Fundamental solution of diffusion equation for Kappa gas: Diffusion length for suprathermal electrons in solar wind

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A recent numerical treatment of data obtained by the Parker Solar Probe spacecraft describes the electron concentration in solar wind as a function of the heliocentric distance based on a Kappa distribution with spectral index $\kappa = 5$. In this work, we derive and, subsequently, solve an entirely different class of nonlinear partial differential equations describing the one-dimensional diffusion of a suprathermal gas. The theory is applied to describe the aforementioned data and we find a spectral index $\kappa \gtrsim 1.5$ providing the widely acknowledged identification of Kappa electrons in solar wind. We also find that suprathermal effects increase the length scale of classical diffusion by one order of magnitude. Such a result does not depend on the microscopic details of the diffusion coefficient since our theory is based on a macroscopic formulation. Forthcoming extensions of our theory by including magnetic fields and relating our formulation to nonextensive statistics are briefly addressed.

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I. INTRODUCTION

On August 12, 2018, the Parker Solar Probe (PSP) spacecraft [1] was launched. Assisted by Venus gravity, the vehicle orbits the Sun following highly elliptical trajectories. Its closest approach to the center of the Sun (perihelion) varies from 9.86 $R_\odot$ to 35.7 $R_\odot$, with $R_\odot$ denoting the radius of the Sun. Distinct kinds of solar wind and dynamic configurations have been revealed by the first two encounters [2] that occurred in October–November, 2018 (first perihelion) and March–April, 2019 (second perihelion), with perihelions of 35.7 $R_\odot$.

Relying on the power spectra obtained by the low-frequency receiver (LFR) of the radio frequency spectrometer (RFS), a piece of the FIELDS instrument suite [3] on PSP, during the first and second perihelions, Moncuquet *et al.* [4] have deduced the solar wind electron density, and thermal and suprathermal temperatures. Their results followed the application of the technique of quasithermal noise (QTN) spectroscopy [5] to the data acquired by the LFR (10.5 kHz to 1.7 MHz).

A possible anisotropy in the electron temperature T_e (typically, $T_{e,\perp} \geqslant 2T_{e,\parallel}$, where the subscripts $\perp$ and $\parallel$ stand for perpendicular and parallel, respectively, to the magnetic field), usually observed above 50 eV and attributable to the so-called Strahl component [6] (the highly field-aligned, beamlike, suprathermal population in solar wind), has been neglected in Ref. [4]. As a consequence, those authors have been forced to consider a spectral index of $\kappa = 5$ in order to model the suprathermal electrons by a Kappa distribution.

Now, it is widely known that suprathermal electrons in solar wind are identified by a spectral index slightly above its lower bound [7] ($3/2 < \kappa < \infty$). Such a contrast is the main motivation for this work. We perform a numerical treatment of the same data examined in Ref. [4]. However, our analysis is based on the derivation, and subsequent solution, of a differential equation, describing the one-dimensional diffusion of a suprathermal gas. As a result, we actually find $\kappa \gtrsim 1.5$.

This paper is organized as follows. In Sec. II, we derive a differential equation that describes the one-dimensional diffusion of a suprathermal gas. In Sec. III, we find the fundamental solution of the diffusion equation from an ansatz in terms of a deformation parameter β related to the spectral index κ. In Sec. IV, we develop a method to numerically compute the typical diffusion length scale from the fundamental solution. In Sec. V, we apply our method to the same data examined in Ref. [4]. In the concluding section, we summarize our work.

II. DIFFUSION EQUATION

Consider a nonuniform warm plasma, composed of charged and neutral species. The former have mass m, charge q, and concentration n. The frequency ν of the momentum transfer of charged to neutral species is assumed to depend on the Maxwellian temperature T of the first ones. Restricting ourselves to one-dimensional motions of the charged species, in the absence of a magnetic field and on neglect of inertial terms, the steady-state macroscopic force equation is given by [8]

$$\mp|q|n\Phi_x - P_x - \nu m\Gamma = 0, \tag{1}$$

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where the index x stands for a space derivative, Φ is the electric potential, P is the isotropic pressure, and $\Gamma = nv$ is the particle flux, with v denoting the drift velocity. Equation (1) should be supplied with an equation of state. For an isothermal plasma, the latter would imply $P_x = k_B T n_x$, where k_B is the Boltzmann constant. However, we regard the charged fluid as a suprathermal gas. In this case, the aforementioned pressure gradient must be replaced with (see the Appendix)

