$

Hence, we obtain $C_2^{\frac{-1}{p-1}} \leq \|v\|_{p+1}$. Taking the infimum over v, we get $d_\omega > 0$. Next, we prove $d_\omega^0 > d_\omega$. Since $\mathcal{M}_\omega^0$ is not empty, we can take $\phi \in \mathcal{M}_\omega^0$. By *Assumption 3*,

$$I_\omega(\phi) = (V\phi, \phi)_2 < 0.$$

Then, by Lemma 3.1, we obtain

$$d_\omega < \frac{p-1}{2(p+1)} \|\phi\|_{p+1}^{p+1} = d_\omega^0.$$

□

Lemma 3.3. *Let $\{v_n\} \subset H^1(\Gamma)\backslash\{0\}$ be a minimizing sequence for d_ω, i.e., $I_\omega(v_n) = 0$ and $\lim_{n\to\infty} S_\omega(v_n) = d_\omega$. Then, there exist a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ and $v_0 \in H^1(\Gamma)\backslash\{0\}$ such that $\lim_{k\to\infty} \left\|v_{n_k} - v_0\right\|_{H^1(\Gamma)} = 0$, $I_\omega(v_0) = 0$ and $S_\omega(v_0) = d_\omega$. Therefore, $\mathcal{M}_\omega$ is not empty.*

Proof. Since $\omega > \omega_0$ and

$$S_\omega(v_n) = \frac{p-1}{2(p+1)} \left(F_{\gamma,V}(v_n) + \omega \|v_n\|_2^2\right) = \frac{p-1}{2(p+1)} \|v_n\|_{p+1}^{p+1} \underset{n\to\infty}{\longrightarrow} d_\omega, \quad (3.3)$$

the sequence $\{v_n\}$ is bounded in $H^1(\Gamma)$ (see Lemma 4.13-(ii)). Hence, there exist a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ and $v_0 \in H^1(\Gamma)$ such that $\{v_{n_k}\}$ converges weakly to v_0 in $H^1(\Gamma)$. We may assume that $v_{n_k} \neq 0$ and define

$$\lambda_k = \left(\frac{\left\|v_{n_k}'\right\|_2^2 + \omega \left\|v_{n_k}\right\|_2^2 - \gamma \left|v_{n_k,1}(0)\right|^2}{\left\|v_{n_k}\right\|_{p+1}^{p+1}}\right)^{\frac{1}{p-1}} .$$

Notice that $\lambda_k > 0$ and $I_\omega^0(\lambda_k v_{n_k}) = 0$. Therefore, by Lemma 3.2 and the definition of d_ω^0, we obtain

$$d_\omega < d_\omega^0 \leq \frac{p-1}{2(p+1)} \left\|\lambda_k v_{n_k}\right\|_{p+1}^{p+1} = \lambda_k^{p+1} \frac{p-1}{2(p+1)} \left\|v_{n_k}\right\|_{p+1}^{p+1}, \quad \text{for all } k \in \mathbb{N}. \quad (3.4)$$

Furthermore, by $I_\omega(v_{n_k}) = 0$, (3.3) and the weak continuity of $(Vv, v)_2 = \int_\Gamma V(x)|v(x)|^2 \mathrm{d}x$ (see [23, Theorem 11.4]), we get

$$\lim_{k\to\infty} \lambda_k = \lim_{k\to\infty} \left(\frac{\|v_{n_k}\|_{p+1}^{p+1} - (Vv_{n_k}, v_{n_k})_2}{\|v_{n_k}\|_{p+1}^{p+1}} \right)^{\frac{1}{p-1}} = \left(\frac{\mathrm{d}_\omega - \frac{p-1}{2(p+1)}(Vv_0, v_0)_2}{\mathrm{d}_\omega} \right)^{\frac{1}{p-1}}.$$

Taking the limit in (3.4), we obtain $d_\omega < \lim_{k\to\infty} \lambda_k^{p+1} d_\omega$. Since $d_\omega > 0$, we arrive at $\lim_{k\to\infty} \lambda_k > 1$, and consequently, $-(Vv_0, v_0)_2 > 0$. Thus, $v_0 \neq 0$.
By the weak convergence, we obtain

$$\begin{aligned} &\lim_{k\to\infty} \left\{ \left(F_{\gamma,V}(v_{n_k}) - F_{\gamma,V}(v_{n_k} - v_0) \right) + \omega \left(\|v_{n_k}\|_2^2 - \|v_{n_k} - v_0\|_2^2 \right) \right\} \\ &\quad = F_{\gamma,V}(v_0) + \omega \|v_0\|_2^2 . \end{aligned} \tag{3.5}$$

Next, passing to a subsequence of $\{v_{n_k}\}$ if necessary, we may assume that $v_{n_k} \underset{k\to\infty}{\longrightarrow} v_0$ a.e. on Γ. Therefore, by the Brezis–Leib lemma [8],

$$\lim_{k\to\infty} I_\omega(v_{n_k}) - I_\omega(v_{n_k} - v_0) = \lim_{k\to\infty} -I_\omega(v_{n_k} - v_0) = I_\omega(v_0).$$

Since $v_0 \neq 0$, then the right-hand side of (3.5) is positive. It follows from (3.3) and (3.5) that

$$\begin{aligned} &\frac{p-1}{2(p+1)} \lim_{k\to\infty} \left(F_{\gamma,V}(v_{n_k} - v_0) + \omega \|v_{n_k} - v_0\|_2^2 \right) \\ &\quad < \frac{p-1}{2(p+1)} \lim_{k\to\infty} \left(F_{\gamma,V}(v_{n_k}) + \omega \|v_{n_k}\|_2^2 \right) = d_\omega. \end{aligned}$$

Hence, by (3.1), we have $I_\omega(v_{n_k} - v_0) > 0$ for k large enough. Thus, since $-I_\omega(v_{n_k} - v_0) \underset{k\to\infty}{\longrightarrow} I_\omega(v_0)$, we obtain $I_\omega(v_0) \leq 0$. Then, by (3.1) and the weak lower semicontinuity of norms, we see that

$$d_\omega \leq \frac{p-1}{2(p+1)} \left(F_{\gamma,V}(v_0) + \omega \|v_0\|_2^2 \right) \leq \frac{p-1}{2(p+1)} \lim_{k\to\infty} \left(F_{\gamma,V}(v_{n_k}) + \omega \|v_{n_k}\|_2^2 \right) = d_\omega.$$

Therefore, from (3.5) we get

$$\lim_{k\to\infty} F_{\gamma,V}(v_{n_k} - v_0) + \omega \|v_{n_k} - v_0\|_2^2 = 0,$$

and consequently, by Lemma 4.13-(ii), we have $v_{n_k} \underset{k\to\infty}{\longrightarrow} v_0$ in $H^1(\Gamma)$ and $I_\omega(v_0) = 0$. This concludes the proof. □

Proof of Proposition 1.1. Step 1. We prove that $\mathcal{G}_\omega = \mathcal{M}_\omega$. Let $\varphi \in \mathcal{M}_\omega$. Since $I_\omega(\varphi) = 0$, we have

$$\left\langle I'_\omega(\varphi), \varphi \right\rangle = 2 \left(F_{\gamma,V}(\varphi) + \omega \|\varphi\|_2^2 \right) - (p+1) \|\varphi\|_{p+1}^{p+1} = -(p-1) \|\varphi\|_{p+1}^{p+1} < 0. \tag{3.6}$$

There exists a Lagrange multiplier $\mu \in \mathbb{R}$ such that $S'_\omega(\varphi) = \mu I'_\omega(\varphi)$. Furthermore, since

$$\mu \left\langle I'_\omega(\varphi), \varphi \right\rangle = \left\langle S'_\omega(\varphi), \varphi \right\rangle = I_\omega(\varphi) = 0,$$

then, by (3.6), $\mu = 0$. Hence, $S'_\omega(\varphi) = 0$. Moreover, for $v \in H^1(\Gamma)\backslash\{0\}$ satisfying $S'_\omega(v) = 0$, we have $I_\omega(v) = \left\langle S'_\omega(v), v \right\rangle = 0$. Then, from the definition of $\mathcal{M}_\omega$, we get $S_\omega(\varphi) \leq S_\omega(v)$. Hence, we obtain $\varphi \in \mathcal{G}_\omega$. Now, let $\phi \in \mathcal{G}_\omega$. Since $\mathcal{M}_\omega$ is not empty, we take $\varphi \in \mathcal{M}_\omega$. By the first part of the proof, we have $\varphi \in \mathcal{G}_\omega$; therefore, $S_\omega(\phi) = S_\omega(\varphi) = d_\omega$. This implies $\phi \in \mathcal{M}_\omega$.

Step 2. Let $\varphi \in \mathcal{G}_\omega$. Below we show that φ has the form $\varphi(x) = e^{i\theta}\phi(x)$ with positive $\phi(x) \in \mathrm{dom}(H_{\gamma,V})$. Set $\phi := |\varphi|$, then $\|\phi'\|_2^2 \leq \|\varphi'\|_2^2$ and $S_\omega(\phi) \leq S_\omega(\varphi) = d_\omega$. Using $\mathcal{G}_\omega = \mathcal{M}_\omega$, we obtain $I_\omega(\varphi) = 0$, then $I_\omega(\phi) \leq 0$. It follows from Lemma 3.1 that $\phi \in \mathcal{M}_\omega$ and $S_\omega(\varphi) = S_\omega(\phi)$. Observe that this implies

$$\|\phi'\|_2^2 = \sum_{e=1}^{N} \int_0^\infty \left|\phi'_e(x)\right|^2 \mathrm{d}x = \sum_{e=1}^{N} \int_0^\infty \left|\varphi'_e(x)\right|^2 \mathrm{d}x = \|\varphi'\|_2^2. \tag{3.7}$$

From $S'_\omega(\phi) = 0$, repeating the proof of [2, Theorem 4] (see also [5, Lemma 4.1]), one gets $\phi \in \mathrm{dom}(H_{\gamma,V})$ and

$$H_{\gamma,V}\phi + \omega\phi - \phi^p = 0,$$

therefore,

$$-\phi''_e + \omega\phi_e + V_e(x)\phi_e - \phi_e^p = 0, \quad x \in (0,\infty), \ e = 1, \ldots, N.$$

Recalling that $V(x) \leq 0$ a.e. on Γ (see Remark 1.2) and using [28, Theorem 1], we have that ϕ_e is either trivial or strictly positive on $(0,\infty)$. Indeed, to prove that, we need to set $\beta(s) := \omega s - s^p$ and observe that $\beta(s) \in C^1[0,\infty)$ is nondecreasing for s small, and $\beta(0) = \beta(\omega^{\frac{1}{p-1}}) = 0$.

Now assume $\phi_e(0) = \phi'_e(0) = 0$ and put

$$\widetilde{\phi}_e(x) = \begin{cases} \phi_e(x), & x \in [0,\infty) \\ 0, & x \in (-\delta, 0). \end{cases}$$

Then, by the Sobolev extension theorem, we have $\widetilde{\phi}_e \in H^2(-\delta,\infty)$. Moreover,

$$-\widetilde{\phi}''_e + \omega\widetilde{\phi}_e + V_e(x)\widetilde{\phi}_e - \widetilde{\phi}_e^p = 0, \ \text{ on } (-\delta,\infty).$$

Therefore, by [28, Theorem 1], arguing as above, we find that $\widetilde{\phi}_e = 0$ on $(-\delta,\infty)$.

Next assume $\phi(0) = 0$, i.e., $\phi_1(0) = \ldots = \phi_N(0)$. Since $\phi_e \in C^1(0,\infty)$, $\phi_e \geq 0$ and $\phi_e(0) = 0$, then $\phi'_e(0) \geq 0$. By $\sum_{e=1}^N \phi'_e(0) = -\gamma\phi_1(0) = 0$, we get $\phi_e(0) = \phi'_e(0) = 0$. Then, $\phi_e = 0$ on $(0,\infty)$ for all $e = 1, \ldots, N$, and by continuity $\phi = 0$ on

Γ, which is absurd since $\phi \in \mathcal{M}_\omega$. Hence, $\phi_e(0) > 0$ for all $e = 1, \ldots, N$; therefore, $\phi_e > 0$ on $(0, \infty)$ for all $e = 1, \ldots, N$, i.e., $\phi > 0$ on Γ.

Step 3. Now, we can write $\varphi_e(x) = \phi_e(x)\tau_e(x)$, where $\tau_e \in C^1(0, \infty)$, $|\tau_e| = 1$. Then,

$$\varphi_e' = \phi_e'\tau_e + \phi_e\tau_e' = \tau_e(\phi_e' + \phi_e\overline{\tau}_e\tau_e').$$

Using $\mathrm{Re}(\overline{\tau}_e\tau_e') = 0$, we have $\left|\varphi_e'\right|^2 = \left|\phi_e'\right|^2 + \left|\phi_e\tau_e'\right|^2$. Therefore, from (3.7) we obtain

$$\sum_{e=1}^{N}\int_0^\infty \left|\phi_e'\right|^2 \mathrm{d}x = \sum_{e=1}^{N}\int_0^\infty \left|\varphi_e'\right|^2 \mathrm{d}x = \sum_{e=1}^{N}\int_0^\infty \left|\phi_e'\right|^2 \mathrm{d}x + \sum_{e=1}^{N}\int_0^\infty \left|\phi_e\tau_e'\right|^2 \mathrm{d}x.$$

So far as $\phi_e > 0$, we have $\tau_e' = 0$ for all $e = 1, \ldots, N$. Since $\tau_e \in C^1(0, \infty)$, there exists a constant $\theta_e \in \mathbb{R}$ such that $\tau_e(x) = e^{i\theta_e}$ on $(0, \infty)$. By the continuity at the vertex, we obtain $\theta_e = \theta = const$ for all $e = 1, \ldots, N$. This ends the proof. □

4. Instability of standing waves

In this section, we prove Theorem 1.4 and Corollary 1.5.

4.1. Proof of the main result

We begin with the following lemma.

Lemma 4.1. *Let $\varphi_\omega \in \mathcal{M}_\omega$. Then,*

$$(i)\quad \|\varphi_\omega\|_{p+1}^{p+1} = \inf\left\{\|v\|_{p+1}^{p+1} \;:\; v \in H^1(\Gamma)\backslash\{0\},\ I_\omega(v) = 0\right\}$$
$$= \inf\left\{\|v\|_{p+1}^{p+1} \;:\; v \in H^1(\Gamma)\backslash\{0\},\ I_\omega(v) \le 0\right\},$$
$$(ii)\quad S_\omega(\varphi_\omega) = \inf\{S_\omega(v) \;:\; v \in H^1(\Gamma),\ \|v\|_{p+1}^{p+1} = \|\varphi_\omega\|_{p+1}^{p+1}\}.$$

Proof. (i) This is an immediate consequence of Lemma 3.1
(ii) Set $d_\omega^{**} := \inf\{S_\omega(v) \;:\; v \in H^1(\Gamma),\ \|v\|_{p+1}^{p+1} = \|\varphi_\omega\|_{p+1}^{p+1}\}$. As far as $d_\omega^{**} \le S_\omega(\varphi_\omega)$, it suffices to prove $S_\omega(\varphi_\omega) \le d_\omega^{**}$. If $v \in H^1(\Gamma)$ satisfies $\|v\|_{p+1}^{p+1} = \|\varphi_\omega\|_{p+1}^{p+1}$, then, by item (i) and (3.2), we have $I_\omega(v) \ge 0$. Hence, by (3.2),

$$S_\omega(\varphi_\omega) = \frac{p-1}{2(p+1)}\|\varphi_\omega\|_{p+1}^{p+1} = \frac{p-1}{2(p+1)}\|v\|_{p+1}^{p+1} \le S_\omega(v).$$

Thus, we obtain $S_\omega(\varphi_\omega) \le d_\omega^{**}$. □

Recall that

$$P(v) = \left\|v'\right\|_2^2 - \frac{1}{2}\int_\Gamma xV'(x)|v(x)|^2\,\mathrm{d}x - \frac{\gamma}{2}|v_1(0)|^2 - \frac{p-1}{2(p+1)}\|v\|_{p+1}^{p+1}.$$

Lemma 4.2. *If $\partial_\lambda^2 E(\varphi_\omega^\lambda)\,|_{\lambda=1} < 0$, then there exist $\delta > 0$ and $\varepsilon > 0$ such that the following holds: for any $v \in N_\varepsilon(\varphi_\omega)$ satisfying $\|v\|_2^2 \le \|\varphi_\omega\|_2^2$, there exists $\lambda(v) \in (1-\delta, 1+\delta)$ such that $E(\varphi_\omega) \le E(v) + (\lambda(v)-1)P(v)$, where $N_\varepsilon(\varphi_\omega)$ is defined by* (1.7).