$$P_x = n_0 k_B \Theta \left[\left(\frac{n}{n_0}\right)^\beta\right]_x, \tag{2}$$

where n_0 is the equilibrium concentration ($n \to n_0 \Rightarrow P \to$ const), Θ is the Kappa temperature ($\Theta \geqslant T$), and β is a deformation parameter ($0 < \beta < 1$). In the limit $\beta \to 1$, $\Theta \to T$ (see the Appendix). As a consequence, Eq. (2) recovers $P_x = k_B T n_x$ in the same approximation. Solving Eq. (1) for the particle flux, on account of Eq. (2), we get

$$\Gamma = \mp \mu n \Phi_x - n_0 \eta \left[\left(\frac{n}{n_0}\right)^\beta\right]_x, \tag{3}$$

where we have introduced the abbreviations [8]

$$\mu = \frac{|q|}{\nu m}, \quad \eta = \frac{k_B \Theta}{\nu m} \tag{4}$$

for the particle mobility and diffusion coefficient, respectively, with the observation that we shall assume $\nu = \nu(\Theta)$ now. In the steady state, charge cannot build up in the plasma. That should be true even in the presence of ionizing collisions. Those could simply create equal numbers of both species. Once electrons are lighter than ions, in the absence of a magnetic field, the former would tend to flow out faster than the latter. As a consequence, an electric field might spring up to maintain the local balance of particle flow. More precisely, at the initial instant, a few more electrons than ions are left out. As a result of such a charge imbalance, an electric field sets up in the plasma. The above remarks are summed up in the so-called congruence assumption that out of any plasma region, the flux and the concentration of ions and electrons shall be approximately equal, namely [8],

$$\Gamma = \Gamma_i \approx \Gamma_e, \quad n = n_i \approx n_e, \tag{5}$$

where the indexes "i" and "e" stand for ions and electrons, respectively.

On account of Eqs. (5), Eq. (3) leads to

$$-\mu_i n \Phi_x - n_0 \eta_i \left[\left(\frac{n}{n_0}\right)^\beta\right]_x = \mu_e n \Phi_x - n_0 \eta_e \left[\left(\frac{n}{n_0}\right)^\beta\right]_x. \tag{6}$$

Solving Eq. (6) for the electric field $-\Phi_x$, we get

$$-\Phi_x = \frac{n_0}{n}\left(\frac{\eta_i - \eta_e}{\mu_i + \mu_e}\right)\left[\left(\frac{n}{n_0}\right)^\beta\right]_x. \tag{7}$$

Substituting Eq. (7) in Eq. (3), we obtain

$$\Gamma = -n_0 \eta_a \left[\left(\frac{n}{n_0}\right)^\beta\right]_x, \tag{8}$$

where we have introduced the abbreviation [8]

$$\eta_a = \frac{\mu_i \eta_e + \mu_e \eta_i}{\mu_i + \mu_e} \tag{9}$$

for the so-called ambipolar diffusion coefficient. In the absence of a source and/or a sink of matter, conservation of mass, on account of Eq. (8), leads to the differential equation

$$u_t = \eta_a (u^\beta)_{xx}, \tag{10}$$

which describes suprathermal diffusion, where we have introduced the abbreviation $u = n/n_0$ for the normalized concentration and the index t stands for a time derivative. In the limit $\beta \to 1$, Eq. (10) recovers $u_t = \eta_a u_{xx}$, which describes classical diffusion, with the observation that $\eta_a = \eta_a(T_i, T_e)$ in the same approximation. Let us find out the fundamental solution of Eq. (10).

III. FUNDAMENTAL SOLUTION

Let us make the ansatz

$$u = u_0 \left\{1 + \frac{1}{s/r}\left[\left(\frac{\tau}{t}\right)^{1/r'} - 1\right] + \frac{x^2}{s \eta_0 t}\right\}^{s'} \tag{11}$$

for the fundamental solution of Eq. (10). The reason for this is the following. The quantities r, r', s, and s' are assumed to be deformation parameters, which depend on β. As a result, in the limit $\beta \to 1$, we expect to recover the fundamental solution of the classical equation $u_t = \eta_a u_{xx}$. The quantity τ is the typical diffusion timescale, η_0 is a constant with the dimension of diffusion coefficient, and u_0 is a dimensionless constant to be further determined, with all such quantities assumed to be β independent. Relying on Eq. (11), we differentiate u once with respect to t, and $\eta_a u^\beta$ twice with respect to x, to get