Proof. Since $\partial_\lambda^2 E(\varphi_\omega^\lambda)\,|_{\lambda=1} < 0$ and $\partial_\lambda^2 E(v^\lambda)$ is continuous in v (we mean "orbit"-continuity) and λ, there exist positive constants ε and δ such that $\partial_\lambda^2 E(v^\lambda) < 0$ for any $v \in N_\varepsilon(\varphi_\omega)$ and $\lambda \in (1-\delta, 1+\delta)$. Using $P(v) = \partial_\lambda E(v^\lambda)\,|_{\lambda=1}$, the Taylor expansion at $\lambda = 1$ gives

$$E(v^\lambda) \le E(v) + (\lambda - 1)P(v), \quad \lambda \in (1-\delta, 1+\delta), \quad v \in N_\varepsilon(\varphi_\omega). \tag{4.1}$$

Let $v \in N_\varepsilon(\varphi_\omega)$ satisfy $\|v\|_2^2 \le \|\varphi_\omega\|_2^2$. We define

$$\lambda(v) := \left(\frac{\|\varphi_\omega\|_{p+1}^{p+1}}{\|v\|_{p+1}^{p+1}} \right)^{\frac{2}{p-1}}.$$

Then, $\left\|v^{\lambda(v)}\right\|_{p+1}^{p+1} = \|\varphi_\omega\|_{p+1}^{p+1}$ and we can take ε small enough to guarantee $\lambda(v) \in (1-\delta, 1+\delta)$. Since $\left\|v^{\lambda(v)}\right\|_2^2 = \|v\|_2^2 \le \|\varphi_\omega\|_2^2$, by Lemma 4.1-(ii), we have

$$E(v^{\lambda(v)}) = S_\omega(v^{\lambda(v)}) - \frac{\omega}{2}\left\|v^{\lambda(v)}\right\|_2^2 \ge S_\omega(\varphi_\omega) - \frac{\omega}{2}\|\varphi_\omega\|_2^2 = E(\varphi_\omega),$$

which together with (4.1) implies that $E(\varphi_\omega) \le E(v) + (\lambda(v)-1)P(v)$. □

To prove Theorem 1.4, we introduce the following definition.

Definition 4.3. Let ε be the positive constant given by Lemma 4.2. Set

$$\mathcal{Z}_\varepsilon(\varphi_\omega) := \{v \in N_\varepsilon(\varphi_\omega) \; : \; E(v) < E(\varphi_\omega), \; \|v\|_2^2 \le \|\varphi_\omega\|_2^2, \; P(v) < 0\},$$

and for any $u_0 \in N_\varepsilon(\varphi_\omega)$, we define the exit time from $N_\varepsilon(\varphi_\omega)$ by

$$T_\varepsilon(u_0) = \sup\{T > 0 \; : \; u(t) \in N_\varepsilon(\varphi_\omega), \; 0 \le t \le T\},$$

with $u(t)$ being a solution of (1.1).

Lemma 4.4. *Assume $\partial_\lambda^2 E(\varphi_\omega^\lambda)\,|_{\lambda=1} < 0$, then for any $u_0 \in \mathcal{Z}_\varepsilon(\varphi_\omega)$, there exists $b = b(u_0) > 0$ such that $P(u(t)) \le -b$ for $0 \le t < T_\varepsilon(u_0)$.*

Proof. Set $b_0 := E(\varphi_\omega) - E(u_0) > 0$, with $u_0 \in \mathcal{Z}_\varepsilon(\varphi_\omega)$. From the conservation of energy and Lemma 4.2, we have

$$b_0 \le (\lambda(u(t)) - 1)P(u(t)), \quad 0 \le t < T_\varepsilon(u_0). \tag{4.2}$$

Therefore, for $0 \le t < T_\varepsilon(u_0)$ we get $P(u(t)) \ne 0$. Indeed, if $P(u(t_0)) = 0$ for some $t_0 \in [0, T(u_0))$, then from (4.2) it follows $b_0 \le 0$, which contradicts the definition of b_0. Since $P(u_0) < 0$ and the function $t \mapsto P(u(t))$ is continuous, we see that $P(u(t)) < 0$ for $0 \le t < T_\varepsilon(u_0)$, and hence, $\lambda(u(t)) - 1 < 0$ for $0 \le t < T_\varepsilon(u_0)$. Thus, from Lemma 4.2 and (4.2), we have

$$P(u(t)) \le \frac{b_0}{\lambda(u(t)) - 1} \le \frac{-b_0}{\delta}, \quad 0 \le t < T_\varepsilon(u_0).$$

Hence, taking $b = \dfrac{b_0}{\delta}$, we arrive at $P(u(t)) \le -b$ for $0 \le t < T_\varepsilon(u_0)$. □

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Observe that $P(v) = \partial_\lambda S_\omega(v^\lambda)|_{\lambda=1} = \langle S'_\omega(v), \partial_\lambda v^\lambda|_{\lambda=1}\rangle$. Since $S'_\omega(\varphi_\omega) = 0$, we obtain $P(\varphi_\omega) = \partial_\lambda S_\omega(\varphi_\omega^\lambda)\,|_{\lambda=1} = 0$. Moreover, by $P(\varphi_\omega^\lambda) = \lambda \partial_\lambda E(\varphi_\omega^\lambda)$, we have $\partial_\lambda E(\varphi_\omega^\lambda)\,|_{\lambda=1} = 0$. Then, from the assumption $\partial_\lambda^2 E(\varphi_\omega^\lambda)\,|_{\lambda=1} < 0$, we get $E(\varphi_\omega^\lambda) < E(\varphi_\omega)$ and $P(\varphi_\omega^\lambda) < 0$ for $\lambda > 1$ close enough to 1.

Let $\varepsilon > 0$ be given by Lemma 4.2. Since $\lim\limits_{\lambda \to 1} \|\varphi_\omega^\lambda - \varphi_\omega\|_{H^1(\Gamma)} = 0$ and $\|\varphi_\omega^\lambda\|_2^2 = \|\varphi_\omega\|_2^2$, by continuity of E and P, for any $\delta \le \varepsilon$ there exists λ_1 such that $\varphi_\omega^{\lambda_1} \in \mathcal{Z}_{\frac{\delta}{2}}(\varphi_\omega)$.

Suppose that $\chi \in C_c^\infty(\mathbb{R}^+)$ is the function satisfying

$$0 \le \chi \le 1, \quad \chi(x) = 1, \text{ if } x \in [0, 1], \quad \text{and } \chi(x) = 0 \text{ if } x \ge 2.$$

For $a > 0$, we define $\chi_a \in C_c^\infty(\Gamma)$ by

$$(\chi_a)_e(x) = \chi\left(\frac{x}{a}\right), \quad x \in \mathbb{R}^+, \ e = 1, \dots, N.$$

Then, we have $\lim\limits_{a \to \infty} \left\|\chi_a \varphi_\omega^{\lambda_1} - \varphi_\omega^{\lambda_1}\right\|_{H^1(\Gamma)} = 0$ and $\left\|\chi_a \varphi_\omega^{\lambda_1}\right\|_2^2 \le \left\|\varphi_\omega^{\lambda_1}\right\|_2^2 = \|\varphi_\omega\|_2^2$ for all $a > 0$. Thus, by continuity of E and P, for any $\delta \le \varepsilon$ there exists $a_1 > 0$ such that $\chi_{a_1}\varphi_\omega^{\lambda_1} \in \mathcal{Z}_{\frac{\delta}{2}}(\varphi_\omega^{\lambda_1})$, therefore $\chi_{a_1}\varphi_\omega^{\lambda_1} \in \mathcal{Z}_\delta(\varphi_\omega) \subseteq \mathcal{Z}_\varepsilon(\varphi_\omega)$.

Observe that $\chi_{a_1}\varphi_\omega^{\lambda_1} \in \Sigma(\Gamma)$ (see Proposition 2.6 for the definition of $\Sigma(\Gamma)$), and by virial identity (2.4), we see that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\|xu_1(t)\|_2^2 = 8P(u_1(t)), \qquad 0 \le t \le T_\varepsilon(\chi_{a_1}\varphi_\omega^{\lambda_1}), \tag{4.3}$$

where $u_1(t)$ is the solution to (1.1) with $u_1(0) = \chi_{a_1}\varphi_\omega^{\lambda_1}$. From Lemma 4.4, there exists $b = b(\lambda_1, a_1) > 0$ such that

$$P(u_1(t)) \le -b, \qquad 0 \le t < T_\varepsilon(\chi_{a_1}\varphi_\omega^{\lambda_1}). \tag{4.4}$$

Then, from (4.4) and (4.3), we can see that $T_\varepsilon(\chi_{a_1}\varphi_\omega^{\lambda_1}) < \infty$.

Summarizing the above, we affirm: there exists $\varepsilon > 0$ (given by Lemma 4.2) such that for all $\delta > 0$ there exist $u_0 = \chi_{a_1}\varphi_\omega^{\lambda_1} \in N_\delta(\varphi_\omega)$ and $t_1 > 0$ such that the corresponding solution $u_1(t)$ of (1.1) satisfies $u_1(t_1) \notin N_\varepsilon(\varphi_\omega)$. Hence, the standing wave solution $e^{i\omega t}\varphi_\omega$ of (1.1) is orbitally unstable. □

4.2. Rescaled variational problem and proof of Corollary 1.5

Assume that $V(x)=\frac{-\beta}{x^\alpha}$, $\beta>0$, $0<\alpha<1$. Recall that $v^\lambda(x)=\lambda^{1/2}v(\lambda x)$ for $\lambda>0$. By simple computations, we have

$$E(v^\lambda)=\frac{\lambda^2}{2}\|v'\|_2^2+\frac{\lambda^\alpha}{2}(Vv,v)_2-\frac{\lambda}{2}\gamma\,|v_1(0)|^2-\frac{\lambda^{\frac{p-1}{2}}}{p+1}\|v\|_{p+1}^{p+1},$$

$$\partial_\lambda^2 E(v^\lambda)\mid_{\lambda=1}\;=\|v'\|_2^2+\frac{\alpha(\alpha-1)}{2}(Vv,v)_2-\frac{(p-1)(p-3)}{4(p+1)}\|v\|_{p+1}^{p+1}.$$

Since $P(\varphi_\omega)=\partial_\lambda S_\omega(\varphi_\omega^\lambda)\mid_{\lambda=1}\;=0$, then we get

$$\partial_\lambda^2 E(\varphi_\omega^\lambda)\mid_{\lambda=1}\;=-\frac{\alpha(2-\alpha)}{2}(V\varphi_\omega,\varphi_\omega)_2+\frac{\gamma}{2}\left|\varphi_{\omega,1}(0)\right|^2-\frac{(p-1)(p-5)}{4(p+1)}\|\varphi_\omega\|_{p+1}^{p+1},$$

and $\partial_\lambda^2 E(\varphi_\omega^\lambda)\mid_{\lambda=1}\;<0$ is equivalent to

$$\frac{-\alpha(2-\alpha)(V\varphi_\omega,\varphi_\omega)_2+\gamma\left|\varphi_{\omega,1}(0)\right|^2}{\|\varphi_\omega\|_{p+1}^{p+1}}<\frac{(p-1)(p-5)}{2(p+1)}. \tag{4.5}$$

Below we prove that the left-hand side of (4.5) converges to 0 as $\omega\to\infty$. To this end, we consider the following rescaling of $\varphi_\omega\in\mathcal{M}_\omega$:

$$\varphi_\omega(x)=\omega^{\frac{1}{p-1}}\widetilde{\varphi}_\omega(\sqrt{\omega}x),\quad \omega\in(\omega_0,\infty), \tag{4.6}$$

and observe

$$\begin{aligned}&\frac{-\omega^{-\frac{2-\alpha}{2}}\alpha(2-\alpha)(V\widetilde{\varphi}_\omega,\widetilde{\varphi}_\omega)_2+\omega^{-\frac{1}{2}}\gamma\left|\widetilde{\varphi}_{\omega,1}(0)\right|^2}{\|\widetilde{\varphi}_\omega\|_{p+1}^{p+1}}\\&=\frac{-\alpha(2-\alpha)(V\varphi_\omega,\varphi_\omega)_2+\gamma\left|\varphi_{\omega,1}(0)\right|^2}{\|\varphi_\omega\|_{p+1}^{p+1}}.\end{aligned} \tag{4.7}$$

Put

$$\widetilde{I}_\omega(v):=\|v'\|_2^2+\|v\|_2^2-\omega^{-\frac{2-\alpha}{2}}\beta\int_\Gamma\frac{|v(x)|^2}{x^\alpha}\mathrm{d}x-\omega^{-\frac{1}{2}}\gamma\,|v_1(0)|^2-\|v\|_{p+1}^{p+1},$$
$$\widetilde{I}_0(v):=\|v'\|_2^2+\|v\|_2^2-\|v\|_{p+1}^{p+1}.$$

Consider the minimization problem

$$\widetilde{d}_0:=\inf\left\{\|v\|_{p+1}^{p+1}\;:\;v\in H^1(\Gamma)\backslash\{0\},\;\widetilde{I}_0(v)\le 0\right\}. \tag{4.8}$$

In [2, Theorem 3], it was shown that $\widetilde{d}_0>0$. The following lemma is the key result to prove Corollary 1.5.

Lemma 4.5. *Assume that $\gamma > 0$, $\beta > 0$, $0 < \alpha < 1$ and $p > 5$. Let $\varphi_\omega \in \mathcal{M}_\omega$, and $\widetilde{\varphi}_\omega(x)$ be the rescaled function given in (4.6). Then,*

(i) $\lim_{\omega\to\infty} \|\widetilde{\varphi}_\omega\|_{p+1}^{p+1} = \widetilde{d}_0$,

(ii) $\lim_{\omega\to\infty} \widetilde{I}_0(\widetilde{\varphi}_\omega) = 0$,

(iii) $\lim_{\omega\to\infty} \|\widetilde{\varphi}_\omega\|_{H^1(\Gamma)}^2 = \widetilde{d}_0$.

Proof. Notice that

$$\begin{aligned}\|\widetilde{\varphi}_\omega\|_{p+1}^{p+1} &= \inf\left\{\|v\|_{p+1}^{p+1} \;:\; v \in H^1(\Gamma)\setminus\{0\},\ \widetilde{I}_\omega(v) = 0\right\}\\ &= \inf\left\{\|v\|_{p+1}^{p+1} \;:\; v \in H^1(\Gamma)\setminus\{0\},\ \widetilde{I}_\omega(v) \le 0\right\} := \widetilde{d}_\omega.\end{aligned} \tag{4.9}$$

By definition, we have

$$\widetilde{I}_0(v) = \widetilde{I}_\omega(v) - \omega^{-\frac{2-\alpha}{2}}(Vv, v)_2 + \omega^{-\frac{1}{2}}\gamma\,|v_1(0)|^2, \quad \text{and} \tag{4.10}$$

$$\widetilde{I}_0(v) = \lambda^{-2}\widetilde{I}_0(\lambda v) + (\lambda^{p-1} - 1)\|v\|_{p+1}^{p+1}. \tag{4.11}$$

Using, (4.10), (4.11), $\widetilde{I}_\omega(\widetilde{\varphi}_\omega) = 0$, estimate (4.18), and the Sobolev embedding, for any $\lambda > 1$ we get

$$\begin{aligned}\lambda^{-2}\widetilde{I}_0(\lambda\widetilde{\varphi}_\omega) &= -\omega^{-\frac{2-\alpha}{2}}(V\widetilde{\varphi}_\omega, \widetilde{\varphi}_\omega)_2 + \omega^{-\frac{1}{2}}\gamma\left|\widetilde{\varphi}_{\omega,1}(0)\right|^2 - (\lambda^{p-1} - 1)\|\widetilde{\varphi}_\omega\|_{p+1}^{p+1}\\ &\le C_1\omega^{-\frac{2-\alpha}{2}}\|\widetilde{\varphi}_\omega\|_{H^1(\Gamma)}^2 + C_2\omega^{-\frac{1}{2}}\gamma\|\widetilde{\varphi}_\omega\|_{H^1(\Gamma)}^2 - (\lambda^{p-1} - 1)\|\widetilde{\varphi}_\omega\|_{p+1}^{p+1}.\end{aligned} \tag{4.12}$$

Moreover, from $\widetilde{I}_\omega(\widetilde{\varphi}_\omega) = 0$, we deduce

$$\begin{aligned}\|\widetilde{\varphi}_\omega\|_{H^1(\Gamma)}^2 &= -\omega^{-\frac{2-\alpha}{2}}(V\widetilde{\varphi}_\omega, \widetilde{\varphi}_\omega)_2 + \omega^{-\frac{1}{2}}\gamma\left|\widetilde{\varphi}_{\omega,1}(0)\right|^2 + \|\widetilde{\varphi}_\omega\|_{p+1}^{p+1}\\ &\le C_1\omega^{-\frac{2-\alpha}{2}}\|\widetilde{\varphi}_\omega\|_{H^1(\Gamma)}^2 + C_2\omega^{-\frac{1}{2}}\gamma\|\widetilde{\varphi}_\omega\|_{H^1(\Gamma)}^2 + \|\widetilde{\varphi}_\omega\|_{p+1}^{p+1}.\end{aligned}$$

This implies

$$\left(1 - C_1\omega^{-\frac{2-\alpha}{2}} - C_2\omega^{-\frac{1}{2}}\gamma\right)\|\widetilde{\varphi}_\omega\|_{H^1(\Gamma)}^2 \le \|\widetilde{\varphi}_\omega\|_{p+1}^{p+1}.$$

Since for ω sufficiently large $\left(1 - C_1\omega^{-\frac{2-\alpha}{2}} - C_2\omega^{-\frac{1}{2}}\gamma\right) > 0$, from (4.12) we get

$$\lambda^{-2}\widetilde{I}_0(\lambda\widetilde{\varphi}_\omega) \le -\left(\lambda^{p-1} - 1 - \frac{C_1\omega^{-\frac{2-\alpha}{2}} + C_2\omega^{-\frac{1}{2}}\gamma}{1 - C_1\omega^{-\frac{2-\alpha}{2}} - C_2\omega^{-\frac{1}{2}}\gamma}\right)\|\widetilde{\varphi}_\omega\|_{p+1}^{p+1}. \tag{4.13}$$

Hence, for any $\lambda > 1$, there exists $\omega_1 = \omega_1(\lambda) \in (\omega_0, \infty)$ such that $\widetilde{I}_0(\lambda\widetilde{\varphi}_\omega) < 0$ for $\omega \in (\omega_1, \infty)$. Thus, by (4.8), $\widetilde{d}_0 \le \lambda^{p+1}\|\widetilde{\varphi}_\omega\|_{p+1}^{p+1}$ for $\omega \in (\omega_1, \infty)$. Observe that $\widetilde{I}_0(v) \leq 0$ implies $\widetilde{I}_\omega(v) \leq 0$; then, from (4.9) we obtain $\widetilde{d}_\omega = \|\widetilde{\varphi}_\omega\|_{p+1}^{p+1} \leq \widetilde{d}_0$. Therefore,

$$\lambda^{-(p+1)}\widetilde{d}_0 \leq \|\widetilde{\varphi}_\omega\|_{p+1}^{p+1} \leq \widetilde{d}_0, \quad \omega \in (\omega_1, \infty). \tag{4.14}$$

Letting $\lambda \downarrow 1$, we get that $\omega \to \infty$, and from (4.14) it follows (i).