$$\begin{aligned} u_t &= -\frac{u_0}{\tau}\left[\frac{s'/r'}{s/r}\left(\frac{\tau}{t}\right)^{1+1/r'} + \frac{s'\tau x^2}{s\eta_0 t^2}\right]\left[1 - \frac{1}{s/r} + \frac{1}{s/r}\left(\frac{\tau}{t}\right)^{1/r'} + \frac{x^2}{s\eta_0 t}\right]^{s'-1}, \\ \eta_a (u^\beta)_{xx} &= \frac{\eta_a u_0^\beta}{\eta_0 \tau}\left[\frac{2\beta s'\tau}{st}\right]\left[1 - \frac{1}{s/r} + \frac{1}{s/r}\left(\frac{\tau}{t}\right)^{1/r'} + \frac{x^2}{s\eta_0 t}\right]^{\beta s'-1} \\ &\quad + \frac{\eta_a u_0^\beta}{\eta_0 \tau}\left[\frac{4\beta s'(\beta s'-1)\tau x^2}{s^2 \eta_0 t^2}\right]\left[1 - \frac{1}{s/r} + \frac{1}{s/r}\left(\frac{\tau}{t}\right)^{1/r'} + \frac{x^2}{s\eta_0 t}\right]^{\beta s'-2}. \end{aligned} \tag{12}$$

According to Eqs. (12), Eq. (11) is a solution of Eq. (10) provided that the conditions

$$s' = \beta s' - 1, \tag{13}$$

on the powers, and

$$r = s, \quad -u_0^{1-\beta} s\eta_0 = 2\beta r' \eta_a, \quad -u_0^{1-\beta} s\eta_0 = 2\beta\eta_a + 4\beta s' \eta_a, \tag{14}$$

on the coefficients of u_t, and $\eta_a (u^\beta)_{xx}$ are satisfied. Solving Eq. (13) for s', we obtain

$$s' = -1/(1-\beta). \tag{15}$$

Combining the last two equations of Eqs. (14), we get

$$r' = 1 + 2s'. \tag{16}$$

Substituting Eq. (15) in Eq. (16), we obtain

$$r' = -(1+\beta)/(1-\beta). \tag{17}$$

Substituting Eq. (17) in the second equation of Eqs. (14), we get

$$s\eta_0 = \frac{1}{u_0^{1-\beta}} \left(\frac{1+\beta}{1-\beta} \right) (2\beta\eta_a). \tag{18}$$

Substituting the first equation of Eqs. (14), as well as Eqs. (15), (17), and (18) in Eq. (11), we obtain

$$u = \left\{ \left[\frac{\tau/t}{u_0^{-(1+\beta)}} \right]^{-(1-\beta)/(1+\beta)} + \left[\frac{1-\beta}{1+\beta} \right] \left[\frac{x^2/t}{2\beta\eta_a} \right] \right\}^{-1/(1-\beta)}. \tag{19}$$

In the limit $\beta \to 1$, Eq. (19) leads to

$$u = \left(\frac{\tau/t}{u_0^{-2}} \right)^{1/2} \exp\left(-\frac{x^2/t}{4\eta_a} \right), \tag{20}$$

where $\eta_a = \eta_a(T_i, T_e)$ in the same approximation. To determine the constant u_0, we proceed in the following way. Let us take advantage of the assumption that u_0 is a β-independent constant to determine it by making use of Eq. (20). In particular, let us require u_0 to be the normalization parameter in a statement of mass conservation, namely [9],

$$\int_{-\infty}^{\infty} u dx = \chi, \tag{21}$$

where we have introduced the abbreviation [9]

$$\chi = (\eta_a \tau)^{1/2} \tag{22}$$

for the typical diffusion length scale. Substituting Eq. (20) in Eq. (21) and, subsequently, performing the indicated integral, we get

$$u_0 = (4\pi)^{-1/2}. \tag{23}$$

Substituting $u = n/n_0$ and Eq. (23) in Eq. (20), we recover the fundamental solution of $u_t = \eta_a u_{xx}$, namely [9],

$$n = n_0 \left(\frac{\tau/t}{4\pi} \right)^{1/2} \exp\left(-\frac{x^2/t}{4\eta_a} \right). \tag{24}$$

Substituting $u = n/n_0$ and Eq. (23) in Eq. (19), we find the fundamental solution of Eq. (10), namely,

$$n = n_0 \left\{ \left[\frac{\tau/t}{(4\pi)^{(1+\beta)/2}} \right]^{-(1-\beta)/(1+\beta)} + \left[\frac{1-\beta}{1+\beta} \right] \left[\frac{x^2/t}{2\beta\eta_a} \right] \right\}^{-1/(1-\beta)}, \tag{25}$$

where $\eta_a = \eta_a(\Theta_i, \Theta_e)$ now. We next develop a method to compute the diffusion length χ in the realm of solutions (24) and (25).