Now, assume that $\lambda = 1$ in (4.13); then, using (i), we deduce

$$\limsup_{\omega\to\infty} \widetilde{I}_0(\widetilde{\varphi}_\omega) \leq 0. \tag{4.15}$$

Furthermore, define

$$\lambda_1(\omega) = \left(\frac{\|\widetilde{\varphi}'_\omega\|_2^2 + \|\widetilde{\varphi}_\omega\|_2^2}{\|\widetilde{\varphi}_\omega\|_{p+1}^{p+1}}\right)^{\frac{1}{p-1}} > 0,$$

then $\widetilde{I}_0(\lambda_1(\omega)\widetilde{\varphi}_\omega) = 0$. Therefore, we have

$$\widetilde{d}_0 \leq \lambda_1(\omega)^{p+1}\|\widetilde{\varphi}_\omega\|_{p+1}^{p+1}. \tag{4.16}$$

Thus, by (i) and (4.16), we arrive at

$$\liminf_{\omega\to\infty} \lambda_1(\omega) \geq \liminf_{\omega\to\infty} \left(\frac{\widetilde{d}_0}{\|\widetilde{\varphi}_\omega\|_{p+1}^{p+1}}\right)^{\frac{1}{p+1}} = 1.$$

Moreover, by (4.11), $\widetilde{I}_0(\lambda_1(\omega)\widetilde{\varphi}_\omega) = 0$ and (i), we have

$$\liminf_{\omega\to\infty} \widetilde{I}_0(\widetilde{\varphi}_\omega) = \liminf_{\omega\to\infty} (\lambda_1(\omega)^{p-1} - 1)\, \|\widetilde{\varphi}_\omega\|_{p+1}^{p+1} \geq 0,$$

which together with (4.15) implies (ii). Finally, from (i) and (ii), we obtain

$$\widetilde{d}_0 = \lim_{\omega\to\infty} \|\widetilde{\varphi}_\omega\|_{p+1}^{p+1} = \lim_{\omega\to\infty} \|\widetilde{\varphi}_\omega\|_{H^1(\Gamma)}^2,$$

which shows (iii). □

Proof of Corollary 1.5. Recall that, by Theorem 1.4, if $\partial_\lambda^2 E\left(\varphi_\omega^\lambda\right)|_{\lambda=1} < 0$, then $e^{it}\varphi_\omega(x)$ is orbitally unstable. Since

$$\partial_\lambda^2 E\left(\varphi_\omega^\lambda\right)|_{\lambda=1} < 0 \quad \iff \quad \frac{-\alpha(2-\alpha)\,(V\varphi_\omega, \varphi_\omega)_2 + \gamma\left|\varphi_{\omega,1}(0)\right|^2}{\|\varphi_\omega\|_{p+1}^{p+1}} < \frac{(p-1)(p-5)}{2(p+1)},$$

by (4.7), it suffices to prove

$$\lim_{\omega\to\infty} \frac{-\omega^{-\frac{2-\alpha}{2}}\alpha(2-\alpha)(V\widetilde{\varphi}_\omega, \widetilde{\varphi}_\omega)_2 + \omega^{-\frac{1}{2}}\gamma\left|\widetilde{\varphi}_{\omega,1}(0)\right|^2}{\|\widetilde{\varphi}_\omega\|_{p+1}^{p+1}} = 0. \tag{4.17}$$

We have

$$\begin{aligned} 0 &\leq -\omega^{-\frac{2-\alpha}{2}}\alpha(2-\alpha)(V\widetilde{\varphi}_\omega, \widetilde{\varphi}_\omega)_2 + \omega^{-\frac{1}{2}}\gamma\left|\widetilde{\varphi}_{\omega,1}(0)\right|^2 \\ &\leq \left(C_1\omega^{-\frac{2-\alpha}{2}} + C_2\omega^{-\frac{1}{2}}\gamma\right)\|\widetilde{\varphi}_\omega\|_{H^1(\Gamma)}^2. \end{aligned}$$

Hence, by Lemma 4.5-(i), (iii), we obtain (4.17). This concludes the proof. □

4.3. Instability results in $H^1_{\mathrm{eq}}(\Gamma)$

We discuss counterparts of Proposition 1.1, Theorem 1.4, Corollary 1.5 for arbitrary $\gamma\in\mathbb{R}$ and symmetric $V(x)$, i.e., $V_1(x)=\ldots=V_N(x)$, in the space

$$H^1_{\mathrm{eq}}(\Gamma)=\{v\in H^1(\Gamma):\ v_1(x)=\ldots=v_N(x),\ x>0\}.$$

The well-posedness in $H^1_{\mathrm{eq}}(\Gamma)$ follows analogously to [17, Lemma 2.6]. We use index $\cdot_{\mathrm{eq}}$ to denote counterparts of the objects for the space $H^1_{\mathrm{eq}}(\Gamma)$.

It is known that $d^0_{\omega,\mathrm{eq}}=S^0_\omega(\phi_\gamma)$ (see page 12 in [18]) for any $\gamma\in\mathbb{R}$, where

$$\phi_\gamma(x)=\left(\left\{\tfrac{(p+1)\omega}{2}\operatorname{sech}^2\left(\tfrac{(p-1)\sqrt{\omega}}{2}x+\operatorname{arctanh}\left(\tfrac{\gamma}{N\sqrt{\omega}}\right)\right)\right\}^{\frac{1}{p-1}}\right)_{e=1}^N.$$

Then, for $0<\omega_{0,\mathrm{eq}}<\omega$ (observe that $\omega_{0,\mathrm{eq}}\le\omega_0$) one can repeat all the proofs in Sect. 3 and Subsect. 4.1 and 4.2 with $H^1_{\mathrm{eq}}(\Gamma)$ instead of $H^1(\Gamma)$. Thus, we get the following results.

Proposition 4.6. *Let $p>1,\gamma\in\mathbb{R}$, $\omega>\omega_{0,eq}$. If $V(x)=\overline{V(x)}$ is symmetric and satisfies Assumptions 1–3, then the set of ground states $\mathcal{G}_{\omega,eq}$ is not empty, in particular, $\mathcal{G}_{\omega,eq}=\mathcal{M}_{\omega,eq}$. If $\varphi_\omega\in\mathcal{G}_{\omega,eq}$, then there exist $\theta\in\mathbb{R}$ and a positive function $\phi\in H^1_{eq}(\Gamma)$ such that $\varphi_\omega(x)=e^{i\theta}\phi(x)$.*

Theorem 4.7. *Let $p>5$, $\gamma\in\mathbb{R}$, $\omega>\omega_{0,eq}$. If $V(x)=\overline{V(x)}$ is symmetric and satisfies Assumptions 1–4, $\varphi_\omega(x)\in\mathcal{G}_{\omega,eq}$, and $\partial^2_\lambda E(\varphi^\lambda_\omega)\,|_{\lambda=1}<0$, then the standing wave solution $e^{i\omega t}\varphi_\omega(x)$ of (1.1) is orbitally unstable in $H^1_{eq}(\Gamma)$ and therefore in $H^1(\Gamma)$.*

Corollary 4.8. *Assume that $V(x)=\dfrac{-\beta}{x^\alpha}$, $\beta>0$, $0<\alpha<1$, $\gamma\in\mathbb{R}$. Let $p>5$ and $\varphi_\omega(x)\in\mathcal{G}_{\omega,eq}$. Then, there exists $\omega^*_{eq}\in(\omega_{0,eq},\infty)$ such that for any $\omega\in(\omega^*_{eq},\infty)$ the standing wave solution $e^{i\omega t}\varphi_\omega(x)$ of (1.1) is orbitally unstable in $H^1(\Gamma)$.*

Remark 4.9. (i) Observe that when dealing with $H^1_{\mathrm{eq}}(\Gamma)$, no restriction on γ appears. This is due to the fact that the corresponding constrained variational problem is closely related to the one on $\mathbb{R}$, which in turn admits a minimizer for any γ (see [18, Remark 3.1]).