IV. DIFFUSION LENGTH

The linear electron plasma frequency is given by [8]

$$f_{pe} = \frac{\omega_{pe}}{2\pi}, \tag{26}$$

where the angular electron plasma frequency is given by [8]

$$\omega_{pe} = \left(\frac{n_0 e^2}{\epsilon_0 m_e} \right)^{1/2}, \tag{27}$$

with m_e denoting the electron mass, $-e < 0$ the electron charge, n_0 the electron equilibrium concentration, and ϵ_0 the vacuum electric permittivity. Assume that f_{pe} varies in a timescale slow enough to maintain the isothermal regime of the electron gas. Then, the mean value of f_{pe} may be computed from

$$\bar{f}_{pe} = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} f_{pe}(t) dt, \tag{28}$$

where $t_1 < t_2$. According to the mean value theorem of real analysis, there exists at least one time, $\bar{t}$ say, with $t_1 < \bar{t} < t_2$, such that [10] $\bar{f}_{pe} = f_{pe}(\bar{t})$. Thus, substituting Eq. (27) in Eq. (26), and on account of Eq. (28), we may estimate the equilibrium concentration by

$$n_0 = \frac{4\pi^2 \epsilon_0 m_e (\bar{f}_{pe})^2}{e^2}. \tag{29}$$

In view of the reasoning expressed by Eqs. (26) to (29), Eq. (24) can be read as

$$n = a_0 \exp(-b_0 x^2), \tag{30}$$

where we have introduced the abbreviations

$$a_0 = n_0 \left(\frac{\tau/\bar{t}}{4\pi} \right)^{1/2}, \quad b_0 = \frac{1/\bar{t}}{4\eta_a}, \tag{31}$$

with the observation that $\eta_a = \eta_a(T_i, T_e)$. Hence, combining both Eqs. (31), it follows from Eq. (22) that the length scale for the classical diffusion of the electron gas is given by

$$\chi = \frac{a_0 \pi^{1/2}}{n_0 b_0^{1/2}}. \tag{32}$$

Equation (32) shows that χ may be calculated by computing n_0 from Eqs. (28) and (29), and a_0 and b_0 from Eq. (30). The

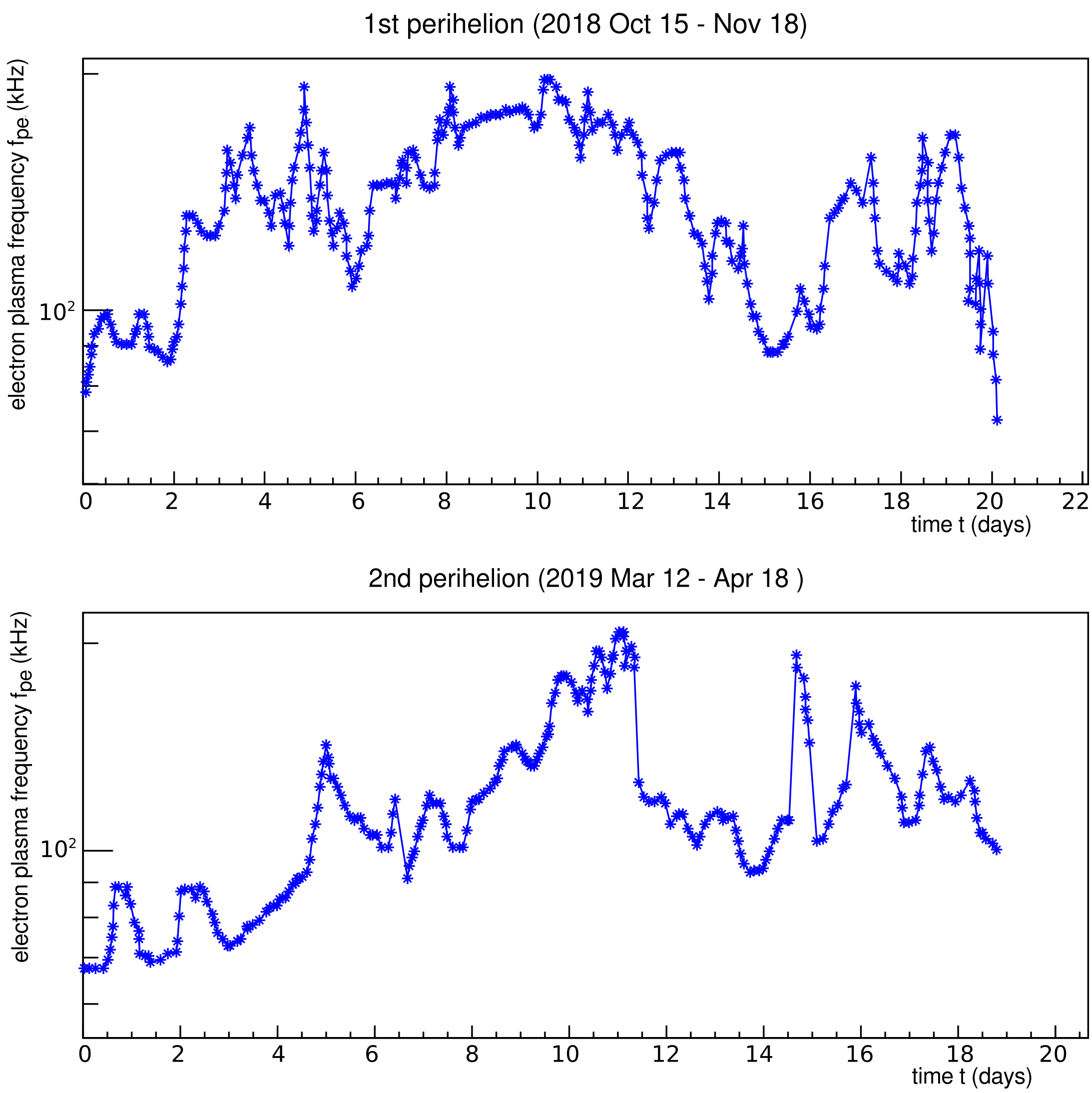

FIG. 1. The linear electron plasma frequency f_{pe} as a function of the slow time scale, deduced from the spectrograms exhibited in Ref. [4]. The data have been collected over twenty-one days of solar encounters, for both perihelions.

same just developed procedure may be pursued for suprathermal electrons.

In view of the reasoning expressed by Eqs. (26) to (29), Eq. (25) can be read as

$$n = \{a + bx^2\}^{-1/(1-\beta)}, \tag{33}$$

where we have introduced the abbreviations

$$a = n_0^{-(1-\beta)}\left[\frac{\tau/\bar{t}}{(4\pi)^{(1+\beta)/2}}\right]^{-(1-\beta)/(1+\beta)},$$
$$b = n_0^{-(1-\beta)}\left[\frac{1-\beta}{1+\beta}\right]\left[\frac{1/\bar{t}}{2\beta\eta_{\mathrm{a}}}\right], \tag{34}$$

with the observation that $\eta_{\mathrm{a}} = \eta_{\mathrm{a}}(\Theta_{\mathrm{i}}, \Theta_{\mathrm{e}})$. Hence, combining both Eqs. (34), it follows from Eq. (22) that the length scale for the suprathermal diffusion of the electron gas is given by

$$\chi = \left[\frac{1-\beta}{1+\beta}\right]^{1/2}\left\{\frac{a^{-(1+\beta)/[2(1-\beta)]}(4\pi)^{(1+\beta)/4}}{n_0(2\beta b)^{1/2}}\right\}. \tag{35}$$

Equation (35) shows that χ may be calculated by computing n_0 from Eqs. (28) and (29), and a, b, and β from Eq. (33). We next contrast the classical with suprathermal diffusion of electrons in solar wind based on Eqs. (30) and (33).

V. SOLAR WIND

In Fig. 1, we show the linear electron plasma frequency f_{pe} as a function of the slow timescale, deduced from the spectrograms exhibited in Ref. [4]. The mean value $\bar{f}_{\mathrm{pe}}$ is computed from Eq. (28) and the electron equilibrium concentration n_0 is estimated by Eq. (29). Those results are displayed in Table I for both perihelions. They are consistent with the estimate for f_{pe} varying between 80 and 200 kHz, given in Ref. [4].