(ii) Consider

$$i\partial_t u(t,x)=-\partial_x^2 u(t,x)-\gamma\delta(x)u(t,x)+V(x)u(t,x)-|u(t,x)|^{p-1}u(t,x),$$

$(t,x)\in\mathbb{R}\times\mathbb{R}$, $\gamma\in\mathbb{R}$. Notice that the above results are valid with $H^1_{\mathrm{eq}}(\Gamma)$ substituted by $H^1_{\mathrm{rad}}(\mathbb{R})=\{f\in H^1(\mathbb{R}):\ f(x)=f(-x)\}$ and analogous assumptions on $V(x)$. One only needs to recall that $d^0_{\omega,\mathrm{rad}}=S^0_\omega(\phi_\gamma)$ (see [14, Theorem 1]), where

$$\phi_\gamma(x)=\left\{\tfrac{(p+1)\omega}{2}\operatorname{sech}^2\left(\tfrac{(p-1)\sqrt{\omega}}{2}|x|+\operatorname{arctanh}\left(\tfrac{\gamma}{2\sqrt{\omega}}\right)\right)\right\}^{\frac{1}{p-1}}.$$

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Appendix

Below we show some properties of the operator $H_{\gamma,V}$ introduced by (1.2).

Lemma 4.10. *Let $\gamma \in \mathbb{R}$ and $V(x) = \overline{V(x)} \in L^1(\Gamma) + L^\infty(\Gamma)$. The quadratic form $F_{\gamma,V}$ given by (1.3) is semibounded and closed, and the operator $H_{\gamma,V}$ defined by*

$$(H_{\gamma,V}v)_e = -v_e'' + V_e v_e,$$
$$dom(H_{\gamma,V}) = \left\{ v \in H^1(\Gamma) : \; -v_e'' + V_e v_e \in L^2(\mathbb{R}^+), \; \sum_{e=1}^{N} v_e'(0) = -\gamma v_1(0) \right\}.$$

is the self-adjoint operator associated with $F_{\gamma,V}$ in $L^2(\Gamma)$.

Proof. We can write $V(x) = V_1(x) + V_2(x)$, with $V_1 \in L^1(\Gamma)$ and $V_2 \in L^\infty(\Gamma)$. Thus, using the Gagliardo–Nirenberg inequality (see formula (2.1) in [10]) and the Young inequality, we have

$$\begin{aligned} \left| \int_\Gamma V(x)\,|v(x)|^2\,\mathrm{d}x \right| &\leq \|V_1\|_1 \, \|v\|_\infty^2 + \|V_2\|_\infty \, \|v\|_2^2 \\ &\leq C \, \|V_1\|_1 \, \left\|v'\right\|_2 \|v\|_2 + \|V_2\|_\infty \, \|v\|_2^2 \\ &\leq \varepsilon \left\|v'\right\|_2^2 + C_\varepsilon \, \|v\|_2^2, \quad \varepsilon > 0. \end{aligned} \tag{4.18}$$

Similarly, by the Sobolev embedding, we obtain

$$\left|\gamma \, |v_1(0)|^2\right| \leq |\gamma| \|v\|_\infty^2 \leq C \|v'\|_2 \, \|v\|_2 \leq \varepsilon \left\|v'\right\|_2^2 + C_\varepsilon \, \|v\|_2^2 .$$

Therefore,

$$\left|\gamma \, |v_1(0)|^2 + \int_\Gamma V(x)|v(x)|^2\mathrm{d}x\right| \leq 2\varepsilon \left\|v'\right\|_2^2 + C_\varepsilon \, \|v\|_2^2, \text{ for every } \epsilon > 0. \tag{4.19}$$

Then, by the KLMN theorem [26, Theorem X.17], we infer that the quadratic form $F_{\gamma,V}$ is associated with a semibounded self-adjoint operator $T_{\gamma,V}$ defined by (observe that $A = H_{0,0}$ in [26, Theorem X.17], i.e., $V \equiv 0$, $\gamma = 0$)

$$\mathrm{dom}(T_{\gamma,V}) = \left\{ u \in H^1(\Gamma) : \exists\, y \in L^2(\Gamma) \, s.t. \forall v \in H^1(\Gamma), \; F_{\gamma,V}(u,v) = (y,v)_2 \right\},$$
$$T_{\gamma,V} u = y.$$

It is easily seen that $\mathrm{dom}(H_{\gamma,V}) \subseteq \mathrm{dom}(T_{\gamma,V})$ and $T_{\gamma,V}u = H_{\gamma,V}u,\;\; u \in \mathrm{dom}(H_{\gamma,V})$. Hence, it is sufficient to prove that $\mathrm{dom}(T_{\gamma,V}) \subseteq \mathrm{dom}(H_{\gamma,V})$.

Let $\tilde{u} \in \mathrm{dom}(T_{\gamma,V})$ and $\tilde{v} \in H^1(\Gamma)$, then there exists $\tilde{y} \in L^2(\Gamma)$ such that

$$F_{\gamma,V}(\tilde{u}, \tilde{v}) = \int_\Gamma (\tilde{u}'\overline{\tilde{v}'} + V\tilde{u}\overline{\tilde{v}})\mathrm{d}x - \gamma\tilde{u}_1(0)\overline{\tilde{v}_1(0)} = (\tilde{y}, \tilde{v})_2. \tag{4.20}$$

Observe that $\tilde{y} - V\tilde{u} \in L^1_{loc}(\Gamma)$ and set

$$z = (z_e)_{e=1}^N, \quad z_e(x) = \int_0^x (\tilde{y}_e(t) - V_e(t)\tilde{u}_e(t))\,\mathrm{d}t.$$

Suppose now additionally that $\tilde{v}$ has a compact support, then

$$\int_\Gamma (\tilde{y} - V\tilde{u})\overline{\tilde{v}}\mathrm{d}x = \int_\Gamma z'\overline{\tilde{v}}\mathrm{d}x = -\overline{\tilde{v}_1(0)}\sum_{e=1}^N z_e(0) - \int_\Gamma z\overline{\tilde{v}'}\mathrm{d}x. \tag{4.21}$$

From (4.20), we deduce

$$\int_\Gamma (\tilde{y} - V\tilde{u})\overline{\tilde{v}}\mathrm{d}x = \int_\Gamma \tilde{u}'\overline{\tilde{v}'}\mathrm{d}x - \gamma\tilde{u}_1(0)\overline{\tilde{v}_1(0)}. \tag{4.22}$$

Combining (4.21) and (4.22), we get

$$\int_\Gamma (\tilde{u}' + z)\overline{\tilde{v}'}\mathrm{d}x + \overline{\tilde{v}_1(0)}\left(-\gamma\tilde{u}_1(0) + \sum_{e=1}^N z_e(0)\right) = 0. \tag{4.23}$$

Choose $\tilde{v} = (\tilde{v}_e)_{e=1}^N$ such that $\tilde{v}_1(x) \in C_0^\infty(\mathbb{R}^+)$ and $\tilde{v}_2(x) \equiv \ldots \equiv \tilde{v}_N(x) \equiv 0$. Then we obtain

$$\int_0^\infty (\tilde{u}_1' + z_1)\overline{\tilde{v}_1'}\mathrm{d}x = 0,$$

therefore $\tilde{u}_1' + z_1 \equiv const \equiv c_1$. We have used that $\tilde{u}_1' + z_1 \in \mathrm{Ran}(A)^\perp$, where $Av = v'$ with $\mathrm{dom}(A) = C_0^\infty(\mathbb{R}^+)$ in $L^2(\mathbb{R}^+)$. Analogously $\tilde{u}_e' + z_e \equiv const \equiv c_e,\;\; e = 2, \ldots, N$. Finally, from (4.23) we deduce

$$\overline{\tilde{v}_1(0)}\left(-\gamma\tilde{u}_1(0) - \sum_{e=1}^N(\tilde{u}_e'(0) + z_e(0)) + \sum_{e=1}^N z_e(0)\right) = 0.$$

Assuming that $\tilde{v}_1(0) \neq 0$, we arrive at $\sum\limits_{e=1}^N \tilde{u}_e'(0) = -\gamma\tilde{u}_1(0)$. Moreover, $-\tilde{u}'' + V\tilde{u} = z' + V\tilde{u} = \tilde{y} - V\tilde{u} + V\tilde{u} = \tilde{y} \in L^2(\Gamma)$. Hence, $\tilde{u} \in \mathrm{dom}(H_{\gamma,V})$ and $\mathrm{dom}(T_{\gamma,V}) \subseteq \mathrm{dom}(H_{\gamma,V})$. $\square$

Lemma 4.11. *Suppose that $V(x) = \overline{V(x)} \in L^2_\varepsilon(\Gamma) + L^\infty(\Gamma)$, i.e., for any $\varepsilon > 0$ and $V \in L^2_\varepsilon(\Gamma) + L^\infty(\Gamma)$ there exists a representation $V = V_1 + V_2$, $V_1 \in L^2(\Gamma)$, $V_2 \in L^\infty(\Gamma)$, with $\|V_1\|_2^2 \leq \varepsilon$. Then, we have*

$$dom(H_{\gamma,V}) = \left\{ v \in H^1(\Gamma) : \quad v_e \in H^2(\mathbb{R}^+), \quad \sum_{e=1}^{N} v_e'(0) = -\gamma v_1(0) \right\} := D_{H^2}. \tag{4.24}$$

Moreover, for m sufficiently large, $H_{\gamma,V}$-norm $\|(H_{\gamma,V} + m) \cdot \|_2$ is equivalent to H^2-norm on Γ.