In Fig. 2, we show the electron concentration n as a function of the heliocentric distance x, exhibited in Ref. [4]. The data are fitted with the least-squares method [11] to Eq. (30) (classical diffusion) and Eq. (33) (suprathermal diffusion). The nonlinear-regression parameters [11], as given by Eqs. (31) (classical diffusion) and Eqs. (34) (suprathermal

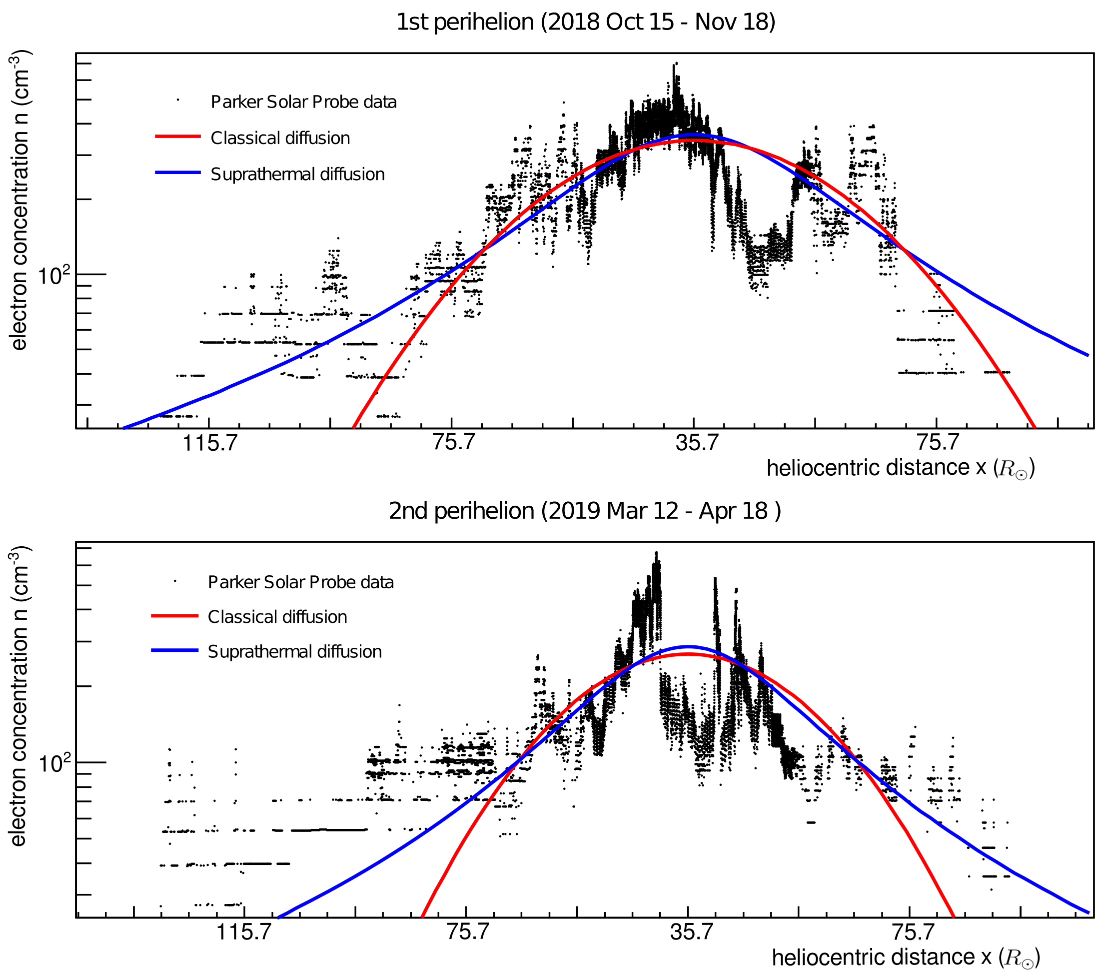

FIG. 2. The electron concentration n as a function of the heliocentric distance x, exhibited in Ref. [4]. The data are fitted with the least-squares method [11] to Eq. (30) (classical diffusion), and Eq. (33) (suprathermal diffusion).

diffusion), are displayed in Table II (first perihelion) and Table III (second perihelion).

It is found that $\beta \simeq 0.002$ ($\kappa \simeq 1.502$) for the first perihelion, and $\beta \simeq 0.003$ ($\kappa \simeq 1.503$) for the second perihelion (see the Appendix). Those results agree with the widely acknowledged value $\kappa \gtrsim 1.5$ of the spectral index for suprathermal electrons in solar wind [7]. It is also found that suprathermal effects increase the length scale of classical diffusion by one order of magnitude for both perihelions.

The coefficient of determination (a quantity that indicates how well a regression equation describes the relationship between a pair of observed variables; see Ref. [11]) r^2 is slightly larger, although still far from unit, for both

TABLE I. The mean value $\bar{f}_{\rm pe}$, computed by Eq. (28), from the data in Fig. 1, and the electron equilibrium concentration n_0, estimated by Eq. (29). Those results are consistent with the estimate for $f_{\rm pe}$ varying between 80 and 200 kHz, given in Ref. [4] for both perihelions.

Gas indexes	First perihelion	Second perihelion
$\bar{f}_{\rm pe}$ (kHz)	139	114
n_0 (cm^{-3})	239	161

TABLE II. The nonlinear-regression parameters [11], as given by Eqs. (31) (classical diffusion) and Eqs. (34) (suprathermal diffusion), for the first perihelion. It is found that $\beta \simeq 0.002$ ($\kappa \simeq 1.502$) (see the Appendix). This result agrees with the widely acknowledged value $\kappa \gtrsim 1.5$ of the spectral index for suprathermal electrons in solar wind [7]. It is also found that suprathermal effects increase the length scale of classical diffusion by one order of magnitude. The coefficient of determination [11] r^2 is slightly larger, although still far from unit, for the suprathermal regression curve. This suggests that the Strahl component [6] plays an important role in the electron diffusion throughout solar wind.

First perihelion	Classical diffusion	Suprathermal diffusion
parameters	$a_0 = 343$ cm^{-3}	$a = 2.77 \times 10^{-3}$ cm$^{3(1-\beta)}$
parameters	$b_0 = 8.40 \times 10^{-4} R_\odot^{-2}$	$b = 4.36 \times 10^{-6}$ cm$^{3(1-\beta)} R_\odot^{-2}$
β	1	0.002
κ	∞	1.502
χ ($R_\odot$)	130	1135
r^2	0.51	0.54

TABLE III. The nonlinear-regression parameters [11], as given by Eqs. (31) (classical diffusion) and Eqs. (34) (suprathermal diffusion), for the second perihelion. It is found that $\beta \simeq 0.003$ ($\kappa \simeq 1.503$) (see the Appendix). This result agrees with the widely acknowledged value $\kappa \gtrsim 1.5$ of the spectral index for suprathermal electrons in solar wind [7]. It is also found that suprathermal effects increase the length scale of classical diffusion by one order of magnitude. The coefficient of determination [11] r^2 is slightly larger, although still far from unit, for the suprathermal regression curve. This suggests that the Strahl component [6] plays an important role in the electron diffusion throughout solar wind.

Second perihelion	Classical diffusion	Suprathermal diffusion
parameters	$a_0 = 267$ cm^{-3}	$a = 3.49 \times 10^{-3}$ cm$^{3(1-\beta)}$
parameters	$b_0 = 10.4 \times 10^{-4} R_\odot^{-2}$	$b = 6.84 \times 10^{-6}$ cm$^{3(1-\beta)} R_\odot^{-2}$
β	1	0.003
κ	∞	1.503
χ ($R_\odot$)	93	995
r^2	0.49	0.53

suprathermal regression curves. This indicates that we are still distant from a satisfactory description of the electron diffusion in the solar wind. Since we have neglected magnetic fields, it suggests that the Strahl component [6] may play a particularly important role in such a phenomenon.

VI. CONCLUSION

We have derived a nonlinear partial differential equation describing the one-dimensional diffusion of an ionized fluid, based on a previously deduced equation of state of a suprathermal gas.

The fundamental solution of the diffusion equation has been found from an ansatz, in terms of a deformation parameter β, related to the spectral index κ. Then, a different class of differential equations has been added to the scope of mathematical physics.

The theory has been applied to describe the electron concentration in solar wind, as a function of the heliocentric distance, recently reported in Ref. [4]. Our main results are the following:

(i) We have found that $\beta \simeq 0.002$ ($\kappa \simeq 1.502$) for the first perihelion, and $\beta \simeq 0.003$ ($\kappa \simeq 1.503$) for the second perihelion. Those results agree with the widely acknowledged value $\kappa \gtrsim 1.5$ of the spectral index for suprathermal electrons in solar wind [7].

(ii) Suprathermal effects increase the length scale of classical diffusion by one order of magnitude for both perihelions. It should be emphasized that such a result does not depend on the microscopic details of the diffusion coefficient because our theory is based on a macroscopic formulation.