Proof. Observe that, by $V(x) \in L^2_\varepsilon(\Gamma) + L^\infty(\Gamma)$, the Sobolev and the Young inequalities we get

$$\|Vv\|_2^2 \leq \|V_1\|_2^2 \|v\|_\infty^2 + \|V_2\|_\infty^2 \|v\|_2^2 \leq \varepsilon \|v\|^2_{H^2(\Gamma)} + C\|v\|_2^2 \tag{4.25}$$

and

$$\begin{aligned} |(v'', Vv)_2| &\leq \|v''\|_2 \|Vv\|_2 \leq \|v''\|_2 \|V_1\|_2 \|v\|_\infty + \|v''\|_2 \|V_2\|_\infty \|v\|_2 \\ &\leq C_1 \|v''\|_2 \|V_1\|_2 \|v\|_{H^2(\Gamma)} + C_2 \|v''\|_2 \|v\|_2 \leq \varepsilon \|v\|^2_{H^2(\Gamma)} + \varepsilon \|v''\|_2^2 \\ &\quad + C_\varepsilon \|v\|_2^2 \leq 2\varepsilon \|v\|^2_{H^2(\Gamma)} + C_\varepsilon \|v\|_2^2. \end{aligned} \tag{4.26}$$

It is immediate from (4.25), (4.26) that

$$\|H_{\gamma,V} v\|_2^2 = \|v''\|_2^2 + 2\mathrm{Re}(v'', Vv)_2 + \|Vv\|_2^2 \leq C_1 \|v\|^2_{H^2(\Gamma)}. \tag{4.27}$$

And for m sufficiently large, inequalities (4.25) and (4.26) imply

$$\begin{aligned} \|H_{\gamma,V} v\|_2^2 + m^2 \|v\|_2^2 &= \|v''\|_2^2 + 2\mathrm{Re}(v'', Vv)_2 + \|Vv\|_2^2 + m^2 \|v\|_2^2 \\ &\geq C_2 \|v\|^2_{H^2(\Gamma)}. \end{aligned} \tag{4.28}$$

Thus, we get (4.24).

The second assertion follows from (4.27),(4.28), and

$$\begin{aligned} &\|(H_{\gamma,V} + m)v\|_2^2 = \|H_{\gamma,V} v\|_2^2 + m^2 \|v\|_2^2 + 2m(H_{\gamma,V} v, v)_2, \\ &|(H_{\gamma,V} v, v)_2| \leq \|H_{\gamma,V} v\|_2 \|v\|_2 \leq \varepsilon \|H_{\gamma,V} v\|_2^2 + C_\varepsilon \|v\|_2^2. \end{aligned}$$

□

Remark 4.12. Observe that there exists potential $V(x)$ satisfying Assumptions 1–4 such that $\mathrm{dom}(H_{\gamma,V}) \neq D_{H^2}$. For example, consider $V(x) = -1/x^\alpha$, $1/2 \leq \alpha < 1$, and $N = \gamma = 2$, then $v = (e^{-x}, e^{-x}) \in D_{H^2}$, but

$$\|H_{\gamma,V} v\|_2^2 = 2\| - v_1'' - \frac{v_1}{x^\alpha}\|_2^2 > 2e^{-2\varepsilon} \int_0^\varepsilon \frac{\mathrm{d}x}{x^{2\alpha}} = \infty.$$

Lemma 4.13. *Let $\gamma > 0$ and $V(x) = \overline{V(x)}$ satisfy Assumptions 1–3. Then, the following assertions hold.*
(i) The number $-\omega_0$ defined by (1.5) is negative.
(ii) Let also $m > \omega_0$, then $\sqrt{F_{\gamma,V}(v) + m\|v\|_2^2}$ defines a norm equivalent to the H^1-norm.
(iii) The number $-\omega_0$ is the first eigenvalue of $H_{\gamma,V}$. Moreover, it is simple, and there exists the corresponding positive eigenfunction $\psi_0 \in dom(H_{\gamma,V})$, i.e., $H_{\gamma,V}\psi_0 = -\omega_0\psi_0$.

Proof. (i) To show $-\omega_0 < 0$, observe that

$$-\omega_0 = \inf \sigma(H_{\gamma,V}) = \inf \left\{ F_{\gamma,V}(v) \,:\, v \in H^1(\Gamma),\ \|v\|_2^2 = 1 \right\}. \tag{4.29}$$

Consider $v^\lambda(x) = \lambda^{\frac{1}{2}} v(\lambda x)$ with $\lambda > 0$. Hence,

$$F_{\gamma,V}(v^\lambda) = \lambda^2 \left\|v'\right\|_2^2 - \lambda\gamma\, |v_1(0)|^2 + (Vv^\lambda, v^\lambda)_2.$$

For λ small enough, we have $F_{\gamma,V}(v^\lambda) < 0$. Finally, $-\omega_0$ is finite since $F_{\gamma,V}(v)$ is lower semibounded.

(ii) Let $\varepsilon > 0$. Firstly, notice that from (4.19) one easily gets

$$F_{\gamma,V}(v) + m\|v\|_2^2 \le (1+2\varepsilon)\|v'\|_2^2 + (C+m)\|v\|_2^2 \le C_1\|v\|_{H^1(\Gamma)}^2.$$

Secondly, for ε and δ sufficiently small,

$$F_{\gamma,V}(v) + m\|v\|_2^2 = \delta\|v'\|_2^2 + (1-\delta)\left(\|v'\|_2^2 + \frac{1}{1-\delta}(Vv, v)_2 - \frac{\gamma}{1-\delta}|v_1(0)|^2\right)$$
$$+ m\|v\|_2^2 \ge \delta\|v'\|_2^2 - (1+\varepsilon)(1-\delta)\omega_0\|v\|_2^2 + m\|v\|_2^2 \ge C_2\|v\|_{H^1(\Gamma)}^2.$$

Indeed, the family of sesquilinear forms

$$\mathrm{t}(\kappa)[u, v] = (u', v')_2 + \frac{1}{1-\kappa}(Vu, v)_2 - \frac{\gamma}{1-\kappa}(u_1(0)\overline{v_1}(0))$$

is holomorphic of type (a) in the sense of Kato in the complex neighborhood of zero (see [21, Chapter VII, §4] for the definition and [21, Chapter VI, §1, Example 1.7] for the proof of sectoriality). Using inequality (4.7) in [21, Chapter VII] with $\kappa = \kappa_2 = 0$, $\kappa_1 = \delta$, we obtain $|\mathrm{t}(\delta)[v] - \mathrm{t}(0)[v]| \le \varepsilon|\mathrm{t}(0)[v]|$. Hence,

$$\mathrm{t}(\delta)[v] \ge \mathrm{t}(0)[v] - \varepsilon|\mathrm{t}(0)[v]| = F_{\gamma,V}(v) - \varepsilon|F_{\gamma,V}(v)| \ge -(1+\varepsilon)\omega_0\|v\|_2^2.$$

(iii) *Step 1.* Let $\{v_n\}$ be a minimizing sequence, that is, $F_{\gamma,V}(v_n) \underset{n\to\infty}{\longrightarrow} -\omega_0$, $\|v_n\|_2^2 = 1$ for all $n \in \mathbb{N}$. From (ii), we deduce that $\{v_n\}$ is bounded in $H^1(\Gamma)$. Then, there exist a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ and $v_0 \in H^1(\Gamma)$ such that $\{v_{n_k}\}$ converges weakly to v_0

in $H^1(\Gamma)$. Observe that, by the weak lower semicontinuity of L^2-norm and $F_{\gamma,V}(\cdot)$, we get $\|v_0\|_2 \le 1$ and

$$F_{\gamma,V}(v_0) \le \lim_{k\to\infty} F_{\gamma,V}(v_{n_k}) = -\omega_0 < 0.$$

We have $\|v_0\|_2 = 1$, since, otherwise, there would exist $\lambda > 1$ such that $\|\lambda v_0\|_2 = 1$ and $F_{\gamma,V}(\lambda v_0) = \lambda^2 F_{\gamma,V}(v_0) < -\omega_0$, which is a contradiction. Consequently, v_0 is a minimizer for (4.29).

Let $\psi_0 = |v_0|$, then $\psi_0 \ge 0$ on Γ and $\|\psi_0\|_2^2 = \|v_0\|_2^2 = 1$. Notice that $\|\psi_0'\|_2^2 \le \|v_0'\|_2^2$, therefore $F_{\gamma,V}(\psi_0) \le F_{\gamma,V}(v_0)$. Then, ψ_0 is a minimizer of (4.29). This implies the existence of the Lagrange multiplier $-\mu$ such that

$$F'_{\gamma,V}(\psi_0) = -\mu Q'(\psi_0), \quad Q(v) = \|v\|_2^2.$$

Repeating the arguments from the proof of [2, Theorem 4], we get $\psi_0 \in \mathrm{dom}(H_{\gamma,V})$ and

$$H_{\gamma,V}\psi_0 = -\mu\psi_0.$$

Multiplying the above equation by $\overline{\psi_0}$ and integrating, we conclude $\mu = \omega_0$. Recalling that $V(x) \le 0$ a.e. on Γ, and arguing as in the proof of Proposition 1.1, one can show that $\psi_0 > 0$ on Γ. Notice that one needs to apply [28, Theorem 1] with $\beta(s) = \omega_0 s$.

Step 2. Suppose that u_0 is a nonnegative solution of

$$H_{\gamma,V}u_0 = -\omega_0 u_0. \tag{4.30}$$

Let us show that there exists $C > 0$ such that $u_0(x) = C\psi_0(x)$. Assume that this is false. Then, there exists $C > 0$ such that $\widetilde{u}_0(x) = u_0(x) - C\psi_0(x)$ takes both positive and negative values. We have $H_{\gamma,V}\widetilde{u}_0 = -\omega_0\widetilde{u}_0$; consequently, $\widetilde{v}_0 = \widetilde{u}_0/\|\widetilde{u}_0\|_2$ is the minimizer of (4.29). Arguing as in *Step 1*, one can show that $|\widetilde{v}_0|$ is also a minimizer and $|\widetilde{v}_0| > 0$. Therefore, $\widetilde{u}_0(x)$ has a constant sign. This is a contradiction.

Suppose now that u_0 is an arbitrary solution to (4.30) such that $\|u_0\|_2^2 = 1$ (that is, u_0 is a minimizer of (4.29)). Define $w_0 = |\mathrm{Re}u_0| + i|\mathrm{Im}u_0|$, then $|w_0| = |u_0|$ and $|w_0'| = |u_0'|$; consequently, $F_{\gamma,V}(u_0) = F_{\gamma,V}(w_0)$ and $\|w_0\|_2^2 = 1$. Therefore, w_0 is a minimizer of (4.29). This implies that w_0 satisfies (4.30), and, in particular, $|\mathrm{Re}u_0|$ and $|\mathrm{Im}u_0|$ satisfy (4.30). Thus, $|\mathrm{Re}u_0| = C_1\psi_0$ and $|\mathrm{Im}u_0| = C_2\psi_0$, $C_1, C_2 > 0$; consequently, $\mathrm{Re}u_0 = \widetilde{C}_1\psi_0$ and $\mathrm{Im}u_0 = \widetilde{C}_2\psi_0$, $\widetilde{C}_1, \widetilde{C}_2 \in \mathbb{R}$, since $\mathrm{Re}u_0$ and $\mathrm{Im}u_0$ do not change the sign. Finally, $u_0 = \widetilde{C}_1\psi_0 + i\widetilde{C}_2\psi_0 = \widetilde{C}\psi_0$, $\widetilde{C} \in \mathbb{C}$, and therefore, $-\omega_0$ is simple.

□

REFERENCES

[1] R. Adami, C. Cacciapuoti, D. Finco, and D. Noja, Constrained energy minimization and orbital stability for the NLS equation on a star graph, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 31(6):1289–1310, 2014.

[2] R. Adami, C. Cacciapuoti, D. Finco, and D. Noja, Variational properties and orbital stability of standing waves for NLS equation on a star graph, *J. Differential Equations*, 257:3738–3777, 2014.

[3] J. Angulo and N. Goloshchapova, Extension theory approach in the stability of the standing waves for the NLS equation with point interactions on a star graph, *Advances in Differential Equations*, 23:793–846, 2018.

[4] J. Angulo and N. Goloshchapova, On the orbital instability of excited states for the NLS equation with the δ-interaction on a star graph, *Discrete Contin. Dyn. Syst.*, 38(10):5039–5066, 2018.

[5] A. H. Ardila, Orbital stability of standing waves for supercritical NLS with potential on graphs, *Applicable Analysis*, https://doi.org/10.1080/00036811.2018.1530763, 2018.

[6] G. Beck, S. Imperiale, and P. Joly, Mathematical modelling of multi conductor cables, *Discrete Contin. Dyn. Syst. Ser. S*, 8:521–546, 2015.

[7] G. Berkolaiko, C. Carlson, S. Fulling, and P. Kuchment, *Quantum Graphs and Their Applications*, *Contemporary Math.,* 415, Amer. Math. Soc., Providence, RI, 2006.

[8] H. Brézis and E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math. Soc.*, 88(3):486–490, 1983.

[9] C. Cacciapuoti, Existence of the ground state for the NLS with potential on graphs, *Mathematical Problems in Quantum Physics, 155–172, Contemporary Math.*, 717, Amer. Math. Soc., Providence, RI, 2018.

[10] C. Cacciapuoti, D. Finco, and D. Noja, Ground state and orbital stability for the NLS equation on a general starlike graph with potentials, *Nonlinearity*, 30:3271–3303, 2017.

[11] T. Cazenave, *Semilinear Schrödinger equations, Courant Lect. Notes in Math.*, 10, New York University, Courant Institute of Mathematical Sciences, New York, Amer. Math. Soc., Providence, RI, 2003.

[12] H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969.

[13] N. Fukaya, M. Ohta, Strong instability of standing waves for nonlinear Schrödinger equations with attractive inverse power potential, *Osaka J. Math.* 56(4):713–726, 2019.

[14] R. Fukuizumi and L. Jeanjean, Stability of standing waves for a nonlinear Schrödinger equation with a repulsive Dirac delta potential, *Discrete Contin. Dyn. Syst.*, 21:121–136, 2008.

[15] R. Fukuizumi and M. Ohta, Stability of standing waves for nonlinear Schrödinger equations with potentials, *Differential and Integral Equations*, 16(1):111–128, 2003.

[16] R. Fukuizumi and M. Ohta, Instability of standing waves for nonlinear Schrödinger equations with potentials, *Differential and Integral Equations*, 16(1):691–706, 2003.

[17] N.Goloshchapova, A nonlinear Klein-Gordon equation on a star graph, arXiv:1912.00884v1.

[18] N.Goloshchapova and M.Ohta, Blow-up and strong instability of standing waves for the NLS-δ equation on a star graph, *Nonlinear Analysis*, 196(111753), 2020.

[19] P. Joly and A. Semin, Mathematical and numerical modeling of wave propagation in fractal trees, *C.R. Math. Acad. Sci. Paris*, 349:1047–1051, 2011.

[20] A. Kairzhan, Orbital instability of standing waves for NLS equation on star graphs, *Proc. Amer. Math. Soc.*, 147:2911–2924, 2019.

[21] T. Kato, *Perturbation theory for linear operators*, Die Grundlehren der mathematischen Wissenschaften, Band 132 Springer-Verlag New York, Inc., New York 1966.

[22] P. Kuchment, Graph models for waves in thin structures, *Waves Random Media*, 12: R1–R24, 2002.

[23] E. H. Lieb and M. Loss, *Analysis, Second edition, Graduate Studies in Mathematics*, 14, American Mathematical Society, Providence, RI, 2001.

[24] Y. Martel, F. Merle, P. Raphaël, J. Szeftel, Near soliton dynamics and singularity formation for L^2 critical problems, *Russian Mathematical Surveys*, 69(2):261, 2014.

[25] M. Ohta, Strong instability of standing waves for nonlinear Schrödinger equations with harmonic potential, *Funkcial. Ekvac.*, 61:135–143, 2018.

[26] M. Reed, B. Simon, *Methods of modern mathematical physics. II. Fourier Analysis, Self-Adjointness,* Academic Press, 1975.

[27] H.A. Rose, M.I. Weinstein, On the bound states of the nonlinear Schrödinger equation with a linear potential, *Physica D: Nonlinear Phenomena*, 30(1-2):207–218, 1988.

[28] J.L. Vázquez. A strong maximum principle for some quasilinear elliptic equations, *Appl. Math. Optim*, 12:191–202, 1984.

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