(iii) The coefficient of determination [11] r^2 is slightly larger, although still far from unit, for both suprathermal regression curves. This suggests that the Strahl component [6] plays an important role in the electron diffusion throughout solar wind.

The Parker Solar Probe data, which we have considered in this work, have been collected with a perihelion at $\sim$35.7 $R_\odot$. For heliocentric distances >3–6 $R_\odot$ (the lower boundary of the solar exosphere, commonly referred to as the thermopause or exobase), the Parker theory of solar wind [12–14] (which proposes a mechanism for the steady conversion of heat energy into kinetic energy) states that

(a) suprathermal electrons behave as an isothermal fluid, and

(b) curvature effects on one-dimensional diffusion are negligible.

Such statements have been continuously confirmed by several numerical treatments of observational data [15–18], including those in Ref. [4] itself. This is why our analysis has been based on an equation of state in an isothermal regime and on a diffusion equation in a flat geometry.

Anisotropic effects portray a vital part in the description of electrostatic solitons [19], electromagnetic instabilities [20], laser beams [21], quantum plasmas [22], and astrophysical plasmas [23]. It is important to further extend our theory by including magnetic fields.

Nonextensivity is intimately related to suprathermality, as may be attested to by studies in a self-similar solution of diffusion equations [24], radio-wave transmission in plasma sheaths [25], energy spectrum in plasma expansions [26], and generation of dispersive shock waves [27]. Hence, it is interesting to further discuss our theory in the context of nonextensive statistics. The aforementioned issues shall be addressed in forthcoming communications.

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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APPENDIX: EQUATION OF STATE OF KAPPA GAS

For a detailed discussion of the method that we have recently developed in order to derive equations of state of non-Maxwellian plasmas, see Refs. [28–33]. Here, we restrict ourselves to a Kappa gas.

The Kappa distribution of particle concentration is given by [34]

$$n = n_0 \left[1 \pm \frac{1}{(\kappa - 3/2)} \left(\frac{|q|\Phi}{k_B \Theta} \right) \right]^{-(\kappa - 1/2)}, \qquad \text{(A1)}$$

where the Kappa Θ temperature is related to the Maxwellian T temperature through

$$\Theta = \left(\frac{\kappa}{\kappa - 3/2} \right) T, \qquad \text{(A2)}$$

with κ denoting the so-called spectral index ($3/2 < \kappa < \infty$). Equation (A2) shows that $\Theta \geqslant T$ for all possible values of κ. This is why particles following Eq. (A1) are said to be suprathermal particles. In the limit $\kappa \to \infty$, Eq. (A2) shows that $\Theta \to T$. As a result, Eq. (A1) recovers the Boltmann relation [35],

$$n = n_0 \exp\left(\mp\frac{|q|\Phi}{k_B T}\right), \tag{A3}$$

in the same approximation.

Neglecting the frequency ν in Eq. (1), we get

$$\mp|q|n\Phi_x - P_x = 0. \tag{A4}$$

Solving Eq. (A1) for $|q|\Phi$, we obtain

$$|q|\Phi = \pm k_B\Theta(\kappa - 3/2)\left[\left(\frac{n}{n_0}\right)^{-1/(\kappa-1/2)} - 1\right]. \tag{A5}$$

Differentiating Eq. (A5) with respect to x, we get

$$|q|\Phi_x = \mp k_B\Theta\left(\frac{\kappa-3/2}{\kappa-1/2}\right)\left(\frac{n}{n_0}\right)^{-(\kappa+1/2)/(\kappa-1/2)}\left(\frac{n}{n_0}\right)_x. \tag{A6}$$

Substituting Eq. (A6) in Eq. (A4), we find

$$P_x = nk_B\Theta\left(\frac{\kappa-3/2}{\kappa-1/2}\right)\left(\frac{n}{n_0}\right)^{-(\kappa+1/2)/(\kappa-1/2)}\left(\frac{n}{n_0}\right)_x, \tag{A7}$$

which may be easily put in the form of Eq. (2), provided that we introduce the abbreviation

$$\beta = \frac{\kappa-3/2}{\kappa-1/2}. \tag{A8}$$

Equation (A8) shows that $0 < \beta < 1$, for $3/2 < \kappa < \infty$. In the limit $\kappa \to \infty$ [$\beta \to 1$], Eq. (A7) [Eq. (2)] recovers the pressure gradient $P_x = k_B T n_x$ (recall $\Theta \to T$ in the same approximation) of a classical gas of charged particles in the isothermal regime [35].

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